

**PHY-1111: Physics**  
**Chapter-2 (Vibrations and Waves)**

**by**

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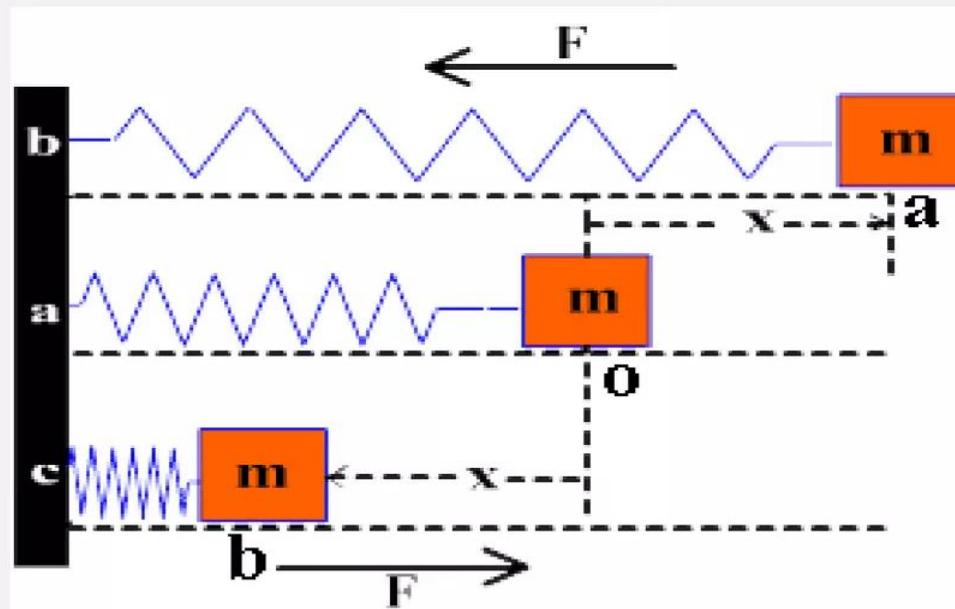
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# Vibrations and Waves

## Simple Harmonic Motion (SHM):

### Definition:

- Such a motion in which acceleration is directly proportional to the displacement and is directed towards the mean position is called **simple harmonic motion(SHM)**.



## Condition FOR SHM:

- The system should have restoring force.
- The system should have inertia.
- The system should be frictionless.

## Hooke's Law

- force that is applied to spring is directly proportional to the displacement.
- If the spring is un stretched, there is no net force on the mass or the system is in equilibrium.
- if the mass is displaced from equilibrium, the spring will exert a restoring force, which is a force that tends to restore it to the equilibrium position.

$$F \propto x$$

where,

F → Elastic force

k → Spring constant

x → Displacement

$$F = kx$$

# Vibrations and Waves

An imaginary circular motion gives a mathematical insight into SHM. Its angular velocity is  $\omega$ .

The connection between SHM and circular motion'

The time period of the motion,  $T = \frac{2\pi}{\omega}$ .

The frequency of the motion,  $f = \frac{1}{T} = \frac{\omega}{2\pi}$ .

Displacement of the SHM,  $s = A \cos(\omega t)$ .

# Vibrations and Waves

## 1.3. Differential Equation of SHM

For a particle vibrating simple harmonically, the general equation of displacement is,

$$y = a \sin (\omega t + \alpha) \quad \dots (1)$$

Here  $y$  is displacement and  $a$  is the amplitude and  $\alpha$  is epoch of the vibrating particle.

Differentiating equation (1) with respect to time

$$\frac{dy}{dt} = a \omega \cos (\omega t + \alpha) \quad \dots (2)$$

Here  $dy/dt$  represents the velocity of the vibrating particle.

Differentiating equation (2) with respect to time

$$\frac{d^2 y}{dt^2} = -a \omega^2 \sin (\omega t + \alpha)$$

But

$$a \sin (\omega t + \alpha) = y$$

∴

$$\frac{d^2 y}{dt^2} = -\omega^2 y$$

or

$$\frac{d^2 y}{dt^2} + \omega^2 y = 0 \quad \dots (3)$$

Here  $d^2 y/dt^2$  represents the acceleration of the particle. Equation (3) represents the differential equation of simple harmonic motion.

## Vibrations and Waves

It also shows that in any phenomenon where an equation similar to equation (3) is obtained, the body executes simple harmonic motion. The general solution of equation (3) is

$$y = a \sin (\omega t + \alpha).$$

Also the time period of a vibrating particle can be calculated from equation (3).

Numerically

$$\omega = \sqrt{\frac{d^2 y/dt^2}{m}}$$

or

$$\omega = \sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$$

or

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

# Vibrations and Waves

## 1.5. Average Kinetic Energy of a Vibrating Particle

The displacement of a vibrating particle is given by

$$y = a \sin (\omega t + \alpha)$$

$$v = \frac{dy}{dt} = a\omega \cos (\omega t + \alpha).$$

If  $m$  is the mass of the vibrating particle, the kinetic energy at any instant

$$= \frac{1}{2} m v^2 = \frac{1}{2} m \cdot a^2 \omega^2 \cos^2 (\omega t + \alpha).$$

The average kinetic energy of the particle in one complete vibration

$$= \frac{1}{T} \int_0^T \frac{1}{2} m a^2 \omega^2 \cos^2 (\omega t + \alpha) dt$$

$$= \frac{1}{T} \cdot \frac{m a^2 \omega^2}{4} \int_0^T 2 \cos^2 (\omega t + \alpha) dt$$

$$= \frac{m a^2 \omega^2}{4T} \int_0^T [1 + \cos 2(\omega t + \alpha)] dt$$

$$= \frac{m a^2 \omega^2}{4T} \left[ \int_0^T dt + \int_0^T \cos 2(\omega t + \alpha) dt \right]$$

## Vibrations and Waves

But

$$\int_0^T \cos 2(\omega t + \alpha) dt = 0$$

$$\therefore \text{Average K.E.} = \frac{ma^2 \omega^2}{4T} \cdot T + 0$$

$$= \frac{ma^2 \omega^2}{4} = \frac{ma^2 (4\pi^2 n^2)}{4}$$
$$= \pi^2 m a^2 n^2$$

where  $m$  is the mass of the vibrating particle,  $a$  is the amplitude of vibration and  $n$  is the frequency of vibration. Also, the average kinetic energy of a vibrating particle is directly proportional to the square of the amplitude.

# Vibrations and Waves

## Total Energy of a Vibrating Particle

$$y = a \sin (\omega t + \alpha)$$

$$\sin (\omega t + \alpha) = \frac{y}{a}$$

$$\cos (\omega t + \alpha) = \sqrt{1 - \frac{y^2}{a^2}} = \sqrt{\frac{a^2 - y^2}{a^2}}$$

$$= \frac{\sqrt{a^2 - y^2}}{a}$$

Velocity       $v = a \omega \cos \omega t = \frac{a \omega \sqrt{a^2 - y^2}}{a}$   
 $= \omega \sqrt{(a^2 - y^2)}$

∴ The kinetic energy of the particle at the instant the displacement is  $y$ ,

$$\begin{aligned} &= \frac{1}{2} m v^2 \\ &= \frac{1}{2} m \cdot \omega^2 (a^2 - y^2) \end{aligned}$$

Potential energy of the vibrating particle is the amount of work done in overcoming the force through a distance  $y$ .

Acceleration       $= -\omega^2 y$

Force                 $= -m\omega^2 y$

(The -ve sign shows that the direction of the acceleration and force are opposite to the direction of motion of the vibrating particle.)

