C.W

Cheapter 03

Complex differentiation

Derivatives (Def^n)

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Example :

Using def^n of derivatives,

find $f'(z)$, where $f(z) = \frac{2z-3i}{3z-2i}$ at the point $z=-i$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Where, $f(z+\Delta z) = \frac{2(z+\Delta z)-3i}{3(z+\Delta z)-2i}$

Now, $$f'(z) = \lim_{\Delta z \to 0} \frac{\frac{2(z+\Delta z)-3i}{3(z+\Delta z)-2i} - \frac{2z-3i}{3z-2i}}{\Delta z}$$

$$f'(i) = \lim_{\Delta z \to 0} \frac{\frac{2(-i+\Delta z)-3i}{3(-i+\Delta z)-2i} - \frac{2(-i)-3i}{3(-i)-2i}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\frac{\frac{-5i + 2\Delta z}{-5i + 3\Delta z} - \frac{-5i}{-5i}}{\Delta z} \right)$$

$$= \lim_{\Delta z \to 0} \left(\frac{\frac{-5i + 2\Delta z}{-5i + 3\Delta z} - 1}{\Delta z} \right)$$

$$= \lim_{\Delta z \to 0} \left(\frac{\cancel{-5i} + 2\Delta z + \cancel{5i} - 3\Delta z}{(-5i + 3\Delta z)\,\Delta z} \right)$$

$$= \lim_{\Delta z \to 0} \frac{-\cancel{\Delta z}}{(-5i + 3\Delta z)\,\cancel{\Delta z}}$$

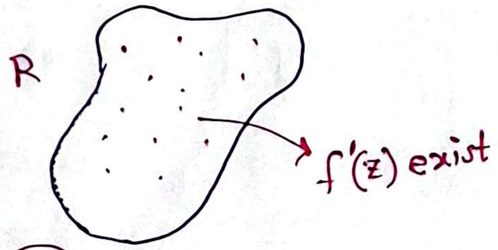
$$= \lim_{\Delta z \to 0} \frac{-1}{-5i + 3\Delta z} = \frac{-1}{-5i} = \frac{1}{5i}$$

$$= \frac{i}{5i^2}$$

$$= -\frac{i}{5} \quad \text{Ans:}$$

Analytic function

If $f'(z)$ exists for all points in z in the region R, then, $f(z)$ is called analytic function.



$f(z)$ → analytic function.

C-R equation

(Cauchy - Riemann Equation)

$W = U + iV$; $U = U(x,y)$; $V = V(x,y)$

$$\boxed{\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \; ; \; \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}}$$

C-R eq^n ⇒ Analytic function ⟹ $f'(z)$ exist.

Example : 2

Show that, $f(z) = z - \bar{z}$ is non-analytic.

or, S.T $f'(z)$ doesn't exist.

Soln :

$$f(z) = z - \bar{z}$$

$$= x+iy - (x-iy) = x+iy-x+iy$$

$$W = f(z) = 2iy$$

$$\Rightarrow u+iv = 2iy$$

$$\therefore u = 0$$

$$v = 2y$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = 2$$

$$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$\therefore$ $f(z)$ is non-analytic and also, $f'(z)$ doesn't exist.

$\nabla \rightarrow$ nabla

Harmonic function :

$\nabla^2 u = 0$ $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \rightarrow$ 'u' is a harmonic function.

$\nabla^2 v = 0$ $\quad \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0 \rightarrow$ 'v' is a harmonic function

$$\because \nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \longrightarrow \text{Laplacian operator.}$$

Must আসবে exam এ.

Example : 3

3.7 + 3.8

P-89

a) Prove that, $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

b) Find v such that $f(z)$ is analytic. $\rightarrow$ C-R eq^n

c) Find $f(z)$ in terms of z.

