Hence the given equation represents two straight line $x + 3y + 5 = 0$ and $x + 3y - 1 = 0$. As these two lines differ only in constant terms, so these form a pair of parallel straight lines. See Art. 26 (1).

**Ex 2. Show that the angle between one of the lines given by $ax^2 + 2hxy + by^2 = 0$ and one of the lines.**

**$ax^2 + 2xy + by^2 + \lambda (x^2 + y^2) = 0$ is equal to the angles between the other two lines of the system.**

The equation of the straight lines bisecting the angles between

$ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

Again, the equation of the straight lines bisecting the angles between $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$

or, $(a + \lambda) x^2 + 2hxy + (a + \lambda)y^2 = 0$ is

$$\frac{x^2 - y^2}{(a + \lambda) - (b + \lambda)} = \frac{xy}{h} = \text{or, } \frac{(x^2 - y^2)}{a - b} = \frac{xy}{h}$$

which is the same as the equation of bisectors for the first pair. Hence the result.

**Ex. 3. If the pair of straight lines $x^2 - 2axy - y^2 = 0$ and $x^2 - 2bxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that $ab = -1$.**

[D. U. 1953, 57, 61, R. U. 1964, 79]

Equation to the bisector of first pair is

$$\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-a} \quad \text{or, } x^2 - y^2 = \frac{2xy}{-a} \quad \ldots \quad (1)$$

By the given condition second pair $x^2 - y^2 = 2bxy$ ... (2)

bisects the angle between the first pair. Hence comparing we have $\frac{1}{-a} = b$

or, $ab = -1$ Proved.

**Ex. 4) Prove that two of the lines represented by the equation $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = 0$ will be at right angles if**

$(b + d)(ad + be) + (a - e)^2 (a + e + c) = 0$

Let $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = (x^2 + kxy - y^2)(ax^2 + k' xy - ey^2) = 0$

$= ax^4 + (k' + ak) x^3y + (-ek - k') xy^3 + (kk' - a - e) x^2 y^2 + ey^4$

Now compare the co-efficients of $x^4$, $x^3y$, $xy^3$, $x^2y^2$ and $y^4$ from both sides, then

$k' + ak = b,$ ... (1)

$-ek - k' = d,$ ... (2)

$-a - e + kk' = c$ ... (3)

Find the value of $k$ and $k'$ from 1st and 3rd equations.

$k = (b + d) / (a - e)$, $k' = -(ad + bc) / (a - e)$

Therefore from (3), we have $kk' = a + e + c$

$$\frac{b+d}{a-e} \times \frac{ad+be}{a-e} = a+e+c$$

or, $(b + d)(ad + be) + (a - e)^2 (a + e + c)$ proved.

**Ex. 5.** **Prove the equation $y^3 - x^3 + 3xy(y - x) = 0$. represents three straight lines equally inclined to one another.** [C. U. (Hons) 1987]

Changing the equation $y^3 - x^3 + 3xy(y - x) = 0$

in polar co-ordinates ; $x = r\cos\theta$, $y = r\sin\theta$, we have $r^3\sin^3\theta - r^3\cos^3\theta + 3r^3\sin\theta.\cos\theta(\sin\theta - \cos\theta) = 0$

or, $\sin^3\theta - \cos^3\theta + 3\sin\theta\cos\theta(\sin\theta - \cos\theta) = 0$

or, $(\sin\theta - \cos\theta)\{\sin^2\theta + \cos^2\theta + 4\sin\theta\cos\theta\} = 0$

or, $(\tan\theta - 1)(\tan^2\theta + 4\tan\theta + 1) = 0$

Either $(\tan\theta - 1 = 0$ *i. e.* $\theta = 45°$. say $\theta_1 = 45°$ ... ... ... (1)

Again $\tan^2\theta + 4\tan\theta + 1 = 0$

or, $\tan\theta = \{-4 \pm \sqrt{(16-4)}\}/2 = -2 \pm \sqrt{3}$

Let $\tan\theta_2 = -2 + (\sqrt{3}) = \tan 165°$

$\therefore \theta_2 = 165°$

$\tan\theta_3 = -2 - \sqrt{3} = -\tan 75° = \tan(-75°) = \tan(360° - 75°) = \tan 285°$

$\therefore \theta_3 = 285°$

Now $\theta_2 - \theta_1 = 165° - 45° = 120°$

$\theta_3 - \theta_2 = 285° - 165° = 120°$

$\theta_3 - \theta_1 = 285° - 45° = 360° - 75° - 45° = -120°$

Hence the straight lines are inclined to each other at an angle of 120°. here RO is produced to OR′.

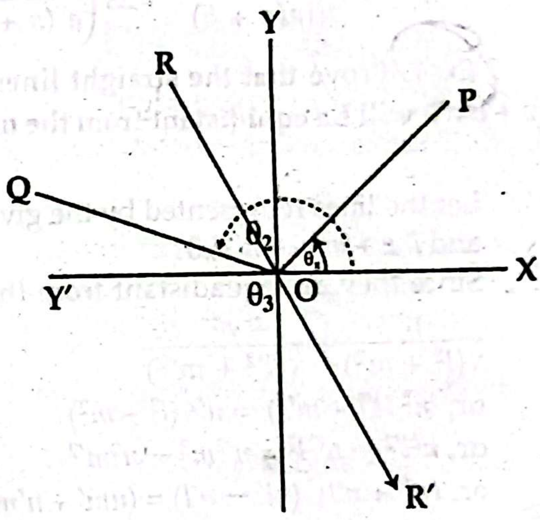


Fig : 14

**Ex. 6.** **Show that the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two parallel lines if $a : h = h : b = g : f$. Also show that the distance between them is $2\sqrt{\left\{\frac{(g^2 - ac)}{a(a+b)}\right\}}$** [N. U. H. 2006]

We know the condition that the general equation represents two straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and it will represent two parallel straight lines if in addition, $ab - h^2 = 0$

Now $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

or, $h^2c + 2fgh - af^2 - bg^2 - ch^2 = 0$, Since $ab = h^2$

or, $bg^2 - 2fgh + af^2 = 0$

or, $bg^2 - 2fg\sqrt{(ab)} + af^2 = 0 \quad (\because h^2 = ab)$

or, $(\sqrt{bg} - \sqrt{af})^2 = 0$ or, $\sqrt{bg} = \sqrt{af}$

or, $\frac{g}{f} = \sqrt{(\frac{a}{b})} = \sqrt{(\frac{ab}{b^2})} = h/b \quad \therefore h^2 = ab.$ or, $\frac{a}{h} = \frac{h}{b}$

$\therefore \frac{a}{h} = \frac{h}{b} = \frac{g}{f}$ which are the equired conditions

Let AB and CD be the lines $lx + my + n = 0$, $lx + my + n_1 = 0$ respectively. Take any point P $(x_1, y_1)$ on AB. Draw PM perpendicular to CD.