## Vibrations and Waves

∴

$$\begin{aligned}\text{P.E.} &= \int_0^y m \cdot \omega^2 y \cdot dy \\ &= m\omega^2 \cdot \frac{y^2}{2} = \frac{1}{2} m\omega^2 y^2.\end{aligned}$$

Total energy of the particle at the instant the displacement is  $y$   
= K.E + P.E.

$$\begin{aligned}&= \frac{1}{2} m\omega^2 (a^2 - y^2) + \frac{1}{2} m\omega^2 y^2 \\ &= \frac{1}{2} m\omega^2 \cdot a^2 \\ &= \frac{1}{2} m (2\pi n)^2 a^2 \\ &= 2\pi^2 m a^2 n^2.\end{aligned}$$

As the average kinetic energy of the vibrating particle  $= \pi^2 m a^2 n^2$ , the average potential energy  $= \pi^2 m a^2 n^2$ . The total energy at any instant is a constant.

## Other symbols, relationships

Displacement,  $s = A \sin(\omega t) = A \sin(2\pi ft)$

At  $t = 0$ , the object passes through its equilibrium position.

$$v = \frac{d}{dt}[A \sin(\omega t)] = \omega A \cos(\omega t)$$

$$a = \frac{d}{dt}[-\omega A \cos(\omega t)] = -\omega^2 A \sin(\omega t)$$

Phase relationship,  $\phi$        $s = A \sin(\omega t + \phi)$

## Vibrations and Waves

**Example 1.1.** For a particle vibrating simple harmonically, the displacement is 12 cm at the instant the velocity is 5 cm/s and the displacement is 5 cm at the instant the velocity is 12 cm/s. Calculate (i) amplitude, (ii) frequency and (iii) time period.

The velocity of a particle executing SHM,

$$v = \frac{dy}{dt} = \omega \sqrt{a^2 - y^2}$$

In the first case,

$$v_1 = \omega \sqrt{a^2 - y_1^2}$$

Here  $v_1 = 5 \text{ cm/s}$ ,  $y_1 = 12 \text{ cm}$ .

$$5 = \omega \sqrt{a^2 - 144} \quad \dots (1)$$

In the second case

$$v_2 = \omega \sqrt{a^2 - y_2^2}$$

Here  $v_2 = 12 \text{ cm/s}$ ,  $y_2 = 5 \text{ cm}$

$$12 = \omega \sqrt{a^2 - 25} \quad \dots (2)$$

## Vibrations and Waves

Dividing (2) by (1) and squaring

$$\frac{144}{25} = \frac{a^2 - 25}{a^2 - 144}$$

$$a = 13 \text{ cm}$$

The amplitude is 13 cm.

Substituting the value of  $a = 13 \text{ cm}$  in equation (1)

$$5 = \omega \sqrt{(13)^2 - 144}$$

or

$$\omega = 1 \text{ radian/s}$$

The frequency  $n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \text{ hertz}$

Time period  $T = \frac{1}{n} = 2\pi \text{ seconds.}$

## 3.1. Free Vibrations

When the bob of a simple pendulum (in vacuum) is displaced from its mean position and left, it executes simple harmonic motion. The time period of oscillation depends only on the length of the pendulum and the acceleration due to gravity at the place. The pendulum will continue to oscillate with the same time period and amplitude for any length of time. In such cases there is no loss of energy by friction or otherwise. In all similar cases, the vibrations will be undamped free vibrations. The amplitude of swing remains constant.

# Vibrations and Waves

## 3.2. Undamped Vibrations

For a simple harmonically vibrating particle, the kinetic energy for displacement  $y$ , is given by

$$\frac{1}{2} m \left( \frac{dy}{dt} \right)^2$$

At the same instant, the potential energy of the particle is  $\frac{1}{2} Ky^2$  where  $K$  is the restoring force per unit displacement.

The total energy at any instant,

$$= \frac{1}{2} m \left( \frac{dy}{dt} \right)^2 + \frac{1}{2} Ky^2$$

For an undamped harmonic oscillator, this total energy remains constant.

$$\therefore \frac{1}{2} m \left( \frac{dy}{dt} \right)^2 + \frac{1}{2} Ky^2 = \text{constant} \quad \dots (1)$$

Differentiating equation (1) with respect to time,

# Vibrations and Waves

$$m \frac{d^2 y}{dt^2} + Ky = 0 \quad \dots (2)$$

$$\frac{d^2 y}{dt^2} + \left(\frac{K}{m}\right)y = 0 \quad \dots (3)$$

Equation (3) is similar to the equation

$$\frac{d^2 y}{dt^2} + \omega^2 y = 0 \quad \dots (4)$$

Here

$$\omega^2 = \left(\frac{K}{m}\right)$$

The solution for equation (4) is

$$y = a \sin (\omega t - \alpha)$$
$$\therefore y = a \sin \left[ \sqrt{\frac{K}{m}} t - \alpha \right]$$

The frequency of oscillation is

$$n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{K}{m}}$$

Thus, in the case of undamped free vibrations, the differential equation is

$$\frac{d^2 y}{dt^2} + \left(\frac{K}{m}\right)y = 0 \quad \dots (5)$$

This is only an ideal case. In the first chapter, for the motion of a pendulum, loaded spring, LC circuit etc., it has been assumed that the vibrations are free and undamped.

# Vibrations and Waves

## 3.3. Damped Vibrations

In actual practice, when the pendulum vibrates in air medium, there are frictional forces and consequently energy is dissipated in each vibration. The amplitude of swing decreases continuously with time and finally the oscillations die out. Such vibrations are called **free damped** vibrations. The dissipated energy appears as heat either within the system itself or in the surrounding medium. The dissipative force due to friction etc. (resistance in LCR circuit) is proportional to the velocity of the particle at that instant. Let  $\mu \frac{dy}{dt}$  be the dissipative force due to friction etc.

This term is to be introduced in equation (2).

Therefore, the differential equation in the case of Free-damped vibrations is,

$$m \frac{d^2 y}{dt^2} + Ky + \mu \frac{dy}{dt} = 0 \quad \dots (6)$$

# Vibrations and Waves

or

$$\frac{d^2y}{dt^2} + \left(\frac{\mu}{m}\right) \frac{dy}{dt} + \left(\frac{K}{m}\right)y = 0 \quad \dots (7)$$

This equation is similar to a general differential equation,

$$\frac{d^2y}{dt^2} + 2b \frac{dy}{dt} + k^2 y = 0 \quad \dots (8)$$

The solution of this equation is

$$y = ae^{-bt} \sin(\omega t - \alpha) \quad \dots (9)$$

The general solution of equation (7) is also given by

$$y = A e^{(-b + \sqrt{b^2 + k^2})t} + B e^{(-b - \sqrt{a^2 - k^2})t}$$

Here

$$b = \frac{\mu}{2m} \text{ and } k^2 = \frac{K}{m}$$

and

$$\omega = \sqrt{k^2 - b^2}$$

or

$$\omega = \sqrt{\frac{K}{m} - \frac{\mu^2}{4m^2}}$$

or

$$n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{k^2 - b^2}$$

# Vibrations and Waves

## 3.5 ✓ Forced Vibrations

The time period of a body executing simple harmonic motion depends on the dimensions of the body and its elastic properties. The vibrations of such a body die out with time due to dissipation of energy. If some external periodic force is constantly applied on the body, it continues to oscillate under the influence of such external forces. Such vibrations of the body are called *forced vibrations*.

Initially, the amplitude of the swing increases, then decreases with time, becomes minimum and again increases. This will be repeated if the external periodic force is constantly applied on the system. In such cases the body will finally be forced to vibrate with the same frequency as that of the applied force.

The frequency of the forced vibration is different from the natural frequency of vibration of the body. The amplitude of the forced vibration of the body depends on the difference between the natural frequency and the frequency of

the applied force. The amplitude will be large if difference in frequencies is small and *vice versa*.

# Vibrations and Waves

## 4.1. Wave Motion

Wave motion is a form of disturbance which travels through the medium due to the repeated periodic motion of the particles of the medium about their mean positions, the disturbance being handed over from one particle to the next. When a stone is dropped into a pond containing water, waves are produced at the point where the stone strikes the water in the pond. The waves travel outward, the particles of water vibrate only up and down about their mean positions. Water particles do not travel along with the wave. Similarly when a tuning fork is set into vibration, it produces waves in air. The wave travels from one particle to the next but the particles of air vibrate about their mean positions.

