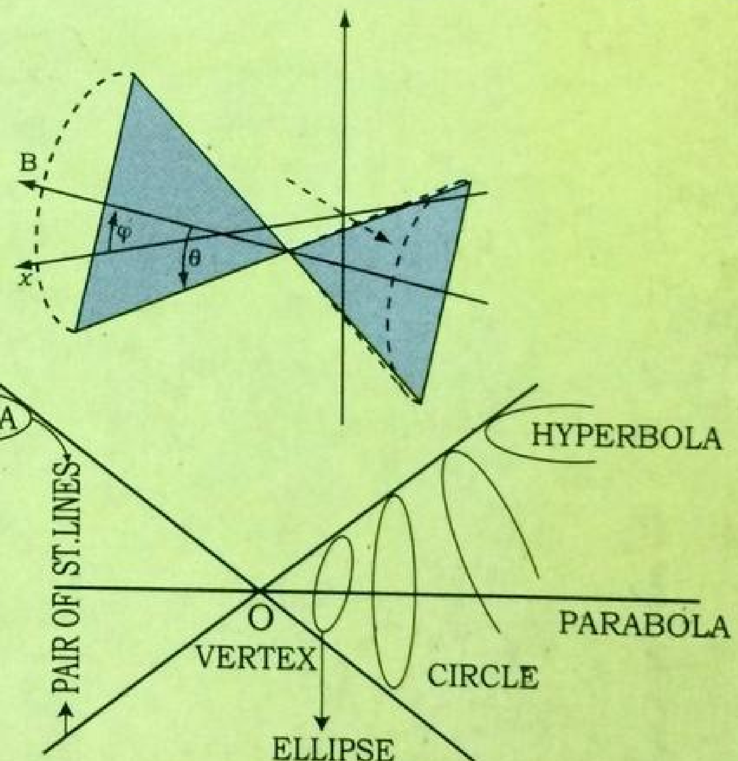
A TEXT BOOK ON CO-ORDINATE GEOMETRY

(TWO AND THREE DIMENSIONS)

WITH VECTOR ANALYSIS



RAHMAN & BHATTACHARJEE

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ON
CO-ORDINATE GEOMETRY

[TWO AND THREE DIMENSIONS]

WITH

VECTOR ANALYSIS

By

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PAIR OF STRAIGHT LINES

Art. 36. Homogeneous Quadratic Equations.

Let us consider the equation $ax^2 + 2hxy + by^2 = 0$ ... ... (1)

On multiplying it by a, it may written in the form

$a^2x^2 + 2ahxy + aby^2 = 0$, if $a \neq 0$

or, $a^2x^2 + 2ahxy + h^2y^2 - (h^2 - ab)\, y^2 = 0$

or, $\{ax + hy\}^2 - \{y \sqrt{h^2 - ah})\}^2 = 0$

or, $(ax + hy) + \sqrt{(h^2 - ab)}\, y\} \{ax + hy - (h^2 - ab)y\} = 0$

The equation (1) therefore, represents the two straight lines whose equations are

$ax + hy + y\sqrt{(h^2 - ab)} = 0$ ... ... (2)

and $ax + hy - y\sqrt{(h^2 - ab)} = 0$ ... ... (3)

each of which passes through the origin. The two lines (2), (3) form the locus of the equation (1) for (1) is satisfied by the co-ordinates of all points which satisfy (2) and (3)

These two straight lines are real and different if $h^2 > ab$, real and coincident if $h^2 = ab$, and imaginary if $h^2 < ab$.

In the case when $h^2 = ab$, the straight lines though themselves imaginary intersect in a real point. For the origin lies on the locus given by (1) since the equation (1) is always satisfied by the values $x = 0$ and $y = 0$.

Alternative method

If $b \pm 0$ dividing both sides of the equation (1) by $x^2 b$ we have

$$\left(\frac{y}{x}\right)^2 + 2\frac{h}{b}\frac{y}{x} + \frac{a}{b} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Let m_1, m_2, be the roots of this quadratic equation in x/y.

Sum of the roots = $m_2 + m_2 = -2\,h/b$

and product of the roots = $m_1 m_2 = a/b$ ... ... (5)

The Eq. (4) must be equivalent to

$$\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

The two lines represented by (4) *i.e.* (1) are given

$\frac{y}{x} - m_1 = 0$, and $\frac{y}{x} - m_2 = 0$.

i.e. $y - m_1x = 0$ and $y - m_2x = 0$ which pass through the origin.

Thus the homogeneous quadratic equation $ax^2 + 2hxy + by^2 = 0$ *always represents a pair of straight lines, real or imaginary, through the origin.*

Art. 37. Prove that a homogeneous equation of the n th degree represents n straight lines, real or, imaginary, which all pass through the origin.

Consider the homogeneous equation

$$y^n + a_1xy^{n-1} + a_2x^2y^{n-2} + a_3x^3y^{n-3} + \dots\ a_rx^ry^{n-r} + a_nx^n = 0 \qquad \dots \qquad \dots \qquad \dots \qquad (1)$$

This can be written as (dividing by x^n)

$$\left(\frac{y}{x}\right)^n + a_1\left(\frac{y}{x}\right)^{n-1} + a_2\left(\frac{y}{x}\right)^{n-2} + a_3\left(\frac{y}{x}\right)^{n-3} + \dots\ a_r\left(\frac{y}{x}\right)^{n-r} + \dots + a_n = 0 \qquad \dots \qquad (2)$$

Since this is an equation of the n th degree in y/x it must have n roots. Let the roots of this equation be $m_1, m_2, m_3, \dots m_r, \dots m_n$.

Then the given equation (1) or (2) must be the same as

$$\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right) - \left(\frac{y}{x} - m_r\right) \dots \left(\frac{y}{x} - m_n\right) = 0$$

or, $(y - m_1x)(y - m_2x) \dots (y - m_rx) \dots (y - m_nx) = 0$

This equation is satisfied by the points which satisfy the separate equations.

$y - m_1x = 0,\ y - m_2x = 0, \dots\ y - m_rx = 0, \dots\ y - m_nx = 0$

which all pass through the origin. Conversely, the co-ordinates of all the points which satisfy these n equation satisfy equations (1) also. Hence the theorem.

Note : 1. An equation of the type

$a_0y^n + a_1y^{n-1}x + a_2y^{n-2}x^2 \dots + a_ry^{n-r}x^r \dots \dots a_n x^n = 0$

in which the sum of the powers of x and y in every term is the same (say n) is called a Homogeneous equation (of degree n)

Note. 2. Since the given equation contains real co-efficients, the roots, if imaginary, must occur in pair.

Note. 3. The relation between the roots and the co-efficients can be determined by the symmetric functions of the roots (see author's Higher Algebra. Art. 53 and 54)

Note. 4. If two or more of the roots are equal the corresponding straight lines are coincident.

Art. 38. Angle between the lines represented by the equation.

$$ax^2 + 2hxy + by^2 = 0 \qquad \dots \qquad \dots \qquad \dots \qquad (1)$$

The axes are assumed to be rectangular. Let the lines represented by (1) be $y - m_1x = 0$ and $y - m_2x = 0$

So that (1) and $(y - m_1x)(y - m_2x)$ are the same.

$\therefore m_1 + m_2 = -2\frac{h}{b}$ and $m_1m_2 = \frac{a}{b}$ by Art 36 ... ... (2)

If θ be the angle between the straight lines

$y - m_1x = 0$ and $y - m_2x = 0$, then by Art. 25.

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{\sqrt{\{(m_1+m_2)^2 - 4m_1m_2\}}}{1 + m_1m_2}$$

$= \sqrt{\{(-2h/b)^2 - 4a/b\}}\,(1 + a/b) = 2\sqrt{(h^2 - ab)}\,/\,(a+b)$ by ... (2).

$$\therefore \tan\theta = \frac{2\sqrt{(h^2 - ab)}}{a + b} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where θ the angle between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$

Cor. 1. Condition of Perpendicularity.

If the straight lines are perpendicular to each other then $\theta = 90°$ hence $\tan\theta = \propto$ then from (3)

$a + b = 0$

i. e. if the sum of the co-efficients of x^2 and y^2 is zero, then the straight lines will be perpendicular to each other.

Cor. 2. Condition of parallelism or Coincidence.

If the lines of $ax^2 + 2hxy + by^2 = 0$ be coincident, $\theta = 0$

From (3) $\tan\theta = 0$

i, e. $h^2 = ab$

which is the required condition for coincidence.

Here lines cannot be parallel as both of the lines pass through the origin, these are coincident straight lines.

Cor. 3. Two lines represented by $ax^2 + 2hxy + by^2 = 0$ will be real if $h^2 > ab$

Cor. 4. Two lines of above equation will be imaginary if $h^2 < ab$.

Art. 39. The bisectors of the angles between the lines represented by $ax^2 + 2hxy + by^2 = 0$

Let the lines $y - m_1x = 0$ and $y - m_2x = 0$ represented by

$ax^2 + 2hxy + by^2 = 0$ ... ... (1) have slopes m_1 and m_2 and make angles θ_1 and θ_2 with OX. Let one of the bisectors make an angle θ with OX. from the diagram we have

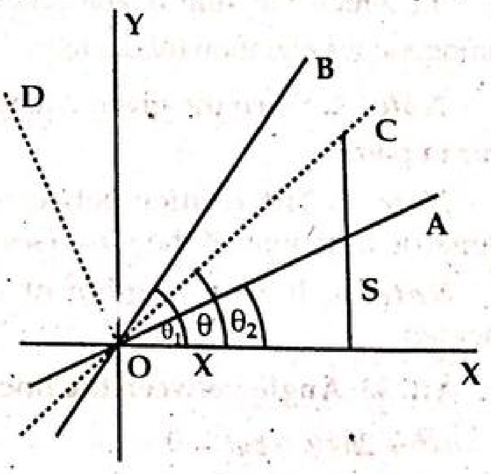

Fig. 12

$\theta = \theta_1 + \frac{1}{2}(\theta_2 - \theta)$ or, $\frac{1}{2}\pi + \theta_1 + \frac{1}{2}(\theta_2 - \theta_1) = \frac{1}{2}(\theta_1 + \theta_2)$

or, $\frac{1}{2}\pi + \frac{1}{2}(\theta_1 + \theta_2)$

$\therefore 2\theta = \theta_1 + \theta_2$

or, $\pi + (\theta_1 + \theta_2)$ ... ... (2)

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In either case

$$\tan 2\theta = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \frac{m_1 + m_2}{1 - m_1 m_2}$$

or, $$\frac{2\tan\theta}{1 - \tan^2\theta} = \frac{m_1 + m_2}{1 - m_1 m_2} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

From Eq. (1), we have

$$b(y - m_1 x)(y - m_2 x) = by^2 + 2hxy + ax^2$$

$$\therefore \; m_1 + m_2 = 2\frac{a}{b}, \; m_1 m_2 = \frac{a}{b} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Also if C (x, y) be any point on one of the bisectors, (see fig)

$$\tan\theta = y/x \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

From (3) by (4) and (5), we have

$$\frac{2xy}{x^2 - y^2} = \frac{2h}{a - b}$$

i. e. the required equation of the bisectors is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Note. **1.** In (2), first value θ is the angle made by the internal bisector OC with OX and the second value of θ is the angle made by the external bisector OD with OX.

Alternative Method :

Let the given equation $ax^2 + 2hxy + by^2 = 0$ represent the line

$y - m_1 x = 0$ and $y - m_2 x = 0$.

$$\therefore m_1 + m_2 = -2\frac{h}{b}, \; m_1 m_2 = \frac{a}{b} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

The equations of the required bisectors are

$$\frac{y - m_1 x}{\sqrt{(1 + m_1^2)}} = \pm\frac{y - m_2 x}{\sqrt{(1 + m_2)}} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Squaring, we have

$$(y - m_1 x)^2 (1 + m_2^2) = (y - m_2 x)^2 (1 + m_1^2)$$

or, $$(m_1^2 - m_2^2)(x^2 - y^2) = 2xy \{(m_1 - m_2) - m_1 m_2 (m_1 - m_2)\}$$

or, $$(m_1 - m_2)(m_1 - m_2)(x^2 - y^2) = 2xy\,(m_1 - m_2)(1 - m_1 m_2)$$

or, $$(m_1 + m_2)(x^2 - y^2) = 2xy(1 - m_1 m_2)$$

or, $$\left(-2\frac{h}{b}\right)(x^2 - y^2) = 2xy\frac{(b - a)}{b}$$

or, $$h(x^2 - y^2) = xy(a - b)$$

or, $$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

GENERAL EQUATION OF SECOND DEGREE

Art. 40. Find the condition that the general equation of the second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... ... (1)

may represent a pair of straight lines.

If we transfer the origin to a point (α, β), the point of intersection of two straight lines and keep the direction of the axes unaltered, the eq. (1) reduces to

$a(x+\alpha)^2 + 2h(x+\alpha)(y+\beta) + b(y+\beta)^2 + 2g(x+\alpha) + 2f(y+\beta) + c = 0$ by Art. 32 (Rule)

or, $ax^2 + 2hxy + by^2 + 2(a\alpha + h\beta + g)x + 2(h\alpha + b\beta + f)y + a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0$ ... ... (2)

The equation (2) may represent a pair of straight lines, if it is reduced to a homogeneous equation in x and y. This is possible if the co-effcients of x and y and the constant terms are saperately zero. *i. e.*

$a\alpha + h\beta + g = 0$ ... ... (3)

$h\alpha + b\beta + f = 0$ ... ... (4)

and $a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2f\alpha + c = 0$ ... ... (5)

The relation (5) may be written as

$\alpha(a\alpha + h\beta + g) + \beta(h\alpha + b\beta + f) + g\alpha + f\beta + c = 0$ ... ... (6)

By (3) and (4), the relation (6) becomes $g\alpha + f\beta + c = 0$ ... ... (7)

Now if we eliminate α and β from (3), (4) and (7), the required condition is

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

or, $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ ... ... (9)

Alternative method — By Calculus

Suppose $F(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$\equiv (lx + my + n)(l' + m'y + n')$ ... ... (1)

This relation being true for all values of x and y, it must admit of partial differentiation w. r. to each of the variables x and y. So we have, by taking partial differentaiation of (1)

$\frac{\delta F}{\delta x} = 2(ax + hy + g) = l'(lx + my + n) + l(l'x + m'y + n)$ ... ... (2)

and $\frac{\delta F}{\delta y} = 2(hx + by + f) = m'(lx + my + n) + m(l' + m'y + n')$ ... ... (3)

Let (α, β) be the point of intersection of the lines represented by the general equation (1)

$\therefore l\alpha + m\beta + n = 0,$ $\quad l'\alpha + m'\beta + n' = 0$

Putting α, β respectively for x ; y in (2) and (3) we have

$a\alpha + h\beta + n = 0$ ... ... (4)

$h\alpha + b\beta + f = 0$ ... ... (5)

Again puttin (α, β) in the left hand side of (1) we have

$a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0$

or, $\alpha(a\alpha + h\beta + g) + \beta(h\alpha + b\beta + f) + (g\alpha + f\beta + c) = 0$

By (4) and (5)

$$g\alpha + f\beta + c = 0 \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Eliminating α, β from (4), (5) and (6) we have

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \quad \text{or, } \Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

which is the required condition. It is both necessary and sufficient.

Cor. 1. Solving (4) and (5), we have

$$\alpha = \frac{hf - bg}{ab - h^2} = \frac{G}{C} \text{ and } \beta = \frac{gh - af}{ab - h^2} = \frac{F}{C}$$

where G, F, C are the co-factors of g, f, c, in

Note : 1. From the above discussions we see that if $F(x, y) = 0$ be the equation of two lines, the point of intersection (α, β) would be obtained by solving the equations.

$$\frac{\delta F(\alpha, \beta)}{\delta\alpha} = 0, \frac{\delta F(\alpha, \beta)}{\delta\beta} = 0$$

Cor. 2. If $a = 0$ and $b \neq 0$, we solve it for y and proceed as before. We thus find the condition $2f\,gh - bg^2 - ch^2 = 0$

Cor. 3. If $a = 0$, $b = 0$ and $h \neq 0$, then the general equations become $2hxy + 2gx + 2fy + c = 0$

or, $xy + (\frac{g}{h})x + (\frac{c}{h})y + \frac{c}{2h} = 0$ $\quad\therefore$ Dividing by $2h$

or, $(x + \frac{f}{h})(y + \frac{g}{h}) = (\frac{fg}{h^2}) - \frac{c}{2h}$

or, $(hx + f)(hy + g) = fg - \frac{1}{2}ch$

and the condition that it may represent a pair of lines is

$fg - \frac{1}{2}ch = 0 \qquad$ or, $2fg - ch = 0$

Note : 2 There is another method of finding the condition that the general second degree equation may represent a pair of straight lines. This process will be demonstrated by a worked out example. See worked sum No. 1.

Art. 41. (A) Angle between the lines given by

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (lx + my + n)(l'x + m'y + n') \quad \ldots \quad (1)$

$ll' = a,\ mm' = b,\ nn' = c$

$lm' + l'm = 2h,\ mn' + m'n = 2f,\ nl' + n'l = 2g \quad \ldots \quad \ldots \quad \ldots \quad (2)$

$lx + my + n = 0$ and $l'x + m'y + n' = 0$

or, $y = -\frac{l}{m}x - \frac{n}{m}$ and $y = -\frac{l'}{m'} - \frac{x - n'}{m}$

If θ be the angle between the lines then

$$\text{an } \theta = \frac{-l/m - (-l'/m')}{1 + (-l/m\ l'/m')}$$

$$= \frac{l'm - lm'}{l'l + mm'}$$

$$\frac{\sqrt{(lm' + l'm)^2 - 4\ ll'mn'}}{ll' + mm'} = \frac{\sqrt{2h^2 - ab}}{a + b} \quad \ldots \quad \ldots \quad \text{by (2)}$$

On account of relation (2), we haye $(lx + my)(l'x + m'y) = ll'x^2 + 2(lm' + l'm)xy + mm'\ y^2 = ax^2 + 2hxy + by^2$

hence it follows that the lines represented bv

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

are parallel to the lines given by $ax^2 + 2hxy + by^2 = 0$

(B) Condition for perpendicularity of lines

The lines will be perpendicular of $\theta = 90°$ *i. e.* $\tan\theta = \infty$

From (3), we have $ll' + mm' = 0$: *i. e.* if $a + b = 0$

This also follows from (4). Hence the given equation represents a pair of perpendicular lines, if $\Delta = 0$ and $a + b = 0$

(C) Condition for parallelism of the lines.

The lines in (4) will be parallel if $\frac{l}{l'} = \frac{m}{m'}$ or, $lm' = l'm$

or, $lm' - l'm = 0$

or, $(lm' + l'm) - 4ll'\ mm' = 0$ or $h^2 - ab = 0$ by (4).

This also follows from (4). Hence the given equation represents a pair of parallel lines is $\Delta = 0$ and $h^2 = ab$

(D) Condition for coincidence of the lines.

The line in (1) will be concident if $\frac{l}{l'} = \frac{m}{m'} = \frac{n}{n'}$

i. e. $lm' - l'm = 0,\ mn' - m'n = 0,\ n'l - nl' = 0$

or, $(lm' + l'm)^2 - 4ll'\ mm' = 0,\ (mm' + m'n)^2 - 4mm'\ nn' = 0,\ (nl' + n'l)^2 - 4nn'll' = 0$

or, $h^2 - ab = 0, f^2 - bc = 0, g^2 - ac = 0$ by (2)

In such a case, the given equation becomes a perfect square. Hence the given equation will represent a pair of coincident lines if

$\Delta = 0, f^2 - bc = 0, g^2 - \alpha c = 0, h^2 - ab = 0$

which are equavalent to $gh = af, hf = bg, fg = ch$

(E) Bisectors of the angles between the lines.

Let (α, β) be the points of intersection of the two lines represented by the equation.

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

If we transfer the origin to (α, β) without changing the direction of axes, the first degree terms and the term independent of x and y will vanish. Then the transformed equation will be a homogeneous equation and will be a form $ax^2 + 2hxy + by^2 = 0$

The equations of the bisectors of the angle between the lines with (α, β) as origin by

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

If we go back to our old axes, the above equations, of the bisectors are given by

$$\frac{(x-\alpha)^2 - (y-\beta)^2}{(a-b)} = \frac{(x-\alpha)(y-\beta)}{h}$$

which is the required (See Art. 39 Rule) equation of the bisectors of the angles between the given lines.

It should be noted that with reference to the new origin the co-ordinates of the old origin are $(-\alpha, -\beta)$

Art. 42. Lines Joining the origin to the intersection of a curve and a line.

