

# The Jacobi and Gauss-Seidel Iterative Methods

Numerical Analysis

# Background

- For small linear systems direct methods are often as efficient (or even more efficient) than the iterative methods to be discussed today.
- For large linear systems particularly those with sparse matrix representations (matrices with many zero entries), the iterative methods can be more efficient than the direct methods.

# Jacobi Method

- Solve systems of linear equations.
- It is useful when dealing with large systems, where Gaussian elimination is computationally expensive.
- Breaking down a complex set of equations into simpler parts.
- Leverages the properties of matrices to find solutions efficiently.

# Initial Approximation

Consider the linear system  $A\mathbf{x} = \mathbf{b}$  where  $A$  is an  $n \times n$  matrix and  $\mathbf{b} \in \mathbb{R}^n$ .

Given an initial approximation  $\mathbf{x}^{(0)}$  to the solution of the linear system  $\mathbf{x}$ , iterative techniques generate a sequence of vectors  $\{\mathbf{x}^{(k)}\}_{k=0}^{\infty}$  which converge to the solution  $\mathbf{x}$ .

# Jacobi's Method

Given the linear system  $A\mathbf{x} = \mathbf{b}$ , if  $a_{ii} \neq 0$  solve the  $i$ th equation of the system for  $x_i$ .

$$\begin{aligned} b_i &= a_{i1}x_1 + \cdots + a_{ii}x_i + \cdots + a_{in}x_n \\ x_i &= \frac{b_i}{a_{ii}} - \sum_{j=1, j \neq i}^n \frac{a_{ij}x_j}{a_{ii}} \end{aligned}$$

We will have  $n$  equations of this form ( $1 \leq i \leq n$ ).

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We will have  $n$  equations of this form ( $1 \leq i \leq n$ ).

Given  $\mathbf{x}^{(k)}$  then

$$x_i^{(k+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1, j \neq i}^n \frac{a_{ij}x_j^{(k)}}{a_{ii}}$$

for  $1 \leq i \leq n$ . The process can be repeated until

$$\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_{\infty}}{\|\mathbf{x}^{(k)}\|_{\infty}} < \epsilon.$$

## Example

Use Jacobi's method to approximate the solution to the following linear system. Use  $\mathbf{x}^{(0)} = \mathbf{0}$  and let  $\epsilon = 10^{-3}$ .

$$\begin{aligned} -2x_1 + x_2 + \frac{1}{2}x_3 &= 4 \\ x_1 - 2x_2 - \frac{1}{2}x_3 &= -4 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

For purposes of comparison, the exact solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -16/11 \\ 16/11 \\ -8/11 \end{bmatrix} \approx \begin{bmatrix} -1.454545 \\ 1.454545 \\ -0.727273 \end{bmatrix}.$$

# Solution

$$x_1 = -2 + \frac{1}{2}x_2 + \frac{1}{4}x_3$$

$$x_2 = 2 + \frac{1}{2}x_1 - \frac{1}{4}x_3$$

$$x_3 = -\frac{1}{2}x_2$$

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$k$	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0.0000	0.0000	0.0000
1	-2.0000	2.0000	0.0000
2	-1.0000	1.0000	-1.0000
3	-1.2500	1.2500	-0.8750
4	-1.5938	1.5938	-0.6250
$\vdots$	$\vdots$	$\vdots$	$\vdots$
19	-1.4552	1.4552	-0.7268
20	-1.4541	1.4541	-0.7276

# Matrix Notation for Jacobi's Method (1 of 2)

Matrix  $A$  can be decomposed as  $A = D - L - U$  where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ 0 & 0 & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

## Matrix Notation for Jacobi's Method (2 of 2)

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$(\mathbf{D} - \mathbf{L} - \mathbf{U})\mathbf{x} = \mathbf{b}$$

$$\mathbf{D}\mathbf{x} = (\mathbf{L} + \mathbf{U})\mathbf{x} + \mathbf{b}$$

$$\mathbf{x} = \mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})\mathbf{x} + \mathbf{D}^{-1}\mathbf{b}$$

assuming  $a_{ii} \neq 0$  for  $1 \leq i \leq n$ .

## Matrix Notation for Jacobi's Method (2 of 2)

$$\begin{aligned}A\mathbf{x} &= \mathbf{b} \\(D - L - U)\mathbf{x} &= \mathbf{b} \\D\mathbf{x} &= (L + U)\mathbf{x} + \mathbf{b} \\\mathbf{x} &= D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}\end{aligned}$$

assuming  $a_{ii} \neq 0$  for  $1 \leq i \leq n$ .

- ▶ Define  $T_j = D^{-1}(L + U)$  and  $\mathbf{c}_j = D^{-1}\mathbf{b}$ .
- ▶ The Jacobi method can be expressed in matrix notation as

$$\mathbf{x}^{(k)} = T_j \mathbf{x}^{(k-1)} + \mathbf{c}_j.$$

## Example

Express the following linear system in the Jacobi matrix notation.

$$\begin{aligned} -2x_1 + x_2 + \frac{1}{2}x_3 &= 4 \\ x_1 - 2x_2 - \frac{1}{2}x_3 &= -4 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

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**Solution**

$$\text{Let } A = \begin{bmatrix} -2 & 1 & 1/2 \\ 1 & -2 & -1/2 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}.$$

# Solution

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$L + U = \begin{bmatrix} 0 & -1 & -1/2 \\ -1 & 0 & 1/2 \\ 0 & -1 & 0 \end{bmatrix}$$

$$T_j = D^{-1}(L + U) = \begin{bmatrix} 0 & 1/2 & 1/4 \\ 1/2 & 0 & -1/4 \\ 0 & -1/2 & 0 \end{bmatrix}$$

$$\mathbf{c}_j = D^{-1}\mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

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$$\mathbf{c}_j = D^{-1}\mathbf{b} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1/4 \\ 1/2 & 0 & -1/4 \\ 0 & -1/2 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

