Ontario Tech Science

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Lecture 11: Shortest Path

CSCI 3070U: Design and Analysis of Algorithms

Learning Outcomes

- Single source shortest path problem:
  - Dijkstra
  - Bellman-Ford

Shortest-path Problem

- Problem: given a weighted directed graph G, find the minimum-weight path from a given source vertex s to another vertex v
  - Weight of path $p = \langle v_0, v_1, \ldots, v_k \rangle$ is:

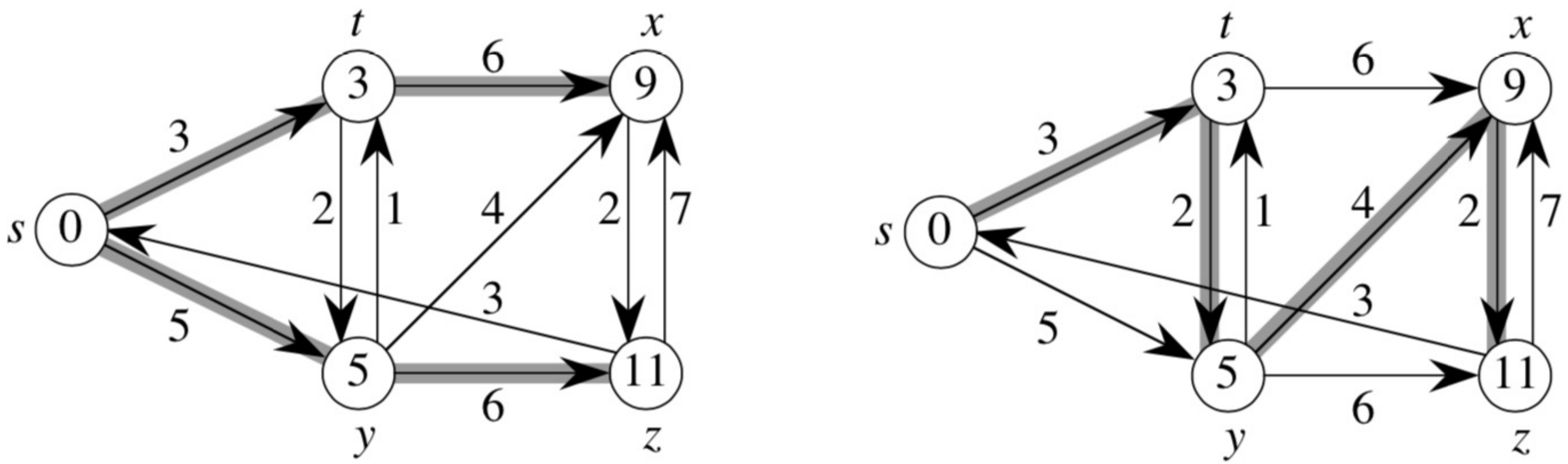
$$\sum_{i=1}^{k} w(v_{i-1}, v_i)$$

  - Shortest-path weight u to v:

$$\delta(u,v) = \begin{cases} \min\left\{ w(p) : u \overset{p}{\leadsto} v \right\} & \text{if there exists a path } u \leadsto v\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

Shortest-path Problem

- Problem: given a weighted directed graph G, find the minimum-weight path from a given source vertex s to another vertex v

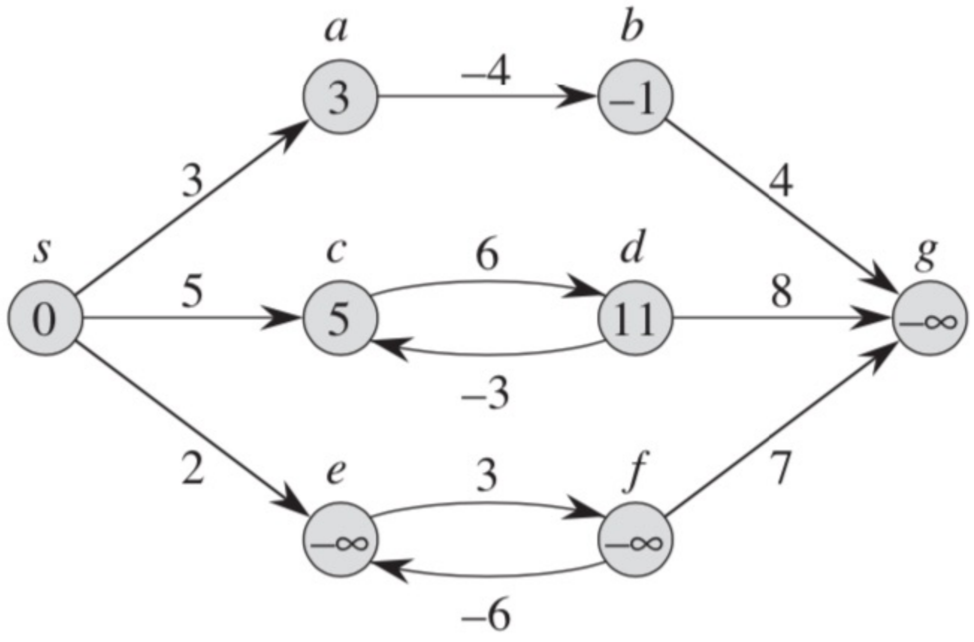


Shortest-path Problem

- Variants:
  - Single-source:
    - Find shortest paths from a given source vertex $s \in \mathrm{V}$ to every vertex $v \in \mathrm{V}$
  - Single-destination:
    - Find shortest paths to a given destination vertex.
  - Single-pair:
    - Find shortest path from u to v.
  - All-pairs:
    - Find shortest path from u to v for all $u, v \in \mathrm{V}$. We'll see algorithms

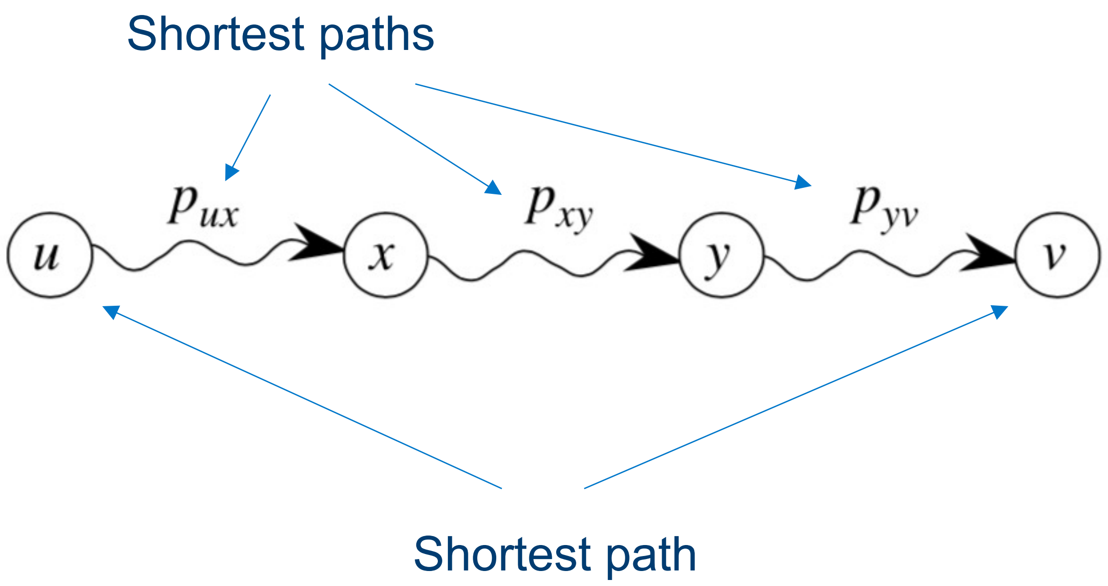
Negative-Weight Edges

- OK, as long as no negative-weight cycles are reachable from the source.
  - If we have a negative-weight cycle, we can just keep going around it, and get $w(s, v) = -\infty$ for all v on the cycle



Optimal Substructure

- Shortest path problem has optimal substructure
  - Any subpath of a shortest path is a shortest path.



Dijkstra’s algorithm

- No negative-weight edges !
- Have two sets of vertices:
  - S : vertices whose final shortest-path weights are determined
  - Q: priority queue = $V - S$

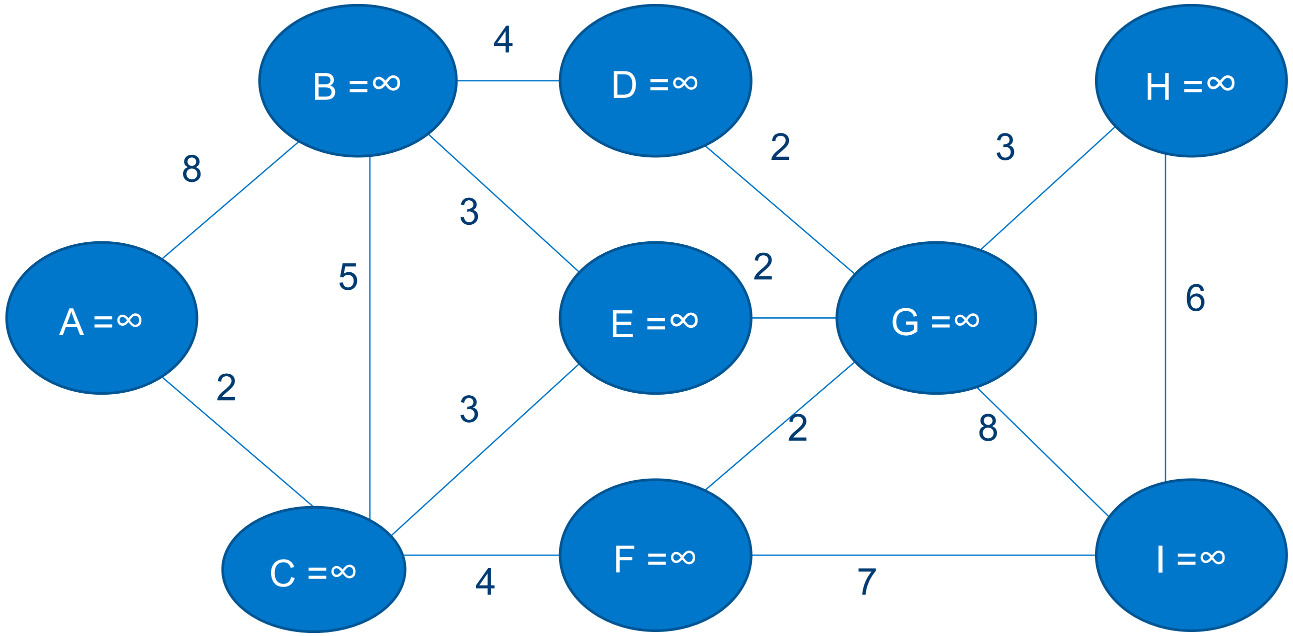
- Looks a lot like Prim’s algorithm, but computing $v.d$, and using shortest-path weights as keys.

- Dijkstra’s algorithm can be viewed as greedy, since it always chooses the closest vertex in $V - S$ to add to S

Dijkstra's algorithm

$\text{DIJKSTRA}(G, w, s)$

$\quad \text{INIT-SINGLE-SOURCE}(G, s)$
$\quad S = \emptyset$
$\quad$ **for** each vertex $u \in G.V$
$\qquad \text{INSERT}(Q, u)$
$\quad$ **while** $Q \neq \emptyset$
$\qquad u = \text{EXTRACT-MIN}(Q)$
$\qquad S = S \cup \{u\}$
$\qquad$ **for** each vertex $v \in G.Adj[u]$
$\qquad\quad$ **if** $v.d > u.d + w(u, v)$
$\qquad\qquad v.d = u.d + w(u, v)$
$\qquad\qquad v.\pi = u$
$\qquad\quad$ **if** $v.d$ changed
$\qquad\qquad \text{DECREASE-KEY}(Q, v, v.d)$



Q: A(∞), B(∞), C(∞), D(∞), E(∞), F(∞), G(∞), H(∞), I(∞)

