Integral calculus

Introduction : The integral is an important concept in Mathematics. Integration is one of the two main operations in calculus. So we may consider integration as the inverse of differentiation.

Antiderivative : If $f(x)$ be a function of x defined on the interval $[a,b]$ and if another function $F(x)$ be obtained such that its differential co-efficient with respect to x equal to $f(x)$ that is $F'(x) = f(x)$, $\forall x \in [a,b]$, then $F(x)$ is called an antiderivative of $f(x)$.

Ex: The function $F(x) = \frac{1}{3}x^3$ is an antiderivative of the function $f(x) = x^2$ on the interval $(-\infty, \infty)$, because for each x in $(-\infty, \infty)$

$$F'(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2 = f(x).$$

However, the function $F(x) = \frac{1}{3}x^3$ is not only an antiderivative of $f(x)$ on this interval, if we add any constant c to $\frac{1}{3}x^3$, then the function $G(x) = \frac{1}{3}x^3 + c$ is also an antiderivative of $f(x)$ on $(-\infty, \infty)$, since

$$G'(x) = \frac{d}{dx}G(x) = \frac{d}{dx}\left(\frac{1}{3}x^3 + c\right) = x^2 = f(x).$$

Indefinite integral : The process of finding antiderivatives is called antidifferentiation or integration.

Thus, if

$$\frac{d}{dx}F(x) = f(x) \quad \text{—— (1), then}$$

$$\int f(x)\,dx = F(x) + c$$

Here the expression $\int f(x)\,dx$ is called an indefinite integral.

(i) $\int 2x\,dx = 2\int x\,dx = 2\frac{x^2}{2} + c = x^2 + c$

(ii) $\int (x+x^2)\,dx = \int x\,dx + \int x^2\,dx$

$$= \frac{x^2}{2} + \frac{x^3}{3} + c$$

(iii) $\int (3x^6 - 2x^2 + 7x + 1)\,dx = 3\int x^6\,dx - 2\int x^2\,dx + 7\int x\,dx + \int 1\,dx$

$$= 3.\frac{x^7}{7} - 2.\frac{x^3}{3} + 7\frac{x^2}{2} + x + C$$

Standard methods of integration :

The are two principal processes for integration.

(i) The method of substitution, that is, a change of the independent variable.

(ii) Integration by parts.

In some cases when the integrand is a rational fraction, it may be broken into partial fractions by the rules of Algebra and then each part may be integrated by one of the above methods.

Properties of indefinite integral :

Let $F(x)$ & $G(x)$ are antiderivatives of $f(x)$ & $g(x)$ respectively, and let c is a constant, then

(i) A constant factor can be removed through an integral sign, that is,

$$\int c\, f(x)\, dx = cF(x) + C.$$

(ii) An antiderivative of a sum is the sum of antiderivatives, that is

$$\int [f(x) + g(x)]\, dx = F(x) + G(x) + C.$$

(iii) An antiderivative of a difference is the difference of antiderivatives, that is,

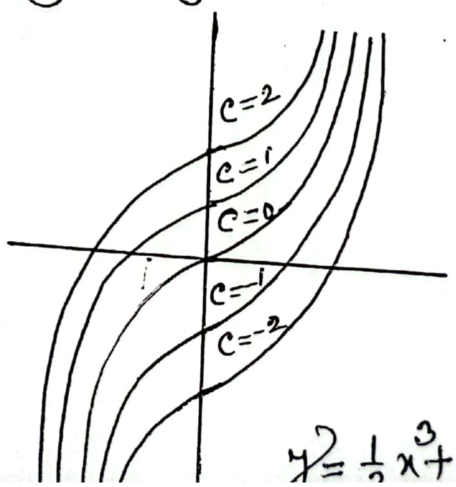
$$\int [f(x) - g(x)]\, dx = F(x) - G(x) + C.$$

Integral Curves:

Graphs of antiderivatives of a function f are called integral curves of f. So, if $y = F(x)$ is any integral curve of $f(x)$, then all other integral curves are vertical translations of this curve, since they have eqns of the form $y = F(x) + C$.

For example, $y = \frac{1}{3}x^3$ is one integral curve for $f(x) = x^2$, so all other integral curves have eqns of the form $y = \frac{1}{3}x^3 + C$.

Thus the graph of any eqn of this form is an integral curve.



$y = \frac{1}{3}x^3 + C.$

(i) $\int 2x\,dx = 2\int x\,dx = 2\frac{x^2}{2} + c = x^2 + C$

(ii) $\int (x+x^2)\,dx = \int x\,dx + \int x^2\,dx$

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Method of substitution : In calculus, integration by substitution, also known as u-substitution, is a method for finding integrals. It is the counterpart to the chain rule of differentiation. This technique involves making a substitution in order to simply an integral before evaluating it. We let a new variable, u say, equal to a more complicated part of the function we are trying to integrate. The choice of which substitution to make often relies upon experience.

For this purpose, let F is an antiderivative of f and that g is a differentiable function. The chain rule implies that the derivative of $F(g(x))$ can be expressed as

$$\frac{d}{dx}[F(g(x))] = F'(g(x))\, g'(x).$$

Which we can write in integral form

$$\int F'(g(x))\, g'(x)\, dx = F(g(x)) + C \quad \text{--- (1)}$$

or since F is an antiderivative of f.

$$\int f(g(x))\, g'(x)\, dx = F(g(x)) + C \quad \text{--- (2)}$$

Let $u = g(x)$ then $\frac{du}{dx} = g'(x)$, in the differential form $du = g'(x)\, dx$. So (2) $\Rightarrow$

$$\boxed{\int f(u)\, du = F(u) + C} \quad \text{--- (3)}$$

The process of evaluating an integral of the form (2) by converting it into form (3) with the substitution,

$$u = g(x) \quad \text{and} \quad du = g'(x)\,dx$$

is called the method of substitution. Note that after evaluating the integral we are to replace u by the equivalent expression in x.

— x —

<u>Guidelines for u-substitution:</u>

<u>Step 1</u>: Look for some composition $f(g(x))$ within the integrand for which the substitution

$$u = g(x), \quad du = g'(x)\,dx,$$

produces an integral that is expressed in terms of u & du. This may or may not be possible.

<u>Step 2</u>: If you are successful in step 1, then try to evaluate the resulting integral in terms of u. Again, this may or may not be possible.

<u>Step 3</u>: If you are successful in step 2, then replace u by $g(x)$ to express your final answer in ~~tear~~ terms of x.

