ICT-1101 Math (Differential and Intregal Calculas)

Book referrences:

1. Differential calculus by Motin & Chakraborty.
2. Integral calculus by "
3. Calculus by Howard Anton.

Function

$A = \{a, b\}$

$B = \{1, 2, 3\}$

$\therefore A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

$R = \{(x, y) : x \in A, y \in B, x = a\}$

$R = \{(a,1), (a,2), (a,3)\}$

$\therefore R \subseteq A \times B$

Function is a subset of relation.

Function: If A and B are two non-empty set and f is a value or correspondence between A and B such that for every element in A there is a f- correspondence to an unique element in B, then f is said to be a function from A to B and denoted by $f : A \longrightarrow B$.

If $y = f(x)$ $\forall x \in A, y \in B$ then x is called the argument and y is called image of x. In otherwords x is called preimage of y.

Domain: If $f: A \longrightarrow B$ is a function then A is called domain of f.

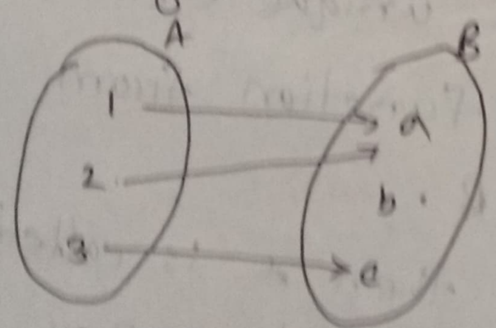
~~Rang~~

Codomain: If $f: A \longrightarrow B$ is a function then B is called co-domain of f.

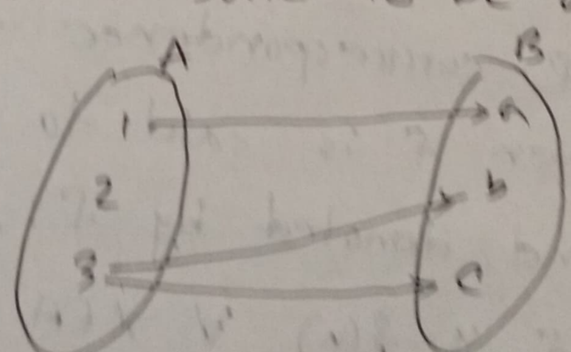
Range: If B is the set of all values of y or $f(x)$ corresponding each of the values or points x in domain A of the formula $y = f(x)$ then B is called range of the formula.

One-one function: If $f: A \longrightarrow B$ is a function and every element in A have different images in B, then f is said to be one-one function.

~~If~~ Again, if for every element in B have preimges in A, then f is said to be onto functio



function



not function

A function $f(x)$ is defined as follows: $f(x) = \begin{cases} x+1, & \text{when } x \geq 0 \\ 1-x, & \text{when } x < 0 \end{cases}$

Draw the graph of the function $f(x)$ and find domain and range of $f(x)$. Is $f(x)$ one-one?

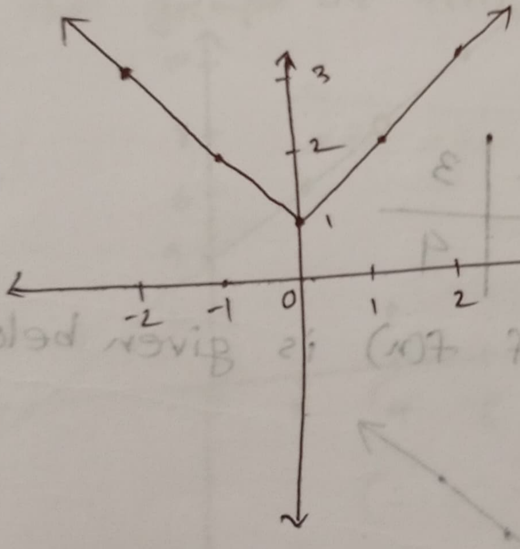
Sol^n: Given,

$$f(x) = \begin{cases} x+1, & \text{when } x \geq 0 \\ 1-x, & \text{when } x < 0 \end{cases}$$

We construct a chart for different values of x:

x	-2	-1	0	1	2
$f(x)$	3	2	1	2	3

So, the graph of $f(x)$ is given below:



domain, $f = \mathbb{R}$

range, $f = [1, \infty)$

Since, every element in domain have not different images in range f, so $f(x)$ is not one-one.

(Ans)

A function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} x, & \text{when } x<1 \\ 1+x, & \text{when } x>1 \\ 3/2, & \text{when } x=1 \end{cases}$$

Draw the ~~functio~~ graph of the function $f(x)$ and find domain and range of $f(x)$. Is $f(x)$ ~~is~~ one-one?

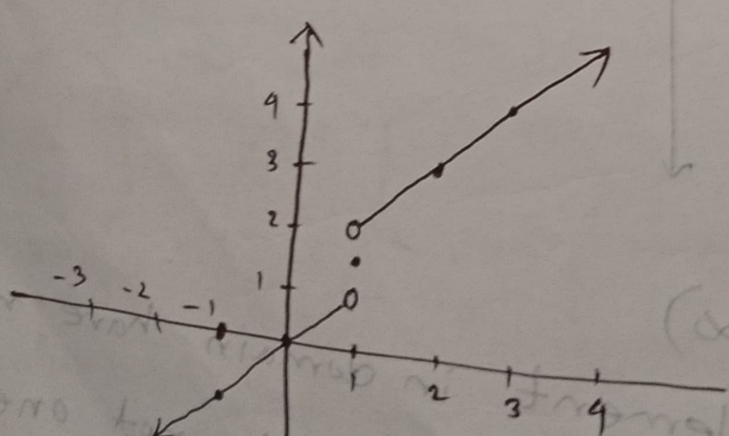
Sol^n: Given,

$$fx = \begin{cases} x, & \text{when } x<1 \\ 1+x, & \text{when } x>1 \\ 3/2, & \text{when } x=1 \end{cases}$$

