Preliminaries

Overview This chapter reviews the main things you need to know to start calculus. The topics include the real number system, Cartesian coordinates in the plane, straight lines, parabolas, circles, functions, and trigonometry.

1 Real Numbers and the Real Line

This section reviews real numbers, inequalities, intervals, and absolute values.

Real Numbers and the Real Line

Much of calculus is based on properties of the real number system. **Real numbers** are numbers that can be expressed as decimals, such as

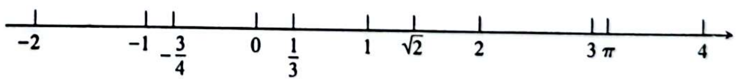
$$-\frac{3}{4} = -0.75000\ldots$$

$$\frac{1}{3} = 0.33333\ldots$$

$$\sqrt{2} = 1.4142\ldots$$

The dots ... in each case indicate that the sequence of decimal digits goes on forever.

The real numbers can be represented geometrically as points on a number line called the **real line**.



The symbol $\mathbb{R}$ donotes either the real number system or, equivalently, the real line.

Properties of Real Numbers

The properties of the real number system fall into three categories: algebraic properties, order properties, and completeness. The algebraic properties say that the real numbers can be added, subtracted, multiplied, and divided (except by 0) to produce more real numbers under the usual rules of arithmetic. *You can never divide by* 0.

The order properties of real numbers are summarized in the following list.

The symbol $\Rightarrow$ means "implies."

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \dfrac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \dfrac{1}{b} < \dfrac{1}{a}$

Notice the rules for multiplying an inequality by a number. Multiplying by a positive number preserves the inequality; multiplying by a negative number reverses the inequality. Also, reciprocation reverses the inequality for numbers of the same sign.

The completeness property of the real number system is deeper and harder to define precisely. Roughly speaking, it says that there are enough real numbers to "complete" the real number line, in the sense that there are no "holes" or "gaps" in it. Many of the theorems of calculus would fail if the real number system were not complete, and the nature of the connection is important. The topic is best saved for a more advanced course, however, and we will not pursue it.

Subsets of $\mathbb{R}$

We distinguish three special subsets of real numbers.

1. The **natural numbers**, namely $1, 2, 3, 4, \ldots$
2. The **integers**, namely $0, \pm 1, \pm 2, \pm 3, \ldots$
3. The **rational numbers**, namely the numbers that can be expressed in the form of a fraction m/n, where m and n are integers and $n \neq 0$. Examples are
$$\frac{1}{3}, \quad -\frac{4}{9}, \quad \frac{200}{13}, \quad \text{and} \quad 57 = \frac{57}{1}.$$

The rational numbers are precisely the real numbers with decimal expansions that are either

a) terminating (ending in an infinite string of zeros), for example,
$$\frac{3}{4} = 0.75000\ldots = 0.75 \qquad \text{or}$$

b) repeating (ending with a block of digits that repeats over and over), for example
$$\frac{23}{11} = 2.090909\ldots = 2.\overline{09}.$$
The bar indicates the block of repeating digits.

The set of rational numbers has all the algebraic and order properties of the real numbers but lacks the completeness property. For example, there is no rational number whose square is 2; there is a "hole" in the rational line where $\sqrt{2}$ should be.

Real numbers that are not rational are called **irrational numbers.** They are characterized by having nonterminating and nonrepeating decimal expansions. Examples are π, $\sqrt{2}$, $\sqrt[3]{5}$, and $\log_{10} 3$.

Intervals

A subset of the real line is called an **interval** if it contains at least two numbers and contains all the real numbers lying between any two of its elements. For example, the set of all real numbers x such that $x > 6$ is an interval, as is the set of all x such that $-2 \leq x \leq 5$. The set of all nonzero real numbers is not an interval; since 0 is absent, the set fails to contain every real number between -1 and 1 (for example).

Geometrically, intervals correspond to rays and line segments on the real line, along with the real line itself. Intervals of numbers corresponding to line segments are **finite intervals**; intervals corresponding to rays and the real line are **infinite intervals.**

A finite interval is said to be **closed** if it contains both of its endpoints, **half-open** if it contains one endpoint but not the other, and **open** if it contains neither endpoint. The endpoints are also called **boundary points;** they make up the interval's **boundary.** The remaining points of the interval are **interior points** and together make up what is called the interval's **interior.**

Table 1 Types of intervals

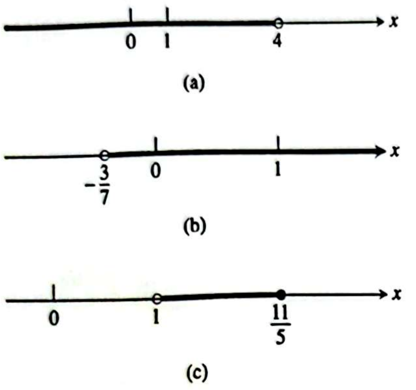
	Notation	Set	Graph
Finite:	(a, b)	$\{x \mid a < x < b\}$	
	$[a, b]$	$\{x \mid a \leq x \leq b\}$	
	$[a, b)$	$\{x \mid a \leq x < b\}$	
	$(a, b]$	$\{x \mid a < x \leq b\}$	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	
	$[a, \infty)$	$\{x \mid x \geq a\}$	
	$(-\infty, b)$	$\{x \mid x < b\}$	
	$(-\infty, b]$	$\{x \mid x \leq b\}$	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

Solving Inequalities

The process of finding the interval or intervals of numbers that satisfy an inequality in x is called **solving** the inequality.

EXAMPLE 1 Solve the following inequalities and graph their solution sets on the real line.

a) $2x - 1 < x + 3$ b) $-\frac{x}{3} < 2x + 1$ c) $\frac{6}{x-1} \geq 5$



1 Solutions for Example 1.

Solution

a)

$$
\begin{aligned}
2x - 1 &< x + 3 \\
2x &< x + 4 && \text{Add 1 to both sides.} \\
x &< 4 && \text{Subtract } x \text{ from both sides.}
\end{aligned}
$$

The solution set is the interval $(-\infty, 4)$ (Fig. 1a).

b)

$$
\begin{aligned}
-\frac{x}{3} &< 2x + 1 \\
-x &< 6x + 3 && \text{Multiply both sides by 3.} \\
0 &< 7x + 3 && \text{Add } x \text{ to both sides.} \\
-3 &< 7x && \text{Subtract 3 from both sides.} \\
-\frac{3}{7} &< x && \text{Divide by 7.}
\end{aligned}
$$

The solution set is the interval $(-3/7, \infty)$ (Fig. 1b).

c) The inequality $6/(x-1) \geq 5$ can hold only if $x > 1$, because otherwise $6/(x-1)$ is undefined or negative. Therefore, the inequality will be preserved if we multiply both sides by $(x-1)$, and we have

$$
\begin{aligned}
\frac{6}{x-1} &\geq 5 \\
6 &\geq 5x - 5 && \text{Multiply both sides by } (x-1). \\
11 &\geq 5x && \text{Add 5 to both sides.} \\
\frac{11}{5} &\geq x. && \text{Or } x \leq \frac{11}{5}.
\end{aligned}
$$

The solution set is the half-open interval $(1, 11/5]$ (Fig. 1c). ❑

Absolute Value

The **absolute value** of a number x, denoted by $|x|$, is defined by the formula

$$
|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}
$$

EXAMPLE 2 $|3| = 3,\quad |0| = 0,\quad |-5| = -(-5) = 5,\quad \big|-|a|\big| = |a|$ ❑

Notice that $|x| \geq 0$ for every real number x, and $|x| = 0$ if and only if $x = 0$.

Since the symbol $\sqrt{a}$ always denotes the *nonnegative* square root of a, an alternate definition of $|x|$ is

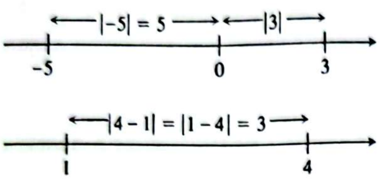
$$|x| = \sqrt{x^2}.$$

It is important to remember that $\sqrt{a^2} = |a|$. Do not write $\sqrt{a^2} = a$ unless you already know that $a \geq 0$.

Geometrically, $|x|$ represents the distance from x to the origin 0 on the real line. More generally (Fig. 2)

$$|x - y| = \text{the distance between } x \text{ and } y.$$

The absolute value has the following properties.



2 Absolute values give distances between points on the number line.

Absolute Value Properties

1. $\lvert -a \rvert = \lvert a \rvert$	A number and its negative have the same absolute value.
2. $\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	The absolute value of a product is the product of the absolute values.
3. $\left\lvert \dfrac{a}{b} \right\rvert = \dfrac{\lvert a \rvert}{\lvert b \rvert}$	The absolute value of a quotient is the quotient of the absolute values.
4. $\lvert a + b \rvert \leq \lvert a \rvert + \lvert b \rvert$	**The triangle inequality** The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

If a and b differ in sign, then $|a + b|$ is less than $|a| + |b|$. In all other cases, $|a + b|$ equals $|a| + |b|$.

Notice that absolute value bars in expressions like $|-3+5|$ also work like parentheses: We do the arithmetic inside *before* taking the absolute value.

EXAMPLE 3

$$|-3+5| = |2| = 2 < |-3| + |5| = 8$$

$$|3+5| = |8| = |3| + |5|$$

$$|-3-5| = |-8| = 8 = |-3| + |-5| \quad \square$$

EXAMPLE 4 Solve the equation $|2x - 3| = 7$.

Solution The equation says that $2x - 3 = \pm 7$, so there are two possibilities:

$2x - 3 = 7$	$2x - 3 = -7$	Equivalent equations without absolute values
$2x = 10$	$2x = -4$	Solve as usual.
$x = 5$	$x = -2$	

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$. $\square$

Inequalities Involving Absolute Values

The inequality $|a| < D$ says that the distance from a to 0 is less than D. Therefore, a must lie between D and $-D$.

The symbol $\Leftrightarrow$ means "if and only if," or "implies and is implied by."

Intervals and Absolute Values

If D is any positive number, then

$$|a| < D \quad \Leftrightarrow \quad -D < a < D, \tag{1}$$

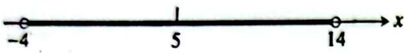
$$|a| \leq D \quad \Leftrightarrow \quad -D \leq a \leq D. \tag{2}$$

EXAMPLE 5 Solve the inequality $|x - 5| < 9$ and graph the solution set on the real line.

Solution

$$\begin{aligned} |x-5| &< 9 \\ -9 < x - 5 &< 9 && \text{Eq. (1)} \\ -9 + 5 < x &< 9 + 5 && \text{Add 5 to each part to isolate } x. \\ -4 < x &< 14 \end{aligned}$$

The solution set is the open interval $(-4, 14)$ (Fig. 3). ❑



3 The solution set of the inequality $|x - 5| < 9$ is the interval $(-4, 14)$ graphed here (Example 5).

EXAMPLE 6 Solve the inequality $\left|5 - \dfrac{2}{x}\right| < 1$.

Solution We have

$$\begin{aligned} \left|5 - \frac{2}{x}\right| < 1 \quad &\Leftrightarrow \quad -1 < 5 - \frac{2}{x} < 1 && \text{Eq. (1)} \\ &\Leftrightarrow \quad -6 < -\frac{2}{x} < -4 && \text{Subtract 5.} \\ &\Leftrightarrow \quad 3 > \frac{1}{x} > 2 && \text{Multiply by } -\tfrac{1}{2}. \\ &\Leftrightarrow \quad \frac{1}{3} < x < \frac{1}{2}. && \text{Take reciprocals.} \end{aligned}$$

Notice how the various rules for inequalities were used here. Multiplying by a negative number reverses the inequality. So does taking reciprocals in an inequality in which both sides are positive. The original inequality holds if and only if $(1/3) < x < (1/2)$. The solution set is the open interval $(1/3, 1/2)$. ❑

EXAMPLE 7 Solve the inequality and graph the solution set:

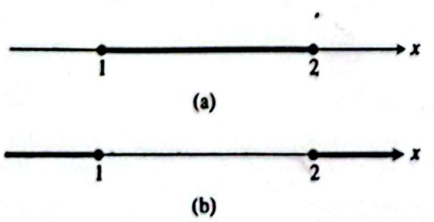
a) $|2x - 3| \leq 1$ b) $|2x - 3| \geq 1$

Solution

a)

$$\begin{aligned} |2x - 3| &\leq 1 \\ -1 \leq 2x - 3 &\leq 1 && \text{Eq. (2)} \\ 2 \leq 2x &\leq 4 && \text{Add 3.} \\ 1 \leq x &\leq 2 && \text{Divide by 2.} \end{aligned}$$

The solution set is the closed interval $[1, 2]$ (Fig. 4a).



4 Graphs of the solution sets (a) $[1, 2]$ and (b) $(-\infty, 1] \cup [2, \infty)$ in Example 7.

Union and intersection

Notice the use of the symbol $\cup$ to denote the union of intervals. A number lies in the **union** of two sets if it lies in either set. Similarly we use the symbol $\cap$ to denote intersection. A number lies in the **intersection** $I \cap J$ of two sets if it lies in *both* sets I and J. For example, $[1, 3) \cap [2, 4] = [2, 3)$.

b) $$|2x - 3| \geq 1$$

$2x - 3 \geq 1$	or	$-(2x - 3) \geq 1$	
$2x - 3 \geq 1$	or	$2x - 3 \leq -1$	Multiply second inequality by -1.
$x - \frac{3}{2} \geq \frac{1}{2}$	or	$x - \frac{3}{2} \leq -\frac{1}{2}$	Divide by 2.
$x \geq 2$	or	$x \leq 1$	Add $\frac{3}{2}$.

The solution set is $(-\infty, 1] \cup [2, \infty)$ (Fig. 4b). ❑

Exercises 1

Decimal Representations

1. Express 1/9 as a repeating decimal, using a bar to indicate the repeating digits. What are the decimal representations of 2/9? 3/9? 8/9?

2. Express 1/11 as a repeating decimal, using a bar to indicate the repeating digits. What are the decimal representations of 2/11? 3/11? 9/11?

Inequalities

3. If $2 < x < 6$, which of the following statements about x are necessarily true, and which are not necessarily true?

   a) $0 < x < 4$ b) $0 < x - 2 < 4$

   c) $1 < \frac{x}{2} < 3$ d) $\frac{1}{6} < \frac{1}{x} < \frac{1}{2}$

   e) $1 < \frac{6}{x} < 3$ f) $|x - 4| < 2$

   g) $-6 < -x < 2$ h) $-6 < -x < -2$

4. If $-1 < y - 5 < 1$, which of the following statements about y are necessarily true, and which are not necessarily true?

   a) $4 < y < 6$ b) $-6 < y < -4$

   c) $y > 4$ d) $y < 6$

   e) $0 < y - 4 < 2$ f) $2 < \frac{y}{2} < 3$

   g) $\frac{1}{6} < \frac{1}{y} < \frac{1}{4}$ h) $|y - 5| < 1$

In Exercises 5–12, solve the inequalities and graph the solution sets.

5. $-2x > 4$
6. $8 - 3x \geq 5$
7. $5x - 3 \leq 7 - 3x$
8. $3(2 - x) > 2(3 + x)$
9. $2x - \frac{1}{2} \geq 7x + \frac{7}{6}$
10. $\frac{6 - x}{4} < \frac{3x - 4}{2}$
11. $\frac{4}{5}(x - 2) < \frac{1}{3}(x - 6)$
12. $-\frac{x + 5}{2} \leq \frac{12 + 3x}{4}$

Absolute Value

Solve the equations in Exercises 13–18.

13. $|y| = 3$
14. $|y - 3| = 7$
15. $|2t + 5| = 4$
16. $|1 - t| = 1$
17. $|8 - 3s| = \frac{9}{2}$
18. $\left|\frac{s}{2} - 1\right| = 1$

Solve the inequalities in Exercises 19–34, expressing the solution sets as intervals or unions of intervals. Also, graph each solution set on the real line.

19. $|x| < 2$
20. $|x| \leq 2$
21. $|t - 1| \leq 3$
22. $|t + 2| < 1$
23. $|3y - 7| < 4$
24. $|2y + 5| < 1$
25. $\left|\frac{z}{5} - 1\right| \leq 1$
26. $\left|\frac{3}{2}z - 1\right| \leq 2$
27. $\left|3 - \frac{1}{x}\right| < \frac{1}{2}$
28. $\left|\frac{2}{x} - 4\right| < 3$
29. $|2s| \geq 4$
30. $|s + 3| \geq \frac{1}{2}$
31. $|1 - x| > 1$
32. $|2 - 3x| > 5$
33. $\left|\frac{r + 1}{2}\right| \geq 1$
34. $\left|\frac{3r}{5} - 1\right| > \frac{2}{5}$

Quadratic Inequalities

Solve the inequalities in Exercises 35–42. Express the solution sets as intervals or unions of intervals and graph them. Use the result $\sqrt{a^2} = |a|$ as appropriate.

35. $x^2 < 2$
36. $4 \leq x^2$
37. $4 < x^2 < 9$
38. $\frac{1}{9} < x^2 < \frac{1}{4}$
39. $(x - 1)^2 < 4$
40. $(x + 3)^2 < 2$
41. $x^2 - x < 0$
42. $x^2 - x - 2 \geq 0$

Theory and Examples

43. Do not fall into the trap $|-a| = a$. For what real numbers a is this equation true? For what real numbers is it false?

44. Solve the equation $|x - 1| = 1 - x$.

45. *A proof of the triangle inequality.* Give the reason justifying each of the numbered steps in the following proof of the triangle inequality.

$$\begin{aligned} |a+b|^2 &= (a+b)^2 && (1)\\ &= a^2 + 2ab + b^2 \\ &\le a^2 + 2|a||b| + b^2 && (2)\\ &\le |a|^2 + 2|a||b| + |b|^2 && (3)\\ &= (|a| + |b|)^2 \\ |a+b| &\le |a| + |b| && (4) \end{aligned}$$

46. Prove that $|ab| = |a||b|$ for any numbers a and b.

47. If $|x| \le 3$ and $x > -1/2$, what can you say about x?

48. Graph the inequality $|x| + |y| \le 1$.

49. GRAPHER

a) Graph the functions $f(x) = x/2$ and $g(x) = 1 + (4/x)$ together to identify the values of x for which

$$\frac{x}{2} > 1 + \frac{4}{x}.$$

b) Confirm your findings in (a) algebraically.

50. GRAPHER

a) Graph the functions $f(x) = 3/(x-1)$ and $g(x) = 2/(x+1)$ together to identify the values of x for which

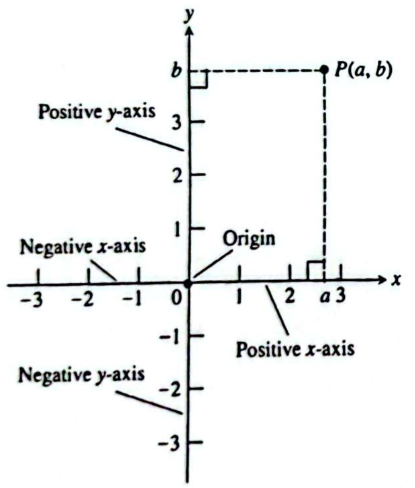
$$\frac{3}{x-1} < \frac{2}{x+1}.$$

b) Confirm your findings in (a) algebraically.

2 Coordinates, Lines, and Increments

This section reviews coordinates and lines and discusses the notion of increment.

Cartesian Coordinates in the Plane



5 Cartesian coordinates.

The positions of all points in the plane can be measured with respect to two perpendicular real lines in the plane intersecting in the 0-point of each (Fig. 5). These lines are called **coordinate axes** in the plane. On the horizontal x-axis, numbers are denoted by x and increase to the right. On the vertical y-axis, numbers are denoted by y and increase upward. The point where x and y are both 0 is the **origin** of the coordinate system, often denoted by the letter O.

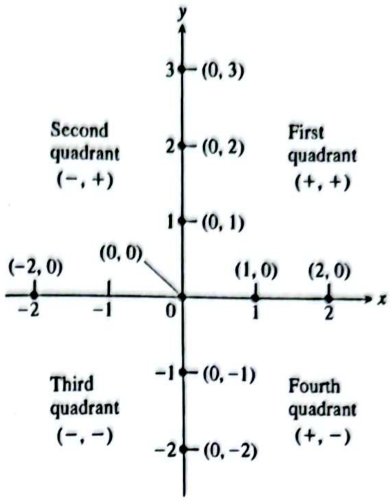
If P is any point in the plane, we can draw lines through P perpendicular to the two coordinate axes. If the lines meet the x-axis at a and the y-axis at b, then a is the **x-coordinate** of P, and b is the **y-coordinate**. The ordered pair (a, b) is the point's **coordinate pair**. The x-coordinate of every point on the y-axis is 0. The y-coordinate of every point on the x-axis is 0. The origin is the point $(0, 0)$.

