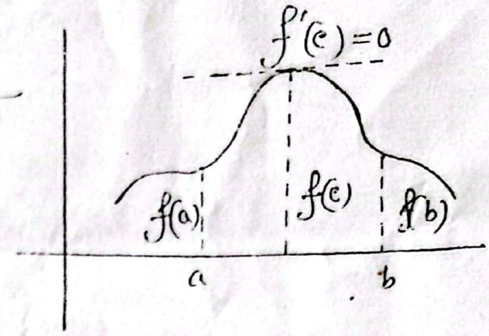
Rolle's th^m & Mean Value th^m

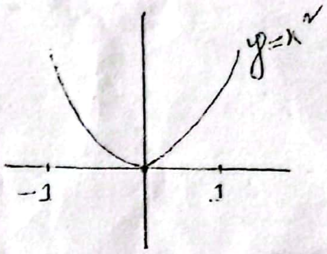
Marcuf

Rolle's th^m: Let f is differentiable on (a,b) & continuous on $[a,b]$. If $f(a)=f(b)$, then there exists at least one number c in (a,b) such that

$$f'(c)=0.$$



Prob: Discuss the application of Rolle's th^m to the function $f(x)=x^2$ on $(-1,1)$.



Soln: Given, $f(x)=x^2$ on $(-1,1)$

Here $f(a)=f(-1)=1$

$f(b)=f(1)=1$ $\therefore \boxed{f(-1)=f(1)}$

Again

$$Rf'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$
$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$
$$=\lim_{h\to 0}\frac{h(2x+h)}{h}$$
$$=\lim_{h\to 0}(2x+h)=2x$$

& $$Lf'(x)=\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$$
$$=\lim_{h\to 0}\frac{(x-h)^2-x^2}{-h}$$
$$=\lim_{h\to 0}\frac{x^2-2xh+h^2-x^2}{-h}$$
$$=\lim_{h\to 0}\frac{-h(2x-h)}{-h}$$
$$=\lim_{h\to 0}(2x-h)=2x.$$

Here $Rf'(x)=Lf'(x)$

So $f'(x)$ exists on $(-1,1)$.

Since $f(x)$ is diff^{ble} on $(-1,1)$, it is also continuous on the same interval.

AJ

So the conditions of Rolle's thm are satisfied.
Thus there exist a number c in $(-1,1)$ such that.

$$f'(c) = 0$$
$$\Rightarrow 2c = 0$$
$$\Rightarrow \boxed{c = 0}$$

Here $0 \in (-1,1)$

So the thm is varrified.

Prob: Verify Rolle's thm for $f(x) = 4x^2 - 20x + 29$ in $(1,4)$.

Soln: Given, $f(x) = 4x^2 - 20x + 29$ in $(1,4)$

Here $f(1) = 13$

$f(4) = 13$

$$\therefore \boxed{f(1) = f(4)}$$

Here

$$Rf'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

& $Lf'(x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h}$

$= 8x - 20$

$$= \lim_{h \to 0} \frac{4(x+h)^2 - 20(x+h) + 29 - 4x^2 + 20x - 29}{h}$$

$$= \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 - 20x - 20h - 4x^2 + 20x}{h}$$

$$= \lim_{h \to 0} \frac{h(8x + 4h - 20)}{h}$$

$$= 8x - 20$$

$\therefore Rf'(x) = Lf'(x)$

So $f'(x)$ exists on $(1,4)$.

Since $f(x)$ is diffble on $(1,4)$, it is also continuous on the same interval.

So f satisfies the all conditions of Rolle's thm.