Some important forms:

(i) Integrate $\int (a+bx)^n\,dx$

Let $\boxed{u = a+bx} \Rightarrow \dfrac{du}{dx} = b$

$\Rightarrow \boxed{\dfrac{1}{b}\,du = dx}$

$$\therefore\ I = \int (a+bx)^n\,dx = \int u^n \cdot \frac{1}{b}\,du$$

$$= \frac{1}{b}\,\frac{u^{n+1}}{n+1} + c$$

$$= \frac{(a+bx)^{n+1}}{b(n+1)} + c.$$

(ii) Integrate $\int [f(x)]^n f'(x)\,dx$

Let $f(x) = u$

$\Rightarrow f'(x) = \dfrac{du}{dx}$

$\Rightarrow f'(x)\,dx = du$

$$\therefore\ I = \int [f(x)]^n f'(x)\,dx$$

$$= \int u^n\,du$$

$$= \frac{u^{n+1}}{n+1} + c$$

$$= \frac{[f(x)]^{n+1}}{n+1} + c.$$

(iii) Integrate $\int \frac{f'(x)}{f(x)}\,dx$

Let $f(x) = u$

$\Rightarrow f'(x) = \frac{du}{dx}$

$\Rightarrow f'(x)\,dx = du$

$\therefore I = \int \frac{f'(x)}{f(x)}\,dx = \int \frac{du}{u}$

$= \log_e |u| + c = \log_e |f(x)| + c$

<u>Ex</u>: Evaluate (i) $\int (1+3x)^2\,dx$

(ii) $\int \frac{1+\cos x}{\sqrt[3]{x+\sin x}}\,dx$

(iii) $\int \frac{dx}{x\log x}$.

<u>Soln</u>: (i) Given $I = \int (1+3x)^2\,dx$

Let $1+3x = z$

$\therefore 3\,dx = dz$

$\Rightarrow dx = \frac{1}{3}dz$

$= \int z^2 \frac{1}{3}\,dz$

$= \frac{1}{3} \cdot \frac{z^3}{3} + c$

$= \frac{(1+3x)^3}{9} + c$ <u>Ans</u>

(ii) Given $I = \int \frac{1+\cos x}{\sqrt[3]{x+\sin x}}\,dx$

Let $x+\sin x = z$

$\therefore 1+\cos x = \frac{dz}{dx}$

$\Rightarrow (1+\cos x)\,dx = dz$

$= \int \frac{dz}{\sqrt[3]{z}}$

$= \int z^{-1/3}\,dz$

$= \frac{z^{2/3}}{2/3} + c = \frac{3}{2}(x+\sin x)^{2/3} + c.$

(iii) Given, $I = \int \frac{dx}{x \log x}$

Let $\log x = z$

$\therefore \frac{1}{x} = \frac{dz}{dx}$

$\Rightarrow \frac{dx}{x} = dz$

$= \int \frac{dz}{z}$

$= \ln(z) + C$

$= \ln(\ln x) + C$ Ans

Ex: Evaluate $\int \frac{\sin 2x \, dx}{a \sin^2 x + b \cos^2 x}$.

Soln: Let $I = \int \frac{\sin 2x \, dx}{a \sin^2 x + b \cos^2 x}$

Let $a \sin^2 x + b \cos^2 x = z$

$\Rightarrow 2a \sin x \cos x + 2b \cos x(-\sin x) = \frac{dz}{dx}$

$\Rightarrow a \sin 2x - b \sin 2x = \frac{dz}{dx}$

$\Rightarrow (a-b) \sin 2x = \frac{dz}{dx}$

$\Rightarrow \sin 2x \, dx = \frac{dz}{a-b}$

$= \int \frac{1}{z} \frac{dz}{a-b}$

$= \frac{1}{a-b} \cdot \ln z + C$

$= \frac{1}{a-b} \ln(a \sin^2 x + b \cos^2 x) + C$ Ans

Ex: Evaluate (i) $\int \frac{\tan x \sec^2 x \, dx}{(a^2 + b^2 \tan^2 x)^2}$ (ii) $\int \frac{\sin 2x}{(a + b\cos x)^2} dx$

Soln (i) Let $I = \int \frac{\tan x \sec^2 x \, dx}{(a^2 + b^2 \tan^2 x)^2}$

Let $a^2 + b^2 \tan^2 x = z$

$\therefore 2b^2 \tan x \sec^2 x = \frac{dz}{dx}$

$\Rightarrow \tan x \sec^2 x \, dx = \frac{dz}{2b^2}$

$= \int \frac{1}{z^2} \frac{dz}{2b^2}$

$= \frac{1}{2b^2} \int z^{-2} dz$

$= \frac{1}{2b^2} \cdot \frac{z^{-1}}{-1} + C$

$= \frac{-1}{2b^2} \frac{1}{a^2 + b^2 \tan^2 x} + C$ Ans

Soln : (ii) Let $I = \int \frac{\sin 2x}{(a+b\cos x)^2} dx$

Let, $a + b\cos x = z$

$\Rightarrow -b\sin x\, dx = dz$

$\Rightarrow \sin x\, dx = -\frac{1}{b} dz$

& $\cos x = \frac{z-a}{b}$

$$= \int \frac{2\sin x \cos x}{(a+b\cos x)^2} dx$$

$$= 2\int \frac{z-a}{b} \cdot \frac{1}{z^2} \left(-\frac{1}{b}\right) dz$$

$$= \frac{-2}{b^2} \int \frac{z-a}{z^2} dz$$

$$= \frac{-2}{b^2} \int \left(\frac{1}{z} - az^{-2}\right) dz$$

$$= \frac{-2}{b^2} \left(\ln z + \frac{a}{z}\right) + c$$

$$= \frac{-2}{b^2} \left[\ln(a+b\cos x) + \frac{a}{a+b\cos x}\right] + c \quad \text{Ans}$$

Integrate $\int \frac{dx}{x^m (a+bx)^n}$.