The origin divides the x-axis into the **positive x-axis** to the right and the **negative x-axis** to the left. It divides the y-axis into the **positive** and **negative y-axis** above and below. The axes divide the plane into four regions called **quadrants**, numbered counterclockwise as in Fig. 6.

A Word About Scales

When we plot data in the coordinate plane or graph formulas whose variables have different units of measure, we do not need to use the same scale on the two axes. If we plot time vs. thrust for a rocket motor, for example, there is no reason to place the mark that shows 1 sec on the time axis the same distance from the origin as the mark that shows 1 lb on the thrust axis.

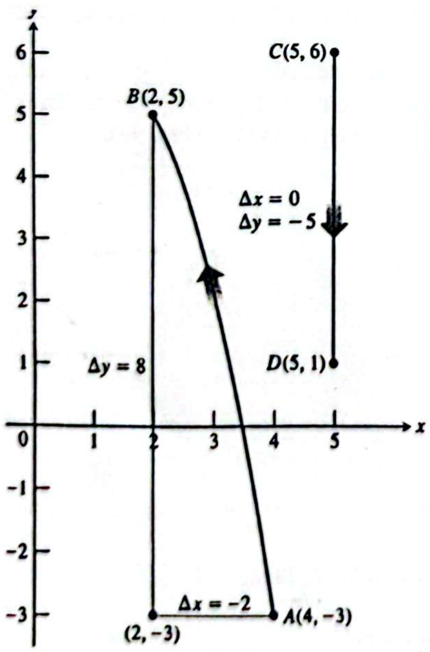
When we graph functions whose variables do not represent physical measurements and when we draw figures in the coordinate plane to study their geometry and trigonometry, we try to make the scales on the axes identical. A vertical unit



6 The points on the axes all have coordinate pairs, but we usually label them with single numbers. Notice the coordinate sign patterns in the quadrants.

of distance then looks the same as a horizontal unit. As on a surveyor's map or a scale drawing, line segments that are supposed to have the same length will look as if they do and angles that are supposed to be congruent will look congruent.

Computer displays and calculator displays are another matter. The vertical and horizontal scales on machine-generated graphs usually differ, and there are corresponding distortions in distances, slopes, and angles. Circles may look like ellipses, rectangles may look like squares, right angles may appear to be acute or obtuse, and so on. Circumstances like these require us to take extra care in interpreting what we see. High-quality computer software usually allows you to compensate for such scale problems by adjusting the *aspect ratio* (ratio of vertical to horizontal scale). Some computer screens also allow adjustment within a narrow range. When you use a grapher, try to make the aspect ratio 1, or close to it.



7 Coordinate increments may be positive, negative, or zero.

Increments and Distance

When a particle moves from one point in the plane to another, the net changes in its coordinates are called *increments*. They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point.

EXAMPLE 1 In going from the point $A(4, -3)$ to the point $B(2, 5)$ (Fig. 7), the increments in the x- and y-coordinates are

$$\Delta x = 2 - 4 = -2, \qquad \Delta y = 5 - (-3) = 8.$$

❑

Definition

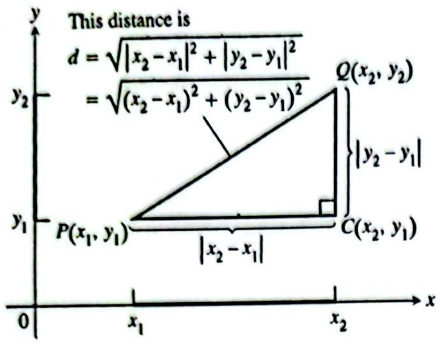
An **increment** in a variable is a net change in that variable. If x changes from x_1 to x_2, the increment in x is

$$\Delta x = x_2 - x_1.$$

EXAMPLE 2 From $C(5, 6)$ to $D(5, 1)$ (Fig. 7) the coordinate increments are

$$\Delta x = 5 - 5 = 0, \qquad \Delta y = 1 - 6 = -5.$$

❑



8 To calculate the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$, apply the Pythagorean theorem to triangle *PCQ*.

The distance between points in the plane is calculated with a formula that comes from the Pythagorean theorem (Fig. 8).

Distance Formula for Points in the Plane

The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 3

a) The distance between $P(-1, 2)$ and $Q(3, 4)$ is

$$\sqrt{(3 - (-1))^2 + (4 - 2)^2} = \sqrt{(4)^2 + (2)^2} = \sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}.$$

b) The distance from the origin to $P(x, y)$ is

$$\sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}.$$

❑

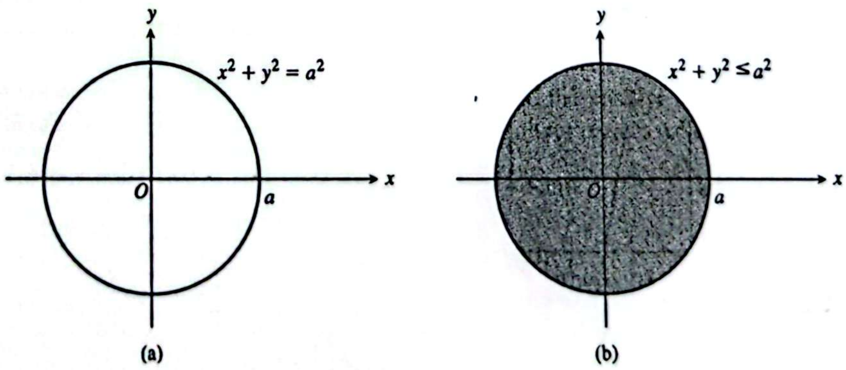
Graphs

The graph of an equation or inequality involving the variables x and y is the set of all points $P(x, y)$ whose coordinates satisfy the equation or inequality.

EXAMPLE 4 *Circles centered at the origin*

a) If $a > 0$, the equation $x^2 + y^2 = a^2$ represents all points $P(x, y)$ whose distance from the origin is $\sqrt{x^2 + y^2} = \sqrt{a^2} = a$. These points lie on the circle of radius a centered at the origin. This circle is the graph of the equation $x^2 + y^2 = a^2$ (Fig. 9a).

b) Points (x, y) whose coordinates satisfy the inequality $x^2 + y^2 \leq a^2$ all have distance $\leq a$ from the origin. The graph of the inequality is therefore the circle of radius a centered at the origin together with its interior (Fig. 9b).

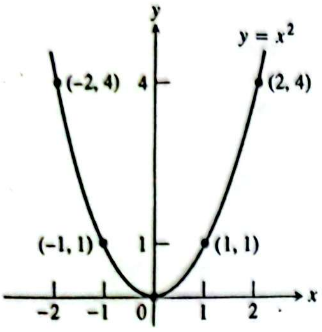


9 Graphs of (a) the equation and (b) the inequality in Example 4.

❑

The circle of radius 1 unit centered at the origin is called the unit circle.

EXAMPLE 5 Consider the equation $y = x^2$. Some points whose coordinates satisfy this equation are $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(2, 4)$, and $(-2, 4)$. These points (and all others satisfying the equation) make up a smooth curve called a parabola (Fig. 10). ❑



10 The parabola $y = x^2$.

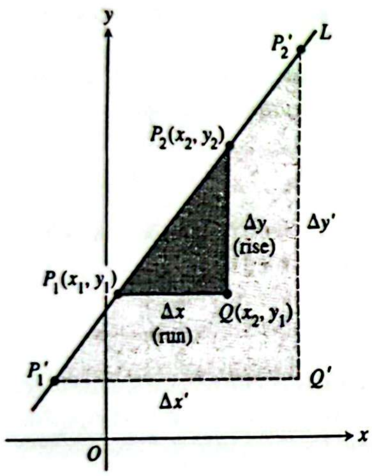
Straight Lines

Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the plane, we call the increments $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ the **run** and the **rise**, respectively, between P_1 and P_2. Two such points always determine a unique straight line (usually called simply a line) passing through them both. We call the line P_1P_2.

Any nonvertical line in the plane has the property that the ratio

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

has the same value for every choice of the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the line (Fig. 11).



11 Triangles P_1QP_2 and $P_1'Q'P_2'$ are similar, so

$$\frac{\Delta y'}{\Delta x'} = \frac{\Delta y}{\Delta x} = m.$$

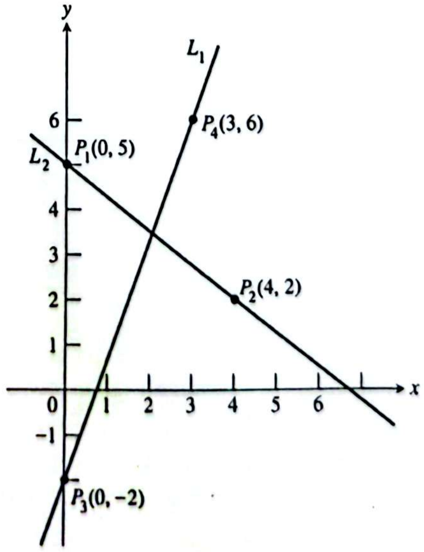
Definition

The constant

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

is the **slope** of the nonvertical line P_1P_2.

The slope tells us the direction (uphill, downhill) and steepness of a line. A line with positive slope rises uphill to the right; one with negative slope falls downhill to the right (Fig. 12). The greater the absolute value of the slope, the more rapid the rise or fall. The slope of a vertical line is *undefined*. Since the run Δx is zero for a vertical line, we cannot form the ratio m.



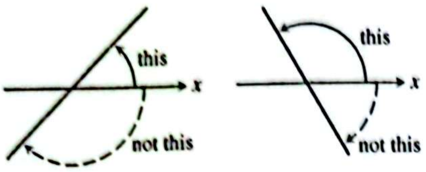
12 The slope of L_1 is

$$m = \frac{\Delta y}{\Delta x} = \frac{6 - (-2)}{3 - 0} = \frac{8}{3}.$$

That is, y increases 8 units every time x increases 3 units. The slope of L_2 is

$$m = \frac{\Delta y}{\Delta x} = \frac{2 - 5}{4 - 0} = \frac{-3}{4}.$$

That is, y decreases 3 units every time x increases 4 units.

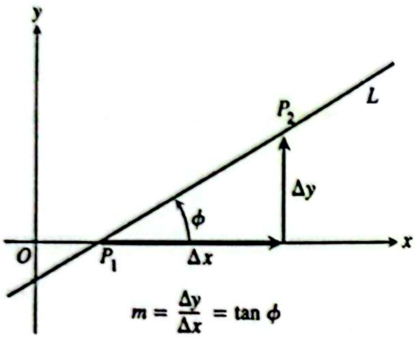


13 Angles of inclination are measured counterclockwise from the x-axis.

The direction and steepness of a line can also be measured with an angle. The **angle of inclination (inclination)** of a line that crosses the x-axis is the smallest counterclockwise angle from the x-axis to the line (Fig. 13). The inclination of a horizontal line is 0°. The inclination of a vertical line is 90°. If ϕ (the Greek letter phi) is the inclination of a line, then $0 \le \phi < 180°$.

The relationship between the slope m of a nonvertical line and the line's inclination ϕ is shown in Fig. 14:

$$\boxed{m = \tan \phi.}$$



14 The slope of a nonvertical line is the tangent of its angle of inclination.

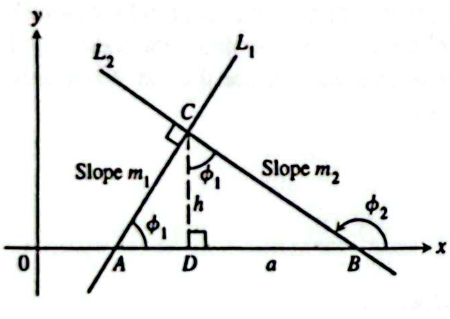
Parallel and Perpendicular Lines

Lines that are parallel have equal angles of inclination. Hence, they have the same slope (if they are not vertical). Conversely, lines with equal slopes have equal angles of inclination and so are parallel.

If two nonvertical lines L_1 and L_2 are perpendicular, their slopes m_1 and m_2 satisfy $m_1 m_2 = -1$, so each slope is the *negative reciprocal* of the other:

$$m_1 = -\frac{1}{m_2}, \qquad m_2 = -\frac{1}{m_1}.$$

The argument goes like this: In the notation of Fig. 15, $m_1 = \tan \phi_1 = a/h$, while $m_2 = \tan \phi_2 = -h/a$. Hence, $m_1 m_2 = (a/h)(-h/a) = -1$.

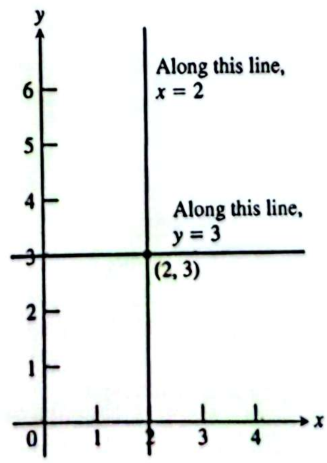


15 ΔADC is similar to ΔCDB. Hence ϕ_1 is also the upper angle in ΔCDB. From the sides of ΔCDB, we read $\tan \phi_1 = a/h$.

Equations of Lines

Straight lines have relatively simple equations. All points on the *vertical line* through the point a on the x-axis have x-coordinates equal to a. Thus, $x = a$ is an equation for the vertical line. Similarly, $y = b$ is an equation for the *horizontal line* meeting the y-axis at b.

EXAMPLE 6 The vertical and horizontal lines through the point $(2, 3)$ have equations $x = 2$ and $y = 3$, respectively (Fig. 16).



16 The standard equations for the vertical and horizontal lines through $(2, 3)$ are $x = 2$ and $y = 3$. ❑

We can write an equation for a nonvertical straight line L if we know its slope m and the coordinates of one point $P_1(x_1, y_1)$ on it. If $P(x, y)$ is *any* other point on L, then

$$\frac{y - y_1}{x - x_1} = m,$$

so that

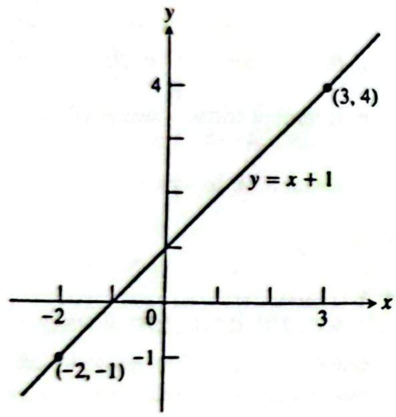
$$y - y_1 = m(x - x_1) \qquad \text{or} \qquad y = y_1 + m(x - x_1).$$

Definition

The equation

$$y = y_1 + m(x - x_1)$$

is the **point–slope equation** of the line that passes through the point (x_1, y_1) and has slope m.



16 **The line in Example 8.**

EXAMPLE 7 Write an equation for the line through the point (2, 3) with slope −3/2.

Solution We substitute $x_1 = 2$, $y_1 = 3$, and $m = -3/2$ into the point–slope equation and obtain

$$y = 3 - \frac{3}{2}(x - 2), \qquad \text{or} \qquad y = -\frac{3}{2}x + 6.$$ ❑

EXAMPLE 8 Write an equation for the line through (−2, −1) and (3, 4).

Solution The line's slope is

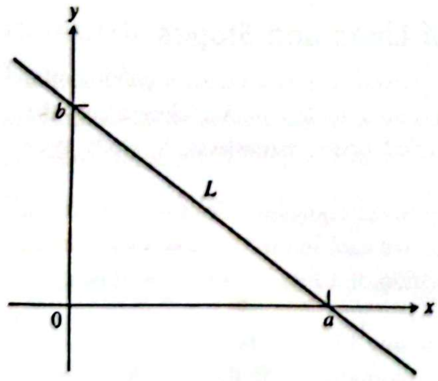
$$m = \frac{-1 - 4}{-2 - 3} = \frac{-5}{-5} = 1.$$

We can use this slope with either of the two given points in the point–slope equation:

With $(x_1, y_1) = (-2, -1)$	With $(x_1, y_1) = (3, 4)$
$y = -1 + 1 \cdot (x - (-2))$	$y = 4 + 1 \cdot (x - 3)$
$y = -1 + x + 2$	$y = 4 + x - 3$
$y = x + 1$	$y = x + 1$

Same result

Either way, $y = x + 1$ is an equation for the line (Fig. 17). ❑



17 ***Line L has x-intercept a and y-intercept b.***

The y-coordinate of the point where a nonvertical line intersects the y-axis is called the **y-intercept** of the line. Similarly, the **x-intercept** of a nonhorizontal line is the x-coordinate of the point where it crosses the x-axis (Fig. 18). A line with slope m and y-intercept b passes through the point $(0, b)$, so it has equation

$$y = b + m(x - 0), \qquad \text{or, more simply,} \qquad y = mx + b.$$

Definition

The equation

$$y = mx + b$$

is called the **slope–intercept equation** of the line with slope m and y-intercept b.

EXAMPLE 9 The line $y = 2x - 5$ has slope 2 and y-intercept -5.

The equation

$$Ax + By = C \qquad (A \text{ and } B \text{ not both } 0)$$

is called the **general linear equation** in x and y because its graph always represents a line and every line has an equation in this form (including lines with undefined slope).

EXAMPLE 10 Find the slope and y-intercept of the line $8x + 5y = 20$.

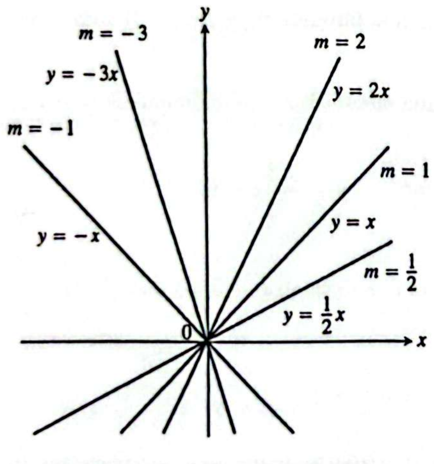
Solution Solve the equation for y to put it in slope–intercept form. Then read the slope and y-intercept from the equation:

$$\begin{aligned} 8x + 5y &= 20 \\ 5y &= -8x + 20 \\ y &= -\frac{8}{5}x + 4. \end{aligned}$$

The slope is $m = -8/5$. The y-intercept is $b = 4$.

EXAMPLE 11 *Lines through the origin*

Lines with equations of the form $y = mx$ have y-intercept 0 and so pass through the origin. Several examples are shown in Fig. 19.



19 **The line $y = mx$ has slope m and passes through the origin.**

Applications—The Importance of Lines and Slopes

Light travels along lines, as do bodies falling from rest in a planet's gravitational field or coasting under their own momentum (like a hockey puck gliding across the ice). We often use the equations of lines (called **linear equations**) to study such motions.

Many important quantities are related by linear equations. Once we know that a relationship between two variables is linear, we can find it from any two pairs of corresponding values just as we find the equation of a line from the coordinates of two points.

Slope is important because it gives us a way to say how steep something is (roadbeds, roofs, stairs). The notion of slope also enables us to describe how rapidly things are changing. For this reason it will play an important role in calculus.

EXAMPLE 12 *Celsius vs. Fahrenheit*

Fahrenheit temperature (F) and Celsius temperature (C) are related by a linear equation of the form $F = mC + b$. The freezing point of water is $F = 32°$ or $C = 0°$, while the boiling point is $F = 212°$ or $C = 100°$. Thus

$$32 = 0m + b, \quad \text{and} \quad 212 = 100m + b,$$

so $b = 32$ and $m = (212 - 32)/100 = 9/5$. Therefore,

$$F = \frac{9}{5}C + 32, \quad \text{or} \quad C = \frac{5}{9}(F - 32).$$

Exercises 2

Increments and Distance

In Exercises 1–4, a particle moves from A to B in the coordinate plane. Find the increments Δx and Δy in the particle's coordinates. Also find the distance from A to B.

1. $A(-3, 2)$, $B(-1, -2)$
2. $A(-1, -2)$, $B(-3, 2)$
3. $A(-3.2, -2)$, $B(-8.1, -2)$
4. $A(\sqrt{2}, 4)$, $B(0, 1.5)$

Describe the graphs of the equations in Exercises 5–8.

5. $x^2 + y^2 = 1$
6. $x^2 + y^2 = 2$
7. $x^2 + y^2 \le 3$
8. $x^2 + y^2 = 0$

Slopes, Lines, and Intercepts

Plot the points in Exercises 9–12 and find the slope (if any) of the line they determine. Also find the common slope (if any) of the lines perpendicular to line AB.