Dijkstra's algorithm

INIT-SINGLE-SOURCE(G, s)

for each $v \in G.V$
    $v.d = \infty$
    $v.\pi = \text{NIL}$
$s.d = 0$

DIJKSTRA(G, w, s)

```
INIT-SINGLE-SOURCE(G, s)
S = Ø
for each vertex u ∈ G.V
    INSERT(Q, u)
while Q ≠ Ø
    u = EXTRACT-MIN(Q)
    S = S ∪ {u}
    for each vertex v ∈ G.Adj[u]
        if v.d > u.d + w(u, v)
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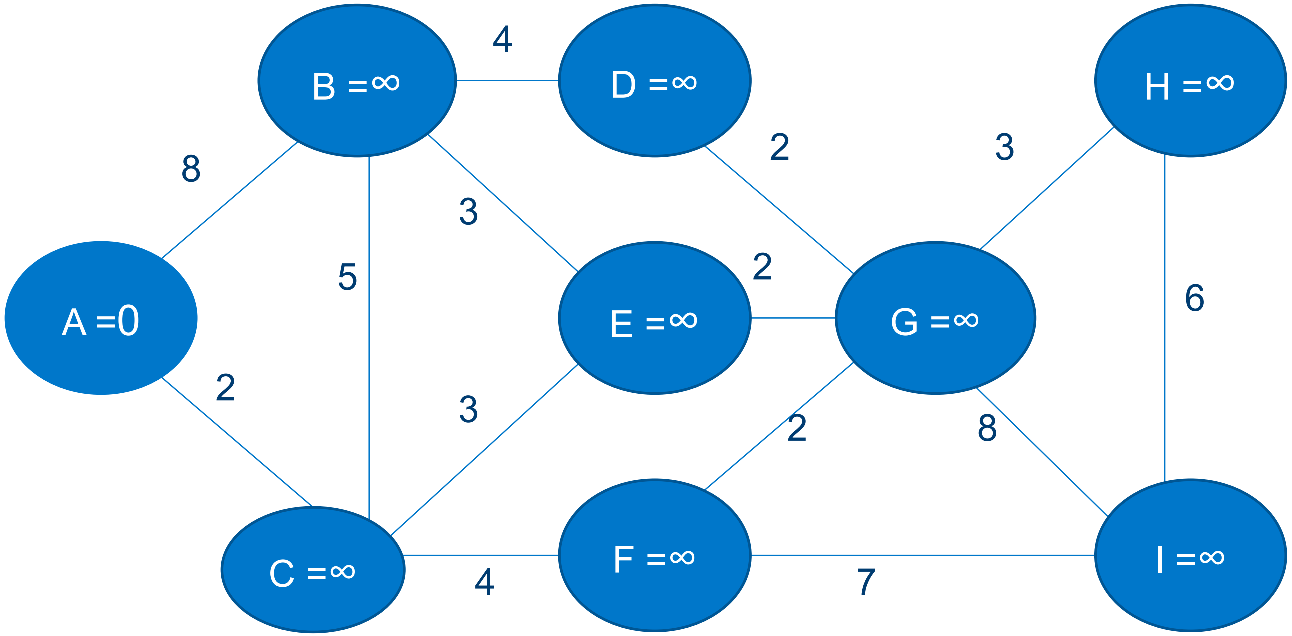

Q: A(0), B(∞), C(∞), D(∞), E(∞), F(∞), G(∞), H(∞), I(∞)

Dijkstra’s algorithm

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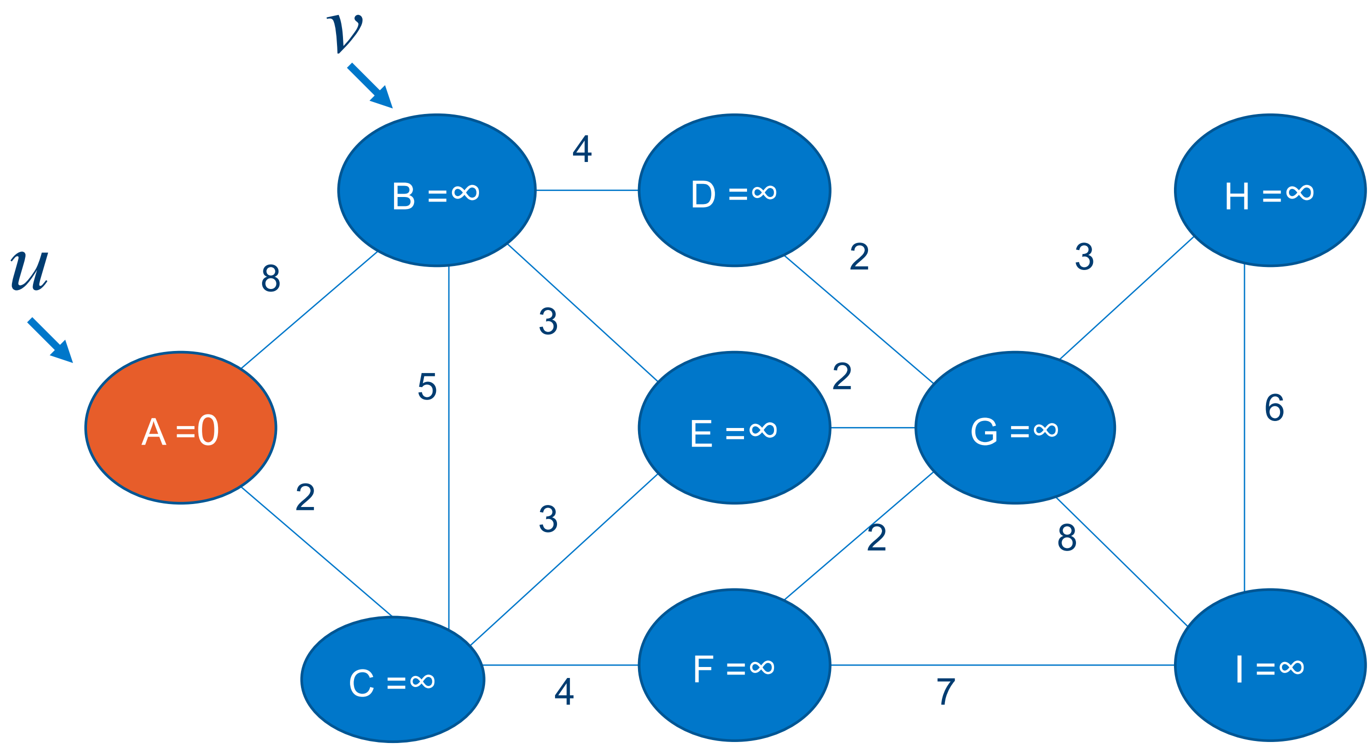

Q: ~~A(0)~~, B(8), C(∞), D(∞), E(∞), F(∞), G(∞), H(∞), I(∞)

After update

Dijkstra's algorithm

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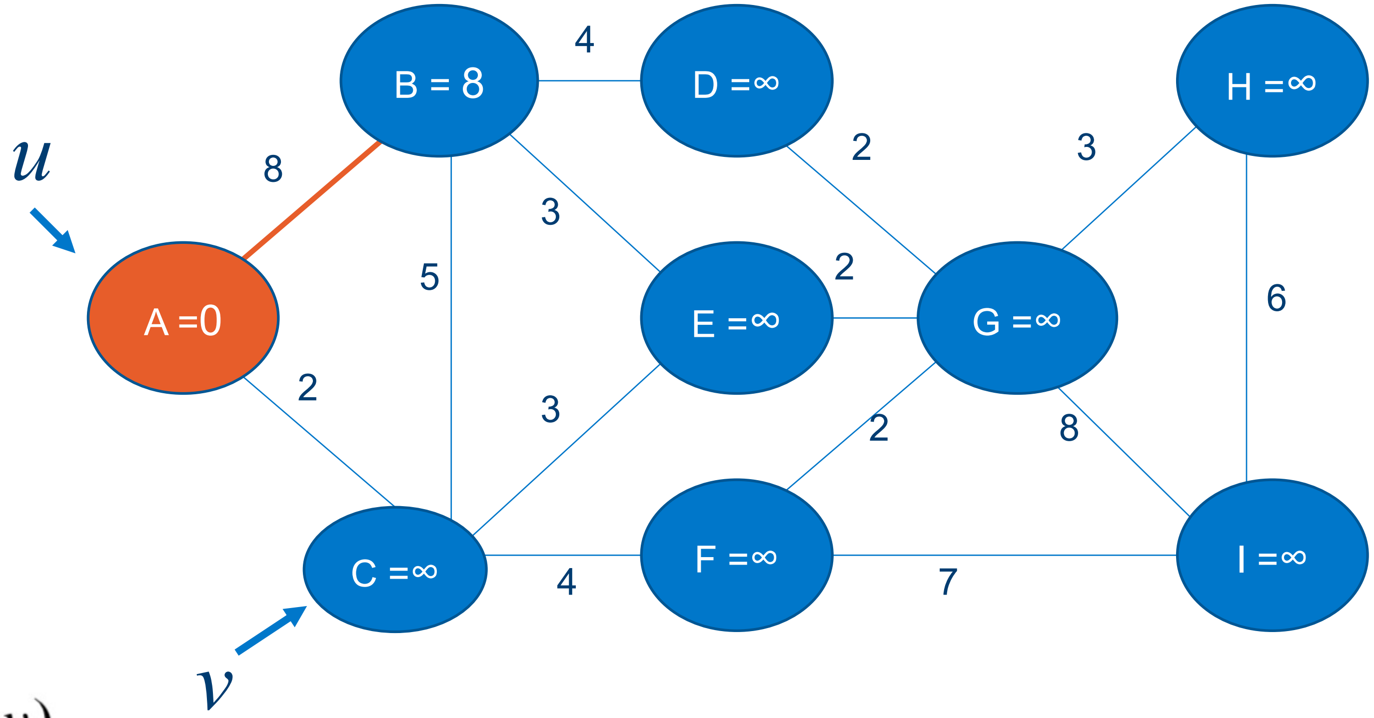

Q: ~~A(0)~~, B(8), C(2), D(∞), E(∞), F(∞), G(∞), H(∞), I(∞)

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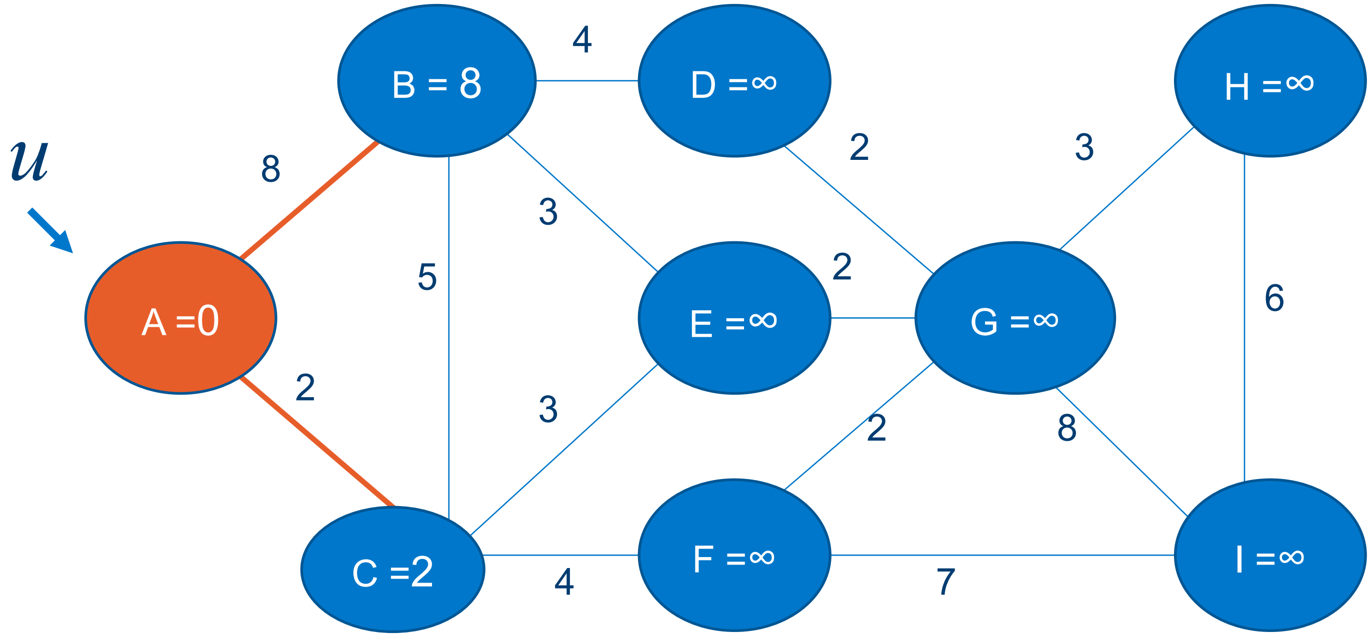

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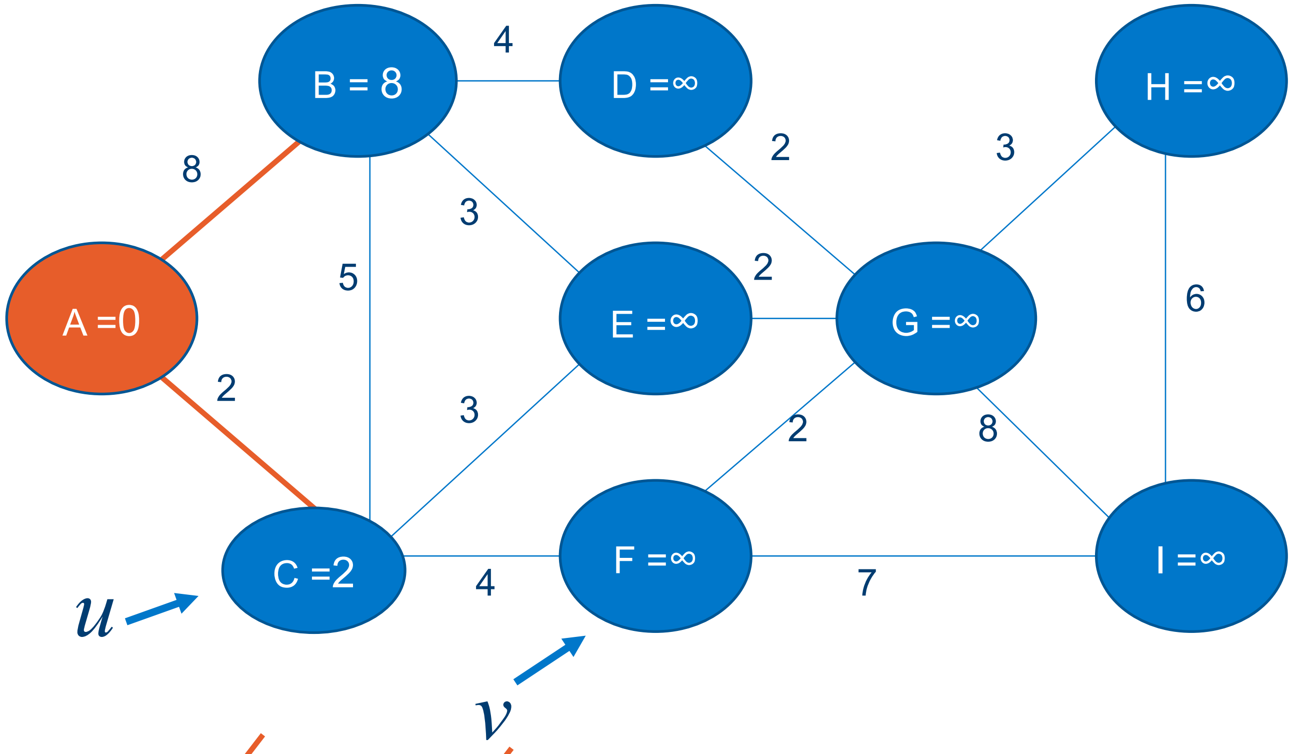

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Dijkstra’s algorithm