We construct a chart for different values of x:

x	-1	0	1	2	3
$f(x)$	-1	0	$3/2$	3	4

So, the graph of $f(x)$ is given below:



domain, $f = (-\infty, 1) \cup (1, \infty) \cup \{1\} = \mathbb{R}$

range, $f = (-\infty, 1) \cup (2, \infty) \cup \{3/2\}$

$\therefore f(x)$

03-11-2024

Draw the graph of the following function:

$$f(x) = \begin{cases} \frac{x^2-16}{x-4}, & \text{when } x \neq 4 \\ 2, & \text{when } x = 4 \end{cases}$$

Also find the domain and range of $f(x)$. Is $f(x)$ one-one?

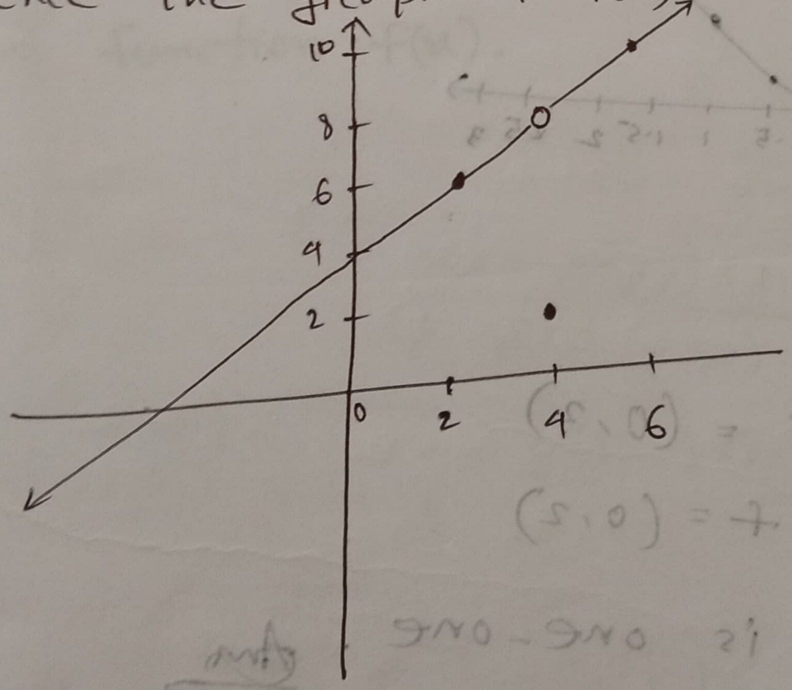
Solⁿ: Construct a chart for different values of x and $f(x)$.

when $x \neq 4$ and, when $x = 4$

$\therefore f(x) = 2$

x	2	4	6
$f(x)$	6	8	10

Hence the graph of $f(x)$ is given below:



$\therefore$ domain, $f = \mathbb{R}$

$\therefore$ range, $f = \mathbb{R} - \{8\}$

Also, $f(x)$ is not one-one. Ans

$f(x) = \begin{cases} x^2, & \text{when } 0<x<1 \\ x, & \text{when } 1\le x<2 \end{cases}$

Draw the graph of the function f(x) and find the domain and range of f(x). Is f(x) one-one

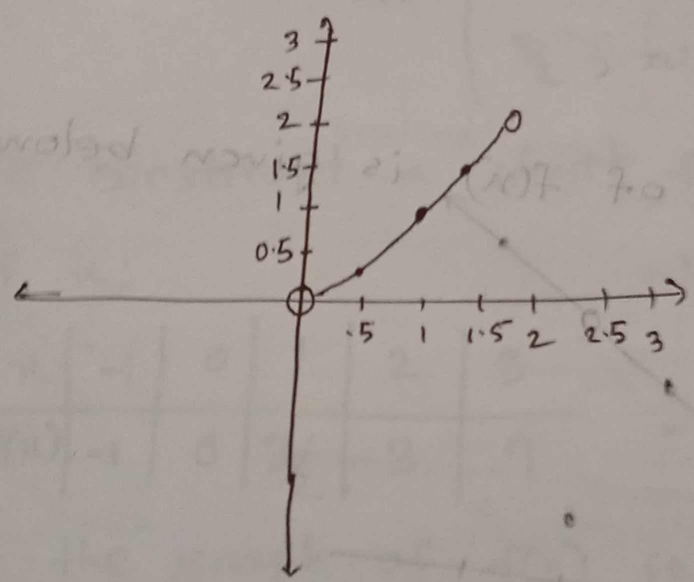
Solⁿ: Construct a chart ~~for~~ for different values of

when $0<x<1$

x	0	0.5	1
$f(x)$	0	0.25	1

when $1\le x<2$

x	1	1.5	2
$f(x)$	1	1.5	2



∴ domain, f = (0, 2)

∴ range, f = (0, 2)

Also, f(x) is one-one. Ans

Limit

If the values of $f(x)$ become arbitrarily close to a single number l as the values of a variable x approaches to a from both sides of a (Right and Left) then l is called the limit of the function $f(x)$. It is denoted by $\lim_{x \to a} f(x) = l$]

Or,

If for each number $\epsilon > 0$, there corresponds a small positive number δ such that $|f(x) - l| < \epsilon$ when $0 < |x-a| < \delta$ then l is called the limit of the function $f(x)$.