9. $A(-1, 2)$, $B(-2, -1)$
10. $A(-2, 1)$, $B(2, -2)$
11. $A(2, 3)$, $B(-1, 3)$
12. $A(-2, 0)$, $B(-2, -2)$

In Exercises 13–16, find an equation for (a) the vertical line and (b) the horizontal line through the given point.

13. $(-1, 4/3)$
14. $(\sqrt{2}, -1.3)$
15. $(0, -\sqrt{2})$
16. $(-\pi, 0)$

In Exercises 17–30, write an equation for each line described.

17. Passes through $(-1, 1)$ with slope -1
18. Passes through $(2, -3)$ with slope $1/2$
19. Passes through $(3, 4)$ and $(-2, 5)$
20. Passes through $(-8, 0)$ and $(-1, 3)$
21. Has slope $-5/4$ and y-intercept 6
22. Has slope $1/2$ and y-intercept -3
23. Passes through $(-12, -9)$ and has slope 0
24. Passes through $(1/3, 4)$ and has no slope
25. Has y-intercept 4 and x-intercept -1
26. Has y-intercept -6 and x-intercept 2
27. Passes through $(5, -1)$ and is parallel to the line $2x + 5y = 15$
28. Passes through $(-\sqrt{2}, 2)$ parallel to the line $\sqrt{2}x + 5y = \sqrt{3}$
29. Passes through $(4, 10)$ and is perpendicular to the line $6x - 3y = 5$
30. Passes through $(0, 1)$ and is perpendicular to the line $8x - 13y = 13$

In Exercises 31–34, find the line's x- and y-intercepts and use this information to graph the line.

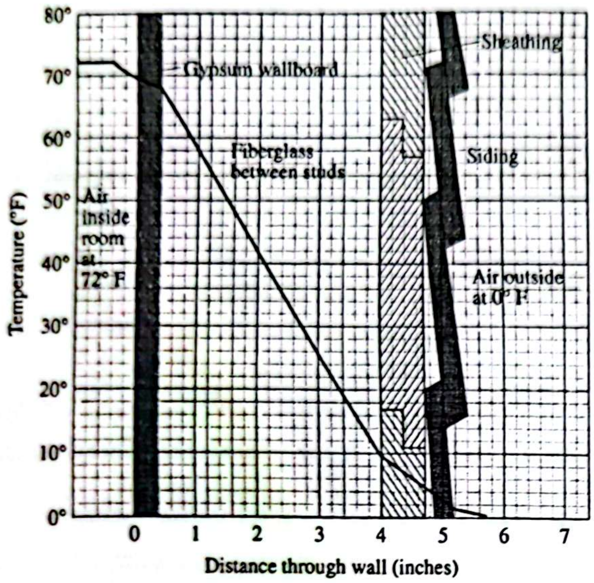
31. $3x + 4y = 12$
32. $x + 2y = -4$
33. $\sqrt{2}x - \sqrt{3}y = \sqrt{6}$
34. $1.5x - y = -3$
35. Is there anything special about the relationship between the lines $Ax + By = C_1$ and $Bx - Ay = C_2$ $(A \ne 0, B \ne 0)$? Give reasons for your answer.
36. Is there anything special about the relationship between the lines $Ax + By = C_1$ and $Ax + By = C_2$ $(A \ne 0, B \ne 0)$? Give reasons for your answer.

Increments and Motion

37. A particle starts at $A(-2, 3)$ and its coordinates change by increments $\Delta x = 5$, $\Delta y = -6$. Find its new position.
38. A particle starts at $A(6, 0)$ and its coordinates change by increments $\Delta x = -6$, $\Delta y = 0$. Find its new position.
39. The coordinates of a particle change by $\Delta x = 5$ and $\Delta y = 6$ as it moves from $A(x, y)$ to $B(3, -3)$. Find x and y.
40. A particle started at $A(1, 0)$, circled the origin once counterclockwise, and returned to $A(1, 0)$. What were the net changes in its coordinates?

Applications

41. *Insulation.* By measuring slopes in Fig. 20, estimate the temperature change in degrees per inch for (a) the gypsum wallboard; (b) the fiberglass insulation; (c) the wood sheathing. (Graphs can shift in printing, so your answers may differ slightly from those in the back of the book.)



20 The temperature changes in the wall in Exercises 41 and 42. (Source: *Differentiation*, by W. U. Walton et al., Project CALC, Education Development Center, Inc., Newton, Mass. [1975], p. 25.)

42. *Insulation.* According to Fig. 20, which of the materials in Exercise 41 is the best insulator? the poorest? Explain.

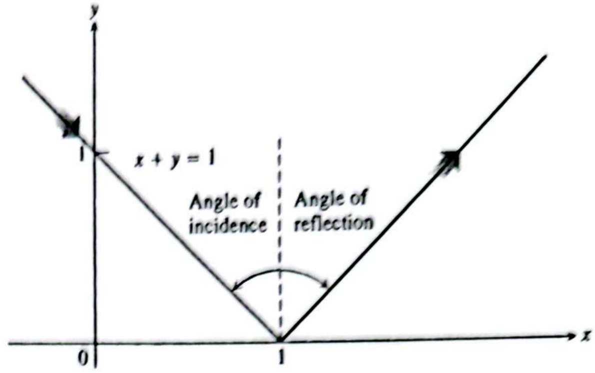
43. *Pressure under water.* The pressure p experienced by a diver under water is related to the diver's depth d by an equation of the form $p = kd + 1$ (k a constant). At the surface, the pressure is 1 atmosphere. The pressure at 100 meters is about 10.94 atmospheres. Find the pressure at 50 meters.

44. *Reflected light.* A ray of light comes in along the line $x + y = 1$ from the second quadrant and reflects off the x-axis (Fig. 21). The angle of incidence is equal to the angle of reflection. Write an equation for the line along which the departing light travels.

45. *Fahrenheit vs. Celsius.* In the FC-plane, sketch the graph of the equation

$$C = \frac{5}{9}(F - 32)$$

linking Fahrenheit and Celsius temperatures (Example 12). On the same graph sketch the line $C = F$. Is there a temperature at which a Celsius thermometer gives the same numerical reading as a Fahrenheit thermometer? If so, find it.



21 The path of the light ray in Exercise 44. Angles of incidence and reflection are measured from the perpendicular.

46. *The Mt. Washington Cog Railway.* Civil engineers calculate the slope of roadbed as the ratio of the distance it rises or falls to the distance it runs horizontally. They call this ratio the **grade** of the roadbed, usually written as a percentage. Along the coast, commercial railroad grades are usually less than 2%. In the mountains, they may go as high as 4%. Highway grades are usually less than 5%.

    The steepest part of the Mt. Washington Cog Railway in New Hampshire has an exceptional 37.1% grade. Along this part of the track, the seats in the front of the car are 14 ft above those in the rear. About how far apart are the front and rear rows of seats?

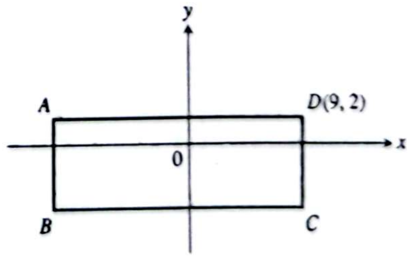
Theory and Examples

47. By calculating the lengths of its sides, show that the triangle with vertices at the points $A(1, 2)$, $B(5, 5)$, and $C(4, -2)$ is isosceles but not equilateral.

48. Show that the triangle with vertices $A(0, 0)$, $B(1, \sqrt{3})$, and $C(2, 0)$ is equilateral.

49. Show that the points $A(2, -1)$, $B(1, 3)$, and $C(-3, 2)$ are vertices of a square, and find the fourth vertex.

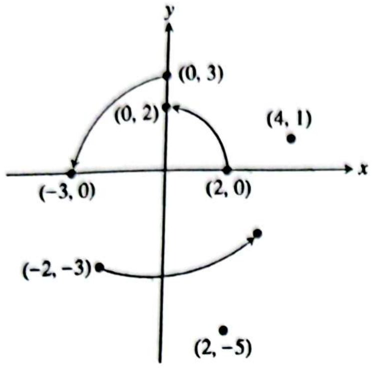
50. The rectangle shown here has sides parallel to the axes. It is three times as long as it is wide, and its perimeter is 56 units. Find the coordinates of the vertices A, B, and C.



51. Three different parallelograms have vertices at $(-1, 1)$, $(2, 0)$, and $(2, 3)$. Sketch them and find the coordinates of the fourth vertex of each.

52. A 90° rotation counterclockwise about the origin takes (2, 0) to (0, 2), and (0, 3) to (−3, 0), as shown in Fig. 22. Where does it take each of the following points?

a) (4, 1) b) (−2, −3) c) (2, −5)
d) $(x, 0)$ e) $(0, y)$ f) (x, y)
g) What point is taken to (10, 3)?



22 The points moved by the 90° rotation in Exercise 52.

53. For what value of k is the line $2x + ky = 3$ perpendicular to the line $4x + y = 1$? For what value of k are the lines parallel?

54. Find the line that passes through the point (1, 2) and through the point of intersection of the two lines $x + 2y = 3$ and $2x - 3y = -1$.

55. Show that the point with coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

is the midpoint of the line segment joining $P(x_1, y_1)$ to $Q(x_2, y_2)$.

56. *The distance from a point to a line.* We can find the distance from a point $P(x_0, y_0)$ to a line $L: Ax + By = C$ by taking the following steps (there is a somewhat faster method in Section 10.5):

1. Find an equation for the line M through P perpendicular to L.
2. Find the coordinates of the point Q in which M and L intersect.
3. Find the distance from P to Q.

Use these steps to find the distance from P to L in each of the following cases.

a) $P(2, 1)$, $L: y = x + 2$
b) $P(4, 6)$, $L: 4x + 3y = 12$
c) $P(a, b)$, $L: x = -1$
d) $P(x_0, y_0)$, $L: Ax + By = C$

3 Functions

Functions are the major tools for describing the real world in mathematical terms. This section reviews the notion of function and discusses some of the functions that arise in calculus.

Functions

The temperature at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). The interest paid on a cash investment depends on the length of time the investment is held. In each case, the value of one variable quantity, which we might call y, depends on the value of another variable quantity, which we might call x. Since the value of y is completely determined by the value of x, we say that y is a function of x.

The letters used for variable quantities may come from what is being described. When we study circles, we usually call the area A and the radius r. Since $A = \pi r^2$, we say that A is a function of r. The equation $A = \pi r^2$ is a *rule* that tells how to calculate a ***unique*** (single) output value of A for each possible input value of the radius r.

The set of all possible input values for the radius is called the **domain** of the function. The set of all output values of the area is the **range** of the function. Since circles cannot have negative radii or areas, the domain and range of the circle area function are both the interval $[0, \infty)$, consisting of all nonnegative real numbers.

The domain and range of a mathematical function can be any sets of objects; they do not have to consist of numbers. Most of the domains and ranges we will encounter in this book, however, will be sets of real numbers.

Leonhard Euler (1707–1783)

Leonhard Euler, the dominant mathematical figure of his century and the most prolific mathematician who ever lived, was also an astronomer, physicist, botanist, chemist, and expert in Oriental languages. He was the first scientist to give the function concept the prominence in his work that it has in mathematics today. Euler's collected books and papers fill 70 volumes. His introductory algebra text, written originally in German (Euler was Swiss), is still read in English translation.

In calculus we often want to refer to a generic function without having any particular formula in mind. Euler invented a symbolic way to say "y is a function of x" by writing

$$y = f(x) \qquad (\text{“}y \text{ equals } f \text{ of } x\text{”})$$

In this notation, the symbol f represents the function. The letter x, called the **independent variable,** represents an input value from the domain of f, and y, the **dependent variable,** represents the corresponding output value $f(x)$ in the range of f. Here is the formal definition of *function.*

Definition

A **function** from a set D to a set R is a rule that assigns a *unique* element $f(x)$ in R to each element x in D.

23 A function from a set D to a set R assigns a unique element of R to each element in D.

In this definition, $D = D(f)$ (read "D of f") is the domain of the function f and R is a set *containing* the range of f. See Fig. 23.

Think of a function f as a kind of machine that produces an output value $f(x)$ in its range whenever we feed it an input value x from its domain (Fig. 24).

In this book we will usually define functions in one of two ways:

1. by giving a formula such as $y = x^2$ that uses a dependent variable y to denote the value of the function, or
2. by giving a formula such as $f(x) = x^2$ that defines a function symbol f to name the function.

Strictly speaking, we should call the function f and not $f(x)$, as the latter denotes the value of the function at the point x. However, as is common usage, we will often refer to the function as $f(x)$ in order to name the variable on which f depends.

It is sometimes convenient to use a single letter to denote both a function and its dependent variable. For instance, we might say that the area A of a circle of radius r is given by the function $A(r) = \pi r^2$.

24 A "machine" diagram for a function.

Evaluation

As we said earlier, most of the functions in this book will be **real-valued functions of a real variable,** functions whose domains and ranges are sets of real numbers. We evaluate such functions by substituting particular values from the domain into the function's defining rule to calculate the corresponding values in the range.

EXAMPLE 1 The volume V of a ball (solid sphere) of radius r is given by the function

$$V(r) = \frac{4}{3}\pi r^3.$$

The volume of a ball of radius 3 m is

$$V(3) = \frac{4}{3}\pi(3)^3 = 36\pi \text{ m}^3. \qquad \square$$

EXAMPLE 2 Suppose that the function F is defined for all real numbers t by the formula

$$F(t) = 2(t - 1) + 3.$$

Evaluate F at the input values 0, 2, $x + 2$, and $F(2)$.

Solution In each case we substitute the given input value for t into the formula for F:

$$\begin{aligned} F(0) &= 2(0 - 1) + 3 = -2 + 3 = 1 \\ F(2) &= 2(2 - 1) + 3 = 2 + 3 = 5 \\ F(x + 2) &= 2(x + 2 - 1) + 3 = 2x + 5 \\ F(F(2)) &= F(5) = 2(5 - 1) + 3 = 11. \end{aligned}$$ □

The Domain Convention

When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly, the domain is assumed to be the largest set of x-values for which the formula gives real y-values. This is the function's so-called **natural domain.** If we want the domain to be restricted in some way, we must say so.

The domain of the function $y = x^2$ is understood to be the entire set of real numbers. The formula gives a real y-value for every real number x. If we want to restrict the domain to values of x greater than or equal to 2, we must write "$y = x^2, x \geq 2$."

Changing the domain to which we apply a formula usually changes the range as well. The range of $y = x^2$ is $[0, \infty)$. The range of $y = x^2$, $x \geq 2$, is the set of all numbers obtained by squaring numbers greater than or equal to 2. In symbols, the range is $\{x^2 | x \geq 2\}$ or $\{y | y \geq 4\}$ or $[4, \infty)$.

Most of the functions we encounter will have domains that are either intervals or unions of intervals.

EXAMPLE 3

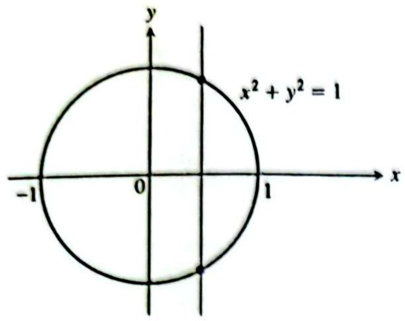
Function	Domain (x)	Range (y)
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$
$y = \dfrac{1}{x}$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$

The formula $y = \sqrt{1 - x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Beyond this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $\sqrt{1 - x^2}$ is $[0, 1]$.

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. *We cannot divide any number by zero.* The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is precisely the set of all nonzero real numbers.

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

In $y = \sqrt{4-x}$, the quantity $4-x$ cannot be negative. That is, $4-x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4-x}$ is $[0, \infty)$, the set of all square roots of nonnegative numbers. ❑



25 This circle is not the graph of a function $y = f(x)$; it fails the vertical line test.

Graphs of Functions

The **graph** of a function f is the graph of the equation $y = f(x)$. It consists of the points in the Cartesian plane whose coordinates (x, y) are input–output pairs for f.

Not every curve you draw is the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so no *vertical line* can intersect the graph of a function more than once. Thus, a circle cannot be the graph of a function since some vertical lines intersect the circle twice (Fig. 25). If a is in the domain of a function f, then the vertical line $x = a$ will intersect the graph of f in the single point $(a, f(a))$.

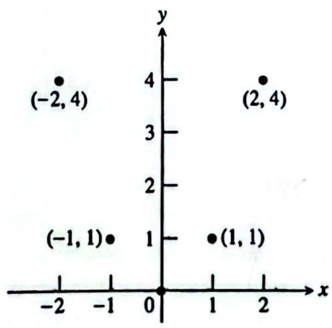
EXAMPLE 4 Graph the function $y = x^2$ over the interval $[-2, 2]$.

Solution

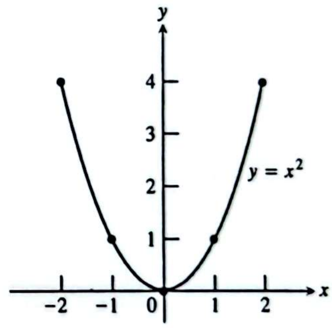
Step 1: Make a table of xy-pairs that satisfy the function rule, in this case the equation $y = x^2$.

x	$y = x^2$
−2	4
−1	1
0	0
1	1
2	4

Step 2: Plot the points (x, y) whose coordinates appear in the table.



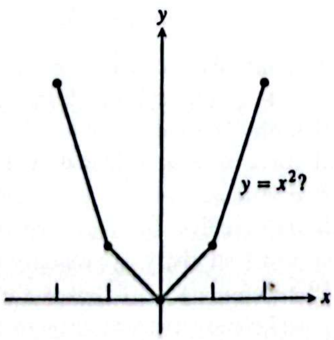
Step 3: Draw a smooth curve through the plotted points. Label the curve with its equation.

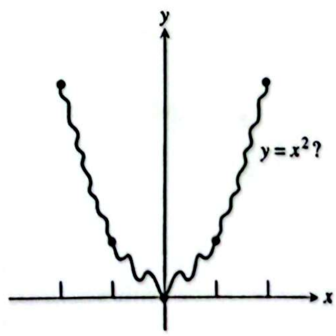


❑

Computers and graphing calculators graph functions in much this way—by stringing together plotted points—and the same question arises.

How do we know that the graph of $y = x^2$ doesn't look like one of these curves?

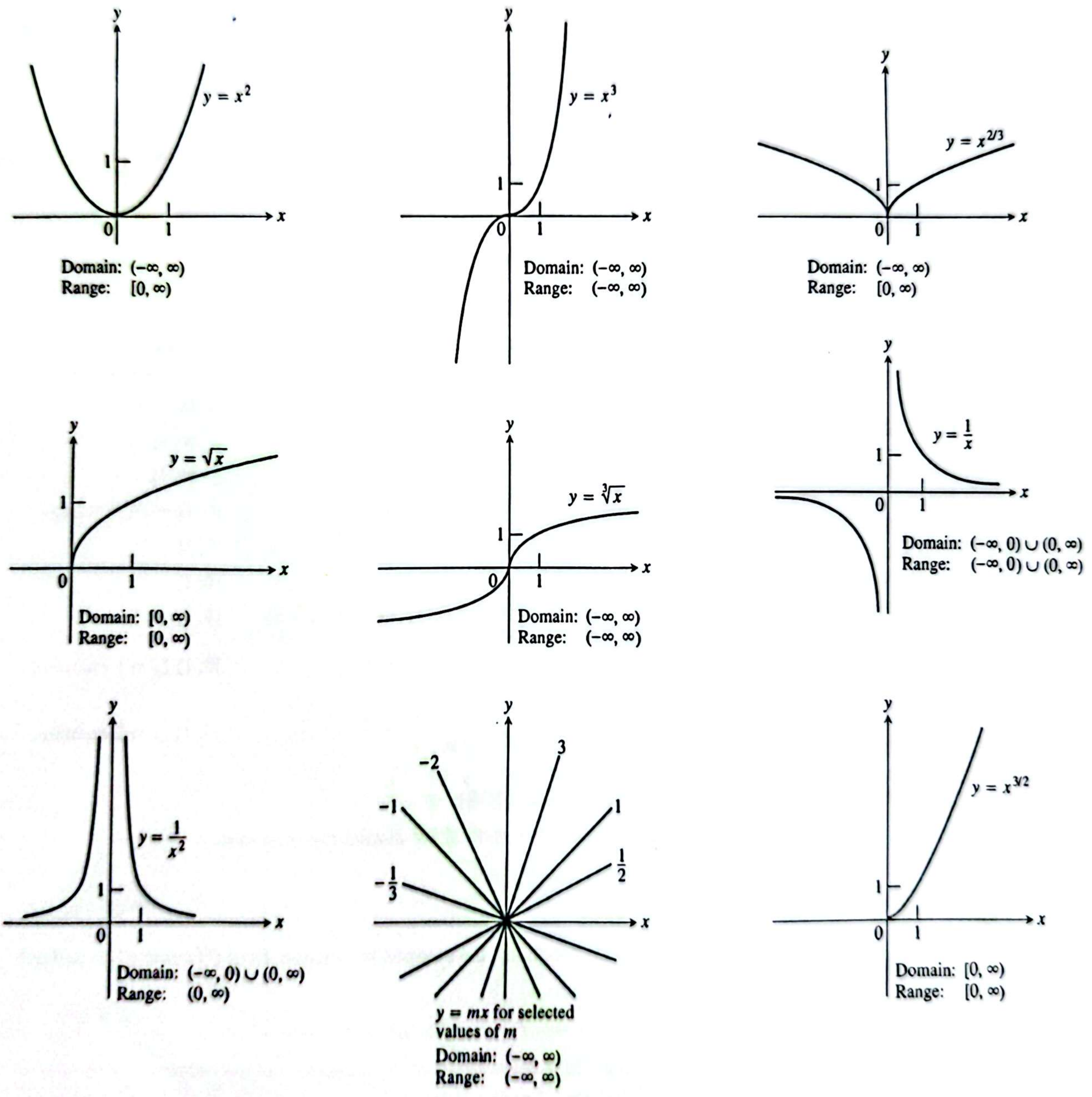




To find out, we could plot more points. But how would we then connect *them*? The basic question still remains: How do we know for sure what the graph looks like between the points we plot? The answer lies in calculus, as we will see in Chapter 3. There we will use a marvelous mathematical tool called the *derivative* to find a curve's shape between plotted points. Meanwhile we will have to settle for plotting points and connecting them as best we can.