$l^2 = a$, $m^2 = bn_1 = c$, $l(n + n_1) = 2g$.

$\therefore lx_1 + my_1 + n_1 = 0 \qquad \ldots \qquad \ldots \qquad \ldots (1)$

$$\therefore \text{PM} = \frac{lx_1 + my_1 + n_1}{\sqrt{(l^2 + m^2)}} = \frac{n_1 - n}{\sqrt{(l^2 + m^2)}} \text{ from (1)}$$

$$= \frac{\sqrt{\{(n + n_1)^2 - 4nn_1\}}}{\sqrt{(l^2 + m^2)}} = \frac{\sqrt{(4g^2/l^2 - 4c)}}{\sqrt{(a + b)}} = \frac{\sqrt{(4g^2/a - 4c)}}{\sqrt{(a + b)}}$$

$$= \frac{2\sqrt{\{(g^2 - ac)}}{\sqrt{\{a(a + b)}} = 2\sqrt{\left(\frac{g^2 - ac}{a\,(a + b)}\right)}$$

**Ex. 7. Prove that the straight lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will be equidistant from the origin, if $f^4 - g^4 = (bf^2 - ag^2)$**

[D. U. 1956, C. U. (Hons) 1977, C. U. 1981]

Let the lines represented by the given equation be $lx + my + n = 0$ and $l'x + m'y + n' = 0$

Since they are equadistant from the origin.

$$\frac{n}{\sqrt{(l^2 + m^2)}} = \frac{n'}{\sqrt{(l'^2 + m'^2)}}$$

or, $n^2(l'^2 + m'^2) = n'^2(l^2 + m^2)$

or, $n^2l'^2 - n'^2l^2 = n'^2m^2 - n^2m'^2$

or, $(nl' + n'l)\ (nl' - n'l) = (nm' + n'm)\ (n'm - nm')$

or, $(nl' + n'l)^2\ \{(nl' + n'l)^2 - 4nn'll'\} = (nm' + n'm)\}^2$

$\{(n'm + nm')^2 - 4nn'mm'\} \qquad \ldots \qquad \ldots \qquad \ldots (2)$

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (lx + my + n)\ (l'x + m'y + n')$ comparing the co-efficients we have

$ll' = a$, $mm' = b$, $nn' = c$, $mn' + m'n = 2f$, $nl' + n'l = 2g$, $lm' + l'm = 2h$

Putting the values in (2) we have

$g^2(g^2 - ac) = f^2(f^2 - bc)$ or, $f^4 - g^4 = c\,(bf^2 - ag^2)$

**Ex. 8. Show that the lines joining the origin to the intersection of $7x^2 + 8x^2 - 7x^2 + 6x - 12y = 0$ and $2x + y - 1 = 0$ are at right angles.**

Making the first equation homogeneus with the help of $2x + y = 1$, we have

$7x^2 + 8xy - 7x^2 + (6x - 12y)\ (2x + y) = 0$ or, $19x^2 - 10xy - 19y^2 = 0$ which is the equation of the required straight lines.

Since the sum of the co-efficients of $x^2$ and $y^2$ is equal to zero, the straight lines are at right angles.

$$\left(\frac{-n}{1+nm_1}, \frac{-nm_1}{1+n_1m_2}\right) \text{and} \left(\frac{-n}{1+mm_2} \frac{-nm_2}{1+mm_2}\right)$$

which will be two vertices of the triangle and the third vertex is clearly (0, 0). Hence the area of the triangle,

$$\Delta = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{-n}{l+mm_1} & \frac{-nm_1}{1+mm_1} & 1 \\ \frac{-n}{l+mm_2} & \frac{-nm_2}{1+mm_2} & 1 \end{vmatrix} = \frac{n^2(m_2-m_1)}{2(l+mm_1)(l+mm_2)}$$

$$= \frac{\frac{n^2}{2}\sqrt{[(m_1+m_2)^2-4m_1m_2]}}{l^2+lm(m_1+m_2)+m^2m_2m_2}$$

$$= \frac{n^2\sqrt{(h^2-2ab)}}{(am^2+bl^2-2hlm)}$$ by (1) Ans.

**Ex. 12. The circle $x^2+y^2=a^2$ cuts off an intercept on the straight line $lx+my=1$, which subtends an angle of 45° at the origin, show that $4[a^2(l^2+m^2)-1]=[a^2(l^2+m^2)-2]^2$**

The equation of the straight lines joining the origing to the points of intersection of

$x^2+y^2=a^2$ ... ... ... (1)

and $lx+my=1$ ... ... ... (2)

is $x^2+y^2=a^2(lx+my)^2$

or $x^2(1-a^2l^2)+y^2(1-a^2m^2)-2a^2lmxy=0$

Let the angle between them be $\phi = 45°$ (given)

$$\tan\phi = \frac{2\sqrt{\{l^2m^2a^4-(1-a^2l^2)(1-a^2m^2)\}}}{(1-a^2l^2)+(1-a^2m^2)}$$ by Art. 41

or, $(1-a^2l^2)+(1-a^2m^2)=2\sqrt{\{l^2m^2a^4-(1-a^2l^2)(1-a^2m^2)\}}$

or, $(2-a^2l^2-a^2m^2)^2=4(a^2l^2+a^2m^2-1)$

or, $4[a^2l^2+m^2)-1] = [a^2(l^2+m^2)-2]^2$ Proved.