It is essential to understand the concept of wave motion in the study of various branches in Physics. Wave motion, in general, refers to the transfer of energy from one point to another point of the medium. Transference of various forms of energy like sound, heat, light, X-rays,  $\gamma$ -rays, radio-waves etc. takes place in the form of wave motion. For the transference of energy through a medium, the medium must possess the properties of *elasticity, inertia and negligible frictional resistance*.

# Vibrations and Waves

## 4.3. Characteristics of Wave Motion

1. Wave motion is a disturbance produced in the medium by the repeated periodic motion of the particles of the medium.
2. Only the wave travels forward whereas the particles of the medium vibrate about their mean positions.
3. There is a regular phase change between the various particles of the medium. The particle ahead starts vibrating a little later than a particle just preceding it.
4. The velocity of the wave is different from the velocity with which the particles of the medium are vibrating about their mean positions. The wave travels with a uniform velocity whereas the velocity of the particles is different at different positions. It is maximum at the mean position and zero at the extreme position of the particles.

There are two types of wave motions :

(i) Transverse and (ii) Longitudinal.

Sound waves are longitudinal waves and light waves are transverse waves.

# Vibrations and Waves

## 4.4. Transverse Wave Motion

In this type of wave motion, the particles of the medium vibrate at right angles to the direction of propagation of the wave.

To understand the propagation of transverse waves in a medium consider nine particles of the medium and the circle of reference (Fig. 4.1). The particles are vibrating about their mean positions up and down and the wave is travelling from left to right. The disturbance takes  $T/8$  seconds to travel from one particle to the next.

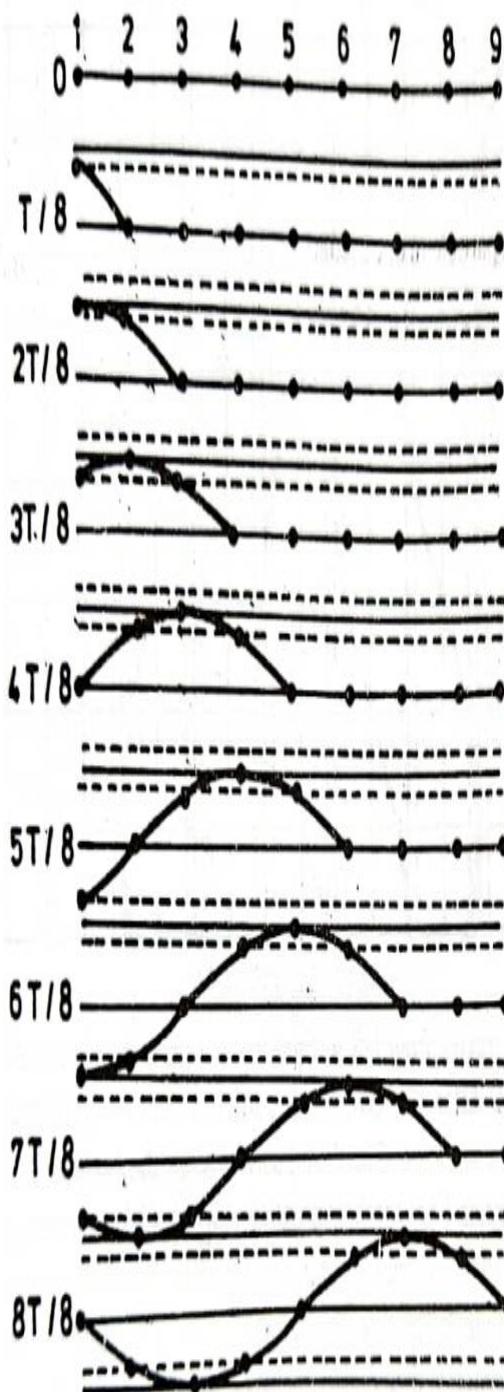
(1) At  $t = 0$ , all the particles are at their mean position.

(2) After  $T/8$  seconds, particle 1 travels a certain distance upward and the disturbance reaches particle 2.

(3) After  $2T/8$  seconds, particle 1 has reached its extreme position and the disturbance has reached particle 3.

(4) After  $3T/8$  seconds, particle 1 has completed  $3/8$  of its vibration and the disturbance has reached particle 4. The positions of particles 2 and 3 are also shown in Fig. 4.1.

(5) In this way after  $T/2$  seconds, particle 1 has come back to its mean position and the particles 2, 3 and 4 are at the positions shown in the diagram.



# Vibrations and Waves

The disturbance has reached particle 5.

In this way the process continues and the positions of the particles after  $5T/8$ ,  $6T/8$ ,  $7T/8$  and  $T$  seconds are shown in the diagram.

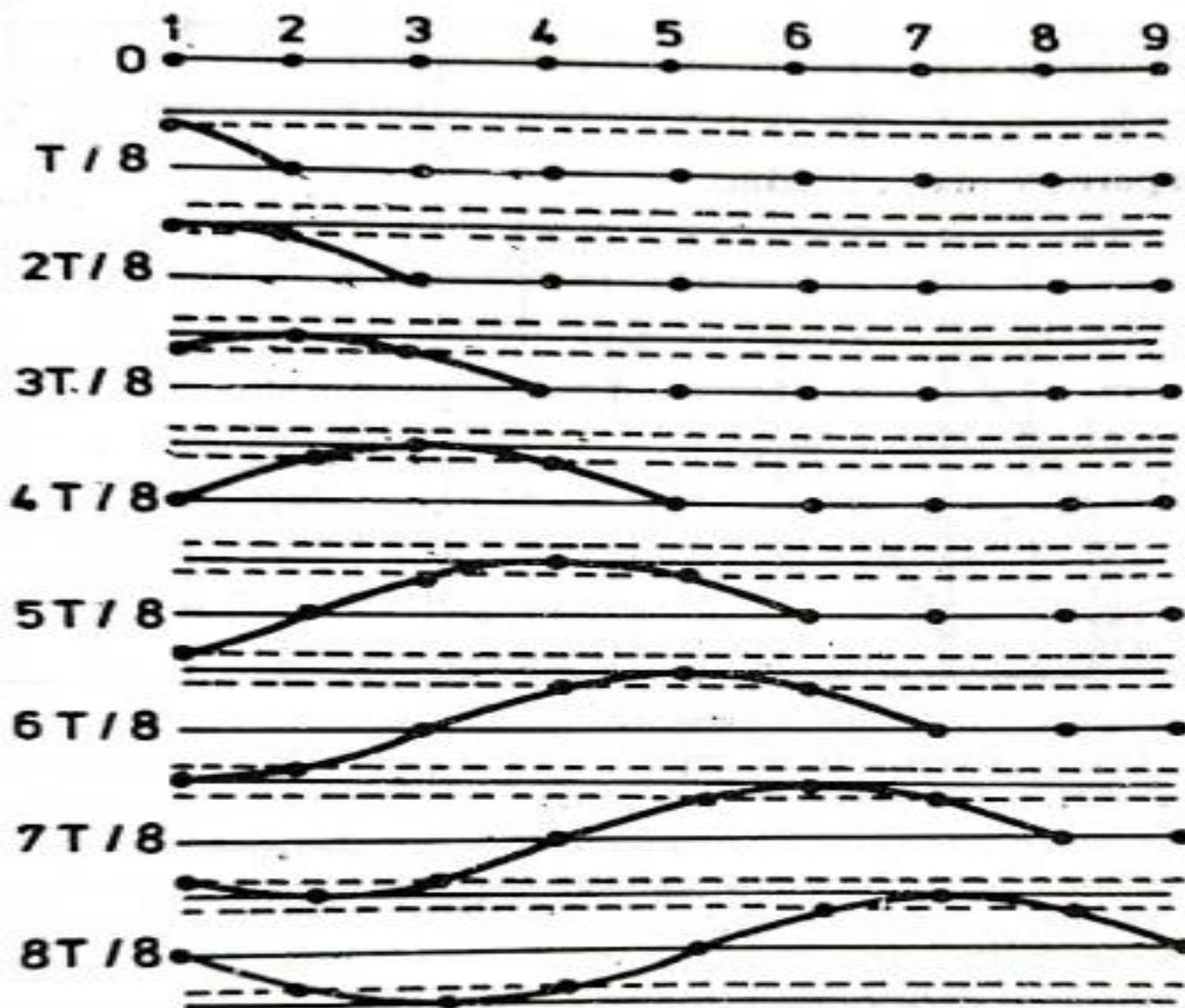


Fig. 4.1.

