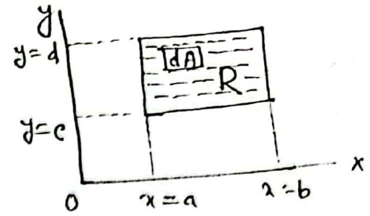
2nd Sheet

Multiple integral (বহুযোজিক সমাকলন)

Double integral: (i) Let R be the rectangular region bounded by $x=a$, $x=b$, $y=c$ and $y=d$. If $f(x,y)$ is continuous on this region then

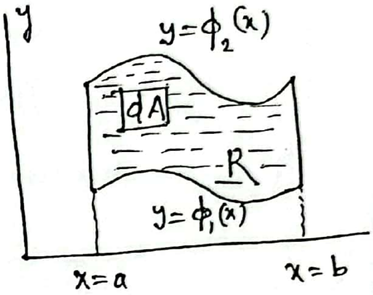
$$\iint_R f(x,y)\,dA = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dy\,dx$$



Where dA is the elementary area of R.

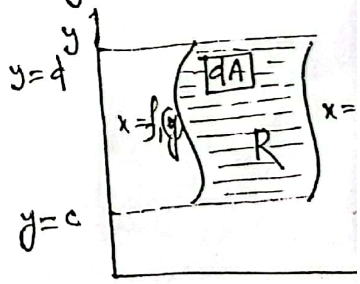
(ii) Let R be a region bounded by $x=a$, $x=b$, $y=\phi_1(x)$ & $y=\phi_2(x)$. If $f(x,y)$ is continuous on this region then

$$\iint_R f(x,y)\,dA = \int_{x=a}^{b}\int_{y=\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$



(iii) Let R be the a region bounded by $x=f_1(y)$, $x=f_2(y)$, $y=c$ and $y=d$. If $f(x,y)$ is continuous on this region then

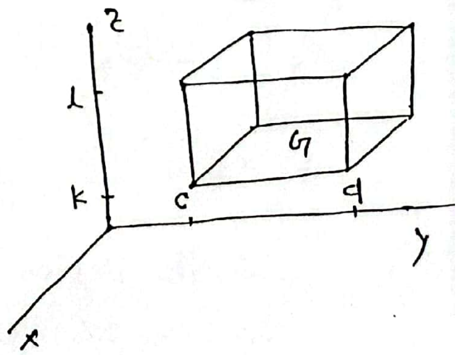
$$\iint_R f(x,y)\,dA = \int_{y=c}^{d}\int_{x=f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy$$



Triple integral:

Let G be a rectangular solid bounded by $a \le x \le b$, $c \le y \le d$, & $k \le z \le l$. If $f(x,y,z)$ is continuous on the region G then

$$\iiint_G f(x,y,z)\,dV = \int_a^b\int_c^d\int_k^l f(x,y,z)\,dz\,dy\,dx$$



Relation bet^n Polar co-ordinates (r,θ) and Cartesian co-ordinates

We know, $x = r\cos\theta$ & $y = r\sin\theta$

Here $J(x,y) = \dfrac{\partial(x,y)}{\partial(r,\theta)} = d\circ = r = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$\therefore dx\,dy = |J(x,y)|\,dr\,d\theta$ $\qquad = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

$\Rightarrow \boxed{dx\,dy = r\,dr\,d\theta}$

Prob: If R be a region bounded by $x=1, x=4, y=-1$ and $y=2$ then evaluate $\iint_R (2x+6x^2y)\,dy\,dx$

Sol^n: Here R be the region bounded by $x=1, x=4, y=-1$ & $y=2$.

we know $\iint_R (2x+6x^2y)\,dy\,dx = \int_{x=1}^{4}\int_{y=-1}^{2}(2x+6x^2y)\,dy\,dx$