Here $f'(x) = 8x - 20$

$\therefore f'(c) = 8c - 20.$

By thm there exist c in $(1,4)$ such that

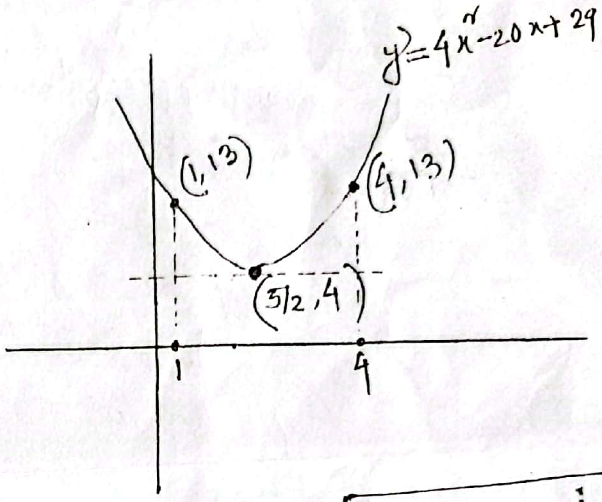
$f'(c) = 0$

$\Rightarrow 8c - 20 = 0$

$\Rightarrow \boxed{c = 5/2}$

Here $c = \frac{5}{2} \in (1,4)$.

So the thm is varified.



$$\boxed{\begin{aligned} y &= 4x^2 - 20x + 29 \\ &= 4x^2 - 2\cdot 2x \cdot 5 + 25 + 4 \\ &= (2x-5)^2 + 4 \end{aligned}}$$

Prob: Verify Rolle's thm for $f(x) = x^2 - 5x + 4$ in $(1,4)$.

Prob: Verify Rolle's thm for $f(x) = x^2 + 5x - 6$ in $(-6,1)$.

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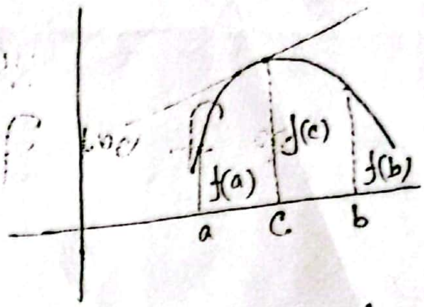
State & prove MVT

Statement: Let f be diff^{ble} on (a,b) & cont^s on $[a,b]$, then there is atleast one number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof: Let $\phi(x) = f(x) - Ax$ ---(1), be a fn which follows the conditions of Rolle's thm.



Here $\phi(a) = f(a) - Aa$

& $\phi(b) = f(b) - Ab$

By Rolle's thm

$$\phi(a) = \phi(b)$$

$$\Rightarrow f(a) - Aa = f(b) - Ab \Rightarrow Ab - Aa = f(b) - f(a)$$

$$\Rightarrow A = \frac{f(b)-f(a)}{b-a} \quad \text{---(2)}$$

Putting (2) in (1) we get

$$\phi(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}\right]x \quad \text{---(3)}$$

since $\phi(x)$ satisfies the Rolle's thm, thus there is a number c in (a,b) such that $\phi'(c) = 0$

Differentiating (3) with respect to x we get

$$\phi'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \phi'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} \quad [\text{Putting } x=c]$$

$$\Rightarrow 0 = f'(c) - \frac{f(b)-f(a)}{b-a} \quad [\because \phi'(c) = 0]$$

$$\Rightarrow \boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$

So the MVT is proved

Prob: Verify MVT for $f(x)=\frac{1}{4}x^2+1$ in $(-1,4)$.

Soln: Given, $f(x)=\frac{1}{4}x^2+1$.

Here $f'(x)=\frac{1}{2}x$

$\therefore f'(c)=\frac{1}{2}c$.

& $f(-1)=5/4$ & $f(4)=5$

since $f(x)$ is a polynomial fn, it is diffble & continuous.

By MVT, we have

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \frac{1}{2}c=\frac{5-5/4}{4+1}=\frac{3}{4}$$

$$\Rightarrow \boxed{c=3/2} \in (-1,4)$$

So the MVT is varified for $c=3/2$. Ans

Prob: Verify MVT for $f(x)=x^3-8x-5$ in $(1,4)$.