Evaluate the following limits:

(*) $\lim_{x \to \infty} \frac{1^2+2^2+3^2+\dots+n^2}{n^3}$

$= \lim_{x \to \infty} \frac{\frac{n(n+1)(2n+1)}{6}}{n^3}$

$= \lim_{n \to \infty} \frac{(n+1)(2n+1)}{6n^2}$

$= \frac{1}{6} \lim_{x \to \infty} \left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

$= \frac{1}{6} \times \left(1+\frac{1}{\infty}\right) \times \left(2+\frac{1}{\infty}\right)$

$= \frac{1}{3}$ Ans

(*) $\lim_{x \to 2} \frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}$

$= \lim_{x \to 2} \frac{(x^2-4)(\sqrt{x+2}+\sqrt{3x-2})}{(x+2)-(3x-2)}$

$= \lim_{x \to 2} \frac{(x+2)(x-2)(\sqrt{x+2}+\sqrt{3x-2})}{-2(x-2)}$

$= -\frac{1}{2} \times (2+2) \times \left(\sqrt{2+2}+\sqrt{3\times 2-2}\right)$

$= -8$

Ans

(*) $\lim\limits_{x \to a} \dfrac{x^{9/2} - a^{9/2}}{x^{1/2} - a^{1/2}}$ $\left[\dfrac{0}{0} \text{ form}\right]$

$= \lim\limits_{x \to a} \dfrac{\frac{9}{2} x^{9/2 - 1}}{\frac{1}{2} x^{1/2 - 1}}$

$= 9 \lim\limits_{x \to a} \dfrac{x^{7/2}}{x^{-\frac{1}{2}}}$

$= 9 \lim\limits_{x \to a} x^{7/2 + \frac{1}{2}}$

$= 9a^4$ Ans

05-11-2024

Evaluate the following limits:

(*) $\lim\limits_{x \to \infty} (3^n + 2^n)^{1/n}$

$= \lim\limits_{n \to \infty} \left\{3^n \left(1 + \dfrac{2^n}{3^n}\right)\right\}^{1/n}$

$= 3 \lim\limits_{n \to \infty} \left\{1 + \left(\dfrac{2}{3}\right)^n\right\}^{1/n}$

$= 3 \lim\limits_{n \to \infty} \left[1 + \dfrac{1}{n}\left(\dfrac{2}{3}\right)^n + \dfrac{\frac{1}{n}\left(\frac{1}{n} - 1\right)}{\lfloor 2} \left(\dfrac{2}{3}\right)^{2n} + \dfrac{\frac{1}{n}\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)}{\lfloor 3} \left(\dfrac{2}{3}\right)^{3n} + \cdots\right.$

$= 3(1 + 0)$ $\left[\text{since } n \to \infty,\ \dfrac{1}{n} \to 0,\ \left(\dfrac{2}{3}\right)^n \to 0\right]$

$= 3$

Ans

(*) $\lim\limits_{x \to 0} \dfrac{x^2}{1-\cos x}$ $\left[\dfrac{0}{0} \text{ form}\right]$

$= \lim\limits_{x \to 0} \dfrac{2x}{\sin x}$

$= \lim\limits_{x \to 0} \dfrac{2}{\cos x}$

$= \dfrac{2}{\cos 0^\circ}$

$= \dfrac{2}{1}$

$= 2$ Ans

(*) $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x}$ $\left[\dfrac{0}{0} \text{ form}\right]$

$= \lim\limits_{x \to 0} \dfrac{\sin x}{1}$

$= \dfrac{\sin 0^\circ}{1}$

$= 1$

Find the limiting value if $\lim_{x \to a} f(x)$ exists; where

i) $f(x) \begin{cases} 2-3x, & \text{when } x < 0 \\ 3x-2, & \text{whe } x \geq 0 \end{cases}$ and $a = 0$.

Sol^n: Here, $R.H.L = \lim_{x \to 0^+} f(x)$

$$= \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} (3h-2)$$

$$= -2$$

$$L.H.L = \lim_{x \to 0^-} f(x)$$

$$= \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} (2+3h)$$

$$= 2$$

Since $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$

So, $\lim_{x \to 0} f(x)$ does not exists.

Find the limiting value if $\lim_{x \to a} f(x)$ exists; when

i) $f(x) = \begin{cases} \tan x/2, & \text{when } x < \frac{\pi}{2} \\ 3 - \frac{\pi}{2}, & \text{when } x = \frac{\pi}{2} \\ \frac{x^3 - \frac{\pi^3}{8}}{x - \frac{\pi}{2}}, & \text{when } x > \frac{\pi}{2} \end{cases}$ and $a = \frac{\pi}{2}$

Sol^n: Here,

$$R.H.L = \lim_{x \to \pi/2^+} f(x)$$

$$= \lim_{h \to 0} f\left(\frac{\pi}{2} + h\right)$$

$$= \lim_{h \to 0} \frac{\left(\frac{\pi}{2} + h\right)^3 - \frac{\pi^3}{8}}{\frac{\pi}{2} + h - \frac{\pi}{2}} \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{h \to 0} \frac{3\left(\frac{\pi}{2} + h\right)^2 \cdot 1 - 0}{1}$$

$$= 3\left(\frac{\pi}{2} + 0\right)^2$$

$$= \frac{3\pi^2}{4}$$

$$L.H.L = \lim_{x \to \pi/2^-} f(x)$$

$$= \lim_{h \to 0} f\left(\frac{\pi}{2} - h\right)$$

$$= \lim_{h \to 0} \tan\left(\frac{\pi}{4} - h\right)$$

$$= \cancel{\lim_{h \to 0} \cot \frac{\pi}{2}}$$

$$= \tan\left(\frac{\pi}{4} - 0\right)$$

$$= 1$$

~~in~~ since, $\lim\limits_{x \to \frac{\pi}{2}^{+}} f(x) \neq \lim\limits_{x \to \frac{\pi}{2}^{-}} f(x)$

So, $\lim\limits_{x \to \frac{\pi}{2}} f(x)$ does not exists.

