CSE-330 Chapter-02

Polynomials

$3x^2 + 5x + 3 = 0$

$\Rightarrow \underbrace{3}_{a_0}x^0 + \underbrace{5}_{a_1}x^1 + \underbrace{3}_{a_2}x^2 \rightarrow$ In general form

Highest power : 02
Degree - 02

General form

$$P_n(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_nx^n$$

Degree

Co-efficients

$$P_3(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3$$

Highest power of the polynomial is the degree

$$\text{Coeff} = [a_0, a_1, a_2, a_3] = 4$$

$$\left[\text{Number of co-efficient} = \text{No. of degree} + 1 \right]$$

Number of coefficients is always one more than the degree

$$P_{25}(x) = 2x^0 + 3x^1 + \dots 25x^{25}$$

Number of coefficients $= 25 + 1 = 26$

Vector Space : A region where we can add ($\pm$) or multiply with scalars.

Polynomial : $P_2(x) = 1 + x + 2x^2$ ← Highest power == Degree

$P_3(x) = x^3$

$\langle P_2(x) + P_3(x) = 1 + x + 2x^2 + x^3 \rangle$ Adding these two gives us another polynomial.

$P_3(x) * 5 = 5x^3$

$P_2(x) * 3 = 3 + 3x + 6x^2$

Multiplication

Basis is a set of vectors that spans the place!

Polynomial:

$$P_3(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3$$

Co-efficients

$$\text{Basis} = \{x^0, x^1, x^2, x^3\}$$

Natural basis

Dimensional Space:

How many dimensions ?

3 degree polynomial, basis = $\underbrace{\{1, x, x^2, x^3\}}_{4}$

Basis's dimension : 4

Basis's dimension = degree + 1

Functional Space

→ Natural functions → has ∞ degree, ∞ dimension

$f(x) = 2 + 3x + 10x^2 + 14x^3 + \ldots$

Basis : $\{1, x, x^2, x^3, \ldots\}$ → Infinite values.

We can reproduce it using polynomials with some error.

$P_2(x) = 2 + 3x + 10x^2$

$f(x) \in V^{\infty}$ → infinite dimension vector space ← Function belongs to ∞ dimension vector space.

$P_n(x) \in V^{n+1}$ ← Polynomial with degree n, belongs to (n+1) dimension vector space.

Can be represented

$f(x) = 2 + 3x + 10x^2 + 14x^3 + \ldots$

→ Error ; Truncation Error

আমরা এটাকে represent করতে পারি না

$P_2(x) = a_0 + a_1 x + a_2 x^2$

$P_5(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$

if we increase the degree, we will be able to reduce the error.

This is called

"Weierstrass Approximation Theorem"

Error = $|f(x) - P_n(x)|$

$n \uparrow$

Error $\downarrow$

(i) Example

$P_{37}(x) = \ldots$

Basis = $\{\ldots\}$ = 38

Dimension = Degree + 1
= 37 + 1
= 38

Degree = 37

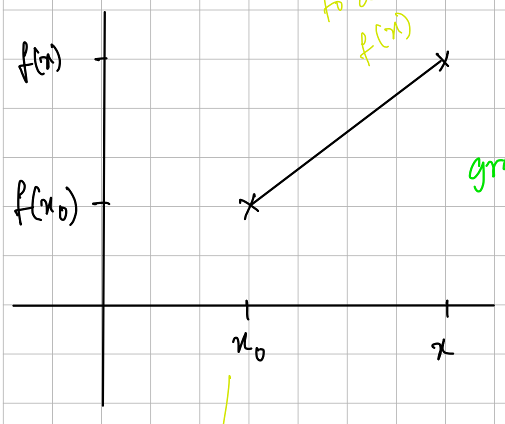
Basis = $\{1, x, x^2, \ldots, x^{37}\}$

$$f(x) \simeq p_n(x)$$

↳ for a very large value of n.

Taylor Series

Taylor Series



We have to determine $f(x)$

given

$$\text{gradient, } f'(x) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow f'(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow f'(x)(x - x_0) = f(x) - f(x_0)$$

$$\Rightarrow f(x) = f'(x)(x - x_0) + f(x_0)$$

↳ Taylor series for straight line

Actual taylor series → more complex & infinitely large

Taylor series, (Not finite)

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x - x_0)^2}{2!} + \frac{f'''(x_0)(x - x_0)^3}{3!} + \dots$$

Proof of Taylor Series

(Remember!)

(Might come in the exam) → ASK JTTSC

Step 01 :

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots$$

$$f'(x) = 0 + a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \dots$$

$$f''(x) = 0 + 0 + 2a_2 + 3 \times 2\, a_3(x - x_0) + \dots$$

$$f'''(x) = 0 + 0 + 0 + 3 \times 2 a_3 + \dots$$

Step 02: Let, $x = x_0$,

$$f(x) = a_0 + a_1(x_0 - x_0) + a_2(x_0 - x_0)^2 + a_3(x_0 - x_0)^2 + \dots$$

$$= a_0 + 0 + 0 + 0$$

$$f(x) = a_0$$

Now.

$$f'(x) = a_1$$

$$f''(x) = 2a_2$$

$$f'''(x) = 3 * 2 a_3$$

$$a_0 = f(x)$$

$$a_1 = f'(x)$$

$$a_2 = \frac{f''(x)}{2} \rightarrow \frac{f''(x)}{2!}$$

$$a_3 = \frac{f'''(x)}{3*2} \rightarrow \frac{f'''(x)}{3!}$$

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots$$

$$= f(x) + f'(x)(x - x_0) + \frac{f''(x)}{2!}(x - x_0)^2 + \frac{f'''(x)}{3!}(x - x_0)^3 + \dots$$ (showed).