Hinds:
$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow 3c^2-8=\frac{27-(-12)}{4-1}=\frac{27+12}{3}$$

$$\Rightarrow c^2=7 \Rightarrow c=\pm\sqrt{7}$$

but $\boxed{c=\sqrt{7}\in(1,4)}$

So for $c=\sqrt{7}$, the MVT is varified.

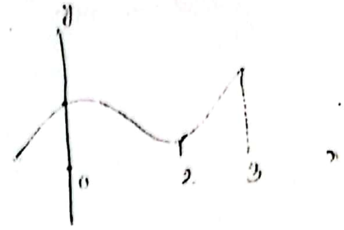
Prob: Verify MVT for $f(x)=3+2x-x^2$ in $(0,1)$

Ans $\boxed{c=1/2}$

Maximum & Minimum

Manuf

The terms 'increasing', 'decreasing', & 'constant' are used to describe the behaviour of a function over an interval as we travel left to right along its graph. For example, the fn graphed in the figure can be described as increasing on the interval $(-\infty, 0]$, decreasing on the interval $[0, 2]$, increasing again on the interval $[2, 3]$ and constant on the interval $[3, \infty)$.

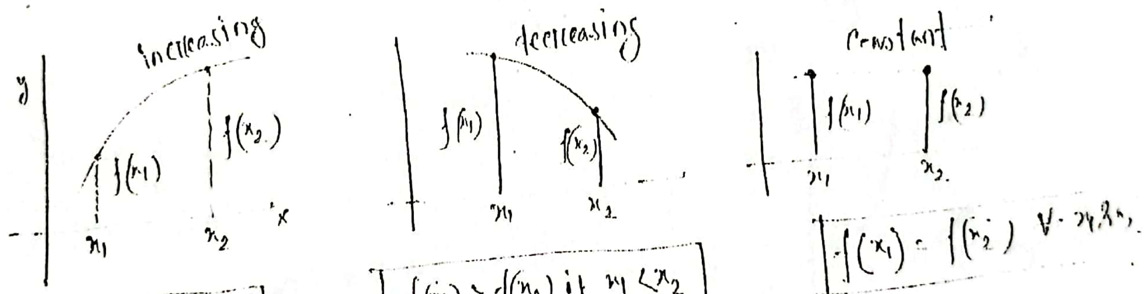


Defn. Let $f(x)$ be defined on an interval, and let x_1 & x_2 denote two points in that interval then

(a) $f(x)$ is increasing on that interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$

(b) $f(x)$ is decreasing " " " " $f(x_1) > f(x_2)$ " $x_1 < x_2$

(c) $f(x)$ is constant " " " " $f(x_1) = f(x_2)$ for all points x_1 & x_2



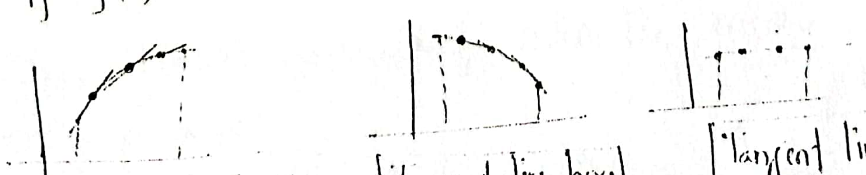
$\boxed{f(x_1) < f(x_2) \text{ if } x_1 < x_2}$ $\boxed{f(x_1) > f(x_2) \text{ if } x_1 < x_2}$ $\boxed{f(x_1) = f(x_2)\ \forall\ x_1, x_2}$

Thm Let $f(x)$ be a continuous fn on a closed interval $[a, b]$ & differentiable on the open interval (a, b) then

(a) If $f'(x) > 0$ for every value of x in (a, b), then $f(x)$ is increasing on $[a, b]$

(b) If $f'(x) < 0$ " " " " " " " " $f(x)$ " decreasing on $[a, b]$

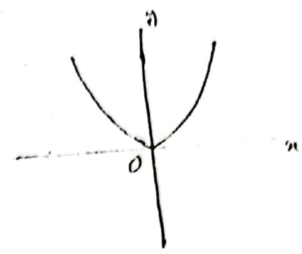
(c) If $f'(x) = 0$ " " " " " " " " $f(x)$ is constant on $[a, b]$



| Tangent lines have +ve slope | | Tangent lines have -ve slope | | Tangent lines have zero slope |

Prob: Find the intervals on which the following fns are increasing and the intervals on which they are decreasing.