Let the second degree equation representing the curve be

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ .... (1)

and the first degree equation representing the line be

$lx + my = n$ ... ... or $\dfrac{lx}{n} + \dfrac{my}{n} = 1$ ... ... (2)

Let the straight line meet the curve (1) at P and Q. We are to find the equation of the pair of lines OP and OQ.

Now making (1) homogeneous with the help of (3) we get

$$ax^2 + 2hxy + by^2 (gx + fy) \left(\frac{lx + my}{n}\right) + c\left(\frac{lx + my}{n}\right)^2 = 0$$

Which, on simplification, reduces to an equation of the form

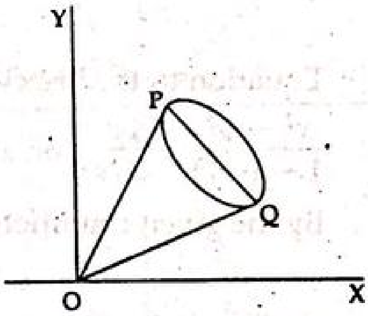

Fig. 13

$Ax^2 + 2Hxy + By^2 = 0$ ... ... (4)

This is a Homogeneous equation of 2nd degree. It represents a pair of straight lines through the origin. Moreover the co-ordinates of P and Q satisfy (4), because the co-ordinates of P and Q satisfy (1) and (3). Therefore, (4) must represent OP and OQ and is our required equation.

Ex. 1. Prove that the equation $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$ represents a pair of parallel lines. [R. U. 1997; D. U. 1986]

$x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$

or, $x^2 + x(6y + 4) + (9y^2 + 12y - 5) = 0$ quadratic in x

$$x = \frac{-(6y-4) \pm \sqrt{(6y+4)^2 - 4 \cdot 1 (9y^2 + 12y - 5)\}}}{2}$$

$= (-6x - 4 \pm 6)/2 = 3y - 2 \pm 3$

$x + 3y + 5 = 0$ and $x + 3y - 1 = 0$

Hence the given equation represents two straight line $x + 3y + 5 = 0$ and $x + 3y - 1 = 0$. As these two lines differ only in constant terms, so these form a pair of parallel straight lines. See Art. 26 (1).

Ex 2. Show that the angle between one of the lines given by $ax^2 + 2hxy + by^2 = 0$ and one of the lines.

$ax^2 + 2xy + by^2 + \lambda (x^2 + y^2) = 0$ is equal to the angles between the other two lines of the system.

The equation of the straight lines bisecting the angles between

$ax^2 + 2hxy + by^2 = 0$ is $\dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}$

Again, the equation of the straight lines bisecting the angles between $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$

or, $(a + \lambda) x^2 + 2hxy + (a + \lambda)y^2 = 0$ is

$$\frac{x^2 - y^2}{(a + \lambda) - (b + \lambda)} = \frac{xy}{h} = \text{or, } \frac{(x^2 - y^2)}{a - b} = \frac{xy}{h}$$

which is the same as the equation of bisectors for the first pair. Hence the result.

Ex. 3. If the pair of straight lines $x^2 - 2axy - y^2 = 0$ and $x^2 - 2bxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that $ab = -1$.

[D. U. 1953, 57, 61, R. U. 1964, 79]

Equation to the bisector of first pair is

$$\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-la} \text{ or, } x^2 - y^2 = \frac{2xy}{a} \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

By the given condition second pair $x^2 - y^2 = 2bxy$ $\quad \cdots \quad \cdots \quad (2)$

bisects the angle between the first pair. Hence comparing we have $1 = -\dfrac{\frac{2}{a}}{2b}$

or, $ab = -1$ Proved.

Ex. 4 Prove that two of the lines represented by the equation $ax^4 + by^3x + cx^2y^2 + dxy^3 + ey^4 = 0$ will be at right angles if

$(b + d) (ad + be) + (a - e)^2 (a + e + c) = 0$

Let $ax^4 + bxy^3 + cx^2y^2 + dxy^3 + ey^4 = (x^2 + kxy - y^2) (ax^2 + k' xy - ey^2) = 0$

$= ax^4 + (k' + ak) x^3y + (-ek - k') xy^3 + (kk' - a - e) x^2 y^2 + ey^4$

Now compare the co-efficients of x^4, x^3y, xy^3, x^2y^2 and y^4 from both sides, then

$k' + ak = b, \quad \cdots \quad \cdots \quad \cdots \quad (1)$

$-ek - k' = d, \quad \cdots \quad \cdots \quad \cdots \quad (2)$

$-a - e + kk' = c \quad \cdots \quad \cdots \quad \cdots \quad (3)$

Find the value of k and k' from 1st and 3rd equations.

$k = (b + d) / (a - e)$, $k = -(ad + bc) / (a - e)$

Therefore from (3), we have $kk' = a + e + c$

$$\frac{b+d}{a-e} \times \frac{ad+be}{a-e} = a+e+c$$

or, $(b + d)(ad + be) + (a - e)^2 (a + e + c)$ proved.

Ex. 5. Prove the equation $y^3 - x^3 + 3xy(y - x) = 0$. represents three straight lines equally inclined to one another. [C. U. (Hons) 1987]

Changing the equation $y^3 - x^3 + 3xy(y - x) = 0$

in polar co-ordinates ; $x = r\cos\theta$, $y = r\sin\theta$, we have $r^3\sin^3\theta - r^3\cos^3\theta + 3r^3\sin\theta.\cos\theta(\sin\theta - \cos\theta) = 0$

or, $\sin^3\theta - \cos^3\theta + 3\sin\theta\cos\theta(\sin\theta - \cos\theta) = 0$

or, $(\sin\theta - \cos\theta)\{\sin^2\theta + \cos^2\theta + 4\sin\theta\cos\theta\} = 0$

or, $(\tan\theta - 1)(\tan^2\theta + 4\tan\theta + 1) = 0$

Either $(\tan\theta - 1 = 0$ *i.e.* $\theta = 45°$. say $\theta_1 = 45°$ ... ... (1)

Again $\tan^2\theta + 4\tan\theta + 1 = 0$

or, $\tan\theta = \{-4 \pm \sqrt{(16-4)}\}/2 = 2 \pm \sqrt{3}$

Let $\tan\theta_2 = -2 + (\sqrt{3}) = \tan 165°$

$\therefore \theta_2 = 165°$

$\tan\theta_2 = -2 - \sqrt{3} = -\tan 75° = \tan(-75°) = \tan(360° - 75°) = \tan 285°$

$\therefore \theta_3 = 285°$

Now $\theta_2 - \theta_1 = 165° - 45° = 120°$

$\theta_3 - \theta_2 = 285° - 165° = 120°$

$\theta_3 - \theta_1 = 75° - 45° = -120°$

Hence the straight lines are inclined to each other at an angle of 120°. here RO is produced to OR′.

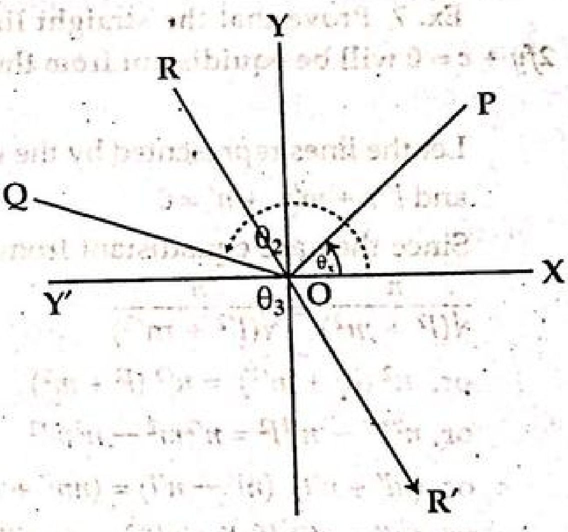

Fig : 14

Ex. 6. Show that the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two parallel lines if $a : h = h : b = g : f$. Also show that the distance between them is $\frac{2\sqrt{\{(g^2 - ac)}}{a(a+b)\}}$.

We know the condition that the general equation represents two straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and it will represent two parallel straight lines if in addition, $ab - h^2 = 0$

Now $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

or, $h^2c + 2fgh - af^2 - bg^2 - ch^2 = 0$, Since $ab = h^2$

or, $bg^2 - 2fgh + af^2 = 0$

or, $bg^2 - 2fg\sqrt{(ab)} + af^2 = 0$ $(\therefore h^2 = ab)$

or, $(\sqrt{bg} - \sqrt{af})^2 = 0$ or, $\sqrt{bg} = \sqrt{af}$

or, $\frac{g}{f} = \sqrt{(\frac{a}{b})} = \sqrt{(\frac{ab}{b^2})} = h/b \therefore h^2 = ab.$, or, $\frac{a}{h} = \frac{h}{b}$

$\therefore \frac{a}{g} = \frac{h}{b} = \frac{g}{f}$ which are the equired conditions

Let AB and CD be the lines $lx + my + n = 0$, $lx + my + n_1 = 0$ respectively. Take any point P (x_1, y) on AB. Draw PM perpendicular to CD.

$l^2 = a, m^2 = bnn_1 = c, l(n + n_1) = 2g.$

$\therefore lx_1 + my_1 + n_1 = 0$ ... ... (1)

$\therefore$ PM $= \frac{lx_1 + my_1 + n_1}{\sqrt{(l^2 + m^2)}} = \frac{n_1 - n}{\sqrt{(l^2 + m^2)}}$ from (1)

$= \frac{\sqrt{\{(n + n_1)^2 - 4nn_1\}}}{\sqrt{(l^2 + m^2)}} = \frac{\sqrt{(4g^2/l^2 - 4c)}}{\sqrt{(a + b)}} = \frac{\sqrt{(4g^2/a - 4c)}}{\sqrt{(a + b)}}$

$= \frac{2\sqrt{\{(g^2 - ac)}}{\sqrt{\{a(a + b)}} = 2\sqrt{\left(\frac{g^2 - ac}{a(a + b)}\right)}$

Ex. 7. Prove that the straight lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will be equidistant from the origin, if $f^4 - g^4 = (bf^2 - ag^2)$

[D. U. 1956, C. U. (Hons) 1977, C. U. 1981]

Let the lines represented by the given equation be $lx + my + n = 0$

and $l'x + m'y + n' = 0$

Since they are equadistant from the origin.

$\frac{n}{\sqrt{(l^2 + m^2)}} = \frac{n'}{\sqrt{(l'^2 + m'^2)}}$

or, $n^2(l'^2 + m'^2) = n'^2(l^2 + m^2)$

or, $n^2l'^2 - n'^2l^2 = n'^2m^2 - n^2m'^2$

or, $(nl' + n'l)(nl' - n'l) = (nm' + n'm)(n'm - nm')$

or, $(nl' + n'l)\{(nl' + n'l)^2 - 4nn'll'\} = (nm' + n'm)^2$

$\{(n'm + nm')^2 - 4nn'mm'\}$ ... ... (2)

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (lx + my + n)(l'x + m'y + n')$ comparing the co-efficients we have

$ll' = a, mm' = b, nn' = c, mn' + m'n = 2f, nl' + n'l = 2g, lm' + l'm = 2h$

Putting the values in (2) we have

$g^2(g^2 - ac) = f^2(f^2 - bc)$ or, $f^4 - g^4 = c(bf^2 - ag^2)$

Ex. 8. Show that the lines joining the origin to the intersection of $7x^2 + 8x^2 - 7x^2 + 6x - 12y = 0$ and $2x + y - 1 = 0$ are at right angles.

Making the first equation homogeneus with the help of

$2x + y = 1$, we have

$7x^2 + 8xy - 7x^2 + (6x - 12y)(2x + y) = 0$ or, $19x^2 - 10xy - 19y^2 = 0$ which is the equation of the required straight lines.

Since the sum of the co-efficients of x^2 and y^2 is equal to zero, the straight lines are at right angles.

Ex. 9. The axes being rectangular, find the equation to the pair of sraight lines meeting at the origin which are perpendicular to the pair given by the equation $ax^2 + 2hxy + by^2 = 0$

If the lines represented by $ax^2 + 2hxy + by^2 = 0$ be $y - m_1x = 0$ and $y - m_2x = 0$, then we have $m_1 + m_2 = 2\frac{h}{b}$ and $m_1m_2 = \frac{a}{b}$ ... ... (1)

Now the equations of the lines perpendicular to the above lines and passing through the origin are $m_1y + x = 0$ and $m_2y + x = 0$

Their joint equation is

$(m_1y + x)(m_2y + x) = 0$, or $m_1m_2y^2 + (m_1 + m_2)xy + x^2 = 0$

or, $(\frac{a}{b})y^2 - (\frac{2h}{b})xy + x^2 = 0$ by (1)

or, $bx^2 - 2hxy + ay^2 = 0$ Ans.

Ex. 10. The line joining the origin to the point P (2,3) is the diameter of a circle which cuts the lines $5x^2 - 12xy + 3y^2 = 0$ at Q and R. Find the combined equation of the lines PQ, PR.

P is a point (2, 3) and O is the origin. The circle on OP as diameter cuts at Q and R, the lines OQ & OR and whose combined equations are $5x^2 - 12xy + 3y^2 = 0$. PQ and PR are perpendicular to the chord OQ, OR respectively.

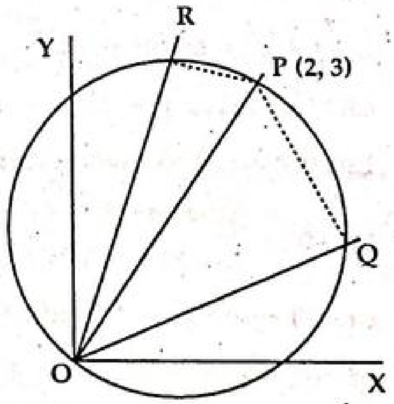

Fig. 15

Let $5x^2 - 12xy + 3y^2$

$= 3(y - m_1x)(y - m_2x)$

$\therefore m_1 + m_2 = \frac{12}{3}, m_1m_2 = \frac{5}{3}$

The combined equation of the perpendiculars through O to OQ, OR is

$(m_1y + x)(m_2y + x) = 0$

or, $m_1m_2y^2 + (m_1 + m_2)xy + x^2 = 0$

by (1) ... ... (2)

or, $5y^2 + 12xy + 4x^2 = 0$

Now the combined equation of the lines PR, PQ parallel to the lines represented by (2) is obtained by changing the origin O to

P (2, 3) or by putting $x - 2$ for x and $y - 3$ for y in (2)

Therefore, $5(y - 3)^2 + 12(x - 2)(y - 3) + 3(x - 2)^2 = 0$

or, $3x^2 + 12xy + 5y^2 - 48x - 54y + 129 = 0$

Ex. 11. Find the area of the triangle formed by the lines.

$ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ [R. U. Hons. 1978; D. U. '60; C. U. Hons. 1991]

Let the lines represented by the given pair be $y = m_1x$ and $y = m_2x$, so that $m_1 + m_2 = -2\frac{h}{b}$ and $m_1m_2 = \frac{a}{b}$. ... ... (1)

These lines will meet the, hird line at points.

$$\left(\frac{-n}{1+nm_1}\right),\left(\frac{-nm_1}{1+mm_2}\right) \text{ and } \left(\frac{-n}{1+mm_2}\right)\left(\frac{-nm_2}{1+mm_2}\right)$$

which will be two vertices of the triangle and the third vertex is clearly (0, 0). Hence the area of the triangle,

$$\Delta = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{-n}{l+mm_1} & \frac{-nm_1}{1+mm_1} & 1 \\ \frac{-n}{l+mm_2} & \frac{-nm_2}{1+mm_2} & 1 \end{vmatrix} = \frac{n^2(m_2-m_1)}{2(l+mm_1)(l+mm_2)}$$

$$= \frac{\frac{n^2}{2}[(m_1+m_2)^2-4m_1m_2]}{l^2+lm(m_1+m_2)+m^2m_2m_2}$$

$$= \frac{n^2\sqrt{(h^2-2b)}}{2(am^2+bl^2-2hlm)}$$ by (1) Ans.

Ex. 12. The circle $x^2+y^2=a^2$ cuts off an intercept on the straight line $lx+my=1$, which subtends an angle of 45° at the origin, show that $4[a^2(l^2+m^2)-1]=[a^2(l^2+m^2)-2]^2$

The equation of the straight lines joining the origing to the points of intersection of

$x^2+y^2=a^2$ ... ... (1)

and $lx+my=1$ ... ... (2)

is $x^2+y^2=a^2(lx+my)^2$

or $x^2(1-a^2l^2)+y^2(1-a^2m^2)-2a^2lmxy=0$

Let the angle between them be $\phi=45°$ (given)

$\tan\phi = \dfrac{2\sqrt{\{l^2m^2a^4-(1-a^2l^2)(1-a^2m^2)\}}}{(1-a^2l^2)+(1-a^2m^2)}$ by Art. 41

or, $(1-a^2l^2)+(1-a^2m^2)=2\sqrt{\{l^2m^2a^4-(1-a^2l^2)(1-a^2m^2)\}}$

or, $(2-a^2l^2-a^2m^2)^2=4(a^2l^2+a^2m^2-1)$

or, $4[a^2l^2+m^2)-1]=[a^2(l^2+m^2)-2]^2$ Proved.

EXERCISE V

[তারকা চিহ্নের অঙ্কগুলি অনার্স ছাত্রদের জন্য]

Find the lines represented by (নিম্নে সমীকরণের দ্বারা প্রকাশিত সরলরেখাগুলো নির্ণয় কর।)

1. $x^2\cos 2\theta-4xy\cos\theta+2y^2+x^2=0$. Ans. $y=x\cos\theta$, $y=x\cos\theta$
2. $3x^2-16xy+5y^2=0$ Ans. $x-5y=0; 3x-y=0$
3. $y^3-xy^2-14x^2y+24x^3=0$ Ans. $y=-4x, y=2x, y=3x$

Prove that the following equations represent two straight lines; find also their point of intersection and the angles between them.

4. $3y^2-8xy-3x^2-29x+3y-18=0$ Ans. (3/2, 5/2)) ; 90°

5. $2y^2 - xy - x^2 + y + 2x - 1 = 0$ Ans. $(1, 0)$, $\tan^{-1}3$ [R. U. 1960]

6. $x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ Ans. $(\frac{-11}{13}, \frac{10}{13}; 90°)$

7. $2y^2 + 3xy - 5y + 6x + 2 = 0$ Ans. $(-1, 2, \tan^{-1}\frac{3}{2})$ [D. U. 1952]

Find the value of λ or k so that the following equations may represent pairs of straight lines.

8. $\lambda x^2 + 4xy + y^2 - 4x - 2y - 3 = 0$ Ans. 4 [D. U. 1960]
9. $2x^2 - y^2 + xy - 2x - 5y + k = 0$ Ans. -4
10. $6x^2 + xy + ky^2 - 11x + 43y - 35 = 0$ Ans. -12
11. $x^2 - \lambda xy + 2y^2 + 3x - 5y + 2 = 0$ Ans. $3, \frac{9}{2}$
12. $kxy - 8x + 9y - 12 = 0$ Ans. 6
13. $6x^2 + 2kxy + 12y^2 + 22x + 31y + 20 = 0$ Ans. $\frac{171}{20}, \frac{17}{2}$ [D. U. 1962]
14. $6x^2 - 7xy + 16x - 3y^2 - 2y + k = 0$ Ans. 8 [D.U. (Hons) 1962]
15. $12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0$ Ans. $\lambda = 2$ [D. U. 1962]

16. **Find the angle between the straight lines.**
$(x^2 + y^2)(\cos^2\theta \sin^2\alpha + \sin^2\theta) = (x\tan\theta - y\sin\alpha)^2$ Ans. 2θ [R. U. 1961]

16. Show that $2x^2 - 7xy + 3y^2 + x + 7y - 6 = 0$ equation representing two straight lines whose included angle is 45° (দেখাও যে $2x^2 - 7xy + 3y^2 + x + 7y - 6 = 0$ দুটি সরলরেখা প্রকাশ করে এবং এদের অন্তর্ভুক্ত কোণ 45°)

17. If the two straight line represented by $x^2(\tan^2\phi + \cos^2\phi) - 2xy\tan\phi + y^2\sin^2\phi = 0$ make angles α and β with the axis of x, then show that $\tan\alpha - \tan\beta = 2$. ($x^2(\tan^2\phi + \cos^2\phi) - 2xy\tan\phi + y^2\sin^2\phi = 0$ এর দুটি সরলরেখা x অক্ষরেখার সাথে αও β কোণ করলে দেখাও যে, $\tan\alpha - \tan\beta = 2$.)

