Start

Standard integrals

Manuf

Prob: Show that $\int \sec x\, dx = \log (\sec x + \tan x) + c$

$= \log \left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right] + c.$

Soln: Let $I = \int \sec x\, dx$

$$= \int \frac{\sec x(\sec x + \tan x)\, dx}{\sec x + \tan x}$$

$$= \int \frac{(\sec^2 x + \sec x \tan x)\, dx}{\sec x + \tan x}$$

Let $\sec x + \tan x = z$

$\therefore (\sec x \tan x + \sec^2 x)\, dx = dz$

$$= \int \frac{dz}{z}$$

$$= \log z + c$$

$$\Rightarrow \boxed{I = \log(\sec x + \tan x) + c} \quad \text{---}(1)$$

Again, $\sec x + \tan x = \frac{1}{\cos x} + \frac{\sin x}{\cos x}$

$$= \frac{1 + \sin x}{\cos x}$$

$$= \frac{\sin^2 x/2 + \cos^2 x/2 + 2 \sin x/2 \cos x/2}{\cos^2 x/2 - \sin^2 x/2}$$

$$= \frac{(\cos x/2 + \sin x/2)^2}{(\cos x/2 - \sin x/2)(\cos x/2 + \sin x/2)}$$

$$= \frac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2}$$

$$= \frac{1 + \tan x/2}{1 - \tan x/2}$$

$$= \frac{\tan \pi/4 + \tan x/2}{1 - \tan \pi/4 \tan x/2} \qquad [\because \tan \pi/4 = 1]$$

$$= \tan(\pi/4 + x/2)$$

$$\therefore (1) \Rightarrow \boxed{I = \log[\tan(\pi/4 + x/2)] + c} \quad \text{---}(2)$$

So from (1) & (2), we get

$$\int \sec x \, dx = \log(\sec x + \tan x) + c$$
$$= \log\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right] + c$$

proved.

Prob: Show that $\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$.

Soln: Let $I = \int \sqrt{a^2 - x^2}\, dx$ Let $x = a\sin\theta \Rightarrow \theta = \sin^{-1} x/a$

$= \int \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\, d\theta$ $\therefore dx = a\cos\theta\, d\theta$

$= \int \sqrt{a^2(1-\sin^2\theta)}\; a\cos\theta\, d\theta$

$= \int \sqrt{a^2\cos^2\theta}\; a\cos\theta\, d\theta$

$= \int a\cos\theta \cdot a\cos\theta\, d\theta$

$= a^2 \int \cos^2\theta\, d\theta$

$= \frac{a^2}{2}\int 2\cos^2\theta\, d\theta$

$= \frac{a^2}{2}\int (1+\cos 2\theta)\, d\theta$

$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$

$= \frac{a^2}{2}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right] + c$

$= \frac{a^2}{2}\left[\theta + \sin\theta\sqrt{1-\sin^2\theta}\right] + c$

$= \frac{a^2}{2}\left[\sin^{-1} x/a + \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right] + c$

$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}\right] + c$

$= \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

Ans

$$\Rightarrow I\left(\frac{a^2+b^2}{a^2}\right) = \frac{e^{ax}}{a^2}\left[a\sin bx - b\cos bx\right]$$

$$\Rightarrow \boxed{I = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)}$$

proved

Prob: (i) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}, \ (a \neq 0).$

(ii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|, \ (|x|>|a|)$

Soln: (i) Let $I = \int \frac{dx}{x^2+a^2}$

Let $x = a\tan\theta$

$\therefore dx = a\sec^2\theta\, d\theta$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2\tan^2\theta + a^2}$$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2(\tan^2\theta+1)}$$

$$= \frac{1}{a}\int \frac{\sec^2\theta\, d\theta}{\sec^2\theta}$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\Rightarrow \boxed{I = \frac{1}{a}\tan^{-1}\frac{x}{a}}$$

(ii) Let $I = \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int \left[\frac{1}{x-a} - \frac{1}{x+a}\right]dx$

$$= \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right)$$

$$\Rightarrow \boxed{I = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|}.$$

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Standard integral of the form :

$$\int \frac{dx}{ax^2+bx+c}, \quad \int \frac{dx}{\sqrt{ax^2+bx+c}}, \quad \int \sqrt{ax^2+bx+c}\, dx.$$