$= \int_1^4 \left[2xy + 6x^2\frac{y^2}{2}\right]_{-1}^{2} dx$

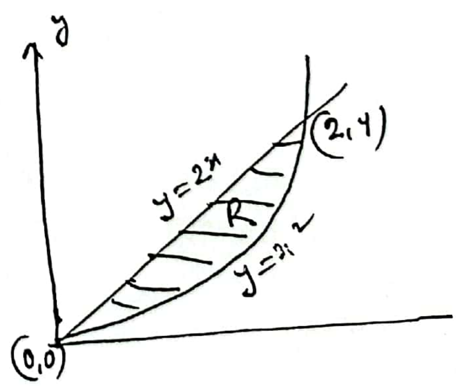
$= \int_1^4 [2x(2+1) + 3x^2(4-1)]\,dx$

$= \int_1^4 (6x+9x^2)\,dx = 6\left[\frac{x^2}{2}\right]_1^4 + 9\left[\frac{x^3}{3}\right]_1^4$

$= 3(16-1) + 3(64-1)$

$= 3\times 15 + 3\times 63 = 234.$ Ans

Prob: Evaluate $\iint_R (x^3+4y)\,dy\,dx$ over the region R bounded by the lines $y=2x$ & $y=x^2$.



Sol^n: Given $y = 2x$ — (I) & $y = x^2$ — (II)

(I) & (II) $\Rightarrow x^2 = 2x$

$\Rightarrow x(x-2) = 0$

$\Rightarrow \boxed{x=0,\ x=2}$

$\Rightarrow \boxed{y=0,\ y=4}$

So the region is bounded by $x=0, x=2, y=2x$ & $y=x^2$

Thus $\iint\limits_R (x-1)\,dA = \int_{x=0}^{1}\int_{y=x^3}^{x} (x-1)\,dy\,dx$

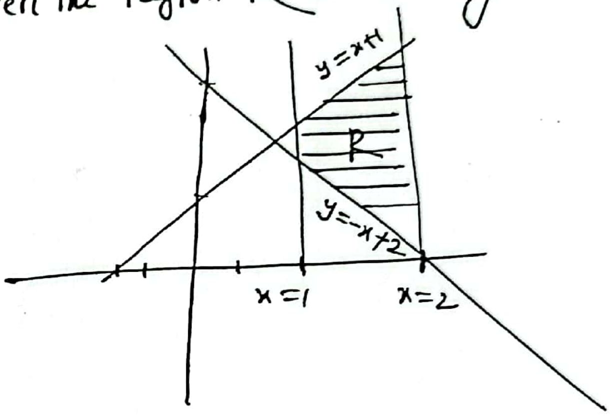
$= \int_{x=0}^{1} (x-1)\,[y]_{x^3}^{x}\,dx = \int_0^1 (x-1)(x-x^3)\,dx$

$= \int_0^1 (x^2 - x^4 - x + x^3)\,dx$

$= \left[\frac{x^3}{3} - \frac{x^5}{5} - \frac{x^2}{2} + \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{5} - \frac{1}{2} + \frac{1}{4}$

$= \boxed{-7/20 \text{ Ans}} = \frac{20-12-30+15}{60} = -7/60$ Ans

Prob: Evaluate $\iint\limits_R (x+y)\,dy\,dx$ over the region R enclosed by the lines $x=1$, $x=2$, $y=-x+2$ and $y=x+1$



Soln: Given $x=1$ —(i)

$x=2$ —(ii)

$y=-x+2$

$\Rightarrow \frac{x}{2}+\frac{y}{2}=1$ —(iii)

$y=x+1$

$\Rightarrow -\frac{x}{1}+\frac{y}{1}=1$ —(iv)

Here the region R is given in the figure.