# Vibrations and Waves

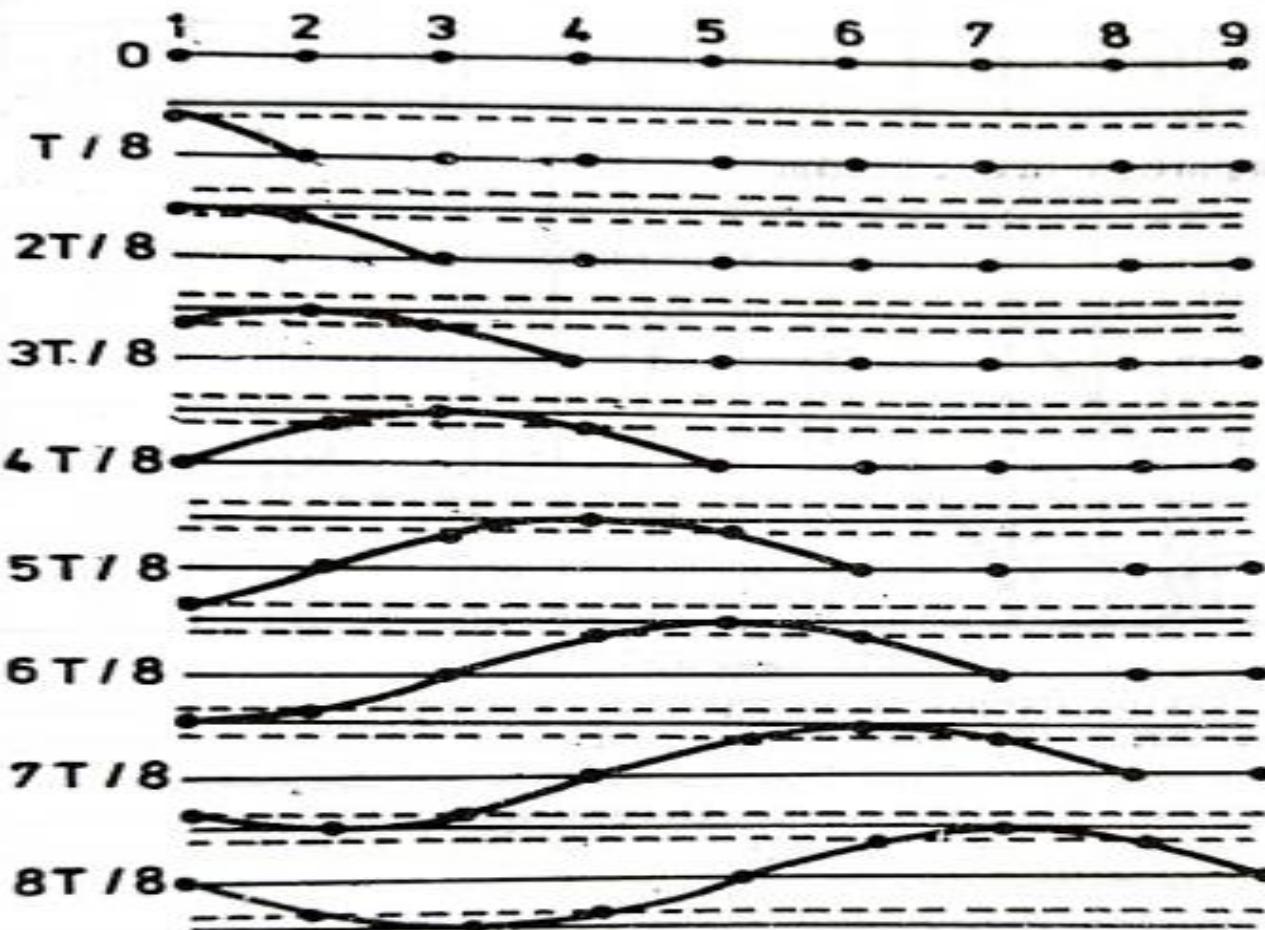


Fig. 4.1.

After  $T$  seconds, the particles 1, 5 and 9 are at their mean positions. The wave has reached particle 9. Particles 1 and 9 are in the same phase. The wave has travelled a distance between particles 1 and 9 in the time in which the particle 1 has completed one vibration.

The top point on the wave at the maximum distance from the mean position is called *crest*, while the point at the maximum distance below the mean position is called *trough*. Thus in a transverse wave, crests and troughs are alternately formed. The contour of the displaced particles of the medium represents the wave. In the case of transverse (or longitudinal) progressive waves, this contour continuously changes position in space and the wave seems to advance in the direction of propagation.

# Vibrations and Waves

## 4.5. Longitudinal Wave Motion

In this type of wave motion, particles of the medium vibrate along the direction of propagation of the wave.

Consider nine particles of the medium and the circle of reference (Fig. 4.2).