13. Prove that the equation $m(x^3-3xy^2)+y^3-3x^2y=0$ represent three straight lines equally inclined to one another. [N.U.H 1999]

**Solution :** Given that

$m(x^3-3xy^2)+y^3-3x^2y=0$ ... ... ... (1)

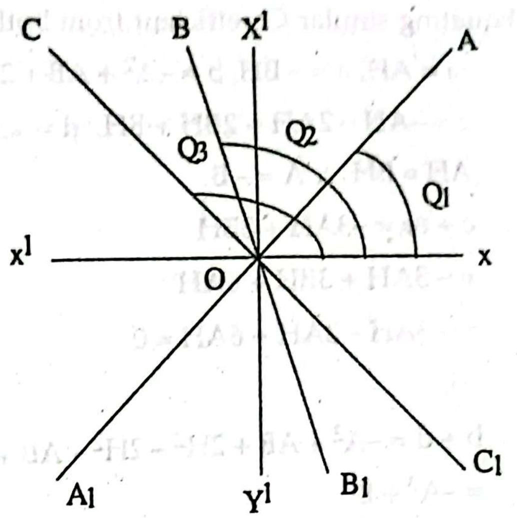
Since the equation (1) is a homogeneous is of degree 3. So it represents straight lines through origin.

Let $y = x\tan\theta$ represented the equation (1) whene $\theta$ is the slope of the line represented by (1)

$\therefore$ (1) $\Rightarrow m(x^3-3x.x^2\text{ten}^2\theta)+x^3\tan^3\theta-3x^2.x\tan\theta=0$

$\Rightarrow mx^3-3x^3m\tan^2\theta+x^3\tan^3\theta-3x^3\tan\theta=0$

$\Rightarrow m-3m\tan^2+\tan^3\theta-3\tan\theta=0$




$\therefore\ m = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\text{ten}^2\theta}$

$\Rightarrow m = \tan 3\theta$

$\therefore\ \tan 3\theta = \tan\alpha$ let $m = \tan\alpha$

$\therefore\ 3\theta = n\pi + \alpha$

$\therefore\ \theta = \dfrac{n\pi}{3} + \dfrac{\alpha}{3},$ $n = 0, 1, 2$

Let A $A_1$ B $B_1$ and $CC_1$ makes an angles $\theta_1, \theta_2$ and $\theta_3$ with x – axis.

$\theta_1 = \alpha/3, n = 0$

$\theta_2 = \dfrac{\pi}{3} + \dfrac{\alpha}{3}, n = 1$

$\theta_3 = \dfrac{2\pi}{3} + \dfrac{\alpha}{3}; n = 2$

Angle between the lines $AA_1$ and $BB_1$ is $\theta_2 - \theta_1 = \dfrac{\pi}{3}$

Angle between lines $BB_1$ and $CC_1$ is $\theta_3 - \theta_2 = \dfrac{2\pi}{3} + \dfrac{\alpha}{3} - \dfrac{\pi}{3}\dfrac{d}{3} = \pi/3$

Angle between lines $CC_1$ and $AA_1$ is $\pi - (\theta_3 - \theta_1)$

$= \pi - \theta_1 + \theta_1$

$= \pi - \dfrac{2\pi}{3} - \dfrac{\alpha}{3} + \dfrac{\alpha}{3}$

$= \pi/3$

So, three lines equally inclined to one another.

14. Prove that two of the lines represented by equation $ax^4 + bx^3y + cx^2y + dxy^3 + ay^4 = 0$ will bisect angle between the other tow if $c + 6a = 0, b + d = 0$ [N.U.H. 2000,'02,'09]

**Solution :** Given that

$ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4 = 0$ ... ... ... (1)

Since equation (1) is a homogeneous equation of degree 4. So, it represent 4 straight lines through origin.

Let $Ax^2 + 2Hxy + By^2 = 0$ ... ... ... (2)

be the two of the lines represented by (1).

According to the problem, other two lines represented by (1) is the bisectors of the lines represented by (2)

Equation of the bisectors of the two lines represented by (2)

$\dfrac{x^2 - y^2}{A - B} = \dfrac{xy}{H}$ ... (3)

$\Rightarrow Hx^2 - Hy^2 - Axy + Bxy = 0$ ... ...

$ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4 = (Ax^2 + 2Hxy + By^2)(Hx^2 - Hy^2 - Axy + Bxy)$

$\Rightarrow ax^4 + bx^3y + cx^2y^2 + xy^4 = AHx^4 - AHx^2y^2 - A^2x^3y + ABx^3y$

$+ 2H^2x^3y - 2H^2xy^3 - 2AHx^2y^2 + 2BHx^2y^2 + BHx^2y^2 - B_1 + y^4 - BAxy^3 + Bxy^3$

Equating similar Co-efficient from both sides

$\therefore \quad a = AH, a = -BH, b = -A^2 + AB + 2H^2,$

$c = -AH - 2AH + 2BH + BH, \; d = -2H^2 - AB + B^2$

$\therefore \quad AH = BH \;\therefore A = -B$

$\therefore \quad c + 6a = -3AH + 3BH$

$= -3AH + 3BH + 6AH$

$= -3AH - 3AH + 6AH = 0$

$= 0$

$b + d = -A^2 + AB + 2H^2 - 2H^2 - AB + B^2$

$= -A^2 + B^2$

$= -A^2 + A^2 \; [\therefore B = A]$

$= 0$ Proved.

15. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2gy + c = 0$ represents two straight lines. Prove that sauare of the distance of their point of intersection from the origin is $\dfrac{c(a+b) - f^2 - g^2}{ab - h^2}$.