Let $\boxed{a+bx = zx}$

Ex: Evaluate $\int \frac{dx}{x^2 (a+bx)^2}$

Soln: Let $I = \int \frac{dx}{x^2 (a+bx)^2}$

Let $\boxed{a+bx = zx}$

$\Rightarrow \frac{a}{x} + b = z$

$\Rightarrow -\frac{a}{x^2} dx = dz$

$\Rightarrow \boxed{\frac{dx}{x^2} = -\frac{dz}{a}}$

$$= \int \frac{1}{(zx)^2} \cdot -\frac{dz}{a}$$

$$= -\frac{1}{a} \int \frac{dz}{z^2 x^2}$$

$$= -\frac{1}{a}\int \frac{(z-b)^2}{z^2 a^2}\,dz$$

$$= -\frac{1}{a^3}\int \frac{z^2-2zb+b^2}{z^2}\,dz$$

$$= -\frac{1}{a^3}\int \left(1-2b\cdot\frac{1}{z}+b^2 z^{-2}\right)dz$$

$$= -\frac{1}{a^3}\left(z-2b\ln z-\frac{b^2}{z}\right)+c$$

$$= -\frac{1}{a^3}\left(z-\frac{b^2}{z}\right)+\frac{2b}{a^3}\ln z+c$$

$$= \frac{2b}{a^3}\cdot\ln\left(\frac{a+bx}{x}\right) \cancel{+} -\frac{1}{a^3}\cdot\left(\frac{a+bx}{x}-\frac{b^2x}{a+bx}\right)+c$$

$$= \frac{2b}{a^3}\ln\left(\frac{a+bx}{x}\right)-\frac{1}{a^3}\,\frac{a^2+2abx+\cancel{b^2x^2}-\cancel{b^2x^2}}{x(a+bx)}+c$$

$$= \frac{2b}{a^3}\ln\frac{a+bx}{x}-\frac{a+2bx}{a^2x(a+bx)}+c$$

Ans

The product rule and integration by parts:

Functions often arise as products of other functions, and we may be required to integrate these products. For example, we may be asked to determine

$$\int x \cos x \, dx$$

Here the integrand is the product of the functions x & $\cos x$. So integration by parts is a rule/method for integrating products of two functions.

To develop a general rule for finding the integrals of the form $\int f(x) g(x) \, dx$

Let $G(x)$ be any antiderivative of $g(x)$, that is

$$G'(x) = g(x)$$

$$\Rightarrow \frac{d}{dx}[G(x)] = g(x)$$

From the product rule in differential calculus, we have

$$\frac{d}{dx}[f(x)G(x)] = f(x)\frac{d}{dx}[G(x)] + \frac{d}{dx}[f(x)]\, G(x)$$

$$\frac{d}{dx}[f(x)G(x)] = f(x)g(x) + \frac{d}{dx}[f(x)]\, G(x) \quad \text{——(1)}$$

$$\Rightarrow \frac{d}{dx}[f(x)G(x)] = f(x)g(x) + \frac{d}{dx}[f(x)]\, G(x).$$

So $f(x)G(x)$ is an antiderivative of $f(x)g(x) + \frac{d}{dx}[f(x)]\, G(x)$

Thus eq^n (1) can be expressed as

$$\int \frac{d}{dx}[f(x)G(x)]\, dx = \int f(x) g(x) \, dx + \int \frac{d}{dx}[f(x)]\, G(x) \, dx$$

$$\Rightarrow f(x)G(x) = \int f(x)g(x)\, dx + \int \frac{d}{dx}[f(x)]\, G(x)\, dx$$

$$\Rightarrow \boxed{\int f(x) g(x)\, dx = f(x)G(x) - \int \frac{d}{dx}[f(x)]\, G(x)\, dx} \quad \text{—(2)}$$

If we consider, $G'(x) = g(x)$, then $G(x) = \int g(x)$ [cut off]

eqn (2) $\Rightarrow$

$$\boxed{\int f(x) g(x)\,dx = f(x)\int g(x)\,dx - \int \left\{ \frac{d}{dx}[f(x)] \int g(x)\,dx \right\} dx}$$

<u>Guidelines</u>: The main goal in integration by parts is to choose first and second functions to obtain a new integral that is easier to evaluate than the original. Usually we choose $f(x)$ so that $f(x)$ becomes simpler when differentiated and $g(x)$ is chosen so that it can be easily integrated. The useful strategy for first and second functions ($f(x)$ & $g(x)$) is :

Logarithmic, Inverse Trigonometric, Algebraic, Trigonometric, Exponential.

The acronym : $\boxed{\text{LIATE}}$

L : $\ln x$, $\ln(1+x)$, ... etc

I : $\sin^{-1} x$, $\cos^{-1} x$, ... etc

A : x, $1+x^2$, x^3 ... etc

T : $\sin x$, $\cos x$, ... etc

E : e^x, e^{-5x}, ... etc.

III If u & v are two functions of x, then

$$\boxed{\int uv\,dx = u\int v\,dx - \int \left\{ \frac{d}{dx}u \int v\,dx \right\} dx}$$

Integration by parts

Let u & v are two functions of x, then

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}u\int v\,dx\right\}dx.$$

Prob: Evaluate (i) $\int x\ln x\,dx$ (ii) $\int \ln x\,dx$ (iii) $\int \tan^{-1}x$.

Soln: (i) Let $I = \int x\ln x\,dx$

$$= \ln x\int x\,dx - \int\left\{\frac{d}{dx}\ln x\int x\,dx\right\}dx$$

$$= \frac{x^2}{2}\cdot\ln x - \int\frac{1}{x}\cdot\frac{x^2}{2}dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + c \quad \text{Ans}$$

(ii) Let $I = \int \ln x\,dx$

$$= \int \ln x\cdot 1\,dx$$

$$= \ln x\int 1\cdot dx - \int\left\{\frac{d}{dx}\ln x\int 1\,dx\right\}dx$$

$$= x\ln x - \int\frac{1}{x}\cdot x\,dx$$

$$= x\ln x - \int 1\,dx$$

$$= x\ln x - x + c \quad \text{Ans}$$

(iii) Let $I = \int \tan^{-1}x\,dx$

$$= \tan^{-1}x\int 1\cdot dx - \int\left\{\frac{d}{dx}\tan^{-1}x\int 1\cdot dx\right\}dx$$

$$= x\tan^{-1}x - \int\frac{1}{1+x^2}x\,dx.$$

$$= x\tan^{-1}x - \frac{1}{2}\cdot\int\frac{2x}{1+x^2}dx = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c$$

Ans

Prob: Integrate (i) $\int x^3 e^x dx$ (ii) $\int e^{ax} x^2 dx$

(iii) $\int x^2 (\ln x)^2 dx$ (iv) $\int e^x \cos x \, dx$ (v) $\int e^x (\cos x + \sin x) dx$.