Find the limiting value if $\lim\limits_{x \to a} f(x)$ exists, where

i) $f(x) = \begin{cases} x^2+1, & \text{when } x > 0 \\ 1, & \text{when } x = 0 \\ 1+x, & \text{when } x < 0 \end{cases}$ and $a = 0$.

Sol^n: Here, $R.H.L = \lim\limits_{x \to 0^{+}} f(x)$

$= \lim\limits_{h \to 0} f(0+h)$

$= \lim\limits_{h \to 0} (h^2+1)$

$= 1$

$L.H.L = \lim\limits_{x \to 0^{-}} f(x)$

$= \lim\limits_{h \to 0} f(0-h)$

$= \lim\limits_{h \to 0} (1-h)$

$= 1$

Since, $\lim\limits_{x \to 0^{+}} f(x) = \lim\limits_{x \to 0^{-}} f(x)$

So, $\lim\limits_{x \to 0} f(x)$ exists and $\lim\limits_{x \to 0} f(x) = 1$ Ans.

অবিচ্ছিন্ন

Continuity: A function $f(x)$ is said to continuous at $x=a$ if limiting value of $f(x)$ at $x=a$ exists and $f(x)$ tends to $f(a)$ if x tends to a.

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$$

✻ Show that, $f(x) = \begin{cases} x^2+1, & \text{when } x>0 \\ 1, & \text{when } x=0 \\ x+1, & \text{when } x<0 \end{cases}$ will be continuous at $x=0$.

Sol^n: Here,

$$\cancel{R.H.L} \lim_{x \to 0^+} f(x) = 1$$

$$\lim_{x \to 0^-} f(x) = 1$$

and $f(0) = 1$

So, $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = f(0)$, hence $f(x)$ is continuous at $x=0$.

Tes the continuity of $f(x)$ at $x=a$, where

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases} \quad \text{and } a = 0.$$

Sol^n: Here,

$$R.H.L. = \lim_{x \to 0^+} f(x)$$
$$= \lim_{h \to 0} f(0+h)$$
$$= \lim_{h \to 0} e^{-\frac{1}{h^2}}$$
$$= e^{-\infty}$$
$$= 0$$

~~L.H.L~~

$$L.H.L = \lim_{x \to 0^-} f(x)$$
$$= \lim_{h \to 0} f(0-h)$$
$$= \lim_{h \to 0} e^{-\frac{1}{h^2}}$$
$$= e^{-\infty}$$
$$= 0$$

and $f(a) = 1$

So, since $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) \neq f(a)$

... discontinuous at $x = 0$.

Differentiability: A function $f(x)$ is said to be differentiable at $x=a$ if

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$$

Test the differentiability of $f(x)$ at $x=0$, where $f(x)=\begin{cases} e^{-\frac{1}{x^2}}, & \text{when } x\neq 0 \\ 1, & \text{when } x=0 \end{cases}$

Soln: Here,

$$Rf'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{e^{-1/h^2}-1}{h}$$

$$= \frac{0-1}{0}$$

$$= -\infty$$

$$Lf'(0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{e^{-\frac{1}{h^2}}-1}{-h}$$

$$= \frac{0-1}{-0}$$

$$= \infty$$

since, $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \neq \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$

So, $f(x)$ does not differentiable.

Sketch the graph of f(x), also find the domain and range of f(x). Is f(x) is one-one? Test the continuity and differentiability of f(x) at x = a. when $f(x) = \begin{cases} -x, & \text{when } x \le 0 \\ x, & \text{when } 0 < x \le 1 \\ 2-x, & \text{when } x > 1 \end{cases}$

and a = 0.

<u>Soln:</u> Construct a chart for different values of x:

when, $x \le 0$

x	-1	-2	0	1
$f(x)$	1	2	0	-1

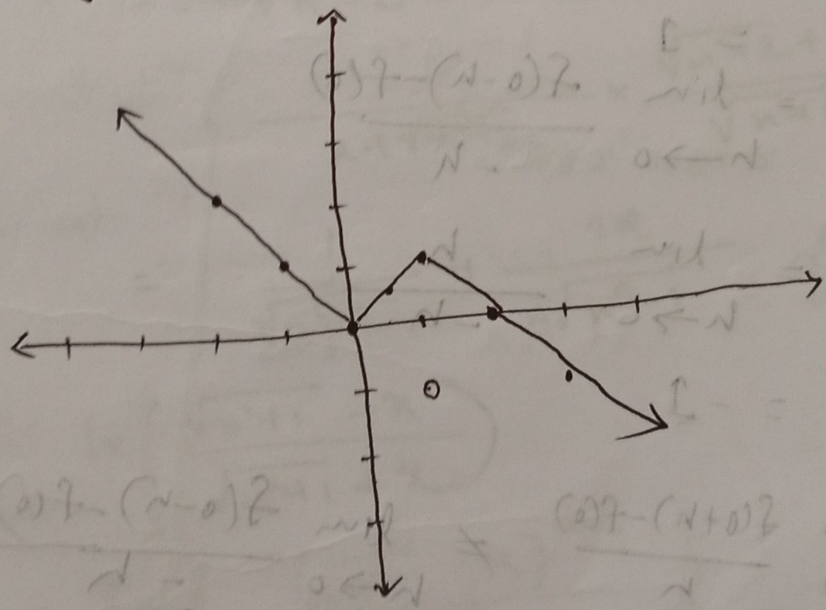
when, $0 < x \le 1$

x	0	0.5	1
$f(x)$	0	0.5	1

when, $x > 1$

x	1	2	3
$f(x)$	1	0	-1

So, the graph of f(x) is given below.



domain = $\mathbb{R}$

range = $\mathbb{R}$

Now,

$L.H.L = \lim_{x \to 0^-} f(x)$

$= \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} h$

$= 0$

$R.H.L = \lim_{x \to 0^+} f(x)$

$= \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} h$

$= 0$

and, $f(0) = 0$

Hence

$L.H.L = R.H.L = f(0)$

So, $f(x)$ is continuous at $x=0$.