[N.U.H -2001]

**Solution :** Given that

$ax^2 + 2hxy + by^2 + 2gx + 2gy + c = 0$ ... ... (1)

Since equation (1) reprsents pair of st-lines

Let $l_1x + m_1y + n_1 = 0$ ... ... ... (2)

and $l_2x + m_2y + n_2 = 0$ ... ... ... (3)

by the two lines represented by (1)

$\therefore \quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$= (l_1x + m_1y + n_1). \quad (l_2x + m_2y + n_2)$

$\Rightarrow \quad ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$= l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2 + (l_1n_2 + l_2n_1)x + (m_1n_2 + m_2n_1)y + n_1n_2$

Equating the similar Co-efficients from both sides

$\therefore \quad l_1l_2 = a, m_1m_2 = b, l_1m_2 + l_2m_1 = 2h, l_1n_2 + l_2n_1 = 2g$

$m_1n_2 + m_2n_1 = 2f, n_1n_2 = c.$

Now, $\quad l_1x + m_1y + n_1 = 0$

$l_2x + m_2y + n_2 = 0$

$$\therefore \frac{x}{m_1n_2 - m_2n_1} = \frac{y}{n_1l_2 - n_2l_1} = \frac{1}{l_1m_2 - l_2m_1}$$

$$\therefore x = \frac{m_1n_2 - m_2n_1}{l_1m_2 - l_1m_1}, y = \frac{n_1l_2 - n_2l_1}{l_1m_2 - l_2m_1}$$

$\therefore$ Point of intersection of the lines (2) and (3) say A,

$$A\left(\frac{m_1n_2 - m_2n_1}{l_1m_2 - l_2m_1}, \frac{n_1l_2 - n_2l_1}{l_1m_2 - l_2m_1}\right)$$

$$\therefore \quad OA^2 = \left(\frac{m_1n_2 - m_2n_1}{l_1m_2 - l_2m_1} - 0\right)^2 + \left(\frac{n_1l_2 - n_2l_1}{l_1m_2 - l_2m_1} - 0\right)^2$$

$$= \frac{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2}{(l_1m_2 - l_2m_1)^2}$$

$$= \frac{(m_1n_2 + m_2n_1)^2 - 4m_1m_2\, n_1n_2 + (n_1l_2 + n_2l_1)^2 - 4l_1l_1\, n_1n_2}{(l_1m_2 + l_2m_1)^2 - 4l_1l_2\, m_1m_2}$$

$$= \frac{4f^2 - 4bc + 4g^2 - 4ca}{4h^2 - 4ab}$$

$$= \frac{f^2 + g^2 - bc - ca}{h^2 - ab}$$

$$= \frac{f^2 + g^2 - c(a + b)}{h^2 - ab} \quad \text{(Proved)}$$

15 If two straight lines represented by the equation $x^2(\tan^2\varphi + \cos^2\varphi) - 2xy\tan\varphi + y^2\sin^2\varphi = 0$ makes an angle $\alpha$ and $\beta$ with $x$ – axis respectively, then show that $\tan\alpha - \tan\beta = 2$.

[N.U.H-2003, 2008]

**Solution :** Given equation

$x^2(\tan^2\varphi + \cos^2\varphi) - 2xy\tan\varphi + y^2\sin^2\varphi = 0$ ... ... ... (1)

since the two lines represented by the equation (1)

makes an angle $\alpha$ and $\beta$ with $x$ – axis.

So their equation are

$y = x\tan\alpha$ and $y = x\tan\beta$

$$\therefore \tan\alpha + \tan\beta = \frac{2\tan\varphi}{\sin^2\varphi}$$

$$= \frac{2\sin\varphi}{\cos\varphi} \times \frac{1}{\sin^2\varphi}$$

$$= 2\text{sce}\varphi\ \text{cosec}\varphi$$

$$\text{and } \tan\alpha.\ \tan\beta = \frac{\tan^2\varphi + \cos^2\varphi}{\sin^2\varphi}$$

$$= \frac{\sin^2\varphi}{\cos^2\varphi}.\frac{1}{\sin^2\varphi} + \frac{\cos^2\varphi}{\sin^2\varphi}$$

$$= \sin^2\varphi + \cot^2\varphi$$

Now, $(\tan\alpha - \tan\beta)^2 = (\tan\alpha + \tan\beta)^2 - 4\tan\alpha.\ \tan\beta$

$$= (2\sec\varphi\ \text{coce}\varphi)^2 - 4.\ (\sec^2\varphi + \cot^2\varphi)$$

$= 4\sec^2\varphi \csc^2\varphi - 4\sec^2\varphi - 4\cot^2\varphi$

$= 4\{\sec^2\varphi(\csc^2\varphi - 1) - \cot^2\varphi\}$

$= 4\{\sec^2\varphi.\cot^2\varphi - \cot^2\varphi\}$

$= 4\{\cot^2\varphi(\sec^2 - 1)\}$

$= 4\{\cot^2\varphi.\tan^2\varphi\}$

$\therefore (\tan\alpha - \tan\beta)^2 = 4$

$\therefore \tan\alpha - \tan\beta = 2$ (Proved)

17. If one of the straigh line given the equation $ax^2 + 2hxy + by^2 = 0$ concides with one of those given by $a_1x^2 + 2h_1xy + b_1y^2 = 0$ and the other lines represented them be perpendicular. Then show that $\frac{ha_1b_1}{b_1 - a_1} = \frac{h_1ab}{b - a} = \frac{1}{2}\sqrt{-aa_1bb_1}$ [N.U.H 1998, 2004]

**Solution :** Given equations

$$ax^2 + 2hxy + by^2 = 0 \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

$$a_1x^2 + 2h_1xy + b_1y^2 = 0 \quad \cdots \quad \cdots \quad \cdots \quad (2)$$

Let $y = m_1x$ and $y = m_2x$ be the two lines represented by (1)

$$\therefore \quad m_1 + m_2 = -\frac{2h}{b} \quad \cdots \quad \cdots \quad \cdots \quad (3)$$

$$m_1m_2 = \frac{a}{b} \quad \cdots \quad \cdots \quad \cdots \quad (4)$$

According to the problem, $y = m_1x$ and $y = -\frac{1}{m_2}x$ be the two lines represented by (2)

$$\therefore \quad m_1 - \frac{1}{m_2} = -\frac{2h_1}{b_1} \quad \cdots \quad \cdots \quad \cdots \quad (5)$$

$$m_1.-\frac{1}{m_2} = \frac{a_1}{b_1} \quad \cdots \quad \cdots \quad \cdots \quad (6)$$