Soln: (i) Let $I = \int x^3 e^x dx$

$$= x^3 \int e^x dx - \int \left\{ \frac{d}{dx} x^3 \cdot \int e^x dx \right\} dx$$

$$= x^3 e^x - \int 3x^2 \, e^x dx$$

$$= x^3 e^x - 3\left[x^2 \int e^x dx - \int \left\{ \frac{d}{dx} x^2 \int e^x dx \right\} dx \right]$$

$$= x^3 e^x - 3\left[x^2 e^x - \int 2x e^x dx \right]$$

$$= x^3 e^x - 3x^2 e^x + 6\left[x \int e^x - \int \left\{ \frac{d}{dx} x \int e^x dx \right\} dx \right]$$

$$= x^3 e^x - 3x^2 e^x + 6x e^x - 6 \int e^x dx$$

$$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + c$$

$$= e^x (x^3 - 3x^2 + 6x - 6) + c \quad \text{Ans}$$

Soln: (ii) Let $I = \int x^2 e^{ax} dx$

$$= x^2 \int e^{ax} dx - \int \left\{ \frac{d}{dx} x^2 \int e^{ax} dx \right\} dx$$

$$= x^2 \frac{e^{ax}}{a} - \int 2x \cdot \frac{e^{ax}}{a} dx$$

$$= \frac{1}{a} x^2 e^{ax} - \frac{2}{a}\left[x \int e^{ax} dx - \int \left\{ \frac{d}{dx} x \int e^{ax} dx \right] dx \right.$$

$$= \frac{1}{a} x^2 e^{ax} - \frac{2}{a}\left[x \cdot \frac{e^{ax}}{a} - \int 1 \cdot \frac{e^{ax}}{a} dx \right]$$

$$= \frac{1}{a} x^2 e^{ax} - \frac{2}{a^2} x e^{ax} + \frac{2}{a^3} \cdot e^{ax} + c \quad \text{Ans}$$

Sol^n: (III) Let $I = \int x^2 (\ln x)^2 \, dx$

$$= (\ln x)^2 \int x^2 dx - \int \left\{ \frac{d}{dx} (\ln x)^2 \int x^2 dx \right\} dx$$

$$= (\ln x)^2 \cdot \frac{x^3}{3} - \int 2 \ln x \cdot \frac{1}{x} \cdot \frac{x^3}{3} dx.$$

$$= \frac{x^3}{3} (\ln x)^2 - \frac{2}{3} \int \ln x \cdot x^2 dx.$$

$$= \frac{x^3}{3} (\ln x)^2 - \frac{2}{3} \left[\ln x \int x^2 dx - \int \left\{ \frac{d}{dx} \ln x \int x^2 dx \right\} dx \right]$$

$$= \frac{x^3}{3} (\ln x)^2 - \frac{2}{3} \left[\ln x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} dx \right]$$

$$= \frac{x^3}{3} (\ln x)^2 - \frac{2}{9} x^3 \ln x + \frac{2}{9} \int x^2 dx$$

$$= \frac{x^3}{3} (\ln x)^2 - \frac{2}{9} x^3 \ln x + \frac{2}{27} x^3 + c \quad \underline{\text{Ans}}$$

Sol^n: (IV) Let $I = \int e^x \cos x \, dx$

$$= \cos x \int e^x dx - \int \left\{ \frac{d}{dx} \cos x \int e^x dx \right\} dx$$

$$= \cos x \, e^x + \int \sin x \, e^x dx$$

$$= \cos x \, e^x + \sin x \int e^x dx - \int \left\{ \frac{d}{dx} \sin x \int e^x dx \right\} dx$$

$$\Rightarrow I = e^x \cos x + \sin x . e^x - \int \cos x . e^x dx$$

$$\Rightarrow I = e^x (\cos x + \sin x) - I$$

$$\Rightarrow 2I = e^x (\cos x + \sin x)$$

$$\Rightarrow I = \frac{e^x}{2} (\cos x + \sin x) + c$$

Ans.

(v) Let $I = \int e^x (\cos x + \sin x)\,dx$

$= \int e^x \cos x\,dx + \int e^x \sin x\,dx$

$= \int e^x \cos x\,dx + \sin x \int e^x\,dx - \int \left\{ \frac{d}{dx} \sin x \int e^x\,dx \right\} dx$

$= \cancel{\int e^x \cos x\,dx} + \sin x . e^x - \cancel{\int \cos x . e^x\,dx}$

$= e^x . \sin x + c$ Ans

Integration by partial fraction

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Partial fractions:

In algebra, we sometimes combine two or more fractions into a single fraction by finding a common denominator. For example,

$$\frac{2}{x-4}+\frac{3}{x+1}=\frac{2(x+1)+3(x-4)}{(x-4)(x+1)}=\frac{5x-10}{x^2-3x-4} \quad \text{---(1)}$$

However, for the purpose of integration, the leftside of (1) is preferable to the right side since each of the terms is easy to integrate:

$$\int\frac{5x-10}{x^2-3x-4}dx=\int\frac{2}{x-4}dx+\int\frac{3}{x+1}dx=2\ln|x-4|+3\ln|x+1|+c$$

Thus, it is desirable to have some method that will enable us to obtain the left side of (1), starting with the right side.

Integration of rational fractions:

When we have to integrate a rational fraction, say $\frac{P(x)}{Q(x)}$, if $P(x)$ be not a lower degree than $Q(x)$, we will first express it in the form

$$\frac{P(x)}{Q(x)}=F_1(x)+F_2(x)+\dots+F_n(x) \quad \text{---(1)}$$

where, $F_1(x), F_2(x), \dots, F_n(x)$ are rational functions, in which the denominators are factors of $Q(x)$. Here the sum is called the partial fraction decomposition of $\frac{P(x)}{Q(x)}$ and the term are called partial fractions.

Then $\displaystyle\int\frac{P(x)}{Q(x)}dx=\int F_1(x)dx+\int F_2(x)dx+\dots+\int F_n(x)dx$

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Case I : When the the denominator contains factors real, linear & none repeated.

To each non-repeated linear factor of the denominator, such as $x-a$, there corresponds a partial fraction of the form $\frac{A}{x-a}$, where A is a constant. The given fraction can be expressed as a sum of fractions of this type & the unknown constant A's can be determined.

Thus if we have to evaluate $\int \frac{px^2+qx+r}{(ax+\alpha)(bx+\beta)(cx+\gamma)}dx$

then

$$\boxed{\frac{px^2+qx+r}{(ax+\alpha)(bx+\beta)(cx+\gamma)} = \frac{A}{ax+\alpha} + \frac{B}{bx+\beta} + \frac{C}{cx+\gamma}}$$

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Integration by partial fraction

Evaluate $\int \frac{x-1}{(x-2)(x-3)}dx$

Soln: Let $I = \int \frac{x-1}{(x-2)(x-3)}dx$ —— (1)

Let $\frac{x-1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$ —— (2)

$\Rightarrow x-1 = A(x-3) + B(x-2)$ —— (3)

When $x=2$ then (3) $\Rightarrow 1 = A(2-3) \Rightarrow \boxed{A=-1}$

When $x=3$ then (3) $\Rightarrow \boxed{2=B}$

$\therefore$ (2) $\Rightarrow \frac{x-1}{(x-2)(x-3)} = \frac{-1}{x-2} + \frac{2}{x-3}$