```
DIJKSTRA(G, w, s)
  INIT-SINGLE-SOURCE(G, s)
  S = ∅
  for each vertex u ∈ G.V
      INSERT(Q, u)
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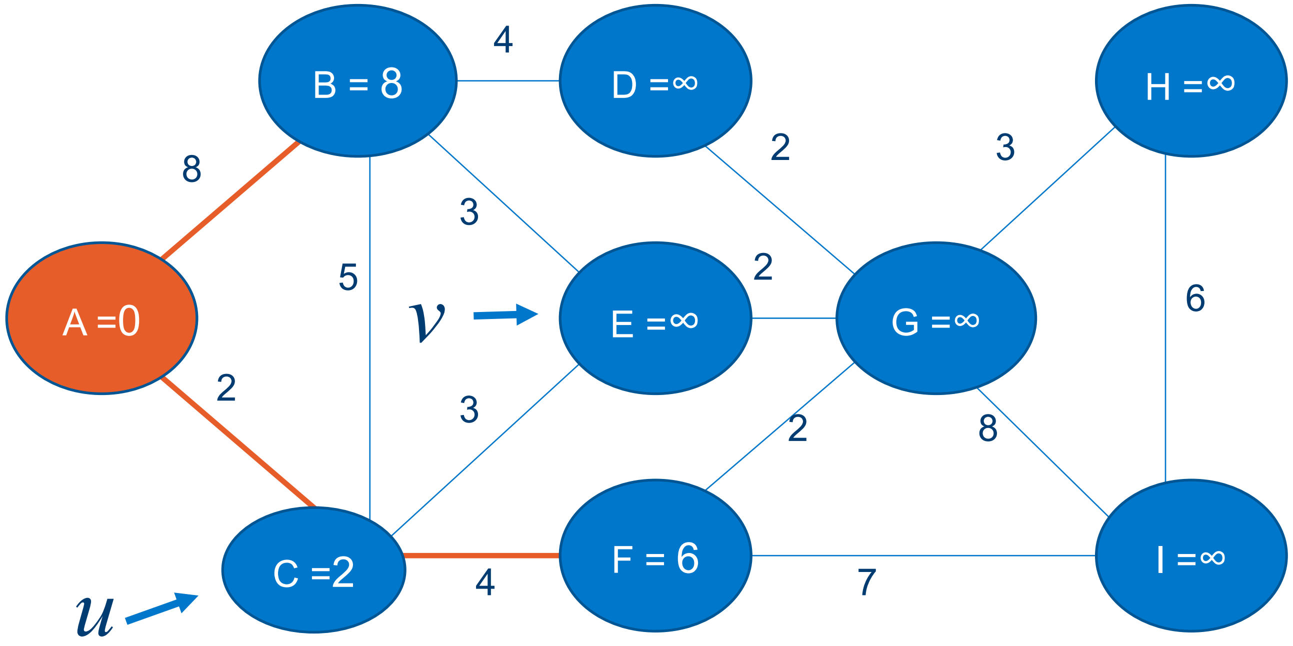

Q: ~~A(0)~~, B(8), ~~C(2)~~, D(∞), E(5), F(6), G(∞), H(∞), I(∞)

After update

Dijkstra’s algorithm

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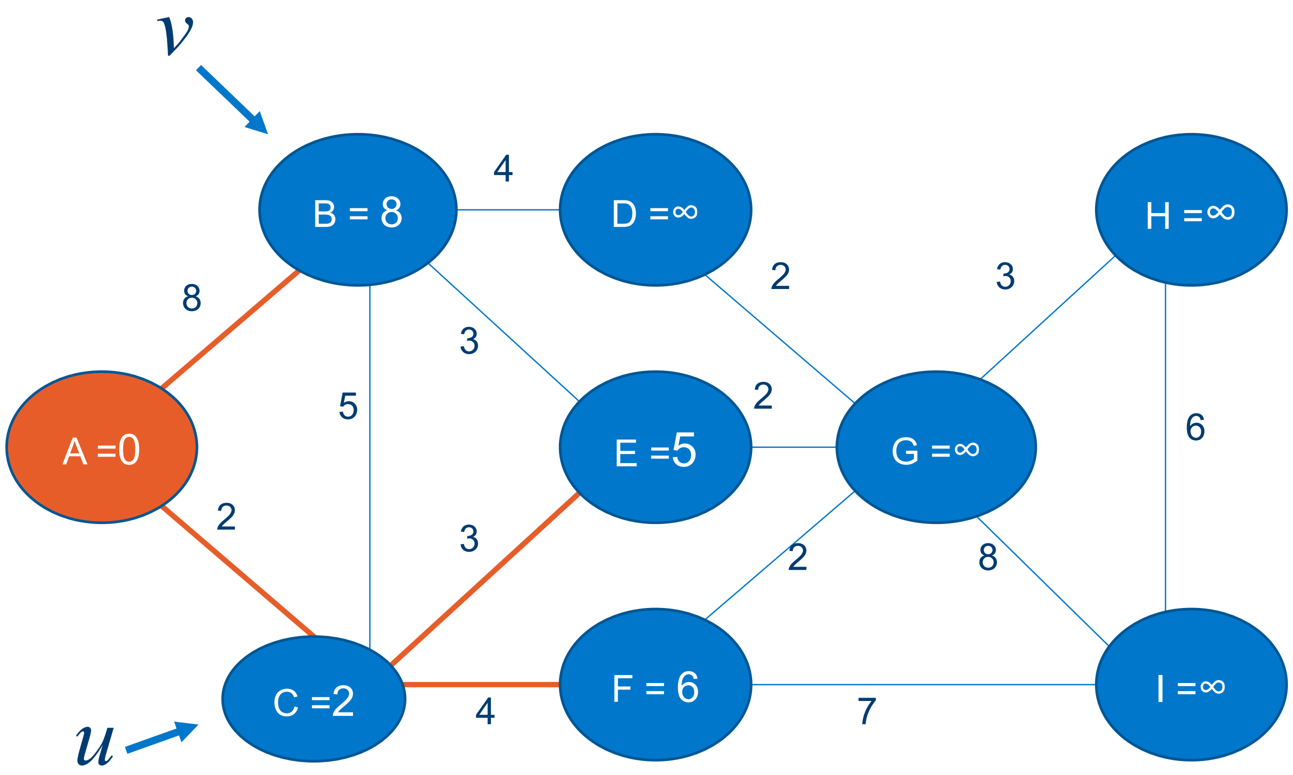

Q: ~~A(0~~), B(7), ~~C(2~~), D(∞), E(5), F(6), G(∞), H(∞), I(∞)

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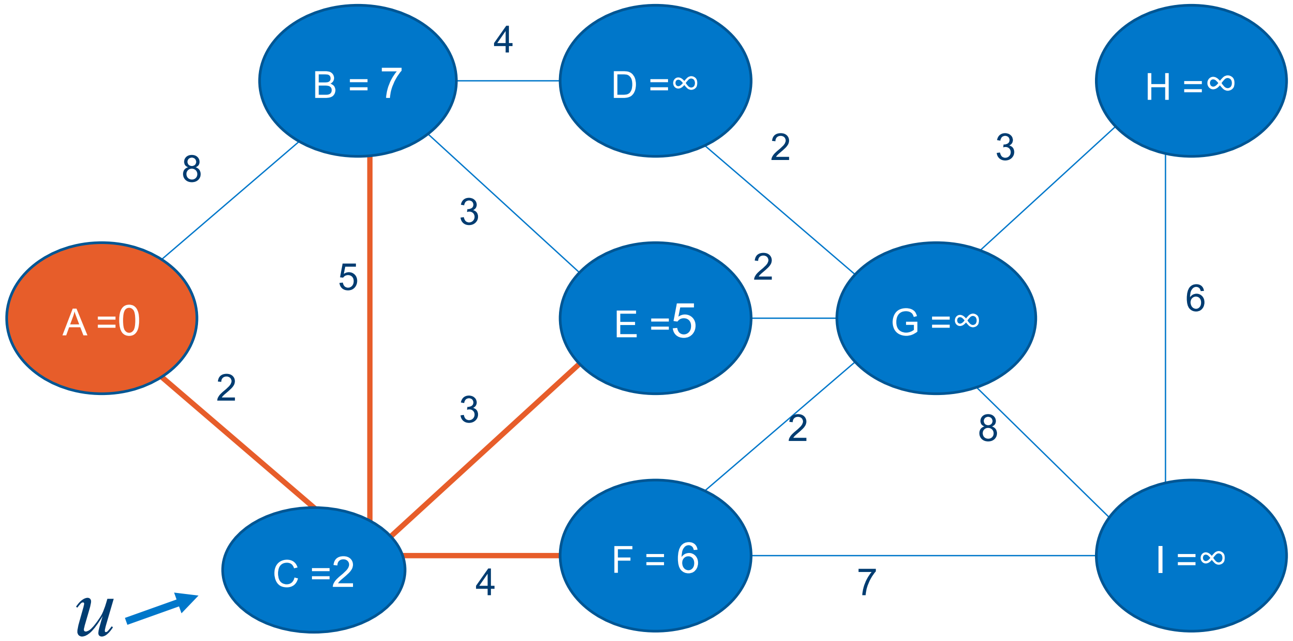

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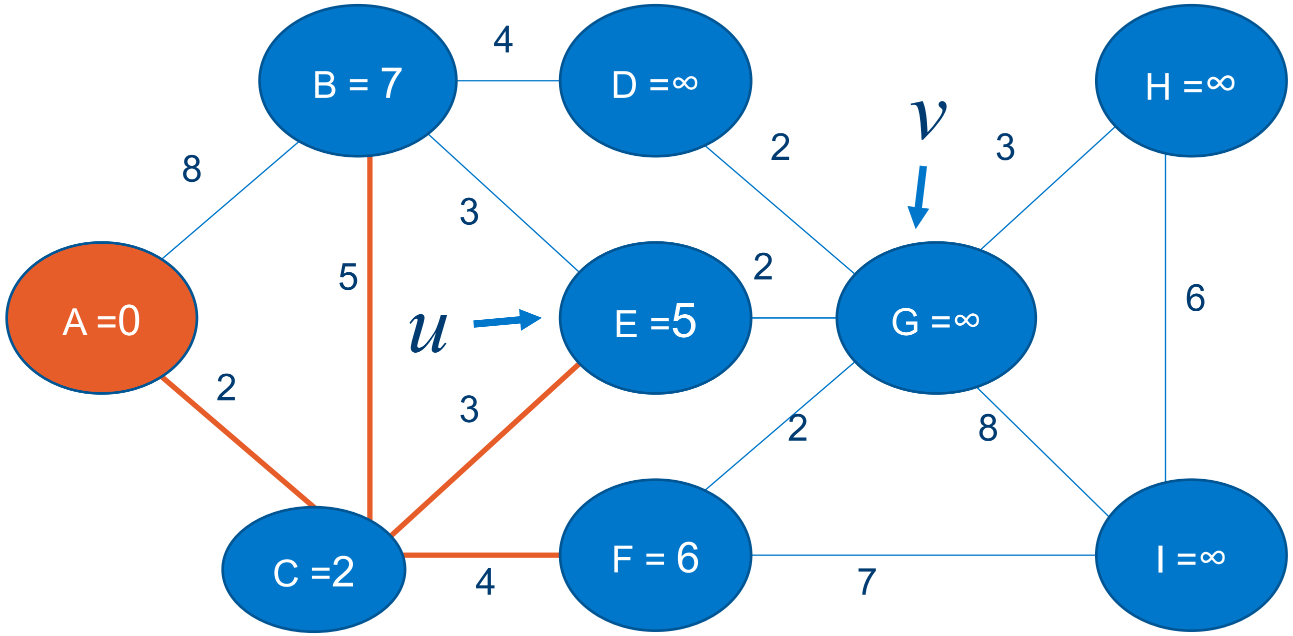

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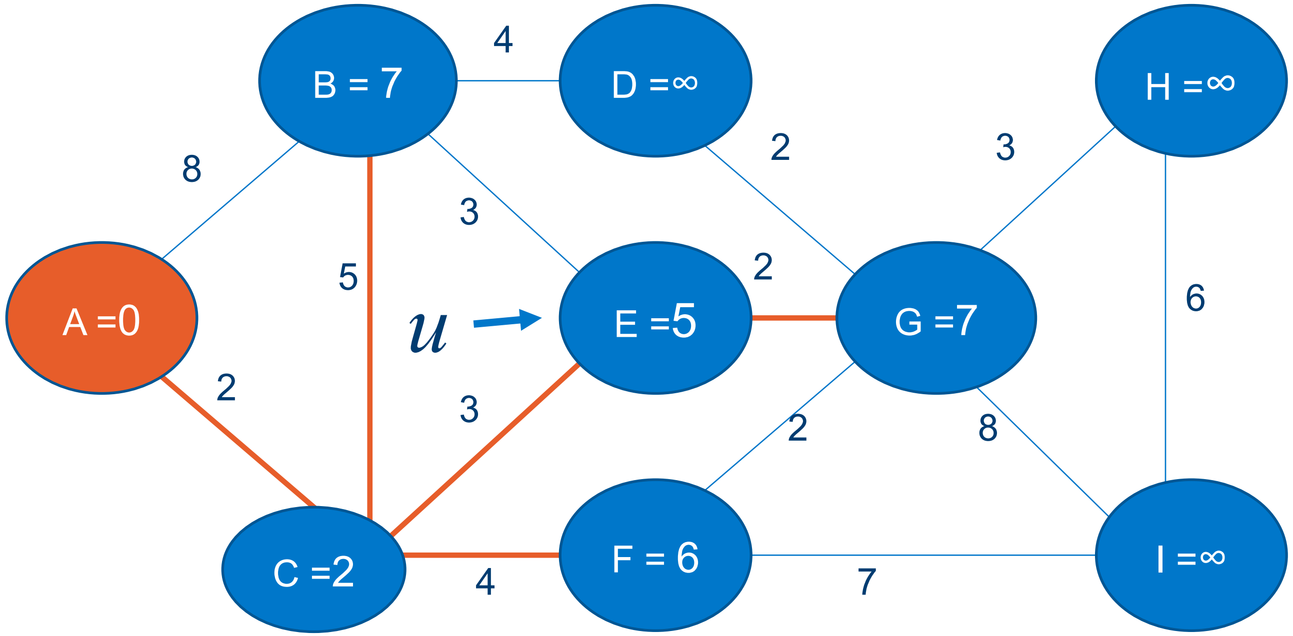

