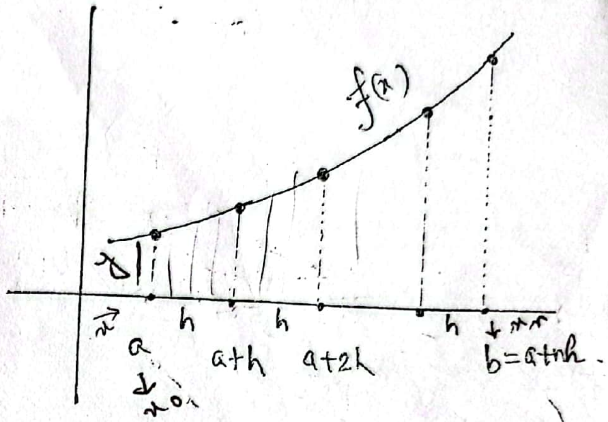
Definite integral

Definite integral as the limit of a sum :

Let $f(x)$ be a single-valued function which is continuous on $[a,b]$ with $b>a$. Divide the interval $[a,b]$ into n equal sub-intervals, so we get

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$



Let h be the length of each subinterval, then

$$nh = b-a \Rightarrow \boxed{h = \frac{b-a}{n}}$$

Thus $a < a+h < a+2h < \cdots < a+nh = b$

Here $h \to 0$ as $n \to \infty$.

From the figure,

$$S_n = h f(a+h) + h f(a+2h) + \cdots + h f(a+nh)$$

$$\Rightarrow S_n = \sum_{r=1}^{n} h f(a+rh)$$

$$\Rightarrow S_n = h \sum_{r=1}^{n} f(a+rh) \Rightarrow \lim_{n\to\infty} S_n = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh).$$

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left(a + \frac{(b-a)r}{n}\right),$$ when $h = \frac{b-a}{n}$, if $h \to 0$ then $n \to \infty$

If $\lim_{n\to\infty} S_n$ exists when $h \to 0$ then $\lim_{n\to\infty} S_n = S$ is defined as the definite integral of $f(x)$ with respect to x between a & b and we write

$$\boxed{S = \int_a^b f(x)\,dx}$$

Definition: A function $f(x)$ is said to be integrable $[a, b]$ if

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left(a+\frac{r}{n}(b-a)\right)$$

exists. When this is the case, we denote this limit by the symbol

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left(a+\frac{r(b-a)}{n}\right)$$

Which is called the definite integral of $f(x)$ from a to b. Here the numbers a & b are called the lower limit & upper limit of the integration and the function $f(x)$ is called the integrand.

Prob: Using definition, evaluate

(i) $\int_a^b x\,dx$ (ii) $\int_a^b \sin x\,dx$ (iii) $\int_0^1 x^2\,dx$.

Soln: (i) Here $f(x) = x$

$\therefore f(a+rh) = a+rh$

We know, $\int_a^b f(x)\,dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh)$, where $nh = b-a$

$\Rightarrow \int_a^b x\,dx = \lim_{h\to 0} h[(a+h)+(a+2h)+(a+3h)+\cdots+(a+nh)]$

$[\because f(x)=x]$

$= \lim_{h\to 0} h[na+h(1+2+3+\cdots+n)]$

$= \lim_{h\to 0} h\left[na+h\cdot\frac{n(n+1)}{2}\right]$

$= \lim_{h\to 0}\left[hna+\frac{nh(nh+h)}{2}\right]$

$= \lim_{h\to 0}\left[(b-a)a+\frac{(b-a)(b-a+h)}{2}\right]$

$= ab-a^2+\frac{(b-a)(b-a)}{2}$

$= \frac{2ab-2a^2+b^2-2ab+a^2}{2}$

$= \frac{b^2-a^2}{2}$.

(iii) Here $f(x) = x^2$

Given $a=0$ & $b=1$

$\therefore nh = b-a = 1$

We know $\int_a^b f(x)\,dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh)$

$\Rightarrow \int_0^1 x^2\,dx = \lim_{h\to 0} h\sum_{r=1}^{n}(rh)^2$ $[\because a=0]$