From (3) and (6) we get

$$m_1 + m_2 = -\frac{2h}{b}$$

$$\text{and } -m_1 - \frac{a_1}{b_1}m_2 = 0$$

$$(+) \quad m_2 - \frac{a_1}{b_1}m_2 = -\frac{2h}{b}$$

$$\Rightarrow \quad m_2\left(1 - \frac{a_1}{b_1}\right) = -\frac{2h}{b}$$

$$\Rightarrow \quad m_2.\frac{b_1 - a_1}{b_1} = -\frac{2h}{b}$$

$$\therefore \quad m_2 = -\frac{2hb_1}{b(b_1 - a_1)}$$

From (3); $m_1 + m_2 = -\frac{2h}{b}$

$$m_1 = -\frac{2h}{b} - m_2$$

$$= -\frac{2h}{b} + \frac{2hb_1}{b(b_1 - a_1)}$$

$$= \frac{2h}{b}\left\{\frac{b_1}{b_1 - a_1} - 1\right\}$$

$$= \frac{2h}{b} \cdot \frac{b_1 - b_1 + a_1}{b_1 - a_1}$$

$$m_1 = \frac{2ha_1}{b(b_1 - a_1)}$$

From (4) ; $m_1 m_2 = \frac{a}{b}$

$$\Rightarrow \frac{2ha_1}{b(b_1 - a_1)} \times \frac{-2hb_1}{b(b_1 - a_1)} = \frac{a}{b}$$

$$\Rightarrow \frac{4h^2 a_1 b_1}{b^2(b_1 - a_1)} = -\frac{a}{b}$$

$$\Rightarrow \frac{h^2 a_1 b_1}{b_1 - a_1} = -\frac{1}{4} ab$$

$$\Rightarrow \frac{h^2 a_1^2 b_1^2}{b_1 - a_1} = -\frac{1}{4} aa_1 bb_1$$

$$\therefore \quad \frac{ha_1 b_1}{b_1 - a_1} = \frac{1}{2}\sqrt{-aa_1 bb_1} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (7)$$

Again From (4) ; $m_1 m_2 = \frac{a}{b} \therefore m_1 = \frac{a}{bm_2}$

From (5); $m_1 - \frac{1}{m_2} = -\frac{2h_1}{b_1}$

$$\frac{1}{m_2} - m_1 = \frac{2h_1}{b_1}$$

$$\Rightarrow \frac{1}{m_2} - \frac{a}{bm_2} = \frac{2h_1}{b_1}$$

$$\Rightarrow \frac{b - a}{bm_2} = \frac{2h_1}{b_1}$$

$$\Rightarrow \quad m_2 = \frac{b_1(b - a)}{2h_1 b}$$

$$m_1 = \frac{a}{bm_2}$$

$$= \frac{a}{b}, \frac{2h_1b}{b_1(b-a)} = \frac{2h_1a}{b_1(b-a)}$$

From (6); $m_1 . \frac{-1}{m_2} = \frac{a_1}{b_1}$

$$\Rightarrow -\frac{2h_1a}{b_1(b-a)} \times \frac{2h_1b}{b_1(b-a)} = \frac{a_1}{b_1}$$

$$\Rightarrow -\frac{4h_1^2ab}{b_1^2(b-a)^2} = \frac{a_1}{b_1}$$

$$\Rightarrow \frac{4h_1^2a^2b^2}{(b-a)^2} = -aa_1bb_1 \quad \ldots \quad (৪)$$

$$\therefore \frac{h_1ab}{b-a} = \frac{1}{2}\sqrt{-aa_1bb_1} \quad \ldots$$

From (7) and (8) we get

$$\frac{ha_1b_1}{b_1-a_1} = \frac{h_1ab}{b-a} = \frac{1}{2}\sqrt{-aa_1bb_1} \text{ (Proved)}$$

## EXERCISE V

[ তারকা চিহ্নের অঙ্কগুলি অনার্স ছাত্রদের জন্য ]

**Find the lines represented by (নিম্নে সমীকরণের দ্বারা প্রকাশিত সরলরেখাগুলো নির্ণয় কর।)**

1. $x^2\cos 2\theta - 4xy\cos\theta + 2y^2 + x^2 = 0.$ Ans. $y = x\cos\theta,\ y = x\cos\theta$
2. $3x^2 - 16xy + 5y^2 = 0$ Ans. $x - 5y = 0; 3x - y = 0$
3. $y^3 - xy^2 - 14x^2y + 24x^3 = 0$ Ans. $y = -4x,\ y = 2x,\ y = 3x$

**Prove that the following equations represent two straight lines; find also their point of intersection and the angles between them.**

4. $3y^2 - 8xy - 3x^2 - 29x + 3y - 18 = 0$ Ans. (৩/2, –5/2)) ; 90°
5. $2y^2 - xy - x^2 + y + 2x - 1 = 0$ Ans. (1, 0), $\tan^{-1}3$ [R. U. 1960]
6. $x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ Ans. $(\frac{-11}{13}, \frac{10}{13}; 90°)$
7. $2y^2 + 3xy - 5y - 6x + 2 = 0$ Ans. $(-1, 2,\ \tan^{-1}\frac{3}{2})$ [D. U. 1952]

**Find the value of λ or k so that the following equations may represent pairs of straight lines.**

8. $\lambda x^2 + 4xy + y^2 - 4x - 2y - 3 = 0$ Ans. 4 [D. U. 1960]
9. $2x^2 - y^2 + xy - 2x - 5y + k = 0$ Ans. –4
10. $6x^2 + xy + ky^2 - 11x + 43y - 35 = 0$ Ans. + 12
11. $x^2 - \lambda xy + 2y^2 + 3x - 5y + 2 = 0$ Ans. $3, \frac{9}{2}$
12. $kxy - 8x + 9y - 12 = 0$ Ans. 6

13. $6x^2 + 2kxy + 12y^2 + 22x + 31y + 20 = 0$ Ans. $\frac{171}{20}, \frac{17}{2}$ [D. U. 1962]

14. $6x^2 - 7xy + 16x - 3y^2 - 2y + k = 0$ Ans. 8 [D.U. (Hons) 1962]