Example : $\sin(x) = x - \frac{x^3}{3!} + \frac{5x^5}{5!} + \dots$

Step 01 : Start with Taylor Series equation

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x - x_0)^2}{2!} + \frac{f'''(x_0)(x - x_0)^3}{3!} + \frac{f''''(x_0)(x - x_0)^4}{4!}$$

$$+ \frac{f^5(x_0)(x - x_0)}{5!}$$

Step 02:

Check Question! If the value of x_0 is given, use that or $x_0 = 0$

$$f(x) = \sin(0) + \cos(0)(x) + \frac{-\sin(0)x^2}{2!} + \frac{-\cos(0)(x^3)}{3!} + \frac{\sin(0)x^4}{4!} + \frac{\cos(0)x^5}{5!}$$

$\sin(0) = 0$ && $\cos(0) = 1$

$$= 0 + x + 0 + \frac{-x^3}{3!} + 0 + \frac{x^5}{5!} \rightarrow x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

(x) Example : Find $\sin(0.1)$ using Taylor Series

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

Upto 1 term

$$\sin(0.1) \simeq 0.1$$

Upto 2^{nd} term

$$\sin(0.1) \simeq 0.1 - \frac{(0.1)^3}{3!} = 0.099833\dots$$

Upto 3^{rd} term

$$\sin(0.1) \simeq 0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} = 0.0998334167\dots$$

Actual value

$$\sin(0.1) = 0.0998334166\mathbf{5}$$

↳ If we consider upto the 9th term, we get the correct ans upto 9 significant no.

—— NHL 1st video ——

ITTSC 4th class – 18.10.25

$P_3(x) = x^3 + 3x^2 + 3x + 2 = 0 \longrightarrow$ 3rd degree polynomial.

$P_n(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$

↳ n-degree polynomial

$$\begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \\ \end{array}$$
$$\begin{bmatrix} 2, & 3, & 3, & 1 \end{bmatrix}$$

• No. of co-efficients = Degree + 1

Basis of a polynomial

↳ Dimensional space = Degree + 1

Interpolation

Taylor Series

Taylor Series is a polynomial

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \dots$$

Remember the formula!

Question

Taylor Series

$f(x) = \sin x$

$x_0 = 0$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$f(x_0) = \sin(0) = 0$

$f'(x_0) = \cos(0) = 1$

$f''(x_0) = -\sin(0) = 0$

$f'''(x_0) = -\cos(0) = -1$

$f^4(x_0) = \sin(0) = 0$

↳ we reached the first term

- Just calculate the values of the derivative
- Substitute them inside the formula
- Take 4/5 values.

$\left(\underline{e^x \sin x} \; , \; \underline{\cos x} \right) \longrightarrow$ Check practice problems

Size (sqft)	Price
1000	60
1200	68
1800	90

We have the data but we do not know the function

$f(x) \longrightarrow P_n(x)$

we find the polynomial of the function.

- Make sure the polynomial matches with the function for nodal points

$$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$P_3(1000) = a_0 + a_1 \cdot 1000 + a_2(1000)^2 + a_3(1000)^3 = 60$

$P_3(1200) = a_0 + a_1(1200) + a_2(1200)^2 + a_3(1200)^3 = 68$

$P_3(1800) = a_0 + a_1(1800) + a_2(1800)^2 + a_3(1800)^3 = 90$

n, We took a three degree polynomial but there are 04 variables.

যতগুলো points থাকবে তার চেয়ে $(n-1)$ polynomial solve করতে পারবো।

↳ এটার কম ও করতে পারি

$$P_2(x) = 50 + 0.049x - 4.16\times10^{-6}x^2$$

If they ask for a 1 deg polynomial

$x = 1700 \rightarrow x_1 = 1200,\ x_2 = 1800$ — Use these two

$x = 1100 \rightarrow x_0 = 1000,\ x_1 = 1200$ — Use these two

↳ Writing it in $Ax = B$ form

$$\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix}$$

General form ← Vandermonde's matrix

$$\begin{pmatrix} 1 & 1000 & 1000^2 \\ 1 & 1200 & 1200^2 \\ 1 & 1800 & 1800^2 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 60 \\ 68 \\ 90 \end{pmatrix}$$

First, they will ask us to find the polynomial.

↳ They will then ask to find the vandermonde's matrix

$$x = A^{-1}B$$

(Just inverse করে দেওয়া লাগবে)

— ITTSe —

Lagrange Polynomial

$$P_n(x) = \sum_{k=0}^{n} f(x_k)\underbrace{l_k(x)}$$

→ Lagrange Basis

$$= f(x_0)l_0(x) + f(x_1)l_1(x) + f(x_2)l_2(x)$$

যেটার টা বের করবো ওইটা বাদে বাকিগুলার multiply

		x	$f(x)$	$l_0(x)$
x_0	0	1	6	$\frac{x-x_1}{x_0-x_1}\times\frac{x-x_2}{x_0-x_2} = \frac{(x-2)(x-3)}{(1-2)(1-3)}$
x_1	1	2	-1	$\frac{x-x_0}{x_1-x_0}\times\frac{x-x_2}{x_1-x_2} = \frac{(x-1)(x-3)}{(2-1)(2-3)}$
x_2	2	3	8	$\frac{x-x_0}{x_2-x_0}\times\frac{x-x_1}{x_2-x_1} = \frac{(x-1)(x-2)}{(3-1)(3-2)}$