18. Show that the straight lines represented by $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ are perpendicular to each other. (দেখাও যে, $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ এর দুটি সরলরেখা পরস্পরের উপর লম্ব হবে।)

19. If the origin is joined by the two points ofintersection of the line $lx + my + n = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ there will be two straight lines show that they will be coincident. (প্রমাণ কর যে, $lx + my + n = 0$ এবং $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ বক্ররেখার ছেদ বিন্দুদ্বয়কে মূল বিন্দুতে সংযোগ করলে যে সরলরেখা দুটি উৎপন্ন হবে তারা সমপতিত হবে যদি $a^2l^2 + b^2m^2 = n^2$) [R. U. 1980]

20. The straight line $y = mx$ bisects the angle between two straight lines represented by $ax^2 - 2hxy + by^2 = 0$ if $h(1 - m^2) + m(a - b) = 0$. (দেখাও যে, $y = mx$ সরলরেখা $ax^2 - 2hxy + by^2 = 0$ দ্বারা প্রকাশিত সরলরেখাদ্বয়ের অন্তর্ভুক্ত কোণকে দ্বিখণ্ডিত করবে যদি $h(1 - m^2) + m(a - b) = 0$) [D. U. 1977]

21. Show that the two straight lines represented by $ax^2 + 2\lambda xy - ay^2 = 0$ will be perpendicular to each other. (দেখাও যে, $ax^2 + 2\lambda xy - ay^2 = 0$ পরস্পরের উপর লম্ব দুটি সরলরেখা প্রকাশ করে।)

22. Show that the four straight lines represented by $12x^2 + 7xy - 12y^2 = 0$ and $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$ will be on the sides of a square. (দেখাও যে, $12x^2 + 7xy - 12y^2 = 0$. এবং $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$. এদের দ্বারা প্রকাশিত চারটি সরলরেখা একটি বর্গক্ষেত্রের বাহুর উপর থাকবে।)

23. Show that the four straight lines represented by $4xy(x^2 - y^2) - \tan\alpha(x^2 + 2xy - y^2)(x^2 - 2xy - y^2) = 0$ will make equal angles with each other. (দেখাও যে, $4xy(x^2 - y^2) - \tan\alpha(x^2 + 2xy - y^2)(x^2 - 2xy - y^2) = 0$ এর দ্বারা প্রকাশিত চারটি সরলরেখা একে অন্যের সাথে সমান কোণে থাকবে।)

24. Show that the four straight lines represented by $y^2 - 4y + 3 = 0$ and $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$ (দেখাও যে, $y^2 - 4y + 3 = 0$ এবং $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$ will form a parallelogram. এদের দ্বারা প্রকাশিত চারটি সরলরেখা একটি সামান্তরিক উৎপন্ন করবে।)

25. Show the four straight lines represented by $4xy - 2y^2 + 6x - 12y - 15 = 0$ and $x^2 + 4xy - 2y^2 = 0$ will form a rhombus through the origin. (দেখাও যে, $x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$ এবং $x^2 + 4xy - 2y^2 = 0$ এদের দ্বারা প্রকাশিত চারটি সরলরেখা একটি রম্বস প্রকাশ করবে।)

26. If the two straight lines through the origin represented by $2x^2 + 5xy + 2y^2 + 10x + 5y = 0$ are perpendicular to each other then find the combined equation of straight lines passing through the origin. (মূলবিন্দুগামী সরলরেখা দুটি $2x^2 + 5xy + 2y^2 + 10x + 5y = 0$ দ্বারা প্রকাশিত সরলরেখা দুটির উপর পরস্পর লম্ব হলে মূলবিন্দুগামী সরলরেখাদ্বয়ের যুগ্ম সমীকরণ নির্ণয় কর।) Ans. $2y^2 - 5xy + 2x^2 = 0$

[C. U. 1990]

27. Prove that the angles between the straight lines. $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ have the same pair of bisectors. Interpret the case when $\lambda = -(a + b)$ (দেখাও যে $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ দ্বারা প্রকাশিত সরলরেখা দুইটির কোণের দ্বিখণ্ডকদ্বয় একই হবে। যখন $\lambda = -(a + b)$ তখন এটা কি প্রকাশ করে ?) [R. U. Hons. 1977; '80]

28. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines equidistant from the origin. show that $h(g^2 - f^2) = fg(a - b)$ (যদি মূল বিন্দু হতে সমান দূরত্বে দুইটি সরলরেখা $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ সমীকরণ দ্বারা প্রকাশিত হয় তখন দেখাও যে, $h(g^2 - f^2) = fg(a - b)$) [D. U.(Hons.) 1959; C. U. Hons. 1989]

29. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines, prove that the square of the distance of their point of intersection from the origin is $\dfrac{c(a + b) - f^2 - g^2}{ab - h^2}$ (যদি মূল বিন্দু হতে সমান দূরত্বে দুইটি সরলরেখা $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ সমীকরণ দ্বারা প্রকাশিত হয় তখন দেখাও যে সরলরেখা দুইটির ছেদবিন্দু হতে মূলবিন্দুর দূরত্ব $\dfrac{c(a + b) - f^2 - g^2}{ab - h^2}$ হবে।)

30. Show that the pair of straight lines joining the origin O to the intersections A and B of the line $lx + my = 1$ with the conic. $ax^2 + by^2 = 1$ has the equation. $(a - l^2)x^2 - 2lmxy + (b - m^2)y^2 = 0$ ($lx + my = 1$ এবং $ax^2 + by^2 = 1$ কণিকের ছেদবিন্দু দুইটি A এবং B হলে, দেখাও যে সরলরেখা দুইটির সংযুক্ত সমীকরণ হবে $(a - l^2)x^2 - 2lmxy + (b - m^2)y^2 = 0$) [R. U. (Pass) 1985; R. U. 1962, '82]

31. দেখাও যে, $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$ এবং $4x^2 - 4xy + y^2 + 4x - 2y - 3 = 0$ দ্বারা প্রকাশিত চারটি সরলরেখা একটি সামান্তরিক প্রকাশ করে।

32. Prove that two of the lines represented by the equation $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4 = 0$ will bisect angle between the other two if $c + 6a = 0$, $b + d = 0$. [C. H. 1979, 78]

33. Find the equations of the bisectors of the angles between the lines represented by

(a) $12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$ [D. U. 1962]

Ans. $7(23x + 25y)^2 - 7(23y - 43)^2 = 44(23x + 25)(23y - 43)$

(b) $2x^2 + 7xy + 6y^2 + 13x + 22y + 20 = 0$ [R. U. 1962]

Ans. $7x^2 - 7y^2 + 8xy - 4x - 58y - 83 = 0$

34. Prove that the angles between the lines joining the origin to the intersections of the line $y = 3x + 2$ with the curve.

$x^2 + 3y^2 + 2xy + 4x + 8y - 11 = 0$ is $\tan^{-1}(2\sqrt{2})/3$

35. Prove that one of the line represented by the equations $ax^3 + bx^2y + cxy^2 + dy^3 = 0$ will bisect the angle between the other two if $(3a + c)^2(bc + 2cd - 3ad) = (b + 3d)^2(bc + 2ab - 3ad)$

36. Show that the lines joining the origin to the intersection of $x^2 + y^2 + 19x + 4y - 3 = 0$ and $3x + 4y = 1$ are at right angles. [R. H. 1988; D. U. 1961]

37. দেখাও যে $3x^2 + 3xy - 3y + 2x + 5y = 0$ এবং $3x - 2y = 1$

সরলরেখার ছেদবিন্দু দুটির সাথে মূলবিন্দুর সংযোগ সরলরেখা দুটি পরস্পর লম্ব হবে। [C. U. 1987]

37. (a) Find the angle between the lines joining the origin to the pt of intersection of $y = x + 1$ with $x^2 - 3y^2 + 2xy - 3x + 3y + 1 = 0$ [R. U. 1981] Ans. $\tan^{-1} 2\sqrt{5}/5$.

38. দেখাও যে $a(x^4 + y^4) - 4bxy(x^2 - y^2) + 6cx^2y^2 = 0$ দ্বারা প্রকাশিত দু' জোড়া পরস্পর লম্ব সরলরেখা প্রকাশ করে। যদি দু' জোড় সরলরেখা সমাপতিত হয় তা হবে যদি $2b^2 = a^2 + 3ac$

38. (a) Prove that $x^4 + y^4 - 4xy(x^2 - y^2) + 2cx^2y^2 = 0$ represents a pair of pair of perpendicular straight lines, also show that if $c = 1$ then those of straight lines will be coincident. [C. U(P) 1989]

39. Prove that two of the straight lines represented by the equation $ax^3 + bx^2y + cxy^2 + dy^3 = 0$. will be at right angles if. $a^2 + ac + bd + d^2 = 0$. [C. H. 1980]

40. Show that lines joining the origin to the point of intersections of $y^2 - 4ax = 0$ and $y = mx + e$ will be coincident of $c = \frac{a}{m}$

41. Prove that the straight lines represented by the equation $\cos 3\alpha (x^3 - 3xy^2) + \sin 3\alpha (y^3 - 3x^2y) + 3a(x^2 + y^2) - 4a^3 = 0$ form an equailateral triangle whose area is $3\sqrt{3}a^2$. [C.H.1989]

42. Prove that the equation $m(x^3 - 3xy^2) + y^3 - 3x^2y = 0$ represents three straight lines equally inclined to one another. [R. U. 1987]

[প্রমাণ কর যে $m(x^3 - 3xy^2) + y^3 - 3x^2y = 0$ দ্বারা তিনটি সরলরেখা প্রকাশ করে এবং এরা একে অন্যের সাথে একই কোণে থাকবে।]

43. Prove that the straight lines joining the origin to the points of intersection of the straight line $kx + hy = 2hk$ with the curve

$(x-h)^2 + (y-k)^2 = c^2$ are at right angles if $h^2 + k^2 = c^2$.

(প্রমাণ কর যে $kx + hy = 2hk$ এবং বক্ররেখা $(x-h)^2 + (y-k)^2 = c^2$ ছেদবিন্দুর সাথে মূলবিন্দুর সংযোগ সরলরেখাদ্বয় পরস্পরের উপর লম্ব হবে যদি $h^2 + k^2 = c^2$)

44. If one of the straight line given by the equation $ax^2 + 2hxy + by^2 = 0$ coincides with one of those given by $a_1x^2 + 2h_1xy + b_1y^2 = 0$ and the other lines represented by them be perpendicular, prove that $\frac{ha_1b_1}{b_1-a_1} = \frac{h_1ab}{b-a} = \frac{1}{2}\sqrt{(-aa_1bb_1)}$.

(যদি $ax^2 + 2hxy + by^2 = 0$ রেখাদ্বয়ের একটি রেখা $a_1x^2 + 2h_1xy + b_1y^2 = 0$ রেখাদ্বয়ের একটি রেখার সাথে সমাপতিত হয় এবং অপর দুটি রেখা পরস্পরের সাথে লম্ব হয়, তবে দেখাও যে $\frac{ha_1\,b_1}{b_1-a_1} = \frac{h_1ab}{b-a} = \frac{1}{2}\sqrt{(-aa_1bb_1)}$.

45. Find the equation of the line through the origin and perpendicular to the lines $5x^2 - 7xy - 3y^2 = 0$

($5x^2 - 7xy - 3y^2 = 0$ রেখাদ্বয়ের উপর লম্ব এবং মূলবিন্দুগামী সরলরেখাদ্বয়ের সমীকরণ নির্ণয় কর।

Ans. $3x^2 + 7xy + 5y^2 = 0$) [D. U. H. 1979]

46. Show that $x^2 + 4xy^2 - 2y^2 + 6x - 12y - 15 = 0$ represents pair of straight lines.

(দেখাও যে, $x^2 + 4xy^2 - 2y^2 + 6x - 12y - 15 = 0$ সমীকরণ দুটি সরলরেখা প্রকাশ করে।) [D. U. 1979]

47. Show that the st. lines joining the origin to the points of intersection of the curves $ax^2 + 2hxy^2 + by^2 + 2gx = 0$ and $a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x = 0$ will be at right angles. if $g(a_1 + b_1) = g\,(a + b)$ [D. U. '76]

48. Show that the equation $bx^2 - 2hxy \quad ay^2 = 0$ represents a pair of st. lines which are at right angle to the pair given by equation $ax^2 + 2hxy^2 + by^2 = 0$ [R. U. 1985]

(দেখাও যে, $bx^2 - 2hxy + ay^2 = 0$ এর সরলরেখা দুটি $ax^2 + 2hxy + by^2 = 0$ সরলরেখাদ্বয়ের সাথে পরস্পর লম্ব হবে।) [D. U. 1976, R. U. 1978, '85]

49. If one of the lines $ax^2 + 2hxy + by^2 = 0$ be perpendicular to one of the lines $a'x^2 + 2h'xy + b'y^2 = 0$, prove that $(aa' - bb')^2 + 4\,(a'h + bh')\,(ah' + b'h) = 0$. [R.H. 1977]

(যদি $ax^2 + 2hxy^2 + by^2 = 0$ সরলরেখার একটি $a'x^2 + 2h'xy^2 + b'y^2 = 0$ এই সরলরেখার একটির উপর পরস্পর লম্ব হয়, তা হলে প্রমাণ কর যে $(aa' - bb')^2 + 4\,(a'h + bh')\,(ah' + b'h) = 0$.) [R.H. 1997]

50. Prove that $(ax + by)\,(\alpha x + \beta y) + kxy - (a + \alpha)\,x - (b + \beta)\,y + 1 = 0$ represents a pair of straight lines if $k = (a - \alpha)\,(b - \beta)$. Find the co-ordinates of their point of intersection.

(প্রমাণ কর যে $(ax + by)\,(\alpha x + \beta y) + kxy - (a + \alpha)\,x - (b + \beta)\,y + 1 = 0$ দুটি সরলরেখা প্রকাশ করবে, যদি $k = (a - \alpha)\,(b - \beta)$. হয় সরলরেখা দুটির ছেদবিন্দু নির্ণয় কর।) [D. U. 1975]

51. Prove that product of the perpendiculars from the point (x_1, y_1) on the lines $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{\{(a-b)^2 + 4h^2\}}}$ [C. U. 1986; R. U. 1979]

* **52.** দেখাও যে $ax^2 + 2hxy^2 + by^2 = 0$ সরলরেখাদ্বয়ে একটি রেখা $a_1x^2 + 2h_1xy + b_1y^2 = 0$ রেখা দুটির একটি রেখার সাথে সমাপতিত হবে যদি

$(ab_1 - a_1b)^2 = 4\ (a_1h - ah_1)\ (bh_1 - b_1h)$. [R. U. 1988; D.U. 1989]

* **53.** If $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines, the area of the triangle formed by their bisectors and the axis of x is

$\Delta = \dfrac{\sqrt{\{(a-b)^2 + 4h^2\}}}{2h}\,\dfrac{ca - g^2}{ab - h^2}$ [C.H. 1989; R.U. 1982]

(S দ্বারা দুটি সরলরেখা সূচিত হলে, দেখাও যে তাদের অন্তর্গত কোণের দ্বিখণ্ডকদ্বয় x অক্ষের সাথে ত্রিভুজ উৎপন্ন করে, তার ক্ষেত্রফল Δ হবে)

* **54.** Show that four lines given by the equations

$(y - mx)^2 = c^2\ (1 + m^2)$; $(y - nx)^2 = c^2\ (1 + n^2)$ form a rombus.

* **55.** The vertices of a triangle lie on the lines $y = m_1x$, $y = m_2x$, $y = m_3x$ the circumcentre being at the origin, prove that the locus of the other centre is the line $L \equiv x\ (\sin\theta_1 + \sin\theta_2 + \sin\theta_3) - y(\cos\theta_1 + \cos\theta_2 + \cos\theta_3) = 0$ (একটি ত্রিভুজের শীর্ষবিন্দুগুলো যথাক্রমে $y = m_1x$, $y = m_2x$, $y = m_3x$ সরলরেখার উপর থাকবে। এদের পরিলিখিত বৃত্তের কেন্দ্র মূল বিন্দুতে হলে দেখাও যে অন্য কেন্দ্রগুলোর সঞ্চারপথ একটি সরল রেখা L হবে।)

* **56.** The distance from the origin to orthocentre of the triangle formed by the lines $x/\alpha + y/\beta = 1$ and $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{(a + b)\ \alpha\beta\sqrt{(\alpha^2 + \beta^2)}}{a\alpha^2 - 2h\alpha\beta + b\beta^2} = L$$

($x/\alpha + y/\beta = 1$ এবং $ax^2 + 2hxy + by^2 = 0$ সরলরেখার দ্বারা সৃষ্ট ত্রিভুজের লম্ববিন্দু হতে মূল বিন্দুর দূরত্ব L হবে।)

* **57.** Show that the centroids of the triangles of which the three perpendiculars lie along the lines. $y - m_1x = 0$, $y - m_2x = 0$, $y - m_3x = 0$ lie on

$y\ (3 + m_2m_3 + m_3m_1 + m_1m_2) = x\ (m_1 + m_2 + m_3 + 3m_1m_2m_3)$

(দেখাও যে কোন ত্রিভুজের লম্বত্রয় $y - m_1x = 0$, $y - m_2x = 0$, $y - m_3x = 0$ সরলরেখা বরাবর থাকলে ত্রিভুজের ভরকেন্দ্রে $y\ (3 + m_2m_3 + m_3m_1 + m_1m_2) = x(m_1 + m_2 + m_3 + 3m_1m_2m_3)$ এর উপর থাকবে।)

* **58.** The base of a triangle passes through a fixed point (f, g) and its sides are respectively bisected at right angles of by the lines $ax^2 + 2hxy + by^2 = 0$. Prove that the locus of its vertex is.

$S = (a + b)\ (x^2 + y^2) + 2h\ (fy + gx) + (a - b)\ (fx - gy) = 0$

(কোন ত্রিভুজের ভূমি (f, g) একটি নির্দিষ্ট বিন্দুগামী এবং এর বাহুগুলো $ax^2 + 2hxy + by^2 = 0$ সরলরেখার দ্বারা লম্বভাবে দ্বিখণ্ডিত হয়। দেখাও যে ত্রিভুজের শীর্ষবিন্দুর সঞ্চারপথ S হবে।)

* 59. Show that the equation $E = ax^2 + 2hxy + by^2 = \{(a + b) \sin^2\theta + \sqrt{(h^2 - ab)} \sin 2\theta\}$ $(x^2 + y^2)$ represents two straight lines having the same bisectors as $ax^2 + 2hxy + by^2 = 0$ and making equal angles θ with them respectively. (দেখাও যে সমীকরণ E দুটি সরলরেখা প্রকাশ করে যাদের দ্বিখণ্ডকদ্বয় $ax^2 + 2hxy + by^2 = 0$ সরলরেখাদ্বয়ের দ্বিখণ্ডকদ্বয় একই হয় এবং যথাক্রমে তাদের সাথে θ সমান কোণ উৎপন্ন করে।)

* 60. Show that the equation $S = (ab - h^2)(ax^2 + 2hxy + by^2 + 2gx + 2fy) + af^2 + bg^2 - 2fgh = 0$ represents a pair of straight lines, and that these straight lines form a rhombus with the lines $ax^2 + 2hxy + by^2 = 0$ provided that $(a - b) fg + h(f^2 - g^2) = 0$ (দেখাও যে S সমীকরণ দুটি সরলরেখা প্রকাশ করে এবং দেখাও যে S সমীকরণ এর দুটি সরলরেখা এবং $ax^2 + 2hxy + by^2 = 0$ সরলরেখা দুটি একত্রে একটি রম্বস উৎপন্ন করবে যদি $(a - b) fg + h(f^2 - g^2) = 0$ হয়।)

61. $ax^2 + 2hxy + by^2 = 0$ দ্বারা নির্ধারিত সরলরেখাযুগল এবং $x + y = 1$ সরলরেখা দ্বারা উৎপন্ন ত্রিভুজের ক্ষেত্রফল নির্ণয় কর। Ans. $\sqrt{(h^2 - ab)}/(a + b - 2h)$ [D. U. 1987]

62. Find the condition that two of the lines given by $ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$ may be at right angles (কোন শর্তে $ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$ সমীকরণের তিনটি সরলরেখার মধ্যে দুটি সরলরেখা পরস্পর লম্ব হবে) Ans. $a^2 + 3bd + 3ac + d^2 = 0$

63. Show that the or the centre of the triangle formed by the lines $ax^2 + 2hxy + by^2 =$ and $lx + my = 1$ is a point (x_1, y_1) such that $\frac{x_1}{l} = \frac{y}{m} = \frac{a + b}{am^2 - 2hlm + bl^2}$

1

RECTANGULAR CO-ORDINATES

Art. 1. Introduction : The position of a point in space may be defined by refering it to three fixed planes. These planes have a common point, known as the **Origin** and intersect in pairs in three straight lines, known as **Co-ordinate Axes.** The planes themselves are called the **Co-ordinate planes.** When the three Co-ordinate planes and consequently the three co-ordinate axes are mutually at right angles, the axes are said to be **Rectangular.** The three planes divides the space into eight parts known as the **Octants.**

Art. 2. Co-ordinates of a point in space : Let P be any point in space. Through P draw a plane PMAN parallel to yz plane and hence perpendicular to XOX′ cutting in the point A, then $OX = x$ is called the x co-ordinate of the point P. Again through P draw another plane PNBL parallel to zx plane and hence perpendicular to YOY′ cutting it in B then $ON = y$ is called the y-co-ordinate of the point P. Similarly $OC = z$ is the z-co-ordinate of the point P where C is the

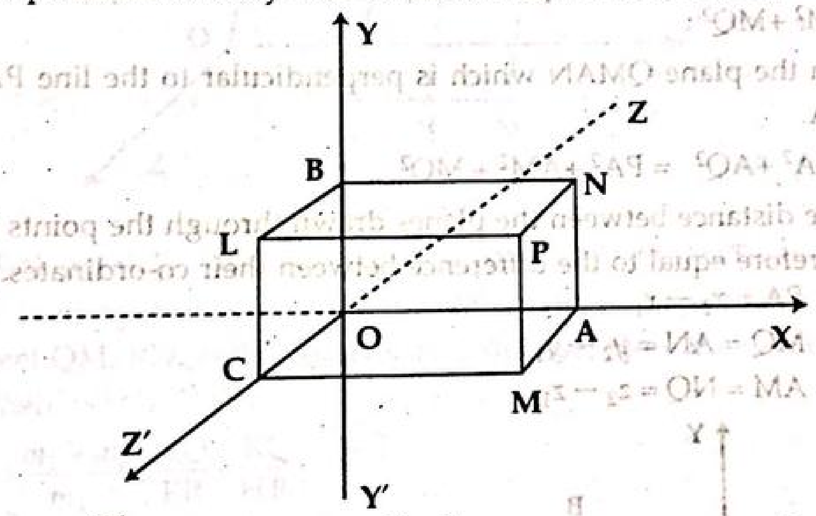

Fig : 1

point in which the plane PLCM through P parallel to xy plane cuts the z-axis ZOZ′.