Here
$$ax^2+bx+c = a\left(x^2+\frac{bx}{a}+\frac{c}{a}\right)$$
$$= a\left[x^2+2.x.\frac{b}{2a}+\left(\frac{b}{2a}\right)^2+\frac{c}{a}-\left(\frac{b}{2a}\right)^2\right]$$
$$= a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$$

$$\therefore I = \int \frac{dx}{ax^2+bx+c}$$

$$\Rightarrow I = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}} \quad \text{——— (1)}$$

$$= \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2+\left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2} \quad \text{if } 4ac-b^2>0 \Rightarrow 4ac>b^2$$

$$= \frac{1}{a}\int \frac{dz}{z^2+k^2}$$

Let $x+\frac{b}{2a} = z$

$\therefore dx = dz$ &

$\frac{\sqrt{4ac-b^2}}{2a} = k$

$$\Rightarrow \boxed{I = \frac{1}{a}\,\frac{1}{k}\tan^{-1}\frac{z}{k}}$$

Again, (1) $\Rightarrow$

$$I = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}} \quad \text{if } 4ac-b^2<0 \Rightarrow 4ac<b^2$$

$$= \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2}$$

$$= \frac{1}{a} \int \frac{dz}{z^2 - k^2}$$

$$\Rightarrow \boxed{I = \frac{1}{a} \frac{1}{2k} \ln \frac{z-k}{z+k}}$$

Ex: Evaluate $\int \frac{dx}{2x^2+x+1}$

Soln: Let $I = \int \frac{dx}{2x^2+x+1}$

$$= \frac{1}{2} \int \frac{dx}{x^2 + \frac{x}{2} + \frac{1}{2}}$$

$$= \frac{1}{2} \int \frac{dx}{(x+\frac{1}{4})^2 + \frac{1}{2} - \frac{1}{16}}$$

$$= \frac{1}{2} \int \frac{dx}{(x+\frac{1}{4})^2 + \frac{7}{16}}$$

$$= \frac{1}{2} \int \frac{dx}{(x+\frac{1}{4})^2 + (\frac{\sqrt{7}}{4})^2}$$

$$= \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{7}}{4}} \cdot \tan^{-1} \frac{x+\frac{1}{4}}{\frac{\sqrt{7}}{4}} + c$$

$$= \frac{2}{\sqrt{7}} \tan^{-1} \frac{4x+1}{\sqrt{7}} + c$$ Ans

Prob: Integrate $\int \frac{dx}{3+4\cos x}$

Sol^n: Let $I = \int \frac{dx}{3+4\cos x}$

$$= \int \frac{dx}{3(\sin^2 x/2 + \cos^2 x/2) + 4(\cos^2 x/2 - \sin^2 x/2)}$$

$$= \int \frac{\sec^2 x\, dx}{3(1+\tan^2 x/2) + 4(1-\tan^2 x/2)}$$

$$= \int \frac{\sec^2 x\, dx}{7 - \tan^2 x/2}$$

Let $\tan x/2 = z$

$\therefore \sec^2 \frac{x}{2} \cdot \frac{1}{2}\, dx = dz$

$\Rightarrow \sec^2 x/2\, dx = 2\,dz$

$$= \int \frac{2\,dz}{7 - z^2}$$

$$= 2\int \frac{dz}{(\sqrt{7})^2 - z^2}$$

$$= 2 \cdot \frac{1}{2\sqrt{7}} \ln \frac{\sqrt{7}+z}{\sqrt{7}-z} + c \qquad \left[\because \int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \frac{a+x}{a-x}\right]$$

$$= \frac{1}{\sqrt{7}} \ln \frac{\sqrt{7}+\tan x/2}{\sqrt{7}-\tan x/2} + c$$

Prob: Evaluate $\int \frac{dx}{5-3\cos x}$

Sol^n: Let $I = \int \frac{dx}{5-3\cos x}$

$$= \int \frac{dx}{5(\sin^2 x/2 + \cos^2 x/2) - 3(\cos^2 x/2 - \sin^2 x/2)}$$

$$= \int \frac{\sec^2 x/2\, dx}{5(1+\tan^2 x/2) - 3(1-\tan^2 x/2)}$$

$$= \int \frac{\sec^2 x/2\, dx}{2 + 8\tan^2 x/2}$$

$$= \int \frac{\sec^2 x/2 \, dx}{8\left[\frac{1}{4} + \tan^2 x/2\right]}$$

$$= \frac{1}{8}\int \frac{\sec^2 x/2 \, dx}{\frac{1}{4} + \tan^2 x/2}$$

Let $\tan x/2 = z$

$\Rightarrow \sec^2 x/2 \cdot \frac{1}{2} dx = dz$

$\Rightarrow \sec^2 x/2 \, dx = 2dz$

$$= \frac{1}{8}\int \frac{2dz}{\left(\frac{1}{2}\right)^2 + z^2}$$

$$= \frac{1}{4} \cdot \frac{1}{\frac{1}{2}} \tan^{-1} \frac{z}{1/2} + c \qquad \left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{4} \cdot 2 \tan^{-1} 2z + c$$