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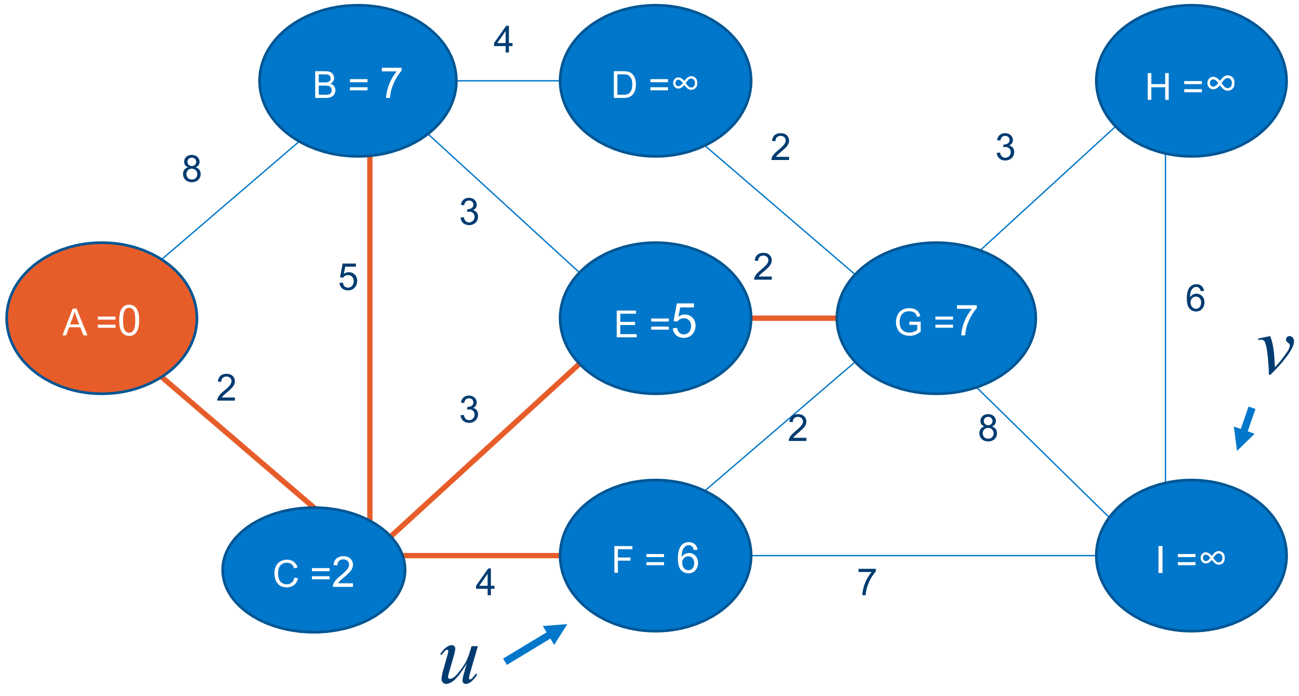

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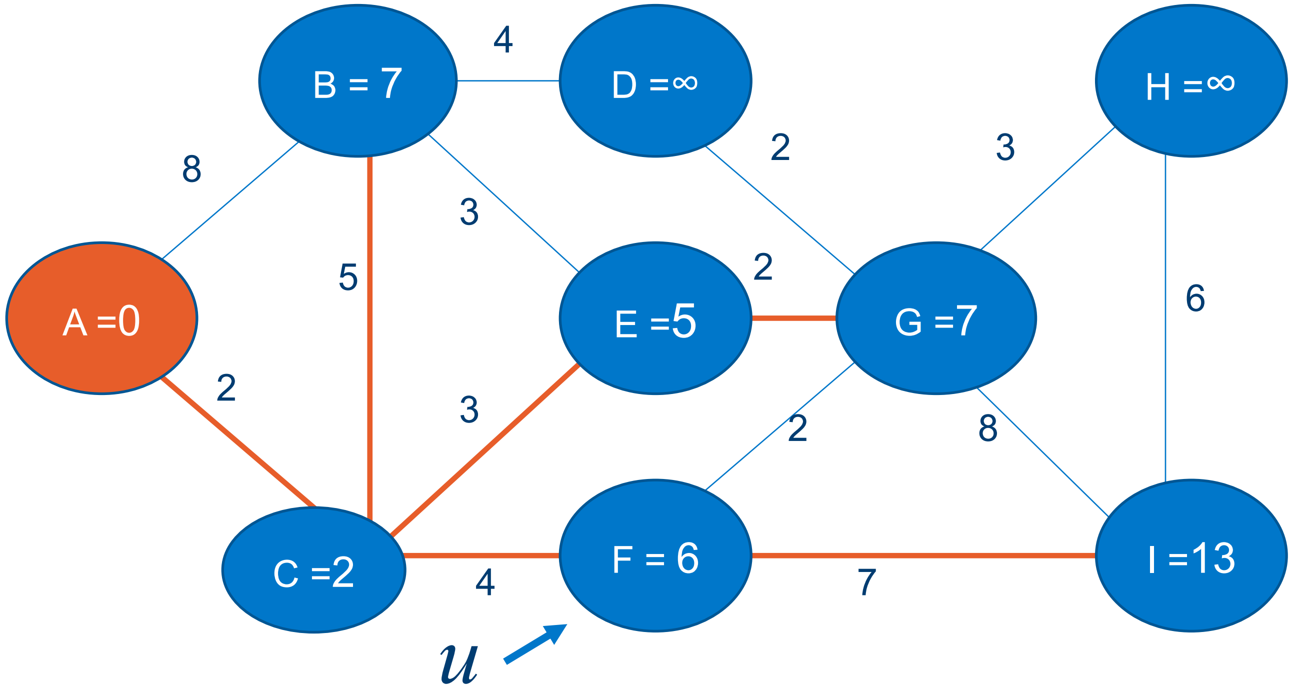

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Dijkstra's algorithm

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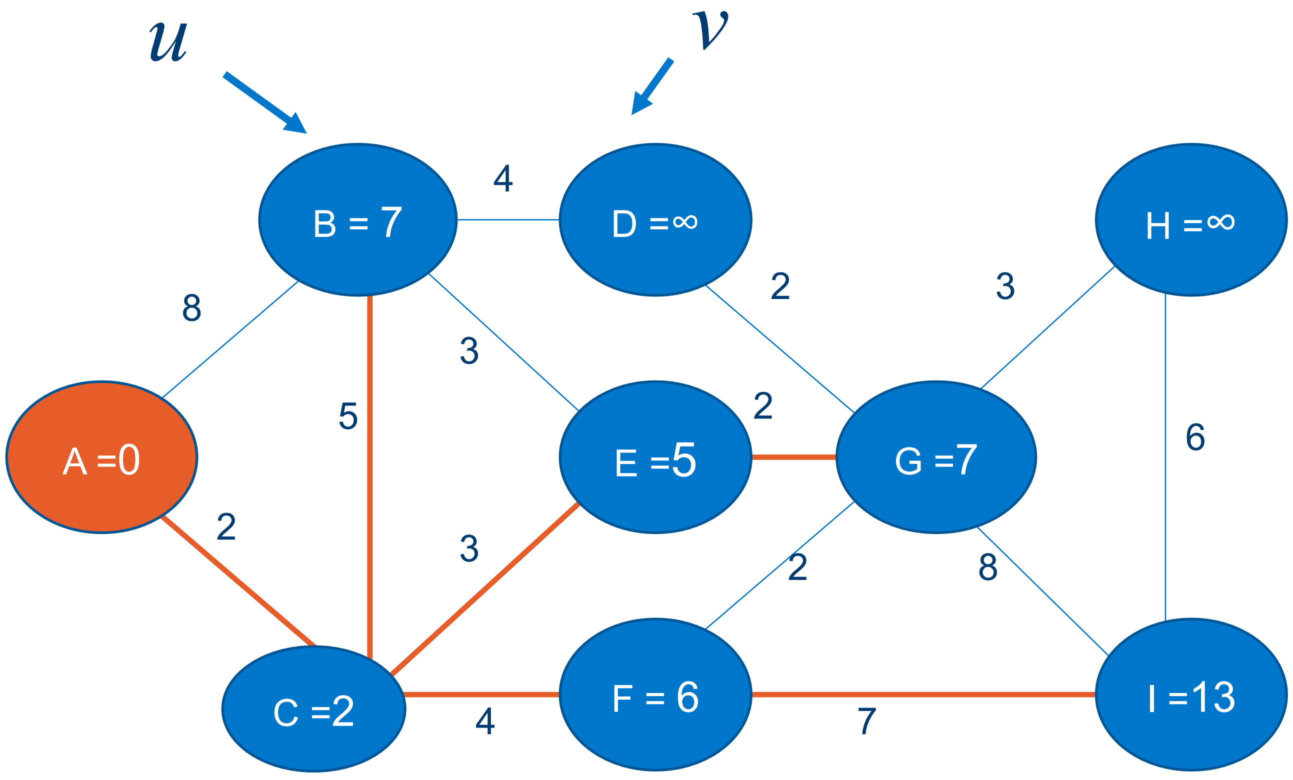
$$



Q: ~~A(0)~~, ~~B(7)~~, ~~C(2)~~, D(11), ~~E(5)~~, ~~F(6)~~, G(7), H(∞), I(13)

After update

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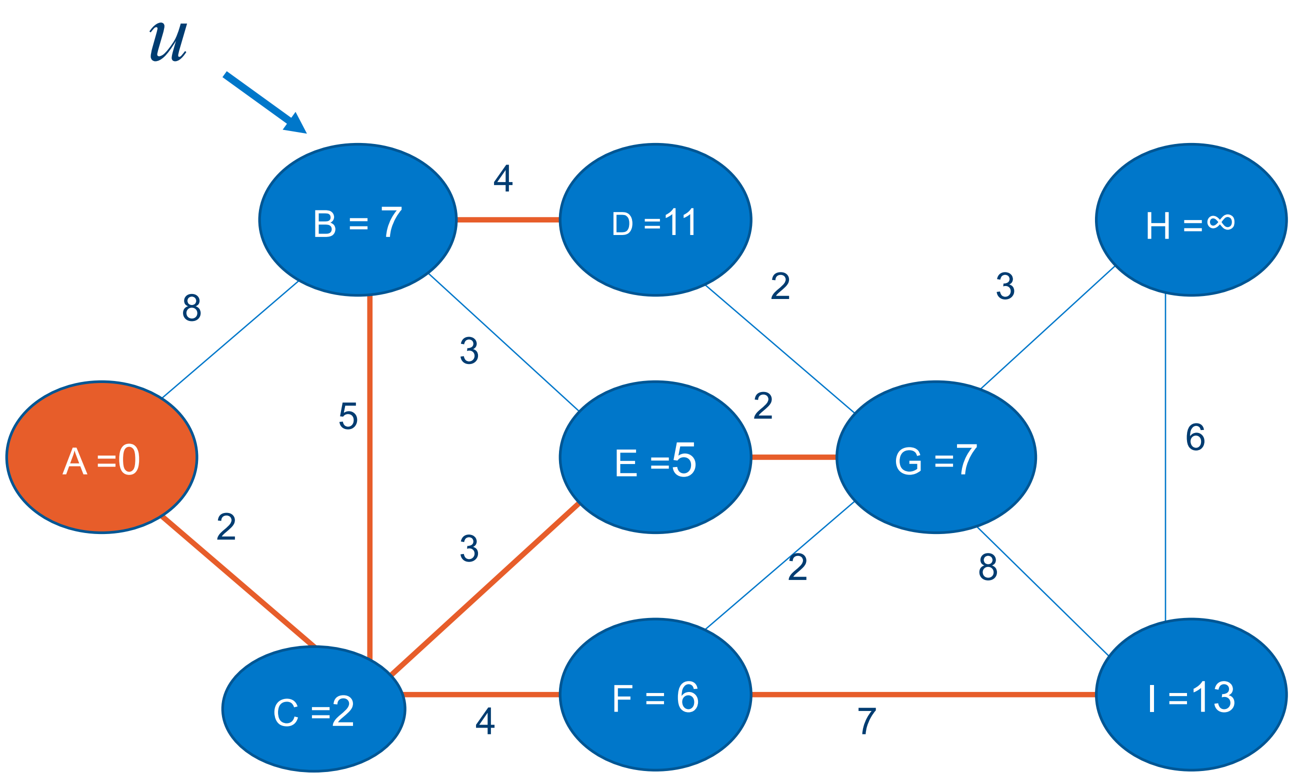

Q: ~~A(0)~~, ~~B(7)~~, ~~C(2)~~, D(11), ~~E(5)~~, ~~F(6)~~, G(7), H(∞), I(13)