Figure 26 shows the graphs of several functions frequently encountered in calculus. It is a good idea to learn the shapes of these graphs so that you can recognize them or sketch them when the need arises.

26 Useful graphs.



Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If f and g are functions, then for every x that belongs to the domains of both f and g, we define functions $f+g$, $f-g$, and fg by the formulas

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x).$$

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{(where } g(x) \neq 0\text{)}.$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

EXAMPLE 5

Function	Formula	Domain
f	$f(x) = \sqrt{x}$	$[0, \infty)$
g	$g(x) = \sqrt{1-x}$	$(-\infty, 1]$
$3g$	$3g(x) = 3\sqrt{1-x}$	$(-\infty, 1]$
$f+g$	$(f+g)(x) = \sqrt{x} + \sqrt{1-x}$	$[0, 1] = D(f) \cap D(g)$
$f-g$	$(f-g)(x) = \sqrt{x} - \sqrt{1-x}$	$[0, 1]$
$g-f$	$(g-f)(x) = \sqrt{1-x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1-x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

❑

Composite Functions

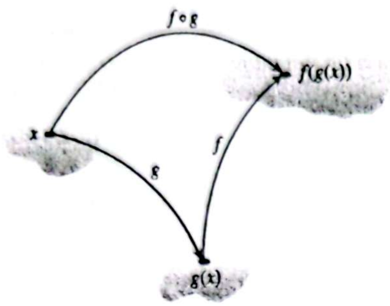
Composition is another method for combining functions.

Definition

If f and g are functions, the **composite function** $f \circ g$ ("f circle g") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.



27 The relation of $f \circ g$ to g and f.

The definition says that two functions can be composed when the range of the first lies in the domain of the second (Fig. 27). To find $(f \circ g)(x)$, we *first find* $g(x)$ and *second find* $f(g(x))$.

To evaluate the composite function $g \circ f$ (when defined), we reverse the order, finding $f(x)$ first and then $g(f(x))$. The domain of $g \circ f$ is the set of numbers x in the domain of f such that $f(x)$ lies in the domain of g.

The functions $f \circ g$ and $g \circ f$ are usually quite different.

EXAMPLE 6 If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

a) $(f \circ g)(x)$ b) $(g \circ f)(x)$ c) $(f \circ f)(x)$ d) $(g \circ g)(x)$.

Solution

	Composite	Domain
a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x+1) + 1 = x + 2$	$\mathbb{R}$ or $(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is to say, if $x \geq -1$. ❑

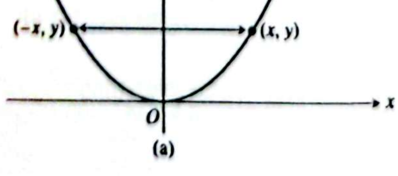
Even Functions and Odd Functions—Symmetry

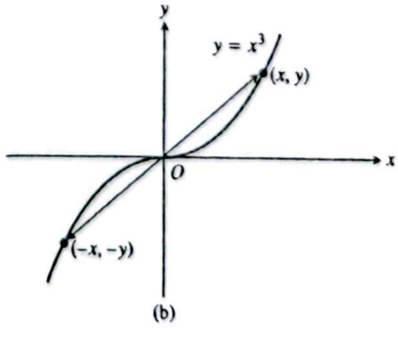
A function $y = f(x)$ is **even** if $f(-x) = f(x)$ for every number x in the domain of f. Notice that this implies that both x and $-x$ must be in the domain of f. The function $f(x) = x^2$ is even because $f(-x) = (-x)^2 = x^2 = f(x)$.

The graph of an even function $y = f(x)$ is symmetric about the y-axis. Since $f(-x) = f(x)$, the point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph (Fig. 28a). Once we know the graph on one side of the y-axis, we automatically know it on the other side.

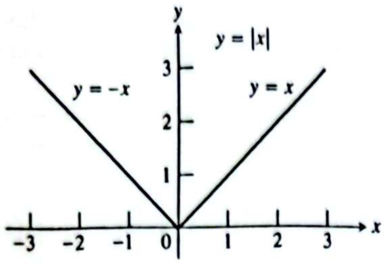
A function $y = f(x)$ is **odd** if $f(-x) = -f(x)$ for every number x in the domain of f. Again, both x and $-x$ must lie in the domain of f. The function $f(x) = x^3$ is odd because $f(-x) = (-x)^3 = -x^3 = -f(x)$.

The graph of an odd function $y = f(x)$ is symmetric about the origin. Since $f(-x) = -f(x)$, the point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph (Fig. 28b). Here again, once we know the graph of f on one side of the y-axis, we know it on both sides.





28 **(a) Symmetry about the y-axis. If (x, y) is on the graph, so is $(-x, y)$. (b) Symmetry about the origin. If (x, y) is on the graph, so is $(-x, -y)$.**



29 The absolute value function.

Piecewise Defined Functions

Sometimes a function uses different formulas on different parts of its domain. One example is the absolute value function

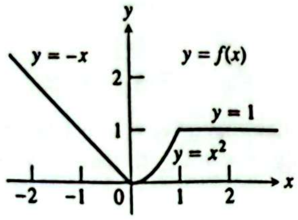
$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

whose graph is given in Fig. 29. Here are some examples.

EXAMPLE 7 The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

is defined on the entire real line but has values given by different formulas depending on the position of x (Fig. 30). ❑



30 To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain (Example 7).

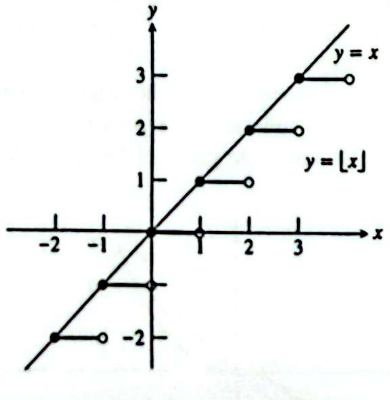
EXAMPLE 8 *The greatest integer function*

The function whose value at any number x is the *greatest integer less than or equal to x* is called the **greatest integer function** or the **integer floor function.** It is denoted $\lfloor x \rfloor$, or, in some books, $[x]$ or $[[x]]$. Figure 31 shows the graph. Observe that

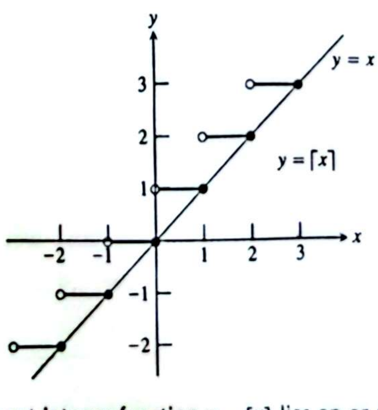
$$\lfloor 2.4 \rfloor = 2, \quad \lfloor 1.9 \rfloor = 1, \quad \lfloor 0 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 2 \rfloor = 2, \quad \lfloor 0.2 \rfloor = 0, \quad \lfloor -0.3 \rfloor = -1 \quad \lfloor -2 \rfloor = -2.$$ ❑

EXAMPLE 9 *The least integer function*

The function whose value at any number x is the *smallest integer greater than or equal to x* is called the **least integer function** or the **integer ceiling function.** It is denoted $\lceil x \rceil$. Figure 32 shows the graph. For positive values of x, this function might represent, for example, the cost of parking x hours in a parking lot which charges $1 for each hour or part of an hour.



31 The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for x.



32 The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for x. ❑

Exercises 3

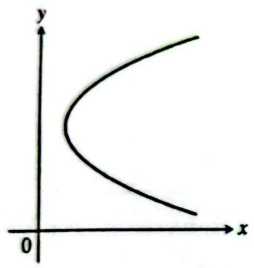
Functions

In Exercises 1–6, find the domain and range of each function.

1. $f(x) = 1 + x^2$
2. $f(x) = 1 - \sqrt{x}$
3. $F(t) = \dfrac{1}{\sqrt{t}}$
4. $F(t) = \dfrac{1}{1 + \sqrt{t}}$
5. $g(z) = \sqrt{4 - z^2}$
6. $g(z) = \dfrac{1}{\sqrt{4 - z^2}}$

In Exercises 7 and 8, which of the graphs are graphs of functions of x, and which are not? Give reasons for your answers.

7. a)



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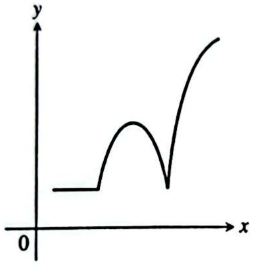
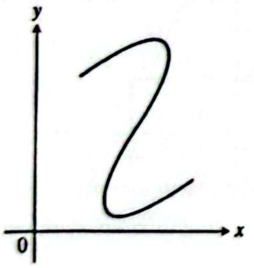


8. a)



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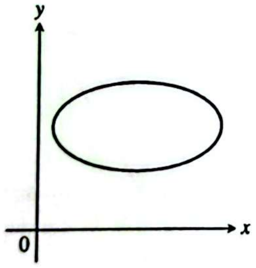


Finding Formulas for Functions

9. Express the area and perimeter of an equilateral triangle as a function of the triangle's side length x.

10. Express the side length of a square as a function of the length d of the square's diagonal. Then express the area as a function of the diagonal length.

11. Express the edge length of a cube as a function of the cube's diagonal length d. Then express the surface area and volume of the cube as a function of the diagonal length.

12. A point P in the first quadrant lies on the graph of the function $f(x) = \sqrt{x}$. Express the coordinates of P as functions of the slope of the line joining P to the origin.

Functions and Graphs

Graph the functions in Exercises 13–24. What symmetries, if any, do the graphs have? Use the graphs in Fig. 26 for guidance, as needed.

13. $y = -x^3$
14. $y = -\dfrac{1}{x^2}$
15. $y = -\dfrac{1}{x}$
16. $y = \dfrac{1}{|x|}$
17. $y = \sqrt{|x|}$
18. $y = \sqrt{-x}$
19. $y = x^3/8$
20. $y = -4\sqrt{x}$
21. $y = -x^{3/2}$
22. $y = (-x)^{3/2}$
23. $y = (-x)^{2/3}$
24. $y = -x^{2/3}$

25. Graph the following equations and explain why they are not graphs of functions of x.

    a) $|y| = x$ b) $y^2 = x^2$

26. Graph the following equations and explain why they are not graphs of functions of x.

    a) $|x| + |y| = 1$ b) $|x + y| = 1$

Even and Odd Functions

In Exercises 27–38, say whether the function is even, odd, or neither.

27. $f(x) = 3$
28. $f(x) = x^{-5}$
29. $f(x) = x^2 + 1$
30. $f(x) = x^2 + x$
31. $g(x) = x^3 + x$
32. $g(x) = x^4 + 3x^2 - 1$
33. $g(x) = \dfrac{1}{x^2 - 1}$
34. $g(x) = \dfrac{x}{x^2 - 1}$
35. $h(t) = \dfrac{1}{t - 1}$
36. $h(t) = |t^3|$
37. $h(t) = 2t + 1$
38. $h(t) = 2|t| + 1$

Sums, Differences, Products, and Quotients

In Exercises 39 and 40, find the domains and ranges of f, g, $f + g$, and $f \cdot g$.

39. $f(x) = x, \quad g(x) = \sqrt{x - 1}$
40. $f(x) = \sqrt{x + 1}, \quad g(x) = \sqrt{x - 1}$

In Exercises 41 and 42, find the domains and ranges of f, g, f/g, and g/f.

41. $f(x) = 2, \quad g(x) = x^2 + 1$
42. $f(x) = 1, \quad g(x) = 1 + \sqrt{x}$

Composites of Functions

43. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, find the following.

a) $f(g(0))$ b) $g(f(0))$
c) $f(g(x))$ d) $g(f(x))$
e) $f(f(-5))$ f) $g(g(2))$
g) $f(f(x))$ h) $g(g(x))$

44. If $f(x) = x - 1$ and $g(x) = 1/(x + 1)$, find the following.

a) $f(g(1/2))$ b) $g(f(1/2))$
c) $f(g(x))$ d) $g(f(x))$
e) $f(f(2))$ f) $g(g(2))$
g) $f(f(x))$ h) $g(g(x))$

45. If $u(x) = 4x - 5$, $v(x) = x^2$, and $f(x) = 1/x$, find formulas for the following.

a) $u(v(f(x)))$ b) $u(f(v(x)))$
c) $v(u(f(x)))$ d) $v(f(u(x)))$
e) $f(u(v(x)))$ f) $f(v(u(x)))$

46. If $f(x) = \sqrt{x}$, $g(x) = x/4$, and $h(x) = 4x - 8$, find formulas for the following.

a) $h(g(f(x)))$ b) $h(f(g(x)))$
c) $g(h(f(x)))$ d) $g(f(h(x)))$
e) $f(g(h(x)))$ f) $f(h(g(x)))$

Let $f(x) = x - 3$, $g(x) = \sqrt{x}$, $h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in Exercises 47 and 48 as a composite involving one or more of f, g, h, and j.

47. a) $y = \sqrt{x} - 3$ b) $y = 2\sqrt{x}$
c) $y = x^{1/4}$ d) $y = 4x$
e) $y = \sqrt{(x-3)^3}$ f) $y = (2x-6)^3$

48. a) $y = 2x - 3$ b) $y = x^{3/2}$
c) $y = x^9$ d) $y = x - 6$
e) $y = 2\sqrt{x-3}$ f) $y = \sqrt{x^3 - 3}$

49. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a)	$x - 7$	$\sqrt{x}$	
b)	$x + 2$	$3x$	
c)		$\sqrt{x-5}$	$\sqrt{x^2-5}$
d)	$\dfrac{x}{x-1}$	$\dfrac{x}{x-1}$	
e)		$1 + \dfrac{1}{x}$	x
f)	$\dfrac{1}{x}$		x

50. *A magic trick.* You may have heard of a magic trick that goes like this: Take any number. Add 5. Double the result. Subtract 6. Divide by 2. Subtract 2. Now tell me your answer, and I'll tell you what you started with.

Pick a number and try it.

You can see what is going on if you let x be your original number and follow the steps to make a formula $f(x)$ for the number you end up with.

Piecewise Defined Functions

Graph the functions in Exercises 51–54.

51. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

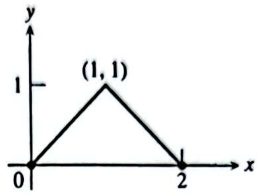
52. $g(x) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \end{cases}$

53. $F(x) = \begin{cases} 3 - x, & x \le 1 \\ 2x, & x > 1 \end{cases}$

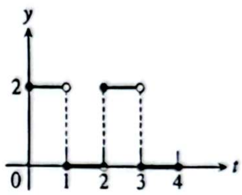
54. $G(x) = \begin{cases} 1/x, & x < 0 \\ x, & 0 \le x \end{cases}$

55. Find a formula for each function graphed.

a)

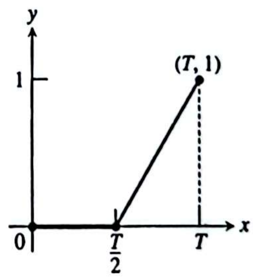


b)



56. Find a formula for each function graphed.

a)



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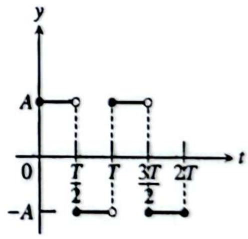


The Greatest and Least Integer Functions

57. For what values of x is (a) $\lfloor x \rfloor = 0$? (b) $\lceil x \rceil = 0$?

58. What real numbers x satisfy the equation $\lfloor x \rfloor = \lceil x \rceil$?

59. Does $\lceil -x \rceil = -\lfloor x \rfloor$ for all real x? Give reasons for your answer.

60. Graph the function

$$f(x) = \begin{cases} \lfloor x \rfloor, & x \ge 0 \\ \lceil x \rceil, & x < 0 \end{cases}$$

Why is $f(x)$ called the *integer part* of x?

Even and Odd Functions

61. Assume that f is an even function, g is an odd function, and both f and g are defined on the entire real line $\mathbb{R}$. Which of the following (where defined) are even? odd?

a) fg b) f/g c) g/f
d) $f^2 = ff$ e) $g^2 = gg$ f) $f \circ g$
g) $g \circ f$ h) $f \circ f$ i) $g \circ g$

62. Can a function be both even and odd? Give reasons for your answer.

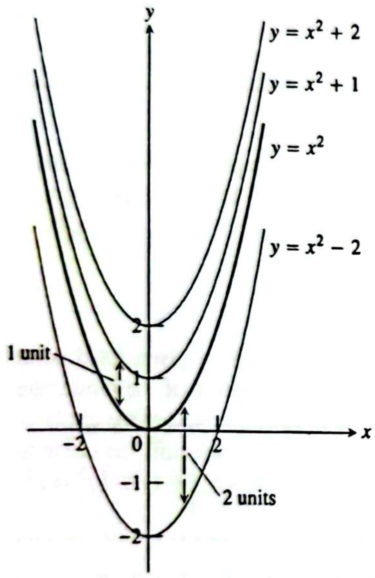
Grapher

63. (*Continuation of Example 5.*) Graph the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$ together with their (a) sum, (b) product, (c) two differences, (d) two quotients.

64. Let $f(x) = x - 7$ and $g(x) = x^2$. Graph f and g together with $f \circ g$ and $g \circ f$.

4 Shifting Graphs

This section shows how to change an equation to shift its graph up or down or to the right or left. Knowing about this can help us spot familiar graphs in new locations. It can also help us graph unfamiliar equations more quickly. We practice mostly with circles and parabolas (because they make useful examples in calculus), but the methods apply to other curves as well. We will revisit parabolas and circles in Chapter 9.



33 To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for f.

How to Shift a Graph

To shift the graph of a function $y = f(x)$ straight up, we add a positive constant to the right-hand side of the formula $y = f(x)$.

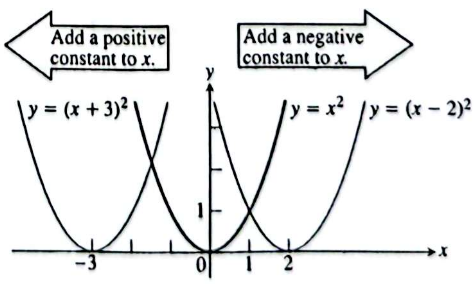
EXAMPLE 1 Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Fig. 33). ❑

To shift the graph of a function $y = f(x)$ straight down, we add a negative constant to the right-hand side of the formula $y = f(x)$.