Again, $Rf'(0) = \lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h \to 0} \frac{h}{h}$

$= 1$

$Lf'(0) = \lim_{h \to 0} \frac{f(0-h)-f(0)}{-h}$

$= \lim_{h \to 0} \frac{h}{-h}$

$= -1$

Hence,

$$\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} \neq \lim_{h \to 0} \frac{f(0-h)-f(0)}{-h}$$

So, $f(x)$ does not diffentiable. Ans

Find $\frac{dy}{dx}$, where

i) $y = \ln(\sec x + \tan x)$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\{\ln(\sec x + \tan x)\}$

$= \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x)$

$= \sec x$ Ans

ii) $y = \ln(x + \sqrt{x^2+2}) + \sec^{-1} x^2 + \sqrt{5}$

$\therefore \frac{dy}{dx} = \frac{dy}{dx}\{\ln(x + \sqrt{x^2+2}) + \sec^{-1} x^2 + \sqrt{5}\}$

$= \frac{1}{x + \sqrt{x^2+2}}\left(1 + \frac{1}{2\sqrt{x^2+2}} \cdot 2x \cdot 1\right) + \frac{2x}{x^2\sqrt{x^4-1}}$

$= \frac{1}{x + \sqrt{x^2+2}} \times \frac{\sqrt{x^2+2} + x}{\sqrt{x^2+2}} + \frac{2x}{x^2\sqrt{x^4-1}}$

$= \frac{1}{\sqrt{x^2+2}} + \frac{2x}{x\sqrt{x^4-1}}$ Ans

iii) $y = \ln\left(\frac{\sqrt{x^2+1} - x}{\sqrt{x^2+1} + x}\right)$

$= \ln\left\{\frac{(\sqrt{x+1} - x)^2}{(\sqrt{x^2+1})^2 - x^2}\right\}$

$= \ln\left\{\frac{(\sqrt{x+1} - x)^2}{x^2+1 - x^2}\right\}$

$\Rightarrow y = 2\ln(\sqrt{x^2+1} - x)$

$\therefore \frac{dy}{dx} = 2\frac{d}{dx}\{\ln(\sqrt{x^2+1} - x)\}$

$= 2 \cdot \frac{1}{\sqrt{x^2+1} - x} \cdot \left(\frac{2x}{2\sqrt{x^2+1}} - 1\right)$

$= 2 \cdot \frac{1}{\sqrt{x^2+1} - x} \cdot \frac{x - \sqrt{x^2+1}}{\sqrt{x^2+1}}$

$= \frac{-2}{\sqrt{x^2+1}}$ Ans

iv) $y = \tan^{-1}\frac{4\sqrt{x}}{1-4x}$

$= \tan^{-1}\frac{2 \cdot 2\sqrt{x}}{1-(2\sqrt{x})^2}$

$= 2\tan^{-1}2\sqrt{x}$

$\therefore \frac{dy}{dx} = 2 \cdot \frac{1}{1+4x} \cdot \frac{2}{2\sqrt{x}}$

$=$

or,

let $2\sqrt{x} = \tan\theta$

$\Rightarrow \theta = \tan^{-1}(2\sqrt{x})$

$\therefore y = \tan^{-1}\frac{4\sqrt{x}}{1-4x}$

$= \tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta}$

$= \tan^{-1}\tan 2\theta$

$= 2\theta$

$= 2\tan^{-1}(2\sqrt{x})$

v) $y = \cos^{-1}(2x\sqrt{1-x^2})$ let, $x = \sin\theta$

$\Rightarrow \theta = \sin^{-1}x$

$= \cos^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$

$= \cos^{-1}(\sin 2\theta)$

$= \cos^{-1}\cos\left(\frac{\pi}{2} - 2\theta\right)$

$= \frac{\pi}{2} - 2\theta$

$= \frac{\pi}{2} - 2\sin^{-1}x$

$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 2\sin^{-1}x\right)$

$= 0 - \frac{2}{\sqrt{1-x^2}}$

$= -\frac{2}{\sqrt{1-x^2}}$ (Ans)

45. $y = \sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$ let, $z = \frac{a+b\cos x}{b+a\cos x}$

$\Rightarrow y = \sin^{-1}z$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-z^2}} \cdot \frac{dz}{dx}$ ......

Ex - 5(B)

11

$$y = \left\{ \frac{x}{x + \sqrt{a^2 - x^2}} \right\}^n$$

$$\Rightarrow \frac{dy}{dx} = n\left(\frac{x}{x + \sqrt{a^2 - x^2}} \right)^{n-1} \cdot \frac{d}{dx}\left(\frac{x}{x + \sqrt{a^2 - x^2}} \right)$$

$$\cancel{y = \frac{x(x - \sqrt{a^2 - x^2})}{-a^2}}$$

20) $y = x^{x^x}$

15

$$y = (\sin x)^{\cos x}$$

$$\Rightarrow \ln y = \cos x \ln \sin x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos x \cdot \frac{1}{\sin x} \cdot \cos x + \ln \sin x\,(-\sin x)$$

Ex - 5(c)

5. $\ln(xy) = x^2+y^2$

Differentiating both sides with respect to x,

$$\frac{1}{xy}\left(x\frac{dy}{dx}+y\right) = 2x+2y\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x - \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1-2y^2}{y}\right) = \frac{2x^2-1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(2x^2-1)}{x(1-2y^2)} \quad \text{Ans}$$

Differentiate $\tan^{-1}\frac{2x}{1-x^2}$ with respect to $\sin^{-1}\frac{2x}{1+x^2}$

