Our equation is

$$
\begin{aligned}
&\left(K_X + \frac{\pi'^2}{a^2} K_{XX}\right) \pi'' = \\
&\qquad a^2 \left(K_\pi - \frac{\pi'^2}{a^2} K_{\pi X}\right) - \mathcal{H} \left(2K_X - \frac{\pi'^2}{a^2} K_{XX}\right) \pi' \\
&\qquad + K_X \nabla^2 \pi + K_{\pi X} \partial_i \pi \partial^i \pi + 2 \frac{K_{XX}}{a^2} \pi' \partial_i \pi \partial^i \pi' - \mathcal{H} \frac{K_{XX}}{a^2} \pi' \partial_i \pi \partial^i \pi \\
&\qquad - \frac{K_{XX}}{a^2} \partial_i \pi \partial_j \pi \partial^j \partial^i \pi
\end{aligned}
\tag{0.1}
$$

where

$$
\begin{aligned}
K &= -g_0 + g_2 (X - \hat{X})^2 + g_4 (X - \hat{X})^4 \\
K_X &= 2 g_2 (X - \hat{X}) + 4 g_4 (X - \hat{X})^3 \\
K_{XX} &= 2 g_2 + 12 g_4 (X - \hat{X})^2 \\
K_\pi &= 0 \\
K_{\pi X} &= 0
\end{aligned}
$$

where

$$
X = \frac{1}{2a^2} \left(\pi'^2 - \partial_i \pi \partial^i \pi\right)
$$

1 Rewriting the PDE

We write down the PDE in the following form to simplify our numerics implementation,

$$
\begin{aligned}
\pi'' = \frac{1}{\left(K_X + \frac{\pi'^2}{a^2} K_{XX}\right)} \Bigg[&- \mathcal{H} \left(2K_X - \frac{\pi'^2}{a^2} K_{XX}\right) \pi' + K_X \nabla^2 \pi + K_{\pi X} \partial_i \pi \partial^i \pi \\
&+ 2 \frac{K_{XX}}{a^2} \pi' \partial_i \pi \partial^i \pi' - \mathcal{H} \frac{K_{XX}}{a^2} \pi' \partial_i \pi \partial^i \pi - \frac{K_{XX}}{a^2} \partial_i \pi \partial_j \pi \partial^j \partial^i \pi \Bigg]
\end{aligned}
\tag{1.1}
$$

2 Numerical implementation

2.1 Euler method

$$
\pi^{\mathrm{n+1}}_{(\mathrm{i,j,k})} = \pi^{\mathrm{n}}_{(\mathrm{i,j,k})} + \pi'^{\ \mathrm{n}}_{(\mathrm{i,j,k})} \ \Delta\tau \tag{2.1}
$$

$$
\pi'^{\mathrm{n+1}}_{(\mathrm{i,j,k})} = \pi'^{\mathrm{n}}_{(\mathrm{i,j,k})} + \pi''^{\ \mathrm{n}}_{(\mathrm{i,j,k})} \ \Delta\tau \tag{2.2}
$$

$$
\pi^{\mathrm{n+2}}_{(\mathrm{i,j,k})} = \pi^{\mathrm{n+1}}_{(\mathrm{i,j,k})} + \pi'^{\ \mathrm{n+1}}_{(\mathrm{i,j,k})} \ \Delta\tau \tag{2.3}
$$

where the superscript n and subscript $(\mathrm{i,j,k})$ shows respectively the time step and the position on the lattice, i.e $\pi'^{\mathrm{n}}_{(\mathrm{i,j,k})}$ is the field π' at discrete time step (n) and point $((\mathrm{i,j,k}))$ on the lattice. To find $\pi''^{\ \mathrm{n}}_{(\mathrm{i,j,k})}$ we discretize our equation as

$$
\begin{aligned}
\pi''^{\,\mathrm{n}}_{(\mathrm{i,j,k})} = & \frac{1}{\left(K_X + \frac{(\pi'^{\,n})^2}{a^2} K_{XX}\right)} \left[a^2 \left(K_\pi - \frac{(\pi'^{\,\mathrm{n}})^2}{a^2} K_{\pi X} \right) - \mathcal{H} \left(2K_X - \frac{(\pi'^{\,\mathrm{n}})^2}{a^2} K_{XX} \right) \pi'^{\,\mathrm{n}} \right. \\
& + K_X \nabla^2 \pi^{\mathrm{n}} + K_{\pi X} \partial_i \pi^{\mathrm{n}} \partial^i \pi^{\mathrm{n}} + 2 \frac{K_{XX}}{a^2} \pi'^{\,\mathrm{n}} \partial_i \pi^{\mathrm{n}} \partial^i \pi'^{\,\mathrm{n}} - \mathcal{H} \frac{K_{XX}}{a^2} \pi'^{\,\mathrm{n}} \partial_i \pi^{\mathrm{n}} \partial^i \pi^{\mathrm{n}} \\
& \left. - \frac{K_{XX}}{a^2} \partial_i \pi^{\mathrm{n}} \partial_j \pi^{\mathrm{n}} \partial^j \partial^i \pi^{\mathrm{n}} \right]
\end{aligned} \tag{2.4}
$$

2.2 The leap-frog method

We use the Newton-Stormer-Verlet-leapfrog method [**?**] to solve the two first order partial differential equations for the linear k-essence scalar field on the lattice,

$$
\pi'^{\,\mathrm{n}+\frac{1}{2}}_{(\mathrm{i,j,k})} = \pi'^{\,\mathrm{n}-\frac{1}{2}}_{(\mathrm{i,j,k})} + \pi''^{\,\mathrm{n}}_{(\mathrm{i,j,k})} \, \Delta\tau \tag{2.5}
$$

$$
\pi^{\mathrm{n}+1}_{(\mathrm{i,j,k})} = \pi^{\mathrm{n}}_{(\mathrm{i,j,k})} + \pi'^{\,\mathrm{n}+\frac{1}{2}}_{(\mathrm{i,j,k})} \Delta\tau \tag{2.6}
$$

It is important to note that in our scheme we have split the background from perturbations, as a result we have access to the $\mathcal{H}^{\mathrm{n}+\frac{1}{2}}$ independently of the value of the fields. Moreover we compute $\pi^{\mathrm{n}+\frac{1}{2}}_{(\mathrm{i,j,k})}$ as,

$$
\pi'^{\mathrm{n}}_{(\mathrm{i,j,k})} = \pi'^{\mathrm{n}-\frac{1}{2}}_{(\mathrm{i,j,k})} + \pi''^{\mathrm{n}}_{(\mathrm{i,j,k})} \frac{\Delta\tau}{2} \tag{2.7}
$$

To do this in the code we need to also define the $\pi''^{\mathrm{n}}_{(\mathrm{i,j,k})}$ as a new field to make sure that we update the π' field correctly.