$\therefore$ (1) $\Rightarrow I = \int \frac{x-1}{(x-2)(x-3)}dx$

$= \int \left(\frac{-1}{x-2} + \frac{2}{x-3}\right)dx$

$= -\ln(x-2) + 2\ln(x-3) + c$ Ans

Prob: Evaluate $\int \frac{x+35}{x^2-25}dx$

Soln: Let $I = \int \frac{x+35}{x^2-25}dx$ —— (1)

Let $\frac{x+35}{x^2-25} = \frac{x+35}{(x+5)(x-5)} = \frac{A}{x+5} + \frac{B}{x-5}$ —— (2)

$\Rightarrow x+35 = A(x-5) + B(x+5)$ —— (3)

When $x=-5$ then (3) $\Rightarrow 30 = A(-10) \Rightarrow \boxed{A=-3}$

When $x=5$ then (3) $\Rightarrow 40 = B.10 \Rightarrow \boxed{B=4}$

$\therefore (2) \Rightarrow \frac{x+35}{x^2-25} = \frac{-3}{x+5} + \frac{4}{x-5}$

& $(3) \Rightarrow I = \int \frac{x+35}{x^2-25}\,dx = \int \left[\frac{-3}{x+5} + \frac{4}{x-5}\right] dx$

$= -3\ln(x+5) + 4\ln(x-5) + c$ Ans

Prob: Evaluate (i) $\int \frac{x\,dx}{x^2-5x+6}$ (ii) $\int \frac{dx}{x^2-9}$ (iii) $\int \frac{x^2+x-1}{x^3+x^2-6x}\,dx$

Soln: (iii) Let $I = \int \frac{x^2+x-1}{x^3+x^2-6x}\,dx$ —— (1)

Here $x^3+x^2-6x = x(x^2+x-6) = x(x+3)(x-2)$

$\therefore$ Let $\frac{x^2+x-1}{x^3+x^2-6x} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2}$ —— (2)

$\Rightarrow x^2+x-1 = A(x+3)(x-2) + Bx(x-2) + Cx(x+3)$ —— (3)

When $x = 0$ then $(3) \Rightarrow$ $\boxed{A = 1/6}$

When $x = -3$ then $(3) \Rightarrow$ $\boxed{B = 1/3}$

When $x = 2$ then $(3) \Rightarrow$ $\boxed{C = 1/2}$

$\therefore (2) = \frac{x^2+x-1}{x^3+x^2-6x} = \frac{1}{6}\cdot\frac{1}{x} + \frac{1}{3}\frac{1}{x+3} + \frac{1}{2}\frac{1}{x-2}$

$\therefore (1) \Rightarrow I = \int \frac{x^2+x-1}{x^3+x^2-6x}\,dx = \int \left[\frac{1}{6}\frac{1}{x} + \frac{1}{3}\frac{1}{x+3} + \frac{1}{2}\frac{1}{x-2}\right] dx$

$= \frac{1}{6}\ln x + \frac{1}{3}\ln(x+3) + \frac{1}{2}\ln(x-2) + c$ Ans

Case II: When the denominator contains factors real, linear but some repeated.

To each p-fold linear factor, such as $(x-a)^p$, there will corresponds the sum of p partial fractions of the form

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_p}{(x-a)^p},$$

where the constants $A_1, A_2, \ldots, A_p$ can be evaluated easily.

Thus to evaluate $\int \frac{px^2+qx+r}{(ax+b)^2(lx+m)}\,dx$,

$$\boxed{\frac{px^2+qx+r}{(ax+b)^2(lx+m)} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{lx+m}}$$

Ex: Evaluate (i) $\int \frac{x}{(x-1)^2(x+2)}\,dx$. (ii) $\int \frac{x^2}{(x+1)^2(x+2)}\,dx$

Sol^n: (i) Let $I = \int \frac{x\,dx}{(x-1)^2(x+2)}$ ——(I)

Let $\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$ ——(II)

$\Rightarrow x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$ ——(III)

When $x=1$ then (III) $\Rightarrow 1 = B(1+2) \Rightarrow \boxed{B = 1/3}$.

When $x=-2$ then (III) $\Rightarrow -2 = C(-2-1)^2 \Rightarrow \boxed{C = -2/9}$

Equating the co-efficient of x^2 from (III), we get

$$0 = A + C \Rightarrow \boxed{A = 2/9}$$

$\therefore$ (II) $\Rightarrow \frac{x}{(x-1)^2(x+2)} = \frac{2/9}{x-1} + \frac{1/3}{(x-1)^2} + \frac{-2/9}{x+2}$

$\therefore (1) \Rightarrow I = \int \frac{x\,dx}{(x-1)^2(x+2)}$

$= \frac{2}{9}\int \frac{dx}{x-1} + \frac{1}{3}\int \frac{dx}{(x-1)^2} - \frac{2}{9}\int \frac{dx}{x+2}$

$= 2/9 \cdot \ln|x-1| + \frac{1}{3}\int (x-1)^{-2}dx - \frac{2}{9}\ln|x+2|$

$= \frac{2}{9}\left(\ln|x-1| - \ln|x+2|\right) + \frac{1}{3}\frac{(x-1)^{-1}}{-1} + c$

$= \frac{2}{9}\ln\left|\frac{x-1}{x+2}\right| - \frac{1}{3}\frac{1}{(x-1)} + c$ Ans

Case III: When the denominator contains factors, real, quadratic but none repeated.

To each non-repeated quadratic factor, such as ax^2+bx+c (or $ax^2+c, c\neq 0$) there corresponds a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$.

Thus to evaluate $\int \frac{px^2+qx+r}{(ax^2+bx+c)(lx+m)}dx$, let

$$\boxed{\frac{px^2+qx+r}{(ax^2+bx+c)(lx+m)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{lx+m}}$$

Ex: Evaluate (i) $\int \frac{x}{(x-1)(x^2+4)}dx$. (ii) $\int \frac{dx}{x^3+1}$

Soln: Let $I = \int \frac{x\,dx}{(x-1)(x^2+4)}$ —— (1)

Let $\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$ —— (II)

$\Rightarrow x = A(x^2+4) + (Bx+C)(x-1)$ —— (III)

When $x=1$ then (III) $\Rightarrow 1 = A(1+4) \Rightarrow \boxed{A = 1/5}$

Equating the co-efficients of x^2 & x on both sides of (III), we get

$0 = A + B$ & $C - B = 1$

$\Rightarrow \boxed{B = -1/5}$ $\Rightarrow \boxed{C = 1 + B = 4/5}$

$\therefore$ (II) $\Rightarrow \dfrac{x}{(x-1)(x^2+4)} = \dfrac{1/5}{x-1} + \dfrac{-\frac{1}{5}x + \frac{4}{5}}{x^2+4}$

$$\Rightarrow \int \frac{x\,dx}{(x-1)(x^2+4)} = \frac{1}{5}\int \frac{dx}{x-1} - \frac{1}{5}\int \frac{x-4}{x^2+4}\,dx$$

$$= \frac{1}{5}\ln|x-1| - \frac{1}{5}\left[\int \frac{x\,dx}{x^2+4} - \int \frac{4}{x^2+4}\,dx\right]$$

$$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\int \frac{2x\,dx}{x^2+4} + \frac{4}{5}\int \frac{dx}{x^2+2^2}$$

$$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln|x^2+4| + \frac{4}{5}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + c.$$

$$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln|x^2+4| + \frac{2}{5}\tan^{-1}\frac{x}{2} + c$$

Ans

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}}$$

Case IV : When the degree of numerator is equal or greater than denominator.

