CHAPTER-2

Functions, Limits, Continuity.

(1) Single valued function : gives a single output.

eg → $\omega = z$.

Multiple valued function: gives more than one outputs.

eg → $\omega = z^{1/2}$, $\omega = \ln|z|$. $\omega = z^{1/3}$

Prb-1) Let, $\underset{f(z)}{\underline{\omega = e^z}}$, find (i) f(vertical line)
(ii) f(horizontal line).

$\text{vertical line} \longrightarrow \boxed{e^z} \longrightarrow f(\text{vertical line})$.

$\boxed{\begin{array}{l} z \rightarrow \text{input} \\ f(z) \rightarrow \text{output} \end{array}}$

Soln: (i) Now, $\omega = e^z = f(z) = e^{x+iy}$ $[z = x+iy]$.

$\therefore f(z) = e^x \cdot e^{iy}$

Polar form of $f(z)$.

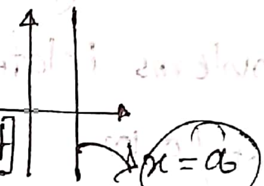
$$\Rightarrow re^{i\theta} = e^x \cdot e^{iy} = e^x \cdot e^{i(y+2\pi k)}.$$

By equating, $r = e^x$ and $\theta = y + 2\pi k$.

$$\therefore re^{i\theta} = e^x \times e^{iy} = \boxed{e^x \cdot e^{i(y+2\pi k)}}$$

$\hookrightarrow$ in term of general argument.

Vertical line: eq^n of vertical line, $x = a$ [a is a constant]



$\therefore r = e^a = c_1$. ($e, a, c_1 \rightarrow$ constants).

$\therefore \sqrt{x^2+y^2} = c_1$

$\Rightarrow x^2 + y^2 = c_1^2$

$\therefore$ Input vertical line $\rightarrow$ Output circle.

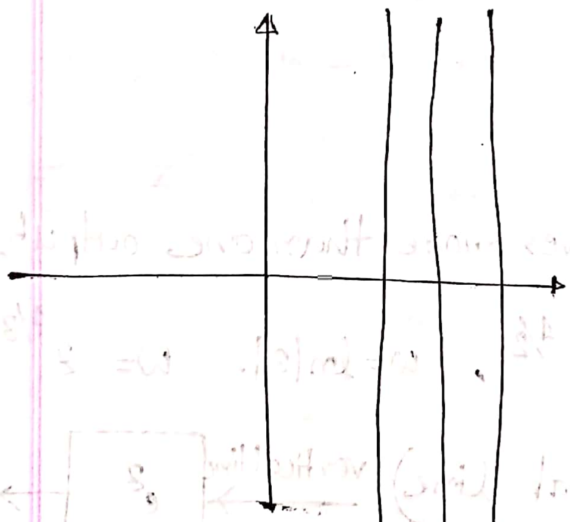
$r = a$

$\sqrt{x^2+y^2} = a$

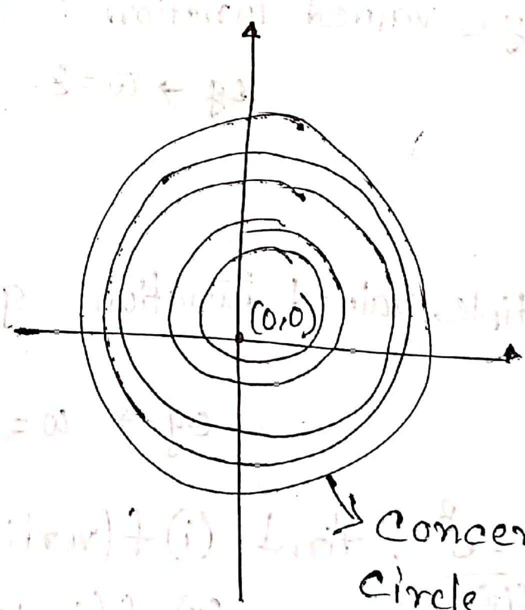
$\Rightarrow \left(\sqrt{x^2+y^2}\right)^2 = (a)^2$

$\Rightarrow x^2 + y^2 = a^2$.

$x = 1$. $x = 2$



z-plane.



Concentric Circle

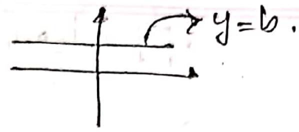
w-plane.

center same, radius different.

*if we change the value of a, we get a concentric circle in a complex plane.

