Newton's Divided Difference Polynomial Method

1 Introduction

Newton's Divided Difference Polynomial Method is an interpolation technique used to construct a polynomial that passes through a given set of data points. Unlike Lagrange interpolation, this method allows new data points to be added without recomputing the entire polynomial, making it computationally efficient.

2 Problem Statement

Given a set of $n+1$ distinct data points:

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$$

determine an interpolating polynomial $P_n(x)$ of degree at most n such that:

$$P_n(x_i) = y_i, \quad i = 0, 1, 2, \ldots, n$$

3 Newton's Divided Difference Polynomial

The Newton interpolating polynomial is given by:

$$P_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \cdots + b_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

where the coefficients $b_0, b_1, \ldots, b_n$ are called **divided differences**.

4 Divided Differences

4.1 Zeroth Divided Difference

$$b_0 = f[x_0] = y_0$$

4.2 First Divided Difference

$$b_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

4.3 Second Divided Difference

Consider Newton's interpolating polynomial of degree two:

$$P_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

Since the polynomial must interpolate the function at x_2, we impose the condition:

$$P_2(x_2) = f(x_2)$$

Substituting $x = x_2$:

$$f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

From previous steps, we know:

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Substituting these values:

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

Rearranging terms to isolate a_2:

$$f(x_2) - f(x_0) - \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) = b_2(x_2 - x_0)(x_2 - x_1)$$

Dividing both sides by $(x_2 - x_0)(x_2 - x_1)$:

$$b_2 = \frac{f(x_2) - f(x_0) - \frac{f(x_1)-f(x_0)}{x_1-x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

Rewriting in divided difference form:

$$b_2 = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2 - x_0}$$

Thus,

$$\boxed{b_2 = f[x_0, x_1, x_2]}$$

which is the **second divided difference**.

$$b_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

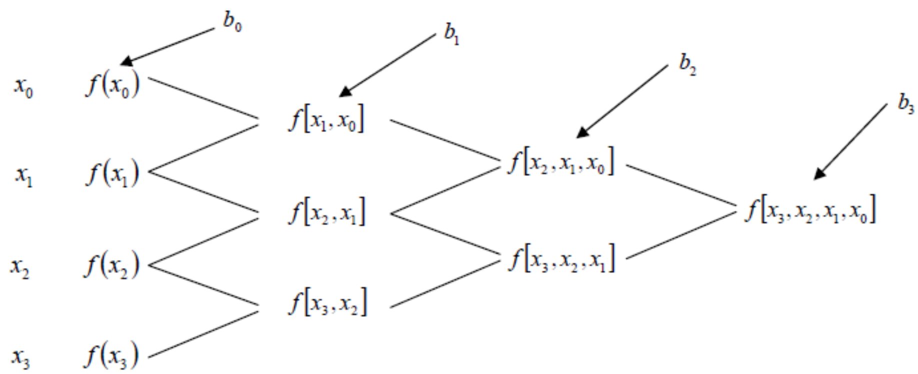

Figure 1: Table of divided differences for a cubic polynomial.

4.4 General Divided Difference

The m-th divided difference is defined recursively as:

$$f[x_0, x_1, \ldots, x_m] = \frac{f[x_1, x_2, \ldots, x_m] - f[x_0, x_1, \ldots, x_{m-1}]}{x_m - x_0}$$

5 Divided Difference Table

Divided differences are conveniently computed using a table:

6 Construction of the Polynomial

Once the divided differences are obtained, substitute them into the Newton polynomial:

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots$$

7 Advantages of Newton's Divided Difference Method

- Easy to add new data points without recomputing the entire polynomial
- Computationally efficient compared to Lagrange interpolation
- Suitable for unequal spacing of data points
- Polynomial can be built incrementally

8 Disadvantages

- More complex than Lagrange form conceptually
- Numerical instability may occur for high-degree polynomials

9 Error Term

The interpolation error is given by:

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n), \quad \xi \in [x_0, x_n]$$

This shows that the accuracy depends on the smoothness of $f(x)$ and the distribution of nodes.

10 Conclusion

Newton's Divided Difference Polynomial Method provides a flexible and efficient way to perform interpolation, especially when data points are added sequentially. Its recursive nature and incremental polynomial construction make it highly valuable in numerical analysis.

Table 3 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t = 16$ seconds with third order polynomial interpolation using Newton's divided difference polynomial method.
b) Using the third order polynomial interpolant for velocity, find the distance covered by the rocket from $t = 11\,\text{s}$ to $t = 16\,\text{s}$.
c) Using the third order polynomial interpolant for velocity, find the acceleration of the rocket at $t = 16\,\text{s}$.