$= \lim_{h\to 0} h(1^2h^2+2^2h^2+3^2h^2+\cdots+n^2h^2)$

$$= \lim_{h \to 0} h^3 \left(1^2+2^2+3^2+\dots+n^2\right)$$

$$= \lim_{h \to 0} h^3 \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{h \to 0} h^3 \frac{2n^3+3n^2+n}{6}$$

$$= \lim_{h \to 0} \frac{2n^3h^3+3.n^2h^2.h+nh.h^2}{6}$$

$$= \lim_{h \to 0} \frac{2+3h+h^2}{6} \qquad [\because nh=1]$$

$$= \frac{1}{3} \text{ Ans.}$$

(ii) Given $f(x) = \sin x$

$\therefore f(a+rh) = \sin(a+rh)$

Here $\boxed{nh = b-a}$

We know $\int_a^b f(x)\,dx = \lim_{h \to 0} h \sum_{r=1}^{n} f(a+rh)$

$$\Rightarrow \int_a^b \sin x \, dx = \lim_{h \to 0} h \sum_{r=1}^{n} \sin(a+rh)$$

$$= \lim_{h \to 0} h \left[\sin(a+h)+\sin(a+2h)+\dots+\sin(a+nh)\right]$$

$$= \lim_{h \to 0} h \frac{\sin\frac{nh}{2}}{\sin\frac{h}{2}} \sin\left(\frac{a+h+a+nh}{2}\right)$$

$$= \lim_{h \to 0} \left(\frac{h/2}{\sin h/2}\right) . 2\sin\left(\frac{a+h+a+nh}{2}\right) . \sin\frac{nh}{2}$$

$$= \lim_{h \to 0} \left(\frac{h/2}{\sin h/2}\right) 2.\sin\left(\frac{a+h+b}{2}\right) . \sin\frac{b-a}{2}$$

$$= 1.2.\sin\frac{a+b}{2} . \sin\frac{b-a}{2}$$

$$= \cos a - \cos b \quad \text{Ans.}$$

$\sin(a+h)+\sin(a+2h)+\dots+\sin(a+nh) = \dfrac{\sin\left(\frac{\text{no. of terms} \times \text{angle difference}}{2}\right)}{\sin\left(\frac{\text{angle difference}}{2}\right)} . \sin\left(\frac{\text{1st angle} + \text{last angle}}{2}\right)$

$= \dfrac{\sin\frac{n \times h}{2}}{\sin\frac{h}{2}} \sin\left(\frac{a+h+a+nh}{2}\right)$

$\cos A - \cos B = 2\sin\frac{A+B}{2} \sin\frac{B-A}{2}$

Series represented by definite integrals:

The definition of a definite integral as the limit of a sum enables us to evaluate ~~est~~ easily the limits of the sums of certain series, when the number of terms tends to infinity by identifying them with some definite integrals.

By definition, we know

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(a+rh), \text{ when } nh = b-a.$$

$$= \lim_{n\to\infty} \frac{b-a}{n}\sum_{r=1}^{n} f\left(a + r\,\frac{b-a}{n}\right)$$

In special case, when $a=0$, $b=1$, then $h=\frac{1}{n}$ & we get

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

[As if we write x for $\frac{r}{n}$ & dx for $1/n$]

or, putting $h = 1/n \Rightarrow \int_0^1 f(x)\,dx = \lim_{h\to 0} h\sum_{r=1}^{n} f(rh)$.

[As if we write x for rh & dx for h].

Prob: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right]$

Soln: Given, $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right]$

$$= \lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{1}{n+r}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{1}{1+r/n}\right)$$

$$= \int_0^1 \frac{1}{1+x}\,dx = \ln(1+x)\Big]_0^1 = \ln 2. \quad \text{Ans}$$

Prob: Evaluate $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right]$

Soln: Given, $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{n}{n^2+r^2}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left[\frac{1}{1+(r/n)^2}\right]$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}x\Big]_0^1 = \frac{\pi}{4}. \quad \text{Ans}$$

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Prob: Evaluate (i) $\lim_{n\to\infty}\left[\frac{n+2}{n^2+1}+\frac{n+4}{n^2+4}+\frac{n+6}{n^2+9}+\cdots+\frac{n+2n}{n^2+n^2}\right]$

(ii) $\lim_{n\to\infty}\left[\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\cdots+\frac{n^2}{n^3+n^3}\right]$

Soln: (i) Given $\lim_{n\to\infty}\left[\frac{n+2}{n^2+1}+\frac{n+4}{n^2+4}+\frac{n+6}{n^2+9}+\cdots+\frac{n+2n}{n^2+n^2}\right]$

$$= \lim_{n\to\infty}\left[\frac{n+2.1}{n^2+1^2}+\frac{n+2.2}{n^2+2^2}+\frac{n+2.3}{n^2+3^2}+\cdots+\frac{n+2.n}{n^2+n^2}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n+2r}{n^2+r^2}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n\left(1+2.\frac{r}{n}\right)}{n^2\left(1+\frac{r^2}{n^2}\right)}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1+2\,r/n}{1+r^2/n^2}$$