$\therefore \iint\limits_R (x+y)\,dy\,dx = \int_{x=1}^{2}\int_{y=-x+2}^{x+1} (x+y)\,dy\,dx$

$= \int_1^2 \left[xy + \frac{y^2}{2}\right]_{-x+2}^{x+1} dx = \int_0^1 \left[x\{(x+1)-(-x+2)\} + \frac{1}{2}\{(x+1)^2 - (-x+2)^2\}\right]$

$= \int_1^2 \left[x^2 + x + x^2 - 2x + \frac{x^2+2x+1}{2} - \frac{x^2-4x+4}{2}\right] dx$

$= \frac{1}{2}\int_1^2 (2x^2 + 2x + 2x^2 - 4x + \cancel{x^2} + 2x + 1 - \cancel{x^2} + 4x - 4)\,dx$

$= \frac{1}{2}\int_1^2 (4x^2 + 4x - 3)\,dx = \frac{1}{2}\left[4\frac{x^3}{3} + 4\frac{x^2}{2} - 3x\right]_1^2$

$= \frac{1}{2}\left[\frac{4}{3}(8-1) + 2(4-1) - 3(2-1)\right]$

$= \frac{1}{2}\left[\frac{28}{3} + 6 - 3\right] = \frac{1}{2}\,\frac{28+18-9}{3} = \frac{37}{6}$ Ans

$$\therefore \iint_R (x^3+4y)\,dy\,dx = \int_{x=0}^{2}\int_{y=x^2}^{2x} (x^3+4y)\,dy\,dx$$

$$= \int_0^2 \left[x^3 y + 4\frac{y^2}{2}\right]_{x^2}^{2x} dx$$

$$= \int_0^2 \left[x^3(2x-x^2)+2(4x^2-x^4)\right] dx$$

$$= \int_0^2 (2x^4 - x^5 + 8x^2 - \cancel{2x^4})\,dx$$

$$= \int_0^2 (8x^2-x^5)\,dx = \left[8\frac{x^3}{3} - \frac{x^6}{6}\right]_0^2$$

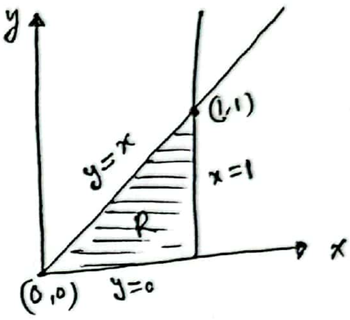
$$= \frac{8}{3}\times 8 - \frac{1}{6}.64 = \frac{64.2-64}{6} = \frac{64}{6} = \frac{32}{3}$$

Prob: Evaluate $\iint_R e^{y/x}\,dA$ over the region R bounded by the lines $y=x$, $y=0$, & $x=1$.

Soln: Given $y=x$ —(1)

$y=0$ —(2)

$x=1$ —(3)



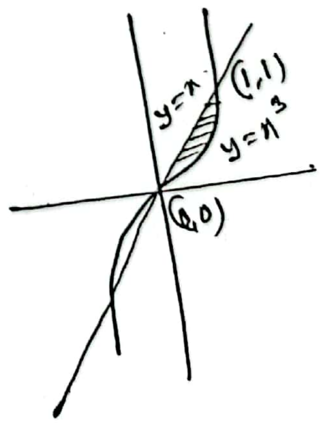
Here the region is shown in the figure.

Thus
$$\iint_R e^{y/x}\,dA = \int_{x=0}^{1}\int_{y=0}^{x} e^{y/x}\,dy\,dx$$

$$= \int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^x dx = \int_0^1 x(e^1-1)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 (e-1) = \frac{1}{2}(e-1)$$ Ans

Prob: Evaluate $\iint_R (x-1)\,dA$, where R is the region in the first quadrant enclosed betn $y=x$ & $y=x^3$.