$x = x_0,\ l_0(x) = 1$

$x = x_1,\ l_0(x) = 1$

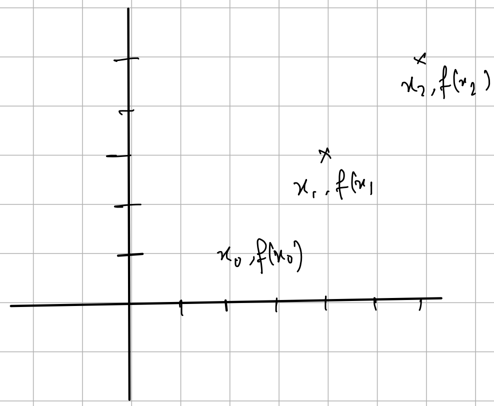
$x = x_2,\ l_0(x) = 2$

$$P_n(x_0) = 61$$

$$P_n(x_1) = 0 + (-1)(1)$$

Polynomial Interpolation - NHL Playlist

Given $(n+1)$ datapoints



		house size	Rent	
$n+1$	x_0	20	200	$f(x_0)$
	x_1	30	250	$f(x_1)$
	x_2	40	275	$f(x_2)$

$P_n(x)$

$$P_n(x) = P_2(x) = a_0 x^0 + a_1 x^1 + a_2 x^2$$ → Polynomial generate bolte co-efficient figure out kora bhujay

(i) Vandermond Matrix

Given,

	x time	$f(x) = y$ velocity	
$n+1$	1 x_0	40	y_0
	2 x_1	50	y_1
	3 x_2	55	y_2

Vandermond Matrix

$$P_n(x) = P_2(x) = a_0 x^0 + a_1 x^1 + a_2 x^2$$

$$= a_0 1^0 + a_1 1^1 + a_2 1^2 = 40 \quad -(i)$$

$$= a_0 2^0 + a_1 2^1 + a_2 2^2 = 50 \quad -(ii)$$

$$= a_0 3^0 + a_1 3^1 + a_2 3^2 = 55 \quad -(iii)$$

Separating the values of x

$$\underbrace{\begin{bmatrix} 1^0 & 1^1 & 1^2 \\ 2^0 & 2^1 & 2^2 \\ 3^0 & 3^1 & 3^2 \end{bmatrix}}_{X} \times \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 40 \\ 50 \\ 55 \end{bmatrix}}_{Y}$$

$$X \cdot A = Y$$

$$A = X^{-1} Y$$

↳ Vandermond Matrix

$$\begin{bmatrix} x_0^0 & x_0^1 & x_0^2 \\ x_1^0 & x_1^1 & x_1^2 \\ x_2^0 & x_2^1 & x_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \longrightarrow \text{General form}$$

Example

Time	Velocity
15	362.8
20	517.3

node = n+1
= 2

Degree = 1

$$P_1(x) = a_0 x^0 + a_1 x^1$$

$$P_1(15) = a_0 15^0 + a_1 15 = 362.8$$
$$P_2(20) = a_0 + a_1 \cdot 20 = 517.3$$

$$\boxed{XA = Y}$$

$$\begin{bmatrix} x_0^0 & x_0^1 \\ x_1^0 & x_1^1 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 362.8 \\ 517.3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 15 \\ 1 & 20 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 362.8 \\ 517.3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 15 \\ 1 & 20 \end{bmatrix}^{-1} \begin{bmatrix} 362.8 \\ 517.3 \end{bmatrix}$$

$$\boxed{A = X^{-1} Y}$$

We want this because A contains the value of the co-efficients.

$$a_0 = \begin{bmatrix} -100.85 \\ 30.91 \end{bmatrix}$$

$$P_2(x) = -100.85 + 30.91x$$

Example

Example: Given the time and velocity v(t) , find an interpolating Polynomial of velocity that goes through the datapoints using Vandermond Matrix. Also, find the approx. value of acceleration at Time t = 7 second.

Nodes = 3
Degree = 3-1
= 2

	time(t)	Velocity, v(t)	
x_0	3	11	y_0
x_1	5	21	y_1
x_2	7	26	y_2

$$P_2(x) = a_0x_0 + a_1x_1 + a_2x_2$$

$$XA = Y \longrightarrow \text{Vandermond Matrix Formula}$$

$$\begin{pmatrix} 3^0 & 3^1 & 3^2 \\ 5^0 & 5^1 & 5^2 \\ 7^0 & 7^1 & 7^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 11 \\ 21 \\ 26 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 3^0 & 3^1 & 3^2 \\ 5^0 & 5^1 & 5^2 \\ 7^0 & 7^1 & 7^2 \end{pmatrix}^{-1} \begin{pmatrix} 11 \\ 21 \\ 26 \end{pmatrix}$$

$$= \begin{pmatrix} 4.375 & -5.25 & 1.875 \\ -1.5 & 2.5 & -1 \\ 0.125 & -0.25 & 0.125 \end{pmatrix} \begin{pmatrix} 11 \\ 21 \\ 26 \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -13.37 \\ 10 \\ 0.0625 \end{pmatrix}$$

$$P_2(x) = -13.37 + 10x + 0.0625x^2 \longrightarrow \text{velocity}$$

$$\text{Acceleration} = \frac{dP_2(x)}{dt} = 0 + 10 + 0.0625 \times 2x$$

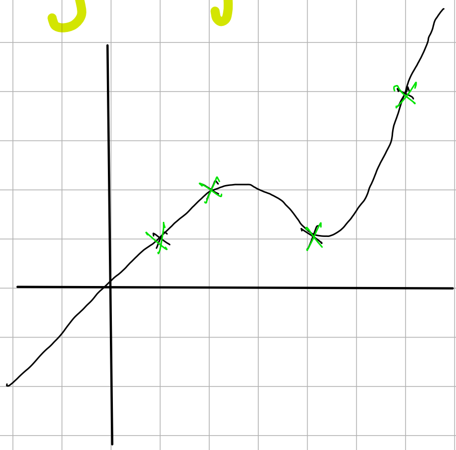
$$A(7) = 10 + 0.0625(7)$$
$$= 10.0625\ m/s^2$$