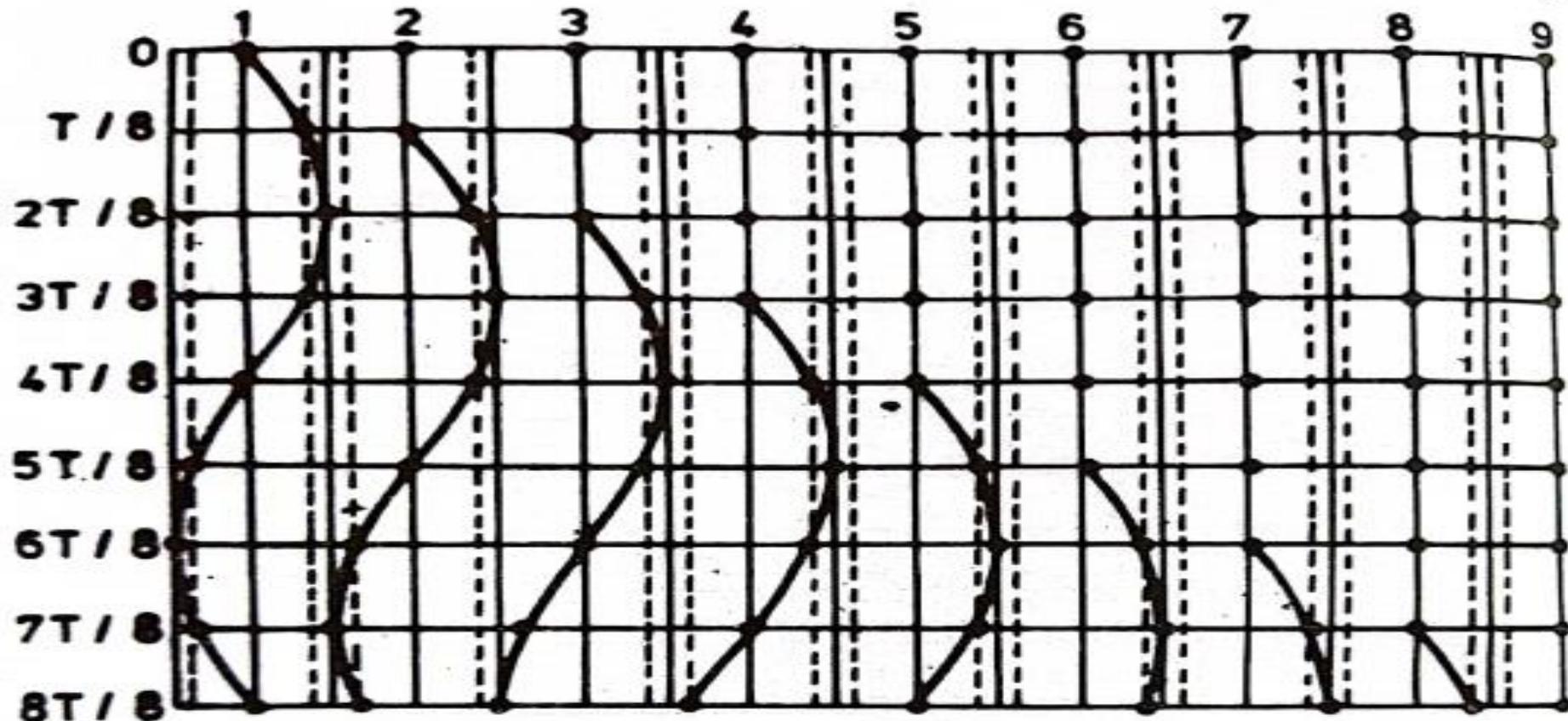
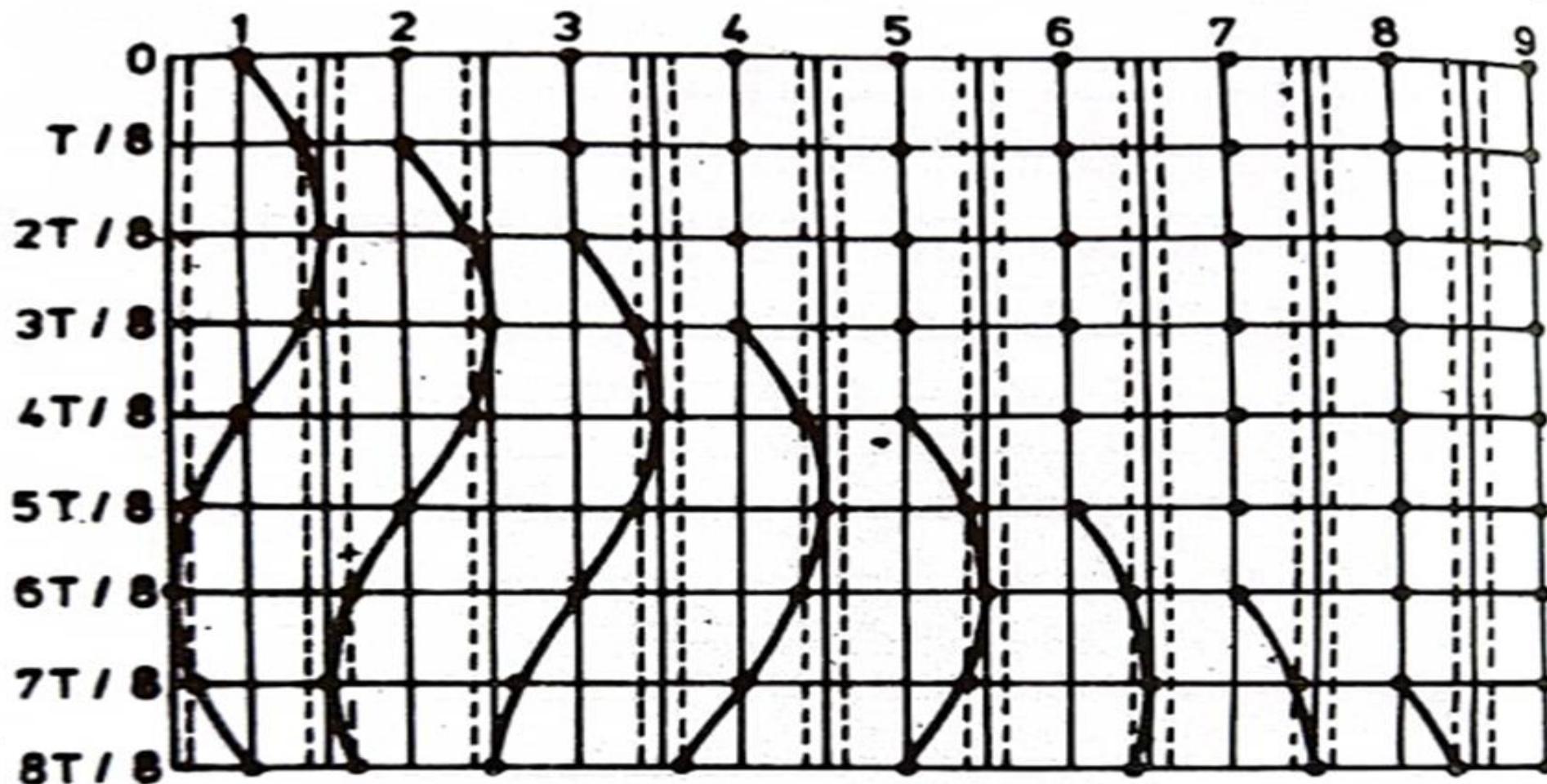


Fig. 4.2.

The wave travels from left to right and the particles vibrate about their mean positions. After  $T/8$  seconds, the particle 1 goes to the right and completes  $1/8$  of its vibration. The disturbance reaches the particle 2. After  $T/4$  seconds the particle 1 has reached its extreme right position and completes  $1/4$  of its vibration and the particle 2 completes  $1/8$  of its vibration. The disturbance reaches the particle 3. The process continues.

# Vibrations and Waves



After one complete time period, the positions of the various particles is as shown in the diagram. The wave has reached particle 9. Here 1 and 9 are again in the same phase. Here particles 1, 5 and 9 are at their mean positions. The particles 1 and 3 are close to the particle 2. This is the position of condensation. Similarly particles 9 and 8 are close to the particle 7. This is also the position of condensation or compression. On the other hand, particles 4 and 6 are far away from the particle 5. This is the position of rarefaction. Hence in a longitudinal wave motion, condensations and rarefactions are alternately formed.

# Vibrations and Waves

## 4.6. Definitions

**Wavelength.** It is the distance travelled by the wave in the time in which the particle of the medium completes one vibration. It is also defined as the distance between two nearest particles in the same phase.

The distance  $AB$  (Fig. 4.3) is equal to the wavelength  $\lambda$ .

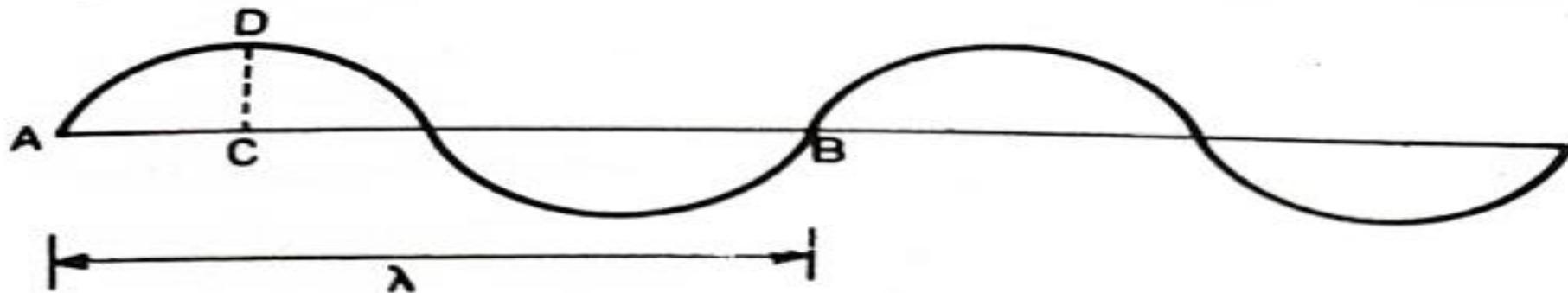


Fig. 4.3.

**Frequency.** It is the number of vibrations made by a particle in one second.

**Amplitude.** It is the maximum displacement of the particle from its mean position of rest. In the diagram  $CD$  is the amplitude.