$$= \frac{1}{2}\tan^{-1}\{2\tan x/2\} + c \quad \text{Ans}$$

Prob: Evaluate $\int_0^{\pi/2} \frac{\tan x \, dx}{1 + m^2 \tan^2 x}$

Soln: Let $I = \int_0^{\pi/2} \frac{\tan x \, dx}{1 + m^2 \tan^2 x}$

$$= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{1 + m^2 \frac{\sin^2 x}{\cos^2 x}} dx$$

$$= \int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + m^2 \sin^2 x} dx$$

$$= \int_0^{\pi/2} \frac{\sin x \cos x \, dx}{1 - \sin^2 x + m^2 \sin^2 x}$$

$$= \int_0^{\pi/2} \frac{\sin x \cos x \, dx}{1 + (m^2 - 1)\sin^2 x}$$

$$= \int_1^{m^2} \frac{1}{z} \cdot \frac{dz}{2(m^2-1)}$$

$$= \frac{1}{2(m^2-1)} \cdot [\ln z]_1^{m^2}$$

$$= \frac{1}{2(m^2-1)} [\ln m^2 - \ln 1]$$

$$= \frac{2 \ln m - 0}{2(m^2-1)}$$

$$= \frac{\ln m^2}{m^2-1} \quad \underline{\text{Ans}}$$

Let
$1 + (m^2-1)\sin^2 x = z$

$\Rightarrow 0 + (m^2-1) \cdot 2 \sin x \cos x \, dx = dz$

$\Rightarrow \sin x \cos x \, dx = \dfrac{dz}{2(m^2-1)}$

x	0	$\pi/2$
z	1	m^2

Prob: Evaluate $\displaystyle\int \frac{x^2 - x + 1}{x^2 + x + 1} dx$

Sol^n: Let $I = \displaystyle\int \frac{x^2 - x + 1}{x^2 + x + 1} dx$

$$= \int \frac{x^2 + x + 1 - 2x}{x^2 + x + 1} dx$$

$$= \int dx - \int \frac{2x}{x^2 + x + 1} dx$$

$$= x - \int \frac{2x + 1 - 1}{x^2 + x + 1} dx$$

$$= x - \left[\int \frac{2x+1}{x^2+x+1} dx - \int \frac{dx}{x^2+x+1} \right]$$

$$= x - \ln(x^2+x+1) + \int \frac{dx}{x^2 + 2 \cdot x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1}$$

$$= x - \ln(x^2+x+1) + \int \frac{dx}{3/4 + (x + 1/2)^2}$$

$$= x - \ln(x^2+x+1) + \int \frac{dx}{(\sqrt{3}/2)^2 + (x + 1/2)^2}$$

Prob. Evaluate $\int \frac{5x^2+3x+17}{x^3-x^2+4x-4}\,dx$

Sol^n. Let $I = \int \frac{5x^2+3x+17}{x^3-x^2+4x-4}\,dx$

$$= \int \frac{5x^2+3x+17}{x(x^2+4)-1\cdot(x^2+4)}\,dx$$

$$\Rightarrow I = \int \frac{5x^2+3x+17}{(x^2+4)(x-1)}\,dx \text{ ------ (i)}$$

Let $\frac{5x^2+3x+17}{(x^2+4)(x-1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-1}$ ------ (ii)

$\Rightarrow 5x^2+3x+17 = (Ax+B)(x-1) + C(x^2+4)$ ------ (iii)

$\Rightarrow 5x^2+3x+17 = Ax^2+Bx-Ax-B+Cx^2+4C$

Equating the co-efficient of x^2, we get

$5 = A+C$ ------ (iv)

Equating the co-efficient of x, we get

$3 = B-A$ ------ (v)

Equating the constant, we get

$17 = -B+4C$ ------ (vi)

using (iv), (v) & (vi) we get

$\boxed{A=0}$ $\boxed{B=3}$ & $\boxed{C=5}$

$\therefore$ (ii) $\Rightarrow \frac{5x^2+3x+17}{(x^2+4)(x-1)} = \frac{3}{x^2+4} + \frac{5}{x-1}$