In this case the numerator is divided by the denominator and the fraction would be of the form

$1 + \frac{P}{Q}$, where Q = denominator & P is of a lower degree than Q.

Ex: Evaluate (i) $\int \frac{x^2-6x+5}{(x-2)(x-4)}dx$ (ii) $\int \frac{x^3\,dx}{(x-1)(x-2)(x-3)}$.

Soln: (i) Given, $I = \int \frac{x^2-6x+5}{(x-2)(x-4)}dx$ ——(i)

Let $\frac{x^2-6x+5}{(x-2)(x-4)} = 1 + \frac{A}{x-2} + \frac{B}{x-4}$ ——(ii)

$\Rightarrow x^2-6x+5 = (x-2)(x-4) + A(x-4) + B(x-2)$ ——(iii)

When $x=2$ then (iii) $\Rightarrow 2^2-6.2+5 = A(2-4) \Rightarrow \boxed{A = 3/2}$

When $x=4$ then (iii) $\Rightarrow 4^2-6.4+5 = B(4-2) \Rightarrow \boxed{B = -3/2}$

$\therefore (1) \Rightarrow I = \int \frac{x^2-6x+5}{(x-2)(x-4)}dx = \int dx + \frac{3}{2}\int \frac{dx}{x-2} - \frac{3}{2}\int \frac{dx}{x-4}$.

$= x + \frac{3}{2}\ln|x-2| - \frac{3}{2}\ln|x-4| + C$

$= x + \frac{3}{2}\ln\left|\frac{x-2}{x-4}\right| + C$ Ans

Ex: Evaluate

(i) $\int \frac{x^3+x+6}{(x-1)(x-2)(x-3)}dx$

(ii) $\int \frac{(2x-3)\,dx}{(x-1)(x+1)(2x+3)}$

(iii) $\int \frac{e^x\,dx}{e^x-3e^{-x}+2}$

Standard Integrals

Manuf

Prob: Show that $\int \sec x \, dx = \log(\sec x + \tan x) + c$
$= \log\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right] + c.$

Sol^n: Let $I = \int \sec x \, dx$

$$= \int \frac{\sec x(\sec x + \tan x)\,dx}{\sec x + \tan x}$$

$$= \int \frac{(\sec^2 x + \sec x \tan x)\,dx}{\sec x + \tan x}$$

Let $\sec x + \tan x = z$

$\therefore (\sec x \tan x + \sec^2 x)\,dx = dz$

$$= \int \frac{dz}{z}$$

$$= \log z + c$$

$$\Rightarrow \boxed{I = \log(\sec x + \tan x) + c} \quad \text{——(1)}$$

Again, $\sec x + \tan x = \frac{1}{\cos x} + \frac{\sin x}{\cos x}$

$$= \frac{1 + \sin x}{\cos x}$$

$$= \frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

$$= \frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)}$$

$$= \frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}$$

$$= \frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}$$

$$= \frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}} \quad \left[\because \tan\frac{\pi}{4} = 1\right]$$

$$= \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$\therefore (1) \Rightarrow \boxed{I = \log\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right] + c} \quad \text{——(2)}$$

So from (1) & (2), we get

$$\int \sec x\, dx = \log(\sec x + \tan x) + c$$
$$= \log\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right] + c$$

proved.

Prob: Show that $\int \sqrt{a^2 - x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$.

soln: Let $I = \int \sqrt{a^2 - x^2}\, dx$

Let $x = a\sin\theta \Rightarrow \theta = \sin^{-1} x/a$

$$= \int \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\, d\theta \quad \because dx = a\cos\theta\, d\theta$$
$$= \int \sqrt{a^2(1-\sin^2\theta)}\; a\cos\theta\, d\theta$$
$$= \int \sqrt{a^2\cos^2\theta}\; a\cos\theta\, d\theta$$
$$= \int a\cos\theta \cdot a\cos\theta\, d\theta$$
$$= a^2 \int \cos^2\theta\, d\theta$$
$$= \frac{a^2}{2}\int 2\cos^2\theta\, d\theta$$
$$= \frac{a^2}{2}\int (1+\cos 2\theta)\, d\theta$$
$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$$
$$= \frac{a^2}{2}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right] + c$$
$$= \frac{a^2}{2}\left[\theta + \sin\theta\sqrt{1-\sin^2\theta}\right] + c$$
$$= \frac{a^2}{2}\left[\sin^{-1} x/a + \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right] + c$$
$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a}\,\frac{\sqrt{a^2-x^2}}{a}\right] + c$$
$$= \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

Ans

Prob: Show that $\int \operatorname{cosec} x\, dx = \log(\operatorname{cosec} x - \cot x) + c$.

Soln: Let $I = \int \operatorname{cosec} x\, dx$

$$= \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x}\, dx$$

$$= \int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{\operatorname{cosec} x - \cot x}\, dx$$

Let $\operatorname{cosec} x - \cot x = z$

$\therefore (-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x)\, dx = dz$

$$= \int \frac{dz}{z}$$

$$= \log z + c$$

$$= \log(\operatorname{cosec} x - \cot x) + c \quad \text{Ans}$$

Prob: Show that $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$.