**Time period.** It is the time taken by a particle to complete one vibration.

Suppose frequency =  $n$

Time taken to complete  $n$  vibrations = 1 second.

Time taken to complete 1 vibration =  $\frac{1}{n}$  second.

From the definition of time period, time taken to complete one vibration is the time period ( $T$ )

∴

$$T = \frac{1}{n} \quad \text{or} \quad nT = 1$$

∴

$$\text{Frequency} \times \text{Time period} = 1$$

# Vibrations and Waves

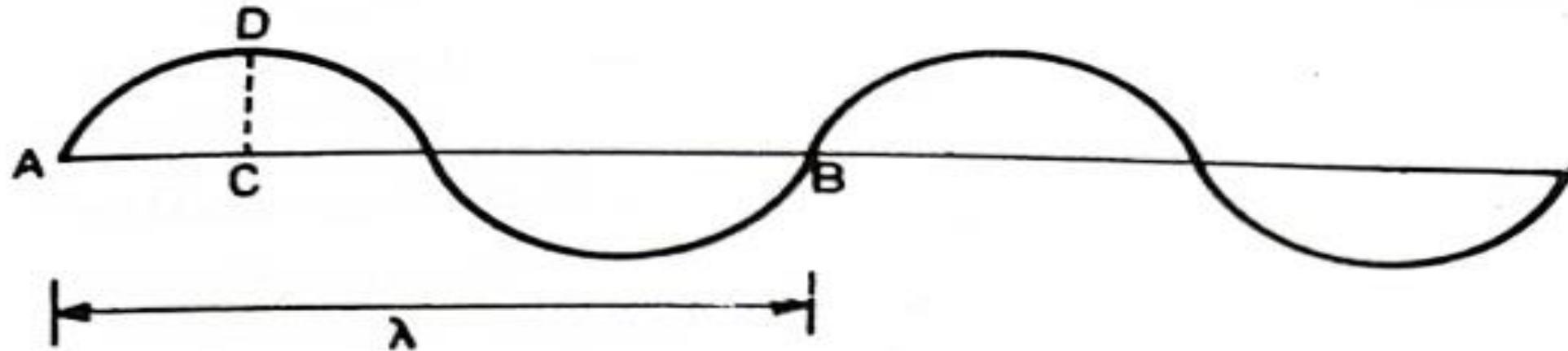


Fig. 4.3.

**Vibration.** It is the to and fro motion of a particle from one extreme position to the other and back again. It is also equal to the motion of a particle from the mean position to one extreme position, then to the other extreme position and finally back to the mean position.

## Vibrations and Waves

### 4.7. Relation between Frequency and Wavelength

Velocity of the wave is the distance travelled by the wave in one second.

$$\text{Velocity} = \frac{\text{Distance}}{\text{Time}}$$

Wavelength ( $\lambda$ ) is the distance travelled by the wave in one time period ( $T$ ).

$$\text{Velocity} = \frac{\text{Wavelength}}{\text{Time period}} = \frac{\lambda}{T}$$

**But, frequency  $\times$  time period = 1**

$$n \times T = 1$$

$$T = \frac{1}{n}$$

$$v = \frac{\lambda}{T} = \frac{\lambda}{\frac{1}{n}}$$

$$v = n \lambda$$

## Vibrations and Waves

**Example 4.1.** If the frequency of a tuning fork is 400 and the velocity of sound in air is 320 metres/s, find how far sound travels while the fork completes 30 vibrations.

Here,  $n = 400$ ,  $v = 320$  metres/second,  $\lambda = ?$

$$v = n \lambda$$

or 
$$\lambda = \frac{v}{n} = \frac{320}{400} = 0.8 \text{ metre}$$

$\therefore$  Distance travelled by the wave when the fork completes 1 vibration  
 $= 0.8 \text{ metre}$

Distance travelled by the wave when the fork completes 30 vibrations  
 $= 0.8 \times 30 = 24 \text{ metres.}$

## Vibrations and Waves

### 4.8. Properties of Longitudinal Progressive Waves

1. The particles of the medium vibrate simple harmonically along the direction of propagation of the wave.
2. All the particles have the same amplitude, frequency and time period.
3. There is a gradual phase difference between the successive particles.
4. All the particles vibrating in phase will be at a distance equal to  $n\lambda$ . Here  $n = 1, 2, 3$  etc. It means the minimum distance between two particles vibrating in phase is equal to the wavelength.
5. The velocity of the particle is maximum at their mean position and it is zero at their extreme positions.
6. When the particle moves in the same direction as the propagation of the wave, it is in a region of compression.
7. When the particle moves in a direction opposite to the direction of propagation of the wave, it is in a region of rarefaction.
8. When the particle is at the mean position, it is a region of maximum compression or rarefaction.

## Vibrations and Waves

### 4.12. Differential Equation of Wave Motion

The general equation of a simple harmonic wave is,

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad \dots (1)$$

Differentiating equation (1) with respect to time,

$$\frac{dy}{dt} = \frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \quad \dots (2)$$

Differentiating equation (2) with respect to time,

$$\frac{d^2y}{dt^2} = -\frac{4\pi^2 av^2}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt - x) \quad \dots (3)$$

To find the value of compression, differentiate equation (1) with respect to  $x$ ,

$$\frac{dy}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \quad \dots (4)$$

To find the rate of change of compression with respect to distance, differentiate equation (4) with respect to  $x$ ,

$$\frac{d^2y}{dx^2} = -\frac{4\pi^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt - x) \quad \dots (5)$$

# Vibrations and Waves

From equations (2) and (4)

$$\frac{dy}{dt} = -v \frac{dy}{dx} \dots (6)$$

From equations (3) and (5)

$$\frac{d^2 y}{dt^2} = v^2 \frac{d^2 y}{dx^2} \dots (7)$$

Equation (7) represents the differential equation of wave motion.

The general differential equation of wave motion can be written as

$$\frac{d^2 y}{dt^2} = K \frac{d^2 y}{dx^2} \dots (8)$$

Here

$$K = v^2$$

or

$$v = \sqrt{K}$$

Thus, knowing the value of  $K$ , the value of the wave velocity can be calculated.

\*\*Where  $K$  is the restoring force.

# Vibrations and Waves

**Particle velocity:** Particle velocity is the velocity with which the individual particles of a medium move when traversed by a wave.

## How do you find a particles velocity?

The instantaneous velocity  $v(t)$  of a particle is the derivative of the position with respect to time. That is,  $v(t)=\frac{dx}{dt}$ . This derivative is often written as  $\dot{x}(t)$ , or simply as  $\dot{x}$ .

**wave velocity:** distance traversed by a periodic, or cyclic, motion per unit time (in any direction). Wave velocity in common usage refers to speed, although, properly, velocity implies both speed and direction.

## Difference between wave velocity and particle velocity:

When the tiny particle is said to travel in a medium, then the velocity associated with the particle is known as the particle velocity. On the other hand, the waves are associated with the oscillatory motion through the medium; therefore, the velocity associated with such waves in space is called wave velocity.

## What is difference between wave velocity and particle velocity?

- The velocity with which the wave travels in space is called the wave velocity. It is defined as  $v=frequency \times wavelength$ .
- Particle velocity is the velocity with which the particles are vibrating to transfer the energy in form of a wave.
- The wave velocity remains constant (provided the density of medium and frequency of source is constant), whereas particle velocity depends on the time (for a particular particle) or depends on where the particle is (for a particular time).

# Vibrations and Waves

## 4.13. Particle Velocity and Wave Velocity

The equation for a simple harmonic wave is given by

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad \dots (1)$$

Here  $v$  is the velocity of the wave and  $y$  is the displacement of the particle.  
The velocity of the particle  $U = dy/dt$ .