$$= \int_0^1 \frac{1+2x}{1+x^2}\,dx$$

$$= \int_0^1\left[\frac{1}{1+x^2}+\frac{2x}{1+x^2}\right]dx$$

$$= \left[\tan^{-1}x+\ln(1+x^2)\right]_0^1$$

$$= \tan^{-1}(1)+\ln 2-\tan^{-1}(0)-\ln 1$$

$$= \pi/4+\ln 2.$$ Ans

Solⁿ: (ii) Given $\lim_{n\to\infty}\left[\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\cdots+\frac{n^2}{n^3+n^3}\right]$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{n^3+r^3}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{n^3(1+r^3/n^3)}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{r^2/n^2}{1+r^3/n^3}$$

$$=\int_0^1\frac{x^2}{1+x^3}\,dx$$

$$=\frac{1}{3}\int_0^1\frac{3x^2}{1+x^3}\,dx$$

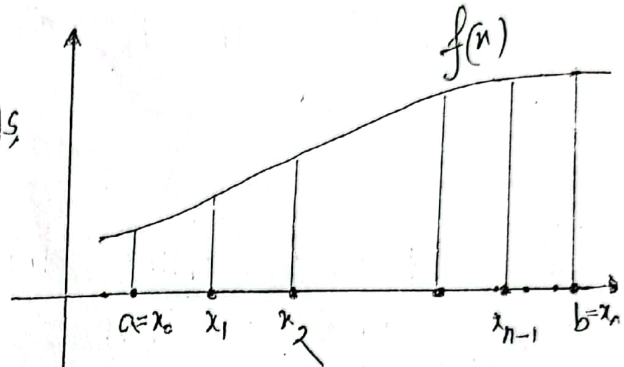
$$=\frac{1}{3}\left[\ln(1+x^3)\right]_0^1$$

$$=\frac{1}{3}\ln 2$$ Ans

Fundamental theorem of calculus:

If f is continuous on $[a,b]$ and if F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b)-F(a).$$



Proof: Divide the interval $[a,b]$ into n equal subintervals, so we get

$a=x_0<x_1<x_2<\cdots<x_n=b.$

Here the subintervals are $[x_0,x_1], [x_1,x_2], \dots, [x_{n-1},x_n]$, whose lengths, as usual, we denote by $\Delta x_1, \Delta x_2, \dots, \Delta x_n$.

Since F is an antiderivative of f then $F'(x)=f(x)$ for all x in $[a,b]$, so F satisfies the conditions of Mean Value theorem on each subinterval. Hence we can find the points $x_1^*, x_2^*, \dots, x_n^*$ in the respective subintervals such that

$$\frac{f(B)-f(A)}{B-A}=f'$$

$$F(x_1)-F(x_0)=F'(x_1^*)(x_1-x_0)=f(x_1^*)\Delta x_1$$

$$F(x_2)-F(x_1)=F'(x_2^*)(x_2-x_1)=f(x_2^*)\Delta x_2$$

$$\vdots$$

$$F(x_n)-F(x_{n-1})=F'(x_n^*)(x_n-x_{n-1})=f(x_n^*)\Delta x_n$$

Adding all these, we get

$$F(x_n)-F(x_0)=\sum_{r=1}^{n} f(x_r^*)\Delta x_r$$

$$\Rightarrow \boxed{F(b)-F(a)=\sum_{r=1}^{n} f(x_r^*)\Delta x_r} \quad \text{---}(1)$$

Now let us increase n in such a way that $\Delta x_r \to 0$. Then the right hand side of (1) approaches to $\int_a^b f(x)dx$.
Here the left hand side of (1) is independent of n, that is, the left hand side of (1) remains constant as n increase.

$$\boxed{F(b) - F(a) = \int_a^b f(x)\,dx}$$

Proved.

Note: Sometimes the difference $F(b)-F(a)$ can be written as $[F(x)]_a^b$, that is, $[F(b)]_a^b = F(b) - F(a)$.

Thus $\boxed{\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)}$

Ex: Evaluate $\int_1^2 x\,dx$

Soln: Here $F(x) = \frac{x^2}{2}$ is an antiderivative of $f(x) = x$.

$\therefore \int_1^2 x\,dx = [F(x)]_1^2 = F(2) - F(1) = \frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}$

Ans

Note: (I) The above thm establishes a connection between the integration as a particular kind of summation and the integration as an operation inverse to differentiation.

(II) From the above theorem it is clear that the definite integral is a function of its upper & lower limits and not of the independent variable x.

Some properties of definite integral:

(i) $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$.

Proof: We know $\int_a^b f(x)\,dx = F(b) - F(a)$

similarly, $\int_a^b f(z)\,dz = F(b) - F(a)$

$\therefore \boxed{\int_a^b f(x)\,dx = \int_a^b f(z)\,dz}$

(ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Proof: We know, $\int_a^b f(x)\,dx = F(b) - F(a)$

similarly, $-\int_b^a f(x)\,dx = -[F(a) - F(b)]$
$= F(b) - F(a)$

So $\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$

That is, an interchange of the limits changes the sign of the integral.