The three numbers x, y, z are called the **co-ordinates of the point P and the point is represented as P(x, y, z).**

To determine the position of any point P it is not sufficient merely to know the absolute lengths of the lines LP, MP, NP we must also know the directions in which they are drawn. If lines drawn in one directions be considered as positive, those drawn in the opppsite direction must be considered negative. **We shall consider the direction as OX, OY, OZ are positive.**

The three co-ordinate planes divide the whole of the space into eight compartments which are called **octants**. The octant OXYZ in which the point P lies is called **first octant** and any point in this has each of its co-ordinates positive. Again since each of the co-ordinates of the point P may be either positive or negative we shall have **eight such point P(x, y, z)** whose co-ordinates may be represented numerically by the three numbers x, y and z but have different signs. Below we give the eight octants and the co-ordinates of the point P in each of them situated with respect to axes just P is situated in the Octant OXYZ and has co-ordinates (x, y, z)

1. OXYZ (x, y, z)
2. OX′YZ $(-x, y, z)$
3. OXY′Z $(x, -y, z)$
4. OXYZ′ $(x, y, -z)$
5. OXYZ′ $(x - y, - z)$
6. OX′Y′Z′ $(-x, y, -z)$
7. OX′Y′Z $(-x, -y, z)$
8. OX′Y′Z′ $(-x, -y, -z)$

OCT	OXYZ	OX′YZ	OXY′Z	OX′YZ′	OXY′Z′	OX′YZ′	OX′Y′Z	OX′Y′Z″
x	+	−	+	+	+	−	−	−
y	+	+	−	+	−	+	−	−
z	+	+	+	−	−	−	+	−

Art. 3. Distance between two points: To find the distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

Through the points P, Q draw planes parallel to the co-ordinate planes to form a rectangular parallelopiped whose one diagonal is PQ. Then APCM, NBLQ, LCPB, QMAN, BPAN, LCMQ are the three pairs of parallel faces of this parallelopiped. Now $\angle AMQ = 90°$. Therefore $AQ^2 = AM^2 + MQ^2$

Also Q lies in the plane QMAN which is perpendicular to the line PA. Therefore AQ is perpendicular to PA.

Hence $PQ^2 = PA^2 + AQ^2 \;= PA^2 + AM^2 + MQ^2$

Now PA is the distance between the planes drawn through the points P and Q parallel to YZ plane and is therefore equal to the difference between their co-ordinates.

$$PA = x_2 - x_1$$

Similarly $\quad MQ = AN = y_2 - y_1$

and $\quad AM = NQ = z_2 - z_1$

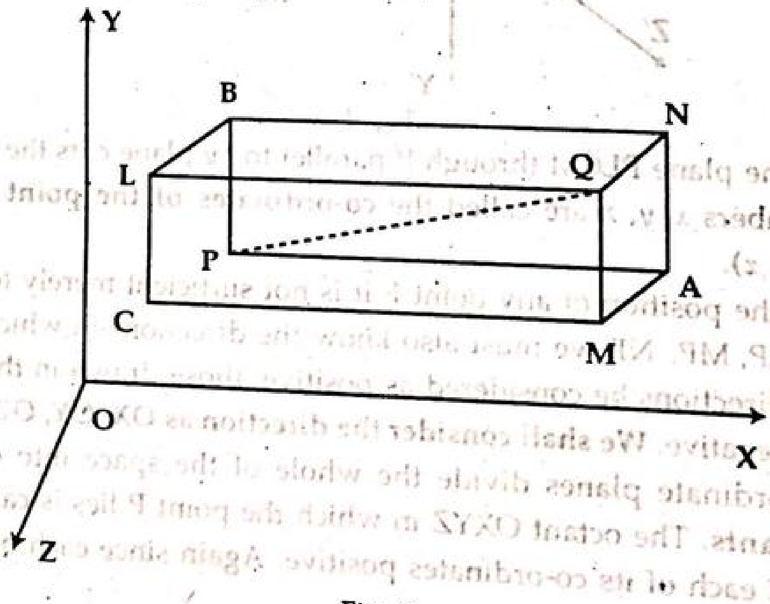

Fig : 2

Hence $BO^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

$\therefore$ The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) $PQ = \sqrt{\{(x_2 - x_2)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}}$ ... ... (1)

in magnitude.

Cor: Distance from the origin

when P coincides with the origin Q, we have $x_1 = y_1 = z_1 = 0$ so from (1), the distance OQ $= \sqrt{(x_2 + y_2{}^2 + z_2{}^2)}$ ... ... (2)

Art. 4. Section Ratio : To find the co-ordinates of the point which divides the straight line joining two given points in a given ratio.

Let the two given points be $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ and let $R(x, y, z)$ be a point in PQ, such that $PR : RQ = m_1 : m_2$

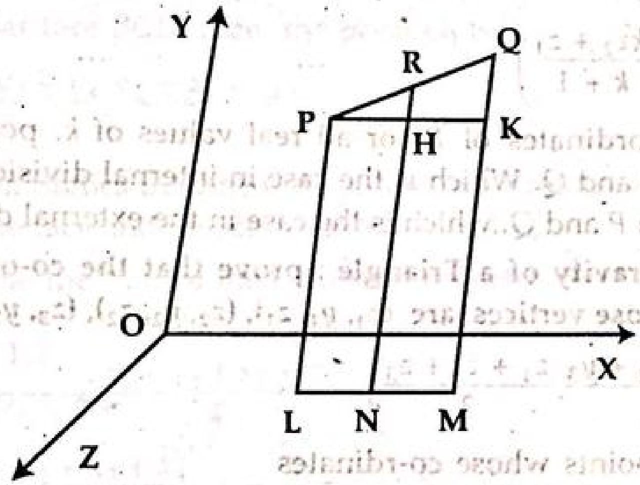

Fig : 3

Draw PL, QM, RN, paralell to OY meeting ZOX plane in L, M, N. Then the points P, Q, R, L, M, N are clearly all in one plane,. and a line through P parallel to LM will be in that plane, and will therefore, meet QM, RN, in the points K, H. (suppose) Now the two triangle ΔHPR and ΔKPQ, are similar. Then

$$\frac{m_2}{m_1} = \frac{RQ}{PR} \quad \text{or,} \quad \frac{m_1 + m_2}{m_1} = \frac{PQ}{PR} = \frac{KQ}{HR}$$

$$\frac{MQ - MK}{NR - NH} = \frac{MQ - LP}{NR - LP} = \frac{y_2 - y_1}{y - y_1} \quad \therefore y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$$

Similarly by drawing perpendiculars on XOY and YOZ and proceeding as above we can prove that

$$z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2} \text{ and } x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$$

Thus the co-ordinates of a point which divides the join of the point (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $m_1 : m_2$ are

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 - m_2}, \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}\right) \quad \cdots \cdots \quad (3)$$

Note : If $\frac{m_1}{m_1}$ is positive, then R divides the line PQ internally.

Cor. 1. If $\frac{m_1}{m_1}$ is negative, then R divides the line PQ externaly

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \frac{m_1z_2 - m_2z_1}{m_1 - m_2}\right) \ldots \quad \ldots \quad \ldots \quad (4)$$

Cor. 2. Co-ordinates of the Middle Point : If R bisects the line PQ then $m_1 = m_2$ and the co-ordinates of R are

$$\left\{\tfrac{1}{2}(x_1 + x_2), \tfrac{1}{2}(y_1 + y_2), \tfrac{1}{2}(z_1 + z_2)\right\} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Cor. 3. General co-ordinates of a point on the line PQ.

Put $\frac{m_1}{m_2} = k$ where k is called the **position** Ratio of R with regard to P and Q. The co-ordinates of R are

$$\left(\frac{kx_2 + x_1}{k + 1}, \frac{ky_2 + y_1}{k + 1}, \frac{kz_2 + z_1}{k + 1}\right) \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

These give the co-ordinates of R for all real values of k, positive or negative. If k is positive, R lies between P and Q. Which is the case in internal division : and if k is negative, R is on the same side of both P and Q, which is the case in the external division.

Art. 5. Centre of Gravity of a Triangle : prove that the co-ordinates of the centre of gravity of the triangle whose vertices, are $(z_1, y_1, z_1), (z_2, y_2, z_2), (z_3, y_3, z_3)$ are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

Let A, B, C be the points whose co-rdinates are $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ respectively.

Then D, the mid point of BC is

$$\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}, \frac{z_2 + z_3}{2}\right)$$

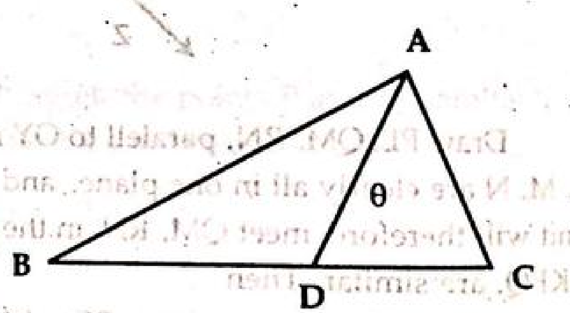

Fig : 4

Let (x, y, z) be the required co-ordinates of G, the centre of gravity of the ΔABC which divides AD in the ratio $2 : 1$.

Then $x = \dfrac{2.\frac{x_2 + x_3}{2} + 1.x_1}{2 + 1} = \dfrac{x_1 + x_2 + x_3}{3}$

similarly $y = \dfrac{y_1 + y_2 + a_3}{3}$ and $z = \dfrac{z_1 + z_2 + z_3}{3}$

Hence the co-ordinates of the centre of gravity of a ΔABC are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right) \quad (7)$$

Art. 6. Tetrahedron : ***Definition-*** *A tetrahedron is a solid bounded by four triangular faces. It has four vertices, each vertex arising as a point of intersection of three of the four faces (planes). It has six edges arising as the line of intersection of two of the four faces (edges, ${}^4C_2 = 6$).*

Thus $ABCD$ is a tetrahedron having for triangular faces, the Δ's ABC, ACD, ADB, BCD. The points A, B, C, D, are called the VERTICES of the tetrahedron. The edges $A\,B$ and CD; BC and AD; CA and BD which do not meet are called the pairs of opposite edges of the tetrahedron.

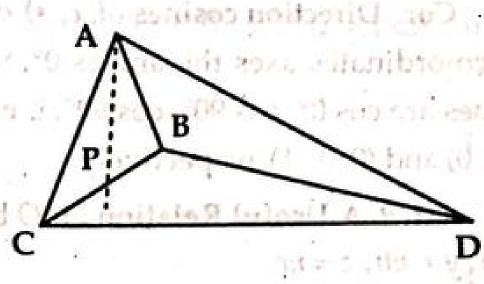

Fig : 5

Art. 7. To prove that the centre of gravity of a tetrahedron the co-ordinates of whose vertices are (x_r, y_r, z_r) where $r = 1, 2, 3, 4$ is the point.

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

Let A, B, C, D be the points (x_1, y_1, z_1); (x_2, y_2, z_2); (x_3, y_3, z_3) (x_4, y_4, z_4) If G_1 be the centre of gravity of the triangular face BCD, then the point G_1 is

$$\left(\frac{x_2 + x_3 + x_4}{3}, \frac{y_2 + y_3 + y_4}{3}, \frac{z_2 + z_3 + z_4}{3}\right)$$

Now we know from Statics that the C. G of a tetrahedron is on the line joining any vertex to the C. G of the opposite face and divides it in the ratio 3 : 1.

Hence if G (x, y, z) be the C. G of the tetrahedron; then

$$x = \frac{3.\frac{x_2 + x_3 + x_4}{3} + 1.x_1}{4} = \frac{x_1 + x_2 + x_3 + x_4}{4}$$

Similarly $y = \dfrac{x_1 + x_2 + x_3 + x_4}{4}$, $z = \dfrac{y_1 + y_2 + y_3 + y_4}{4}$ (8)

The C. G of the tetrahedron is therefore,

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

Art. 8. Direction Cosines of a Line.

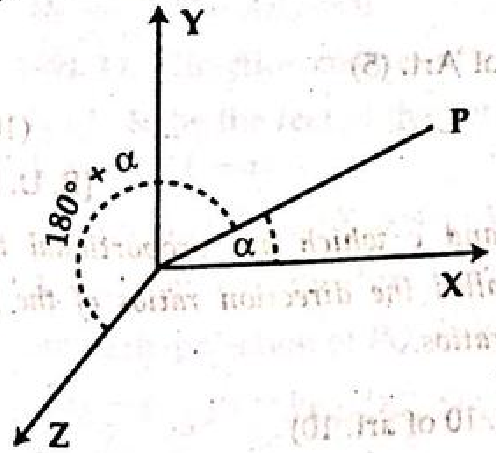

Fig : 6

If a given line OP makes angles α, β, γ, with the positive direction of axes of x, y and z respectively then cosα, cosβ, cosγ are the **Diretion Cosines** of the line OP and are generally denoted by the letters, l, m, n respectively. Again the direction cosines of PO will be cos (180 + α), cos (180 + β), cos (180 + γ) because the line through O, *i, e*, OP will now make angles 180 + α, 180 + β, 180 + γ with the axes.

The angles α, β, γ, of the line OP are called the **Direction Angle of OP.**

Note : *If a line does not pass through the origin O, then its direction angles are the angles between the axes and a line drawn through O parallel to the given line (and having the same direction.)*

Cor. Direction cosines (d, c, s) of the Axes of References : Now the axis of x makes with the co-ordinates axes the angles $0°, 90°, 90°$, respectively and hence by definition its direction cosines are $\cos 0°. \cos 90°, \cos 90°$. i, e, $(1, 0, 0)$. Similarly the d, c, s of the axes of y and z are $(0, 1, 0)$ and $(0, 0, 1)$ respectively.

Art. 9. A Useful Relation : If O be the orgin and (x, y, z) the co-ordinates of a point P, then $x = lr, y = mr, z = nr$

Where l, m, n, are direction cosines of OP and r, is the length of OP.

Through p draw PL perpendicular to x-axis so that $OL = x$. From the right angled triangle OPL, we have

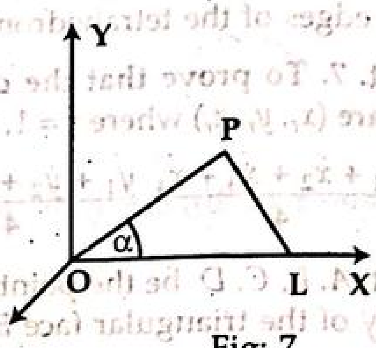

Fig: 7

$$\frac{OL}{OP} = \cos \angle LOP = \cos \alpha \text{ or, } \frac{x}{r} = l \text{ or, } x = lr$$

Similarly we have $y = mr, z = nr$,

Therefore the co-ordinates of P are $x = lr, y = mr, z = nr$ ... ... (9)

Note : $\frac{x}{l} = \frac{yi}{m} = \frac{z}{n} = r$ ... ... 9(a)

Art. 10. *Relation between direction cosines. To prove that if l, m, n, be the direction cosines of any line then $l^2 + m^2 + n^2 = 1$, i, e. the sum of the squares of the direction cosines of every line is one.*

Let OP be drawn through the origin parallel to the given line so that l, m, n, are the cosines of angles which makes with OX, OY, OZ respectively.

Let (x, y, z) be the co-ordinates of any point P on this line

Let $OP = r$

$\therefore x = lr, y = m, r, z = nr$ (By eq, 9 of Art. 9)

Squaring and adding, we obtain

$x^2 + y^2 + z^2 = (l^2 + m^2 + n^2) r^2$

or, $r^2 = (l^2 + m^2 + n^2) r^2$ (By eq (2) of Art. (5)

or, $l^2 + m^2 + n^2 = 1$

or, $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$ ... ... (10)

[R. U. 1980]

* **Art. 11. Direction Ratios :** *Any three numbers a, b, and c which are proportional to the direction cosines l, m, n, respectively of a given line are called the direction ratios of the given line (Rectangular axes). Now from definition of direction ratios.*

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{1}{\sqrt{(a^2 + b^2 + c^2)}} \quad \text{(By eq. 10 of art. 10)}$$

$$l = \pm \frac{a}{\sqrt{(a^2 + b^2 + c^2)}}, \; m = \pm \frac{b}{\sqrt{(a^2 + b^2 + c^2)}}, \; n = \pm \frac{c}{\sqrt{(a^2 + b^2 + c^2)}}$$

Where the same sign either + ve or – ve is to be chosen throughout.

Hence of (a, b, c), be three numbers proprtional to the d c's of OP, then the actual direction cosines of OP are

$$\frac{a}{\sqrt{(a^2+b^2+c^2)}}, \frac{b}{\sqrt{(a^2+b^2+c^2)}}, \frac{c}{\sqrt{(a^2+b^2+c^2)}} \quad \ldots \quad \ldots \quad (11)$$

and that of PO are

$$\frac{-a}{\sqrt{(a^2+b^2+c^2)}}, \frac{-b}{\sqrt{(a^2+b^2+c^2)}}, \frac{-c}{\sqrt{(a^2+b^2+c^2)}} \quad \ldots \quad \ldots \quad (12)$$

Art. 12. Projection of a point on a straight line.

The projection of a point A on a given line CD is the point B which is the foot of the perpendicular from A on CD.

This the point in which the plane through A and perpendicular to CD meets it.

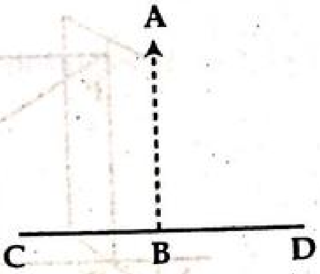

Fig : 8

Art. 13. The orthogonal projection of a given segment AB of a line on any line CD is $AB \cos\theta$ where θ is the angle between AB and CD.

Let the planes through A and B perpendicular to the line CD meet it in A', B' respectively so that A', B' is the projection of AB on CD.

Through A draw a line AP parallel to CD to meet the plane through B at P.