# Improving the Jacobi Method

Recall that in the Jacobi method,

$$x_i^{(k)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1, j \neq i}^n a_{ij} x_j^{(k-1)} \right).$$

- ▶ As designed all the components of  $\mathbf{x}^{(k-1)}$  are used to calculate  $x_i^{(k)}$ .
- ▶ When  $i > 1$  the components  $x_j^{(k)}$  for  $1 \leq j < i$  have already been calculated and should be more accurate than the components  $x_j^{(k-1)}$  for  $1 \leq j < i$ .
- ▶ We can modify the Jacobi method to use  $x_j^{(k)}$  for  $1 \leq j < i$  in place of  $x_j^{(k-1)}$  to improve the convergence of the algorithm. This modification is known as the **Gauss-Seidel iterative technique**.

# Gauss-Seidel Method

$$x_i^{(k)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^n a_{ij} x_j^{(k-1)} \right).$$

## Example

Use the Gauss-Seidel method to approximate the solution to the following linear system. Use  $\mathbf{x}^{(0)} = \mathbf{0}$  and let  $\epsilon = 10^{-3}$ .

$$\begin{aligned} -2x_1 + x_2 + \frac{1}{2}x_3 &= 4 \\ x_1 - 2x_2 - \frac{1}{2}x_3 &= -4 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

# Solution

$$x_1^{(k)} = -2 + \frac{1}{2}x_2^{(k-1)} + \frac{1}{4}x_3^{(k-1)}$$

$$x_2^{(k)} = 2 + \frac{1}{2}x_1^{(k)} - \frac{1}{4}x_3^{(k-1)}$$

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$$x_3^{(k)} = -\frac{1}{2}x_2^{(k)}$$

$k$	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0.0000	0.0000	0.0000
1	-2.0000	1.0000	-0.5000
2	-1.6250	1.3125	-0.6523
3	-1.5078	1.4102	-0.7051
4	-1.4712	1.4407	-0.7203
5	-1.4598	1.4502	-0.7251
6	-1.4562	1.4532	-0.7266
7	-1.4551	1.4541	-0.7271

# Gauss-Seidel Method in Matrix Form (1 of 2)

$$x_i^{(k)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^n a_{ij} x_j^{(k-1)} \right)$$

$$a_{ii} x_i^{(k)} = b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^n a_{ij} x_j^{(k-1)}$$

$$a_{ii} x_i^{(k)} + \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} = b_i - \sum_{j=i+1}^n a_{ij} x_j^{(k-1)}$$

# Gauss-Seidel Method in Matrix Form (2 of 2)

Since for  $i = 1, 2, \dots, n$ ,

$$a_{ii}x_i^{(k)} + \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} = b_i - \sum_{j=i+1}^n a_{ij}x_j^{(k-1)},$$

we can express the linear system as follows:

$$\begin{aligned} a_{11}x_1^{(k)} &= b_1 - a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \dots - a_{1n}x_n^{(k-1)} \\ a_{21}x_1^{(k)} + a_{22}x_2^{(k)} &= b_2 - a_{23}x_3^{(k-1)} - a_{24}x_4^{(k-1)} - \dots - a_{2n}x_n^{(k-1)} \\ &\vdots \\ a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \dots + a_{nn}x_n^{(k)} &= b_n \end{aligned}$$

This is equivalent to the matrix form

$$\begin{aligned} (D - L)\mathbf{x}^{(k)} &= \mathbf{b} + U\mathbf{x}^{(k-1)} \\ \mathbf{x}^{(k)} &= (D - L)^{-1}\mathbf{b} + (D - L)^{-1}U\mathbf{x}^{(k-1)} \\ \mathbf{x}^{(k)} &= \mathbf{c}_g + T_g\mathbf{x}^{(k-1)}. \end{aligned}$$

# Example

Express the following linear system in the Gauss-Seidel matrix notation.

$$\begin{aligned}-2x_1 + x_2 + \frac{1}{2}x_3 &= 4 \\ x_1 - 2x_2 - \frac{1}{2}x_3 &= -4 \\ x_2 + 2x_3 &= 0\end{aligned}$$

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### Solution

$$\text{Let } A = \begin{bmatrix} -2 & 1 & 1/2 \\ 1 & -2 & -1/2 \\ 0 & 1 & 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}.$$

# Solution

$$D - L = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$(D - L)^{-1} = \begin{bmatrix} -1/2 & 0 & 0 \\ -1/4 & -1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{bmatrix}$$

$$T_g = (D - L)^{-1}U = \begin{bmatrix} 0 & 1/2 & 1/4 \\ 0 & 1/4 & -1/8 \\ 0 & -1/8 & 1/16 \end{bmatrix}$$

$$\mathbf{c}_g = (D - L)^{-1}\mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ -1/2 \end{bmatrix}$$

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$$T_g = (D - L)^{-1}U = \begin{bmatrix} 0 & 1/2 & 1/4 \\ 0 & 1/4 & -1/8 \\ 0 & -1/8 & 1/16 \end{bmatrix}$$

$$\mathbf{c}_g = (D - L)^{-1}\mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ -1/2 \end{bmatrix}$$

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1/4 \\ 0 & 1/4 & -1/8 \\ 0 & -1/8 & 1/16 \end{bmatrix} \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ -1/2 \end{bmatrix}$$