∴ Differentiating equation (1) with respect to time  $t$ ,

$$U = \frac{dy}{dt} = \frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \quad \dots (2)$$

The maximum value of the particle velocity is

$$U_{\max} = \frac{2\pi av}{\lambda} \quad \dots (3)$$

$$\therefore [\text{Maximum Particle Velocity}] = \frac{2\pi a}{\lambda} [\text{Wave Velocity}]$$

To find the particle acceleration, differentiate equation (2) with respect to time

$$f = \frac{d^2 y}{dt^2} = -\frac{4\pi^2 av^2}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt - x) \quad \dots (4)$$

$$f = -\frac{4\pi^2 v^2}{\lambda^2} \left[ a \sin \frac{2\pi}{\lambda} (vt - x) \right]$$

$$\therefore f = -\left( \frac{4\pi^2 v^2}{\lambda^2} \right) y$$

# Vibrations and Waves

The acceleration is maximum when  $y = a$

$$\therefore f_{\max} = - \left( \frac{4\pi^2 v^2}{\lambda^2} \right) a \quad \dots (5)$$

The negative sign shows that the acceleration of the particle is directed towards its mean position.

Differentiating equation (1) with respect to  $x$ :

$$\frac{dy}{dx} = - \frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \quad \dots (6)$$

$dy/dx$  represents the slope of the displacement curve.

From equations (2) and (6)

$$U = \frac{dy}{dt} = -v \left( \frac{dy}{dx} \right) \quad \dots (7)$$

$$\therefore \begin{bmatrix} \text{Particle velocity} \\ \text{at any instant} \end{bmatrix} = \begin{bmatrix} \text{Wave velocity} \end{bmatrix} \begin{bmatrix} \text{Slope of the displacement} \\ \text{curve at that instant} \end{bmatrix}$$

## Vibrations and Waves

The acceleration is maximum when  $y = a$

∴

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## Vibrations and Waves

### 4.15. Energy of a Progressive Wave

In the case of a progressive wave, new waves are continuously formed at the head of the wave. This means that there is continuous transfer of energy in the direction of propagation of the wave. This energy is supplied from the source. The energy transferred per second also corresponds to the energy possessed by the particles in a length  $v$ , where  $v$  is the velocity of the wave. The energy of the particles is partly kinetic and partly potential. The kinetic energy is due to the velocity of the vibrating particles. For a particle executing simple harmonic motion, the velocity is maximum at the mean position and it is zero at the extreme positions. Consequently the kinetic energy of the particle at the mean position is maximum and zero at the extreme positions. Similarly the particles also possess potential energy due to their displacements from their mean positions. At the extreme positions, the potential energy of a particle is maximum and at the mean position the potential energy is minimum.

# Vibrations and Waves

In longitudinal wave motion, compressions and rarefactions are formed. The energy distribution is not uniform over the wave. At points of no velocity there is no compression and the particles do not possess energy at these points. At points of maximum velocity there is compression and the particles possess maximum energy. However, in the case of progressive wave motion, *there is no transfer of the medium in the direction of propagation of the wave, but there is always transfer of energy in the direction of propagation of the wave.*

## Analytical Treatment .

The equation of a simple harmonic wave is

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad \dots (1)$$

The particle velocity  $U$  at any instant can be obtained by differentiating equation (1) with respect to time

$$\therefore U = \frac{dy}{dt} = \frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \quad \dots (2)$$

The acceleration of the particle at that instant,

$$f = \frac{dU}{dt}$$

$$\therefore f = \frac{d^2 y}{dt^2}$$

Differentiating equation (2) with respect to time,

$$f = \frac{d^2 y}{dt^2} = - \frac{4\pi^2 av^2}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt - x) \quad \dots (3)$$

The -ve sign shows that acceleration is directed towards the mean position.

# Vibrations and Waves

## Potential Energy

To move the particle from its mean position to a distance  $y$ , work has to be done against acceleration.

Work done for a displacement  $dy$   
 $= F dy$

Let  $\rho$  be the density of the medium

Work done per unit volume for a displacement  $dy$

$$= \rho \left( + \frac{4\pi^2 a v^2}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt - x) \right) dy$$

Total work done for a displacement  $y$

$$= \int_0^y \rho \left( + \frac{4\pi^2 a v^2}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt - x) \right) dy$$

But

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

∴ Potential energy per unit volume

$$= \left( \frac{4 \pi^2 \rho v^2}{\lambda^2} \right) \int_0^y y dy$$

# Vibrations and Waves

$$\begin{aligned} &= \frac{4\pi^2 \rho v^2 y^2}{2\lambda^2} \\ \text{P.E.} &= \frac{2\pi^2 \rho v^2 y^2}{\lambda^2} \\ \text{P.E.} &= \frac{2\pi^2 \rho v^2 \cdot a^2}{\lambda^2} \sin^2 \left[ \frac{2\pi}{\lambda} (vt - x) \right] \quad \dots (4) \end{aligned}$$

K.E. per unit volume  $= \frac{1}{2} \rho U^2$

$$\begin{aligned} \text{K.E.} &= \frac{1}{2} \rho \left[ \frac{2\pi a v}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \right]^2 \\ \text{K.E.} &= \frac{2\pi^2 \rho v^2 a^2}{\lambda^2} \cos^2 \left[ \frac{2\pi}{\lambda} (vt - x) \right] \quad \dots (5) \end{aligned}$$

Total energy per unit volume,

$$E = \text{P. E.} + \text{K.E.}$$

$$= \frac{2\pi^2 \rho v^2 a^2}{\lambda^2} \left[ \sin^2 \frac{2\pi}{\lambda} (vt - x) + \cos^2 \frac{2\pi}{\lambda} (vt - x) \right]$$

$$E = \frac{2\pi^2 \rho v^2 a^2}{\lambda^2} \quad \dots (6)$$

But

$$v = n\lambda$$

$$\therefore E = 2\pi^2 \rho n^2 a^2 \quad \dots (7)$$

The average kinetic energy per unit volume and the average potential energy per unit volume are equal and each is equal to half the total energy per unit volume.

## Vibrations and Waves

The average kinetic energy per unit volume and the average potential energy per unit volume are equal and each is equal to half the total energy per unit volume.

$$\text{Average K.E. per unit volume} = \frac{1}{2} \rho n^2 a^2 v^2 \quad \dots (8)$$

$$\text{Average P.E. per unit volume} = \frac{1}{2} \rho n^2 a^2 v^2 \quad \dots (9)$$

Suppose that the area of cross section of a parallel beam of radiation is 1 unit and the velocity of the wave is  $v$ .

$$\text{Volume} = 1 \times v = v$$

∴ Energy transfer per unit area per second.

$$\begin{aligned} &= E \times v \\ &= 2\pi^2 \rho n^2 a^2 v \end{aligned} \quad \dots (10)$$

Energy transfer per second is also called energy current per unit area of cross section.

## Vibrations and Waves

**Example 4.2.** When a simple harmonic wave is propagated through a medium, the displacement of a particle (in cm) at any instant of time is given by

$$y = 10 \sin \frac{2\pi}{100} (36000 t - 20)$$

Calculate, the amplitude of the vibrating particle, wave velocity, wavelength, frequency and time period.

The general equation of a simple harmonic wave is,

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad \dots (1)$$

Here  $y = 10 \sin \frac{2\pi}{100} (36000 t - 20) \quad \dots (2)$

Comparing equations (1) and (2)

$$a = 10 \text{ cm}$$

$$\lambda = 100 \text{ cm}$$

Wave velocity  $v = 36000 \text{ cm/s}$

Frequency  $n = \frac{v}{\lambda} = \frac{36000}{100}$

or

$$n = 360 \text{ hertz}$$

Time period  $T = \frac{1}{n} = \frac{1}{360} \text{ second.}$

# Vibrations and Waves

**Example 4.4.** The velocity of a simple harmonic wave is 30 cm/s. At a time  $t=0$  the displacement of a particle is given by

$$y = 4 \sin 2\pi \left( \frac{x}{100} \right)$$

Find the equation for the displacement at a time  $t=2$  s.