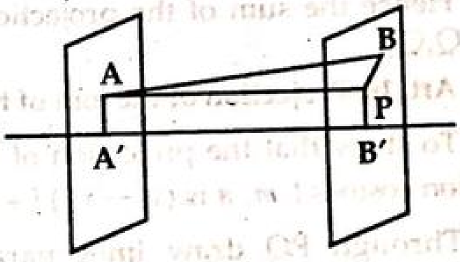

Fig : 9

Now AP is parallel to CD; $\angle PAB = \theta$

Also PB lies in the plane which is perpendicular to AP.

$$\angle APB = 90^\circ$$

$$\text{Hence } AP = AB\cos\theta,$$

Clearly $A'B'PA$ is a rectangle so that we have $AP = A'B'$

Hence $A'B' = AB\cos\theta$

Art. 14. Direction cosines of a line joining two point $P(x_1, y_1, z_1)$ **and** $Q(x_2, y_2, z_2)$

Let L, M be the feet of the perpendiculars drawn from P, Q. to the X-axis respectively so that $OL = x_1$, $OM = x_2$

Projection of PQ on X axis $= LM = OM - OL = x_2 - x_1$

Also if l, m, n be the direction cosines of PQ the projection of PQ on X-axis $= l\,PQ$.

Similarly pojection of PQ on Y- axis abd Z - axis, are $m\,PQ = y_2 - y_1$, $n PQ = z_2 - z_1$

$$\therefore \frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} = PQ \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

This the direction cosines of the line joining the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) are projectional to $\ldots \quad \ldots \quad \ldots \quad (14)$

$$x_2 - x_1 : y_2 - y_1 : z_2 - z_1$$

Art. 15. Projection of a broken line conising of continuos segments on a given line.

Let $P_1, P_2, P_3 \ldots \ldots P_n$ be any number of points in space and $Q_1, Q_2, Q_3 \ldots \ldots Q_n$ be their projections on the line AB, and these points being on the same straight line, we have

Q_1Q_n = Projection of $P_1P_n = Q_1Q_2 + Q_2Q_3 + Q_3Q_4 + \ldots + Q_n^{-1} Q_n$ Where Q_1Q_2 = Projection of P_1P_2

Q_2Q_3 = Projection of P_2P_3 and so on

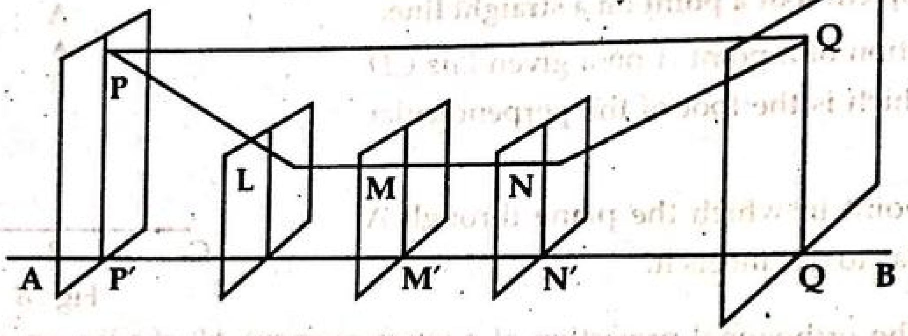

Fig : 10

Hence the sum of the projection of $P_1P_2, P_2P_3 \ldots P_n^{-1} P_n$ on AB = Projection of P_1P_n on AB = Q_1Q_n ... ... (15)

Art. 16. Projection of the join of two points on a line :

To show that the projection of the line joinig $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on a line with direction cosines l, m, n is $(x_2 - x_1) l + (y_2 - y_1) m + (z_2 - z_1)n$.

Through PQ draw lines parallel to the co-ordinates planes to form rectangular parallelopiped whose one diagonal is PQ (Fig. 3)

Now $PA = x_2 - x_1$, $AN = y_2 - y_1$, $NQ = z_2 - z_1$

The lines PA, AN, NQ are respectively parallel to x-axis, y-axis and z-axis. Therefore, their respective projections on the line with direction cosines l, m, n are

$(x_2 - x_1)\, l, \quad (y_2 - y_1)\, m, \quad (z_2 - z_1)\, n_1$

As the projection of PQ on any two lines is equal to the sum of the projection of PA. AN, NQ on that line therefore the required projection is $(x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n$ ...(16)

Art. 17. Angle between two lines : If (l_1, m_1, n_1) and (l_2, m_2, n_2) be the direction cosines of any two lines AB and CD and θ be the angle between them, then

$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

First Method : Let OP_1, OP_2 be lines through the origin parallel to the given lines so that the cosines of the angles which OP_1 and OP_2 make with the axes are l_1, m_1, n_1 and l_2, m_2, n_2 respectively and the angle between the line is the angle between OP_1 and OP_2. Let the angle be θ.

The projection of the line OP_2 joining O (0, 0, 0) and $P_2(x_2, y_2, z_2)$ on the line OP_1 whose direction cosines are, l_1, m_1, n_1 is $(x_2 - 0)\, l_1 + (y_2 - 0)\, m_1 + (z_2 - 0)n_1$

$s = l_1x_2 + m_1y_2 + n_1z_2$ ... ... (1)

But the projection of OP_2 on OP_1 is $OP_2 \cos\theta$

$\therefore OP_2 \cos\theta = l_1x_2 + m_1y_2 + n_1z_2$ ... ... (2)

But $x_2 = l_2\, OP_2$; $y_2 = m_2\, OP_2$, $z_2 = n_2\, OP_2$

Then from (ii), we have

$OP_2 \cos\theta = l_1l_2\, OP_2 + m_1m_2\, OP_2 + n_1n_2\, OP_2$

or, $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$ ... ... (17)

Second method : Let $OP_1 = r_1$, $OP_2 = r_2$, $P_1P_2 = d$ and the co-ordinates of P_1, P_2 be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

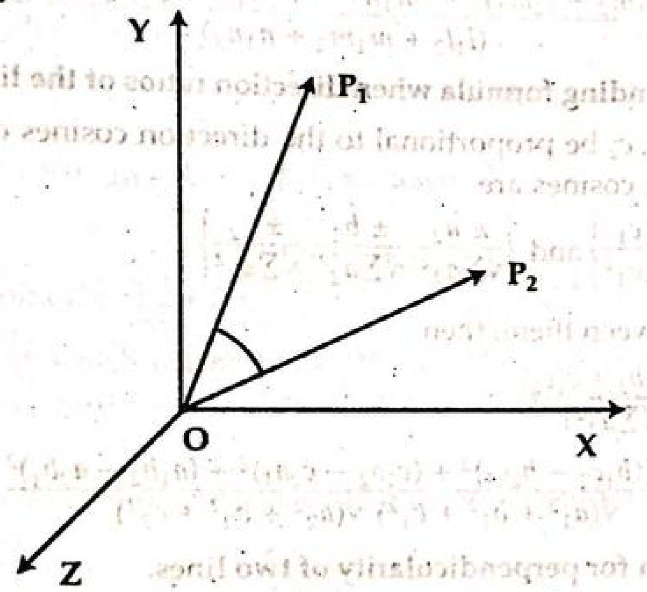

Fig : 11

Then $x_1 = r_1l_1$; $y_1 = r_1m_1$; $z_1 = r_1n_1$ ... ... (n)

and $x_2 = r_2l_2$; $y_2 = r_2m_2$; $z_2 = r_2n_2$

then by geometry, we have

$OP_1^2 + OP_2^2 - P_1P_2^2 = 2\, OP_1\, OP_2 \cos\theta$

or, $\cos\theta = \dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$

Now $r_1^2 = x_1^2 + y_1^2 + z_1^2$, $r_2^2 = x_2^2 + y_2^2 + z_2^2$

and $P_1P_2^2 = d_2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

$= r_1^2 + r_2^2 - 2r_1r_2(l_1l_2 + m_1m_2 + n_1n_2)$ by eq. (3)

$\therefore \cos\theta = \dfrac{r_1^2 + r_2^2 - r^2{}_1 - r_2^2 + 2r_1r_2\,(l_1l_2 + m_1m_2 + n_1n_2)}{2r_1r_2}$

or, $\cos\theta = l_1l_2 + m_1m + n_1n_2$ ... ... (18)

Art.17. (a) Expression for $\sin\theta$ and $\tan\theta$.

$\sin^2\theta = 1 - \cos^2\theta = 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2$

$= (l^2{}_1 + m_1^2 + n_1^2)\,(l^2{}_2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1\,n_2)^2$

$= (l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2$ ... ... (19)

by Lagranges Identity

$$\text{or, } \sin^2\theta = \begin{vmatrix} l_1 m_1{}^2 \\ l_2 m_2{}^2 \end{vmatrix} + \begin{vmatrix} m_1 n_1{}^2 \\ m_2 n_2{}^2 \end{vmatrix} + \begin{vmatrix} n_1 l_1{}^2 \\ n_2 l_2 \end{vmatrix} \quad \ldots \quad \ldots \quad (20)$$

$$\text{or, } \sin\theta = \pm\sqrt{\{(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2\}}$$

$$\text{Again } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \pm\frac{\sqrt{\{(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2\}}}{(l_1l_2 + m_1m_2 + n_1n_2)} \quad \ldots \quad \ldots \quad (21)$$

Art. 17. (b) Corresponding formula when direction ratios of the lines are given.

If a_1, b_1, c_1 and a_2, b_2, c_2 be proportional to the direction cosines of the lines OP_1 and OP_2, then their actual direction cosines are

$$\left\{\frac{\pm a_1}{\sqrt{\Sigma a_1{}^2}}, \frac{\pm b_1}{\sqrt{\Sigma a_1{}^2}}, \frac{\pm c_1}{\sqrt{\Sigma a_1{}^2}}\right\} \text{ and } \left\{\frac{\pm a_2}{\sqrt{\Sigma a_2{}^2}}, \frac{\pm b_2}{\sqrt{\Sigma a_2{}^2}}, \frac{\pm c_2}{\sqrt{\Sigma a_2{}^2}}\right\}$$

If θ be the angle between them, then

$$\cos\theta = \frac{\pm(a_1a_2 + b_1b_2 + c_1c_2)}{\sqrt{(\Sigma a_1{}^2)}\sqrt{(\Sigma a_2{}^2)}} \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

$$\text{and} \quad \sin\theta = \pm\frac{\sqrt{\{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}{\sqrt{(a_1{}^2 + b_1{}^2 + c_1{}^2)}\sqrt{(a_2{}^2 + b_2{}^2 + c_2{}^2)}} \quad \ldots \quad (23)$$

Art. 17. (c) Condition for perpendicularity of two lines.

If the two lines OP_1 and OP_2 are perpendicular to each other, then **$\cos\theta = 0$ as $\theta = 90°$ then** the relation becomes.

$$l_1l_2 + m_1m_2 + n_1n_2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

If we use direction ratios, then we get the required condition from eq (22)

$$a_1a_2 + b_1b_2 + c_1c_2 = 0 \quad \ldots \quad \ldots \quad (25)$$

Art. 17. (d) Condition for parallelism of two lines.

If the two lines OP_1 and OP_2 are parallel to each other, then sin **$\theta = 0$, as $\theta = 0°$**

Form eq (19) we have the condition

$$(l_1m_2 - l_2m_2)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 = 0$$

Hence we have, $l_1m_2 - l_2m_1 = 0$; $m_1n_2 - m_2n_1 = 0$ and $n_1m_2 - n_2m_1 = 0$

$$\text{or, } \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{\sqrt{(l_1{}^2 + m_1{}^2 + n_1{}^2)}}{\sqrt{(l_2{}^2 + m_2{}^2 + n_2{}^2)}} = 1$$

$$l_1 = l_2\,;\ m_1 = m_2\ ;\ n_1 = n_2 \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

i. e. The Direction Cosines of two lines are equal. From the eq (23). we have

$$(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2 = 0$$

$$b_1c_2 - b_2c_1 = 0;\ c_1a_2 - c_2a_1 = 0\ ;\ a_1b_2 - a_2b_1 = 0$$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

The direction ratios of the two lines are proportional.

EXAMPLES

Ex. 1. Find the distance between the points P_1 (4, 3, – 6) and P_2(– 2, 1, – 3)

$\therefore\ P_1P_2^2 = (4+2)^2 + (3-1)^2 + (-6+3) = 49$

$\therefore\ P_1P_2 = 7$ Ans.

Ex. 2. A,B, C are three points on the axes x, y, and z respectively at distance a, b, c from origin O, find the co-ordinates of the point which is equidistant form A, B, C and O.

Let P be the requred point (x, y, z) and the points A, B, C and O are $(a, 0, 0)$, $(0, b, o)$ (o, o, c) and (o, o, o) we are given that $PO = PA = PB = PC$

Taking $PO = PA$ or, $PO^2 = PA^2$, we have

$x^2 + y^2 + z^2 = (x-a)^2 + y^2 + z^2$

or, $2ax = a^2$ or $x = a/2$

Similarly taking $PO^2 = PB^2$ and $PO^2 = PC^2$ we have

$y = b/2$ and $z = c/2$

Hence the required point P is $\left(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c\right)$

Ex. 3. Find the ratio in which the yz plane divides the joint of the points (– 2, 4, 7) and (3, – 5, 8) and also find the co-ordiantes of the point of intersection of this line with the yz plane.

The co-ordiantes of any point on the line joining the two points are

$$\left(\frac{3\lambda-2}{\lambda+1}, \frac{-5\lambda+4}{\lambda+1}, \frac{8\lambda+7}{\lambda+1}\right)$$

If the points in the yz-plane, then its x co-ordinate should be zero.

$\frac{3\lambda-2}{\lambda-1} = 0$ or, $3\lambda - 2 = 0$ or, $\lambda = \frac{2}{3}$

Hence the required ratio is 2 : 3 and putting $\lambda = 2/3$ the required point is (0, 2/5, 37/15)

Ex 4. Find the dirction cosines of the line drawn from the origin to the point (– 6, 2, 3)

$$\cos\theta = \frac{x_2 - x_1}{\sqrt{\{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2\}}}$$

$$= \frac{-6-0}{\sqrt{\{(-6-0)^2 + (2-0)^2 + (3-0)^2}} = \frac{6}{7}$$

Similarly, $\cos\beta = 2/7$ and $\cos\gamma = 3/7$

The direction cosines are $-6/7, 2/7, 3/7$

Ex. 5. Find the direction cosines of the line which is equally inclined to the axes.

If the line makes angles α, β, γ with the axes then

$\cos\alpha = \cos\beta = \cos\gamma \therefore \alpha = \beta = \gamma$ (given)

or $\frac{l}{1} = \frac{m}{1} = \frac{n}{1} = \pm\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(1+1+1)}} = \pm\frac{1}{\sqrt{3}}$

$\therefore$ d, c, s are $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{3}, \pm\frac{1}{\sqrt{3}}\right)$

Ex. 6. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube, prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$ [C.U. Hons. 1984, 1985]

Let the $d.c,s$ of the line be l, m, n. The eight vertices can taken as $O(o, o, o)$, $P(a, a, a)$ $A(a, o, o)$, $B(o, a, o)$, $C(o, o, a)$, $L(o, a, a)$, $M(a, o, a)$, $N(a, a, o)$

AL, BM, CN and OP are the diagonals of the cube Fig. (1) Art. (2). the d, c's of the four diagonals are

$(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

$(1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$, $(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$

The direction cosines of the line which is inclined at angles $\alpha, \beta, \gamma, \delta$ respectively to the four diagonals are

$$\cos\alpha = \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} = \frac{l+m+n}{\sqrt{3}}$$

Similarly $\cos\beta = \dfrac{-l+m+n}{\sqrt{3}}$, $\cos\gamma = \dfrac{l-m+n}{\sqrt{3}}$, $\cos\delta = \dfrac{l+m-n}{\sqrt{3}}$

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{1}{3} \cdot 4(l^2 + m^2 + n^2) = \frac{4}{3}$ proved.

Ex. 7. If a line makes angles α, β, γ with the axes, show that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

since $l^2 + m^2 + n^2 = 1$

or, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

or, $1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$

or, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$ Proved.

Ex. 8 Find the condition so that the three concurrent lines with diresction cosines (l_1, m_1, n_1), (l_2, m_2, n_2), (l_2, m_3, n_3) are co-planar

Let (l, m, n) be the direction cosines of a line which is normal to the plane that contains the two lines whose $d\ c$'s are (l_1, m_1, n_1) and (l_2, m_2, n_2)

$ll_1 + mm_1 + nn_1 = 0$ ... ... (1)

$ll_2 + mm_2 + nn_2 = 0$ ... ... (2)

Normal to a plane is perpendicular to every line in that plane. Again if the line whose $d\ c$'s (l_3, m_3, n_3) also lies in plane, then the line with d, c's (l, m, n) is also perpendicular to it.

$\therefore\ ll_3 + mm_3 + nn_3 = 0$ ... ... (3)

Eliminate l, m, n from (1), (2), (3), then we have

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$$

which is the required condition.

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Ex. 9. If the edges of a rectangular parallelopiped are a, b, c show that the angles between the four diagonals are given by $\cos^{-1}\left(\dfrac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$ [C.U. 1984]

Take one of the vertices O of the parallelopied as origin and three rectangulr faces through it as the three rectangular co-ordinate planes. See fig (1). Art 2.

Let $OA = a$, $OB = b$, $OC = c$,

The lines OP, AL, BM, CN are the four diagonals.

The co-ordinates of A, B, C are $(a, 0, 0), (0, b, 0), (0, 0, c)$

The co-ordinate of L, M, N are $(0, b, c), (a, 0, c), (a, b, 0)$

The co-ordinates of O, P are $(0, 0, 0); (a, b, c)$

Direction cosines of OP are $\dfrac{a}{\sqrt{\Sigma a^2}}, \dfrac{b}{\sqrt{\Sigma a^2}}, \dfrac{c}{\sqrt{\Sigma a^2}}$

Direction cosines of AL are $\dfrac{-a}{\sqrt{\Sigma a^2}}, \dfrac{b}{\sqrt{\Sigma a^2}}, \dfrac{c}{\sqrt{\Sigma a^2}}$

Direction cosines of BM are $\dfrac{a}{\sqrt{\Sigma a^2}}, \dfrac{-b}{\sqrt{\Sigma a^2}}, \dfrac{c}{\sqrt{\Sigma a^2}}$

Direction cosines of CN are $\dfrac{a}{\sqrt{\Sigma a^2}}, \dfrac{b}{\sqrt{\Sigma a^2}}, \dfrac{-c}{\sqrt{\Sigma a^2}}$

The angle between OP and Al, therefore. is

$$\cos\theta = \frac{a(-a)+b(b)+c(c)}{\sqrt{\Sigma a^2}\sqrt{\Sigma a^2}} = \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}\right)$$

Similarly the angle between any one of the six pairs of diagonals can be found

Hence the angles in general between the six pairs are given by

$$\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$$

Ex. 10. If $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ are the $d.c,s$ of three mutually perpendicular lines whose direction cosines are proportional to $l_1+l_2+l_3$; $m_1+m_2+m_3$, $n_1+n_2+n_3$ make equal angles with them.

we have

$$l_1l_2 + m_1m_2 + n_1n_2 = 0, l_2l_3 + m_2m_3 + n_2n_3 = 0, \quad \ldots \quad (1)$$

$$l_3l_1 + m_1m_3 + n_3n_1 = 0 \quad \ldots \quad (2)$$

Also, $l_1^2+m_1^2+n_1^2 = 1, l_2^2+m_2^2+n_2^2 = 1, l_3^2+m_3^2+n_3^2 = 1$

Therefore, the actual direction cosines of the lines are

$$\left(\frac{l_1+l_2+l_3}{\sqrt{3}}, \frac{m_1+m_2+m_3}{\sqrt{3}}, \frac{n_1+n_2+n_3}{\sqrt{3}}\right)$$

N. B. $\sqrt{[(l_1+l_2+l_3)^2+(m_1+m_2+m_3)^2+(n_1+n_2+n_3)^2]} = \sqrt{3}$

If the angle between the first line and the other line θ then

$$\cos\theta = [l_1(l_1 + l_2 + l_3) + m_1(m_1 + m_2 + m_3) + n_1(n_1 + n_2 + n_3)]/\sqrt{3}$$
$$= 1/\sqrt{3} \therefore \theta = \cos^{-1} 1/\sqrt{3}$$

Similarly, we can show that the angle between each of the other two line and the line with $d.c$,s given by (3) is also $\cos^{-1} 1/\sqrt{3}$

Hence proved

Ex. 11. The direction cosines of a moving line in two adjacent positions are l, m, n and $l + \delta l$, $m + \delta m$, $n + \delta n$. Show that the small angle $\delta\theta$ between the positions is given by $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$

Now $\cos\delta\theta = l(l + \delta l) + m(m + \delta n) + n(n + \delta n)$

$\therefore 1 - \cos\delta\theta = 1 - (l^2 + m^2 + n^2) - (l\delta l + m\delta m + \delta n)$

or, $2\sin^2\left(\frac{1}{2}\delta\theta\right) = -(l\delta l + m\delta m + n\delta n)$

Expand it and take only the first term as $\delta\theta$ is very small.

$$2\left(\frac{1}{2}\delta\theta\right)^2 = -(l\delta l + m\delta m + n\delta n)$$

or, $(\delta\theta)^2 = -2(l\delta l + m\delta m + n\delta n)$ ... ... (1)

Again $l^2 + m^2 + n^2 = 1$ ... ... (2)

and $(l + \delta l)^2 + (m + \delta m)^2 + (n + n\delta)^2 = 1$ ... ... (3)

Subtracting (2) from (3), we have

$(\delta l)^2 + 2l(\delta l) + (\delta m)^2 + 2m(\delta m) + (\delta n)^2 + 2n(\delta n) = 0$

or, $(\delta l)^2 + (\delta m)^2 + (\delta n)^2 = -2(l\delta l + m\delta m + n\delta n)$

Hence from (1), we have $(\delta\theta)^2 + (\delta l)^2 + (\delta m)^2 + (\delta n)^2$ proved.