Soln: Given $y=x$ —(I)

$y=x^3$ —(II)

$(1)\ \&\ (2) \Rightarrow x^3 - x = 0$

$\Rightarrow x(x^2-1)=0$

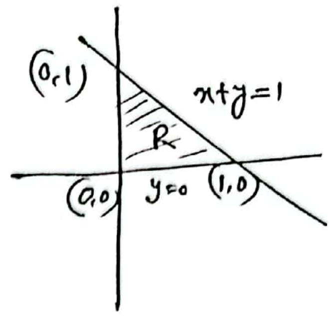
$\Rightarrow \boxed{x=0,\ x=\pm 1}$

$(1) \Rightarrow \boxed{y=0\ \&\ y=\pm 1}$

For this problem $(0,0)$ and $(1,1)$ are intersecting points

Prob: Evaluate $\iint_R (x^2+y^2)\,dx\,dy$, where R is the region enclosed in 1st quadrant by $x+y \le 1$

Sol^n: Here the region R is enclosed in 1st quadrant by $x+y \le 1$.



Here the line $x+y=1$ intersects x & y axes at $(1,0)$ and $(0,1)$ points

So the region is bounded by $x=0$, $x=1$, $y=0$ and $y=1-x$.

Thus $\iint_R (x^2+y^2)\,dx\,dy = \int_{x=0}^{1} \int_{y=0}^{1-x} (x^2+y^2)\,dy\,dx$

$$= \int_0^1 \left[x^2 y + y^3/3\right]_0^{1-x} dx$$

$$= \int_0^1 \left[x^2(1-x) + \frac{1}{3}(1-x)^3\right] dx = \int_0^1 \left[x^2 - x^3 + \frac{1-3x+3x^2-x^3}{3}\right]$$

$$= \frac{1}{3}\int_0^1 \left[1-3x+6x^2-4x^3\right] dx$$

$$= \frac{1}{3}\left[x - \frac{3}{2}x^2 + \frac{6}{3}x^3 - \frac{4}{4}x^4\right]_0^1$$

$$= \frac{1}{3}\left[1 - \frac{3}{2} + 2 - 1\right] = 1/6 \text{ Ans}$$

Change of variables formula:

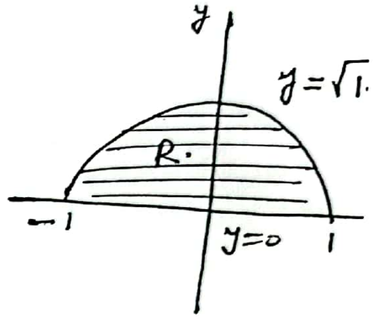
$$\iint_R f(x,y) = \iint_R f(x,y)\,dy\,dx = \iint f(r\cos\theta, r\sin\theta)\, J(x,y)$$

approximate limit of r & θ

$$= \iint_{r\,\theta} f(r\cos\theta, r\sin\theta)\, d\theta\, dr.$$

Prob: Use Polar co-ordinates to evaluate $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx.$

Sol^n: Here R is the region bounded by
$x=-1$, $x=1$, $y=0$ and $y=\sqrt{1-x^2}$.
Thus we get a semicircle as shown in the figure.



We know $x = r\cos\theta$ & $y = r\sin\theta$

and $x^2+y^2 = r^2$

From the figure we see that r varies from 0 to 1
& θ varies from 0 to $\cancel{2}\pi$.

Thus $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} (r^2)^{3/2}\, r\, d\theta\, dr$

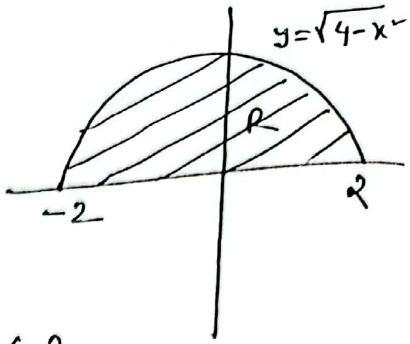
$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} r^3 r\, d\theta\, dr$$

$$= \int_{r=0}^{1} r^4 [\theta]_0^{2\pi}\, dr$$

$$= \int_0^1 \cancel{2}\pi r^4\, dr = \cancel{2}\pi\left[\frac{r^5}{5}\right]_0^1 = \frac{\pi}{5} \text{ Ans}$$

Prob: Use Polar co-ordinate to evaluate $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$.