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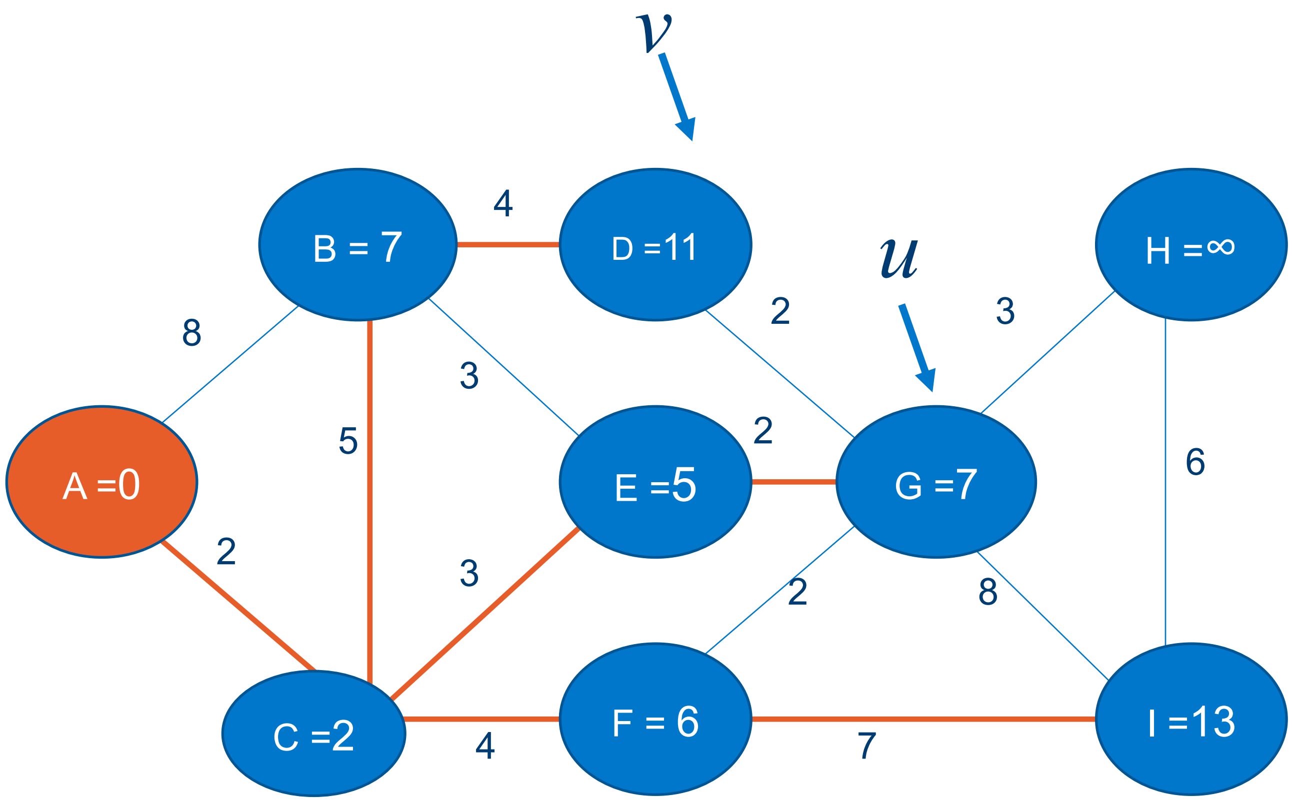

Q: ~~A(0~~), ~~B(7~~), ~~C(2~~), D(9), ~~E(5~~), ~~F(6~~), ~~G(7~~), H(∞), I(13)

After update

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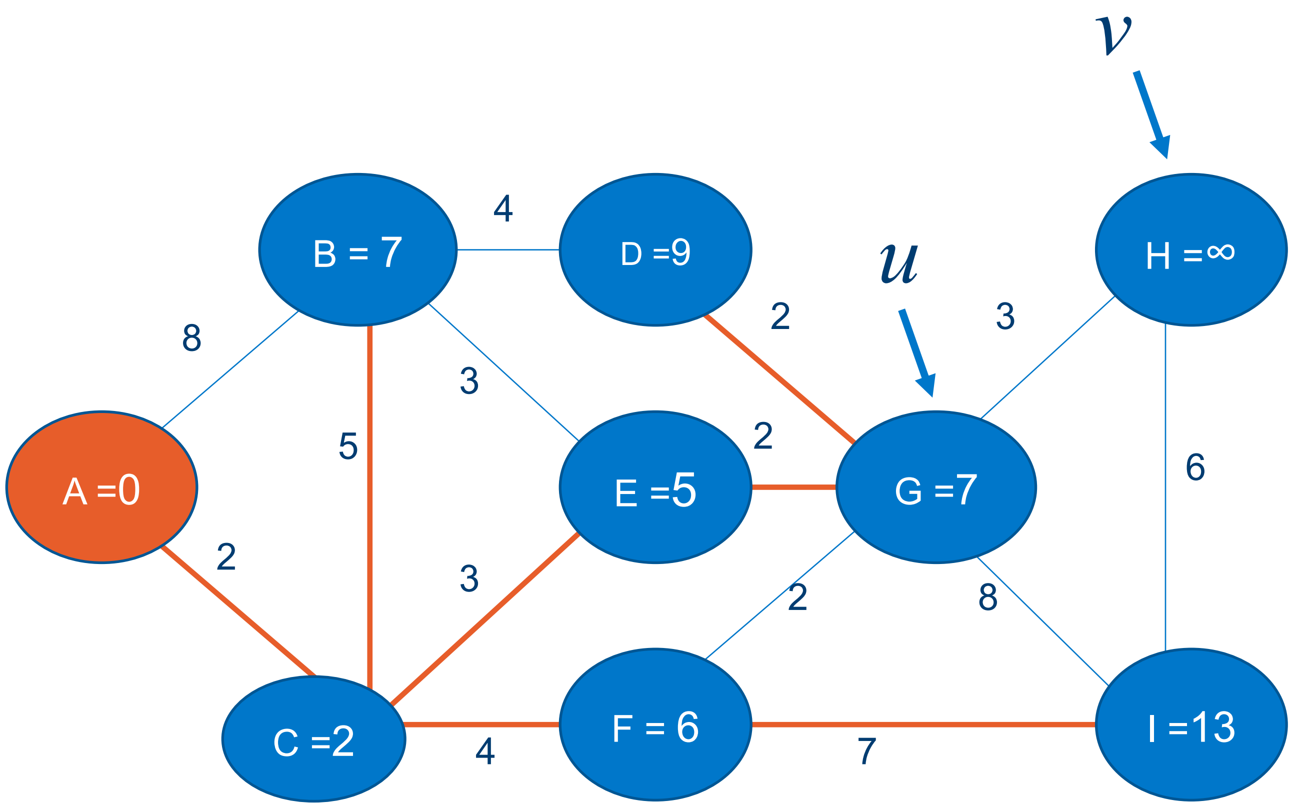

Q: ~~A(0)~~, ~~B(7)~~, ~~C(2)~~, D(9), ~~E(5)~~, ~~F(6)~~, ~~G(7)~~, H(10), I(13)

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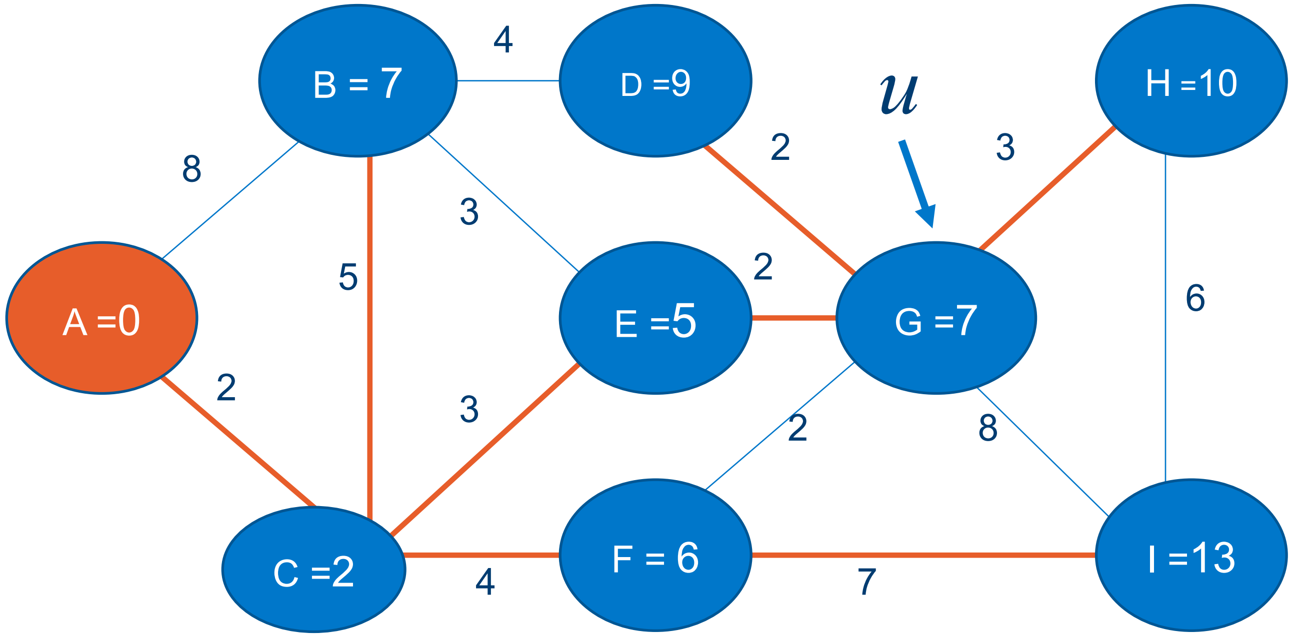

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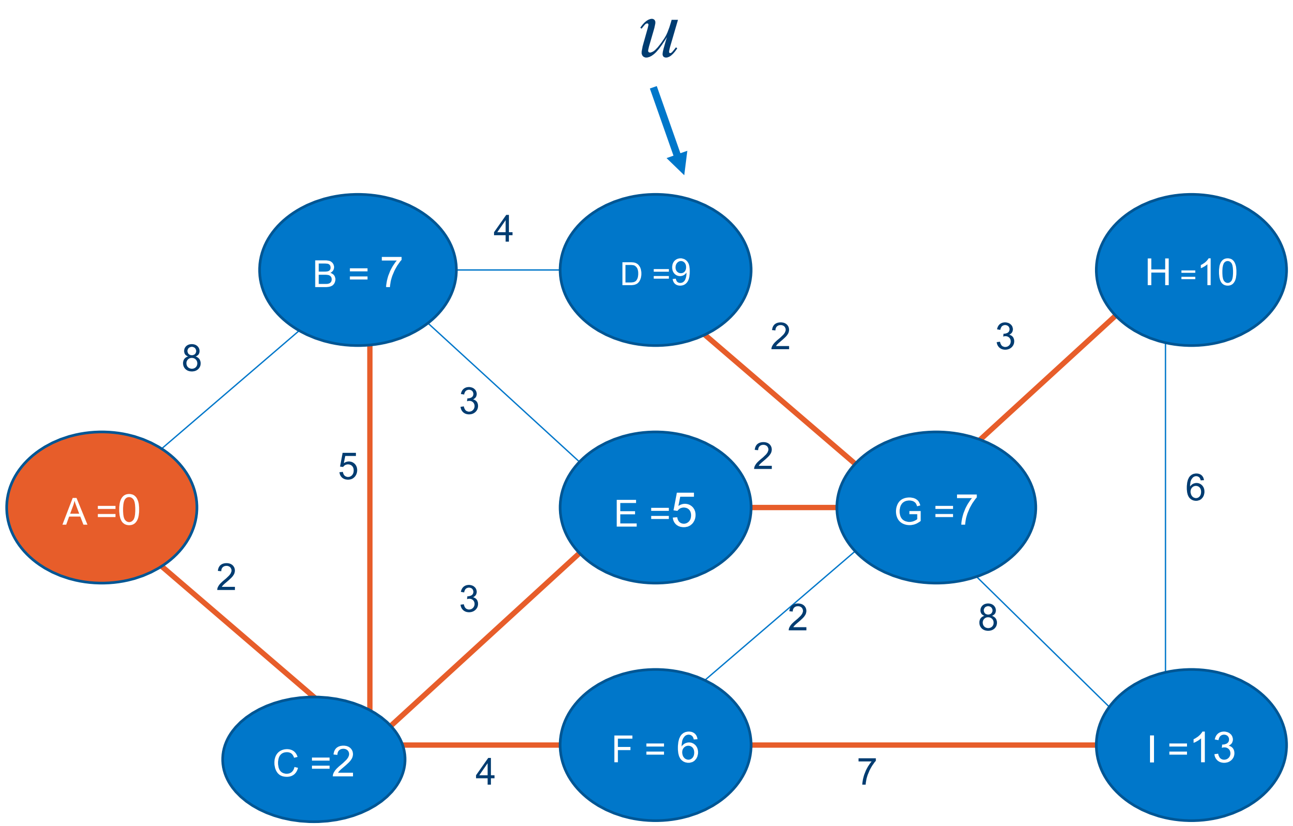

