Assignment 1 Algorithms

October 9, 2022

Farees Siddiqui 100-780-513
Nicholas Kissoon – 100-742-790

Part 1:

a) $T(n) = 2T(\frac{n}{2}) + 3n + 7$ (Master theorem)

$n^{\log_b a} = n^{\log_2 2} = n^1 = n$ Case 2

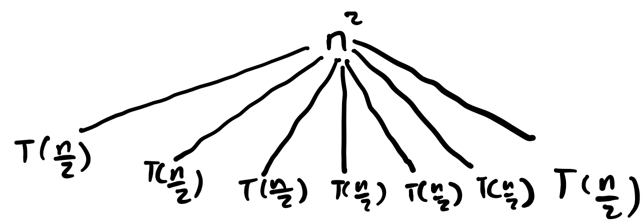
$f(n) = \Theta(n^{\log_b a} \lg^k n)$ where $k=0$

$$\boxed{T(n) = \Theta(n \log n)}$$

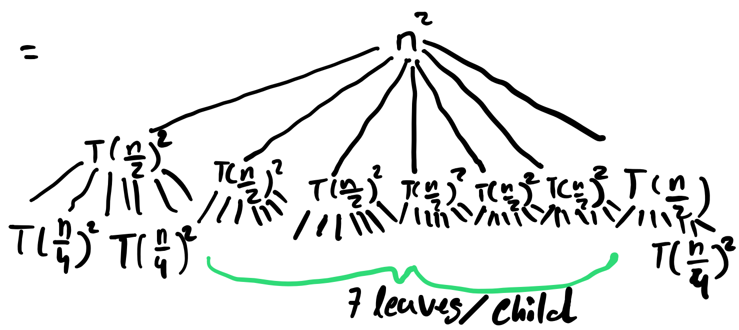
b) $T(n) = 7T(\frac{n}{2}) + n^2$ (recursion tree)

$T(n) = 7T(\frac{n}{2}) + n^2$
$T(1) = 1$

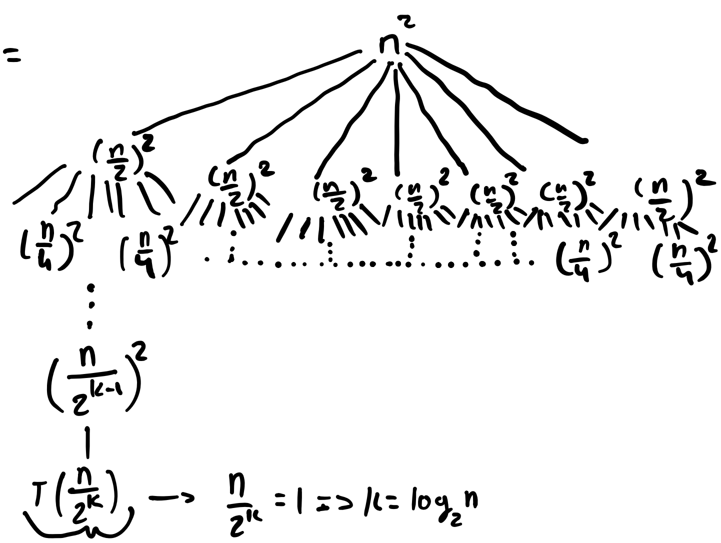
$T(n) =$



$T(n) =$



$T(n) =$



$\rightarrow \frac{n}{2^k} = 1 \Rightarrow k = \log_2 n$

$T(1)$

$$T(n) = n^2 + 7\left(\frac{n}{2^1}\right)^2 + 7^2\left(\frac{n}{2^2}\right)^2 + 7^{k-1} + \dots + 7^k$$

$$\boxed{1 + q + q^2 + \dots + q^k = \frac{1-q^{k+1}}{1-q}}$$

$$= n^2\left(1 + \frac{7^1}{2^2} + \frac{7}{2^4} + \dots + \frac{7^{k-1}}{2^{2(k-1)}}\right) + 7^k$$

$$= n^2\left(\frac{1-\left(\frac{7}{2^2}\right)^{k-1+1}}{1-\left(\frac{7}{2^2}\right)}\right) + 7^k$$

$$= n^2\left(\frac{1-\left(\frac{7}{2^2}\right)^{\log_2 n}}{1-\left(\frac{7}{2^2}\right)}\right) + 7^{\log_2 n}$$

$$= n^2\left(\frac{1-\left(\frac{7}{2^2}\right)^{\log_2 2}}{1-\left(\frac{7}{2^2}\right)}\right) + n^{\log_2 7}$$

$$= n^2\left(\frac{1-\left(\frac{7}{2^2}\right)}{1-\left(\frac{7}{2^2}\right)}\right) + n^{\log_2 7} \in \Theta(n^{\log_2 7})$$

$$\boxed{\therefore T(n) = \Theta(n^{\log_2 7}) \approx \Theta(n^{2.81})}$$

C) $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$ guess $O(n\log n)$

$T(n) = O(n\log n)$

$T(n) \leq Cn\log n$ for some $c > 0$

assume $O(n\log n)$ holds for all positive $m < n$

$$T(n) \leq c\left(\frac{n\log n}{2} + \frac{n\log n}{4} + \frac{n\log n}{8}\right)$$

$$\leq c\left(\frac{7n\log n}{8}\right)$$

$$\leq \frac{c7n\log n}{8}$$

$$\leq c\frac{7}{8}n\log n$$

$$\leq cn\log n$$

$\boxed{\lesssim O(n \log n)}$

d) $T(n) = 3T\left(\frac{n}{2}\right) + \sqrt{\log n}$

$n^{\log_b a} = n^{\log_2 3} = n^{1.5849} > 1$

$f(n) = \theta(n^{\log_2 3 - \log_2 3}) = \theta(1)$

$\boxed{\text{Solution: } T(n) = \theta(n^{\log_2 3})}$

2) $T(n) = T(n-1) + 2^n$, let $m = 2^n \Rightarrow \log_2 m = n$

$T(\log_2 m) = T(\log_2 m - \log_2 2) + m$

$= T(\log_2 m/2) + m$

$S(m) = T(\log_2 m)$

$S(m/2) = T(\log m/2)$

$S(m) = S(m/2) + m$

$a = 1,\ b = 2,\ f(m) = m$

$m^{\log_b a} = m^0 = 1 < f(m)$

$f(m) = \theta(m^{\log_b a + \varepsilon})$ where $\varepsilon > 0$

$= \theta(m^{0+1})$ $\varepsilon = 1 \Rightarrow \theta(m)$

$a f(m/b) \leq c f(m)$

$(m/2) \leq cm$

$\frac{\frac{1}{2}\cancel{m}}{\cancel{m}} \leq \frac{c\cancel{m}}{\cancel{m}}$

$\boxed{c \geq \frac{1}{2}}$

$\therefore S(m) = \theta(m)$

$\therefore$ Subbing back for n:

$T(n) = \theta(2^n)$