$\therefore$ (i) $\Rightarrow I = \int \frac{5x^2+3x+17}{(x^2+4)(x-1)}\,dx = \int \frac{3\,dx}{x^2+2^2} + \int \frac{5}{x-1}\,dx$

$$= 3\cdot\frac{1}{2}\cdot \tan^{-1}\frac{x}{2} + 5\ln(x-1) + C$$

Ans

$$= x - \ln(x^2+x+1) + \frac{1}{\sqrt{3}/2} \tan^{-1} \frac{x+1/2}{\sqrt{3}/2} + c$$

$$= x - \ln(x^2+x+1) + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + c \quad \text{Ans}$$

<u>Prob:</u> Evaluate $\int \frac{x^2-2}{(x+1)(x^2+1)} dx$

<u>Soln:</u> Let $I = \int \frac{x^2-2}{(x+1)(x^2+1)} dx$ ——— (I)

Here, $\frac{x^2-2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$ ——— (II)

$\Rightarrow x^2-2 = A(x^2+1) + (Bx+C)(x+1)$ ——— (III)

Equating the co-efficient of x^2, we get

$1 = A+B$ ——— (IV)

Equating the co-efficient of x, we get

$0 = B+C$ ——— (V)

Equating the constant, we get

$-2 = A+C$ ——— (VI)

Solving (IV), (V) & (VI) we get $\boxed{A=-1/2}$ $\boxed{B=3/2}$ $\boxed{C=-3/2}$

$\therefore$ (II) $\Rightarrow \frac{x^2-2}{(x+1)(x^2+1)} = -\frac{1}{2}\frac{1}{x+1} + \frac{3}{2}\frac{x-1}{x^2+1}$

$\therefore$ (I) $\Rightarrow I = \int \frac{x^2-2}{(x+1)(x^2+1)} dx = -\frac{1}{2}\int \frac{dx}{x+1} + \frac{3}{2}\int \frac{x-1}{x^2+1} dx$

$$= -\frac{1}{2}\ln(x+1) + \frac{3}{2}\int \left[\frac{x}{x^2+1} - \frac{1}{x^2+1}\right] dx$$

$$= -\frac{1}{2}\ln(x+1) + \frac{3}{2}\int \left[\frac{1}{2}\frac{2x}{x^2+1} - \frac{1}{x^2+1}\right] dx$$

$$= -\frac{1}{2}\ln(x+1) + \frac{3}{2}\cdot\frac{1}{2}\int \frac{2x\,dx}{x^2+1} - \frac{3}{2}\int \frac{dx}{1+x^2}$$

$$= -\frac{1}{2}\ln(x+1) + \frac{3}{4}\ln(x^2+1) - \frac{3}{2}\tan^{-1}x + c \quad \text{Ans}$$

Definite integral

Manuf

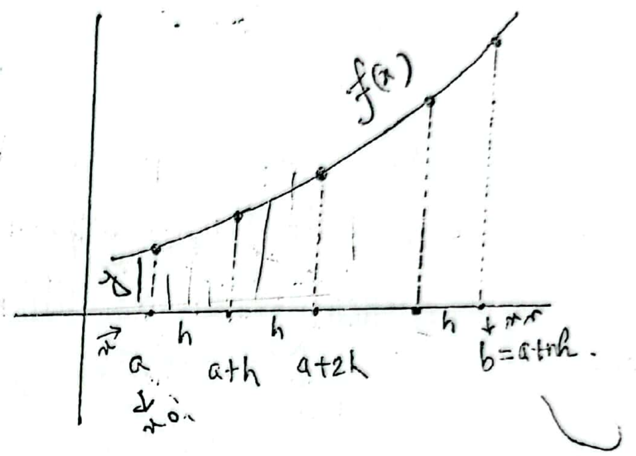
Definite integral as the limit of a sum :

Let $f(x)$ be a single-valued function which is continuous on $[a,b]$ with $b>a$. Divide the interval $[a,b]$ into n equal sub-intervals, so we get

$a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

Let h be the length of each subinterval, then

$nh = b-a \Rightarrow \boxed{h = \frac{b-a}{n}}$



Thus $a < a+h < a+2h < \cdots < a+nh = b$

Here $h \to 0$ as $n \to \infty$.