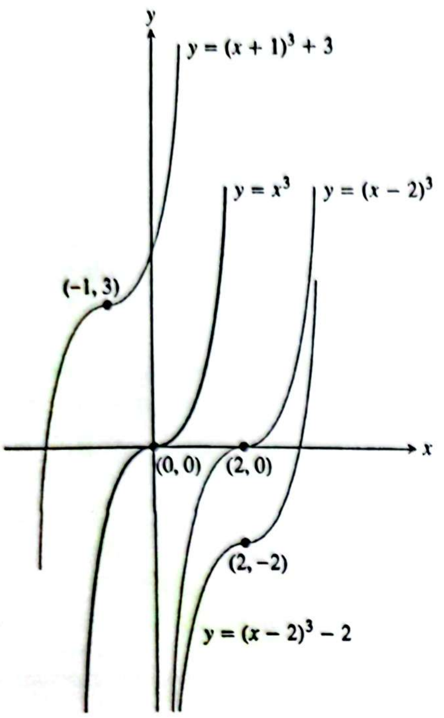
EXAMPLE 2 Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Fig. 33). ❑

To shift the graph of $y = f(x)$ to the left, we add a positive constant to x.

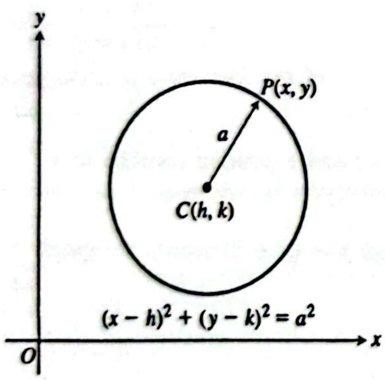
EXAMPLE 3 Adding 3 to x in $y = x^2$ to get $y = (x+3)^2$ shifts the graph 3 units to the left (Fig. 34). ❑



34 To shift the graph of $y = x^2$ to the left, we add a positive constant to x. To shift the graph to the right, we add a negative constant to x.



35 The graph of $y = x^3$ shifted to three new positions in the xy-plane.



36 A circle of radius a in the xy-plane, with center at (h, k).

To shift the graph of $y = f(x)$ to the right, we add a negative constant to x.

EXAMPLE 4 Adding -2 to x in $y = x^2$ to get $y = (x-2)^2$ shifts the graph 2 units to the right (Fig. 34). ❑

Shift Formulas

VERTICAL SHIFTS

$y - k = f(x)$ or	Shifts the graph *up* k units if $k > 0$
$y = f(x) + k$	Shifts it *down* $\lvert k\rvert$ units if $k < 0$

HORIZONTAL SHIFTS

$y = f(x - h)$	Shifts the graph *right* h units *if* $h > 0$
	Shifts it *left* $\lvert h\rvert$ units if $h < 0$

EXAMPLE 5 The graph of $y = (x-2)^3 - 2$ is the graph of $y = x^3$ shifted 2 units to the right and 2 units down. The graph of $y = (x+1)^3 + 3$ is the graph of $y = x^3$ shifted 1 unit to the left and 3 units up (Fig. 35). ❑

Equations for Circles

A **circle** is the set of points in a plane whose distance from a given fixed point in the plane is constant (Fig. 36). The fixed point is the **center** of the circle; the constant distance is the **radius**. We saw in Section 2, Example 4, that the circle of radius a centered at the origin has equation $x^2 + y^2 = a^2$. If we shift the circle to place its center at the point (h, k), its equation becomes $(x-h)^2 + (y-k)^2 = a^2$.

The Standard Equation for the Circle of Radius *a* Centered at the Point (*h*, *k*)

$$(x-h)^2 + (y-k)^2 = a^2 \tag{1}$$

EXAMPLE 6 If the circle $x^2 + y^2 = 25$ is shifted 2 units to the left and 3 units up, its new equation is $(x+2)^2 + (y-3)^2 = 25$. As Eq. (1) says it should be, this is the equation of the circle of radius 5 centered at $(h, k) = (-2, 3)$. ❑

EXAMPLE 7 The standard equation for the circle of radius 2 centered at (3, 4) is

$$(x-3)^2 + (y-4)^2 = (2)^2$$

or

$$(x-3)^2 + (y-4)^2 = 4.$$

There is no need to square out the x- and y-terms in this equation. In fact, it is better not to do so. The present form reveals the circle's center and radius. ❑

EXAMPLE 8 Find the center and radius of the circle

$$(x-1)^2+(y+5)^2=3.$$

Solution Comparing

$$(x-h)^2+(y-k)^2=a^2$$

with

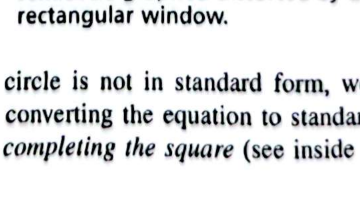
$$(x-1)^2+(y+5)^2=3$$

shows that $h=1, k=-5$, and $a=\sqrt{3}$. The center is the point $(h,k)=(1,-5)$; the radius is $a=\sqrt{3}$. ❑

Technology *Square Windows* We use the term "square window" when the units or scalings on both axes are the same. In a square window graphs are true in shape. They are distorted in a nonsquare window.

The term square window does not refer to the shape of the graphic display. Graphing calculators usually have rectangular displays. The displays of Computer Algebra Systems are usually square. When a graph is displayed, the x-unit may differ from the y-unit in order to fit the graph in the display, resulting in a distorted picture. The graphing window can be made square by shrinking or stretching the units on one axis to match the scale on the other, giving the true graph. Many systems have built-in functions to make the window "square." If yours does not, you will have to do some calculations and set the window size manually to get a square window, or bring to your viewing some foreknowledge of the true picture.

On your graphing utility, compare the perpendicular lines $y_1=x$ and $y_2=-x+4$ in a square window and a nonsquare one such as $[-10,10]$ by $[10,10]$. Graph the semicircle $y=\sqrt{8-x^2}$ in the same windows.

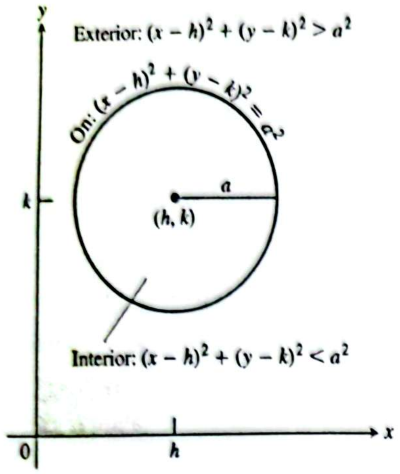


Two perpendicular lines and a semicircle graphed distorted by a rectangular window.

If an equation for a circle is not in standard form, we can find the circle's center and radius by first converting the equation to standard form. The algebraic technique for doing so is *completing the square* (see inside front cover).

EXAMPLE 9 Find the center and radius of the circle

$$x^2+y^2+4x-6y-3=0.$$



37 The interior and exterior of the circle $(x-h)^2+(y-k)^2=a^2$.

Solution We convert the equation to standard form by completing the squares in x and y:

$$x^2+y^2+4x-6y-3=0$$

Start with the given equation.

$$(x^2+4x\quad)+(y^2-6y\quad)=3$$

Gather terms. Move the constant to the right-hand side.

$$\left(x^2+4x+\left(\frac{4}{2}\right)^2\right)+\left(y^2-6y+\left(\frac{-6}{2}\right)^2\right)=3+\left(\frac{4}{2}\right)^2+\left(\frac{-6}{2}\right)^2$$

Add the square of half the coefficient of x to each side of the equation. Do the same for y. The parenthetical expressions on the left-hand side are now perfect squares.

$$(x^2+4x+4)+(y^2-6y+9)=3+4+9$$

$$(x+2)^2+(y-3)^2=16$$

Write each quadratic as a squared linear expression.

With the equation now in standard form, we read off the center's coordinates and the radius: $(h,k)=(-2,3)$ and $a=4$. ❑

Interior and Exterior

The points that lie inside the circle $(x-h)^2+(y-k)^2=a^2$ are the points less than a units from (h,k). They satisfy the inequality

$$(x-h)^2+(y-k)^2<a^2.$$

They make up the region we call the **interior** of the circle (Fig. 37).

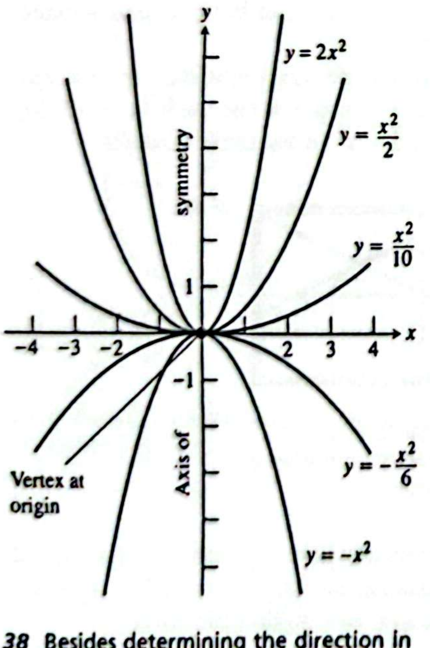
The circle's **exterior** consists of the points that lie more than a units from (h,k). These points satisfy the inequality

$$(x-h)^2+(y-k)^2>a^2.$$

EXAMPLE 10

Inequality	Region
$x^2+y^2<1$	Interior of the unit circle
$x^2+y^2\le 1$	Unit circle plus its interior
$x^2+y^2>1$	Exterior of the unit circle
$x^2+y^2\ge 1$	Unit circle plus its exterior

❑



38 Besides determining the direction in which the parabola $y=ax^2$ opens, the number a is a scaling factor. The parabola widens as a approaches zero and narrows as $|a|$ becomes large.

Parabolic Graphs

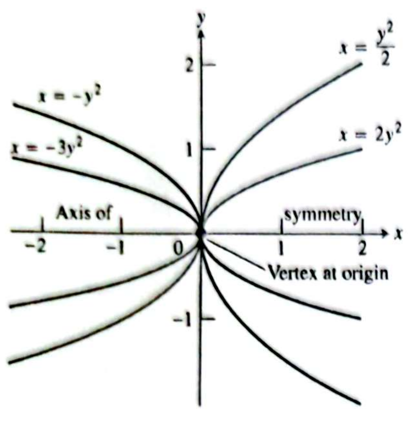
The graph of an equation like $y=3x^2$ or $y=-5x^2$ that has the form

$$y=ax^2$$

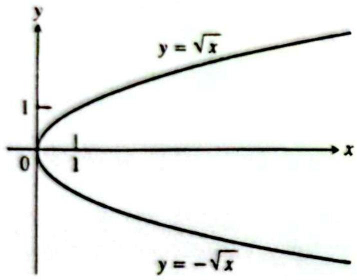
is a **parabola** whose **axis** (axis of symmetry) is the y-axis. The parabola's **vertex** (point where the parabola and axis cross) lies at the origin. The parabola opens upward if $a>0$ and downward if $a<0$. The larger the value of $|a|$, the narrower the parabola (Fig. 38).

If we interchange x and y in the formula $y=ax^2$, we obtain the equation

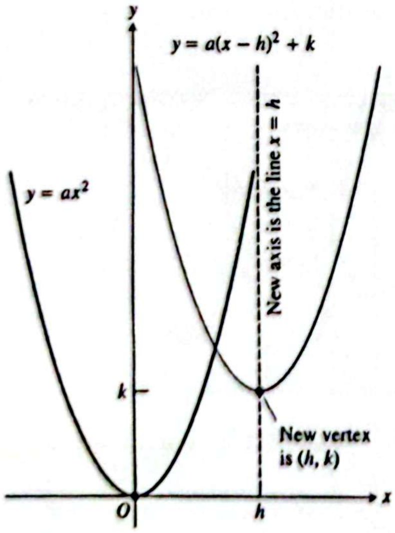
$$x=ay^2.$$



39 The parabola $x = ay^2$ is symmetric about the x-axis. It opens to the right if $a > 0$ and to the left if $a < 0$.



40 The graphs of the functions $y = \sqrt{x}$ and $y = -\sqrt{x}$ join at the origin to make the graph of the equation $x = y^2$ (Example 11).



41 The parabola $y = ax^2$, $a > 0$, shifted h units to the right and k units up.

With x and y now reversed, the graph is a parabola whose axis is the x-axis and whose vertex lies at the origin (Fig. 39).

EXAMPLE 11 The formula $x = y^2$ gives x as a function of y but does *not* give y as a function of x. If we solve for y, we find that $y = \pm\sqrt{x}$. For each positive value of x we get *two* values of y instead of the required single value.

When taken separately, the formulas $y = \sqrt{x}$ and $y = -\sqrt{x}$ do define functions of x. Each formula gives exactly one value of y for each possible value of x. The graph of $y = \sqrt{x}$ is the upper half of the parabola $x = y^2$. The graph of $y = -\sqrt{x}$ is the lower half (Fig. 40). ❑

The Quadratic Equation $y = ax^2 + bx + c$, $a \neq 0$

To shift the parabola $y = ax^2$ horizontally, we rewrite the equation as

$$y = a(x-h)^2.$$

To shift it vertically as well, we change the equation to

$$y - k = a(x-h)^2. \tag{2}$$

The combined shifts place the vertex at the point (h, k) and the axis along the line $x = h$ (Fig. 41).

Normally there would be no point in multiplying out the right-hand side of Eq. (2). In this case, however, we can learn something from doing so because the resulting equation, when rearranged, takes the form

$$y = ax^2 + bx + c. \tag{3}$$

This tells us that the graph of every equation of the form $y = ax^2 + bx + c, a \neq 0$, is the graph of $y = ax^2$ shifted somewhere else. Why? Because the steps that take us from Eq. (2) to Eq. (3) can be reversed to take us from (3) back to (2). The curve $y = ax^2 + bx + c$ has the same shape and orientation as the curve $y = ax^2$.

The axis of the parabola $y = ax^2 + bx + c$ turns out to be the line $x = -b/(2a)$. The y-intercept, $y = c$, is obtained by setting $x = 0$.

The Graph of $y = ax^2 + bx + c$, $a \neq 0$

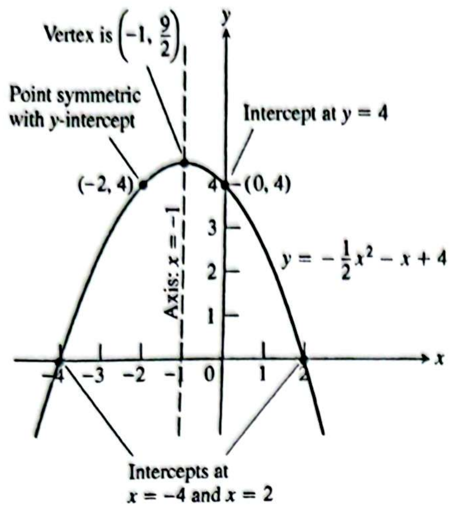
The graph of the equation $y = ax^2 + bx + c, a \neq 0$, is a parabola. The parabola opens upward if $a > 0$ and downward if $a < 0$. The axis is the line

$$x = -\frac{b}{2a}. \tag{4}$$

The vertex of the parabola is the point where the axis and parabola intersect. Its x-coordinate is $x = -b/2a$; its y-coordinate is found by substituting $x = -b/2a$ in the parabola's equation.

EXAMPLE 12 *Graphing a parabola*

Graph the equation $y = -\frac{1}{2}x^2 - x + 4$.



42 The parabola in Example 12.

Solution We take the following steps.

Step 1: *Compare the equation with $y = ax^2 + bx + c$ to identify a, b, and c.*

$$a = -\frac{1}{2}, \qquad b = -1, \qquad c = 4$$

Step 2: *Find the direction of opening.* Down, because $a < 0$.

Step 3: *Find the axis and vertex.* The axis is the line

$$x = -\frac{b}{2a} = -\frac{(-1)}{2(-1/2)} = -1, \qquad \text{Eq. (4)}$$

so the x-coordinate of the vertex is -1. The y-coordinate is

$$y = -\frac{1}{2}(-1)^2 - (-1) + 4 = \frac{9}{2}.$$

The vertex is $(-1, 9/2)$.

Step 4: *Find the x-intercepts (if any).*

$$-\frac{1}{2}x^2 - x + 4 = 0 \qquad \text{Set } y = 0 \text{ in the parabola's equation.}$$

$$x^2 + 2x - 8 = 0 \qquad \text{Solve as usual.}$$

$$(x - 2)(x + 4) = 0$$

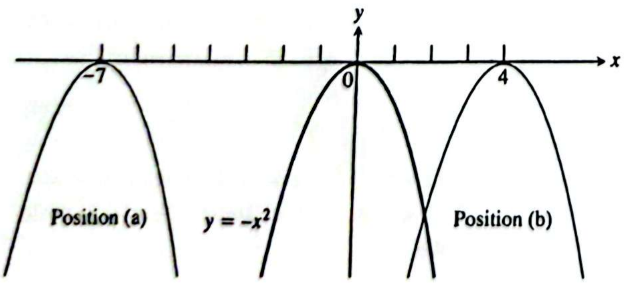
$$x = 2, \qquad x = -4$$

Step 5: *Sketch the graph.* We plot points, sketch the axis (lightly), and use what we know about symmetry and the direction of opening to complete the graph (Fig. 42). ☐

Exercises 4

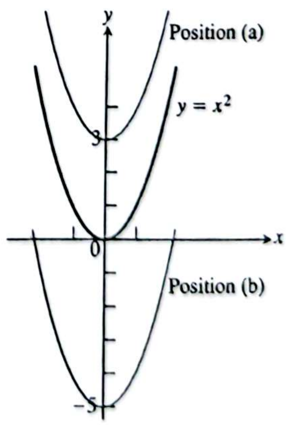
Shifting Graphs

1. Figure 43 shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.



43 The parabolas in Exercise 1.

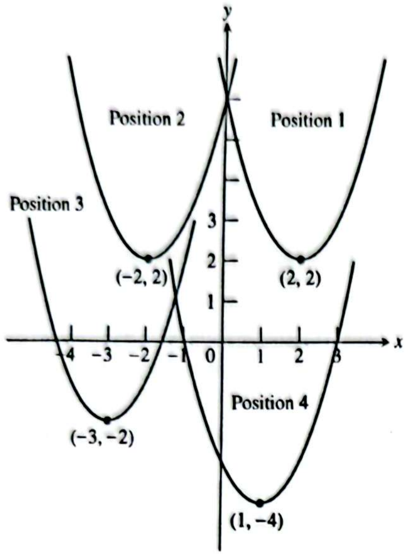
2. Figure 44 shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.



44 The parabolas in Exercise 2.

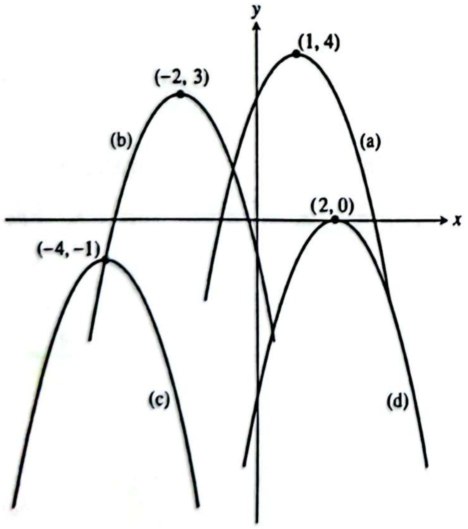
3. Match the equations listed in (a)–(d) to the graphs in Fig. 45.

a) $y = (x-1)^2 - 4$ b) $y = (x-2)^2 + 2$
c) $y = (x+2)^2 + 2$ d) $y = (x+3)^2 - 2$



45 The parabolas in Exercise 3.

4. Figure 46 shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



46 The parabolas in Exercise 4.

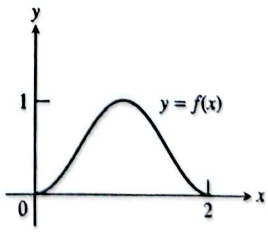
Exercises 5–16 tell how many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation. Use the graphs in Fig. 26 for reference as needed.

5. $x^2 + y^2 = 49$ Down 3, left 2
6. $x^2 + y^2 = 25$ Up 3, left 4
7. $y = x^3$ Left 1, down 1
8. $y = x^{2/3}$ Right 1, down 1
9. $y = \sqrt{x}$ Left 0.81
10. $y = -\sqrt{x}$ Right 3
11. $y = 2x - 7$ Up 7
12. $y = \dfrac{1}{2}(x+1) + 5$ Down 5, right 1
13. $x = y^2$ Left 1
14. $x = -3y^2$ Up 2, right 3
15. $y = 1/x$ Up 1, right 1
16. $y = 1/x^2$ Left 2, down 1

Graph the functions in Exercises 17–36. Use the graphs in Fig. 26 for reference as needed.