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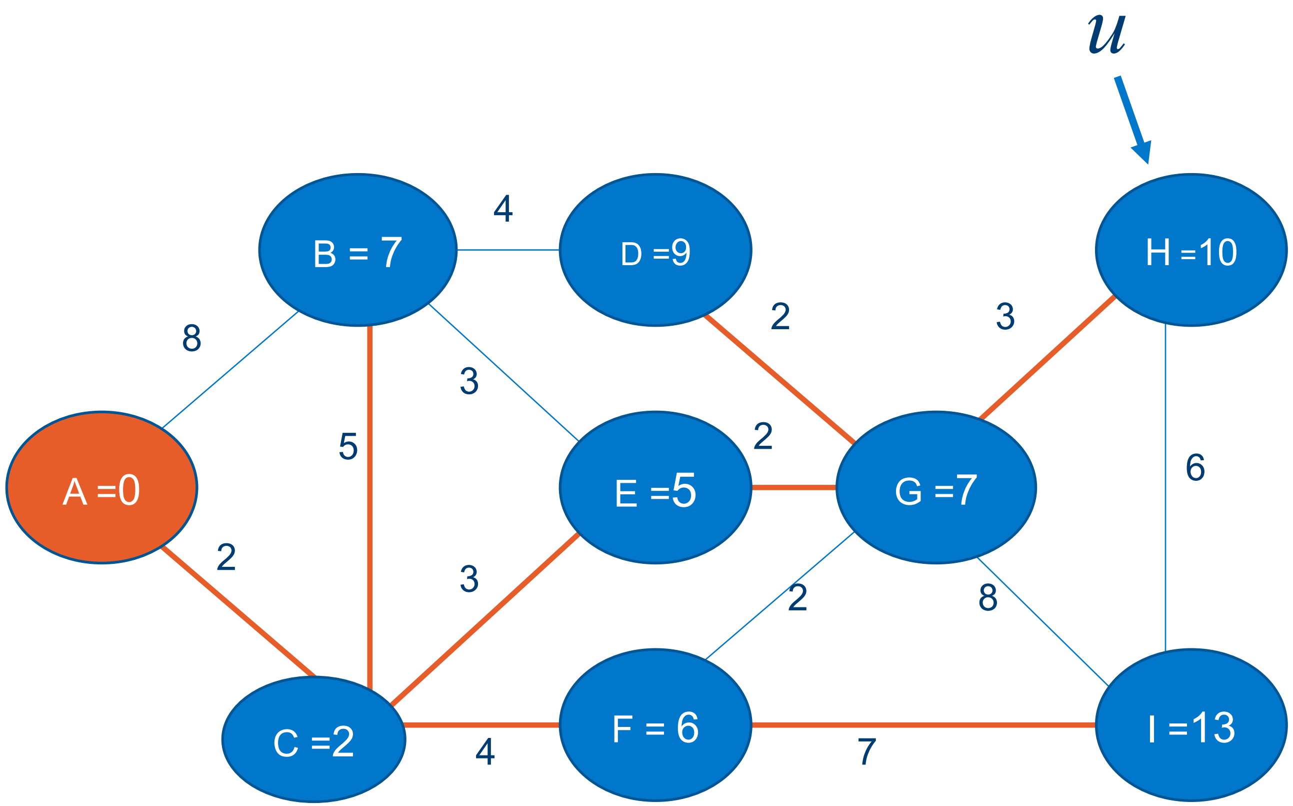
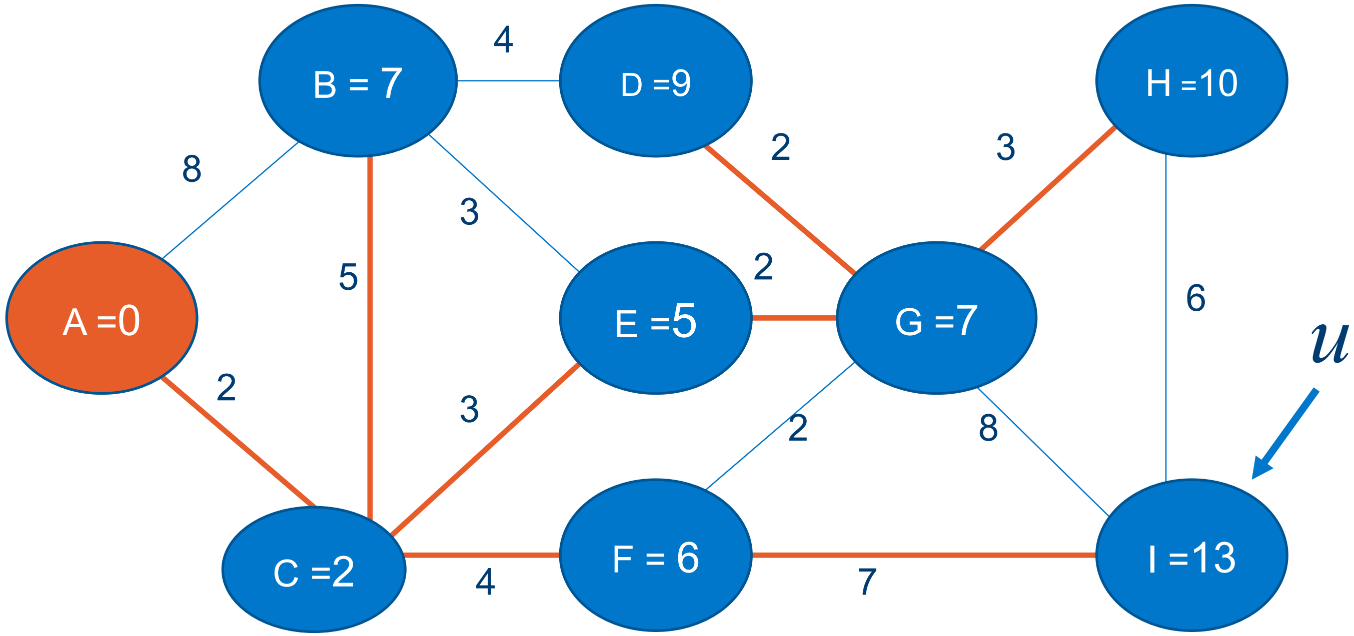


Q: ~~A(0)~~, ~~B(7)~~, ~~C(2)~~, ~~D(9)~~, ~~E(5)~~, ~~F(6)~~, ~~G(7)~~, ~~H(10)~~, I(13)

Dijkstra’s algorithm

DIJKSTRA(G, w, s)

  INIT-SINGLE-SOURCE(G, s)
  $S = \emptyset$
  for each vertex $u \in G.V$
      INSERT(Q, u)
  while $Q \neq \emptyset$
      $u = $ EXTRACT-MIN(Q)
      $S = S \cup \{u\}$
      for each vertex $v \in G.Adj[u]$
          if $v.d > u.d + w(u, v)$
              $v.d = u.d + w(u, v)$
              $v.\pi = u$
          if $v.d$ changed
              DECREASE-KEY$(Q, v, v.d)$



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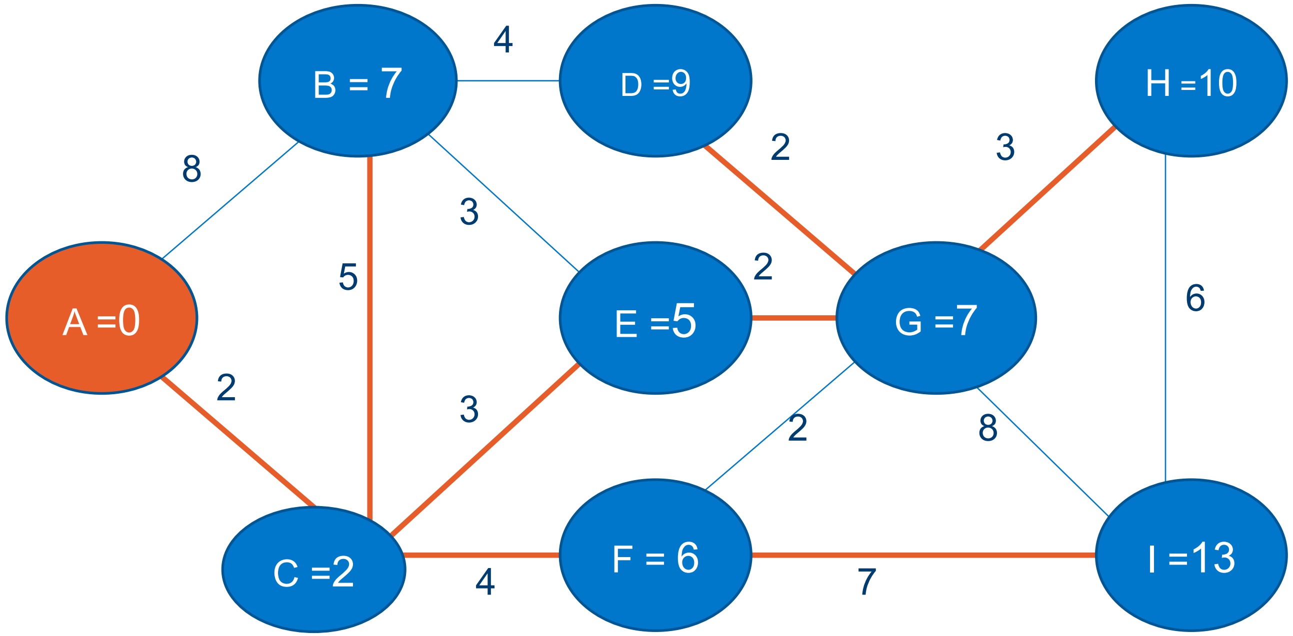
Dijkstra's algorithm

- You can draw a tree whose root is A



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$\Theta(V \log V)$

$\Theta(E \log V)$

Involves an implicit DECREASE-KEY operation on the min-heap, which a binary min-heap supports in O(log V)

Time Complexity

$\Theta(E \log V)$

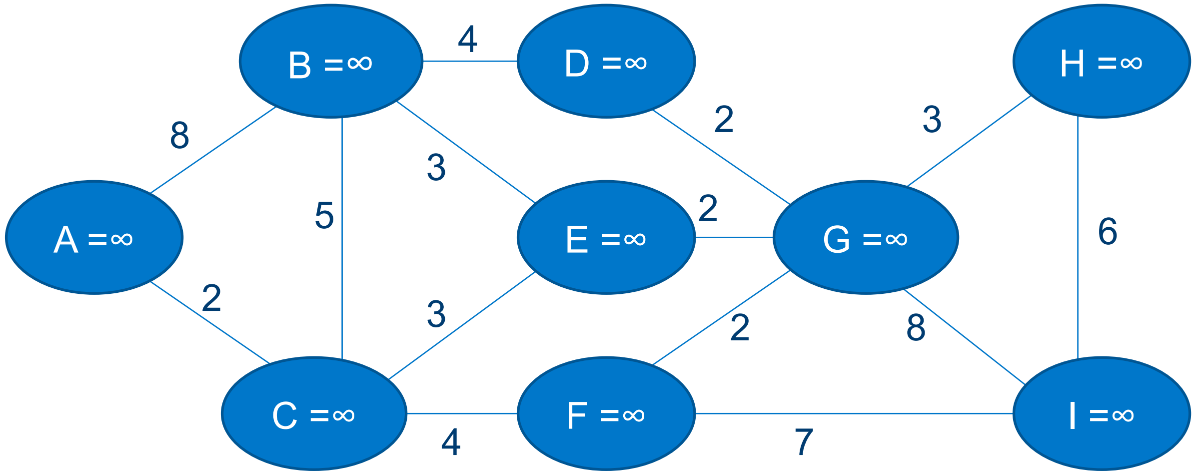
Bellman-Ford Algorithm

- Allows negative-weight edges
- Returns TRUE if no negative-weight cycles reachable from s, FALSE otherwise
- Main Idea:
  - Scan all edges |V|-1 times and update the shortest paths accordingly.
  - It should converge after |V|—1, if not there is negative-weight cycle(s)

Bellman-Ford Algorithm

BELLMAN-FORD(G, w, s)

for each $v \in G.V$
    $v.d = \infty$
    $v.\pi = \text{NIL}$
$s.d = 0$
for $i = 1$ to $|G.V| - 1$
    for each edge $(u, v) \in G.E$
        if $v.d > u.d + w(u, v)$
            $v.d = u.d + w(u, v)$
            $v.\pi = u$
for each edge $(u, v) \in G.E$
    if $v.d > u.d + w(u, v)$
        return FALSE
return TRUE



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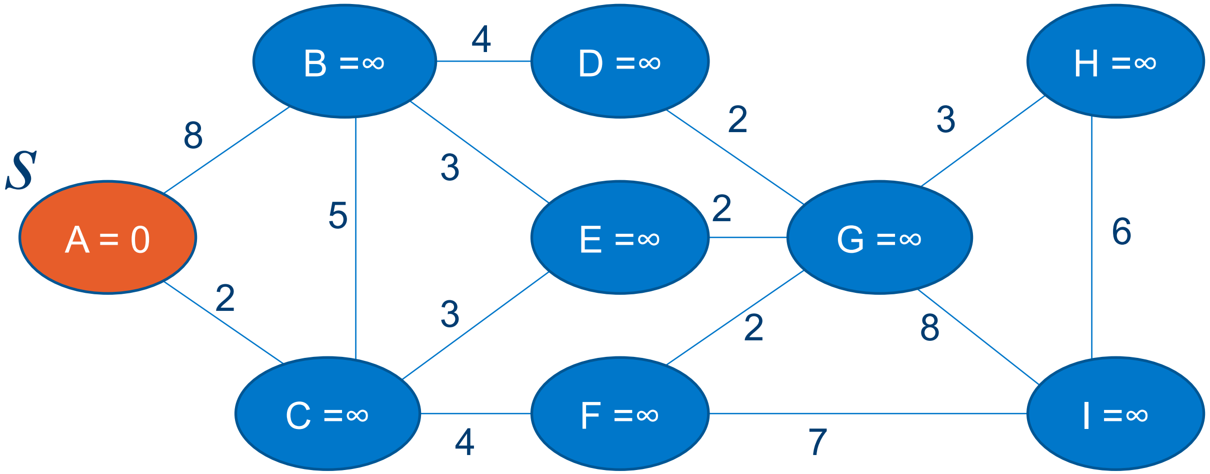

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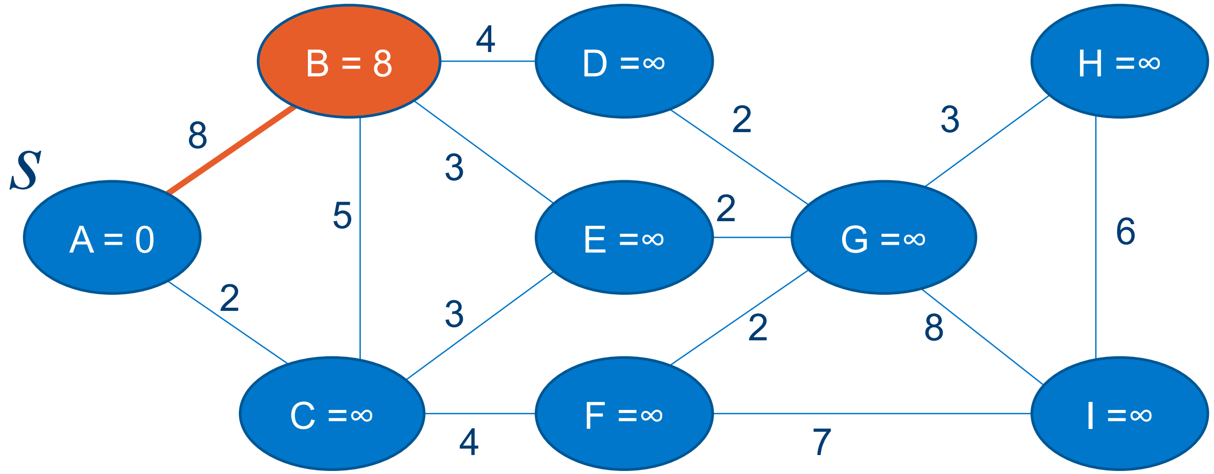