Lagrange

(ii) Lagrange Interpolation

x_0	$f(x_0)$
x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$

(n+1) nodes



degree = n = 3

— Vandermond Matrix

- Takes a lot of time, lengthy and slow.
- One of the biggest drawbacks for bigger nodes

$$P_3(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3$$

$$P_3(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2) + l_3(x)f(x_3)$$

Langrange Basis

$$l_0(x) = \frac{x-x_1}{x_0-x_1}\cdot\frac{x-x_2}{x_0-x_2}\cdot\frac{x-x_3}{x_0-x_3}$$

They follow a pattern:
- x on top and the one we are calculating at the bottom

$$l_1(x) = \frac{x-x_0}{x_1-x_0}\cdot\frac{x-x_2}{x_1-x_2}\cdot\frac{x-x_3}{x_1-x_3}$$

$$l_2(x) = \frac{x-x_0}{x_2-x_0}\cdot\frac{x-x_1}{x_2-x_1}\cdot\frac{x-x_3}{x_2-x_3}$$

$$l_3(x) = \frac{x-x_0}{x_3-x_0}\cdot\frac{x-x_1}{x_3-x_1}\cdot\frac{x-x_2}{x_3-x_2}$$

Example

Node = 3

Degree = 2

	Time (x)	Velocity (x)	
x_0	15	227.04	$f(x_0)$
x_1	20	362.78	$f(x_1)$
x_2	22.5	517.38	$f(x_2)$

$x = 17$

$P_n(x) = ?$

$$P_2(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2)$$

$$l_0(x) = \frac{x-x_1}{x_0-x_1} \cdot \frac{x-x_2}{x_0-x_2} = \frac{x-20}{15-20} * \frac{x-22.5}{15-22.5} = \frac{2(x-20)(x-22.5)}{75}$$

$$l_1(x) = \frac{x-x_0}{x_1-x_0} \cdot \frac{x-x_2}{x_1-x_2} = \frac{x-15}{20-15} * \frac{x-22.5}{20-22.5} = \frac{-2(x-15)(x-22.5)}{25}$$

$$l_2(x) = \frac{x-x_0}{x_2-x_0} \cdot \frac{x-x_1}{x_2-x_1} = \frac{x-15}{22.5-15} * \frac{x-20}{22.5-20} = \frac{4(x-15)(x-20)}{75}$$

$$P_2(x) = \frac{2}{75}(x-20)(x-22.5)\cdot(227.04) - \frac{2}{25}(x-15)(x-22.5)\cdot(362.78)$$
$$+ \frac{4}{75}(x-15)(x-20) * (517.35)$$

$$P_2(17) = \frac{2}{75}(17-20)(17-22.5)\cdot(227.04) - \frac{2}{25}(17-15)(17-22.5)\cdot(362.78)$$
$$+ \frac{4}{75}(17-15)(17-20)\cdot(517.35) = 197.228$$

Example

x	$f(x)$
0	0
10	227.4
15	362.4
20	517.35
22.5	602.97
30	90.67

$x = 19$

$x = 19$, $P_n(x)$

You are only allowed to use a polynomial of degree 2.

Datapoint = 2+1 = 3

$$P_2(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2)$$

$$l_0 = \frac{x-x_1}{x_0-x_1} \cdot \frac{x-x_2}{x_0-x_2}$$

$$l_1 = \frac{x-x_0}{x_1-x_0} \cdot \frac{x-x_2}{x_1-x_2}$$

$$l_2 = \frac{x-x_0}{x_2-x_0} \cdot \frac{x-x_1}{x_2-x_1}$$

(ii) Newton's Divided Difference Error

Newton's Divided

x	$f(x)$
x_0	$f(x_0)$
x_1	$f(x_1)$
$\vdots$	
x_n	$f(x_n)$

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \dots + a_n(x-x_0)(x-x_1)(x-x_2)\dots(x-x_n)$$

Here,

$$\left.\begin{aligned} a_0 &= f[x_0] \\ a_1 &= f[x_0, x_1] \\ a_2 &= f[x_0, x_1, x_2] \\ &\vdots \\ a_n &= f[x_0, x_1, \dots x_n] \end{aligned}\right]$$

$$P_n(x) = \underline{f[x_0]} + \underline{f[x_0,x_1]}(x-x_0) + \underline{f[x_0,x_1,x_2]}(x-x_0)(x-x_1) + \underline{f[x_0,\dots x_3]}(x-x_0)(x-x_1)(x-x_2) + \dots$$

Example

	x	$f(x)$	
x_0	-1	5	$f(x_0)$
x_1	0	1	$f(x_1)$
x_2	1	3	$f(x_2)$
x_3	2	11	$f(x_3)$
x_4	4	20	$f(x_4)$

Nodes: 4
Degree: 3

$$P_n(x) = \underline{f[x_0]} + \underline{f[x_0,x_1]}(x-x_0) + \underline{f[x_0,x_1,x_2]}(x-x_0)(x-x_1) + \underline{f[x_0,x_1,x_2,x_3]}(x-x_0)(x-x_1)(x-x_2) + \underline{f[x_0,x_1,x_2,x_3,x_4]}(x-x_0)(x-x_1)(x-x_2)(x-x_3)$$

