UNIT 3

ERROR ANALYSIS

Errors come in a variety of ways; some are avoidable while some are not. How errors occur and how they affect the accuracy of calculation is very essential in understanding numerical methods. In applied mathematics, error is the difference between a true value and an estimate or approximation of that value.

E = True value – Approximate value

Source of Error

- Modeling Error: a wrong or inappropriate choice of model
- Measurement Error: incorrect or poor measurements
- Implementation Error: incorrect or poor choice of algorithms
- Simulation Error: error accumulated due to the execution of our model

Example

The value of π is 3.14159265….

The commonly used approximation to π is 22/7, what is the error in this approximation?

Solution

We must convert 22/7 to decimal form and find the difference.

True value of π is 3.14159265….

Approximated value of π is 3.14285714

Error = 3.14159265 - 3.14285714

= 0.00126449

Generally speaking, error can be introduced into numerical work by the following

- Mistake due to human error
- Error due to given data
- Round off (premature approximation)
- Error due to method employed

Types of Error

(1) **Inherent error**: is that quantity which is already present in the statement of the problem before it solution. The inherent error arises either due to the simplified assumption in the mathematical formulation of the problem or due to the errors in the physical measurements of the parameters of the problem.

(2) **Round off Error:** Rounding means to round by raising the last figure by 1 if the next figure would have been greater or less than 5.

Round-off Error is the quantity R which must be added to the finite representation of a computed number in order to make it the true representation of that number. It is due to representation of a number by a finite number of decimal digits e.g. approximation due to nearest whole number and also approximation to a certain decimal places.

A **round-off error**, also called **rounding error**, is the difference between the calculated approximation of a number and its exact mathematical value. Roundoff error is the difference between an approximation of a number used in computation and its exact (correct) value. Numerical analysis specifically tries to estimate this error when using approximation equations and/or algorithms, especially when using finitely many digits to represent real numbers (which in theory have infinitely many digits). This is a form of quantization error.
When a sequence of calculations subject to rounding error are made, errors may accumulate in certain cases known as ill-conditioned, sometimes to such an extent as to dominate the calculation and make the result meaningless.

Example
The approximation of 9.345 to the nearest whole number is 9
The approximation of 9.345 to 2 decimal points is 9.35
2/3 = 0.6666 rounded to three decimals places is 0.667

Round (17.5) = 18

Representation error

The error introduced by attempting to represent a number on the computer is called *representation error*. Some examples:

Notation	**Represent**	**Approximate**	**Error**
1/7	0.142 857	0.142 857	0.000 000 142 857
ln 2	0.693 147 180 559 945 309 41...	0.693 147	0.000 000 180 559 945 309 41...
$\log_{10} 2$	0.301 029 995 663 981 195 21...	0.3010	0.000 029 995 663 981 195 21...
$\sqrt[3]{2}$	1.259 921 049 894 873 164 76...	1.25992	0.000 001 049 894 873 164 76...
$\sqrt{2}$	1.414 213 562 373 095 048 80...	1.41421	0.000 003 562 373 095 048 80...
e	2.718 281 828 459 045 235 36...	2.718 281 828 459 045	0.000 000 000 000 000 235 36…
π	3.141 592 653 589 793 238 46...	3.141 592 653 589 793	0.000 000 000 000 000 238 46...

(3) **Truncation Error:** This is the quantity T which must be added to the true representation of the quantity in order for the result to be exactly equal to the quantity we are seeking to generate.

Truncate means to cut off and truncation error happen when a fraction is cut off a certain number of decimal or binary places.

In mathematics and computer science, **truncation** is the term for limiting the number of digits right of the decimal point, by discarding the least significant ones.
For example, consider the real numbers

5.6341432543653654
32.438191288
−6.3444444444444

To *truncate* these numbers to 4 decimal digits, we only consider the 4 digits to the right of the decimal point.