Soln :

(a) $U = e^{-x} \cdot x\sin y - e^{-x} y\cos y$

$$\frac{\partial U}{\partial x} = \sin y\left(e^{-x}(1) + x \cdot e^{-x}(-1)\right) - y\cos y \; e^{-x}(-1)$$

$$= e^{-x}\sin y - xe^{-x}\sin y + y\cos y \cdot e^{-x}$$

$$\frac{\partial^2 U}{\partial x^2} = -e^{-x}\sin y - \sin y\left(e^{-x} - xe^{-x}\right) - e^{-x} y\cos y$$

$$= -e^{-x}\sin y - e^{-x}\sin y + x\sin y\, e^{-x} - e^{-x} y\cos y$$

$$\frac{\partial^2 U}{\partial y^2} =$$

$$= 2e^{-x}\sin y - x\sin y\, e^{-x} + e^{-x} y\cos y.$$

$$\because \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ 'U' is a harmonic function.

Soln: (b)

Since analytic,

from C-R eqn we know,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

...... (1) ...... (ii)

from eqn(1),

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial y} = e^{-x}\sin y - e^{-x}x\sin y + e^{-x}y\cos y$$

$$\Rightarrow \int \partial v = \int \left(e^{-x}\sin y - e^{-x}x\sin y + e^{-x}y\cos y\right) dy$$

$$\Rightarrow v = -e^{-x}\cos y + e^{-x}\cos y + e^{-x}\int y\cos y \, dy \quad \text{......(iii)}$$

$$\because \int y\cos y\, dy = y\int \cos y\, dy - \int\left(\frac{d}{dy}(y)\int \cos y\, dy\right) dy$$

$$= y\sin y - \int \sin y\, dy$$

$$= y\sin y + \cos y$$

from eq^n (iii),

$$V = -e^{-x}\cos y + x e^{-x}\cos y + e^{-x}(y\sin y + \cos y) + K(x)$$

2nd Part

from eq^n (ii)

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow x e^{-x}\cos y = -e^{-x}\cos y + e^{-x} y\sin y$$

$$= e^{-x}\cos y + \cos y(-xe^{-x} + e^{-x}) - e^{-x}(y\sin y + \cos y)$$

$$+ K'(x)$$

$$\Rightarrow K'(x) = 0$$

$$\Rightarrow \frac{dK(x)}{dx} = 0$$

$$\Rightarrow \int dk(x) = \int 0\,dx \quad \Rightarrow k(x) = 0 + c$$

$$k(x) = c$$

$$\therefore\ V = xe^{-x}\cos y + e^{-x} y \sin y + c$$

(Ans:)

08.03.25

(e)

Sol^n (e)

$$f(z) = u + iv = e^{-x}(x\sin y - y\cos y) + ie^{-x}(y\sin y + x\cos y)$$

$\boxed{y=0}$

$$f(z) = e^{-x}(\cancel{x\sin y}\ x\sin(0) - 0\cdot\cos(0)) + ie^{-x}(0\cdot\sin(0) + x\cos(0))$$

$$\Rightarrow f(z) = ie^{-x}.x$$

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}$$

Replacing "x" by "z."

$$f(z) = ize^{-z}$$

Ans:

3.24

(P-96) Evaluate $\lim_{z \to 0} (\cos z)^{\frac{1}{z^2}}$

Soln:

let, $y = (\cos z)^{1/z^2}$

$$\Rightarrow \ln y = \frac{1}{z^2} \ln(\cos z)$$

$$\Rightarrow \lim_{z \to 0} \ln y = \lim_{z \to 0} \frac{\ln(\cos z)}{z^2} \quad \left(\frac{0}{0}\right) \text{[indeterminant form]}$$

$$\Rightarrow \lim_{z \to 0} \ln y = \lim_{z \to 0} \frac{\frac{d}{dz}(\ln(\cos z))}{\frac{d}{dz}(z^2)} \quad [\because \text{L. Hospital Rule}]$$

$$= \lim_{z \to 0} \frac{\frac{1}{\cos z} \cdot (-\sin z)}{2z}$$

$$\left[\lim_{z \to 0} \frac{\tan z}{z} = 1\right]$$

$$= -\frac{1}{2} \lim_{z \to 0} \left(\frac{\tan z}{z}\right) \quad \frac{0}{0}$$