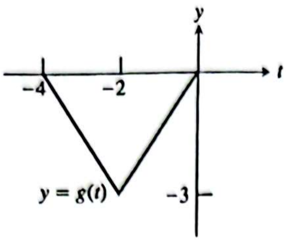
17. $y = \sqrt{x+4}$
18. $y = \sqrt{9-x}$
19. $y = |x-2|$
20. $y = |1-x| - 1$
21. $y = 1 + \sqrt{x-1}$
22. $y = 1 - \sqrt{x}$
23. $y = (x+1)^{2/3}$
24. $y = (x-8)^{2/3}$
25. $y = 1 - x^{2/3}$
26. $y + 4 = x^{2/3}$
27. $y = \sqrt[3]{x-1} - 1$
28. $y = (x+2)^{3/2} + 1$
29. $y = \dfrac{1}{x-2}$
30. $y = \dfrac{1}{x} - 2$
31. $y = \dfrac{1}{x} + 2$
32. $y = \dfrac{1}{x+2}$
33. $y = \dfrac{1}{(x-1)^2}$
34. $y = \dfrac{1}{x^2} - 1$
35. $y = \dfrac{1}{x^2} + 1$
36. $y = \dfrac{1}{(x+1)^2}$

37. The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.



a) $f(x) + 2$ b) $f(x) - 1$
c) $2f(x)$ d) $-f(x)$
e) $f(x+2)$ f) $f(x-1)$
g) $f(-x)$ h) $-f(x+1) + 1$

38. The accompanying figure shows the graph of a function $g(t)$ with domain $[-4, 0]$ and range $[-3, 0]$. Find the domains and ranges of the following functions, and sketch their graphs.



a) $g(-t)$
b) $-g(t)$
c) $g(t)+3$
d) $1-g(t)$
e) $g(-t+2)$
f) $g(t-2)$
g) $g(1-t)$
h) $-g(t-4)$

Circles

In Exercises 39–44, find an equation for the circle with the given center $C(h, k)$ and radius a. Then sketch the circle in the xy-plane. Include the circle's center in your sketch. Also, label the circle's x- and y-intercepts, if any, with their coordinate pairs.

39. $C(0, 2),\quad a=2$
40. $C(-3, 0),\quad a=3$
41. $C(-1, 5),\quad a=\sqrt{10}$
42. $C(1, 1),\quad a=\sqrt{2}$
43. $C(-\sqrt{3}, -2),\quad a=2$
44. $C(3, 1/2),\quad a=5$

Graph the circles whose equations are given in Exercises 45–50. Label each circle's center and intercepts (if any) with their coordinate pairs.

45. $x^2+y^2+4x-4y+4=0$
46. $x^2+y^2-8x+4y+16=0$
47. $x^2+y^2-3y-4=0$
48. $x^2+y^2-4x-(9/4)=0$
49. $x^2+y^2-4x+4y=0$
50. $x^2+y^2+2x=3$

Parabolas

Graph the parabolas in Exercises 51–58. Label the vertex, axis, and intercepts in each case.

51. $y=x^2-2x-3$
52. $y=x^2+4x+3$
53. $y=-x^2+4x$
54. $y=-x^2+4x-5$
55. $y=-x^2-6x-5$
56. $y=2x^2-x+3$
57. $y=\frac{1}{2}x^2+x+4$
58. $y=-\frac{1}{4}x^2+2x+4$
59. Graph the parabola $y=x-x^2$. Then find the domain and range of $f(x)=\sqrt{x-x^2}$.
60. Graph the parabola $y=3-2x-x^2$. Then find the domain and range of $g(x)=\sqrt{3-2x-x^2}$.

Inequalities

Describe the regions defined by the inequalities and pairs of inequalities in Exercises 61–68.

61. $x^2+y^2>7$
62. $x^2+y^2<5$
63. $(x-1)^2+y^2\le 4$
64. $x^2+(y-2)^2\ge 4$
65. $x^2+y^2>1,\quad x^2+y^2<4$
66. $x^2+y^2\le 4,\quad (x+2)^2+y^2\le 4$
67. $x^2+y^2+6y<0,\quad y>-3$
68. $x^2+y^2-4x+2y>4,\quad x>2$
69. Write an inequality that describes the points that lie inside the circle with center $(-2, 1)$ and radius $\sqrt{6}$.
70. Write an inequality that describes the points that lie outside the circle with center $(-4, 2)$ and radius 4.
71. Write a pair of inequalities that describe the points that lie inside or on the circle with center $(0, 0)$ and radius $\sqrt{2}$, and on or to the right of the vertical line through $(1, 0)$.
72. Write a pair of inequalities that describe the points that lie outside the circle with center $(0, 0)$ and radius 2, and inside the circle that has center $(1, 3)$ and passes through the origin.

Shifting Lines

73. The line $y=mx$, which passes through the origin, is shifted vertically and horizontally to pass through the point (x_0, y_0). Find an equation for the new line. (This equation is called the line's *point–slope equation.*)
74. The line $y=mx$ is shifted vertically to pass through the point $(0, b)$. What is the new line's equation?

Intersecting Lines, Circles, and Parabolas

In Exercises 75–82, graph the two equations and find the points in which the graphs intersect.

75. $y=2x,\quad x^2+y^2=1$
76. $x+y=1,\quad (x-1)^2+y^2=1$
77. $y-x=1,\quad y=x^2$
78. $x+y=0,\quad y=-(x-1)^2$
79. $y=-x^2,\quad y=2x^2-1$
80. $y=\frac{1}{4}x^2,\quad y=(x-1)^2$
81. $x^2+y^2=1,\quad (x-1)^2+y^2=1$
82. $x^2+y^2=1,\quad x^2+y=1$

CAS Explorations and Projects

In Exercises 83–86, you will explore graphically what happens to the graph of $y=f(ax)$ as you change the value of the constant a. Use a CAS or computer grapher to perform the following steps.

a) Plot the function $y = f(x)$ together with the function $y = f(ax)$ for $a = 2, 3$, and 10 over the specified interval. Describe what happens to the graph as a increases through positive values.
b) Plot the function $y = f(x)$ and $y = f(ax)$ for the negative values $a = -2, -3$. What happens to the graph in this situation?
c) Plot the function $y = f(x)$ and $y = f(ax)$ for the fractional values $a = 1/2, 1/3, 1/4$. Describe what happens to the graph when $|a| < 1$.

83. $f(x) = \dfrac{5x}{x^2+4}$, $[-10, 10]$

84. $f(x) = \dfrac{2x(x-1)}{x^2+1}$, $[-3, 2]$

85. $f(x) = \dfrac{x+1}{2x^2+1}$, $[-2, 2]$

86. $f(x) = \dfrac{x^4-4x^3+10}{x^2+4}$, $[-1, 4]$

5

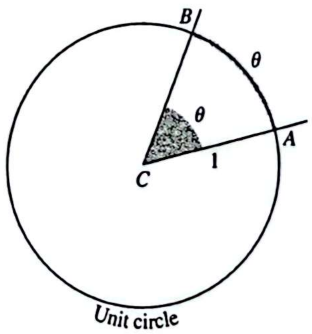
Trigonometric Functions

This section reviews radian measure, trigonometric functions, periodicity, and basic trigonometric identities.

Radian Measure

In navigation and astronomy, angles are measured in degrees, but in calculus it is best to use units called radians because of the way they simplify later calculations (Section 2.4).

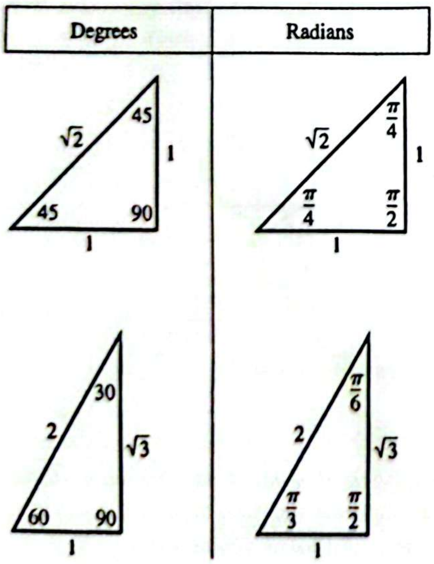
Let ACB be a central angle in a **unit circle** (circle of radius 1), as in Fig. 47.



47 The radian measure of angle *ACB* is the length of the arc *AB*.

The **radian measure** θ of angle ACB is defined to be the length of the circular arc AB. Since the circumference of the circle is 2π and one complete revolution of a circle is 360°, the relation between radians and degrees is given by the following equation.

$$\pi \text{ radians} = 180^\circ$$

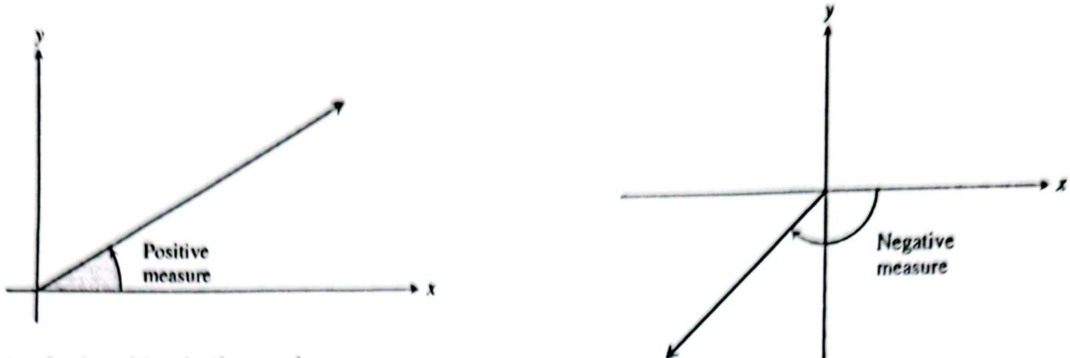


48 The angles of two common triangles, in degrees and radians.

EXAMPLE 1 *Conversions (Fig. 48)*

Convert 45° to radians: $\quad 45 \cdot \dfrac{\pi}{180} = \dfrac{\pi}{4}$ rad

Convert $\dfrac{\pi}{6}$ rad to degrees: $\quad \dfrac{\pi}{6} \cdot \dfrac{180}{\pi} = 30^\circ$ ❑



49 Angles in standard position in the *xy*-plane.

Conversion formulas

1 degree $= \dfrac{\pi}{180}$ (≈ 0.02) radians

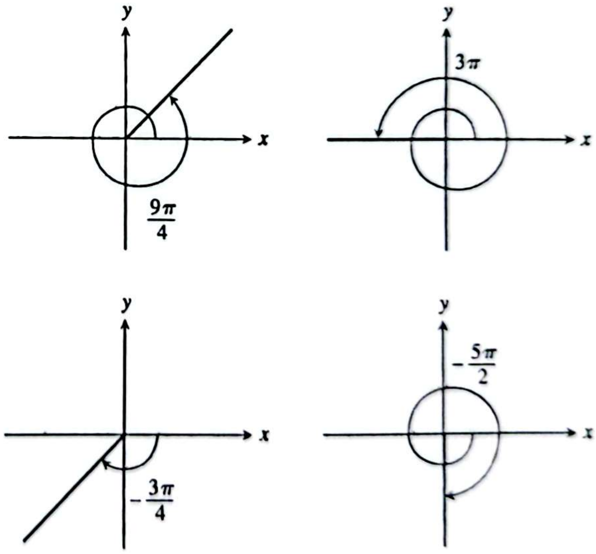
Degrees to radians: multiply by $\dfrac{\pi}{180}$

1 radian $= \dfrac{180}{\pi}$ (≈ 57) degrees

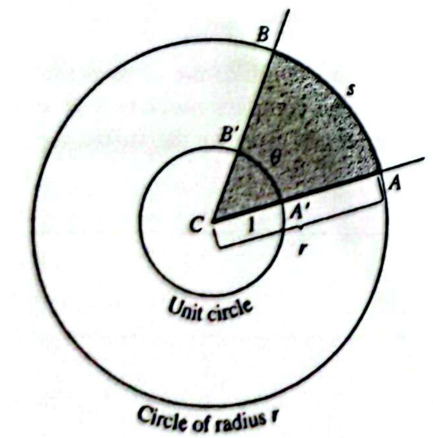
Radians to degrees: multiply by $\dfrac{180}{\pi}$

An angle in the *xy*-plane is said to be in **standard position** if its vertex lies at the origin and its initial ray lies along the positive *x*-axis (Fig. 49). Angles measured counterclockwise from the positive *x*-axis are assigned positive measures; angles measured clockwise are assigned negative measures.

When angles are used to describe counterclockwise rotations, our measurements can go arbitrarily far beyond 2π radians or 360°. Similarly, angles describing clockwise rotations can have negative measures of all sizes (Fig. 50).

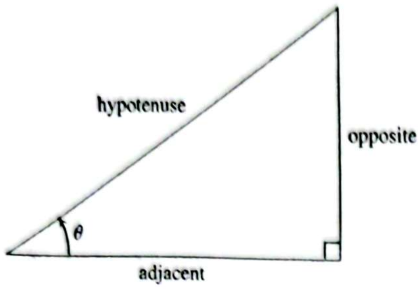


50 Nonzero radian measures can be positive or negative.



***51* The radian measure of angle *ACB* is the length *θ* of arc *A'B'* on the unit circle centered at *C*. The value of *θ* can be found from any other circle as *s/r*.**

There is a useful relationship between the length s of an arc AB on a circle of radius r and the radian measure θ of the angle the arc subtends at the circle's center C (Fig. 51). If we draw a unit circle with the same center C, the arc $A'B'$ cut by the angle will have length θ, by the definition of radian measure. From the similarity of the circular sectors ACB and $A'CB'$, we then have $s/r = \theta/1$.

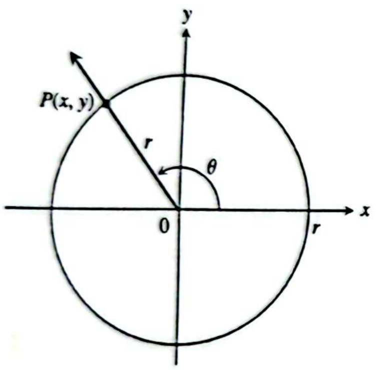


$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

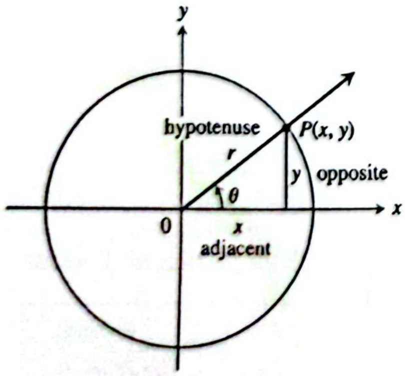
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

52 Trigonometric ratios of an acute angle.



53 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.



54 The new and old definitions agree for acute angles.

Radian Measure and Arc Length

$$\frac{s}{r} = \theta, \qquad \text{or} \qquad s = r\theta$$

Notice that these equalities hold precisely because we are measuring the angle in radians.

Angle Convention: Use Radians

From now on in this book it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees. When you do calculus, keep your calculator in radian mode.

EXAMPLE 2 Consider a circle of radius 8. (a) Find the central angle subtended by an arc of length 2π on the circle. (b) Find the length of an arc subtending a central angle of $3\pi/4$.

Solution

a) $\theta = \dfrac{s}{r} = \dfrac{2\pi}{8} = \dfrac{\pi}{4}$ b) $s = r\theta = 8\left(\dfrac{3\pi}{4}\right) = 6\pi$ ❑

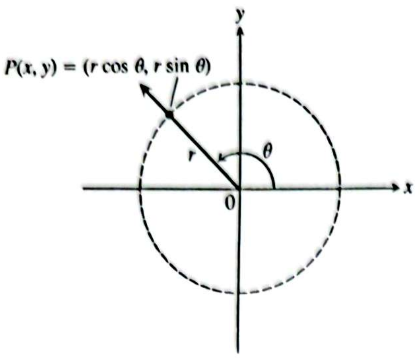
The Six Basic Trigonometric Functions

You are probably familiar with defining the trigonometric functions of an acute angle in terms of the sides of a right triangle (Fig. 52). We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r. We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (Fig. 53).

Sine:	$\sin\theta = \dfrac{y}{r}$	Cosecant:	$\csc\theta = \dfrac{r}{y}$
Cosine:	$\cos\theta = \dfrac{x}{r}$	Secant:	$\sec\theta = \dfrac{r}{x}$
Tangent:	$\tan\theta = \dfrac{y}{x}$	Cotangent:	$\cot\theta = \dfrac{x}{y}$

These extended definitions agree with the right-triangle definitions when the angle is acute (Fig. 54).

As you can see, $\tan\theta$ and $\sec\theta$ are not defined if $x = 0$. This means they are



55 The Cartesian coordinates of a point in the plane expressed in terms of r and θ.

not defined if θ is $\pm\pi/2,\ \pm 3\pi/2, \ldots$. Similarly, $\cot\theta$ and $\csc\theta$ are not defined for values of θ for which $y = 0$, namely $\theta = 0,\ \pm\pi,\ \pm 2\pi, \ldots$.

Notice also the following definitions, whenever the quotients are defined.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

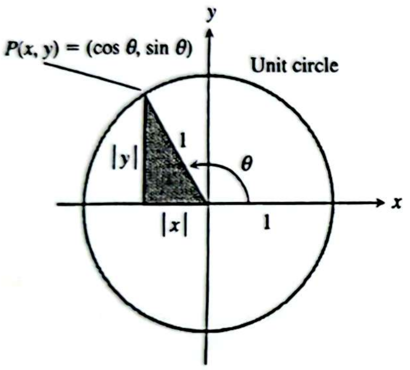
The coordinates of any point $P(x, y)$ in the plane can now be expressed in terms of the point's distance from the origin and the angle that ray OP makes with the positive x-axis (Fig. 55). Since $x/r = \cos\theta$ and $y/r = \sin\theta$, we have

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

Values of Trigonometric Functions

If the circle in Fig. 53 has radius $r = 1$, the equations defining $\sin\theta$ and $\cos\theta$ become

$$\cos\theta = x, \qquad \sin\theta = y.$$



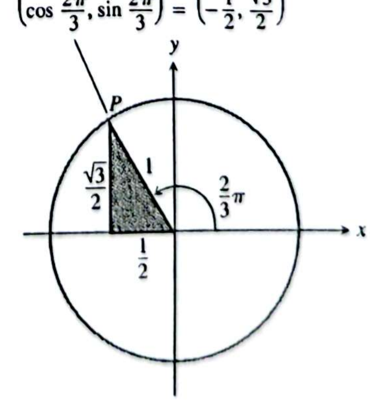
56 The acute reference triangle for an angle θ.

We can then calculate the values of the cosine and sine directly from the coordinates of P, if we happen to know them, or indirectly from the acute reference triangle made by dropping a perpendicular from P to the x-axis (Fig. 56). We read the magnitudes of x and y from the triangle's sides. The signs of x and y are determined by the quadrant in which the triangle lies.

EXAMPLE 3 Find the sine and cosine of $2\pi/3$ radians.

Solution

Step 1: Draw the angle in standard position in the unit circle and write in the lengths of the sides of the reference triangle (Fig. 57).



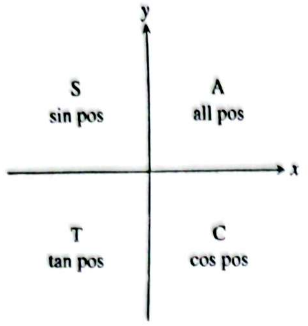
57 The triangle for calculating the sine and cosine of $2\pi/3$ radians (Example 3).

Step 2: Find the coordinates of the point P where the angle's terminal ray cuts the circle:

$$\cos\frac{2\pi}{3} = x\text{-coordinate of } P = -\frac{1}{2}$$

$$\sin\frac{2\pi}{3} = y\text{-coordinate of } P = \frac{\sqrt{3}}{2}. \qquad \square$$

A useful rule for remembering when the basic trigonometric functions are positive and negative is the CAST rule (Fig. 58).

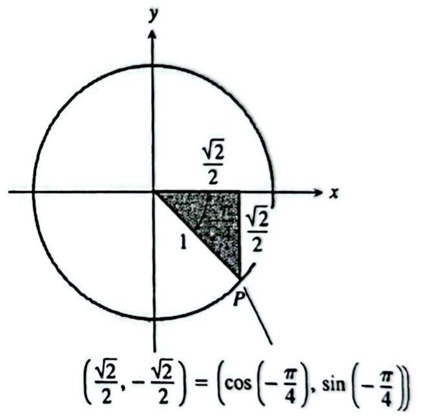


58 The CAST rule.

EXAMPLE 4 Find the sine and cosine of $-\pi/4$ radians.