The result would be:

5.6341
32.4381
−6.3444

Note that in some cases, truncating would yield the same result as rounding, but truncation does not round up or round down the digits; it merely cuts off at the specified digit. The truncation error can be twice the maximum error in rounding.

Whenever a finite, y = f(x) is represented by an infinite series. Truncation error is defined on the error caused by truncating a mathematical procedure. The error in the value of f(x) due to deleting of the series after a finite number of terms is called truncation Error.

Example 2:

$Y = f(x) = x + x^2 + x^3 + x^4\ x^5 + \ldots + x^n + x^{n+1}$

The magnitude of truncation error equal the sum of all the discarded term represented by $x^4\ x^5 + \ldots + x^n + x^{n+1} + \ldots$ may be large and it may even exceed the sum of the term required.

Example 3:

The Maclaurin series for e^x is given as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

The series has infinite number of terms but when using this series to calculate e^x, only a finite number of terms can be used. For example, if one uses three terms to calculate e^x, then

$$e^x \approx 1 + x + \frac{x^2}{2!},$$

The truncation error for such an approximation is

$$\text{Truncation error} = e^x - \left(1 + x + \frac{x^2}{2!}\right),$$

$$= \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

Causes of Truncation

With computers, truncation can occur when a decimal number is typecast as an integer; it is truncated to zero decimal digits because integers cannot store real numbers (that are not themselves integers). Truncation may also occur when a number cannot be fully represented due to memory limitations.

Trunction errors in numerical integration are of two kinds:

- *local truncation errors* – the error caused by one iteration, and
- *global truncation errors* – the cumulative error cause by many iterations.

Local truncation error

The **local truncation error** is the error that the increment function, *A*, causes during a given iteration, assuming perfect knowledge of the true solution at the previous iteration.

More formally, the local truncation error, τ_n, at step n is defined by:

$$\tau_n = |y(t_n) - y(t_{n-1}) - h \cdot A(t_{n-1}, y(t_{n-1}), h, f)|$$

Global truncation error

The **global truncation error** is the accumulation of the *local truncation error* over all of the iterations, assuming perfect knowledge of the true solution at the initial time step.

More formally, the global truncation error, e_n, after N steps is defined by:

$$\begin{aligned} e_N &= |y(t_N) - y_N| \\ &= |y(t_N) - (y_0 + h \cdot A(t_0, y_0, h, f) + h \cdot A(t_1, y_1, h, f) + \cdots + h \cdot A(t_{N-1}, y_{N-1}, h, f))| \end{aligned}$$

Techniques for Measuring Error

The quantity, true value – Approximation value is called the error.

In order to determine the accuracy in an approximate solution to a problem, either we find the bound of the

$$\text{Relative Error} = \frac{|Error|}{|True\ Value|} = \frac{|True\ Value - Approximation\ value|}{|True\ value|}$$

or of the

Absolute error = |Error|

Example:

Define the error of an approximation. The traditional definition is

True value = approximation + error

e.g. $\sqrt{2}$ = 1.414214 + Error

1.414213562373095 = 1.414214 + error

Error = 1.414213562373095 - 1.414214 = - 4.376269049512545e^{-7}

π = 3.1415926536 + Error

Error = True value - approximation

e.g if 36.75 is the exact value of a number and if 37 is the approximated value then the error introduced is

|e | = 36.75 – 37 = |-0.25|

e = 0.25

Relative Error

This is error measure relative to the true value.

$$\text{Relative Error} = \frac{|Error|}{|True\ Value|} = \frac{|True\ Value - Approximation\ value|}{|True\ value|}$$

Example:

A resistor labeled as 240Ω is actually 243.32753Ω. What are the absolute and relative errors of the labeled value?

Solution

Absolute error = |True value – Approximation value|

= 243.32753 – 240

= 3.32753

$$\text{Relative Error} = \frac{|Error|}{|True\,Value|} = \frac{3.32753}{243.32753} = 0.0136751069638524 \approx 0.014$$