Soln: Let $I = \int e^{ax} \sin bx\, dx$

$$= \sin bx \int e^{ax} dx - \int \left\{ \frac{d}{dx} \sin bx \int e^{ax} dx \right\} dx$$

$$= \sin bx \frac{e^{ax}}{a} - \int b \cos bx \frac{e^{ax}}{a}\, dx$$

$$= \frac{e^{ax}}{a} \sin bx - \frac{b}{a} \int \cos bx\, e^{ax}\, dx$$

$$= \frac{e^{ax}}{a} \sin bx - \frac{b}{a}\left[\cos bx \int e^{ax} dx - \int \left\{ \frac{d}{dx} \cos bx \int e^{ax} dx \right\} dx\right]$$

$$= \frac{e^{ax}}{a} \sin bx - \frac{b}{a}\left[\cos bx \frac{e^{ax}}{a} + \int b \sin bx \frac{e^{ax}}{a}\, dx\right]$$

$$= \frac{e^{ax}}{a} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int \sin bx\, e^{ax}\, dx$$

$$\Rightarrow I = \frac{a e^{ax} \sin bx - b e^{ax} \cos bx}{a^2} - \frac{b^2}{a^2} \cdot I$$

$$\Rightarrow I\left[1 + \frac{b^2}{a^2}\right] = \frac{e^{ax}}{a^2}\left[a \sin bx - b \cos bx\right]$$

$$\Rightarrow I\left(\frac{a^2+b^2}{a^2}\right) = \frac{e^{ax}}{a^2}\left[a\sin bx - b\cos bx\right]$$

$$\Rightarrow \boxed{I = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)}$$

proved

Prob: (i) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}, (a\neq 0)$.

(ii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|, (|x|>|a|)$

Soln: (i) Let $I = \int \frac{dx}{x^2+a^2}$

Let $x = a\tan\theta$

$\therefore dx = a\sec^2\theta\, d\theta$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2\tan^2\theta + a^2}$$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2(\tan^2\theta+1)}$$

$$= \frac{1}{a}\int \frac{\sec^2\theta\, d\theta}{\sec^2\theta}$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\Rightarrow \boxed{I = \frac{1}{a}\tan^{-1}\frac{x}{a}}$$

(ii) Let $I = \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int\left[\frac{1}{x-a} - \frac{1}{x+a}\right]dx$

$$= \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right)$$

$$\Rightarrow \boxed{I = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|}.$$

Standard integral of the form:

$$\int \frac{dx}{ax^2+bx+c}, \quad \int \frac{dx}{\sqrt{ax^2+bx+c}}, \quad \int \sqrt{ax^2+bx+c}\,dx.$$

Here $ax^2+bx+c = a\left(x^2+\frac{bx}{a}+\frac{c}{a}\right)$

$$= a\left[x^2+2.x.\frac{b}{2a}+\left(\frac{b}{2a}\right)^2+\frac{c}{a}-\left(\frac{b}{2a}\right)^2\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$$

$$\therefore I = \int \frac{dx}{ax^2+bx+c}$$

$$\Rightarrow I = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}} \quad \text{—— (1)}$$

$$= \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2+\left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2}$$

if $4ac-b^2>0 \Rightarrow 4ac>b^2$

$$= \frac{1}{a}\int \frac{dz}{z^2+k^2}$$

Let $x+\frac{b}{2a} = z$

$\therefore dx = dz$ &

$\frac{\sqrt{4ac-b^2}}{2a} = k$

$$\Rightarrow \boxed{I = \frac{1}{a}\,\frac{1}{k}\tan^{-1}\frac{z}{k}}$$

Again, (1) $\Rightarrow$

$$I = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}}$$

if $4ac-b^2<0 \Rightarrow 4ac<b^2$

$$= \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a}\right)^2}$$

$$= \frac{1}{a} \int \frac{dz}{z^2 - k^2}$$

$$\Rightarrow \boxed{I = \frac{1}{a} \frac{1}{2k} \ln \frac{z-k}{z+k}}$$

1 **Ex:** Evaluate $\int \frac{dx}{2x^2 + x + 1}$

Soln: Let $I = \int \frac{dx}{2x^2 + x + 1}$

$$= \frac{1}{2} \int \frac{dx}{x^2 + \frac{x}{2} + \frac{1}{2}}$$

$$= \frac{1}{2} \int \frac{dx}{\left(x + \frac{1}{4}\right)^2 + \frac{1}{2} - \frac{1}{16}}$$

$$= \frac{1}{2} \int \frac{dx}{\left(x + \frac{1}{4}\right)^2 + \frac{7}{16}}$$

$$= \frac{1}{2} \int \frac{dx}{\left(x + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2}$$

$$= \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{7}}{4}} \cdot \tan^{-1} \frac{x + \frac{1}{4}}{\frac{\sqrt{7}}{4}} + c$$

$$= \frac{2}{\sqrt{7}} \tan^{-1} \frac{4x+1}{\sqrt{7}} + c$$

Ans

CamScanner

Prob: Integrate $\int \frac{dx}{3+4\cos x}$

Soln: Let $I = \int \frac{dx}{3+4\cos x}$

$$= \int \frac{dx}{3(\sin^2 x/2 + \cos^2 x/2) + 4(\cos^2 x/2 - \sin^2 x/2)}$$

$$= \int \frac{\sec^2 x \, dx}{3(1+\tan^2 x/2) + 4(1-\tan^2 x/2)}$$

$$= \int \frac{\sec^2 x \, dx}{7-\tan^2 x/2}$$

Let $\tan x/2 = z$

$\therefore \sec^2 \frac{x}{2} \cdot \frac{1}{2} dx = dz$

$\Rightarrow \sec^2 x/2 \, dx = 2dz$

$$= \int \frac{2dz}{7-z^2}$$

$$= 2\int \frac{dz}{(\sqrt{7})^2 - z^2}$$

$$= 2 \cdot \frac{1}{2\sqrt{7}} \ln \frac{\sqrt{7}+z}{\sqrt{7}-z} + c \qquad \left[\because \int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln \frac{a+x}{a-x}\right]$$

$$= \frac{1}{\sqrt{7}} \ln \frac{\sqrt{7}+\tan x/2}{\sqrt{7}-\tan x/2} + c$$

Prob: Evaluate $\int \frac{dx}{5-3\cos x}$

Soln: Let $I = \int \frac{dx}{5-3\cos x}$

$$= \int \frac{dx}{5(\sin^2 x/2 + \cos^2 x/2) - 3(\cos^2 x/2 - \sin^2 x/2)}$$

$$= \int \frac{\sec^2 x/2 \, dx}{5(1+\tan^2 x/2) - 3(1-\tan^2 x/2)}$$

$$= \int \frac{\sec^2 x/2 \, dx}{2+8\tan^2 x/2}$$

$$= \int \frac{\sec^2 x/2 \, dx}{8\left[\frac{1}{4} + \tan^2 x/2\right]}$$

$$= \frac{1}{8}\int \frac{\sec^2 x/2 \, dx}{\frac{1}{4} + \tan^2 x/2}$$

Let $\tan x/2 = z$

$\Rightarrow \sec^2 x/2 \cdot \frac{1}{2} dx = dz$

$\Rightarrow \sec^2 x/2 \, dx = 2dz$

$$= \frac{1}{8}\int \frac{2dz}{\left(\frac{1}{2}\right)^2 + z^2}$$

$$= \frac{1}{4} \cdot \frac{1}{\frac{1}{2}} \tan^{-1} \frac{z}{1/2} + c \qquad \left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{4} \cdot 2 \tan^{-1} 2z + c$$

$$= \frac{1}{2} \tan^{-1}\{2 \tan x/2\} + c \quad \text{Ans}$$

Prob: Evaluate $\displaystyle\int_0^{\pi/2} \frac{\tan x \, dx}{1+m^2\tan^2 x}$.