15. $12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0$ Ans. $\lambda = 2$ [D. U. 1962]

16. Find the angle between the straight lines.
$(x^2 + y^2)(\cos^2\theta \sin^2\alpha + \sin^2\theta) = (x \tan\theta - y \sin\alpha)^2$ Ans. $2\theta$ [R. U. 1961]

16. Show that $2x^2 - 7xy + 3y^2 + x + 7y - 6 = 0$ equation representing two straight lines whose included angle is 45° (দেখাও যে $2x^2 - 7xy + 3y^2 + x + 7y - 6 = 0$ দুটি সরলরেখা প্রকাশ করে এবং এদের অন্তর্ভুক্ত কোণ 45°)

17. If the two straight line represented by $x^2(\tan^2\phi + \cos^2\phi) - 2xy\tan\phi + y^2\sin^2\phi = 0$ make angles $\alpha$ and $\beta$ with the axis of $x$, then show that $\tan\alpha - \tan\beta = 2$. ($x^2(\tan^2\phi + \cos^2\phi) - 2xy\tan\phi + y^2\sin^2\phi = 0$ এর দুটি সরলরেখা $x$ অক্ষরেখার সাথে $\alpha$ও $\beta$ কোণ করলে দেখাও যে, $\tan\alpha - \tan\beta = 2$.)
[ U. N. H. 2003, 2008 ]

18. Show that the straight lines represented by $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ are perpendicular to each other. (দেখাও যে, $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ এর দুটি সরলরেখা পরস্পরের উপর লম্ব হবে।)

19. If the origin is joined by the two points ofintersection of the line $lx + my + n = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ there will be two straight lines show that they will be coincident. (প্রমাণ কর যে, $lx + my + n = 0$ এবং $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ বক্ররেখার ছেদ বিন্দুদ্বয়কে মূল বিন্দুতে সংযোগ করলে যে সরলরেখা দুটি উৎপন্ন হবে তারা সমপতিত হবে যদি $a^2l^2 + b^2m^2 = n^2$) [R.U. 1980]

20. The straight line $y = mx$ bisects the angle between two straight lines represented by $ax^2 - 2hxy + by^2 = 0$ if $h(1 - m^2) + m(a - b) = 0$. (দেখাও যে, $y = mx$ সরলরেখা $ax^2 - 2hxy + by^2 = 0$ দ্বারা প্রকাশিত সরলরেখাদ্বয়ের অন্তর্ভুক্ত কোণকে দ্বিখণ্ডিত করবে যদি $h(1 - m^2) + m(a - b) = 0$) [D. U. 1977]

21. Show that the two straight lines represented by $ax^2 + 2\lambda xy - ay^2 = 0$ will be perpendicular to each other. ( দেখাও যে, $ax^2 + 2\lambda xy - ay^2 = 0$ পরস্পরের উপর লম্ব দুটি সরলরেখা প্রকাশ করে।)

22. Show that the four straight lines represented by $12x^2 + 7xy - 12y^2 = 0$ and $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$ will be on the sides of a square. (দেখাও যে, $12x^2 + 7xy - 12y^2 = 0$. এবং $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$., এদের দ্বারা প্রকাশিত চারটি সরলরেখা একটি বর্গক্ষেত্রের বাহুর উপর থাকবে।)

23. Show that the four straight lines represented by $4xy(x^2 - y^2) - \tan\alpha(x^2 + 2xy - y^2)(x^2 - 2xy - y^2) = 0$ will make equal angles with each other. (দেখাও যে, $4xy(x^2 - y^2) - \tan\alpha(x^2 + 2xy - y^2)(x^2 - 2xy - y^2) = 0$ এর দ্বারা প্রকাশিত চারটি সরলরেখা একে অন্যের সাথে সমান কোণে থাকবে।)

24. Show that the four straight lines represented by $y^2 - 4y + 3 = 0$ and $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$ (দেখাও যে, $y^2 - 4y + 3 = 0$ এবং $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$ will form a parallelogram. এদের দ্বারা প্রকাশিত চারটি সরলরেখা একটি সামান্তরিক উৎপন্ন করবে।)

25. Show the four straight lines represented by $x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$ and $x^2 + 4xy - 2y^2 = 0$ will form a rhombus through the origin. (দেখাও যে, $x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$ এবং $x^2 + 4xy - 2y^2 = 0$ এদের দ্বারা প্রকাশিত চারটি সরলরেখা একটি রম্বস প্রকাশ করবে।)

**26.** If the two straight lines through the origin represented by $2x^2 + 5xy + 2y^2 + 10x + 5y = 0$ are perpendicular to each other then find the combined equation of straight lines passing through the origin. (মূলবিন্দুগামী সরলরেখা দুটি $2x^2 + 5xy + 2y^2 + 10x + 5y = 0$ দ্বারা প্রকাশিত সরলরেখা দুটির উপর পরস্পর লম্ব হলে মূলবিন্দুগামী সরলরেখাদ্বয়ের যুগ্ম সমীকরণ নির্ণয় কর।) Ans. $2y^2 - 5xy + 2x^2 = 0$

[C. U. 1990]

**27.** Prove that the angles between the straight lines. $ax^2 + 2hxy + by^2 + \lambda (x^2 + y^2) = 0$ have the same pair of bisectors. Interpret the case when $\lambda = -(a + b)$ (দেখাও যে $ax^2 + 2hxy + by^2 + \lambda (x^2 + y^2) = 0$ দ্বারা প্রকাশিত সরলরেখা দুইটির কোণের দ্বিখণ্ডকদ্বয় একই হবে। যখন $\lambda = -(a + b)$ তখন এটা কি প্রকাশ করে?)