$x_0 = -1 \quad f[x_0] = 5$

$f[x_0,x_1] = \dfrac{f[x_1]-f[x_0]}{x_1-x_0} = \dfrac{5-1}{0-(-1)} = -4$

$x_1 = 0 \quad f[x_1] = 1$

$f[x_0,x_1,x_2] = \dfrac{2-(-4)}{1-(-1)} = 3$

$f[x_1,x_2] = \dfrac{f[x_2]-f[x_1]}{x_2-x_1} = \dfrac{3-1}{1-0} = 2$

$f[x_0,x_1,x_2,x_3] = \dfrac{3-3}{2-(-1)} = 0$

$x_2 = 1 \quad f[x_2] = 3$

$f[x_1,x_2,x_3] = \dfrac{8-2}{2-0} = 3$

$f[x_2,x_3] = \dfrac{f[x_3]-f[x_2]}{x_3-x_2} = \dfrac{11-3}{2-1} = 8$

$f[x_1,x_2,x_3,x_4] = \dfrac{20-1}{4-0} = -\dfrac{25}{24}$

$x_3 = 2 \quad f[x_3] = 11$

$f[x_2,x_3,x_4] = \dfrac{20-3}{4-1} = \dfrac{7}{6}$

$f[x_3,x_4] = \dfrac{f[x_4]-f[x_3]}{x_4-x_3} = \dfrac{20-11}{4-2} = \dfrac{9}{2}$

$x_4 = 4 \quad f[x_4] = 20$

⑤ $f[x_0, x_1, x_2, x_3, x_4] = \frac{f[x_4] - f[x_0]}{4-(-1)} = \frac{20-5}{4-(-1)} = -\frac{5}{24}$ (Ans)

$$P_3(x) = 5 - 4(x+1) + 3(x+1)(x) + 0(x+1)(x-0)(x-1) - \frac{5}{24}(x+1)(x)(x-1)(x-2)$$

(iv) Cauchy's Theorem

Cauchy

What can be the maximum possible error from the polynomial?

↳ Upperbound error

x_0	$f(x_0)$
x_1	$f(x_1)$
x_2	$f(x_2)$
$\vdots$	$\vdots$
x_n	$f(x_n)$

$$\underbrace{|f(x) - P_n(x)|}_{\text{Error}} = \frac{f^{n+1}(\varepsilon)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n)$$

To find the upper-bound error / maximum possible Error.

Max error

Example: $f(x) = \cos(x)$, interval $\varepsilon \in [-1, 1]$

	x	$f(x)$	
x_0	$-\pi/4$	$\cos(-\pi/4) = \frac{1}{\sqrt{2}}$	$f(x_0)$
x_1	0	$\cos(0) = 1$	$f(x_1)$
x_2	$\pi/4$	$\cos(\pi/4) = 1/\sqrt{2}$	$f(x_2)$

Node = 3

Degree = 2

Node =

$$|f(x) - P_2(x)| = \left|\frac{f^3(\varepsilon)}{3!}(x+\pi/4)(x-0)(x-\pi/4)\right|$$

$n+1$

↳ $(n+1)!$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x \rightarrow f^3(x)$

$$= \underbrace{\left|\frac{\sin(\varepsilon)}{3!}\right|}_{\sin(x)} \underbrace{\left|(x+\pi/4)(x-0)(x-\pi/4)\right|}_{w(x)}$$

We want the max error → ε is the value for sin for which the error is max → We also want the rest of it to be max error

$\sin(x)$ → Minimum (-1) → Range cannot cross -1 or 1
→ Maximum (1)

$\sin(-1) = -0.8415$

$\sin(1) = 0.8415$ This is maximum so we will take it

$$\boxed{\left|\frac{\sin(\varepsilon)}{3!}\right| = \left|\frac{\sin(1)}{3!}\right| = \frac{0.8415}{6}}$$

$$w(x) = \left|\left(x+\frac{\pi}{4}\right)(x-0)\left(x-\frac{\pi}{4}\right)\right|$$

$$= \left(x+\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)x$$

$a^2-b^2 = (a-b)(a+b)$

To get max value of a function, $\frac{dy}{dx} = 0$

$\therefore$ $w'(x) = 0$, $x \Rightarrow$ maximum

$$w(x) = \left(x^2 - \frac{\pi^2}{4^2}\right)x = x^3 - \frac{\pi^2 x}{16}$$

$$w'(x) = 3x^2 - \frac{\pi^2}{16} = 0 \longrightarrow \text{maximum value} \rightarrow \frac{dy}{dx} = 0$$

$$\Rightarrow x^2 = \frac{\pi^2}{16\times 3} = \left(x = \pm \frac{\pi}{4\sqrt{3}}\right)$$

All the possible values of

x	$w(x) = x^3 - \frac{\pi^2}{16}x$
$\frac{\pi}{4\sqrt{3}}$	$\left(\frac{\pi}{4\sqrt{3}}\right)^3 - \frac{\pi^2}{16}\left(\frac{\pi}{4\sqrt{3}}\right) = -0.186$
$-\frac{\pi}{4\sqrt{3}}$	0.186
-1	-0.3831
1	0.3831 → Maximum

$$\text{Max / Upperbound Error} = \left|\frac{0.8415}{6} \times 0.3831\right|$$

$$= 0.0537 \text{ (Ans)}$$

(vi) Hermite Interpolation

Node = n+1

Degree = n+1-1
= n

x	$f(x)$	$f'(x)$
x_0	$f(x_0)$	$f'(x_0)$
x_1	$f(x_1)$	$f'(x_1)$
$\vdots$		$\vdots$
x_n	$f(x_n)$	$f'(x_n)$

Hermite

$(x_0, f(x_0)), (x_1, f(x_1)), (x_n, f(x_n))$ total $(n+1)$ nodes / conditions

$(x_0, f'(x_0)), (x_1, f'(x_1)), (x_n, f'(x_n))$ total $(n+1)$ nodes / conditions

Total = $2(n+1)$ nodes / conditions

= $2n+2$ nodes / conditions

Hermite polynomial Degree = $(2n+2)-1$
= $(2n+1)$

(*) Hermite Interpolation general formula **Hermite**

$$P_{2n+1}(x) = \sum_{k=0}^{n} h_k(x) \cdot f(x_k) + \hat{h}(x) \cdot f'(x_k)$$

will have x

Remember these 03

$$\boxed{h_k(x) = \left[1 - 2(x-x_k)\{l_k'(x_k)\}\right]\{l_k(x)\}^2}$$

$$\boxed{\hat{h}_k(x) = (x-x_k)\{l_k(x)\}^2}$$

↳ Lagrange basis's derivation
— will always give a value

Example:

Given function, $f(x) = \sin(x)$ and nodes $\{0, \pi/2\}$. Find out the hermite polynomial.

	x	$f(x)$	$f'(x) = \cos x$
x_0	0	0 $f(x_0)$	1 $f'(x_0)$
x_1	$\pi/2$	1 $f(x_1)$	0 $f'(x_1)$

Always calculate this

$$P_{2\cdot 1+1} = P_3 = h_0(x)\underbrace{\cancel{f(x_0)}}_{0} + \hat{h}_0 f'(x_0) +$$
$$h_1(x)f(x_1) + \cancel{\hat{h}_1 f'(x_1)}$$

$$P_{2n+1} = \sum_{k=0}^{n} h_k(x)\cdot f(x_k) + \hat{h}(x) f'(x_k)$$

$\hookrightarrow$ k is with h.

$$= \hat{h}_0 f'(x_0) + h_1(x)f(x_1)$$

$$\hat{h}_0 = (x-x_k)\{l_k(x)\}^2$$
$$= (x-x_0)\{l_0(x)\}^2$$
$$= (x-0)\left\{\frac{x-\pi/2}{-\frac{\pi}{2}}\right\}^2$$

$$l_0(x) = \frac{x-x_1}{x_0-x_1} = \frac{x-\pi/2}{0-\pi/2}$$

$$h_1(x) = \left[1-2(x-x_1)\{\underline{l_1'(x_1)}\}\right]\{l_1(x)\}^2$$

?

$$l_1(x) = \frac{x-x_0}{x_1-x_0} = \frac{x-0}{\pi/2-0} = \frac{2x}{\pi} = l_1(x)$$

$$= \left[1-2\left(x-\frac{\pi}{2}\right)\left(\frac{2}{\pi}\right)\right]\left(\frac{2x}{\pi}\right)^2$$

$$l_1'(x) = \frac{2}{\pi}$$
$$\rightarrow l_1(\pi/2) = \frac{2}{\pi}$$

Important!

In the formula we have $l_1'(x_1)$, so we need to substitute the value of x_1 in the lagrange basis.

$$P_3(x) = \hat{h}_0 f'(x_0) + h_1(x)f(x_1)$$
$$= x\left\{\frac{x-\pi/2}{-\pi/2}\right\}^2(1) + \left[1-2\left(x-\frac{\pi}{2}\right)\left(\frac{2}{\pi}\right)\right]\left(\frac{2x}{\pi}\right)^2(1)$$

$\hookrightarrow f'(x_0)$ $\quad$ $\hookrightarrow f(x_1)$

Example

	x	$f(x)$	$f'(x)$
x_0	-1	1	2
x_1	0	0	2
x_2	1	1	0

Nodes $= 3$

Degree $= 2$

$$P_{2(2)+1} = P_5(x)$$

$$P_5(x) = \underline{h_0} f(x_0) + \hat{h}_0 f'(x_0) + \cancel{h_1 f(x_1)} + \hat{h}_1 f'(x_1) + \underline{h_2} f(x_2) + \cancel{\hat{h}_2 f'(x_2)}$$

0 0

$$= h_0 f(x_0)(1) + \hat{h}_0(2) + \hat{h}_1(2) + h_2(x)(1)$$

$$= h_0(x) f(x_0) + 2\hat{h}_0(x) + 2\hat{h}_1(x) + h_2(x)$$

$$h_0(x) = \left[1 - 2(x+1)\{l_0'(x_0)\}\right]\{l_0(x)\}^2$$

$$h_0(x) = \left[1 - 2(x+1)\{-\tfrac{3}{2}\}\right]\{\tfrac{1}{2}(x^2-x)\}^2$$

$$\hat{h}_0(x) = (x+1)\{\tfrac{1}{2}(x^2-x)\}^2$$

$$\boxed{l_0(x)} = \frac{x-x_1}{x_0-x_1} \cdot \frac{x-x_2}{x_0-x_2}$$

$$= \frac{x-0}{-1-0} \times \frac{x-1}{-1-1}$$

$$= \frac{x(x-1)}{2}$$

$$l_0(x) = \tfrac{1}{2}(x^2-x)$$

$$\boxed{l_0'(x)} = \tfrac{1}{2}(2x-1)$$

$$l_0'(-1) = \tfrac{1}{2}(2(-1)-1)$$

$$= \boxed{-\tfrac{3}{2}}$$

$$\hat{h}_1(x) = (x-0)\{1-x^2\}^2$$

$$= (x)(1-x^2)^2$$

$$\boxed{l_1(x)} = \frac{x-x_0}{x_1-x_0} \times \frac{x-x_2}{x_1-x_2}$$

$$= \frac{x+1}{0+1} \times \frac{x-1}{0-1}$$

$$= \frac{(x+1)(x-1)}{-1} = \frac{x^2-1^2}{-1}$$

$$= 1-x^2$$

$$h_2(x) = \left[1 - 2(x-x_2)\, l_2'(x_2)\right](l_2(x))^2$$

$$= \left[1 - 2(x-1)(\tfrac{3}{2})\right]\left(\frac{x^2+x}{2}\right)^2$$

$$\boxed{l_2(x)} = \frac{x-x_0}{x_2-x_0} \times \frac{x-x_1}{x_2-x_1}$$

$$= \frac{x+1}{1-(-1)} \times \frac{x-0}{1-0}$$

$$= \frac{x+1}{2} \times \frac{x}{1}$$

$$= \frac{x^2+x}{2}$$

$$\boxed{x_2 = 1}$$

$$\frac{2(1)+1}{2} = \tfrac{3}{2}$$

$$\boxed{l_2'(x)} = \frac{2x}{2} + \frac{1}{2} = \frac{2x+1}{2}$$

$$P_5(x) = \left[1 - 2(x+1)(-\tfrac{3}{2})\right] * \left(\tfrac{1}{2}(x^2-x)\right)^2$$
$$+ 2(x+1)\left(\tfrac{1}{2}(x^2-x)\right)^2 + 2x(1-x^2)^2$$
$$+ \left[1 - 2(x-1)\cdot\tfrac{3}{2}\right]\left[\tfrac{1}{2}(x^2+x)\right]^2$$