(i) $f(x) = x^2$ (ii) $f(x) = x^3$



Soln (i) The graph of $f(x)$ in the figure suggests that $f(x)$ is decreasing for $x \leq 0$ and increasing for $x \geq 0$.

To confirm this, now

$f'(x) = 2x$

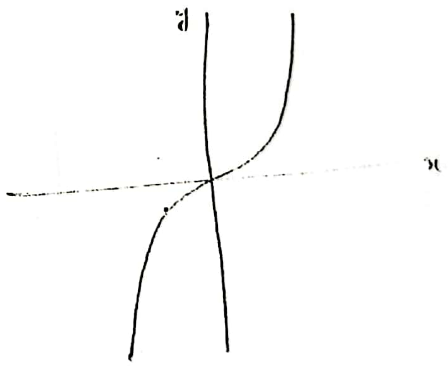
It follows that $f'(x) < 0$ if $-\infty < x < 0$

& $f'(x) > 0$ if $0 < x < \infty$

Since $f(x)$ is continuous at $x=0$, it follows that

$f(x)$ is decreasing on $(-\infty, 0]$

& $f(x)$ " increasing on $[0, \infty)$ ☐



Soln (ii) Given $f(x) = x^3$

$\therefore f'(x) = 3x^2$

Here $f'(x) > 0$ if $-\infty < x < 0$

& $f'(x) > 0$ if $0 < x < \infty$

Since $f(x)$ is continuous at $x=0$,

it follows that

$f(x)$ is increasing on $(-\infty, 0]$

& $f(x)$ is " on $[0, \infty)$

Thus $f(x)$ is increasing over the entire interval $(-\infty, \infty)$.

Prob: find the intervals on which $f(x) = 17 - 15x + 9x^2 - x^3$ is decreasing or increasing.

Soln: Given $f(x) = 17 - 15x + 9x^2 - x^3$

$\therefore f'(x) = -15 + 18x - 3x^2 = -3(x^2 - 6x + 5)$

$\Rightarrow \boxed{f'(x) = -3(x-1)(x-5)}$

Now $f'(x) = 0 \Rightarrow \boxed{x = 1 \;\&\; x = 5}$

Interval	$x-1$	$x-5$	Sign of $f'(x)$	Deci
$-\infty < x < 1$	$-$	$-$	$-$	$f(x)$ is decreasing in $(-\infty, 1]$
$1 < x < 5$	$+$	$-$	$+$	Increasing in $[1, 5]$
$5 < x < \infty$	$+$	$+$	$-$	Decreasing in $[5, \infty)$

Ans

Prob: find the intervals on which $f(x) = 2x^3 - 9x^2 + 12x - 3$ is increasing or decreasing.

Soln: $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$

$\Rightarrow \boxed{f'(x) = 6(x-1)(x-2)}$

Now $f'(x) = 0 \Rightarrow \boxed{x = 1 \;\&\; x = 2}$

Thus

Interval	$x-1$	$x-2$	$f'(x)$	Decision
$-\infty < x < 1$	$-$	$-$	$+$	Increasing
$1 < x < 2$	$+$	$-$	$-$	Decreasing.
$2 < x < \infty$	$+$	$+$	$+$	Increasing

Necessary condition for maximum & minimum:

If $f(x)$ be a maximum, or a minimum at $x = c$, then if $f'(c)$ exists, then $f'(c) = 0$.