[L. Hospital rule]

$$= -\frac{1}{2} \lim_{z \to 0} \left(\frac{\sec^2 z}{1}\right) = -\frac{1}{2}$$

$$\therefore \lim_{z\to 0} \ln y = -\frac{1}{2}$$

$$\Rightarrow \lim_{z\to 0} e^{\ln y} = e^{-\frac{1}{2}}$$

$$\Rightarrow \lim_{z\to 0} y = e^{-\frac{1}{2}}$$

$$\therefore \lim_{z\to 0} (\cos z)^{1/z^2} = e^{-\frac{1}{2}}$$ Ans:

\# Evaluate $\lim_{z\to 0} \left(\frac{\sin z}{z}\right)^{1/z^2}$

Soln:

let, $y = \left(\frac{\sin z}{z}\right)^{1/z^2}$

$$\Rightarrow \ln y = \frac{1}{z^2} \ln\left(\frac{\sin z}{z}\right)$$

$$\Rightarrow \lim_{z\to 0} \ln y = \lim_{z\to 0} \frac{\ln\left(\frac{\sin z}{z}\right)}{z^2}$$

$$\Rightarrow \lim_{z\to 0} \ln y = \lim_{z\to 0} \frac{\frac{d}{dz} \ln\left(\frac{\sin z}{z}\right)}{\frac{d}{dz}(z^2)}$$ [L. Hospital Rule]

rule $\longrightarrow$ $\boxed{\frac{d}{dz}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dv} - u\frac{dv}{dz}}{v^2}}$

$$\boxed{\begin{array}{l}\lim\limits_{z\to 0} \frac{\sin z}{z} = 1 \\ \lim\limits_{z\to 0} \frac{z}{\sin z} = 1\end{array}}$$

$$= \frac{\lim\limits_{z\to 0} \frac{1}{\frac{\sin z}{z}}\left(\frac{z\cos z - \sin z}{z^2}\right)}{2z}$$

$$= \lim_{z\to 0} \frac{\frac{1}{\frac{\sin z}{z}}\left(\frac{z\cos z - \sin z}{z^2}\right)}{2z}$$

$$= \lim_{z\to 0} \frac{z}{\sin z} \cdot \lim_{z\to 0}\left(\frac{z\cos z - \sin z}{2z^3}\right)$$

$$= \lim_{z\to 0}\left(\frac{z\cos z - \sin z}{2z^3}\right) \quad \left[\lim_{z\to 0}\frac{z}{\sin z} = 1\right]$$

$$= \lim_{z\to 0} \frac{\frac{d}{dz}\left(z\cos z - \sin z\right)}{\frac{d}{dz}\left(2z^3\right)}$$

$$= \lim_{z\to 0}\left(\frac{-z\sin z + \cos z - \cos z}{6z^2}\right)$$

$$= \lim_{z\to 0}\left(\frac{-\cancel{z}\sin z}{6\cancel{z^2}z}\right) = -\frac{1}{6}\lim_{z\to 0}\frac{\sin z}{z}$$

$$= -\frac{1}{6}$$

$\therefore \lim_{z \to 0} \ln y = -\frac{1}{6}$

$\Rightarrow \lim_{z \to 0} e^{\ln y} = e^{-\frac{1}{6}}$

$\Rightarrow \lim_{z \to 0} y = e^{-\frac{1}{6}}$

$\Rightarrow \lim_{z \to 0} \left(\frac{\sin z}{z}\right)^{1/z^2} = e^{-\frac{1}{6}}$ Ans:

Ch-3 $\rightarrow$ i) Def^n of derivative

ii) Analytic, harmonic ......

iii) Limit cal^n (using L. Hospital Rule).