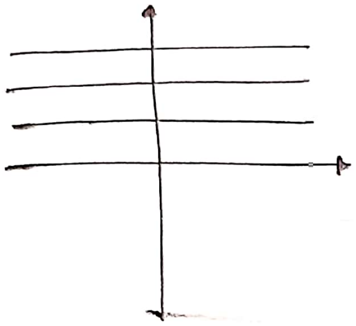
(ii) # Horizontal Line: eq^n of horizontal line.

$y = b.$

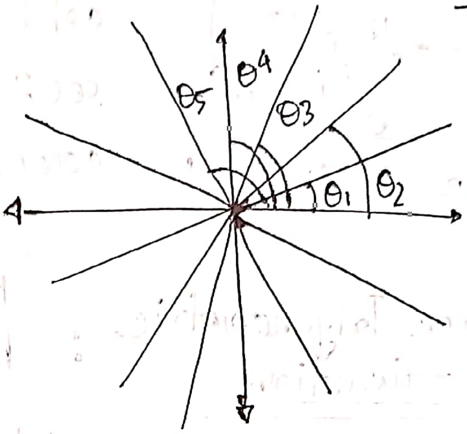


$\theta = y + 2\pi k \Rightarrow \theta = b + 2\pi k.$

Input Horizontal line → Output Different Angles.



z-plane.



w-plane.

* If we change the values of b, we get different angles. in the complex plane.

[H.W.] If c_1 and c_2 are any real constants, then determine the set of all points in the z-plane that map into the lines a) $x = c_1$ and b) $y = c_2$ in the w-plane. by $\boxed{w = z^2}$.

Consider $c_1 = 4\,;\,-2$

$c_2 = 4\,;\,-4.$

Soln: $w = z^2 = (u+iv)^2 = u^2 - v^2 + i\,(2uv).$

$\therefore \boxed{x + iy = u^2 - v^2 + i\,(2uv).}$ — ?

$\therefore\ x = u^2 - v^2 \rightarrow c_1 = u^2 - v^2$

$y = 2uv. \rightarrow c_2 = 2uv.$

...... (should complete).

Elementary Functions:

* $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$
* $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

i) Trigonometry function:

$$\boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i}}$$

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}}$$

$\tan z$

$\cot z$

$\sec z$

$\operatorname{cosec} z$.

ii) Inverse Trigonometric function: $\boxed{\text{Proof}}$ *

$\sin^{-1} z$, $\cos^{-1} z$, $\tan^{-1} z$, $\cot^{-1} z$, $\sec^{-1} z$, $\operatorname{cosec}^{-1} z$.

$$\sin^{-1} z = \frac{1}{i} \ln\left(iz + \sqrt{1 - z^2}\right).$$

iii) Hyperbolic Function:

$\sinh^{-1}(z) \rightarrow$ sin hyperbolic inverse z.

$\sinh(z)$, $\operatorname{sech}(z)$, $\cosh(z)$, $\tanh(z)$, $\coth(z)$, $\operatorname{cosech}(z)$.

$$\sinh(z) = \frac{e^z - e^{-z}}{2}$$

$$\cosh(z) = \frac{e^z + e^{-z}}{2}$$

(iv) Inverse Hyperbolic Function: $\boxed{\text{Proof}}$ *

$\sinh^{-1}(z)$, $\cosh^{-1}(z)$, $\operatorname{sech}^{-1}(z)$, $\tanh^{-1}(z)$.

$\coth^{-1}(z)$, $\operatorname{cosech}^{-1}(z)$.

i) $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ $\tan z = \frac{\sin z}{\cos z}$

ii) $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

i) $\sinh(z) = \frac{e^{z} - e^{-z}}{2}$ $\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$

ii) $\cosh(z) = \frac{e^{z} + e^{-z}}{2}$

Prb-1*: Prove that, $\sin^{-1} z = -i\ln\left(iz + (1-z^2)^{1/2}\right)$. [FALL24, SUMMER 25, SPRING-24].

Proof: Let, $w = \sin^{-1} z$ $[\sin(\sin^{-1} z) = z]$

(i) Apply: Formula
(ii) Extract w.

$\Rightarrow \sin w = z$

$\Rightarrow z = \boxed{\sin w}$

$\Rightarrow z = \frac{e^{iw} - e^{-iw}}{2i}$

$\Rightarrow z = \frac{e^{iw} - \frac{1}{e^{iw}}}{2i}$

$\Rightarrow 2iz = \frac{(e^{iw})^2 - 1}{e^{iw}}$

$\Rightarrow e^{iw} \cdot 2iz = (e^{iw})^2 - 1.$

$\Rightarrow (e^{iw})^2 - 2iz \cdot e^{iw} - 1 = 0$

$ax^2 + bx + c = 0$

$a = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

let, $e^{iw} = a$

$\therefore a^2 - 2iz \cdot a - 1 = 0$

$$\therefore a = \frac{-(-2iz) \pm \sqrt{(-2iz)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1}$$