[R. U. Hons. 1977; '80]

**28.** If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines equidistant from the origin. show that $h (g^2 - f^2) = fg (a - b)$ (যদি মূল বিন্দু হতে সমান দূরত্বে দুইটি সরলরেখা $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ সমীকরণ দ্বারা প্রকাশিত হয় তখন দেখাও যে, $h (g^2 - f^2) = fg (a - b)$)

[D. U.(Hons.) 1959; C. U. Hons.. 1989]

**29.** If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines, prove that the square of the distance of their point of intersection from the origin is $\frac{c(a + b) - f^2 - g^2}{ab - h^2}$ (যদি মূল বিন্দু হতে সমান দূরত্বে দুইটি সরলরেখা $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ সমীকরণ দ্বারা প্রকাশিত হয় তখন দেখাও যে সরলরেখা দুইটির ছেদবিন্দু হতে মূলবিন্দুর দূরত্ব $\frac{c(a + b) - f^2 - g^2}{ab - h^2}$ হবে।) [U. N. H. 2001]

**30.** Show that the pair of straight lines joining the origin O to the intersections A and B of the line $lx + my = 1$ with the conic. $ax^2 + by^2 = 1$ has the equation. $(a - l^2) x^2 - 2lmxy + (b - m^2) y^2 = 0$ ($lx + my = 1$ এবং $ax^2 + by^2 = 1$ কণিকের ছেদবিন্দু দুইটি A এবং B হলে, দেখাও যে সরলরেখা দুইটির সংযুক্ত সমীকরণ হবে $(a - l^2) x^2 - 2lmxy + (b - m^2) y^2 = 0$) [R. U. (Pass) 1985; R. U. 1962, '82]

**31.** দেখাও যে, $x^2 + 4xy + 4y^2 - 5x - 10y + 4 = 0$ এবং $4x^2 - 4xy + y^2 + 4x - 2y - 3 = 0$ দ্বারা প্রকাশিত চারটি সরলরেখা একটি সামান্তরিক প্রকাশ করে।

**32.** Prove that two of the lines represented by the equation $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4 = 0$ will bisect angle between the other two if $c + 6a = 0, b + d = 0$.

[N. U. H. 2000, '02, '09][C. H. 1979, 78]

**33.** Find the equations of the bisectors of the angles between the lines represented by

(a) $12x^2 + 7xy - 10y^2 + 13x + 45y - 35 = 0$

[D. U. 1962]

Ans. $7 (23x + 25)^2 - 7(23y - 43)^2 = 44 (23x + 25) (23y - 43)$

(b) $2x^2 + 7xy + 6y^2 + 13x + 22y + 20 = 0$

Ans. $7x^2 - 7y^2 + 8xy - 4x - 58y - 83 = 0$

[R. U. 1962]

**34.** Prove that the angles between the lines joining the origin to the intersections of the line $y = 3x + 2$ with the curve. $x^2 + 3y^2 + 2xy + 4x + 8y - 11 = 0$ is $\tan^{-1} (2\sqrt{2})/3$

**35.** Prove that one of the line represented by the equations $ax^3 + bx^2y + cxy^2 + dy^3 = 0$ will bisect the angle between the other two if $(3a + c)^2 (bc + 2cd - 3ad) = (b + 3d)^2 (bc + 2ab - 3ad)$

36. Show that the lines joining the origin to the intersection of $x^2 + y^2 + 19x + 4y - 3 = 0$ and $3x + 4y = 1$ are at right angles. [ R. H. 1988; D. U. 1961 ]

37. দেখাও যে $3x^2 + 3xy - 3y^2 + 2x + 5y = 0$ এবং $3x - 2y = 1$
সরলরেখার ছেদবিন্দু দুটির সাথে মূলবিন্দুর সংযোগ সরলরেখা দুটি পরস্পর লম্ব হবে। [C. U. 1987]

37. (a) Find the angle between the lines joining the origin to the pt of intersection of $y = x + 1$ with $x^2 - 3y^2 + 2xy - 3x + 3y + 1 = 0$ [ R. U. 1981 ] Ans. $\tan^{-1} 2\sqrt{5}/5$.

38. দেখাও যে $a(x^4 + y^4) - 4bxy(x^2 - y^2) + 6cx^2y^2 = 0$ দ্বারা প্রকাশিত দু' জোড়া পরস্পর লম্ব সরলরেখা প্রকাশ করে। যদি দু' জোড় সরলরেখা সমাপতিত হয় তা হবে যদি $2b^2 = a^2 + 3ac$

38. (a) Prove that $x^4 + y^4 - 4xy(x^2 - y^2) + 2cx^2y^2 = 0$ represents a pair of pair of perpendicular straight lines, also show that if $c = 1$ then those of straight lines will be coincident. [ C. U(P) 1989]

39. Prove that two of the straight lines represented by the equation $ax^3 + bx^2y + cxy^2 + dy^3 = 0$. will be at right angles if. $a^2 + ac + bd + d^2 = 0$. [ C. H. 1980 ]

40. Show that lines joining the origin to the point of intersections of $y^2 - 4ax = 0$ and $y = mx + c$ will be coincident of $c = \frac{a}{m}$

41. Prove that the straight lines represented by the equation $\cos 3\alpha (x^3 - 3xy^2) + \sin 3\alpha (y^3 - 3x^2y) + 3a(x^2 + y^2) - 4a^3 = 0$ form an equailateral triangle whose area is $3\sqrt{3}a^2$. [C.H.1989 ]

42. Prove that the equation $m(x^3 - 3xy^2) + y^3 - 3x^2y = 0$ represents three straight lines equally inclined to one another. [ N. U. H. 1999 ] [ R. U. 1987 ]
[ প্রমাণ কর যে $m(x^3 - 3xy^2) + y^3 - 3x^2y = 0$ দ্বারা তিনটি সরলরেখা প্রকাশ করে এবং এরা একে অন্যের সাথে একই কোণে থাকবে। ]

43. Prove that the straight lines joining the origin to the points of intersection of the straight line $kx + hy = 2hk$ with the curve
$(x - h)^2 + (y - k)^2 = c^2$ are at right angles if $h^2 + k^2 = c^2$.
(প্রমাণ কর যে $kx + hy = 2hk$ এবং বক্ররেখা $(x - h)^2 + (y - k)^2 = c^2$ ছেদবিন্দুর সাথে মূলবিন্দুর সংযোগ সরলরেখাদ্বয় পরস্পরের উপর লম্ব হবে যদি $h^2 + k^2 = c^2$ )