Sol^n: Here R is the region bounded by $x=-2$, $x=2$, $y=0$ & $y=\sqrt{4-x^2}$. Thus we get a semicircle.



We know $x = r\cos\theta$, $y = r\sin\theta$

& $x^2+y^2 = r^2$

Here $0 \le \theta \le \pi$ & $0 \le r \le 2$

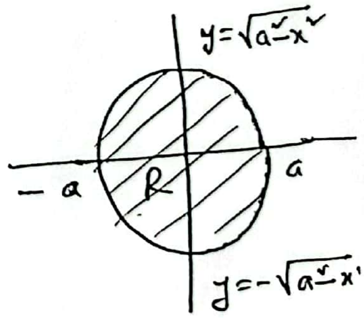
$$\therefore \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx = \int_{r=0}^{2}\int_{\theta=0}^{\pi} r^2\, r\, d\theta\, dr$$

$$= \int_{r=0}^{2} r^3 [\theta]_0^{\pi}\, dr = \int_0^2 r^3 \pi\, dr = \pi\left[\frac{r^4}{4}\right]_0^2$$

$$= 4\pi \text{ Ans}$$

Prob: Evaluate $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2}\,dy\,dx$. by Polar form.

Sol^n: Here the region is bounded by $x=-a$, $x=a$, $y=\sqrt{a^2-x^2}$, $y=-\sqrt{a^2-x^2}$. That R is a circle with radius a.



we know $x = r\cos\theta$, $y = r\sin\theta$

$\therefore x^2+y^2 = r^2$

$0 \le r \le a$ & $0 \le \theta \le 2\pi$

$$\therefore \int_{-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2}\,dy\,dx = \int_{r=0}^{a}\int_{\theta=0}^{2\pi} (r^2)^{3/2}\, r\, d\theta\, dr$$

$$= \int_{r=0}^{a} r^4\, 2\pi\, dr = 2\pi\left[\frac{r^5}{5}\right]_0^a$$

$$= \frac{2\pi a^5}{5}$$

Prob: Evaluate $\iiint_G 12xy^2z^3\,dv$ over the rectangular box G def.^{n} by $-1 \le x \le 2$, $0 \le y \le 3$, $0 \le z \le 2$.

Sol^n: Here $\iiint_G 12xy^2z^3\,dv = \int_{x=-1}^{2}\int_{y=0}^{3}\int_{z=0}^{2} 12xy^2z^3\,dz\,dy\,dx$

$$= \int_{x=-1}^{2}\int_{y=0}^{3} 12xy^2\left[\frac{z^4}{4}\right]_0^2 dy\,dx$$

$$= \int_{x=-1}^{2}\int_{y=0}^{3} 12xy^2\,4\,dy\,dx$$

$$= \int_{x=-1}^{2} 48x\left[\frac{y^3}{3}\right]_0^3 dx$$

$$= \int_{x=-1}^{2} 48\cdot 9\,x\,dx = 432\left[\frac{x^2}{2}\right]_{-1}^{2} = 432\left[\frac{4}{2}-\frac{1}{2}\right]$$

$$= 648$$ Ans

Successive integration

Here, $\int_a^b f(x)\,dx$ means we to integrate $f(x)$ with respect to x from $x=a$ to $x=b$.

But $\int_a^b \left[\int_c^d f(x)\,dx\right] dy$ means first integrate $f(x)$ with respect to x from $x=c$ to $x=d$. Let the result be $F(y)$.

Then $\int_a^b F(y)\,dy$ implies integrate it with respect to y from $y=a$ to $y=b$.

Note: (i) To integrate with respect to x, keep y & z constants.

(ii) To " " " y, keep x & z ".

(iii) To " " " z, keep x & y ".