The general equation of a simple harmonic wave is

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

$$y = a \sin \left( \frac{2\pi t}{T} - \frac{2\pi x}{\lambda} \right)$$

When

$$t = 0$$

$$y = a \sin \left( -\frac{2\pi x}{\lambda} \right)$$

or

$$y = -a \sin \frac{2\pi x}{\lambda}$$

... (1)

At  $t = 0$ , the given equation is

$$y = 4 \sin 2\pi \left( \frac{x}{100} \right)$$

... (2)

Comparing equations (1) and (2)

$$a = -4 \text{ and } \lambda = 100 \text{ cm}$$

At time

$$t = 2 \text{ s}$$

$$y = a \sin \frac{2\pi}{\lambda} (vt - x)$$

Here

$$a = -4 \text{ cm}, \lambda = 100 \text{ cm}, t = 2 \text{ s}$$

$$y = -4 \sin \frac{2\pi}{100} (30 \times 2 - x)$$

$$y = -4 \sin \left[ \frac{6\pi}{5} - 2\pi \left( \frac{x}{100} \right) \right]$$

or

$$y = 4 \sin \left[ 2\pi \left( \frac{x}{100} \right) - \frac{6\pi}{5} \right]$$

## Vibrations and Waves

**Example 4.6.** A source of sound has a frequency of 512 Hz and an amplitude of 0.25 cm. What is the flow of energy across a square cm per second, if the velocity of sound in air is 340 m/s and the density of air is 0.00129 g/cm<sup>3</sup>?

Here

$$n = 512 \text{ Hz}$$

$$a = 0.25 \text{ cm}$$

$$\rho = 0.00129 \text{ g/cm}^3$$

$$v = 340 \text{ m/s} = 34000 \text{ cm/s}$$

Total energy per unit volume

$$= 2\pi^2 \rho n^2 a^2$$

Energy transferred across a sq-cm per second

$$= 2\pi^2 \rho n^2 a^2 v$$

$$= 2 \times (3.14)^2 \times 0.00129 \times (512)^2 \times (0.25)^2 \times 34000$$

$$= 1.417 \times 10^7 \text{ ergs/cm}^2 - \text{s.}$$

# Vibrations and Waves

## 9.1. Doppler Effect

It is commonly observed that the pitch of a note apparently changes when either the source or the observer are in motion relative to each other. When the source approaches the observer or when the observer approaches the source or when both approach each other the apparent pitch is higher than the actual pitch of the sound produced by the source. Similarly, when the source moves away from the observer or when the observer moves away from the source or when both move away from each other, the apparent pitch is lower than the actual pitch of the sound produced by the source.

Suppose a person is standing on a platform. The apparent pitch of the whistle of the engine increases, when the engine is approaching the person. When the engine moves away from the person, the apparent pitch of the whistle of the engine decreases. This apparent change in the pitch due to the relative motion between the source and the observer is called **Doppler Effect**.

Doppler effect in sound is asymmetric. When the source moves towards the observer with a certain velocity, the apparent pitch is different to the case when the observer is moving towards the source with the same velocity. But it is not so in the case of light. Doppler effect in light is symmetric. The apparent pitch in different cases is calculated in the subsequent articles.

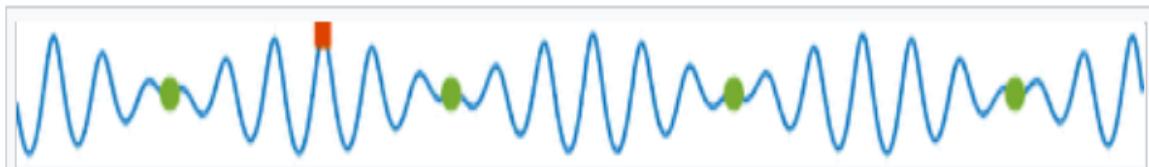
# Vibrations and Waves

The phase velocity of a wave is the rate at which the wave propagates in any medium. This is the velocity at which the phase of any one frequency component of the wave travels. For such a component, any given phase of the wave (for example, the crest) will appear to travel at the phase velocity. The phase velocity is given in terms of the wavelength  $\lambda$  (lambda) and time period  $T$  as

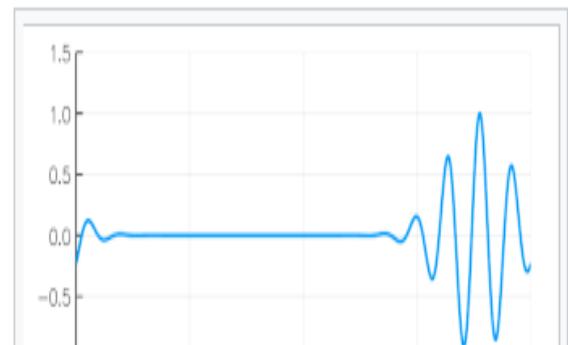
$$v_p = \frac{\lambda}{T}.$$

Equivalently, in terms of the wave's angular frequency  $\omega$ , which specifies angular change per unit of time, and wavenumber (or angular wave number)  $k$ , which represent the angular change per unit of space,

$$v_p = \frac{\omega}{k}.$$



Frequency dispersion in groups of gravity waves on the surface of deep water. The red square moves with the phase velocity, and the green circles propagate with the group velocity. In this deep-water case, the phase velocity is twice the group velocity. The red square overtakes two green circles when moving from the left to the right of the figure.

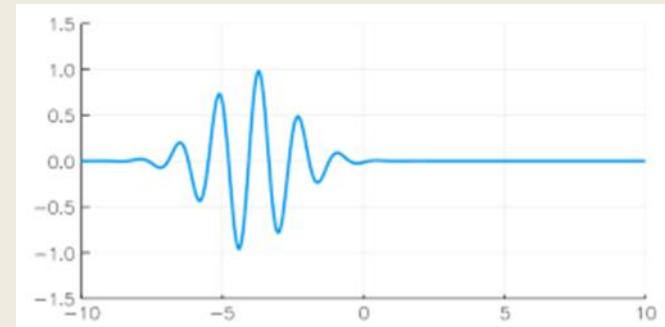


# Vibrations and Waves

What is an example of a group velocity?

What is Group Velocity? The Group Velocity of a Wave is defined as the Velocity at which an entire envelope of Waves moves through a medium. A most common Example, in this case, can be that of throwing stones in a water body which causes multiple Waves on the surface of water.

The group velocity is velocity of the envelope. For two waves group velocity is defined by  $v_g = d\omega/dk$ , where  $d\omega = \omega_1 - \omega_2$  and  $dk = k_1 - k_2$ . This expression for group velocity is the slope of a frequency versus wavenumber graph.



What is called phase velocity?

The phase velocity is defined as the velocity for a single-wavelength wave, whereas the group velocity is defined as the velocity for a packet of waves in which the waves vary in wavelength.

# Vibrations and Waves

## Group Velocity And Phase Velocity Relation

The group **velocity** is directly proportional to phase velocity. This means-

- When group velocity increases, proportionately phase velocity will also increase.
- When phase velocity increases, proportionately group velocity will also increase.

Thus, we see the direct dependence of group velocity on phase velocity and vice-versa.

## Relation Between Group Velocity And Phase Velocity Equation

For the amplitude of wave packet let-

- $\omega$  is the angular velocity given by  $\omega=2\pi f$
- $k$  is the angular **wave number** given by  
$$k = \frac{2\pi}{\lambda}$$
- $t$  is time
- $x$  be the position
- $v_p$  phase velocity
- $v_g$  be the group velocity

# Vibrations and Waves

The phase velocity of a wave is given by the following equation:

$$v_p = \frac{\omega}{k}$$

.....(eqn 1)

Rewriting the above equation, we get:

$$\omega = k v_p$$

.....(eqn 2)

Differentiating (eqn 2) w.r.t k we obtain,

$$\frac{d\omega}{dk} = v_p + k \frac{dv_p}{dk}$$

.....(eqn 3)

As

$$v_g = \frac{d\omega}{dk}$$

(eqn 3) reduces to:

$$v_g = v_p + k \frac{dv_p}{dk}$$

The above equation signifies the relationship between the phase velocity and the group velocity.

# Vibrations and Waves

*Thank you  
for your kind attention*