Numbers in Lambda Calculus

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Numbers

Encoding of numbers with pebbles

Arithmetic

Additions can be done by physically piling up the pebbles together.

Numbers

But pebbles cannot do other arithmetic operations very well

×

= ?

What is the physical process to compute multiplication?

÷

= ?

Numbers

Encoding of numbers with pebbles

Arithmetics

+

=

Numbers

Binary Encoding

0001 0010 0011 0100

*These are **string**.* So we can write them down on the tape of a Turing machine.

Arithmetics

$0010 \times 0011 = ?$

We can use a multiplication TM.
The result is the tape content after it halts.

Numbers: Lambda Calculus Encoding

Remember, everything is a function with a single input.

Numbers: Lambda Calculus Encoding

\f.\x. x

$x \longrightarrow x$

Nothing is applied to the input.

\f.\x. f x

$x \longrightarrow \boxed{f} \longrightarrow f(x)$

*Apply **f** once to the input **x**.*

\f.\x. f (f x)

$x \longrightarrow \boxed{f} \longrightarrow \boxed{f} \longrightarrow f(f(x))$

*Apply **f** twice to the input **x**.*

\f.\x. f (f (f x))

$x \longrightarrow \boxed{f} \longrightarrow \boxed{f} \longrightarrow \boxed{f} \longrightarrow f(f(f(x)))$

*Apply **f** three times to the input **x**.*

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Numbers: Church Encoding

\f.\x. x — 0

\f.\x. f x — 1

\f.\x. f (f x) — 2

\f.\x. f (f (f x)) — 3

...

We need to define addition in lambda calculus

\f.\x. f x	+	\f.\x. f (f x)

→ | \f.\x. f (f (f x)) |

such that its normal form is the encoding of the sum

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Functional Semantics of Church Numbers

3 is \f.\x. f (f (f x))

3 is a function with two inputs*. It applies the first argument to the second argument three times.

All numbers are functions with two arguments. The first argument is repeatedly applied to the second argument *n* times.

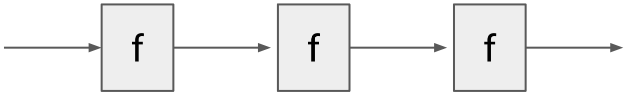
Arithmetics of Church Numbers

Successor function

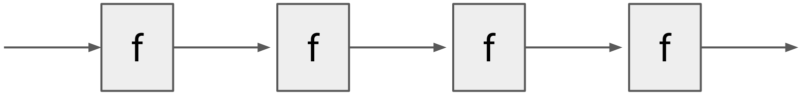
In mathematics:

Succ(n) = n + 1

In Lambda Calculus:



\f.\x. f (f (f x))



\f.\x. f (f (f (f x)))

Succ = \n. (n+1)
= \n. \f.\x. f (n f x)

Keep in mind that n and (n+1) are functions with two inputs.

Understanding the Successor function

```
Succ = \n.\f.\x. f (n f x)
```

*The first parameter is a Church number. So, **n** is a curried function with two inputs.*

*The body evaluates to a Church number, which means that it is a curried function with two parameters: **f** and **x**.*

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Understanding the Successor function

```
Succ = \n.\f.\x. f (n f x)
```

So **(f (n f x))** *applies*
f *to* **x** (n+1)-*times.*

Since **n** *is a Church number, it repeatedly applies* **f** *to* **x** *n-times.*

Lambda Calculus has no variables

```
Succ = \n.\f.\x. f (n f x)
```

Pure LC does not permit variables.
So, here ***Succ*** *is just a short-hand for*
the expression **\n.\f.\x. f (n f x)***.*

```
(Succ (\f.\x. (f x)))

((\n.\f.\x. f (n f x)) (\f.\x. (f x)))
```

How far can we push computation in LC?

We can now compute the **next** Church number from an input Church number using the Succ expression.

What about general programming?

Multiplication	Boolean values: True and False	If-else
Exponentiation	Propositional logic: AND, OR, NOT	For-loop
	Greater than, less than, Equal	While-loop