Advantage → increases

$$|f(x) - P_n(x)| = \text{error} \rightarrow \text{decreases}$$

Polynomial gets bigger, error gets smaller

(vi) Convergence, Runge Phenomenon, Chebyshev's Node.

Runge, Chebyshev

According to Weierstrass Theorem

- Higher the degree of the polynomial, the more accurate the interpolating polynomial will be.

$$\text{Error} = |f(x) - P_n(x)|$$

$n \uparrow$ Error $\downarrow$ → $P_n(x)$ is getting more converged to $f(x)$

$n \to \infty$, error $= 0$

Runge Phenomenon

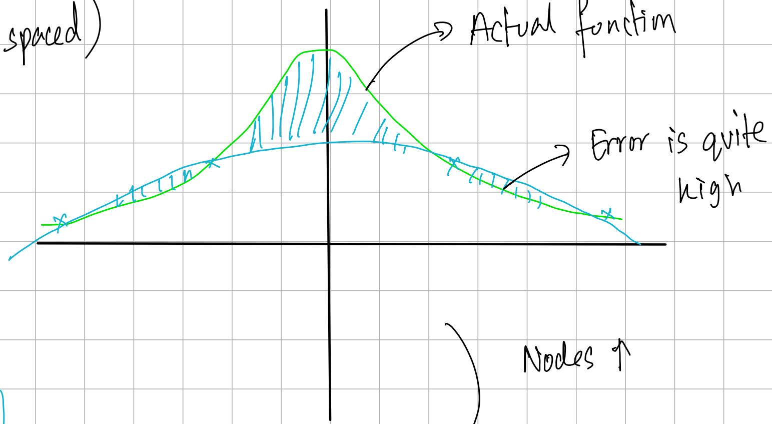
↳ Depends on two things

(1) Depends on the function (Symmetric function) → Mirror functions

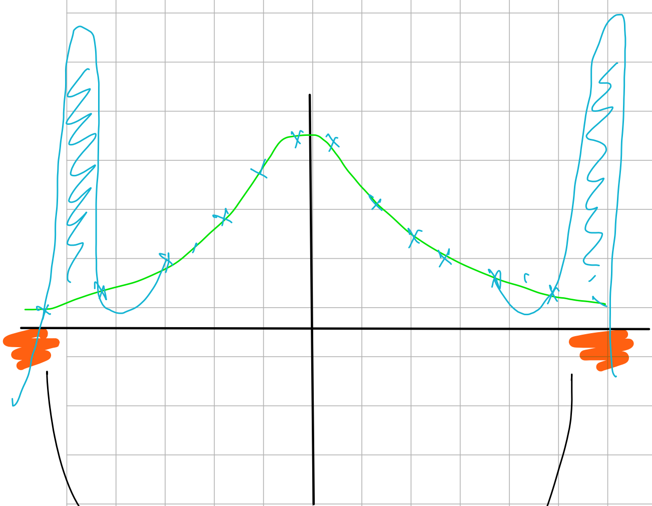
(2) Also depends on the nodes (Equally spaced)

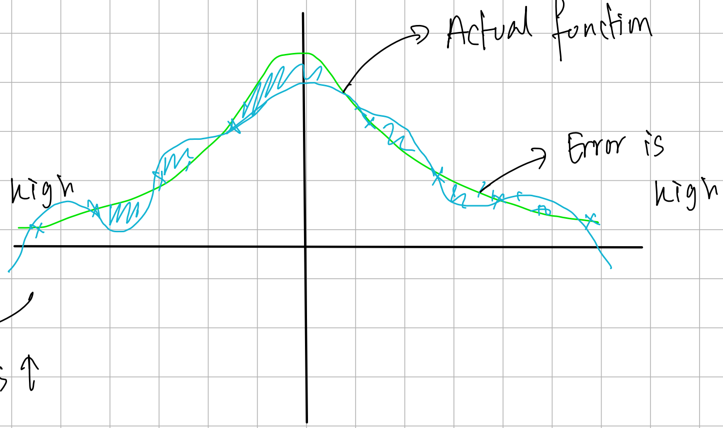
But, it's not always true.

$f(x) = \frac{1}{1+25x^2}$ interval $[-1, 1]$



Nodes ↑





Nodes ↑

Runge Phenomenon → When we take too many nodes close to the intervals the polynomial has a massive spike

↳ For particular functions; Runge functions

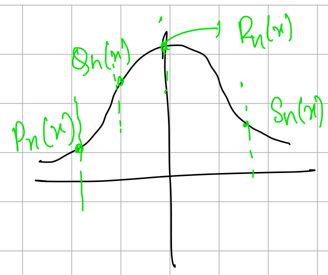
Solution:

(1) Piecewise Interpolation

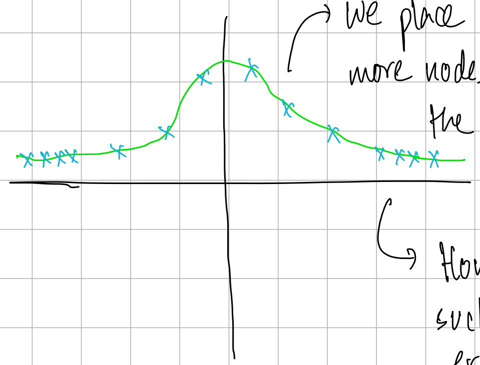
↳ Take smaller intervals and find the interpolating polynomial.