Sol^n: Let, $y = \tan^{-1}\frac{2x}{1-x^2}$ and $z = \sin^{-1}\frac{2x}{1+x^2}$

$$\therefore \frac{dy}{dz} = \frac{dy/dx}{dz/dx} \quad \text{------} \; (1)$$

Now,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{2x}{1-x^2}\right)$$

let $x = \tan\theta$

$\Rightarrow \theta = \tan^{-1}x$

$\Rightarrow \frac{dx}{d\theta} = \sec^2\theta$

$$= \frac{d}{d\theta}\left(\tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta}\right)\cdot\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}(2\theta)\frac{1}{\sec^2\theta}$$

$$= 2\cos^2\theta$$

$2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2} = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2}$

Again,

$$\frac{dz}{dx} = \frac{d}{dx}\sin^{-1}\frac{2x}{1+x^2}$$

let, $x = \tan\theta$

$\Rightarrow \frac{dx}{d\theta} = \sec^2\theta$

$$= \frac{d}{d\theta}\left(\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}\right)\cdot\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}\sin^{-1}\sin 2\theta\cdot\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}(2\theta)\cdot\frac{1}{\sec^2\theta}$$

$$= 2\cos^2\theta$$

So, from equation (i), we get,

$$\frac{dy}{dz} = \frac{2\cos^2\theta}{2\cos^2\theta} = 1 \quad \text{(Ans)}$$

Differentiate $x^{\sin^{-1}x}$ with respect to $\ln x$.

Solⁿ: Let, $y = x^{\sin^{-1}x}$ and $z = \ln x$

$$\therefore \frac{dy}{dz} = \frac{dy/dx}{dz/dx} \quad \text{--- (1)}$$

Now, $y = \cancel{x\sin^{-1}x}\ x^{\sin^{-1}x}$

$$\Rightarrow \ln y = \sin^{-1}x\ \ln x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\sin^{-1}x}{x} + \ln x\frac{1}{\sqrt{1-x^2}} \quad \left[\text{Diffⁿ both sides w.r. to } x\right]$$

$$\Rightarrow \frac{dy}{dx} = x^{\sin^{-1}x}\left(\frac{\sin^{-1}x}{x} + \frac{\ln x}{\sqrt{1-x^2}}\right)$$

Again, $\frac{dz}{dx} = \frac{d}{dx}(\ln x)$

$$= \frac{1}{x}$$

So, from equation (1), we get,

$$\frac{dy}{dz} = x^{\sin^{-1}x + 1}\left(\frac{\ln x}{\sqrt{1-x^2}} + \frac{\sin^{-1}x}{x}\right) \quad \text{Ans}$$

If $x^y = e^{x-y}$, then prove that $\frac{dy}{dx} = \frac{\ln x}{(1+\ln x)^2}$

Sol^n: Given that,

$$x^y = e^{x-y}$$

$$\Rightarrow y \ln x = x - y \quad \text{.......(i)}$$

$$\Rightarrow \frac{y}{x} + \ln x \frac{dy}{dx} = 1 - \frac{dy}{dx} \quad \left[\text{diff}^n \text{ both sides w.r.to } x\right]$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{\ln x}{\cancel{y}} + 1\right) = 1 - \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}(\ln x + 1) = \frac{x-y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x(\ln x + 1)} \quad \text{.......(ii)}$$

From (i), $y \ln x = x - y$

$$\Rightarrow y(\ln x + 1) = x$$

$$\Rightarrow y = \frac{x}{\ln x + 1}$$

→ (এই line $\frac{dy}{dx}$ বসালে proved হয়ে যাবে)

Using this in equation (ii) we get,

$$\frac{dy}{dx} = \frac{x - \frac{x}{1+\ln x}}{x(1+\ln x)}$$

$$= \frac{\ln x}{(1+\ln x)^2}$$ (proved)

Find the average rate of change of the function $y = x^2 - 1$ between $x = 2$ and $x = 3$ and find its rate of change at $x = 3$.

Solution: Let, $y = f(x) = x^2 - 1$

$\therefore f(3) = 3^2 - 1 = 8$

$\therefore f(2) = 2^2 - 1 = 3$

$\therefore$ Average rate of change ~~from~~ between $x=2$ and $x=3$ is, $\frac{f(3) - f(2)}{3-2}$

$$= \frac{8-3}{1}$$

$$= 5$$ Ans

Again, $\frac{dy}{dx} = 2x$

$$\therefore \left.\frac{dy}{dx}\right|_{x=3} = 2 \times 3$$

$$= 6$$ Ans

A point moves on the curve $y = x^3$ in such a way that when $x = 3$, the abscissa is increasing at the rate of 10 cm per second. At what rate is the ordinate increasing at that point?

Solⁿ: Given, $y = x^3$

Differentiate both sides with respect to time t, we get.

$$\frac{dy}{dt} = 3x^2 \frac{dx}{dt}$$

Give, when $x = 3$, $\frac{dx}{dt} = 10$ cm/sec

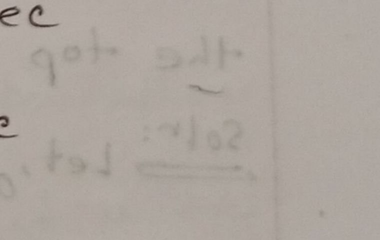
$$\therefore \frac{dy}{dt} = 3 \cdot 3^2 \cdot 10 \text{ cm/sec}$$

$$= 270 \text{ cm/sec} \quad \text{Ans}$$

The botton of a tank is a square of side 50cm. If 250 cubic cm of water is poured into the tank every minute, find the rate of increase in water level in the tank.