Thm (2nd derivative test):

Let $f(x)$ is twice differentiable at the point x_0

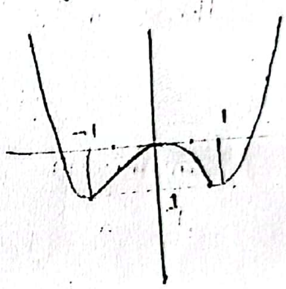
(a) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f(x)$ has a ~~mi~~ relative minimum at x_0

(b) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $f(x)$ has a relative maximum at x_0.

(c) If $f'(x_0) = 0$ & $f''(x_0)$ ~~is~~ $= 0$ then $f(x)$ may have a relative maximum, a relative minimum, or neither at x_0.

Prob: Locate the relative maximum & minimum of $f(x) = x^4 - 2x^2$.

Soln: Given $f(x) = x^4 - 2x^2$



$\therefore f'(x) = 4x^3 - 4x$

$\therefore f''(x) = 12x^2 - 4$

For maximum & minimum,

$f'(x) = 0$

$\Rightarrow 4x(x^2 - 1) = 0$

$\Rightarrow \boxed{x = 0, \pm 1}$

Now $f''(0) = -4 < 0$

$f''(1) = 8 > 0$

$f''(-1) = 8 > 0$

Thus there is a relative maximum ~~at~~ at $x = 0$ and relative minimum at $x = 1$ & $x = -1$

91. Prob: find the maximum & minimum values of $f(x)=2x^3-21x^2+36x-20$

Soln: Given $f(x)=2x^3-21x^2+36x-20$

$\therefore f'(x)=6x^2-42x+36x$ & $f''(x)=12x-42$

For maximum & minimum values,

$f'(x)=0$

$\Rightarrow 6(x^2-7x+6)=0$

$\Rightarrow x^2-6x-x+6=0$

$\Rightarrow \boxed{x=1}\ \boxed{x=6}$

When $x=1$ then $f''(1)=12-42=-30<0$

When $x=6$ then $f''(6)=12.6-42=30>0$

So the fn has a minimum value at $x=6$ and has a maximum value at $x=1$

Here the maximum value is

$f(1)=2(1)^3-21.1^2+36.1-20=-3$

and the minimum value is

$f(6)=2.(6)^3-21.6^2+36.6-20=-128$

in Prob: Discuss the maximum & minimum values of $f(x)=x^3-6x^2+9x+5$

Soln: Given $f(x)=x^3-6x^2+9x+5$

$\therefore f'(x)=3x^2-12x+9$ and $f''(x)=6x-12$

For max^m & min^m values,

$f'(x)=0$

$\Rightarrow 3(x^2-4x+3)=0$

$\Rightarrow (x-1)(x-3)=0$

$\Rightarrow \boxed{x=1,3}$

When $x=1$ then $f''(1)=6.1-12=-6<0$

When $x=3$ then $f''(3)=6.3-12=6>0$

Thus at $x=1$, the ftn has a maximum value & at $x=6$ it has a minimum value.

and the max^m value is $f(1) = 1^3 - 6.1^2 + 9.1 + 5 = 9$
& the min^m " " $f(3) = 3^3 - 6.3^2 + 9.3 + 5 = 5$

Ans

Prob: show that the ft^n $f(x) = x^3 - 3x^2 + 6x + 3$ is neither max^m & or min^m

Sol^n: Given $f(x) =$

$\therefore f'(x) = 3x^2 - 6x + 6$

For max^m or min^m $f'(x) = 0$

$\Rightarrow 3(x^2 - 2x + 2) = 0$

$\Rightarrow (x-1)^2 + 1 = 0$

Here we see that $f'(x)$ is not equal to zero for any real value of x.

Thus $f(x)$ is neither max^m or min^m for any real value. Ans

(*) $f(x) = x^3 - 3x^2 + 9x - 1$

$\boxed{\text{same as above}}$

1. Show that $f(x) = x^3 - 3x^2 + 18x + 15$ is a increasing f^n

Sol^n: $f'(x) = 3x^2 - 6x + 18$

$= 3x^2 - 6x + 3 + 15$

$= 3(x-1)^2 + 15$

Here we see that for all x, $f'(x) > 0$.