Solution

Step 1: Draw the angle in standard position in the unit circle and write in the lengths of the sides of the reference triangle (Fig. 59).



59 The triangle for calculating the sine and cosine of $-\pi/4$ radians (Example 4).

Step 2: Find the coordinates of the point P where the angle's terminal ray cuts the circle:

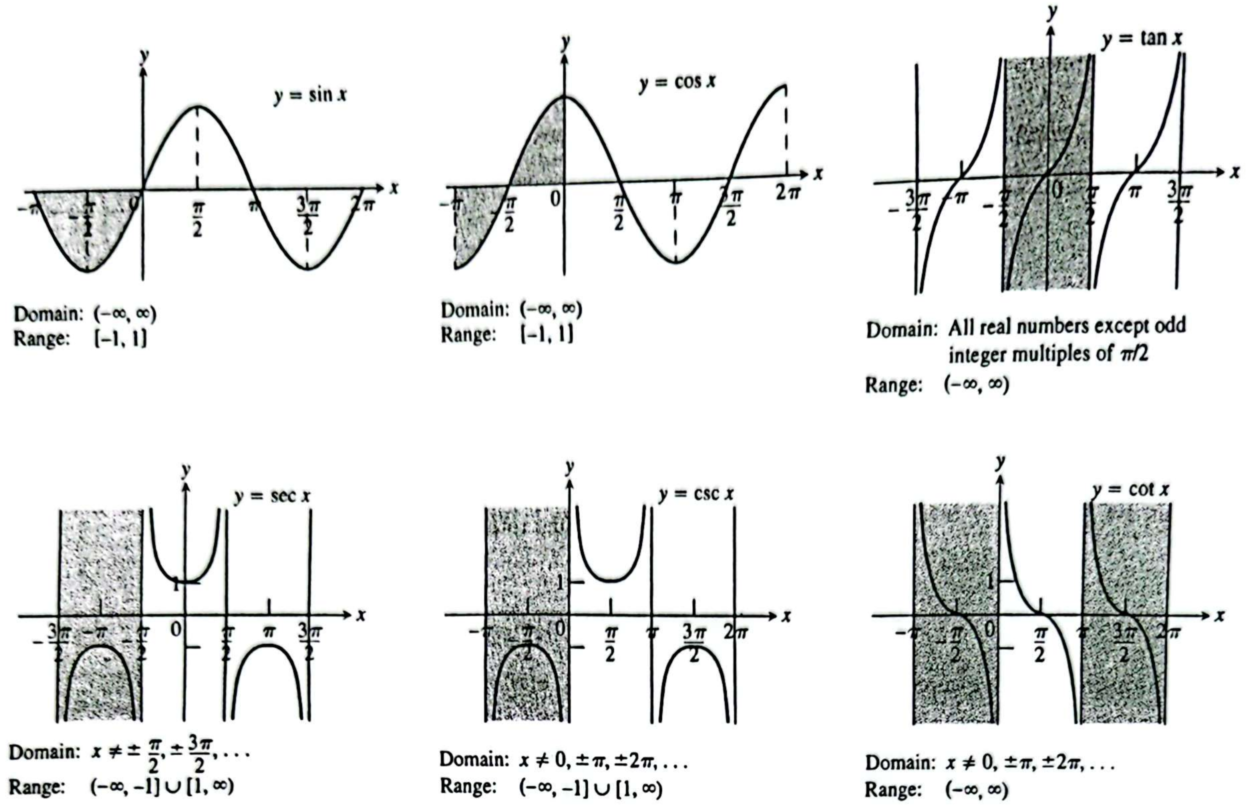
$$\cos\left(-\frac{\pi}{4}\right) = x\text{-coordinate of } P = \frac{\sqrt{2}}{2},$$

$$\sin\left(-\frac{\pi}{4}\right) = y\text{-coordinate of} P = -\frac{\sqrt{2}}{2}. \qquad \square$$

Calculations similar to those in Examples 3 and 4 allow us to fill in Table 2.

Table 2 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	135	180
θ (radians)	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$3\pi/4$	π
$\sin\theta$	0	$-\sqrt{2}/2$	−1	$-\sqrt{2}/2$	0	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{2}/2$	0
$\cos\theta$	−1	$-\sqrt{2}/2$	0	$\sqrt{2}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0	$-\sqrt{2}/2$	−1
$\tan\theta$	0	1		−1	0	$\sqrt{3}/3$	1	$\sqrt{3}$		−1	0



60 The graphs of the six basic trigonometric functions as functions of radian measure. Each function's periodicity shows clearly in its graph.

Graphs

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ. See Fig. 60.

Periodicity

When an angle of measure x and an angle of measure $x + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric values. For example, $\cos(x + 2\pi) = \cos x$. Functions like the trigonometric functions whose values repeat at regular intervals are called periodic.

Definition

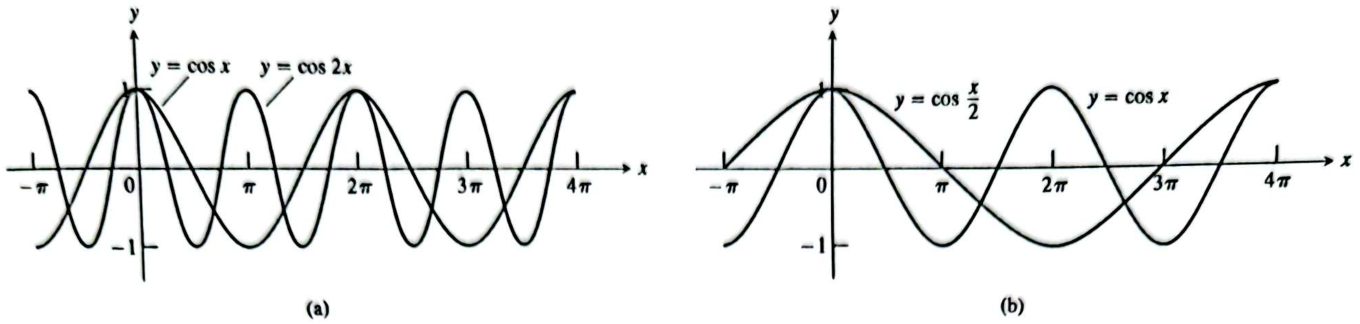
A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for all x. The smallest such value of p is the **period** of f.

Periods of trigonometric functions

Period π:	$\tan(x + \pi) = \tan x$
	$\cot(x + \pi) = \cot x$
Period 2π:	$\sin(x + 2\pi) = \sin x$
	$\cos(x + 2\pi) = \cos x$
	$\sec(x + 2\pi) = \sec x$
	$\csc(x + 2\pi) = \csc x$

As we can see in Fig. 60, the tangent and cotangent functions have period $p = \pi$. The other four functions have period 2π.

Figure 61 shows graphs of $y = \cos 2x$ and $y = \cos(x/2)$ plotted against the graph of $y = \cos x$. Multiplying x by a number greater than 1 speeds up a trigonometric function (increases the frequency) and shortens its period. Multiplying x by a positive number less than 1 slows a trigonometric function down and lengthens its period.



61 (a) Shorter period: $\cos 2x$. (b) Longer period: $\cos(x/2)$

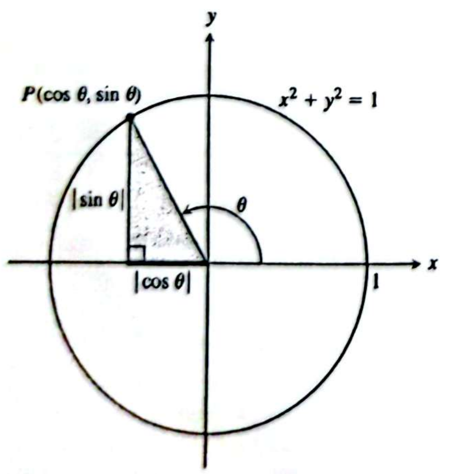
The importance of periodic functions stems from the fact that much of the behavior we study in science is periodic. Brain waves and heartbeats are periodic, as are household voltage and electric current. The electromagnetic field that heats food in a microwave oven is periodic, as are cash flows in seasonal businesses and the behavior of rotational machinery. The seasons are periodic—so is the weather. The phases of the moon are periodic, as are the motions of the planets. There is strong evidence that the ice ages are periodic, with a period of 90,000–100,000 years.

If so many things are periodic, why limit our discussion to trigonometric functions? The answer lies in a surprising and beautiful theorem from advanced calculus that says that every periodic function we want to use in mathematical modeling can be written as an algebraic combination of sines and cosines. Thus, once we learn the calculus of sines and cosines, we will know everything we need to know to model the mathematical behavior of periodic phenomena.

Even vs. Odd

The symmetries in the graphs in Fig. 60 reveal that the cosine and secant functions are even and the other four functions are odd:

Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$



62 The reference triangle for a general angle θ.

Identities

Applying the Pythagorean theorem to the reference right triangle we obtain by dropping a perpendicular from the point $P(\cos\theta, \sin\theta)$ on the unit circle to the x-axis (Fig. 62) gives

$$\cos^2\theta + \sin^2\theta = 1. \tag{2}$$

This equation, true for all values of θ, is probably the most frequently used identity in trigonometry.

Dividing Eq. (2) in turn by $\cos^2\theta$ and $\sin^2\theta$ gives the identities

$$1 + \tan^2\theta = \sec^2\theta,$$
$$1 + \cot^2\theta = \csc^2\theta.$$

You may recall the following identities from an earlier course.

All the trigonometric identities you will need in this book derive from Eqs. (2) and (3).

Angle Sum Formulas

$$\begin{aligned} \cos(A+B) &= \cos A\cos B - \sin A\sin B \\ \sin(A+B) &= \sin A\cos B + \cos A\sin B \end{aligned} \qquad (3)$$

These formulas hold for all angles A and B. There are similar formulas for $\cos(A-B)$ and $\sin(A-B)$ (Exercises 35 and 36).

Substituting θ for both A and B in the angle sum formulas gives two more useful identities:

Instead of memorizing Eqs. (3) you might find it helpful to remember Eqs. (4), and then recall where they came from.

Double-angle Formulas

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta \end{aligned} \qquad (4)$$

Additional formulas come from combining the equations

$$\cos^2\theta + \sin^2\theta = 1, \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta.$$

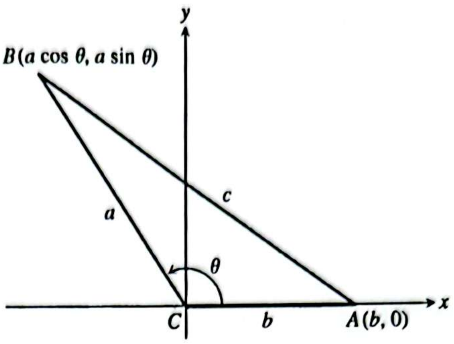
We add the two equations to get $2\cos^2\theta = 1 + \cos 2\theta$ and subtract the second from the first to get $2\sin^2\theta = 1 - \cos 2\theta$.

Additional Double-angle Formulas

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad (5)$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad (6)$$

When θ is replaced by $\theta/2$ in Eqs. (5) and (6), the resulting formulas are called **half-angle** formulas. Some books refer to Eqs. (5) and (6) by this name as well.



63 The square of the distance between A and B gives the law of cosines.

The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{7}$$

This equation is called the **law of cosines.**

We can see why the law holds if we introduce coordinate axes with the origin at C and the positive x-axis along one side of the triangle, as in Fig. 63. The coordinates of A are $(b, 0)$; the coordinates of B are $(a\cos\theta, a\sin\theta)$. The square of the distance between A and B is therefore

$$\begin{aligned} c^2 &= (a\cos\theta - b)^2 + (a\sin\theta)^2 \\ &= a^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + b^2 - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned}$$

Combining these equalities gives the law of cosines.

The law of cosines generalizes the Pythagorean theorem. If $\theta = \pi/2$, then $\cos\theta = 0$ and $c^2 = a^2 + b^2$.

Exercises 5

Radians, Degrees, and Circular Arcs

1. On a circle of radius 10 m, how long is an arc that subtends a central angle of (a) $4\pi/5$ radians? (b) $110°$?
2. A central angle in a circle of radius 8 is subtended by an arc of length 10π. Find the angle's radian and degree measures.
3. CALCULATOR You want to make an $80°$ angle by marking an arc on the perimeter of a 12-in.-diameter disk and drawing lines from the ends of the arc to the disk's center. To the nearest tenth of an inch, how long should the arc be?
4. CALCULATOR If you roll a 1-m-diameter wheel forward 30 cm over level ground, through what angle will the wheel turn? Answer in radians (to the nearest tenth) and degrees (to the nearest degree).

Evaluating Trigonometric Functions

5. Copy and complete the table of function values shown on the following page. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

6. Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-3\pi/2$	$-\pi/3$	$-\pi/6$	$\pi/4$	$5\pi/6$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

In Exercises 7–12, one of $\sin x$, $\cos x$, and $\tan x$ is given. Find the other two if x lies in the specified interval.

7. $\sin x = \dfrac{3}{5}$, x in $\left[\dfrac{\pi}{2}, \pi\right]$

8. $\tan x = 2$, x in $\left[0, \dfrac{\pi}{2}\right]$

9. $\cos x = \dfrac{1}{3}$, x in $\left[-\dfrac{\pi}{2}, 0\right]$

10. $\cos x = -\dfrac{5}{13}$, x in $\left[\dfrac{\pi}{2}, \pi\right]$

11. $\tan x = \dfrac{1}{2}$, x in $\left[\pi, \dfrac{3\pi}{2}\right]$

12. $\sin x = -\dfrac{1}{2}$, x in $\left[\pi, \dfrac{3\pi}{2}\right]$

Graphing Trigonometric Functions

Graph the functions in Exercises 13–22. What is the period of each function?

13. $\sin 2x$

14. $\sin(x/2)$

15. $\cos \pi x$

16. $\cos \dfrac{\pi x}{2}$

17. $-\sin \dfrac{\pi x}{3}$

18. $-\cos 2\pi x$

19. $\cos\left(x - \dfrac{\pi}{2}\right)$

20. $\sin\left(x + \dfrac{\pi}{2}\right)$

21. $\sin\left(x - \dfrac{\pi}{4}\right) + 1$

22. $\cos\left(x + \dfrac{\pi}{4}\right) - 1$

Graph the functions in Exercises 23–26 in the ts-plane (t-axis horizontal, s-axis vertical). What is the period of each function? What symmetries do the graphs have?

23. $s = \cot\ 2t$

24. $s = -\tan\ \pi t$

25. $s = \sec\left(\dfrac{\pi t}{2}\right)$

26. $s = \csc\left(\dfrac{t}{2}\right)$

27. GRAPHER
    a) Graph $y = \cos x$ and $y = \sec x$ together for $-3\pi/2 \le x \le 3\pi/2$. Comment on the behavior of $\sec x$ in relation to the signs and values of $\cos x$.
    b) Graph $y = \sin x$ and $y = \csc x$ together for $-\pi \le x \le 2\pi$. Comment on the behavior of $\csc x$ in relation to the signs and values of $\sin x$.

28. GRAPHER Graph $y = \tan x$ and $y = \cot x$ together for $-7 \le x \le 7$. Comment on the behavior of $\cot x$ in relation to the signs and values of $\tan x$.

29. Graph $y = \sin x$ and $y = \lfloor \sin x \rfloor$ together. What are the domain and range of $\lfloor \sin x \rfloor$?

30. Graph $y = \sin x$ and $y = \lceil \sin x \rceil$ together. What are the domain and range of $\lceil \sin x \rceil$?

Additional Trigonometric Identities

Use the angle sum formulas to derive the identities in Exercises 31–36.

31. $\cos\left(x - \dfrac{\pi}{2}\right) = \sin x$

32. $\cos\left(x + \dfrac{\pi}{2}\right) = -\sin x$

33. $\sin\left(x + \dfrac{\pi}{2}\right) = \cos x$

34. $\sin\left(x - \dfrac{\pi}{2}\right) = -\cos x$

35. $\cos(A - B) = \cos A \cos B + \sin A \sin B$

36. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

37. What happens if you take $B = A$ in the identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$? Does the result agree with something you already know?

38. What happens if you take $B = 2\pi$ in the angle sum formulas? Do the results agree with something you already know?

Using the Angle Sum Formulas

In Exercises 39–42, express the given quantity in terms of $\sin x$ and $\cos x$.

39. $\cos(\pi + x)$

40. $\sin(2\pi - x)$

41. $\sin\left(\dfrac{3\pi}{2} - x\right)$

42. $\cos\left(\dfrac{3\pi}{2} + x\right)$

43. Evaluate $\sin \dfrac{7\pi}{12}$ as $\sin\left(\dfrac{\pi}{4} + \dfrac{\pi}{3}\right)$.

44. Evaluate $\cos \frac{11\pi}{12}$ as $\cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)$.

45. Evaluate $\cos \frac{\pi}{12}$.

46. Evaluate $\sin \frac{5\pi}{12}$.

Using the Double-angle Formulas

Find the function values in Exercises 47–50.

47. $\cos^2 \frac{\pi}{8}$

48. $\cos^2 \frac{\pi}{12}$

49. $\sin^2 \frac{\pi}{12}$

50. $\sin^2 \frac{\pi}{8}$

Theory and Examples

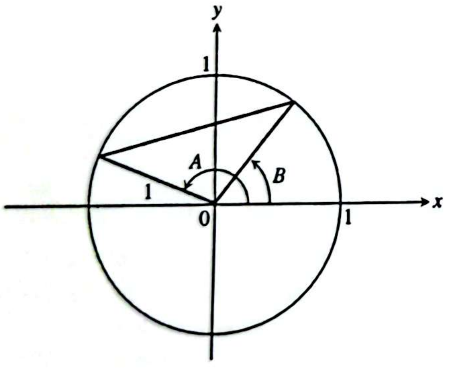
51. *The tangent sum formula.* The standard formula for the tangent of the sum of two angles is

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Derive the formula.

52. *(Continuation of Exercise 51.)* Derive a formula for $\tan(A - B)$.

53. Apply the law of cosines to the triangle in the accompanying figure to derive the formula for $\cos(A - B)$.



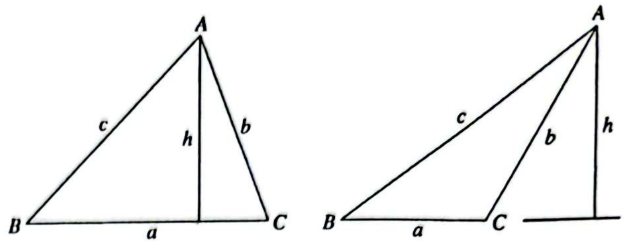
54. When applied to a figure similar to the one in Exercise 53, the law of cosines leads directly to the formula for $\cos(A + B)$. What is that figure and how does the derivation go?

55. CALCULATOR A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$. Find the length of side c.

56. CALCULATOR A triangle has sides $a = 2$ and $b = 3$ and angle $C = 40°$. Find the length of side c.

57. *The law of sines.* The **law of sines** says that if a, b, and c are the sides opposite the angles A, B, and C in a triangle, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



Use the accompanying figures and the identity $\sin(\pi - \theta) = \sin\theta$, if required, to derive the law.

58. CALCULATOR A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$ (as in Exercise 55). Find the sine of angle B using the law of sines.

59. CALCULATOR A triangle has side $c = 2$ and angles $A = \pi/4$ and $B = \pi/3$. Find the length a of the side opposite A.

60. *The approximation $\sin x \approx x$.* It is often useful to know that, when x is measured in radians, $\sin x \approx x$ for numerically small values of x. In Section 3.7, we will see why the approximation holds. The approximation error is less than 1 in 5000 if $|x| < 0.1$.

a) With your grapher in radian mode, graph $y = \sin x$ and $y = x$ together in a viewing window about the origin. What do you see happening as x nears the origin?

b) With your grapher in degree mode, graph $y = \sin x$ and $y = x$ together about the origin again. How is the picture different from the one obtained with radian mode?

c) *A quick radian mode check.* Is your calculator in radian mode? Evaluate $\sin x$ at a value of x near the origin, say $x = 0.1$. If $\sin x \approx x$, the calculator is in radian mode; if not, it isn't. Try it.