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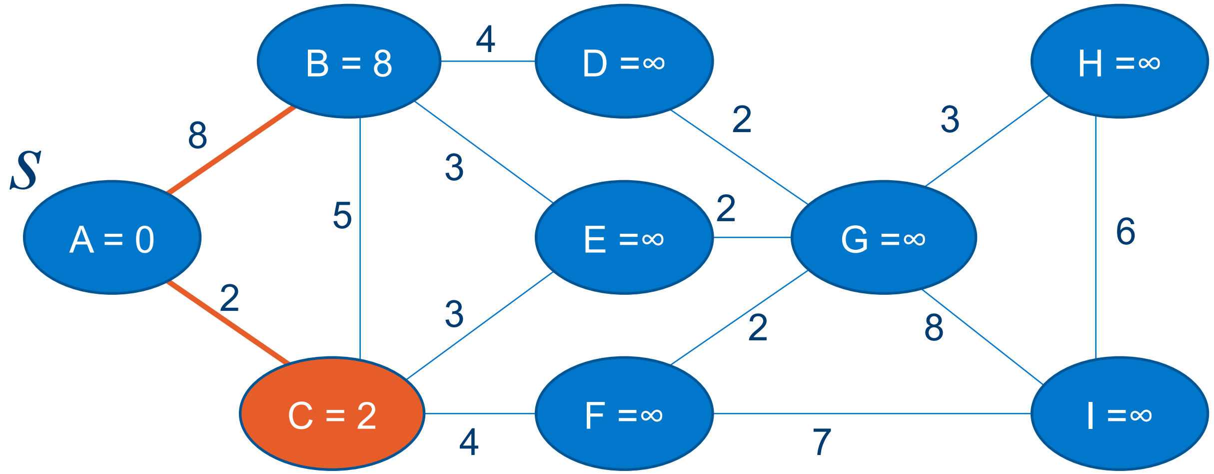

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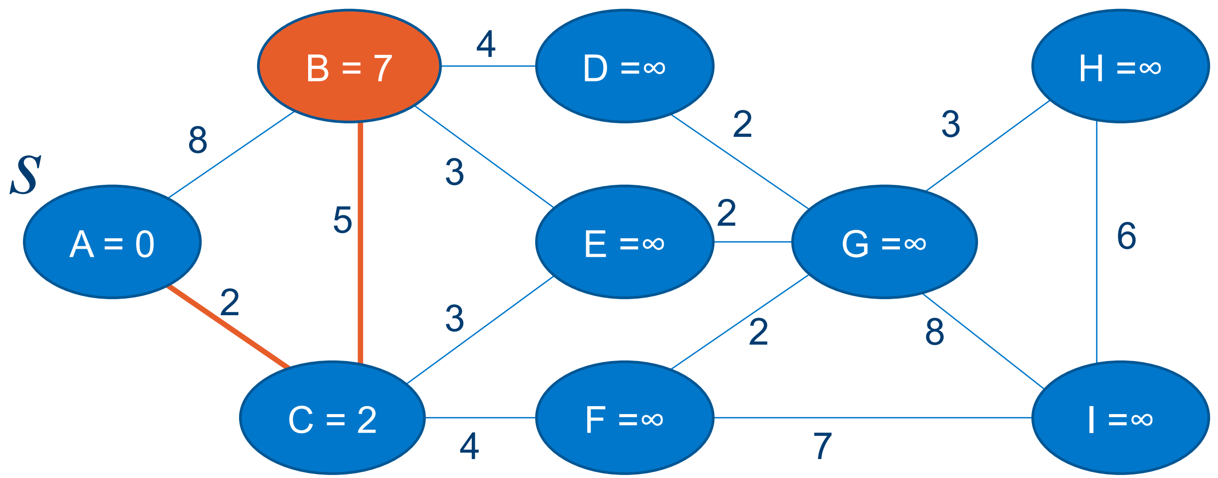

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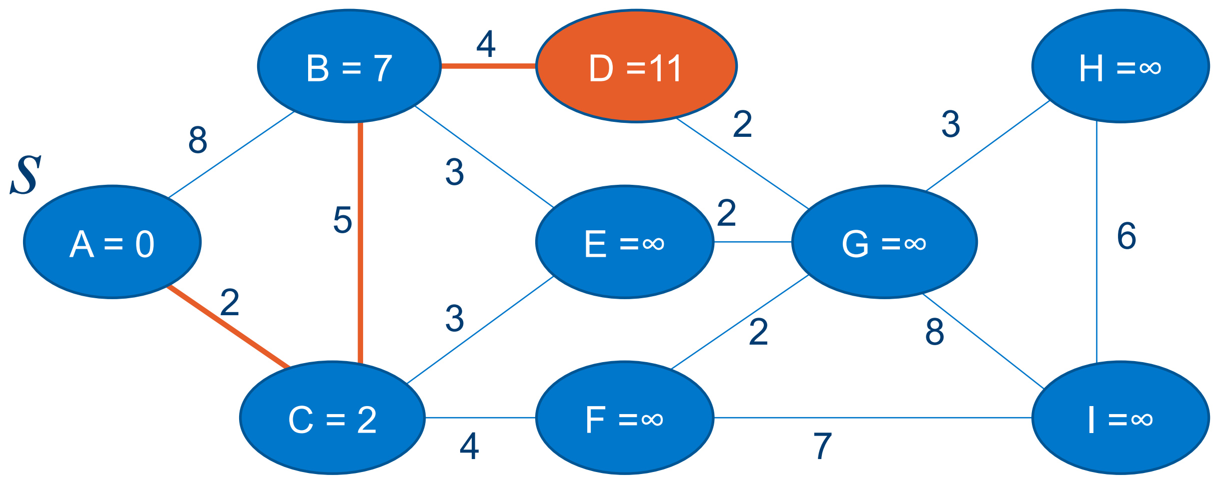

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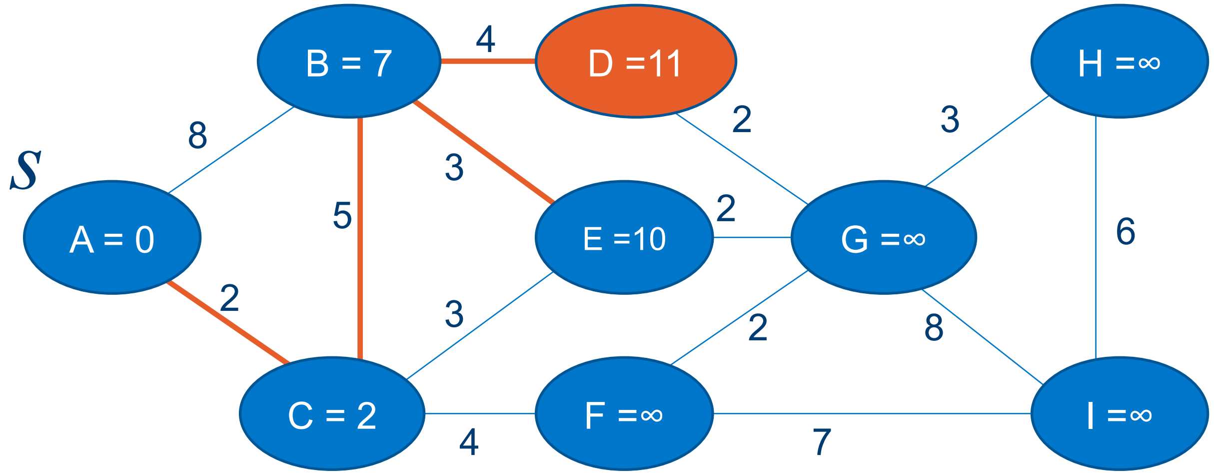

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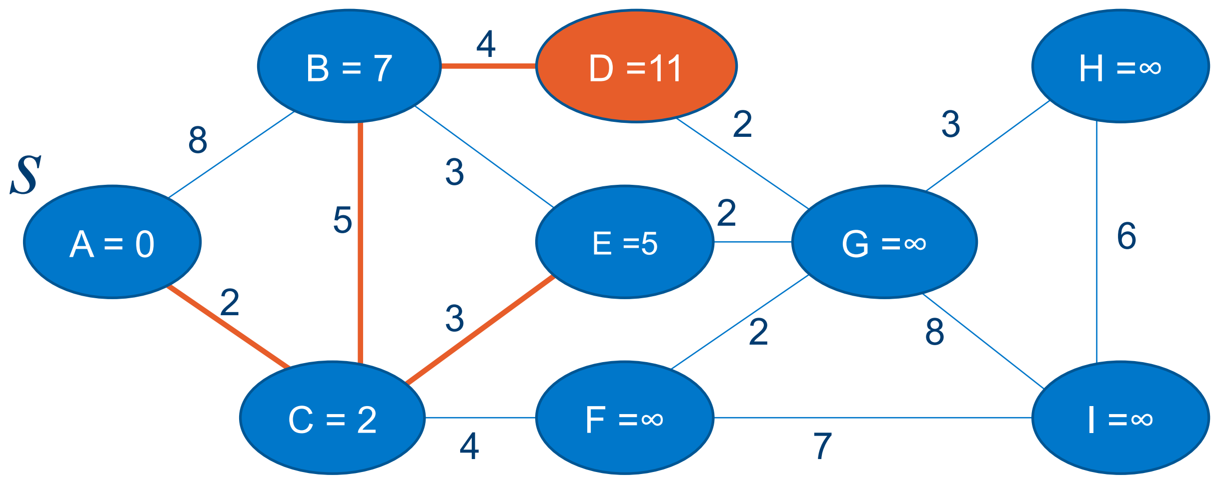

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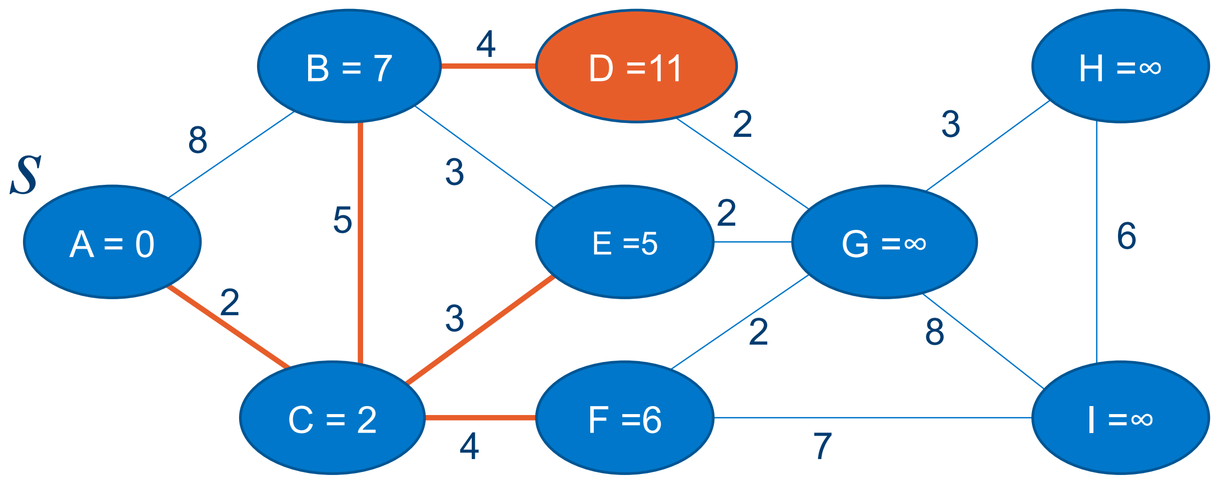

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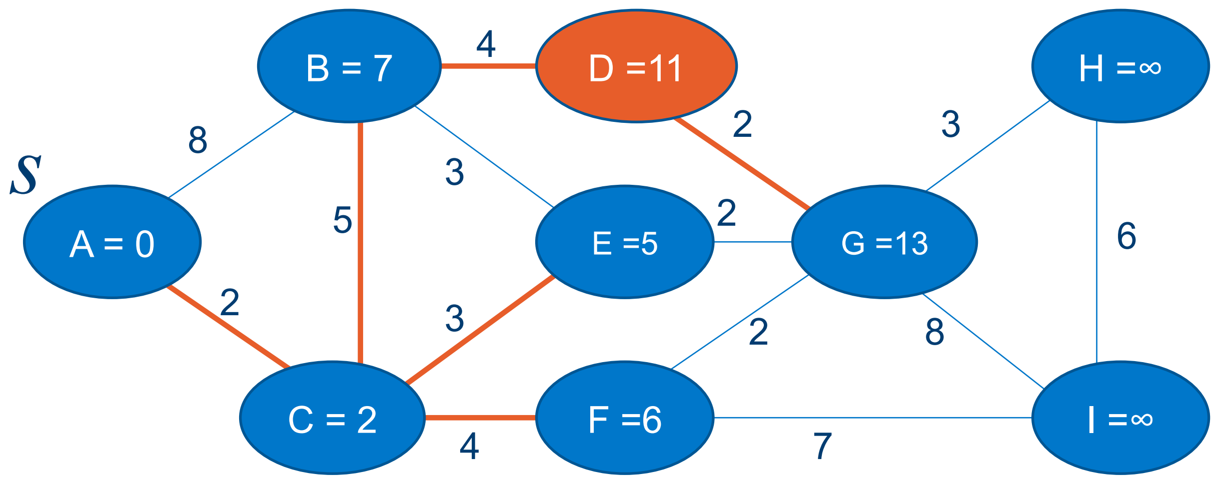