↳ Add them / Merge them

Not so important



(2) No equidistant Nodes



we place more nodes where the error is higher.

↳ How do we find these perfect nodes such that we get least amount of error and reduce the effect of runge phenomenon?

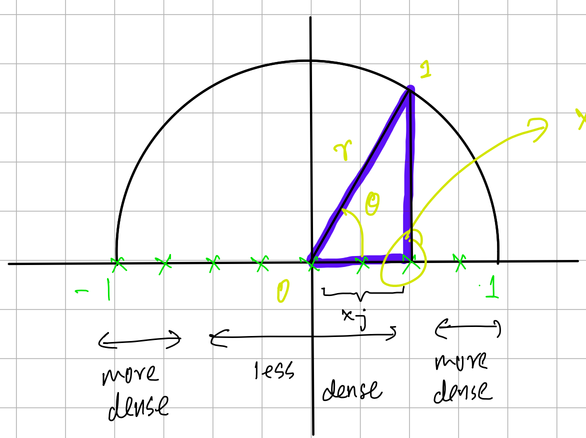
Ans: Chebyshev's Node

Chebyshev's Nodes

Chebyshev

⊗ Takes more nodes at the end point rather than taking equidistant nodes.

The hypotenuse will change



$j = j^{th}$ node

$$\phi_j = \frac{(2j+1)\pi}{2(n+1)}$$

Degree

$$\cos \phi_j = \frac{x_j}{\left[\begin{array}{c}\text{Distance} \\ \text{between} \\ \text{intervals}\end{array}\right] \div 2}$$

$$\cos \phi_j = x_j \quad \Rightarrow \quad x_j = \cos \phi_j$$

Important

$j = 0, 1, 2, 3$

$$\boxed{x_j = \cos\left(\frac{(2j+1)\pi}{2(n+1)}\right)}$$

Chebyshev's Node

Degree

$$\boxed{\text{For other intervals} \quad x_j = \cos\left(\frac{(2j+1)\pi}{2(n+1)}\right) * \text{Radius} + \text{Centre}}$$

In case radius is not 1
& centre is not (0,0)

(A) $f(x) = \dfrac{1}{1+25x^2}$, $[-1,1]$, $n=3$

Find the Chebyshev's node

$n = 3$ means node $= 4$ $\quad j = 0, 1, 2, 3$
x_0, x_1, x_2, x_3

$$x_0 = \cos\left(\frac{(2*0+1)\pi}{2(3+1)}\right) = \cos\frac{\pi}{8}$$

$$x_1 = \cos\left(\frac{(2*1+1)\pi}{2(3+1)}\right) = \cos\frac{3\pi}{8}$$

$$x_2 = \cos\left(\frac{(2*2+1)\pi}{2(3+1)}\right) = \cos\frac{5\pi}{8}$$

$$x_3 = \cos\left(\frac{(2*3+1)\pi}{2(3+1)}\right) = \cos\frac{7\pi}{8}$$

They might then ask us to use these to form a polynomial using Lagrange or whatever

(Chebyshev's Nodes)

Imp

(A) Theory Question

- when we take too many nodes close to the interval, the polynomial has a massive spike. This is due to Runge phenomenon. and to mitigate / lessen the effects and to find the perfect nodes for which error will be minimum, we find Chebyshev's nodes. Chebyshev's nodes take more nodes at the endpoints rather than equidistant nodes.

$j = 0, 1, 2, 3$

Runge Function + Chebyshev's Nodes

(a) Calculate the equal angled points / θ $\longrightarrow$ $\frac{2(j+1)\pi}{2(n+1)}$

(b) Calculate the value of Chebyshev's nodes

$\hookrightarrow \cos\theta_j = \frac{x_j}{r}$

(c) Find the langrange basis, $l_2(x)$

$$f(x) = \frac{3}{4+9x^2} \quad [-4,4] \quad n=3$$

$x_j = \cos\theta_j * r + \text{centre}$

Chebyshev

(a) $\theta_j = \frac{(2j+1)\pi}{2(n+1)}$ $\quad j = 0,1,2,3$

$\theta_0 = \frac{1}{8}\pi$

$\theta_1 = \frac{3}{8}\pi$

$\theta_2 = \frac{5}{8}\pi$

$\theta_3 = \frac{7}{8}\pi$

(b) $\cos\theta_j = \frac{x_j}{4}$

$\therefore x_j = 4\cos\theta_j$

$\to x_0 = 4\cos(1/8\pi)$

$\to x_1 = 4\cos(3/8\pi)$

$\to x_2 = 4\cos(5/8\pi)$

$\to x_3 = 4\cos(7/8\pi)$

(c)

	x	$f(x)$
x_0	$4\cos 1/8\pi$	
x_1	$4\cos 3/8\pi$	
x_2	$4\cos 5/8\pi$	
x_3	$4\cos 7/8\pi$	

Just substitute in the main formula if you ever need it

$$l_2(x) = \frac{x-x_0}{x_2-x_0} \times \frac{x-x_1}{x_2-x_1} \times \frac{x-x_3}{x_2-x_3}$$

$$= \frac{x-4\cos 1/8\pi}{4\cos\frac{5}{8}\pi - 4\cos 1/8\pi} \times \frac{x-4\cos 3/8\pi}{4\cos 5/8\pi - 4\cos 3/8\pi} \times \frac{x-4\cos 7/8\pi}{4\cos 5/8\pi - 4\cos 7/8\pi}$$