General Sine Curves

Figure 64 on the following page shows the graph of a **general sine function** of the form

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D,$$

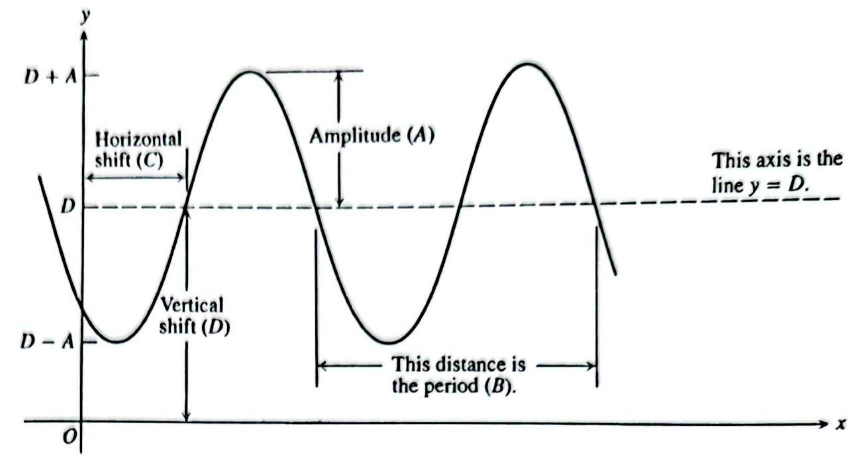
where $|A|$ is the *amplitude*, $|B|$ is the *period*, C is the *horizontal shift*, and D is the *vertical shift*. Identify A, B, C, and D for the sine functions in Exercises 61–64 and sketch their graphs.

61. $y = 2\sin(x + \pi) - 1$

62. $y = \frac{1}{2}\sin(\pi x - \pi) + \frac{1}{2}$

63. $y = -\frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right) + \frac{1}{\pi}$

64. $y = \frac{L}{2\pi}\sin\frac{2\pi t}{L}, \quad L > 0$



64 The general sine curve

$$y = A\sin[(2\pi/B)(x - C)] + D,$$

shown for A, B, C, and D positive.

The Trans-Alaska Pipeline

The builders of the Trans-Alaska Pipeline used insulated pads to keep the heat from the hot oil in the pipeline from melting the permanently frozen soil beneath. To design the pads, it was necessary to take into account the variation in air temperature throughout the year. Figure 65 shows how we can use a general sine function, defined in the introduction to Exercises 61–64, to represent temperature data. The data points in the figure are plots of the mean air temperature for Fairbanks, Alaska, based on records of the National Weather Service from 1941 to 1970. The sine function used to fit the data is

$$f(x) = 37\sin\left(\frac{2\pi}{365}(x - 101)\right) + 25,$$

where f is temperature in degrees Fahrenheit and x is the number of the day counting from the beginning of the year. The fit is remarkably good.

65. *Temperature in Fairbanks, Alaska.* Find the (a) amplitude, (b) period, (c) horizontal shift, and (d) vertical shift of the general sine function

$$f(x) = 37\sin\left(\frac{2\pi}{365}(x - 101)\right) + 25.$$

66. *Temperature in Fairbanks, Alaska.* Use the equation in Exercise 65 to approximate the answers to the following questions about the temperature in Fairbanks, Alaska, shown in Fig. 65. Assume that the year has 365 days.

a) What are the highest and lowest mean daily temperatures shown?

b) What is the average of the highest and lowest mean daily temperatures shown? Why is this average the vertical shift of the function?

CAS Explorations and Projects

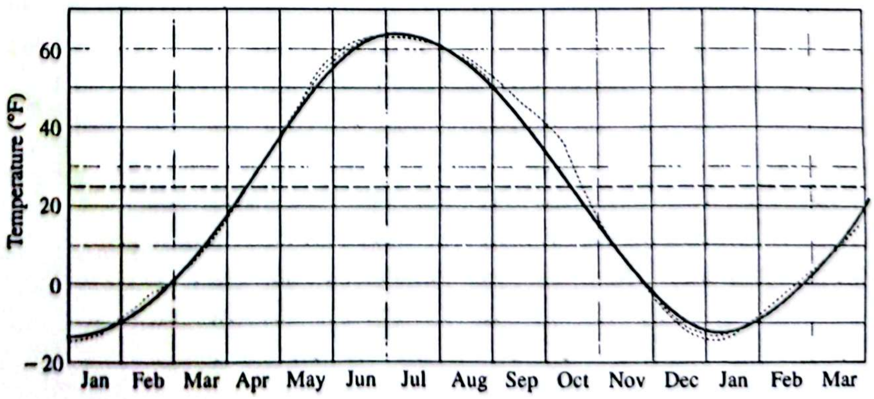
In Exercises 67–70, you will explore graphically the general sine function

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D$$

as you change the values of the constants A, B, C, and D. Use a CAS or computer grapher to perform the steps in the exercises.

67. *The period B.* Set the constants $A = 3$, $C = D = 0$.

a) Plot $f(x)$ for the values $B = 1, 3, 2\pi, 5\pi$ over the interval



65 Normal mean air temperature at Fairbanks, Alaska, plotted as data points. The approximating sine function is

$$f(x) = 37\sin\left(\frac{2\pi}{365}(x - 101)\right) + 25.$$

(Source: "Is the Curve of Temperature Variation a Sine Curve?" by B. M. Lando and C. A. Lando, *The Mathematics Teacher,* 7:6, Fig. 2, p. 535 [September 1977].)

$-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as the period increases.

b) What happens to the graph for negative values of B? Try it with $B = -3$ and $B = -2\pi$.

68. *The horizontal shift C.* Set the constants $A = 3$, $B = 6$, $D = 0$.

a) Plot $f(x)$ for the values $C = 0$, 1, and 2 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as C increases through positive values.

b) What happens to the graph for negative values of C?

c) What smallest positive value should be assigned to C so the graph exhibits no horizontal shift? Confirm your answer with a plot.

69. *The vertical shift D.* Set the constants $A = 3$, $B = 6$, $C = 0$.

a) Plot $f(x)$ for the values $D = 0$, 1, and 3 over the interval $-4\pi \le x \le 4\pi$. Describe what happens to the graph of the general sine function as D increases through positive values.

b) What happens to the graph for negative values of D?

70. *The amplitude A.* Set the constants $B = 6$, $C = D = 0$.

a) Describe what happens to the graph of the general sine function as A increases through positive values. Confirm your answer by plotting $f(x)$ for the values $A = 1$, 5, and 9.

b) What happens to the graph for negative values of A?

PRELIMINARIES — QUESTIONS TO GUIDE YOUR REVIEW

1. What are the order properties of the real numbers? How are they used in solving inequalities?
2. What is a number's absolute value? Give examples. How are $|-a|$, $|ab|$, $|a/b|$, and $|a+b|$ related to $|a|$ and $|b|$?
3. How are absolute values used to describe intervals or unions of intervals? Give examples.
4. How do you find the distance between two points in the coordinate plane?
5. How can you write an equation for a line if you know the coordinates of two points on the line? the line's slope and the coordinates of one point on the line? the line's slope and y-intercept? Give examples.
6. What are the standard equations for lines perpendicular to the coordinate axes?
7. How are the slopes of mutually perpendicular lines related? What about parallel lines? Give examples.
8. When a line is not vertical, what is the relation between its slope and its angle of inclination?
9. What is a function? Give examples. How do you graph a real-valued function of a real variable?
10. Name some typical algebraic and trigonometric functions and draw their graphs.
11. What is an even function? an odd function? What geometric properties do the graphs of such functions have? What advantage can we take of this? Give an example of a function that is neither even nor odd. What, if anything, can you say about sums, products, quotients, and composites involving even and odd functions?
12. If f and g are real-valued functions, how are the domains of $f+g$, $f-g$, fg, and f/g related to the domains of f and g? Give examples.
13. When is it possible to compose one function with another? Give examples of composites and their values at various points. Does the order in which functions are composed ever matter?
14. How do you change the equation $y = f(x)$ to shift its graph up or down? to the left or right? Give examples.
15. Describe the steps you would take to graph the circle $x^2 + y^2 + 4x - 6y + 12 = 0$.
16. If a, b, and c are constants and $a \ne 0$, what can you say about the graph of the equation $y = ax^2 + bx + c$? In particular, how would you go about sketching the curve $y = 2x^2 + 4x$?
17. What inequality describes the points in the coordinate plane that lie inside the circle of radius a centered at the point (h, k)? that lie inside or on the circle? that lie outside the circle? that lie outside or on the circle?
18. What is radian measure? How do you convert from radians to degrees? degrees to radians?
19. Graph the six basic trigonometric functions. What symmetries do the graphs have?
20. How can you sometimes find the values of trigonometric functions from triangles? Give examples.
21. What is a periodic function? Give examples. What are the periods of the six basic trigonometric functions?
22. Starting with the identity $\cos^2\theta + \sin^2\theta = 1$ and the formulas for $\cos(A+B)$ and $\sin(A+B)$, show how a variety of other trigonometric identities may be derived.

PRELIMINARIES PRACTICE EXERCISES

Geometry

1. A particle in the plane moved from $A(-2, 5)$ to the y-axis in such a way that Δy equaled $3\,\Delta x$. What were the particle's new coordinates?

2. a) Plot the points $A(8, 1)$, $B(2, 10)$, $C(-4, 6)$, $D(2, -3)$, and $E(14/3, 6)$.
   b) Find the slopes of the lines AB, BC, CD, DA, CE, and BD.
   c) Do any four of the five points A, B, C, D, and E form a parallelogram?
   d) Are any three of the five points collinear? How do you know?
   e) Which of the lines determined by the five points pass through the origin?

3. Do the points $A(6, 4)$, $B(4, -3)$, and $C(-2, 3)$ form an isosceles triangle? a right triangle? How do you know?

4. Find the coordinates of the point on the line $y = 3x + 1$ that is equidistant from $(0, 0)$ and $(-3, 4)$.

Functions and Graphs

5. Express the area and circumference of a circle as functions of the circle's radius. Then express the area as a function of the circumference.

6. Express the radius of a sphere as a function of the sphere's surface area. Then express the surface area as a function of the volume.

7. A point P in the first quadrant lies on the parabola $y = x^2$. Express the coordinates of P as functions of the angle of inclination of the line joining P to the origin.

8. A hot-air balloon rising straight up from a level field is tracked by a range finder located 500 ft from the point of lift-off. Express the balloon's height as a function of the angle the line from the range finder to the balloon makes with the ground.

Composition with absolute values. In Exercises 9–14, graph f_1 and f_2 together. Then describe how applying the absolute value function before applying f_1 affects the graph.

	$f_1(x)$	$f_2(x) = f_1(\lvert x\rvert)$
9.	x	$\lvert x\rvert$
10.	x^3	$\lvert x\rvert^3$
11.	x^2	$\lvert x\rvert^2$
12.	$\dfrac{1}{x}$	$\dfrac{1}{\lvert x\rvert}$
13.	$\sqrt{x}$	$\sqrt{\lvert x\rvert}$
14.	$\sin x$	$\sin \lvert x\rvert$

Composition with absolute values. In Exercises 15–20, graph g_1 and g_2 together. Then describe how taking absolute values after applying g_1 affects the graph.

	$g_1(x)$	$g_2(x) = \lvert g_1(x)\rvert$
15.	x^3	$\lvert x^3\rvert$
16.	$\sqrt{x}$	$\lvert \sqrt{x}\rvert$
17.	$\dfrac{1}{x}$	$\left\lvert \dfrac{1}{x}\right\rvert$
18.	$4 - x^2$	$\lvert 4 - x^2\rvert$
19.	$x^2 + x$	$\lvert x^2 + x\rvert$
20.	$\sin x$	$\lvert \sin x\rvert$

Trigonometry

In Exercises 21–24, sketch the graph of the given function. What is the period of the function?

21. $y = \cos 2x$

22. $y = \sin \dfrac{x}{2}$

23. $y = \sin \pi x$

24. $y = \cos \dfrac{\pi x}{2}$

25. Sketch the graph $y = 2\cos\left(x - \dfrac{\pi}{3}\right)$.

26. Sketch the graph $y = 1 + \sin\left(x + \dfrac{\pi}{4}\right)$.

In Exercises 27–30, ABC is a right triangle with the right angle at C. The sides opposite angles A, B, and C are a, b, and c, respectively.

27. a) Find a and b if $c = 2$, $B = \pi/3$.
    b) Find a and c if $b = 2$, $B = \pi/3$.

28. a) Express a in terms of A and c.
    b) Express a in terms of A and b.

29. a) Express a in terms of B and b.
    b) Express c in terms of A and a.

30. a) Express $\sin A$ in terms of a and c.
    b) Express $\sin A$ in terms of b and c.

31. CALCULATOR Two guy wires stretch from the top T of a vertical pole to points B and C on the ground, where C is 10 m closer to the base of the pole than is B. If wire BT makes an angle of 35° with the horizontal, and wire CT makes an angle of 50° with the horizontal, how high is the pole?

32. CALCULATOR Observers at positions A and B 2 km apart simultaneously measure the angle of elevation of a weather balloon to

be 40° and 70°, respectively. If the balloon is directly above a point on the line segment between A and B, find the height of the balloon.

33. Express $\sin 3x$ in terms of $\sin x$ and $\cos x$.

34. Express $\cos 3x$ in terms of $\sin x$ and $\cos x$.

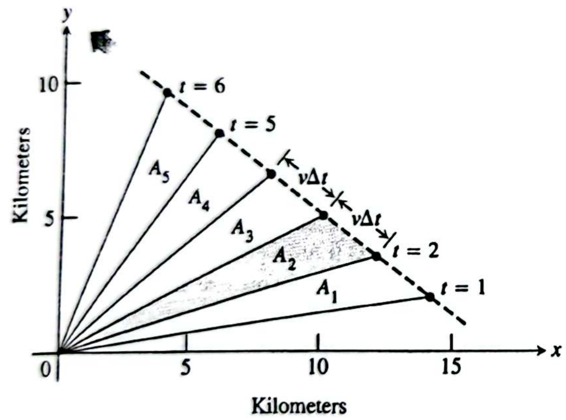
35. a) GRAPHER Graph the function $f(x) = \sin x + \cos(x/2)$.
b) What appears to be the period of this function?
c) Confirm your finding in (b) algebraically.

36. a) GRAPHER Graph $f(x) = \sin(1/x)$.
b) What are the domain and range of f?
c) Is f periodic? Give reasons for your answer.

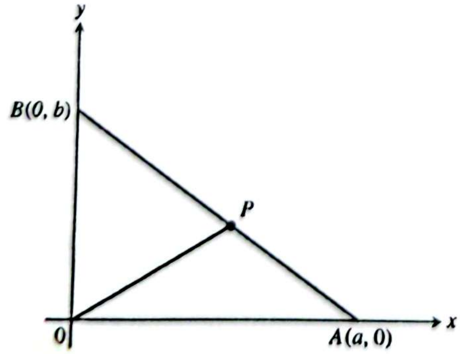
PRELIMINARIES — ADDITIONAL EXERCISES–THEORY, EXAMPLES, APPLICATIONS

Geometry

1. An object's center of mass moves at a constant velocity v along a straight line past the origin. The accompanying figure shows the coordinate system and the line of motion. The dots show positions that are 1 sec apart. Why are the areas $A_1, A_2, \dots, A_5$ in the figure all equal? As in Kepler's equal area law (see Section 11.5), the line that joins the object's center of mass to the origin sweeps out equal areas in equal times.



2. a) Find the slope of the line from the origin to the midpoint P of side AB in the triangle in the accompanying figure $(a, b > 0)$.



b) When is OP perpendicular to AB?

Functions and Graphs

3. Are there two functions f and g such that $f \circ g = g \circ f$? Give reasons for your answer.

4. Are there two functions f and g with the following property? The graphs of f and g are not straight lines but the graph of $f \circ g$ is a straight line. Give reasons for your answer.

5. If $f(x)$ is odd, can anything be said of $g(x) = f(x) - 2$? What if f is even instead? Give reasons for your answer.

6. If $g(x)$ is an odd function defined for all values of x, can anything be said about $g(0)$? Give reasons for your answer.

7. Graph the equation $|x| + |y| = 1 + x$.

8. Graph the equation $y + |y| = x + |x|$.

Trigonometry

In Exercises 9–14, ABC is an arbitrary triangle with sides a, b, and c opposite angles A, B, and C, respectively.

9. Find b if $a = \sqrt{3}$, $A = \pi/3$, $B = \pi/4$.

10. Find $\sin B$ if $a = 4$, $b = 3$, $A = \pi/4$.

11. Find $\cos A$ if $a = 2$, $b = 2$, $c = 3$.

12. Find c if $a = 2$, $b = 3$, $C = \pi/4$.

13. Find $\sin B$ if $a = 2$, $b = 3$, $c = 4$.

14. Find $\sin C$ if $a = 2$, $b = 4$, $c = 5$.

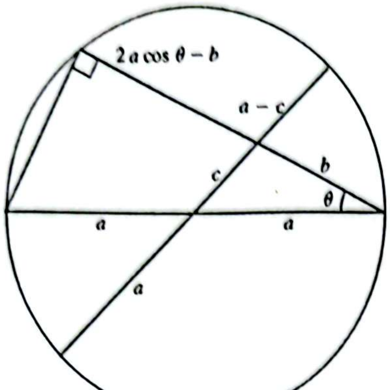
Derivations and Proofs

15. Prove the following identities.

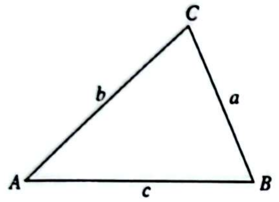
a) $$\frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$$

b) $$\frac{1 - \cos x}{1 + \cos x} = \tan^2 \frac{x}{2}$$

16. Explain the following "proof without words" of the law of cosines. (Source: "Proof without Words: The Law of Cosines," Sidney H. Kung, *Mathematics Magazine*, Vol. 63, No. 5, Dec. 1990, p. 342.)



17. Show that the area of triangle ABC is given by $(1/2)ab\sin C = (1/2)bc\sin A = (1/2)ca\sin B$.



* 18. Show that the area of triangle ABC is given by $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$ is the semiperimeter of the triangle.*

19. *Properties of inequalities.* If a and b are real numbers, we say that **a is less than b** and write $a < b$ if (and only if) $b - a$ is positive. Use this definition to prove the following properties of inequalities.

    If a, b, and c are real numbers, then:

    1. $a < b \implies a + c < b + c$
    2. $a < b \implies a - c < b - c$
    3. $a < b$ and $c > 0 \implies ac < bc$
    4. $a < b$ and $c < 0 \implies bc < ac$
       (Special case: $a < b \implies -b < -a$)
    5. $a > 0 \implies \frac{1}{a} > 0$
    6. $0 < a < b \implies \frac{1}{b} < \frac{1}{a}$
    7. $a < b < 0 \implies \frac{1}{b} < \frac{1}{a}$

20. *Properties of absolute values.* Prove the following properties of absolute values of real numbers.

    a) $|-a| = |a|$

    b) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

21. Prove that the following inequalities hold for any real numbers a and b.

    a) $|a| < |b|$ if and only if $a^2 < b^2$

    b) $|a - b| \geq ||a| - |b||$

22. *Generalizing the triangle inequality.* Prove by mathematical induction that the following inequalities hold for any n real numbers $a_1, a_2, \ldots, a_n$. (Mathematical induction is reviewed in Appendix 1.)

    a) $|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$

    b) $|a_1 + a_2 + \cdots + a_n| \geq |a_1| - |a_2| - \cdots - |a_n|$

23. Show that if f is both even and odd, then $f(x) = 0$ for every x in the domain of f.

24. a) *Even-odd decompositions.* Let f be a function whose domain is symmetric about the origin, that is, $-x$ belongs to the domain whenever x does. Show that f is the sum of an even function and an odd function:

    $$f(x) = E(x) + O(x),$$

    where E is an even function and O is an odd function. (*Hint:* Let $E(x) = (f(x) + f(-x))/2$. Show that $E(-x) = E(x)$, so that E is even. Then show that $O(x) = f(x) - E(x)$ is odd.)

    b) *Uniqueness.* Show that there is only one way to write f as the sum of an even and an odd function. (*Hint:* One way is given in part (a). If also $f(x) = E_1(x) + O_1(x)$ where E_1 is even and O_1 is odd, show that $E - E_1 = O_1 - O$. Then use Exercise 23 to show that $E = E_1$ and $O = O_1$.)

Grapher Explorations—Effects of Parameters

25. What happens to the graph of $y = ax^2 + bx + c$ as

    a) a changes while b and c remain fixed?

    b) b changes (a and c fixed, $a \neq 0$)?

    c) c changes (a and b fixed, $a \neq 0$)?

26. What happens to the graph of $y = a(x + b)^3 + c$ as

    a) a changes while b and c remain fixed?

    b) b changes (a and c fixed, $a \neq 0$)?

    c) c changes (a and b fixed, $a \neq 0$)?

27. Find all values of the slope of the line $y = mx + 2$ for which the x-intercept exceeds $1/2$.

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*Asterisk denotes more challenging problem.

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DIN. XD.