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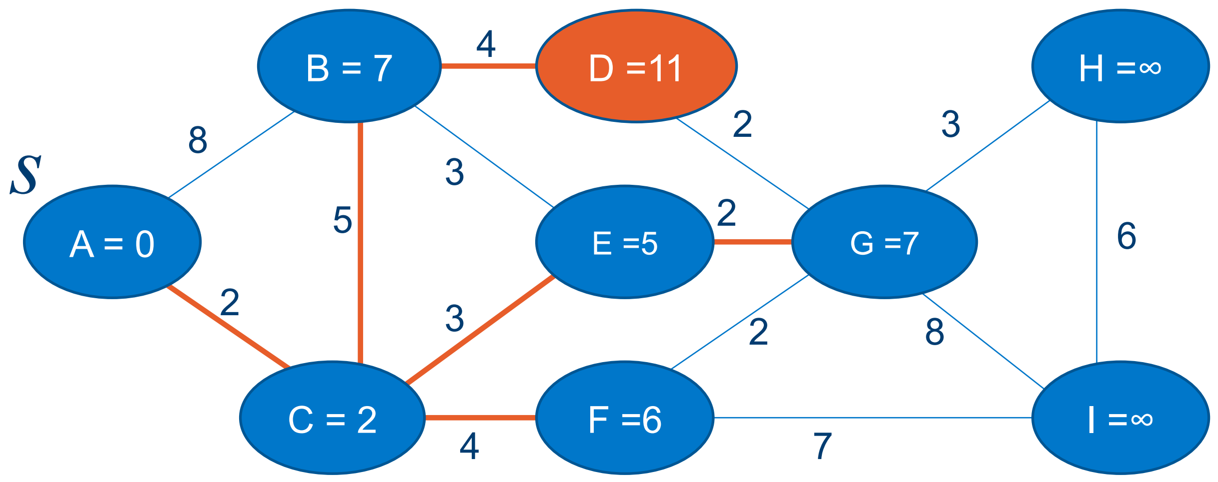

Bellman-Ford

$$
\begin{array}{l}
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\quad \textbf{for } \text{each } v \in G.V \\
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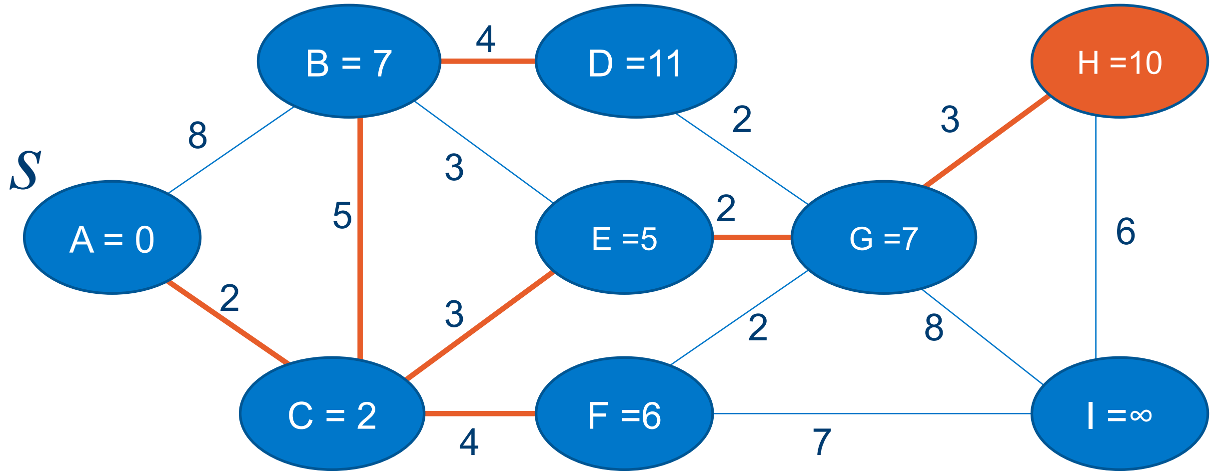

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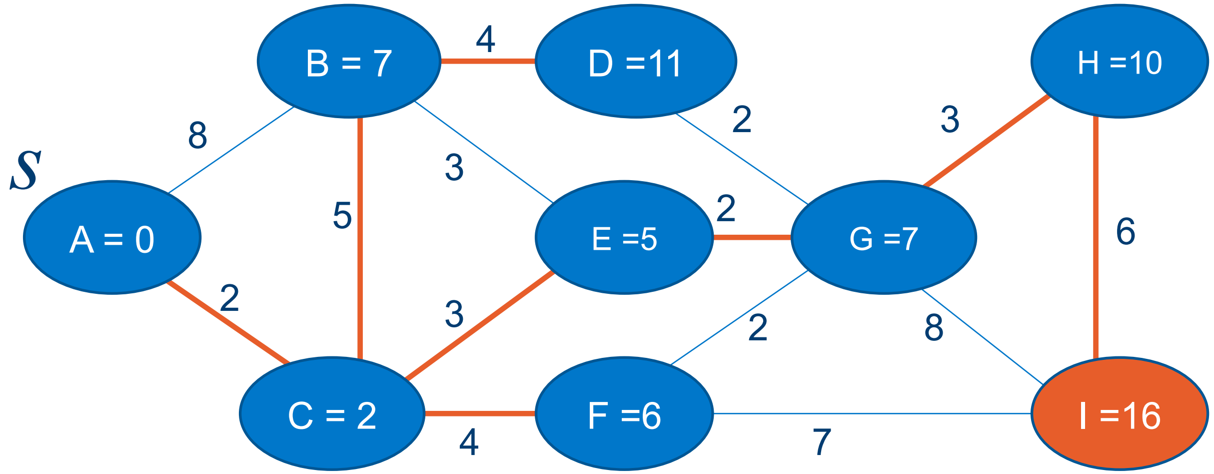

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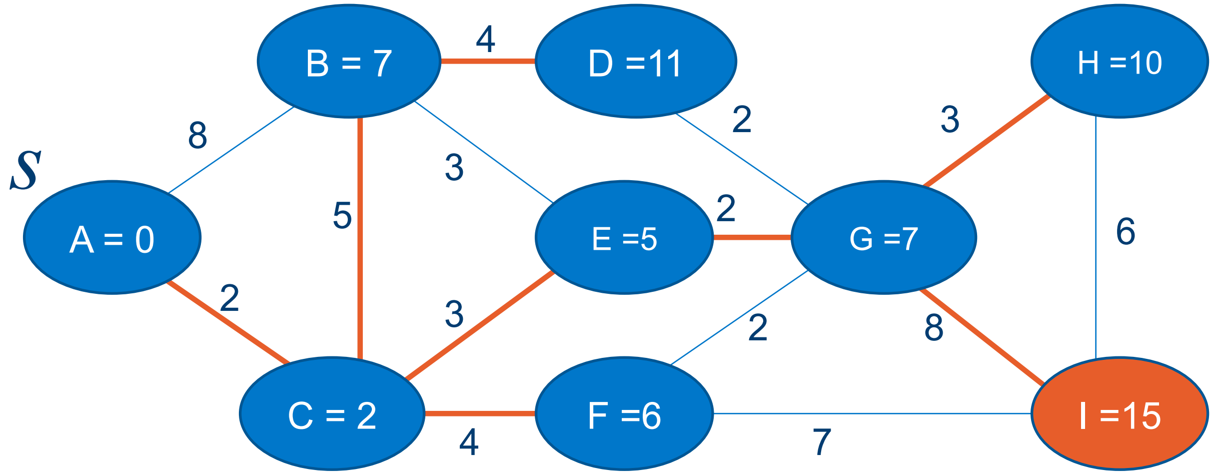

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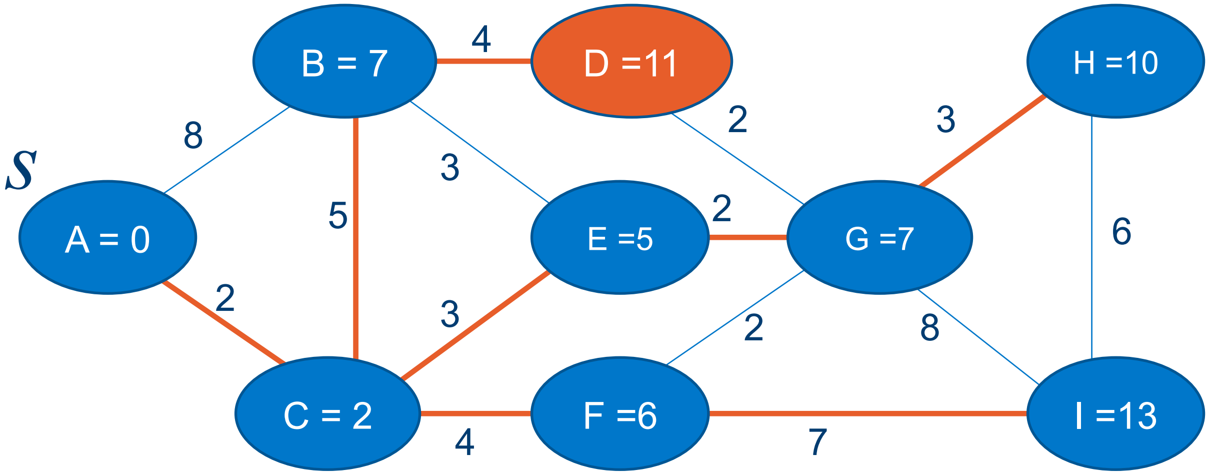

End of pass one (i =1)

Bellman-Ford

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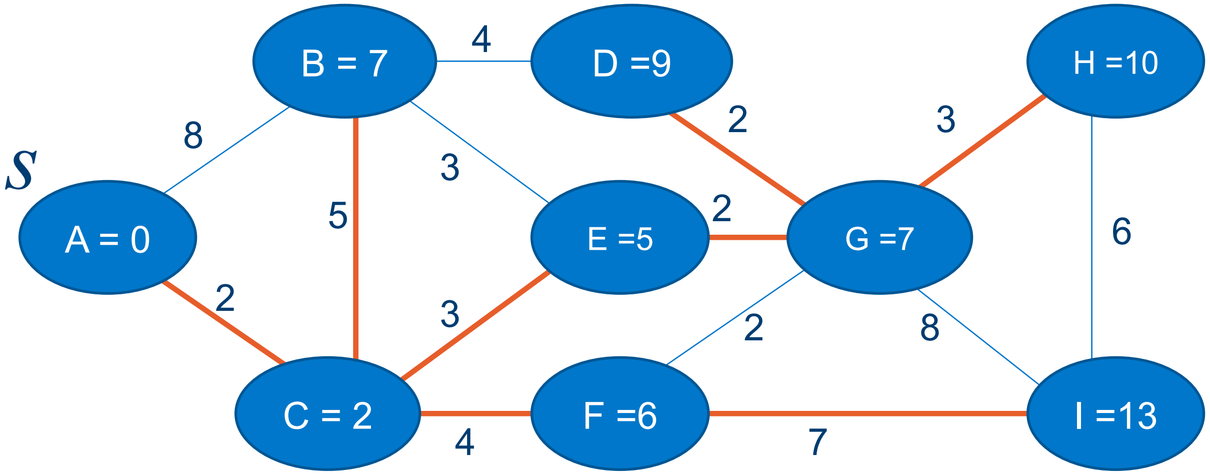
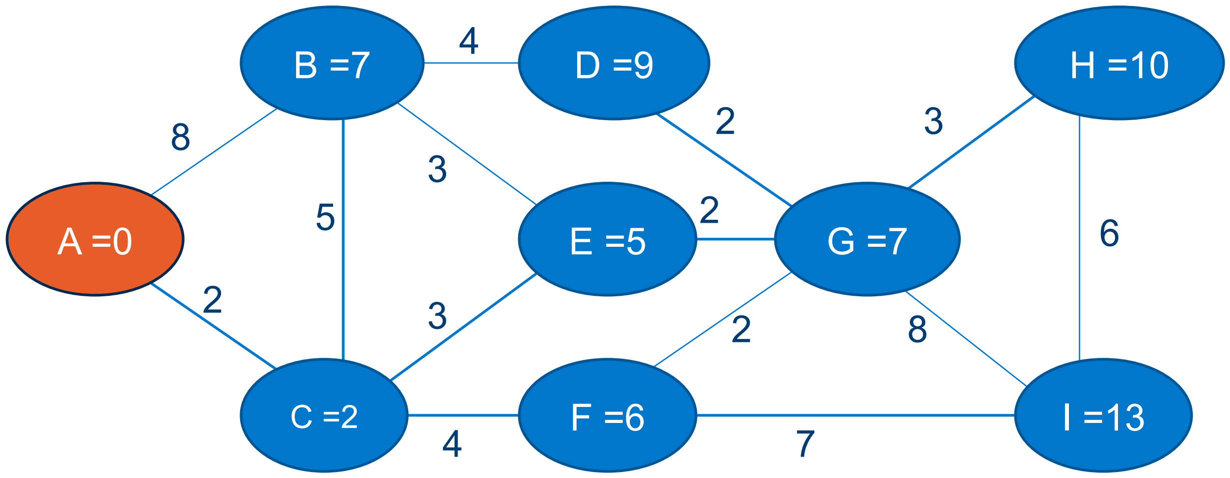


$$\Theta(VE)$$

Bellman-Ford



Vertex	Distance	Path
A	0	A
B	7	A,C,B
C	2	A,C
D	9	A,C,E,G,D
E	5	A,C,E
F	6	A,C,F
G	7	A,C,E,G
H	10	A,C,E,G,H
I	13	A,C,F,I

Wrap-up

- In this week we learned algorithms for finding the shortest paths between a source and other vertices:
    - Dijkstra
    - Bellman-Ford