From the figure,

$$S_n = h f(a+h) + h f(a+2h) + \cdots + h f(a+nh)$$

$$\Rightarrow S_n = \sum_{r=1}^{n} h f(a+rh)$$

$$\Rightarrow S_n = h \sum_{r=1}^{n} f(a+rh) \Rightarrow \lim_{n\to\infty} S_n = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh).$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left(a+\frac{(b-a)r}{n}\right), \text{ When } h = \frac{b-a}{n}, \text{ if } h\to 0 \text{ then } n\to\infty$$

If $\lim_{n\to\infty} S_n$ exists when $h \to 0$ then $\lim_{n\to\infty} S_n = S$ is defined as the definite integral of $f(x)$ with respect to x between a & b and we write

$$\boxed{S = \int_a^b f(x)\,dx}$$

<u>Definition</u>: A function $f(x)$ is said to be integrable [a, b] if

$$\lim_{n\to\infty} \frac{b-a}{n}\sum_{r=1}^{n} f\left(a+\frac{r}{n}(b-a)\right)$$

exists. When this is the case, we denote this limit by the symbol

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n}\sum_{r=1}^{n} f\left(a+\frac{r(b-a)}{n}\right)$$

Which is called the definite integral of $f(x)$ from a to b. Here the numbers a & b are called the lower limit & upper limit of the integration and the function $f(x)$ is called the integrand.

Sheet

Prob: Using definition, evaluate (i) $\int_a^b x\,dx$ (ii) $\int_a^b \sin x\,dx$ (iii) $\int_0^1 x^2\,dx$.

Sol^n: (i) Here $f(x) = x$

$\therefore f(a+rh) = a+rh$

we know, $\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh)$, where $nh = b-a$

$$\Rightarrow \int_a^b x\,dx = \lim_{h\to 0} h[(a+h)+(a+2h)+(a+3h)+\dots+(a+nh)]$$

$[\because f(x)=x]$

$$= \lim_{h\to 0} h[na + h(1+2+3+\dots+n)]$$

$$= \lim_{h\to 0} h\left[na + h\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{h\to 0} \left[hna + \frac{nh(nh+h)}{2}\right]$$

$$= \lim_{h\to 0} \left[(b-a)a + \frac{(b-a)(b-a+h)}{2}\right]$$

$$= ab - a^2 + \frac{(b-a)(b-a)}{2}$$

$$= \frac{2ab-2a^2+b^2-2ab+a^2}{2}$$

$$= \frac{b^2-a^2}{2}.$$

(iii) Here $f(x) = x^2$

Given $a = 0$ & $b = 1$

$\therefore nh = b-a = 1$

we know $\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh)$

$$\Rightarrow \int_0^1 x^2\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} (rh)^2 \quad [\because a=0]$$

$$= \lim_{h\to 0} h\,(1.h^2+2^2h^2+3^2h^2+\dots+n^2h^2)$$

$$= \lim_{h\to 0} h^3 (1^2+2^2+3^2+\cdots+n^2)$$

$$= \lim_{h\to 0} h^3 \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{h\to 0} h^3 \frac{2n^3+3n^2+n}{6}$$

$$= \lim_{h\to 0} \frac{2n^3h^3+3.n^2h^2.h+nh.h^2}{6}$$

$$= \lim_{h\to 0} \frac{2+3h+h^2}{6} \quad [\because nh=1]$$

$$= \frac{1}{3} \text{ Ans.}$$

(ii) Given $f(x) = \sin x$

$\therefore f(a+rh) = \sin(a+rh)$

Here $\boxed{nh = b-a}$

We know $\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh)$

$$\Rightarrow \int_a^b \sin x\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} \sin(a+rh)$$

$$= \lim_{h\to 0} h\,[\sin(a+h)+\sin(a+2h)+\cdots+\sin(a+nh)]$$

$$= \lim_{h\to 0} h \frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}} \sin\left(\frac{a+h+a+nh}{2}\right)$$

$$= \lim_{h\to 0} \left(\frac{h/2}{\sin h/2}\right).2\sin\left(\frac{a+h+a+nh}{2}\right).\sin\frac{nh}{2}$$

$$= \lim_{h\to 0} \left(\frac{h/2}{\sin h/2}\right)2.\sin\left(\frac{a+h+b}{2}\right).\sin\frac{b-a}{2}$$

$$= 1.2.\sin\frac{a+b}{2}.\sin\frac{b-a}{2}$$

$$= \cos a - \cos b$$

Ans

$\sin(a+h)+\sin(a+2h)+\cdots+\sin(a+nh) =$

$$\frac{\sin\left(\frac{\text{no. of terms} \times \text{angle differance}}{2}\right)}{\sin\left(\frac{\text{angle difference}}{2}\right)}.\sin\left(\frac{\text{1st angle}+\text{last angle}}{2}\right)$$

$$= \frac{\sin\frac{n\times h}{2}}{\sin\frac{h}{2}} \sin\left(\frac{a+h+a+nh}{2}\right)$$

$\cos A - \cos B = 2\sin\frac{A+B}{2}.\sin\frac{B-A}{2}$

Series represented by definite integrals:

The definition of a definite integral as the limit of a sum enables us to evaluate ~~est~~ easily the limits of the sums of certain series, when the number of terms tends to infinity by identifying them with some definite integrals.