Ex. 12. Find the angle between two lines whose direction cosines are given by the equations

$l + m + n = 0$ ... ... (1)

and $l^2 + m^2 - n^2 = 0$ ... ... (2)

[D.U. Hons. 1962]

Eliminate n from (1) and (2) then

$l^2 + m^2 - (l + m)^2 = 0$ or $2lm = 0$ ... ... (3)

Either $l = 0$, i.e. $1.l + om + on = 0$, Also $l + m + n = 0$

$\frac{l}{o} = \frac{m}{+1} = \frac{n}{-1}$ ... ... (4)

If $n = 0$ in (3)

$\therefore ol + lm + on = 0$ and $l + m + n = 0$, then

Therefore, the $d.c$'s are proportional to, 0, -1, 1 ; 1, 0, -1. If θ be the angle between them, then

$$\cos\theta = \pm\frac{0(1) + (-1)(0) + (1)(-1)}{\sqrt{[0^2 + (-1)^2 + 1]}\sqrt{[(1)^2 + (0)^2 + (-1)^2]}} = \pm\frac{1}{2}$$

$$\theta = \frac{1}{3}\pi, \text{ or, } \frac{2}{3}\pi$$

Ex. 13 **Find the distance of A (1, − 2, 3) from the line PO through P(2, −3, 5), which makes equal angles with the axes.**

The direction cosines of PQ are $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$

If p be the required perpendicular distance, then

$$p^2 = \{(-3+2)/\sqrt{3} - (5-3)/\sqrt{3}\}^2 + \{5-3)/\sqrt{3} - (2-1)/\sqrt{3}\}^2 + \{(2-1)/\sqrt{3}) - (3+2)/\sqrt{3}\}^2$$

$= 1/3, 14$ or $p = \sqrt{(14/3)}$ (See Art. 33).

EXERCISE—1

1. Find the distance between the origin and point given.

(a) (3, 4, − 5) Ans. $5\sqrt{2}$

(b) (− 4, 0, −2) Ans. $2\sqrt{5}$

2. Find the distance between each of the following pairs of points.

(a) (5, − 2, 3), (− 4, 3, 7) Ans. $\sqrt{(122)}$

(b) (0, 3, 0), (6, 0, 2) Ans. 7

3. Find the co-ordinates of the point which divides the line joining the points (2, − 4, 3), (−4, 5, − 6) in the ratio 1 : − 4 Ans. (4, −7, 6)

4. Find the ratio in which the line joining the points (2, 4, 5) (3, 5, − 4) is divided by the YZ plane. Ans. − 2 : 3, (0, 2, 23)

5. If P and Q are (2, 3, − 6) and (3, − 4, 5) respectively and O be the origin, find the direction cosines of OP, OQ, PQ. [R.U. 1980]

Ans. $2/7, 3/7, -6/7; 3/5\sqrt{2}, -4/5\sqrt{2}, 1/\sqrt{2}, (-1, 7, -11)\sqrt{171}$

6. Find the angle between the lines whose direction ratios are (3, 1, 2) and (2, − 2, 4). Ans. $\cos^{-1}\sqrt{3/7}$

7. Find the direction cosines of the positive z-axis. Ans. 0, 0, 1

8. Find the cosine of the angle between two directed lines having the dirction cosines

(a) $4/9, -8/9, -1/9; -1/3, -2/3, -2/3.$ Ans. $\cos^{-1}\sqrt{14/27}$

(b) $1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}; 1/2, \sqrt{(6/4)}, -1/2$ Ans. $\cos^{-1} 1/4$

9. Show that the lines whole direction cosines are propotional to 2, 1, 1; 4, $\sqrt{3}-1, -\sqrt{3}-1$; 4, $-\sqrt{3}-1, \sqrt{3}-1$, are inclined to the another at angle $\pi/3$.

10. If P, Q are $(x_1, y_1, z_1), (x_2, y_2, z_2)$, the projection of PQ on a line whose direction cosines are $\cos\alpha, \cos\beta, \cos\gamma$ is gven by $(x_2 - x_1)\cos\alpha + (y_2 - y_1)\cos\beta + (z_2 - z_1)\cos\gamma$

11. Find the distance of (−1, 2, 5) from the line through (3, 4, 5) whose direction cosines are proportional to 2, − 3, 6. Ans: $4\sqrt{(61/7)}$

12. (a) Find the distance of $(-2, 3, 4)$ from the line through the point, $(-1, 3, 2)$ whose direction cosines are proportional to $12, 3, -4$. Ans. $\sqrt{445}/13$ [D.U. 1966]

12. (b) The direction cosines l, m, n of two lines are connected by the relations $l + m + n = 0$, $2lm + 2ln - mn = 0$, Find them.

Ans. $1/\sqrt{6}, l/l\sqrt{6}, -2/\sqrt{6}; 1/6, -2/\sqrt{6}, 1/\sqrt{6}$.

13. The direction cosines of two lines are determined by the relations

(a) $l - 5m + 3n = 0, 7l^2 + 5m^2 - 3n^2 = 0$ Ans. $1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14}; -1/\sqrt{6}, 1/\sqrt{6}, 2/\sqrt{6}$

(b) $l + m - n = 0, nm + 6\,ln - 12lm = 0$ Ans. $1/\sqrt{26}, 2/\sqrt{26}, 3/\sqrt{26}; 1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14}$

14. If A, B, C, D, be the four points $(3, 6, 9)$, $(1, 2, 3)$, $(2, 3, 1)$ and $(4, 6, 2)$ respectively, prove that the line joining the points A and B intersects the line joining other two points. Find the co-ordinates of the point and the ratios in which this point of intersection divides the two lines.

Ans. $-3:1$ and $-1:2$

15. Prove by the method of projection that if A, B, C, D be four points whose co-ordinates are $(6, -6, 0)$, $(-1, -7, 6)$, $(3, -4, 4)$ and $(2, -9, 2)$ respectively, then AB is perpendicular to CD.

16. Show that $(4, 3, -1)$, $(-1, 4, 7)$, $(\frac{3}{2}, \frac{7}{7}, 3)$ are three collinear points.

17. Show that A $(3, 6, 9)$, B $(1, 2, 3)$; C $(2, 3, 1)$, D $(4, 6, 2)$ are four points such that AB and CD intersect.

18. Show that the lines from A $(5, 2, -3)$ to B $(6, 1, 4)$ and from C $(-3, -2, -1)$ to D $(-1, -4, 13)$ are parallel.

19. Show that the lines AB and BC are perpendicular to each other, where A $(-11, 8, 4)$, B $(-1, -7, -1)$, C $(9, -2, 4)$

20. Find the direction cosines of the line which is perpendicular to the lines with direction cosines proportional to $(1, -2, -2)$, $(0, 2, 1)$. Ans. $2/3, -1/3, 2/3$.

21. If $l_1, m_1, n_1, l_2, m_2, n_2$, are the direction cosines of two mutually perpendicular lines, show that the cosines of the line perpendicular to them both are $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$

22. Prove that the straight lines whose direction cosines are given by the relations $al + bm + cn = 0$ and $fmn + gnl + hlm = 0$ are perpendicular if $f/a + g/b + h/c = 0$ and parallel if $\sqrt{(af)} \pm \sqrt{(bg)} \pm \sqrt{(ch)} = 0$

[C. U. 1986]

23. Prove that two lines whose direction are cosines are conneceted by the two relations $al - bm + cn = 0$ and $ul^2 + vm^2 + wn^2 = 0$ are perpendicular if

$a^2(v + w) + b^2(w + u) + c^2(u + v) = 0$ and parallel if $\frac{a^2}{u} + \frac{b^2}{u} + \frac{c^2}{w}$ [C. U. 1976, 1980]

24. Prove that the angle between two diagonals of a cube is $\cos^{-1}\cdot\frac{1}{3}$

25. Show that the equation to the right circular cone whose vertex is at the origin whose axis has direction cosines $\cos\alpha, \cos\beta, \cos\gamma$ and whose semi-vertical angle θ. is

$$(y\cos\gamma - z\cos\beta)^2 + (z\cos\alpha - x\cos\gamma)^2 + (x\cos\beta - y\cos\alpha)^2 = (x^2+y^2+z^2)\sin^2\theta$$

26. $l_1, m_1, z_1 ; l_2, m_2, n_2$ are the direction cosines of two concurrent lines: show that the direction cosines of two lines bisecting the angle between them are proportional to $(l_1 \pm l_2, m_1 \pm m_2, n_1 \pm n_2)$ [D.U. 1984]

27. A, B, C, D are four points in space such that AB is perpendicular to CD and AC is perpendicular to BD; prove that AD is perpendicular to BC [R.U. 1962, '64]

28. If $l_1, m_1, n_1, l_2, m_2, n_2$, are two directions cosines of two lines inclned at an angle θ to each other, show that the line with directions cosines.

$\frac{l_1+l_2}{2\cos\theta/2}, \frac{m_1+m_2}{2\cos\theta/2}, \frac{n_1+n_2}{\cos\theta/2}$ bisect the angle between these two lines.

29. Find the point in which the join of $(-9, 4, 5)$ and $(11, 0, -1)$ is met by the perpendicular from the origin, Ans. $(1, 2, 2)$

30. If P and Q are the points $(2, -3, 4)$ and $(-1, 2, 3)$ respectively find the direction cosines of OP, OQ and PQ where O is the point $(0, 0, 0)$. [D. U. 1979]

Ans. $(2/\sqrt{29}, -3/\sqrt{29}, 4/\sqrt{29}, (-1/\sqrt{14}, 2/14, 3/\sqrt{14}), (-3/\sqrt{35}, 5/\sqrt{35}, -1/\sqrt{35})$

31. If the point P and Q are given by $(2, 3, 4)$ and $(1, 1, -1)$ respectively find the angle between OP and OQ. Ans. $1/7\sqrt{3}$. [D. U. 1980]

32. Prove that the joining the points. $(2, 3, -2)$ and $(3, 1, 1)$ is parallel to the line joining the points $(2, 1, -5)$ and $(4, -3, 1)$. [D. U. 1980]

33. The distances of the points $(a_1, a_2, a_3), (a_2, b_2, c_2)$ from the origin K_1 and K_2 respectively. If the straight line passesage through the origin, then show that $a_1a_2 + b_1b_2 + c_1c_2 = k_1k_2$

2

THE PLANE

Art. 18. (a) Definition of a palne : A plane is a surface such that if any two points are taken on it, the straight line joining them lies wholly and the surface i, e, every point of the line joining the two point will be on the plane.

Art. 18. (b) : To prove that the general equation of the first degree in x, y, z, i.e. $ax + by + cz + d = 0$ represents a plane.

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the co-ordinates of any two points lying on the locus whose equation is

$$ax + by + cz + d = 0 \quad \dots \quad \dots \quad \dots \quad (1)$$

$$\therefore \ ax_1 + by_1 + cz_1 + d = 0 \quad \dots \quad \dots \quad \dots \quad (2)$$

and $ax_2 + by_2 + cz_2 + d = 0 \quad \dots \quad \dots \quad \dots \quad (3)$

Multiplying (3) by k and adding to (2), we have

$$a(x_1 + kx_2) + b(y_1 + ky_2) + c(z_1 + kz_2) + d(l + k) = 0$$

Or, $\dfrac{a(x_1 + kx_2)}{l + k} + \dfrac{b(y_1 + ky_1)}{l + k} + \dfrac{c(z_1 + kz_2)}{l + k} + d = 0 \quad \dots \quad \dots \quad (4)$

Result (4) shows that equation (1) is satisfied by the point whose co-ordinates are $\left(\dfrac{x_1 + kx_2}{l + k}, \dfrac{y_1 + ky_2}{l + k}, \dfrac{z_1 + kz_2}{l + k}\right)$

But from equation (6). Art. 4, it represents any point on the line joining (x_1, y_1, z_1), (x_2, y_2, z_2); and giving different value to k. We get all the points on the line. Therefore the line completely lies on the locus; in other words, the locus is a plane.

Conclusion : Hence every equation of the first degree in x, y, z represents a plane.

Cor. 1. General equation of a given plane that passes through a given point.

The general equation of a plane is $ax + by + cz + d = 0 \quad \dots \quad \dots \quad (5)$

Since it passes through the point (x_1, y_1, z_1), we have

$$ax_1 + by_1 + cz_1 + d = 0 \quad \dots \quad \dots \quad \dots \quad (6)$$

Subtracting (6) from (5), we have

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad \dots \quad \dots \quad \dots \quad (28)$$

Cor. 2. General equation of a plane through the origin.

In case the plane passes through origin (0, 0, 0) the general equation of the plane $ax + by + cz + d = 0$ will be satisfied by (0, 0, 0) so that $d = 0$. Hence the equation reduces to the form.

$$ax + by + cz = 0 \quad \dots \quad \dots \quad \dots \quad (29)$$

Cor. 3. The equation of a plane passing through the two points (x_1, y_1, z_1) **and** (x_2, y_2, z_2) **is**

$$L = (x - x_1)(y - y_2) - (x - x_2)(y - y_1) + R\{(x - x_1)(z - z_2) - (x - x_2)(z - z_2)\} = 0 \quad \text{... 29(a)}$$

It may easily be verified that 29 (a) is the equation of a plane passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) because this equation is satisfied by the co-ordinates for all values of R and further in 29 (a) the terms xy cancel out, so that 29(a) is an equation of the first degree in x, y and z and as such it is of a plane.

The equation may also be written as

$$(x - x_1)(y - y_2) - (x - x_2)(y - y_1) = A\{(x - x_1)(z - z_2) - (x - x_2)(z - z_1)\} \quad \text{... ... 29(b)}$$

Art. 19. Plane through three given points. To find the equation of the plane passing through the three non-collinear points.

(x_1, y_1, z_1); (x_2, y_2, z_2), (x_3, y_3, z_3)

Let the equation to the plane be

$$ax + by + cz + d = 0 \quad \text{... ... (1)}$$

As the given points lie on the plane (1), we have

$$ax_1 + by_1 + cz_1 + d = 0 \quad \text{... ... (2)}$$

$$ax_2 + by_2 + cz_2 + d = 0 \quad \text{... ... (3)}$$

$$ax_3 + by_3 + cz_3 + d = 0 \quad \text{... ... (4)}$$

Eliminating a, b, c, d from (1), (2), (3) and (4) we have

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \text{ or, } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{(30)}$$

which is the required equation of the plane.

Art. 20. Standards forms of the equation of a plane.

(a) Intercept form : To find the equation to the plane in terms of the intercepts, which it makes on the axes.

Let the intercepts made by the plane on the axes be a, b, c. Then we can regard it as a plane passing through the points (a, o, o); (o, b, o); (o, o, c). Then by Art: 19 its equation is

$$\begin{vmatrix} x & y & z & 1 \\ a & o & o & 1 \\ o & b & o & 1 \\ o & o & c & 1 \end{vmatrix}$$

$$\text{or, } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{... ... ... (31)}$$

which is the required equation.

(b) Normal form : To find the equation of a plane in terms of p , the length of the perpendicular from orgin on the plane and the direction cosines l, m, n of this perpendicular.

Let p be the length of the perpendicualar ON from the origin on the plane. and let l, m, n be the direction cosines of perpendicualar. Let p be any point on the plane, and draw PL perpendicular on ZOX. and LM perpendicular to OX.

Then the projection of OP on ON is equal to the sum of the projection of OM, ML and LP on ON.

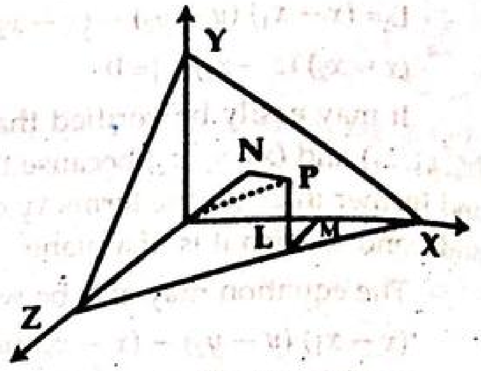

Fig. 12.

Hence if P be (x, y, z), we have

$lx + my + nz = p$ ... ... 31(a)

which is the required equation.

Note: p is always positive.

Art. 21. Reduction of the general equation of the plane

$Ax + By + Cz + D = 0$ to

(i) Intercepet form (ii) Normal form

(i) Intercept form : Transfer D to the right-hand side and divide throughout by-D so as to make R. H. S. equal to one.

$\therefore \dfrac{A}{-Dx} + \dfrac{B}{-Dy} + \dfrac{C}{-Dz} = 1$

or, $\dfrac{x}{-D/A} + \dfrac{y}{-D/B} + \dfrac{z}{-D/C} = 1$

put $-D/A = a, -D/B = b, -D/C = c$

Then $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ which is intercept form of the equation. .

(ii) Normal Form : To transform the equation

$ax + by + cz + d = 0$ ... ... (1)

to the normal form for $lx + my + nz = p$ ... ... (2)

As these two equations are of the same form on comparison we have. $\dfrac{-d}{p} = \dfrac{a}{l} = \dfrac{b}{m} = \dfrac{c}{n}$

or $l = \dfrac{-ap}{d}, m\dfrac{-bp}{d}, n = \dfrac{-cp}{d}$ ... ... (3)

Since l, m, n are the direction consines of the normal to the plane we have. $l^2 + m^2 + n^2 = 1$

or $\dfrac{a^2 + b^2 + c^2}{d^2} p^2 = l$ by ... ... (4)

Now we always keep p positive. Hence if d be negative, then from (i) we must choose $p = \dfrac{d}{\sqrt{(a^2 + b^2 + c^2)}}$ and hence from (3)

$$l = +\frac{a}{\sqrt{(a^2 + b^2 + c^2)}} ; m = +\frac{b}{\sqrt{(a^2 + b^2 + c^2)}} ; n = +\frac{-c}{\sqrt{(a^2 + b^2 + c^2)}}$$

Putting the values of l, m, n, and p in $lx + my + nz = p$ we have

$$\frac{a}{\sqrt{(a^2 + b^2 + c^2)}x} + \frac{b}{\sqrt{(a^2 + b^2 + c^2)}y} + \frac{c}{\sqrt{(a^2 + b^2 + c^2)}z}$$

$$= -\frac{d}{\sqrt{(a^2 + b^2 + c^2)}} \text{ if } d \text{ is negative}$$

Again if d is positive, then from (iv) we must choose

$$p = \frac{d}{\sqrt{(a^2 + b^2 + c^2)}} \text{ and hence from (3) } l = \frac{-a}{\sqrt{(a^2 + b^2 + c^2)}}$$

$$m = \frac{-b}{\sqrt{(a^2 + b^2 + c^2)}n}; = \frac{c}{\sqrt{(a^2 + b^2 + c^2)}}$$

Put the values of l, m, n, and p in (ii), then

$$\frac{-a}{\sqrt{(a^2 + b^2 + c^2)}x} - \frac{b}{\sqrt{(a^2 + b^2 + c)}y} - \frac{c}{\sqrt{(a^2 + b^2 + c^2)}z} = \frac{d}{\sqrt{(a^2 + b^2 \pm c^2)}} \text{ if } d \text{ is positive}$$

Thus the normal form of the equation $ax + by + cz + d = 0$ is

$$\left.\begin{aligned} -\frac{a}{\sqrt{\Sigma a^2}}x - \frac{b}{\Sigma a^2}y - \frac{c}{\sqrt{\Sigma a^2}}z = \frac{d}{\sqrt{\Sigma a^2}} \text{ if } d \text{ is positive} \\ \frac{a}{\sqrt{\Sigma a^2}}x + \frac{b}{\sqrt{\Sigma a^2}}y + \frac{c}{\sqrt{\Sigma a^2}}z = \frac{d}{\Sigma a^2}; \text{ f } d \text{ is negative} \end{aligned}\right\} \quad \dots \quad \dots \quad \dots \quad (32)$$

Note : Since $\frac{a}{\sqrt{\Sigma a^2}}, \frac{b}{\sqrt{\Sigma a^2}}, \frac{c}{\sqrt{\Sigma a^2}}$ are actual direction cosines of the normal to the plane $ax + by + cz + d = 0$, therefore a, b, c, are proportional to the $d.c$'s of the normal to the plane.