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(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a<c<b.$

Proof: we know, $\int_a^b f(x)\,dx = F(b)-F(a).$

Now $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = F(c)-F(a)+F(b)-F(c)$

$= F(b)-F(a)$

$\therefore \boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx}.$

(iv) $\int_0^a f(x)dx = \int_0^a f(a-x)dx.$

Proof: $R.H.S = \int_0^a f(a-x)\,dx$

Let $a-x=z$

$\therefore dx=-dz$

x	0	a
z	a	0

$= \int_a^0 f(z)(-dz)$

$= -\int_a^0 f(z)\,dz$

$= \int_0^a f(z)\,dz \quad [\because \int_a^b f(x)dx = -\int_b^a f(x)dx]$

$= \int_0^a f(x)\,dx \quad [\because \int_a^b f(x)dx = \int_a^b f(z)dz]$

$= L.H.S.$

$\therefore \boxed{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx}$

(v) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx, & \text{when } f(x) \text{ is even} \\ 0, & \text{when } f(x) \text{ is odd} \end{cases}$

Proof: We know,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \quad \text{--- (1)}$$

For 1st integral of R.H.S of (1), let

$$x = -z$$
$$\therefore dx = -dz$$

x	0	$-a$
z	0	a

$\therefore (1) \Rightarrow$

$$\int_{-a}^{a} f(x)\,dx = \int_{a}^{0} f(-z)(-dz) + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{a}^{0} f(-z)\,dz + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(-z)\,dz + \int_{0}^{a} f(x)\,dx \quad \left[\because \int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx\right]$$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \quad \left[\because \int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(z)\,dz\right]$$

$$= \int_{0}^{a} \{f(-x) + f(x)\}\,dx \quad \text{--- (2)}$$

When $f(x)$ is even, then $f(-x) = f(x)$ & (2) $\Rightarrow$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

When $f(x)$ is odd, then $f(-x) = -f(x)$ & (2) $\Rightarrow$

$$\int_{-a}^{a} f(x)\,dx = 0$$

Thus $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x), & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases}$

Prob: Evaluate $\int_{-2}^{2} x^9(1-x^2)^7 dx$.

Soln: Let $I = \int_{-2}^{2} x^9(1-x^2)^7 dx$

$= \int_{-2}^{2} f(x)\, dx$ ——— (1)

where $f(x) = x^9(1-x^2)^7$

Here $f(-x) = (-x)^9\{1-(-x)^2\}^7$

$= -x^9(1-x^2)^7$

$= -f(x)$

So $f(x)$ is odd.

$\therefore (1) \Rightarrow \quad I = \int_{-2}^{2} f(x)\, dx = 0$

Ans

Prob: Evaluate $\int_0^{\pi} \frac{x \sin x\, dx}{1+\cos^2 x}$

Soln: Let $I = \int_0^{\pi} \frac{x \sin x\, dx}{1+\cos^2 x}$ ——— (1)

$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx$

$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$ ——— (11)

Now $(1)+(11) \Rightarrow 2I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x} dx$

Let $\cos x = z$

$\therefore -\sin x\, dx = dz$

$\Rightarrow \sin x\, dx = -dz$

x	0	π
z	1	-1

$= \pi \int_1^{-1} \frac{-dz}{1+z^2}$

$= \pi \int_{-1}^{1} \frac{dz}{1+z^2}$

$\Rightarrow 2I = \pi \left[\tan^{-1} z\right]_{-1}^{1} = \pi\left[\tan^{-1}(1) - \tan^{-1}(-1)\right] = \pi\left[\frac{\pi}{4}+\frac{\pi}{4}\right]$

$\Rightarrow \boxed{I = \frac{\pi^2}{4}}$ Ans

Prob: Evaluate $\int_0^{\pi/2} \frac{dx}{1+\cot x}$

Soln: Let $I = \int_0^{\pi/2} \frac{dx}{1+\cot x}$

$$= \int_0^{\pi/2} \frac{dx}{1+\frac{\cos x}{\sin x}}$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin x \, dx}{\sin x + \cos x} \quad \text{——— (1)}$$

$$= \int_0^{\pi/2} \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \quad \text{——— (2)}$$

Now (1)+(2) $\Rightarrow 2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\cos x + \sin x} dx$

$$= \int_0^{\pi/2} 1 . dx$$

$\int 1 = x + C$

$$\Rightarrow 2I = [x]_0^{\pi/2} = \frac{\pi}{2} - 0$$

$$\Rightarrow \boxed{I = \pi/4} \text{ Ans}$$

Prob: Evaluate $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$