Solution

a) For a third order polynomial, the velocity is given by

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$

Since we want to find the velocity at $t = 16$, and we are using a third order polynomial, we need to choose the four data points that are closest to $t = 16$ that also bracket $t = 16$ to evaluate it. The four data points are $t_0 = 10,\ t_1 = 15,\ t_2 = 20,$ and $t_3 = 22.5$.

Then

$$t_0 = 10, \quad v(t_0) = 227.04$$
$$t_1 = 15, \quad v(t_1) = 362.78$$
$$t_2 = 20, \quad v(t_2) = 517.35$$
$$t_3 = 22.5, \quad v(t_3) = 602.97$$

gives

$$\begin{aligned} b_0 &= v[t_0] \\ &= v(t_0) \\ &= 227.04 \end{aligned}$$

$$\begin{aligned} b_1 &= v[t_1, t_0] \\ &= \frac{v(t_1) - v(t_0)}{t_1 - t_0} \\ &= \frac{362.78 - 227.04}{15 - 10} \\ &= 27.148 \end{aligned}$$

$$\begin{aligned} b_2 &= v[t_2, t_1, t_0] \\ &= \frac{v[t_2, t_1] - v[t_1, t_0]}{t_2 - t_0} \end{aligned}$$

$$v[t_2, t_1] = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$
$$= \frac{517.35 - 362.78}{20 - 15}$$
$$= 30.914$$
$$v[t_1, t_0] = 27.148$$
$$b_2 = \frac{v[t_2, t_1] - v[t_1, t_0]}{t_2 - t_0}$$
$$= \frac{30.914 - 27.148}{20 - 10}$$
$$= 0.37660$$
$$b_3 = v[t_3, t_2, t_1, t_0]$$
$$= \frac{v[t_3, t_2, t_1] - v[t_2, t_1, t_0]}{t_3 - t_0}$$
$$v[t_3, t_2, t_1] = \frac{v[t_3, t_2] - v[t_2, t_1]}{t_3 - t_1}$$
$$v[t_3, t_2] = \frac{v(t_3) - v(t_2)}{t_3 - t_2}$$
$$= \frac{602.97 - 517.35}{22.5 - 20}$$
$$= 34.248$$
$$v[t_2, t_1] = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$
$$= \frac{517.35 - 362.78}{20 - 15}$$
$$= 30.914$$
$$v[t_3, t_2, t_1] = \frac{v[t_3, t_2] - v[t_2, t_1]}{t_3 - t_1}$$
$$= \frac{34.248 - 30.914}{22.5 - 15}$$
$$= 0.44453$$
$$v[t_2, t_1, t_0] = 0.37660$$
$$b_3 = \frac{v[t_3, t_2, t_1] - v[t_2, t_1, t_0]}{t_3 - t_0}$$
$$= \frac{0.44453 - 0.37660}{22.5 - 10}$$
$$= 5.4347 \times 10^{-3}$$

Hence

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$

$$= 227.04 + 27.148(t-10) + 0.37660(t-10)(t-15)$$
$$+ 5.5347 \times 10^{-3}(t-10)(t-15)(t-20)$$

At $t = 16$,

$$v(16) = 227.04 + 27.148(16-10) + 0.37660(16-10)(16-15)$$
$$+ 5.5347 \times 10^{-3}(16-10)(16-15)(16-20)$$
$$= 392.06 \text{ m/s}$$

b) The distance covered by the rocket between $t = 11\text{ s}$ and $t = 16\text{ s}$ can be calculated from the interpolating polynomial

$$v(t) = 227.04 + 27.148(t-10) + 0.37660(t-10)(t-15)$$
$$+ 5.5347 \times 10^{-3}(t-10)(t-15)(t-20)$$
$$= -4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3,\ 10 \le t \le 22.5$$

Note that the polynomial is valid between $t = 10$ and $t = 22.5$ and hence includes the limits of $t = 11$ and $t = 16$.

So

$$s(16) - s(11) = \int_{11}^{16} v(t)dt$$
$$= \int_{11}^{16} (-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3)dt$$
$$= \left[-4.2541t + 21.265\frac{t^2}{2} + 0.13204\frac{t^3}{3} + 0.0054347\frac{t^4}{4}\right]_{11}^{16}$$
$$= 1605 \text{ m}$$

c) The acceleration at $t = 16$ is given by

$$a(16) = \frac{d}{dt}v(t)\Big|_{t=16}$$
$$a(t) = \frac{d}{dt}v(t)$$
$$= \frac{d}{dt}\left(-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3\right)$$
$$= 21.265 + 0.26408t + 0.016304t^2$$
$$a(16) = 21.265 + 0.26408(16) + 0.016304(16)^2$$
$$= 29.664 \text{ m/s}^2$$

INTERPOLATION	
Topic	Newton's Divided Difference Interpolation
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