44. If one of the straight line given by the equation $ax^2 + 2hxy + by^2 = 0$ coincides with one of those given by $a_1x^2 + 2h_1xy + b_1y^2 = 0$ and the other lines represented by them be perpendicular, prove that $\frac{ha_1b_1}{b_1 - a_1} = \frac{h_1ab}{b - a} = \frac{1}{2}\sqrt{(-aa_1bb_1)}$. [ N. U. H. 1998, 2004 ]

(যদি $ax^2 + 2hxy + by^2 = 0$ রেখাদ্বয়ের একটি রেখা $a_1x^2 + 2h_1xy + b_1y^2 = 0$ রেখাদ্বয়ের একটি রেখার সাথে সমাপতিত হয় এবং অপর দুটি রেখা পরস্পরের সাথে লম্ব হয়, তবে দেখাও যে $\frac{ha_1b_1}{b_1 - a_1} = \frac{h_1ab}{b - a} = \frac{1}{2}\sqrt{(-aa_1bb_1)}$.

45. Find the equation of the line through the origin and perpendicular to the lines $5x^2 - 7xy - 3y^2 = 0$
($5x^2 - 7xy - 3y^2 = 0$ রেখাদ্বয়ের উপর লম্ব এবং মূলবিন্দুগামী সরলরেখাদ্বয়ের সমীকরণ নির্ণয় কর।
Ans. $3x^2 + 7xy + 5y^2 = 0$) [D. U. H. 1979 ]

46. Show that $x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$ represents pair of straight lines.
(দেখাও যে, $x^2 + 4xy - 2y^2 + 6x - 12y - 15 = 0$ সমীকরণ দুটি সরলরেখা প্রকাশ করে।) [D. U. 1979]

47. Show that the st. lines joining the origin to the points of intersection of the curves $ax^2 + 2hxy + by^2 + 2gx = 0$ and $a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x = 0$ will be at right angles. if $g(a_1 + b_1) = g(a + b)$ [D. U. '76]

48. Show that the equation $bx^2 - 2hxy\ ay^2 = 0$ represents a pair of st. lines which are at right angle to the pair given by equation $ax^2 + 2hxy + by^2 = 0$ [R. U. 1985]
(দেখাও যে, $bx^2 - 2hxy + ay^2 = 0$ এর সরলরেখা দুটি $ax^2 + 2hxy + by^2 = 0$ সরলরেখাদ্বয়ের সাথে পরস্পর লম্ব হবে।) [D. U. 1976, R. U. 1978, '85]

49. If one of the lines $ax^2 + 2hxy + by^2 = 0$ be perpendicular to one of the lines $a'x^2 + 2h'xy + b'y^2 = 0$, prove that $(aa' - bb')^2 + 4(a'h + bh')(ah' + b'h) = 0$. [R.H. 1977]
(যদি $ax^2 + 2hxy + by^2 = 0$ সরলরেখার একটি $a'x^2 + 2h'xy + b'y^2 = 0$ এই সরলরেখার একটির উপর পরস্পর লম্ব হয়, তা হলে প্রমাণ কর যে $(aa' - bb')^2 + 4(a'h + bh')(ah' + b'h) = 0$.) [R.H. 1997]

50. Prove that $(ax + by)(\alpha x + \beta y) + kxy - (a + \alpha)x - (b + \beta)y + 1 = 0$ represents a pair of straight lines if $k = (a - \alpha)(b - \beta)$. Find the co-ordinates of their point of intersection.
(প্রমাণ কর যে $(ax + by)(\alpha x + \beta y) + kxy - (a + \alpha)x - (b + \beta)y + 1 = 0$ দুটি সরলরেখা প্রকাশ করবে, যদি $k = (a - \alpha)(b - \beta)$. হয় সরলরেখা দুটির ছেদবিন্দু নির্ণয় কর।) [D. U. 1975]

51. Prove that product of the perpendiculars from the point $(x_1, y_1)$ on the lines $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{\{(a - b)^2 + 4h^2\}}}$ [C. U. 1986; R. U. 1979]

* 52. দেখাও যে $ax^2 + 2hxy + by^2 = 0$ সরলরেখাদ্বয়ে একটি রেখা $a_1x^2 + 2h_1xy + b_1y^2 = 0$ রেখা দুটির একটি রেখার সাথে সমাপতিত হবে যদি
$(ab_1 - a_1b)^2 = 4(a_1h - ah_1)(bh_1 - b_1h)$. [খ. ঔ. ১৯৯৭] [R. U. 1988; D.U. 1989]

* 53. If $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines, the area of the triangle formed by their bisectors and the axis of $x$ is

$$\Delta = \frac{\sqrt{\{(a - b)^2 + 4h^2\}}}{2h} \frac{ca - g^2}{ab - h^2}$$ [C.H. 1989; R.U. 1982]

(S দ্বারা দুটি সরলরেখা সূচিত হলে, দেখাও যে তাদের অন্তর্গত কোণের দ্বিখণ্ডকদ্বয় $x$ অক্ষের সাথে ত্রিভুজ উৎপন্ন করে, তার ক্ষেত্রফল $\Delta$ হবে)

* 54. Show that four lines given by the equations
$(y - mx)^2 = c^2(1 + m^2)$, $(y - nx)^2 = c^2(1 + n^2)$ form a rombus.

* 55. The vertices of a triangle lie on the lines $y = m_1x$, $y = m_2x$, $y = m_3x$ the circumcentre being at the origin, prove that the locus of the other centre is the line $L \equiv x(\sin\theta_1 + \sin\theta_2 + \sin\theta_3) - y(\cos\theta_1 + \cos\theta_2 + \cos\theta_3) = 0$ (একটি ত্রিভুজের শীর্ষবিন্দুগুলো যথাক্রমে $y = m_1x$, $y = m_2x$, $y = m_3x$ সরলরেখার উপর থাকবে। এদের পরিলিখিত বৃত্তের কেন্দ্র মূল বিন্দুতে হলে দেখাও যে অন্য কেন্দ্রগুলোর সঞ্চারপথ একটি সরল রেখা L হবে।)