Thus $$\int_{u_1}^{u_2}\int_{v_1}^{v_2}\int_{w_1}^{w_2} f(x,y,z)\,dx\,dy\,dz$$

first

2nd

3rd

Prob: Evaluate $\int_0^1\int_0^2 (x+2)\,dx\,dy$

Soln: Let $I=\int_0^1\int_0^2 (x+2)\,dx\,dy$

$$=\int_0^1 \left[\frac{x^2}{2}+2x\right]_0^2 dy$$

$$=\int_0^1 (2+4)\,dy = 6\,[y]_0^1 = 6 \text{ Ans.}$$

Prob: Evaluate $\int_2^3 \int_1^2 xy^2\,dy\,dx$

Sol^n: $I = \int_2^3 \int_1^2 xy^2\,dy\,dx = \int_2^3 x \left[\frac{y^3}{3}\right]_1^2 dx$

$= \int_2^3 x \left(\frac{8}{3} - \frac{1}{3}\right) dx$

$= \frac{7}{3}\left[\frac{x^2}{2}\right]_2^3 = \cancel{\frac{7}{3}\cdot\frac{9}{2}} =$

$= \frac{7}{3}\left[\frac{9}{2} - \frac{4}{2}\right] = \frac{35}{6}$ Ans

Prob: Evaluate $\int_0^4 \int_0^1 xy(x-y)\,dy\,dx$

Sol^n: $I = \int_0^4 \int_0^1 xy(x-y)\,dy\,dx = \int_0^4 \int_0^1 (x^2y - xy^2)\,dy\,dx$

$= \int_0^4 \left[x^2\frac{y^2}{2} - x\frac{y^3}{3}\right]_0^1 dx$

$= \int_0^4 \left[\frac{x^2}{2} - \frac{x}{3}\right] dx$

$= \left[\frac{1}{2}\frac{x^3}{3} - \frac{1}{3}\cdot\frac{x^2}{2}\right]_0^4$

$= \frac{1}{6}\left[64 - 16\right]$

$= \frac{48}{6} = 8$ Ans

Prob: Evaluate $\int_2^3 \int_1^2 \int_2^5 xy \, dz \, dy \, dx$

$$= \int_2^3 \int_1^2 [xyz]_2^5 \, dy \, dx$$

$$= \int_2^3 \int_1^2 3xy \, dy \, dx$$

$$= \int_2^3 3x \left[\frac{y^2}{2}\right]_1^2 dx = \int_2^3 \frac{3x}{2}(4-1) \, dx$$

$$= \frac{9}{2}\left[\frac{x^2}{2}\right]_2^3 = \frac{9}{4}\cdot(9-4) = 45/4$$ Ans