Solⁿ: Let, height of the tank is h cm,

so, volume of the tank, $V = 50 \times 50 \times h$ cm^3



$$\Rightarrow V = 2500h \text{ cm}^3$$

$$\Rightarrow \frac{dv}{dt} = 2500 \frac{dh}{dt}$$

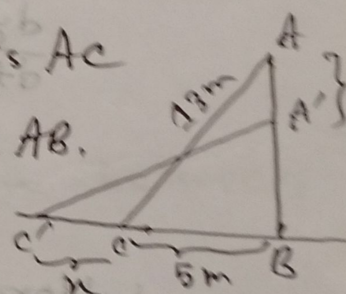
$$\frac{dh}{\dots} = \frac{1}{\dots} \frac{dv}{\dots}$$

$$\Rightarrow \frac{dh}{dt} = \frac{1}{2500} \times 250 \text{ cm}$$

$$= 0.1 \text{ cm}$$

A ladder 13 metres long, lean against a vertical wall. If the lower end of the ladder at a distance of 5 metre from the bottom of the wall is being moved away on the ground from the wall at the rate of 2 metre per minute, find how fast in the top of the ladder discending on the wall.

Solⁿ: Let, a 13 metre long ladder is AC and lean against a vertical wall AB.



So, $AC = 13$ and $BC = 5$

$$\therefore AB = \sqrt{AC^2 - BC^2}$$

$$= \sqrt{13^2 - 5^2}$$

$$= 12$$

Let, the ladder moved away x metre from the a the wall and goes to C' then top of the ladder descending y metre and goes to A'. So $BC' = 5+x$ and $A'B = 12 - y$.

अवरोहण / अवरोहिय

6

39, 40, 43, 44, 49, 50

→ Exer-4

$x=0, \frac{dy}{dt}=$

$\frac{dy}{dt}=1$

From, $\Delta A'BC'$,

$$A'B^2 = A'C'^2 - BC'^2$$

$$\Rightarrow (12-y)^2 = 13^2 - (5+x)^2$$

$$\Rightarrow$$

$$^nc_n = 1, \quad ^nc_1 = n, \quad ^nc_2 = \frac{n(n-1)}{2!}$$

17.11.19

Successive differentiation:

* State and prove Leibnitz's theorem.

<u>Statement:</u> If u and v are two function of x, then ~~$(uv)_n = u_n + {^nc_1} u_{n-1}$~~

$$(uv)_n = u_n v + {^nc_1} u_{n-1} v_1 + {^nc_2} u_{n-2} v_2 + \cdots\cdots + {^nc_n} u v_n$$

<u>Prove:</u> Let us consider,

$$(uv)_n = u_n v + {^nc_1} u_{n-1} v_1 + {^nc_2} u_{n-2} v_2 + \cdots\cdots + {^nc_n} u v_n \cdots\cdots (i)$$

We have show that statement (i) is true by mathematical induction formula.

Let $n=1$, then $(uv)_1 = u_1 v + u v_1$, so statement (i) is true for $n=1$.

Let us consider, statement (i) is true for $n=m$.

so,

$$(uv)_m = u_m v + {^mc_1} u_{m-1} v_1 + {^mc_2} u_{m-2} v_2 + \cdots\cdots + {^mc_m} u v_m \cdots\cdots (ii)$$

Differentiate both sides of equation (ii) with respect to x, we get

$$(uv)_{m+1} = u_{m+1} v + u_m v_1 + m(u_m v_1 + u_{m-1} v_2) + + \frac{m(m-1)}{2!}(u_{m-1} v_2 + u_{m-2} v_3) + \cdots\cdots + u_1 v_m + u v_{m+1}$$

$$= u_{m+1} v + (m+1) u_m v_1 + \left(m + \frac{m^2 - m}{2}\right) u_{m-1} v_2 + \cdots\cdots + u v_{m+1}$$

$\in \rightarrow$ belongs to | $\exists \rightarrow$ there exists | note: $\frac{m(m+1)}{2} = \frac{(m+1)(m+1-1)}{2!} = {}^{m+1}C_2$
$\ni \rightarrow$ such that | $\Rightarrow \rightarrow$ therefore
$\forall \rightarrow$ for all

$\Rightarrow (uv)_{m+1} = u_{m+1}v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + \cdots + {}^{m+1}C_{m+1} u v_{m+1}$

so, statement (i) is true for $n = m+1$.

since, statement (i) is true for $n=1$, hence it is true for $n=2$. Further it is true for $n=3$.

So for $\forall n \in \mathbb{Z}$, statement (i) is true.

(#) If $y = \tan^{-1}x$, then show that, $(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$.

Soln: Given, $y = \tan^{-1}x$

$\Rightarrow y_1 = \frac{1}{1+x^2}$

$\Rightarrow (1+x^2)y_1 = 1$

Again differentiating both sides, we get

$(1+x^2)y_2 + 2xy_1 = 0$

Applying Leibnitz's theorem, we get

$y_{n+2}(1+x^2) + {}^nC_1 y_{n+1} 2x + {}^nC_2 y_n \cdot 2 + {}^nC_3 y_{n-1} \cdot 0$
$+ y_{n+1} \cdot 2x + {}^nC_1 y_n \cdot 2 + {}^nC_2 y_{n-1} \cdot 0 = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2nxy_{n+1} + \frac{n(n-1)}{2} \cdot 2y_n + 2xy_{n+1} + 2ny_n + 0 = 0$

$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$ (Showed)

If $y = (\sin^{-1}x)^2$, then show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y = 0$

Soln: Given,

$$y = (\sin^{-1}x)^2$$

$$\Rightarrow y_1 = 2\sin^{-1}x \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\, y_1 = 2\sin^{-1}x$$

$$\Rightarrow \sqrt{1-x^2}\, y_2 + \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x) \cdot y_1 = \frac{2}{\sqrt{1-x^2}}$$ [di... side ...]