By definition, we know

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh), \text{ when } nh=b-a.$$

$$= \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left(a+r\,\frac{b-a}{n}\right)$$

In special case, when $a=0$, $b=1$, then $h=\frac{1}{n}$ & we get

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

[As if we write x for $\frac{r}{n}$ & dx for $1/n$]

or, putting $h = 1/n \Rightarrow \int_0^1 f(x)\,dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(rh)$.

[As if we write x for rh & dx for h].

Prob: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right]$

Soln: Given, $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right]$

$$= \lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{1}{n+r}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{1}{1+r/n}\right)$$

$$= \int_0^1 \frac{1}{1+x}\,dx = \ln(1+x)\Big]_0^1 = \ln 2.$$ Ans

Prob: Evaluate $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right]$

Soln: Given, $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{n}{n^2+r^2}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left[\frac{1}{1+(r/n)^2}\right]$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}x\Big]_0^1 = \frac{\pi}{4}.$$ Ans

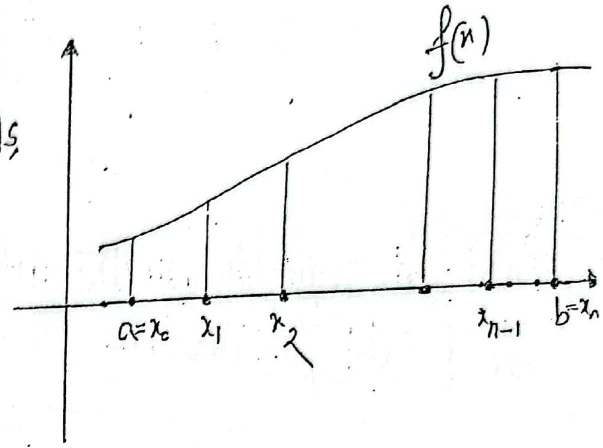
Fundamental theorem of calculus:

If f is continuous on $[a,b]$ and if F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof: Divide the interval $[a,b]$ into n equal subintervals, so we get

$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$



Here the subintervals are $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, whose lengths, as usual, we denote by $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$.

Since F is an antiderivative of f then $F'(x) = f(x)$ for all x in $[a,b]$, so F satisfies the conditions of Mean Value theorem on each subinterval. Hence we can find the points $x_1^*, x_2^*, \ldots, x_n^*$ in the respective subintervals such that

$$\frac{f(B)-f(A)}{B-A} = f$$

$$F(x_1) - F(x_0) = F'(x_1^*)(x_1 - x_0) = f(x_1^*)\Delta x_1$$

$$F(x_2) - F(x_1) = F'(x_2^*)(x_2 - x_1) = f(x_2^*)\Delta x_2$$

$$\vdots$$

$$F(x_n) - F(x_{n-1}) = F'(x_n^*)(x_n - x_{n-1}) = f(x_n^*)\Delta x_n$$

Adding all these, we get

$$F(x_n) - F(x_0) = \sum_{r=1}^{n} f(x_r^*)\Delta x_r$$

$$\Rightarrow \boxed{F(b) - F(a) = \sum_{r=1}^{n} f(x_r^*)\Delta x_r} \quad \text{——(1)}$$

Now let us increase n in such a way that $\Delta x_r \to 0$. Then the right hand side of (1) approaches to $\int_a^b f(x)dx$. Here the left hand side of (1) is independent of n, that is, the left hand side of (1) remains constant as n increase.

$$\boxed{F(b)-F(a) = \int_a^b f(x)\,dx}$$

Proved.

<u>Note</u>: Sometimes the difference $F(b)-F(a)$ can be written as $[F(x)]_a^b$, that is, $[F(b)]_a^b = F(b) - F(a)$.