Art. 22. (a) : Angle between two planes : *Angle between two planes is equal to the angle between the normals to them from any point. Thus the angle between the two planes.*

$$ax + by + cz + d = 0 \quad \dots \quad \dots \quad \dots \quad (1)$$

and $a_1x + b_1y + c_1z + d = 0 \quad \dots \quad \dots \quad \dots \quad (2)$

is equal to the angle between the lines with diretion ratios a, b, c and a_1, b_1, c_1. Hence if θ be the angle between the normals, then

$$\cos\theta = \frac{aa_1 + bb_1 + cc_1}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(a_1^2 + b_1^2 + c_1^2)}}$$

Art. 22. (b) : Condition of perpendicularity of two planes.

If two planes are perpendicualar to each other, so are their normals whose direction cosines are porportional to a_1, b_1, c_1, and a_2, b_2, c_2, Hence the condition of perpendicualarity is

$$a_1a_2 + b_1b_2 + c_1c_2 = 0 \quad \dots \quad (33)$$

Art. 22. (c) : Condition of parallelism of two planes.

Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel if and only if the co-efficients of x, y, z in their equations are proportional. i,e, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \quad \dots \quad \dots \quad \dots \quad (34)$$

Art. 23. Some system of planes.

(i) Plane parallel to a given plane and passing through a given point.

The equation of a plane parallel to a given plane $ax + by + cz + d = 0$ is of the form

$$ax + by + cz + k = 0 \quad \dots \dots \dots (1) \quad \dots \quad \dots \quad \dots \quad (35)$$

where k is an arbitrary constant. If it passes through a point (x_1, y_1, z_1), then (1) is satisfied.

$ax_1 + by_1 + cz_1 + k = 0$ ... ... (2)

From (1) and (2), we have

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ ... ... 35(a)

Which is the required equation of the plane,

(ii) Plane perpendicular to a given plane.

Let the equation of the given plane be

$ax + by + cz + d = 0$ ... ... (1)

and that of the required plane is

$a_1x + b_1y + c_1z + d_1 = 0$ ... ... (2)

Since (1) and (2) are perpendicular to cach other, therefore,

$aa_1 + bb_1 + cc_1 = 0$ ... ... (3)

If the plane (2) passes through a point (x_1, y_1, z_1) then

$a_1x_1 + b_1y_1 + c_1z_1 + d_1 = 0$ ... ... (4)

From (2), (3) and (4) the plane cannot be determined uniquelly since one arbitrary constant remains. Infact we have infinite number of planes by rotating (2) about the normal to (1) which passes through (1)

In order to remove this constant we require another given condition. Let the plane (2) passes through another given point (x_2, y_2, z_2) then eq. (2) in satisfied.

$a_1x_2 + b_1y_2 + c_1z_2 + d_1 = 0$ ... ... (5)

Now eliminant of a_1, b_1, c_1, d_1, from (2), (3), (4) and (5), is the required equation.

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ a & b & c & 1 \end{vmatrix} = 0 \quad \ldots \quad \ldots \quad (36)$$

or

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ a & b & c \end{vmatrix} = 0 \quad (37)$$

Cor. 4. Planes perpendicular to co-ordinates planes.

Now yz plane is $x = 0$ or, $x + oy + oz = 0$ ... (1)

Let the equation to the plane perpendicular to yz plane be

$ax + by + cz + d = 0$ ... ... (2)

The planes (1) and (2) are perpendicular

$\therefore\ 1.a + 0.b + 0.c = 0$ or $a = 0$

Hence the equation of the plane perpendicular to yz plane is

$by + cz + d = 0$

Similarly the equation of the planes perpendicular to zx and xy planes are $ax + cz + d = 0$ and $ax + by + d = 0$ respectively

Rule : Hence we can say that plane $ax + by + cz + d = 0$ represents planes perpendicular to yz, zx, xy planes respectively according as $a = 0$, or $b = 0$, or, $c = 0$ or according as the term of x or of y or of z is missing in its equation.

(iii) Plane Through the line of intersection of given planes.

The equation

$(ax + by + cz + d) + k(a_1x + b_1y + c_1z + d) = 0$ ... ... (1)

represents a system of planes passing through the line of intersection of the planes

$ax + by + cz + d = 0$ ... ... (2)

$a_1x + b_1y + c_1z + d_1 = 0$ ... ... (3)

k being a parameter for the equation is of the first degree in x, y, z and represents a plane : The value of k is obtained if the plane (1) is parallel or perpendicular to given plane.

Art. 24. length of the perpendicular from a point to a plane.

To find the perpendicular distance of the point $P\ (x_1, y_1, z_1)$ from the plane $lx + my + nz = P$ ... ... (1)

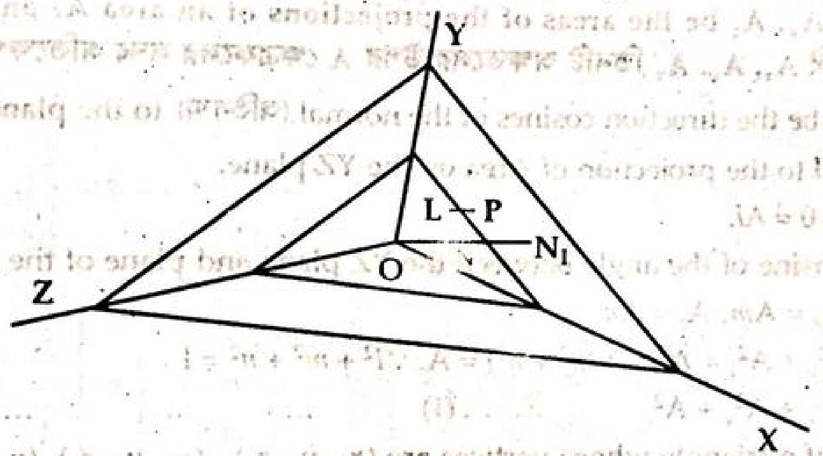

Fig : 13

When P stands for the length of the perpendicular from the origin. $P = ON$.

Let the given point be (x_1, y_1, z_1)

Equation of the plane parallel to (1) is

$lx + my + nz = p_1$ ... ... (39)

and since it passes through (x_1, y_1, z_1) we have

$p_1 = lx_1 + my_1 + nz_1$ ... ... (40)

where $p_1 = ON_1$ (say) is the length of the perpendicular from origin to the plane (2) i.e the plane through p and parallel to the given plane.

Now of $PL = D$ be drawn perpendicular to the given plane from the point P, then

$D = LP = ON_1 - ON = p_1 - p = lx_1 + my_1 + nz_1 - p$ from (3)

Therefore the length of the perpendicular from a given point $P(x_1, y_1, z_1)$ is $D = lx_1 + my_1 + nz_1 - p$ ... ... (41)

Art. 25. To find the length of the perpendicular from (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ ... ... (1)

The normal form of the equation of the plane according as d is $-ve$ or d is $+ve$ is

$$\pm\frac{ax}{\sqrt{(\Sigma a^2)}} \pm \frac{by}{\sqrt{(\Sigma a^2)}} \pm \frac{c_z}{\sqrt{(\Sigma a)^2}} \pm \frac{d}{\sqrt{(\Sigma a)^2}} = 0$$

Hence if D be the length of the perpendicular distance of (x_1, y_1, z_1) then by Art. 25 (a)

$$D = \pm \frac{ax_1 + by_1 + c^2{}_1 + d}{\sqrt{(a^2 + b^2 + c^2)}} \quad \ldots \quad \ldots \quad \ldots \quad (42)$$

Rule : Thus the length of the perpendicular from (x_1, y_1, z_1) to the plane is obtained by substituting (x_1, y_1, z_1) for x, y, z respectively in the expression, $ax + by + cz + d$

and dividing the same by $\sqrt{(a^2 + b^2 + c^2)}$

Art. 26. Projection on a plane : The projection of an area A enclosed by the curve PQR ...... on a plane is the area.

A_1 enclosed by the curve $P_1Q_1R_1$ ...... where P_1, Q_1, R_1 ...... are the feet of the perpendiculars from PQR ...... on the plane of projection. We shall assume here the result of Pure Solid Geometry that the area $A_1 = A \cos\theta$ where θ is the angle between the plane of the area A and the plane of projection.

(a) If A_x, A_y, A_z be the areas of the projections of an area A, on the three co-ordinate planes, then (যদি A_x, A_y, A_z তিনটি অক্ষতলের উপর A ক্ষেত্রফলের লম্ব অভিক্ষেপ ক্ষেত্রফল হয় তা হলে)

Let l, m, n be the direction cosines of the normal (অভিলম্ব) to the plane of the area A.

A_x is equal to the projection of Area on the YZ plane.

$\therefore A_x = \cos\theta = Al.$

l is the consine of the angle between the YZ plane and plane of the area A, therefore

$A_x = Al, A_y = Am, A_z = An$

$\therefore A^2{}_x + A^2{}_y + A^2{}_z = A^2(l^2 + m^2 + n^2) = A \therefore l^2 + m^2 + n^2 = 1$

or, $A^2 = A^2{}_x + A^2{}_y + A^2{}_z \quad \ldots \ldots (1) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (43)$

(b) Area of a triangle whose vertices are $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$

$$A_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_2 & z_2 & 1 \end{vmatrix} ; \quad A_y = \frac{1}{2}\begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}$$

$$A_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Hence A, the area of a triangle is easily obtained by (1).

$A^2 = A^2{}_x + A^2{}_y + A^2{}_z$.

Art. 27. Volume of a tetrahedron (চতুস্তলকের আয়তন) *–To find the volume of a tetrahedron whose four vertices are given.*

Let A, B, C, D be the vertices whose co-ordinates respectively are $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)$

If V be the volume of the tetrahedron $ABCD$.

then $V = \frac{1}{2} p\,\Delta$

$\ldots \quad \ldots \quad \ldots \quad (1)$

where p is the length of the perpendicular (লম্বের দৈর্ঘ্য) from A to the opposite face (বিপরীত তল) BCD and Δ be the area of triangle BCD.

Therefore, $\Delta = \sqrt{(\Delta x^2 + \Delta y^2 + \Delta z^2)}$ ... ... (2)

Where $\Delta_x = \dfrac{1}{2}\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}$; $\Delta_y = \dfrac{1}{2}\begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix}$

$\Delta_z = \dfrac{1}{2}\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$ ... ... (3)

(3) The equation of the plane BCD is

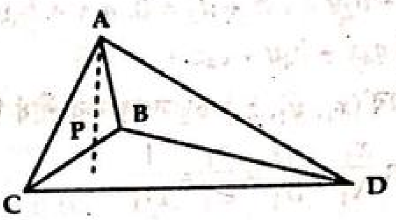

Fig : 14

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

or, $x\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} - y\begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix} + z\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} - \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} = 0$

The length of the perpendicular (লম্বের দৈর্ঘ্য) from A (x_1, y_1, z_1) on the plane (4) is

$2x_1\,\Delta_x - 2y_1\,\Delta_y + 2z_1\,\Delta_z - \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}$ ... ... (4)

$p = \sqrt{\{4(\Delta_x^2 + \Delta_y^2 + \Delta_z^2)\}}$

$P = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \div 2\Delta$ By (2) ... ... (5)

$\therefore \Delta = \dfrac{1}{3}P\,\Delta = \dfrac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$ by (1) ... ... (6)

(6) Find the volume of the tetrahedron when its four faces are given.

Let the four plane faces of the tetrahedron be given by

(চতুস্তলকের চারটি তল দেওয়া থাকলে এর আয়তন নির্ণয়)

$a_r x + b_r y + c_r z + d_r = 0 \quad (r = 1, 2, 3, 4)$

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

be the determinant of the above set of 4 equations.

D_1, D_2, D_3 etc. are the co-factors of d_1, d_2, d_3, etc. respectively in the determinant Δ.

মনে করি (x_1, y_1, z_1) তিনটি তলের ছেদ বিন্দু

$a_2x + b_2y + c_2z + d_2 = 0,\ a_3x + b_3y + c_3z + d_3 = 0$

and $a_4x + b_4y + c_4z + d_4 = 0$

অতএব (x_1, y_1, z_1) চতুস্তলকের শীর্ষ বিন্দু

Then $\dfrac{x_1}{A_1} = \dfrac{y_1}{B_1} = \dfrac{z_1}{C_1} = \dfrac{1}{D_1}$

or, $x_1 = \dfrac{A_1}{D_1},\ y_1 = \dfrac{B_1}{D_1},\ z_1 = \dfrac{C_1}{D_1}$

মনে করি, $(x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)$ অন্য তিন শীর্ষ বিন্দু একইভাবে পাওয়া যায়।

চতুস্তলকের শীর্ষ বিন্দুর পরিপ্রেক্ষিতে এর আয়তন

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} \frac{A_1}{D_1} & \frac{B_1}{D_1} & \frac{C_1}{D_1} & 1 \\ \frac{A_2}{D_2} & \frac{B_2}{D_2} & \frac{C_2}{D_2} & 1 \\ \frac{A_3}{D_3} & \frac{B_3}{D_3} & \frac{C_3}{D_3} & 1 \\ \frac{A_4}{D_4} & \frac{B_4}{D_4} & \frac{G_4}{D_4} & 1 \end{vmatrix}$$

$$= \frac{1}{6D_1D_2D_3D_4}\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}$$

$$\therefore V = \frac{\Delta_1}{6D_1D_2D_3D_4} = \frac{\Delta_3}{6D_1D_2D_3D_4}$$

Note : **If you take the product of Δ and Δ_1 then $\Delta\Delta_1 = \Delta^4$ or $\Delta_1 = \Delta^3$ For detail consult, Author's Higher Algebra.**

EXAMPLES

Ex. 1. Find the equation of the plane through the points (2, 3, 1), (1, 1, 3) and (2, 2, 3). Find also the perpendicular distance from the point (5, 6, 7) to this plane.

First Method : The required plane is

$$\begin{vmatrix} x & y & z & 1 \\ 2 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 2 & 2 & 3 & 1 \end{vmatrix}$$

or, $2x-2y-z+3=0$

2nd Method :

The equation of any plane through (2, 3, 1) is

$a(x-2)+b(y-3)+c(z-1)=0$ ... ... (1)

Since it passes through (1, 1, 3) and (2, 2, 3) we have

$a(1-2)+b(1-3)+c(3-1)=0$ or, $a+2b-2c=0$ ... ... (2)

and $a(2-2)+b(2-3)+c(3-1)=0$ or, $b-2c=0$ ... ... (3)

Solving (2) and (3) for a, b, c by corss multiplication, we have $\dfrac{a}{-2}=\dfrac{b}{2}=\dfrac{c}{1}$.

Putting these value in (1) we have

$-2(x-2)+2(y-3)+1(z-1)=0$ or, $2x-2y-z+3=0$.

The perpendicular distance P is from (5, 6, 7) to the (4) is

$$P=\frac{2.5-2.6-7+3}{\sqrt{(4+4+1)}}=2$$

3rd Method.

See Art. 18. (a) cor. 3

The equation of plane passing through (2, 3, 1) and (1, 1, 3) is by Art. 18 (a) cor. 3, eq. 29 (b)

$(x-2)(y-1)-(x-1)(y-3)=A\{(x-2)(z-3)-(x-1)(z-1)\}$.

Since if passes through (2, 2, 3), hence

$0-(2-1)(2-3)=A\{0(2-1)-(3-1)\}$ or, $A=-1/2$

Now putting the value of A, we have

$$(x-2)(y-1)(x-1)(y-3)=\frac{1}{2}\{(x-2)(z-3)-(x-1)(z-1)\}$$

or $2x-2y-z+3=0$

Ex. 2. Show that the four points (0, − 1, − 1), (4, 5, 1), (3, 9, 4) and (− 4, 4, 4) lie on a plane.

The four points are complanar if the determinant

$$\begin{vmatrix} 0 & -1 & -1 & 1 \\ 4 & 5 & 1 & 1 \\ 3 & 9 & 4 & 1 \\ -4 & 4 & 4 & 1 \end{vmatrix} \text{ is zero}$$

Otherwise we shall find the equation of the plane passing through any three points say $(0, -1, -1)$, $(4, 5, 1)$ and $(3, 9, 4)$ and then show that the fourth point $(-4, 4, 4)$ satisfied the plane.

Equations of any plane through $(0, -1, -1)$, $(4, 5, 1)$ and $(3, 9, 4)$ is

$5x - 7y + 11z + 4 = 0$ by example (1)

The point $(-4, 4, 4)$ clearly satisfies this plane. Hence the four points are coplanar.

N.B : For finding out the equation to the plane use, Cor. 3

Ex. 3. Find the equation of the plane passing through the intersection of the planes $x + 2y + 3z + 4 = 0$ and $4x + 3y + 2z + 1 = 0$ and the point (1, 2, 3)

Any plane through the intersection of the two planes is

$$x + 2y + 3z + 4 + k(4x + 3y + 2z + 1) = 0 \quad \ldots\ldots (1)$$

Since it passes through (1, 2, 3),

$18 + 17k = 0 \quad$ or $k = -18/17$

Putting the value of k in (1) we have

$$x + 2y + 3z + 4 - \left(\frac{18}{17}\right)(4x + 3y + 2z + 1) = 0$$

or, $55x + 20y - 15z - 50 = 0$ or, $11x + 4y - 3z = 10$

which is the required plane.

Ex. 4. Find the angle between the planes $2x - y + z = 6$ and $x + y + 2z = 7$

Let θ be the angle between the planes then

$$\cos\theta = \frac{2.1 + (-1).1 + 1.2}{\sqrt{\{2^2 + (-1^2) + 1^2\}}\ \sqrt{(1^2 + 1^2 + 2^2)}} = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{1}{2} \quad \therefore \theta = \frac{\pi}{3}$$

Ex. 5. Find the equation of the plane through the point (4, — 2, 1) and parallal to the plane whose direction numbers are 7, 2, —3.

The equation to the plane through $(4, -2, 1)$ is $a(x - 4) + b(y + 2) + c(z - 1) = 0$. The condition that the co-efficients a, b, c are proportional to the direction numbers $7, 2, -3$, therefore, the equation of the required plane is $7k(x - 4) + 2k(y + 2) - 3k(z - 1) = 0$ where k is constant.

or, $7(x - 4) + 2(y + 3) - 3(z - 1) = 0$ or, $7x + 2y - 3z - 2l = 0$

Ex. 6. Find the equation of the plane through the point (4, 0, 1) and parallel to the plane $4x + 3y - 12z + 6 = 0$

The equations of the parallel planes differ by a constant only. Hence the equation of the plane parallel to $2x + 3y - 12z + 6 = 0$ is $4x + 4y - 12x + k = 0$ Since it passes through (4, 0, 1).

$16 + 0 - 12 + k = 0.$ or, $k = -4$

The required plane is $4x - 3y - 12z - 4 = 0$

Ex. 7. Find the equation of the plane passing through the lines of intersection of the planes $2x - y = 0$ and $3z - y = 0$ and perpendicular to the plane $4x + 5y - 3z + 1 = 0$

The plane $2x - y + k(3z - y) = 0$

i.e. $2x - (1 + k)\,y + 3kz = 0$

passes through the intersection of the two given planes whatever be the value of k.