$$\Rightarrow (1-x^2)y_2 - xy_1 = 2$$

Applying both sides Leibnitz's theorem, we get,

$$y_{n+2}(1-x^2) + {}^nc_1 y_{n+1}(-2x) + {}^nc_2 y_n(-2) + {}^nc_3 y_{n-1} \cdot 0$$
$$-(y_{n+1} \cdot x + {}^nc_1 y_n \cdot 1 + {}^nc_2 y_{n-1} \cdot 0) = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - 2 \cdot \frac{n(n-1)}{2} y_n - xy_{n+1} - ny_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$$

If $y = \cos\{\ln(1+x)\}$, then show that,

$$(1+x)^2 y_{n+2} + (2n+1)(1+x) y_{n+1} + (n^2+1) y_n = 0$$

Solⁿ:

Given, $y = \cos\{\ln(1+x)\}$

$$\Rightarrow y_1 = -\sin\{\ln(1+x)\} \cdot \frac{1}{1+x} \cdot 1$$

$$\Rightarrow (1+x) y_1 = -\sin\{\ln(1+x)\}$$

$$\Rightarrow (1+x) y_2 + y_1 \cdot 1 = -\cos\{\ln(1+x)\} \cdot \frac{1}{1+x} \cdot 1$$

$$\Rightarrow (1+x)^2 y_2 + (1+x) y_1 = -y$$

$$\Rightarrow (1+x)^2 y_2 + (1+x) y_1 + y = 0$$

Applying both sides Leibnitg's theorem, we get,

$$y_{n+2}(1+x)^2 + {}^nc_1 y_{n+1} \, 2(1+x) + {}^nc_2 y_n \cdot 2 + y_{n+1}(1+x) + {}^nc_1 y_n + y_n \cdot 1 = 0$$

$$\Rightarrow (1+x^2) y_{n+2} + 2n(1+x) y_{n+1} + 2 \cdot \frac{n(n-1)}{2} y_n + (1+x) y_{n+1} + n y_n + y$$

$$\Rightarrow (1+x^2) y_{n+2} + (2n+1)(1+x) y_{n+1} +$$

If $y = e^{a\sin^{-1}x}$, then show that,

i) $(1-x^2)y_2 - xy_1 - a^2y = 0$

ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

Solⁿ: (i) Given,

$$y = e^{a\sin^{-1}x}$$

$$\Rightarrow y_1 = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2} \cdot y_1 = ay$$

$$\Rightarrow (1-x^2)y_1^2 = a^2y^2$$

$$\Rightarrow (1-x^2)2y_1y_2 + y_1^2(-2x) = a^2 2yy_1$$

$$\Rightarrow (1-x^2)y_2 - xy_1 = a^2y$$

$$\Rightarrow (1-x^2)y_2 - xy_1 - a^2y = 0 \quad \text{(proved)}$$

(ii) $(1-x^2)y_2 - xy_1 - a^2y = 0$

Applying both sides Leibnitz's theorem, we get,

$$y_{n+2}(1-x^2) + {}^nc_1 y_{n+1}(-2x) + {}^nc_2 y_n(-2) - (y_{n+1}x + {}^nc_1 y_n \cdot 1) - a^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} = 2nxy_{n+1} - 2\cdot\frac{n(n-1)}{2}y_n - xy_{n+1} - ny_n - a^2y_n$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$$

(proved)

Partial differentiation

$f(x,y) = a_1 xy + a_2 y x$

$\therefore f_x = a_1 y$ $\qquad \therefore f_{xx} = 0$

$\therefore f_y = a_2 x$ $\qquad \therefore f_{yy} = 0$

$\therefore f_{xy} = a_1$

$\therefore f_{yx} = a_2$

$(f_x)_y$

Chain rule: If $u = f(x,y,z)$, where $x = x(t)$, $y = y(t)$ and $z = z(t)$ then,

$$\frac{\delta u}{\partial t} = \frac{\partial u}{\delta t} \cdot \frac{\partial x}{\partial t} + \frac{\delta u}{\partial t} \cdot \frac{\partial y}{\partial t} + \frac{\partial u}{\partial t} \cdot \frac{\partial z}{\partial t}$$

~~Hmo~~

Homogeneous function: A function

$$f(x,y) = a_n x^n + a_{n-1} x^{n-1} y + a_{n-2} x^{n-2} y^2 + \dots\dots + a_0 y^n.$$

where the power of variable in every term is equal is called a homogeneous function. Here $f(x,y)$ is a homogeneous function of order n.

$$f(x,y) = x^n \left(a_n + a_{n-1} \frac{y}{x} + a_{n-2} \frac{y^2}{x^2} + \dots + a_0 \frac{y^n}{x^n} \right)$$

$$= x^n F(y/x)$$

Euler's theorem for 2 variable homogeneous function:

If $f(x,y)$ is a homogeneous function of order n, then $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y)$

Proof: Since $f(x,y)$ is a homogeneous function of order n, the $f(x,y)$ must be written as,

$$f(x,y)=x^n F(y/x)$$

$$\therefore \frac{\partial f}{\partial x}=nx^{n-1}F(y/x)+x^n F'(y/x)\cdot\left(-\frac{y}{x^2}\right)$$

$$\Rightarrow x\frac{\partial f}{\partial x}=nx^n F(y/x)-x^{n-1}y F'(y/x) \quad \text{------ (i)}$$

Again,

$$\frac{\partial f}{\partial y}=x^n F'(y/x)\cdot\frac{1}{x}$$

$$\Rightarrow y\frac{\partial f}{\partial y}=x^{n-1}yF'(y/x) \quad \text{------ (ii)}$$

(i) + (ii) $\Rightarrow$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nx^n F(y/x)$$

$$=nf(x,y)$$