$$= \frac{2iz \pm \sqrt{-4z^2 + 4}}{2}$$

$$= \frac{2iz \pm 2\sqrt{1 - z^2}}{2}$$

$$= \frac{iz \pm \sqrt{1 - z^2}}{\cancel{2}1}$$

$\therefore a = iz \pm \sqrt{1-z^2}$

$\Rightarrow e^{iw} = iz \pm \sqrt{1-z^2}$

$\Rightarrow \ln e^{iw} = \ln\left(iz \pm \sqrt{1-z^2}\right).$

$\Rightarrow iw = \ln\left(iz \pm \sqrt{1-z^2}\right).$

$\Rightarrow w = \frac{1}{i} \ln\left(iz \pm \sqrt{1-z^2}\right).$

$= \frac{i}{i^2} \ln\left(iz \pm \sqrt{1-z^2}\right)$

$w = -i \ln\left(iz \pm \sqrt{1-z^2}\right)$

$\therefore$ ~~sin z~~ $\boxed{\sin^{-1}(z) = -i\ln\left(iz \pm \sqrt{1-z^2}\right)}$ (Proved).

$\therefore \sin^{-1}(z) = -i\ln\left(iz + \sqrt{1-z^2}\right).$

$$\boxed{\begin{array}{l} e^{\ln x} = x \\ \ln e = 1 \end{array}}$$

$$\boxed{\frac{1}{i} = -i}$$

* **Prb-2:** Prove that, $\tanh^{-1}(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right).$

$\sinh(w) = \frac{e^{w} - e^{-w}}{2}$

$\cosh(w) = \frac{e^{w} + e^{-w}}{2}$

$\tanh(w) = \frac{e^{w} - e^{-w}}{e^{w} + e^{-w}}$

Proof: Let, $w = \tanh^{-1}(z)$

$\Rightarrow z = \tanh(w).$

$\Rightarrow z = \frac{e^{w} - e^{-w}}{e^{w} + e^{-w}}$

$\Rightarrow z = \frac{e^{w} - \frac{1}{e^{w}}}{e^{w} + \frac{1}{e^{w}}}$

$\Rightarrow z = \frac{(e^{w})^2 - 1}{(e^{w})^2 + 1}.$

$\Rightarrow z(e^{w})^2 + z = (e^{w})^2 - 1.$

$\Rightarrow z(e^{w})^2 - (e^{w})^2 + z + 1 = 0$

$$\Rightarrow (e^{w})^2(z-1) = -1-z.$$

H.W $\rightarrow$ $\cos^{-1}(z) = \frac{1}{i}\ln\left(z \pm \sqrt{z^2-1}\right).$

$$\Rightarrow (e^{w})^2 = \frac{-1-z}{z-1.}$$

$$\Rightarrow \cancel{e^{w} = \pm\sqrt{\frac{+1-z}{z+1}}}$$

$$\Rightarrow e^{2w} = \frac{1+z}{1-z}$$

$$\Rightarrow \ln e^{2w} = \ln\left(\frac{1+z}{1-z}\right)$$

$$\Rightarrow 2w = \ln\left(\frac{1+z}{1-z}\right)$$

$$\Rightarrow w = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

$$\therefore \boxed{\tanh^{-1}(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)}$$ (Proved).

Limit

Def^n: complex variable

$$\lim_{z\to z_0} f(z) = l$$

$\uparrow$ limiting value.

real variable

$$\lim_{x\to a} f(x) = l.$$

if case I = case II & limit exists otherwise doesn't exist

$\uparrow$

$\boxed{L.H.L = R.H.L}$

Pg 53 ~~Pg 30~~

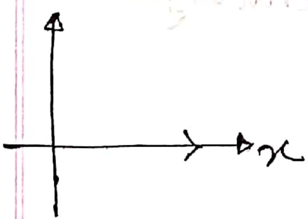
Prob-3: ~~Proof~~ Prove that $\lim_{z\to 0} \frac{\bar{z}}{z}$ doesn't exist.