So it is increasing f^n.

2. $f(x) = 1 - x - x^3$ is decreasing.

$f'(x) = -1 - 3x^2$

$= -(1 + 3x^2)$

$\therefore f'(x) < 0$ for all x.

decreasing.

Find max^m & min of ① $f(x) = 2x^3 - 9x^2 + 12x - 3$

$f'(x) = 6x^2 - 18x + 12$ & $f''(x) = 12x - 18$

For max & min

$f'(x) = 0$

$\Rightarrow 6(x^2 - 3x + 2) = 0$

$\Rightarrow$ ——

$\Rightarrow \boxed{x = 1, 2}$

When $x = 1$ then $f''(1) = \quad = -6 < 0$

So $f(x)$ has max^m value at $x = 1$ & the value is

$f(1) = \text{-----} = 2$

When $x = 2$ $f''(x) = 6 > 0$

$x = 2$ is min

$f(2) = 1$

* <u>Max & Min</u> $f(x) = 2x^3 - 6x^2 - 18x + 7$

$f'(x) = 0$ $\qquad\qquad$ $f''(x) = 12x - 12$

$\Rightarrow 6x^2 - 12x - 18 = 0$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow \boxed{x = 3, -1}$

$\therefore x = -1, \; f''(-1) = -24 < 0$

$\longrightarrow$ max^{m}, $f(-1) = 17$

$x = 3 \quad f''(3) = 24 > 0$

$\longrightarrow$ min $\quad f(3) = -47$

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Max & min

Mark 28.03.0

Prob: Given $f(x) = 5x^6 - 18x^5 + 15x^4 - 10$

$\therefore f'(x) = 30x^5 - 90x^4 + 60x^3$

For max^m and min

$f'(x) = 0 \Rightarrow 30x^3(x^2 - 3x + 2) = 0$

$\Rightarrow \boxed{x = 1, 2, 0}$

$\therefore f''(x) = 150x^4 - 360x^3 + 180x^2$

$= 30(5x^4 - 12x^3 + 6x^2)$

when $x = 1$, then $f''(1) = -30 < 0$

so the f^n has a max^m value at $x = 1$ & the max^m value is $f(1) = -8$

When $x = 2$, then $f''(2) = 240 > 0$

so the f^n has a min value at $x = 2$ & the min^m value is $f(2) = -26$

When $x = 0$ then $f''(0) = 0$

Now $f'''(x) = 30(20x^3 - 36x^2 + 12x)$

& $f^{iv}(x) = 30(60x^2 - 72x + 12)$

Now when $x = 0$ then $f^{iv}(0) = 360 > 0$

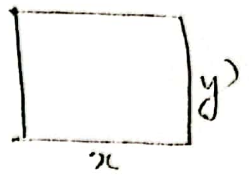
so the f^n has a min^m value at $x = 0$ & the min^m value is $f(0) = -10$

Ans

Prob: A farmer can afford only 800 ft of wire fencing. He wishes to enclose a rectangular field of max^m area. What would be the dimension of the field.

Sol^n: Let length $= x$

& width $= y$



$\therefore$ Perimeter $= 2(x+y)$

Given $2(x+y) = 800 \Rightarrow \boxed{y = 400 - x}$ —— (1)

If the area of the field is A then

$$A = xy$$

$$\Rightarrow A = x\cdot(400-x) = 400x - x^2$$

$$\Rightarrow \frac{dA}{dx} = 400 - 2x$$

$$\& \quad \frac{d^2A}{dx^2} = -2 < 0$$

For max^m & min, $\frac{dA}{dx} = 0$

$$\Rightarrow \boxed{x = 200}$$

So when $x = 200$, we get the max^m area

$$(1) \Rightarrow \boxed{y = 200}$$

$\therefore$ $x = y = 200$ ft Ans

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