Evaluate $\int_1^2 \int_0^1 \int_{-1}^1 (x^2+y^2+z^2) \, dx \, dy \, dz$

$$= \int_1^2 \int_0^1 \left(\frac{x^3}{3} + y^2x + z^2x\right)_{-1}^1 dy \, dz$$

$$= \int_1^2 \int_0^1 \left(\frac{2}{3} + 2y^2 + 2z^2\right) dy \, dz$$

$$= \int_1^2 \left(\frac{2}{3}y + 2\cdot\frac{y^3}{3} + 2z^2y\right)_0^1 dz$$

$$= \int_1^2 \left(\frac{2}{3} + \frac{2}{3} + 2z^2\right) dz$$

$$= \left[\frac{4}{3}z + 2\cdot\frac{z^3}{3}\right]_1^2$$

$$= \frac{4}{3}(2-1) + \frac{2}{3}(8-1)$$

$$= \frac{4}{3} + \frac{14}{3} = 6$$ Ans

Evaluate $\int_0^a \int_0^b \int_0^{2a} x^2y^2z \, dz \, dy \, dx$

$$= \int_0^a \int_0^b \left[x^2y^2\frac{z^2}{2}\right]_0^{2a} dy \, dx$$

$$= \int_0^a \int_0^b \left[x^2y^2\frac{4a^2}{2}\right] dy \, dx$$

$$= \int_0^a \int_0^b 2a^2x^2y^2 \, dy \, dx$$

$$= 2a^2 \int_0^a \left[x^2\frac{y^3}{3}\right]_0^b dx$$

$$= 2a^2 \int_0^a x^2 \cdot \frac{b^3}{3} dx$$

$$= \frac{2a^2b^3}{3}\left[\frac{x^3}{3}\right]_0^a = \frac{2a^2b^3}{3}\cdot\frac{a^3}{3}$$

$$= \frac{2}{9}a^5b^3$$

Ans

Prob: Evaluate $\int_0^1 \int_0^5 (x+y)\,dy\,dx$

Soln: Let, $I = \int_0^1 \int_0^5 (x+y)\,dy\,dx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^5 dx = \int_0^1 \left(5x + \frac{25}{2}\right) dx$

$= \left[5\frac{x^2}{2} + \frac{25}{2}x\right]_0^1 = \frac{5}{2} + \frac{25}{2} = 15$ Ans

Prob: Evaluate $\int_1^2 \int_y^{3y} (3x^2+y^2)\,dx\,dy$

$= \int_1^2 \left[3\frac{x^3}{3} + y^2 x\right]_y^{3y} dy = \int_1^2 \left[\{(3y)^3 - y^3\} + y^2\{3y-y\}\right] dy$

$= \int_1^2 (27y^3 - y^3 + 2y^3)\,dy = \int_1^2 28y^3\,dy = 28\left[\frac{y^4}{4}\right]_1^2$

$= 28 \times \frac{1}{4}[16-1] = 7 \times 15 = 105$ Ans

Prob: Evaluate $\int_{-3}^{3} \int_0^1 \int_1^2 (x+y+z)\,dx\,dy\,dz$

$= \int_{-3}^{3} \int_0^1 \left(\frac{x^2}{2} + yx + zx\right)_1^2 dy\,dz$

$= \int_{-3}^{3} \int_0^1 \left[\frac{1}{2}(4-1) + y(2-1) + z(2-1)\right] dy\,dz$

$= \int_{-3}^{3} \int_0^1 \left(\frac{3}{2} + y + z\right) dy\,dz$

$= \int_{-3}^{3} \left[\frac{3}{2}y + \frac{y^2}{2} + zy\right]_0^1 dz$

$= \int_{-3}^{3} \left(\frac{3}{2} + \frac{1}{2} + z\right) dz$

$= \int_{-3}^{3} (2+z)\,dz = \left[2z + \frac{z^2}{2}\right]_{-3}^{3}$

$= 2(3+3) + \frac{1}{2}(9-9)$

$= 2 \times 6 + 0 = 12$ Ans

Prob: Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-y^2} z \, dz \, dy \, dx$

$$= \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2} \right]_0^{1-y^2} dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2} (1-y^2)^2 \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2} (1-2y^2+y^4) \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \left[y - 2 \cdot \frac{y^3}{3} + \frac{y^5}{5} \right]_0^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 \left[1-x - \frac{2}{3}(1-x)^3 + \frac{1}{5}(1-x)^5 \right] dx$$

$$= \frac{1}{2} \left[x - \frac{x^2}{2} + \frac{2}{3} \frac{(1-x)^4}{4} - \frac{1}{5} \frac{(1-x)^6}{6} \right]_0^1$$

$$= \frac{1}{2} \left[1 - \frac{1}{2} + \frac{2}{12} \left\{ (1-1)^4 - (1-0)^4 \right\} - \frac{1}{30} \left\{ (1-1)^6 - (1-0)^6 \right\} \right]$$

$$= \frac{1}{2} \left[1 - \frac{1}{2} - \frac{1}{6} + \frac{1}{30} \right]$$

$$= \frac{1}{2} \times \frac{11}{30} = 11/60$$