$z = x+iy$

$\bar{z} = x-iy$

Proof: Case 1: (Along x-axis) $\rightarrow$ $\boxed{x\to 0 \text{ and } y=0}$

along x-axis, x changes, but y is always 0



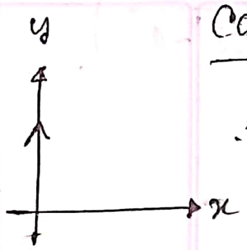
$$\lim_{z\to 0}\frac{\bar{z}}{z} = \lim_{z\to 0}\frac{x-iy}{x+iy}$$

$$= \lim_{\substack{x\to 0\\ y=0}}\frac{x-iy}{x+iy}$$

$$= \lim_{x\to 0}\frac{x-0}{x+0}$$

$$= \lim_{x\to 0}(1)$$

$$= 1.$$



Case-2 : (Along y-axis) $\rightarrow$ $\boxed{x=0 \text{ and } y\to 0}$.

$$\therefore \lim_{z\to 0}\frac{\bar{z}}{z} = \lim_{\substack{y\to 0\\ x=0}} \frac{x-iy}{x+iy} = \lim_{y\to 0}\frac{0-iy}{0+iy} = \lim_{y\to 0}\frac{-iy}{iy} = \lim_{y\to 0}(-1)$$
$$= -1.$$

limiting value along x-axis $\neq$ limiting value along y-axis.

$\therefore \lim_{z\to 0}\frac{\bar{z}}{z}$ doesn't exists. [Proved].

Prb-4: Find all values of z of the following function :

i) $e^{4z} = i$ iii) $\ln\left(i^{1/2}\right) = \left(n+\frac{1}{4}\right)\pi i$

ii) $\sinh(z) = i$ iv) $i^{i} = ?$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$
$= 0 + i\cdot 1 = i$

$e^{i\pi} = \cos\pi + i\sin\pi$
$= -1 + i\cdot 0 = -1.$

$\therefore$ $\boxed{\begin{aligned} e^{i\frac{\pi}{2}} &= i \\ e^{i\pi} &= -1 \end{aligned}}$ *

* For all values of z
↳ general argument.

Soln: i) $e^{4z} = i$

$\Rightarrow e^{4z} = e^{i\left(\frac{\pi}{2}+2\pi k\right)}$
↳ general arg^{t}

By equating.

$4z \Rightarrow i\left(\frac{\pi}{2}+2\pi k\right)$

$\therefore \boxed{z = \frac{i}{4}\left(\frac{\pi}{2}+2\pi k\right)}$; $k\in \mathbb{Z}$.

Ans!

Soln ii) $\sinh(z) = i$
complex

$$\Rightarrow \frac{e^{z}-e^{-z}}{2} = e^{i\frac{\pi}{2}} = e^{i\left(\frac{\pi}{2}+2\pi k\right)}$$

$$\Rightarrow e^{z} - \frac{1}{e^{z}} = 2e^{i\left(\frac{\pi}{2}+2\pi k\right)}$$

$$\Rightarrow \frac{(e^{z})^{2}-1}{e^{z}} = 2e^{i\left(\frac{\pi}{2}+2\pi k\right)}.$$

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another process (easier)

Sol^n ii) $\sinh(z) = i$

$\Rightarrow \frac{e^z - e^{-z}}{2} = i$

$\Rightarrow e^z - \frac{1}{e^z} = 2i$

$\Rightarrow \frac{(e^z)^2 - 1}{e^z} = 2i$

$\Rightarrow (e^z)^2 - 2ie^z - 1 = 0.$

let $e^z = a$

$\therefore a^2 - 2ia - 1 = 0.$

$\therefore a = \frac{-(-2i) \pm \sqrt{(-2i)^2 - 4\cdot1\cdot(-1)}}{2\cdot1}$

$= \frac{2i \pm \sqrt{-4+4}}{2}$

$\therefore a = i$

$\Rightarrow e^z = i$

$\Rightarrow e^z = e^{i\frac{\pi}{2}} = e^{i(\frac{\pi}{2}+2\pi k)}$

By equating. $\boxed{z = i(\frac{\pi}{2}+2\pi k)}$; $k \in \mathbb{Z}$. Ans:

Sol^n iii) $\ln(i^{1/2}) = (n+\frac{1}{4})\pi i$

$i = e^{i\pi/2}$

$\Rightarrow i^{1/2} = e^{i\pi/4}$

$\Rightarrow \ln(i^{1/2}) = \ln(e^{i\pi/4})$

$= i\frac{\pi}{4}$

$= i(\frac{\pi}{4}+2\pi k) \rightarrow$ general argument

$= \pi i(1/4 + 2k)$.

$= \pi i(n+\frac{1}{4})$. [$n \approx 2k$; both are constants].

[Ans:].

Sol^n iv) $i^i = (e^{i\frac{\pi}{2}})^i$

$= e^{i^2\pi/2}$

$= e^{-\frac{\pi}{2}}$

$\approx 0.20788.$

Ans:

ch-2

i) Graph related (function).

ii) $\sin^{-1}z$, $\cos^{-1}z$, ---- } Proof
$\sinh^{-1}z$ ....

iii) limit existence

iv) find all values.