Thus $$\boxed{\int_a^b f(x)\,dx = [F(x)]_a^b = F(b)-F(a)}$$ □

<u>Ex</u>: Evaluate $\int_1^2 x\,dx$

<u>Soln</u>: Here $F(x) = \frac{x^2}{2}$ is an antiderivative of $f(x) = x$.

$$\therefore \int_1^2 x\,dx = [F(x)]_1^2 = F(2)-F(1) = \frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}$$

Ans

<u>Note</u>: (I) The above thm establishes a connection between the integration as a particular kind of summation and the integration as an operation inverse to differentiation.

(II) From the above theorem it is clear that the definite integral is a function of its upper & lower limits and not of the independent variable x.

Some properties of definite integral:

(i) $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$.

Proof: We know $\int_a^b f(x)\,dx = F(b) - F(a)$

Similarly, $\int_a^b f(z)\,dz = F(b) - F(a)$

$$\therefore \boxed{\int_a^b f(x)\,dx = \int_a^b f(z)\,dz}$$

(ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Proof: We know, $\int_a^b f(x)\,dx = F(b) - F(a)$

Similarly, $-\int_b^a f(x)\,dx = -[F(a) - F(b)]$

$= F(b) - F(a)$

So $$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

That is, an interchange of the limits changes the sign of the integral.

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a<c<b.$

Proof: we know, $\int_a^b f(x)\,dx = F(b) - F(a)$.

Now $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = F(c) - F(a) + F(b) - F(c)$

$= F(b) - F(a)$

$$\therefore \boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}$$

(iv) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Proof: R.H.S $= \int_0^a f(a-x)\,dx$

Let $a-x=z$

$\therefore dx = -dz$

x	0	a
z	a	0

$= \int_a^0 f(z)(-dz)$

$= -\int_a^0 f(z)\,dz$

$= \int_0^a f(z)\,dz \quad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$

$= \int_0^a f(x)\,dx \quad \left[\because \int_a^b f(x)\,dx = \int_a^b f(z)\,dz\right]$

$=$ L.H.S.

$$\therefore \boxed{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx}$$

(v) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{When } f(x) \text{ is even} \\ 0, & \text{When } f(x) \text{ is odd} \end{cases}$

Proof: We know,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \quad \text{—— (1)}$$

For 1st integral of R.H.S of (1), let

$x = -z$

$\therefore dx = -dz$

x	0	$-a$
z	0	a

$$\therefore (1) \Rightarrow \int_{-a}^{a} f(x)\,dx = \int_{a}^{0} f(-z)(-dz) + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{a}^{0} f(-z)\,dz + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(-z)\,dz + \int_{0}^{a} f(x)\,dx \quad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \quad \left[\because \int_a^b f(x)\,dx = \int_a^b f(z)\,dz\right]$$

$$= \int_{0}^{a} \{f(-x) + f(x)\}\,dx \quad \text{—— (2)}$$

When $f(x)$ is even, then $f(-x) = f(x)$ & (2) $\Rightarrow$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

When $f(x)$ is odd, then $f(-x) = -f(x)$ & (2) $\Rightarrow$

$$\int_{-a}^{a} f(x)\,dx = 0$$

Thus $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x), & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases}$

Prob: Evaluate $\int_{-2}^{2} x^9(1-x^2)^7 dx$.

Sol^n: Let $I = \int_{-2}^{2} x^9(1-x^2)^7 dx$

$$= \int_{-2}^{2} f(x)\,dx \quad \text{---(i)}$$

where $f(x) = x^9(1-x^2)^7$

Here $f(-x) = (-x)^9\{1-(-x)^2\}^7$

$= -x^9(1-x^2)^7$

$= -f(x)$

So $f(x)$ is odd.

$\therefore$ (i) $\Rightarrow$ $I = \int_{-2}^{2} f(x)\,dx = 0$

Ans

Prob: Evaluate $\int_0^{\pi} \frac{x \sin x\, dx}{1+\cos^2 x}$

Sol^n: Let $I = \int_0^{\pi} \frac{x \sin x\, dx}{1+\cos^2 x}$ ---(i)

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx \quad \text{---(ii)}$$

Now (i)+(ii) $\Rightarrow$ $2I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\,dx$

Let $\cos x = z$

$\therefore -\sin x\,dx = dz$

$\Rightarrow \sin x\,dx = -dz$

x	0	π
z	1	-1

$$= \pi \int_1^{-1} \frac{-dz}{1+z^2}$$

$$= \pi \int_{-1}^{1} \frac{dz}{1+z^2}$$

$\Rightarrow 2I = \pi\left[\tan^{-1} z\right]_{-1}^{1} = \pi\left[\tan^{-1}(1) - \tan^{-1}(-1)\right] = \pi\left[\frac{\pi}{4} + \frac{\pi}{4}\right]$