Soln: Let $\displaystyle I = \int_0^{\pi/2} \frac{\tan x \, dx}{1+m^2 \tan^2 x}$

$$= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{1+m^2 \frac{\sin^2 x}{\cos^2 x}} dx$$

$$= \int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + m^2 \sin^2 x} dx$$

$$= \int_0^{\pi/2} \frac{\sin x \cos x \, dx}{1 - \sin^2 x + m^2 \sin^2 x}$$

$$= \int_0^{\pi/2} \frac{\sin x \cos x \, dx}{1 + (m^2-1)\sin^2 x}$$

$$= \int_1^{m^2} \frac{1}{z} \cdot \frac{dz}{2(m^2-1)}$$

$$= \frac{1}{2(m^2-1)} \cdot [\ln z]_1^{m^2}$$

$$= \frac{1}{2(m^2-1)} [\ln m^2 - \ln 1]$$

$$= \frac{2\ln m - 0}{2(m^2-1)}$$

$$= \frac{\ln m^2}{m^2-1} \quad \underline{\text{Ans}}$$

Let

$1+(m^2-1)\sin^2 x = z$

$\Rightarrow 0 + (m^2-1).2\sin x \cos x \, dx = dz$

$\Rightarrow \sin x \cos x \, dx = \dfrac{dz}{2(m^2-1)}$

x	0	$\pi/2$
z	1	m^2

Prob: Evaluate $\int \frac{x^2-x+1}{x^2+x+1} dx$

Sol^n: Let $I = \int \frac{x^2-x+1}{x^2+x+1} dx$

$$= \int \frac{x^2+x+1-2x}{x^2+x+1} dx$$

$$= \int dx - \int \frac{2x}{x^2+x+1} dx$$

$$= x - \int \frac{2x+1-1}{x^2+x+1} dx$$

$$= x - \left[\int \frac{2x+1}{x^2+x+1} dx - \int \frac{dx}{x^2+x+1} \right]$$

$$= x - \ln(x^2+x+1) + \int \frac{dx}{x^2+2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}$$

$$= x - \ln(x^2+x+1) + \int \frac{dx}{3/4 + (x+1/2)^2}$$

$$= x - \ln(x^2+x+1) + \int \frac{dx}{(\sqrt{3}/2)^2 + (x+1/2)^2}$$

$$= x - \ln(x^2+x+1) + \frac{1}{\sqrt{3}/2} \tan^{-1} \frac{x+1/2}{\sqrt{3}/2} + c$$

$$= x - \ln(x^2+x+1) + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + c$$ Ans

Prob: Evaluate $\int \frac{x^2-2}{(x+1)(x^2+1)} dx$.

Sol^n: Let $I = \int \frac{x^2-2}{(x+1)(x^2+1)} dx$ ——— (I)

Here, $\frac{x^2-2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$ ——— (II)

$\Rightarrow x^2-2 = A(x^2+1) + (Bx+C)(x+1)$ ——— (III)

Equating the co-efficient of x^2, we get

$1 = A+B$ ——— (IV)

Equating the co-efficient of x, we get

$0 = B+C$ ——— (V)

Equating the constant, we get

$-2 = A+C$ ——— (VI)

Solving (IV), (V) & (VI) we get $\boxed{A=-1/2}$ $\boxed{B=3/2}$ $\boxed{C=-3/2}$

$\therefore$ (II) $\Rightarrow \frac{x^2-2}{(x+1)(x^2+1)} = -\frac{1}{2}\frac{1}{x+1} + \frac{3}{2}\frac{x-1}{x^2+1}$

$\therefore$ (I) $\Rightarrow I = \int \frac{x^2-2}{(x+1)(x^2+1)} dx = -\frac{1}{2}\int \frac{dx}{x+1} + \frac{3}{2}\int \frac{x-1}{x^2+1} dx$

$$= -\frac{1}{2}\ln(x+1) + \frac{3}{2}\int \left[\frac{x}{x^2+1} - \frac{1}{x^2+1}\right] dx$$

$$= -\frac{1}{2}\ln(x+1) + \frac{3}{2}\int \left[\frac{1}{2}\frac{2x}{x^2+1} - \frac{1}{x^2+1}\right] dx$$

$$= -\frac{1}{2}\ln(x+1) + \frac{3}{2}\cdot\frac{1}{2}\int \frac{2x\,dx}{x^2+1} - \frac{3}{2}\int \frac{dx}{1+x^2}$$

$$= -\frac{1}{2}\ln(x+1) + \frac{3}{4}\ln(x^2+1) - \frac{3}{2}\tan^{-1}x + c$$ Ans

Prob. Evaluate $\int \frac{5x^2+3x+17}{x^3-x^2+4x-4}dx$

Soln. Let $I = \int \frac{5x^2+3x+17}{x^3-x^2+4x-4}dx$

$$= \int \frac{5x^2+3x+17}{x(x^2+4)-(x^2+4)}dx$$

$$\Rightarrow I = \int \frac{5x^2+3x+17}{(x^2+4)(x-1)}dx \quad \text{——— (I)}$$

Let $\frac{5x^2+3x+17}{(x^2+4)(x-1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-1}$ ——— (II)

$\Rightarrow 5x^2+3x+17 = (Ax+B)(x-1)+C(x^2+4)$ ——— (III)

$\Rightarrow 5x^2+3x+17 = Ax^2+Bx-Ax-B+Cx^2+4C$

Equating the co-efficient of x^2, we get

$5 = A+C$ ——— (IV)

Equating the co-efficient of x, we get

$3 = B-A$ ——— (V)

Equating the constant, we get

$17 = -B+4C$ ——— (VI)

using (IV), (V) & (VI) we get

$\boxed{A=0}$ $\boxed{B=3}$ & $\boxed{C=5}$

$\therefore$ (II) $\Rightarrow \frac{5x^2+3x+17}{(x^2+4)(x-1)} = \frac{3}{x^2+4} + \frac{5}{x-1}$

$\therefore$ (I) $\Rightarrow I = \int \frac{5x^2+3x+17}{(x^2+4)(x-1)}dx = \int \frac{3dx}{x^2+2^2} + \int \frac{5}{x-1}dx$

$= 3\frac{1}{2}.\tan^{-1}x/2 + 5\ln(x-1)+c$

Ans