Master Theorem / Recurrence Equations / Asymptotic Growth

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1. Find the time complexity of following algorithm:

TERNARY-SEARCH(x, A, i, j)

// Assumption: $A[i] \leq x < A[j]$
if $j - i \leq 1$:
    return i
$p = \frac{2}{3}i + \frac{1}{3}j$
$q = \frac{1}{3}i + \frac{2}{3}j$
if $x < A[p]$:
    return TERNARY-SEARCH(x, A, i, p)
elseif $A[p] \leq x < A[q]$:
    return TERNARY-SEARCH(x, A, p, q)
elseif $x \geq A[q]$:
    return TERNARY-SEARCH(x, A, q, j)

2. If $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$, then $h(n) = \Theta(f(n))$. True/False
3. If $f(n) = O(g(n))$ and $g(n) = O(f(n))$ then $f(n) = g(n)$. True/False
4. Solving recurrences: Give solutions to the following.

$$T(n) = 8\ T(n/3) + n^2$$

$$T(n) = 10\ T(n/3) + n^2$$

$$T(n) = 2\ T(n/2) + n$$

5. What is the recurrence relation for the time to naively calculate the *n'th* factorial number? Assume multiplication between two numbers takes constant time. Then solve this recurrence.
6. Find an asymptotic solution of the following functional recurrence. Express your answer using Θ-notation, and give a brief justification.

$$T\ (n) = 16\ T\ (n/4) + n^2 \log^3 n$$

7. Find an asymptotic solution of the following functional recurrence. Express your answer using Θ-notation.

$$T\ (n) = 9\ T\ (n/3) + n^3$$

8. Running merge sort on an array of size *n* which is already correctly sorted takes $O(n)$ time. True/False

9. Use the Master Theorem to find the runtime of a recursive algorithm whose execution time is given by the formula:

$$T(n) = 2\,T(n - 1) + \log n.$$

Hint: The Master Theorem cannot be used on the given formula as it stands. Consider what would happen if you substitute one of $n = 2^m$, $n = \log m$, or $n = m^2$ for n. Identify which substitution allows you to apply the Master Theorem, and use it to find the runtime of T (n).

10. Solve the following recurrences, expressing your solution using asymptotic Θ notation:

$$T(n) = 9\,T(n/3) + \Theta(n \log n)$$

$$T(n) = T(n/2) + \Theta(\log n)$$

11. Is it always true that $f(n) + g(n) = \Theta(\mathit{min}\{f(n), g(n)\})$? If so, prove it. If not, find a counterexample and show that this statement is false.
12. Solve these recurrences:

$$T(n) = 4T(n/2) + \Theta(n^2)$$

$$T(n) = T(4n/5) + \Theta(n)$$

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$$T(n) = T(4n/5) + \Theta(n)$$

14. Find a solution to the recurrence $T(n) = T(n/3) + T(2n/3) + \Theta(n)$

15. Order the functions based on asymptotic growth.

$$f_1(n) = 8\sqrt{n}, \quad f_2(n) = 25^{1000}, \quad f_3(n) = (\sqrt{3})^{\lg n}$$

$$f_1(n) = \frac{1}{100}, \quad f_2(n) = \frac{1}{n}, \quad f_3(n) = \frac{\lg n}{n}$$

$$f_1(n) = 2^{\lg^3 n}, \quad f_2(n) = n^{\lg n}, \quad f_3(n) = \lg n!$$

Reference: From “6.006: Introduction to Algorithms”, MIT.