It will be perpendicular to $4x + 5y - 3z + 1 = 0$

$\therefore\ 2.4 - (1 + k)5 + 3k(-3) = 0$ i.e. $14k = 3$ or, $k = 3/14$

Thus the required plane is $2x - y + 3(3z - y)/14 = 0$

or, $28x - 17y + 9z = 0$

Ex. 8. Find the equation of the plane through the pts. (2, 2, 1) and (9, 3, 6) and perpendicular to the plane $2x + 6y + 6z = 9$

[C.H. 1980; C.U. 1981; R.U. 1962, '78; D.H. 1976] D.U.(P) 1986]

Any plane through (2, 2, 1) is

$a(x-2) + b(y-2) + c(z-1) = 0$

Since it passes through (9, 3, 6) then (1) becomes

$a(9-2) + b(3-2) + c(6-1) = 0$ or, $7a + b + 5c = 0$

The plane (1) is perpendicular to $2x + 6x + 6z - 9 = 0$

Hence $2a + 6b + 6c = 0$

From (2) and (3) by cross multiplication.

$$\frac{a}{-24} = \frac{b}{-32} = \frac{c}{40} \text{ or, } \frac{a}{3} = \frac{b}{4} = \frac{c}{-5} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Putting the values of a, b, c, from (4) in (1), the equation of the required plane is $3(x-2) + 4(y-2) - 5(z-1) = 0$ or, $3x + 4y - 5z = 9$

Alternate Method : The required plane is

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 9-2 & 3-2 & 6-1 \\ 2 & 6 & 6 \end{vmatrix} = 0$$

or, $3x + 4y - 5x = 9$ *by* Art. (11)

Ex. 9. Show that the equation to the plane through P (2, 3, − 1) at right angles to OP is $2x + 3y - z = 14$

The direction cosines of the line OP are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}$

Any plane at right angles to OP can be written as

$$\frac{2}{\sqrt{14}}x + \frac{3}{\sqrt{14}}y - \frac{1}{\sqrt{14}}z = p$$

Where $p = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{14})$

The required plane is $2x + 3y - z = 14$

Ex. 10. A plane meets the Co-ordinate axes in A, B, C such that the centroid of the triangle ABC is the point (p, q, r); show that the equation of the plane is [C.U. 1982]

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$$

Let the equation of the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ... (1)

$\therefore$ A is (a, o, o) B is (o, b, o) and C is (o, o, c)

Centriod of the triangle ABC is

$$G\left(\frac{a+o+o}{3}, \frac{o+b+o}{3}, \frac{o+o+c}{3}\right) \text{ i.e. } \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$$

But we are given that the centroid is the point (p, q, r)

$$\therefore p = \frac{1}{3}a, q = \frac{1}{3}b, r = \frac{1}{3}c$$

$$a = 3q, \ b = 3q, \ c = 3r$$

Therefore eq. (1) is $\frac{x}{3p} + \frac{y}{3q} + \frac{z}{3r} = 1$ or $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$

Ex. 11. Find the distance of the point (2, 0, 1) and (3, — 3, 2) from the plane $x - 2y + z = 6$ and find whether the two points lie on the same side or opposite sides of the plane.

The equation of the plane is $x - 2y + z - 6 = 0$ ... ... (1)

Its distance from (2, 0, 1) is $\frac{2 - 2.0 + 1 - 6}{\sqrt{(1^2 + 2^2 + 1^2)}} = \frac{-3}{\sqrt{6}}$

Its distance from (3, — 3, 2) is $\frac{3 - (2. - 3) + 2 - 6}{\sqrt{(1^2 + 2^2 + 1^2)}} = \frac{5}{\sqrt{6}}$

The distance $3/\sqrt{6}$ and $5/\sqrt{6}$

Put the point (2, 0, 1) in the L.H.S of eq. (1) this gives — 3, i.e negative value.

Put point (3, — 3, 2) in this L.H.S in eq. (1), this gives 5, i.e positive value.

The two results are of opposite signs. Therefore the two points lie on opposite sides of the plane.

Ex. 12. A variable plane is at a constant distance p from the origin and meets the axis in A, B, C. Through A, B, C, planes are drawn parallel to the co-ordinate planes. Show that the locus of their point of intersection is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$

Let the equation of the variable plane be $lx + my + nz = p$.

Here l, m, n vary but p is constant as the plane is always at constant distance from the origin, l, m, n are the actual direction cosines of the plane.

It cuts the axis of x at $A(p/l. \ 0)$ axis of y at $B(0, p/m, 0)$ and axis of z at $C(/0, 0, p/n)$

Now the planes through A, B, C are parallel to the co-ordinate planes, then the co-ordinates of point of intersection $(p/l, p/m, p/n)$.

$\therefore x = p/l, y = p/m, z = p/n$

Therefore, $p^2/x^2 + p^2/y^2 + p^2/z^2 = l^2 + m^2 + n^2 = 1$

or, $x^{-2} + y^{-2} + z^{-2} = p^{-2}$

Ex. 13. A plane makes intercepts OA = a, OB = b and OC = c, on the axes of co-ordinates. Find the area of the trangle ABC made by the plane.

A_x = area of the plane triangle formed by the three points which are the projections of the vertices on the yz plane.

Clearly the points B and C are in the yz plane and the projection of the point $A(a, o, o)$ is (o, o, o)

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A_x = Area of ΔOBC which is right angled at O $A_x = \frac{1}{2} bc$

Similarly $A_y = \frac{1}{2} ca$ and $A_z = ab$ $\therefore$ $A = \frac{1}{2}\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}$

Ex. 14. Find the volume of the tetrahedron formed by planes whose equations are $my + nz = 0$, $nz + lx = 0$, $xl + my = 0$, [D.U. 1979]

$lx + my + nz = p$.

$$\text{Here } \Delta = \begin{vmatrix} o & m & n & o \\ l & o & n & o \\ l & m & o & o \\ l & m & n & -p \end{vmatrix} = -2lmnp$$

Also $D_1 = lmn$, $D_2 = lmn$, $D_3 = lmn$, $D_4 = -2lmm$

Hence the volume $= \dfrac{2p^3}{3lmn}$

Ex. 15. Find the equation of the plane through the line $\frac{x-2}{3} = \frac{y-3}{5} = \frac{z}{7}$ and passing through the point $(1, -2, 3)$

By Cor. 3 Art. 18(a)

The general equation of the plane containing the line (given) is

$5(x-2) - 3(y-3) = k\{7(x-2) - 3z\}$ ... ... (1)

[It may be seen that this equation is obtained by spliting up the equation into two equations $\frac{x-2}{3} = \frac{y-3}{5}$ and $\frac{x-2}{3} = \frac{z}{7}$ then multiplying crosswise and connecting the result by the variable constant k for the general case the value of k to be fixed by the additional condition]

Since this plane passes through $(1, -2, 3)$, we have

$5(-1) - 3(-5) = k\{7(-1) - 9\}$ or $k = -5/8$

Putting this value of k in (1)

$8(5x - 3y - 1) = -5(7x - 3z - 14)$ or, $25x - 8y - 5z - 26 = 0$

EXERCISE II

1. Find the general equations of planes perpendicular to

(1) x-axis (2) y-axis (3) xoy plane (4) xoz plane.

Ans. (1) $x = p$; (2) $y = p$; (3) $lx + my = p$; (4) $lx + nz = p$

2. Find the equation of the plane through the points.

(a) $(2, 1, 3)$, $(-1, -2, 4)$, $(4, 2, 1)$ Ans. $5x - 4y + 3z - 15 = 0$

(b) $(1, 1, -1)$ $(-2, -2, 2)$, $(-1, -1, 2)$ Ans. $x - 3y - 2z = 0$

(c) $(2, 1, -3)$, $(3, -1, 4)$ $(7, 5, 6)$ Ans. $23z - 13y - 7x - 54 = 0$

3. Show that the four points are coplanar

(a) $(0, 4, 3), (-1, -5, -3), (-2, -2, 1), (1, 1, -1)$

(b) $(-6, 3, 2), (3, -2, 4), (5, 7, 3)$ and $(-13, 17, -1)$

4. Show that the line of intersection of the first two planes is co-planar with the line of intersection of the latter two and find the equation of the plane containing two lines.

(a) $x + 2y - z - 3 = 0, 3x - y + 2z - 1 = 0,$

$2x - 2y + 3z - 2 = 0, x - y + z + 1 = 0$ Ans. $7x - 7y + 8z + 3 = 0$

(b) $7x - 4y + 7z + 16 = 0, 4x + 3y - 2z + 3 = 0$

$x - 3y + 4z + 6 = 0, x - y + z + 1 = 0$ Ans. $3x - 7y - 9z + 13 = 0$

5. Find the smallest angle between the planes

(a) $3x + 2y - 5z - 4 = 0, 2x - 3y + 5z - 8 = 0$ Ans. $\cos\theta = 25/38$

(b) $3x - 4y + 5z = 0, 2x - y - 2z = 0$ Ans. $\pi/2$

(c) $2x + 2y - 7 = 0, x + 2y - z + 16 = 0$ Ans. $30°$

6. Find direction cosines of the normals to the planes and length of the normal.

(a) $x + 2y + 2z - 1 = 0$ Ans. $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, p = \frac{1}{3}$

(b) $9x + 6y - 2z + 7 = 0$ Ans. $-\frac{9}{11}, -\frac{6}{11}, \frac{2}{11}, p = \frac{7}{11}$.

7. Find the intercepts made by the plane $2x + y - 2z = 3$ on the co-ordinate axes.

What ar the d.c.,s of the plane. Ans. $\frac{3}{2}, 3, -\frac{3}{2}$; d.c.s, $(\frac{2}{3}, \frac{1}{3}, -\frac{3}{2})$

8. A plane contains the pt. $(1, -1, -2)$. Its normal has d.r's $3, -2, -1$. Find the equation of the plane. Ans. $3x - 2y - z = 7$

9. Find the equations to the planes

(a) Through the intersections of the planes $x - 2y + 3z + 4 = 0$ and $2x - 3y + 4z - 7 = 0$ and the point $(1, -1, 1)$ Ans. $9x - 13y + 17z - 39 = 0$

(b) Through $(1, 1, 2)$ and perpendicular to each of the planes $2x - 2y - 4z - 6 = 0$ and $3x + y + 6z - 4 = 0$ Ans. $x + 3y - z - 2 = 0$

(c) through $(1, 2, 2)$ and parallel to $3x + 2y + z = 7$ Ans. $3x + 2y + z = 0$

(d) through $(-1, 3, 2)$ and perpendicular to the two planes $x + 3y + 2z = 5, 3x + 3y + 2z = 8$ Ans. $2x + 4y + 3z + 8 = 0$

10. Find the equation of the plane which is parallel to the plane

(a) $4x - 4y + 7z - 3 = 0$ and distance 4 units from the point $(3, 1, -2)$.

Ans. $4x - 4y + 7z + 38 = 0, 4x - 4y + 7z - 34 = 0$

(b) $2x - 3y - 6z - 14 = 0$ and distant 5 from the origin. Ans. $2x - 3y - 6z \pm 35 = 0$

11. Find the equation of the plane which passes through the poins $(1, 0, -1)$ and $(2, 1, 3)$ and is perpendicular to the plane $2x + y + z = 1$

Ans. $3x - 7y + z = 2$

[C. U; D. U. 1977]

(a) Find the equation of the plane perpendicular to each of the $x - 4y + z = 0$ and $3x + 4y + z - 2 = 0$ and at a distance unity from the origin. Ans. $-4x + y + 8z + 9 = 0$ [D.U. 1982]

12. Plane is drawn through the line of intersection of the planes $x + 2y + 2z = 1$, $x + y - z + 1 = 0$ and is at distances. 1 (one) from the point $(4, -2, 1)$. Prove that there are two such planes and fine their equations. Ans. $3x + 4y + 1 = 0$, $3x + 2y - 6z + 5 = 0$ [R.U. 63]

13. Find the equation of the plane which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$, and $2x + y - z + 5 = 0$

Ans. $51x + 15y - 50z + 173 = 0$ [C.U. '80; C.H. '80]

14. Find the equation of plane that passes through $(2, -3, 1)$ and is normal to the line joining the points $(3, 4, -1)$ and $(2, -1, 5)$ Ans. $x + 5y - 6z + 19 = 0$ [D.U. '82; C.U. '78].

15. Find the equation of the plane through the intersection of the planes $x + 2y + 3z + 4 = 0$ and $4x + 3y + 2z + 1 = 0$ and perpendicular to the plane $x + y + z + 9 = 0$ and show that it is perpendicular to xz plane. Ans. $x - z = 2$, the Co-efficient of y is zero.

16. Prove that the points $(1, -1, 3)$ and $(3, 3, 3)$ are equidistant from the plane $5x + 2y - 7z + 9 = 0$ and on opposite side of it.

17. Find the distance between the plane $x - 2y + 2z = 3$ and the point $(1, 2, -1)$. Ans. 8/3.

18. Show that distance between the parallel planes

(a) $2x - 2y + z + 3 = 0$ and $4x - 4y + 2z + 5 = 0$ is $\frac{1}{6}$

(b) $2x - 3y - 6z - 14 = 0$ and $2x - 3y - 6z + 7 = 0$ is 3

19. Find the direction cosines of the perpendicular drawn from the origin to the plane $2x + 3y - z + 1 = 0$ [D.U. 1977] Ans. $(-2/\sqrt{14}, -3/\sqrt{14}, 1/\sqrt{14})$

20. Find the distance between parallel planes $x - y - 2z - 3 = 0$, $2x - 2y + 4z - 12 = 0$ Ans. $\sqrt{6}/2$

21. Find the equation of the planes through $(0, 4, -3)$, $(6, -4, 3)$ which cuts off from the axes intercepts whose sum is zero Ans. $6x + 3y - 2z = 18$, $2x - 3y - 6z = 6$

22. Show that the equations $by + cz + d = 0$, $cz + ax + d = 0$, $ax + by + d = 0$ represent planes parallel to OX, OY, OZ respectively. Find the equations to the planes through the point $(2, 3, 1)$, $(4, -5, 3)$ parallel to the co-ordinate planes. Ans. $y + 4z - 7 = 0$, $x - z - 1 = 0$, $4x + y - 11 = 0$

23. Find the point of intersection of the planes.

(a) $x + 2y + 4z - 2 = 0$, $2x + 3y - 2z + 3 = 0$, $2x - y + 4z + 8 = 0$ Ans. $(-4, 2, 1/, 2)$

(b) $x + 2y - z - 6 = 0$, $2x - y + 3z + 13 = 0$, $3x - 2y + 3z + 16 = 0$ Ans. $(-1, 2, -3)$

24. Find the perpendicular distance between the planes Ans. 5/7

$3x + 6y + 2z = 22$ and $3x + 6y + 2z = 27$

25. Find the equation of the plane through the points $(1, -2, 2)$ $(-3, 1, -2)$ and perpendicular to the plane $2x + y - z + 6 = 0$ Ans. $x - 12y - 10z - 5 = 0$ [C.U. 1987]

26. A plane passes through the three points whose rectangular co-ordinates are $(8 - 2, 2)$, $(2, 1, -4)$ and $(2, 4, -6)$. Find the equation to this plane, the length of the perpendicular on it from the origin and the direction cosines of that perpendicular.

Ans. $2x - 2y - 3z = 14, \frac{14\sqrt{17}}{17}, 2/\sqrt{17}, \frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}$

27. Show that equation of the plane through the points $(-1, 3, 2)$ and perpendicular to the planes. $x + 2y + 2z = 5$. and $3x + 3y + 2z = 8$ is $2x - 4y + 3z + 8 = 0$

28. Determine the constant k so that the planes $x - 2y + kz = 0$ and $2x + 5y - z = 0$ are at right angles; find in that case the plane through the pt. $(1, -1, -1)$ and perpendicular to both the given planes. Ans. $k = -8$, $14x - 5y + 3z = 16$

29. A veriable plane is at a constant distance p from the origin and meets the axes in A, B, C. Show that the locus of the centroid of the tetrahedorn OABC is $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$

30. A point P moves on the plane $x/a + y/b + z/c = 1$ which is fixed. The plane through P perpendicular to OP meets the axes in ABC. The plane through ABC parallel to the yz, zx and xy planes intersect in Q. Prove that if the axes be rectangular the locus of Q is
$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{ax} + \frac{1}{by} + \frac{1}{cz}$$

31. Two systems of rectangular axes have the same origin. If a plane cuts, axes at a, b, c and a_1, b_1, c_1 respectively from the origin, prove that [1976, 1980—87]
$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a_1^2} + \frac{1}{b_1^2} + \frac{1}{c_1^2}$$

32. Find the area of the triangle the co-ordinates of whose vertices are $(1, 2, 3)$, $(-2, 1, -4)$, $(3, 4, -2)$

Ans. $\frac{1}{2}\sqrt{(1218)}$

33. A, B, C are $(3, 2, 1)$, $(-2, 0, -3)$, $(0, 0, -2)$.
Find the locus of P of the volume of the tetrahedron PABC = 5. Ans. $2x + 3y - 4z + 22 = 0$

34. Find the volume of the tetrahedron whose vertices are
(a) $(0, 1, 2)$, $(3, 0, 1)$, $(4, 3, 6)$, $(2, 3, 2)$
(b) $(1, 0, 0)$, $(0, 0, 1)$, $(0, 0, 2)$, $(1, 2, 3)$ Ans. 6

Ans. 1/3

35. Find the volume of the tetrahedron formed by the four planse $x + y = 0$, $y + z = 0$, $z + x = 0$, $x + y + z = 1$

Ans. 2/3

36. A variable plane passes through a fixed point (a, b, c) and meets the axes in P, Q, R. Show that the locus of point of intersection of the planes through P, Q, R parallel to the co-ordinate plane is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$

[R. U. 1983]

37. Find the equation of the plane through the point $(2, 3, -1)$ parallel to the plane $3x - 4y + 7z = 0$ and find the distance between the two planes. Find also the angle between this plane and the $8x + 3y - z = 2$.

Ans. $3x - 4y + 7z + 13 = 0$, $13/\sqrt{74}$, $\cos^{-1} 5/74$

38. Find the distance of the point $(2, -1, 5)$ from the plane $3x - 2y + 6z + 8 = 0$

Ans. 46/7 [D. U. 1979]

39. Show that the two point $(-2, 1, 3)$ and $(2, 1, -1)$ are on opposite sides of the plane $3x + 2y - 6z + 4 = 0$ and equidistant from it. [R.U. 1979]

40. Find the locus of the point such that the sum of the squares of its distances from the plane $x + y + z = 0$, $x - z = 0$, $x - 2y + z = 0$ is 9 [D. U. Hons. 1978]

41. Find the equation of a plane passing through the point $(1, -2, 1)$ and perpendicular to the line with direction ratios 2, 3, 5. Ans. $2x + 3y + 5z = 1$

42. Find the eq of the plane passes through the middle point of the join of the points $(2, -3, 1)$ and $(4, 5, -3)$ and is perpendicular to the line joining the points.

Ans. $x + 4y - 2z = 9$. [D. U. 1980]

43. Find the direction cosines of the interior bisectors of the angle $\angle BAC$ and the equations of the plane containing the points A $(1, 2, 3)$, B $(3, 5, -3)$ and C$(-2, 6, 15)$. [$(1, 2, 3)$, $(3, 5, -3)$ এবং $(-2, 6, 15)$ বিন্দুগুলোকে যথাক্রমে A, B এবং C দ্বারা নির্দেশ করা হলে, $\angle BAC$ কোণের অন্তর্দ্বিখণ্ডক এক দিক কোসাইন এবং ABC সমতলের সমীকরণ নির্ণয় কর।]

44. Find the equation of a plane which passes through the intersection of $7x - 4y + 7z + 16 = 0$ and $4x + 3y - 2z + 3 = 0$ and is parallel to $3x - 7y + 9x + 5 = 0$

[$7x - 4y + 7z + 16 = 0$, $4x + 3y - 2z + 3 = 0$ ছেদ বিন্দুগামী $3x - 7y + 9z + 5 = 0$ সমতলের সাথে সমান্তরাল তলের সমীকরণ নির্ণয় কর।] Ans. $3x - 7y + 9z + 13 = 0$ [R. U. 1988]

45. A variable plane makes intercepts on the co-ordinate axes, the the sum of whose square is constant. Find the locus of the foot of the perpendicular from the origin to the plane. (একটি চলমান সমতল অক্ষদলকে ছেদ করে। এদের বর্গের সমষ্টি একটি ধ্রুবক। চলমান তলের উপর মূলবিন্দু হতে লম্বের পাদবিন্দুর সঞ্চারপথ নির্ণয় কর।) Ans. $(x^2 + y^2 + z^2)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = k^2$

Note : General equation of 2nd degree $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ reperesents a pair of planes if $a^2 + b^2 + c^2 = f^2 + g^2 + h^2$