$\Rightarrow \boxed{I = \frac{\pi^2}{4}}$

Ans

Prob: Evaluate $\int_0^{\pi/2} \frac{dx}{1+\cot x}$

Sol^n: Let $I = \int_0^{\pi/2} \frac{dx}{1+\cot x}$

$$= \int_0^{\pi/2} \frac{dx}{1+\frac{\cos x}{\sin x}}$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin x \, dx}{\sin x + \cos x} \quad \text{——— (1)}$$

$$= \int_0^{\pi/2} \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \quad \text{——— (2)}$$

Now (1)+(2) $\Rightarrow$ $2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\cos x + \sin x} dx$

$$= \int_0^{\pi/2} 1 . dx$$

$$\Rightarrow 2I = [x]_0^{\pi/2} = \frac{\pi}{2} - 0$$

$$\Rightarrow \boxed{I = \pi/4} \quad \text{Ans}$$

$\int 1 = x + C$

Prob: Evaluate $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

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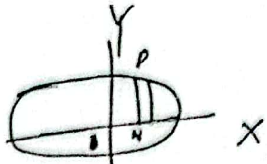
the area of the quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ betw[een the] major and minor axes.



Clearly, the area bounded by the curve, the x-axis & the ordin[ates] $x=0$ and $x=a$, the require area $= \int_0^a y\,dx = \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$= \frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\,d\theta$ putting $x = a\sin\theta$

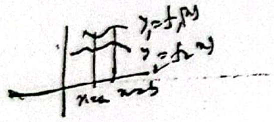
$= \frac{ab}{2}\int_0^{\pi/2} 2\cos^2\theta\,d\theta = \frac{ab}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta = \frac{ab}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{ab}{2}\cdot\frac{\pi}{2} = \frac{1}{4}\pi ab$

area of the whole ellipse is $4\cdot\frac{1}{4}\pi ab = \pi ab$.

area of a quadrant of the circle $x^2+y^2=a^2$ is $\frac{1}{4}\pi a^2$

Putting $b=a$, the area of a quadrant of the circle ~~and the whole circle~~ πr^2. # Find the area above the x-axis included between the parabola $y^2=ax$ & the circle $x^2+y^2=2ax$ | $x=2a$

Area between two given curves and two given ordinates is $\int_a^b (y_1-y_2)\,dx$



{ y axis given on $\int \{f_1(y) - f_2(y)\}\,dy$

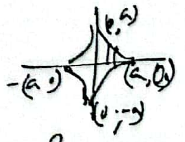
Find the area bounded by the curve $y=\sin x$, the axis x and the straight lines $x=0$ and $x=\pi$. $\int_0^{\pi} y\,dx = \int_0^{\pi}\sin x\,dx = -\cos\pi + \cos 0 = 1+1 = 2$ square unit

Find the whole area of the hypocycloid $x^{2/3}+y^{2/3}=a^{2/3}$.

Here $x^{2/3}+y^{2/3}=a^{2/3} \Rightarrow y = (a^{2/3}-x^{2/3})^{3/2}$ The area of 1st quadrant $\int_0^a y\,dx$

$= \int_0^a (a^{2/3}-x^{2/3})^{3/2}\,dx$ let $x = a\cos^3 t \Rightarrow dx = -3a\cos^2 t\sin t\,dt$

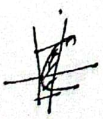
$x=0$ then $t=\pi/2$, $x=a, t=0$



$I = \int_{\pi/2}^{0}(a^{2/3}-a^{2/3}\cos^2 t)^{3/2}(-3a\cos^2 t\sin t)\,dt = -3a^2\int_{\pi/2}^{0}\sin^4 t\cos^2 t\,dt = \frac{3}{32}\pi a^2$

Then whole area of this hypocycloid is $4\cdot\frac{3}{32}\pi a^2 = \frac{3}{8}\pi a^2$

Show that the area of between the parabola $y^2=4x$ & the st. line $y=2x-4$ is 9.



Soln: The intersection point of $y^2=4x$ & $y=2x-4$ are $(1,-2)$ & $(4,4)$. So the area between the parabola $y^2=4x$ & $y=2x-4$ is $\int_{-2}^{4}\{f(y)-f(y)\}\,dy = \int_{-2}^{4}\left(\frac{y+4}{2}-\frac{y^2}{4}\right)dy = 9$