Master Theorem

$T(n) = aT(n/b) + f(n)$ where $a \geq 1, b > 1$, and $f(n) > 0$

Case 1: $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$.
($f(n)$ is polynomially smaller than $n^{\log_b a}$.)

Solution: $T(n) = \Theta(n^{\log_b a})$

Case 2: $f(n) = \Theta(n^{\log_b a} \lg^k n)$, where $k \geq 0$.

Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

Simple case: $k = 0 \Rightarrow f(n) = \Theta(n^{\log_b a}) \Rightarrow T(n) = \Theta(n^{\log_b a} \lg n)$

Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$
$af(n/b) \leq cf(n)$ for some constant $c < 1$
($f(n)$ is polynomially greater than $n^{\log_b a}$.)

Solution: $T(n) = \Theta(f(n))$

1) $T(n) = 8T(n/3) + n^2$

$a = 8$, $b = 3$, $f(n) = n^2$

$\log_b^a = 1.9$

$8(n/3)^2 \leq cn^2$ for some $c < 1$

$8(1/3\, n)^2 \leq cn^2$

$8(1/9\, n^2) \leq cn^2$

$\dfrac{\frac{8}{9}\cancel{n^2}}{\cancel{n^2}} \leq \dfrac{c\cancel{n^2}}{\cancel{n^2}}$

$c > \frac{8}{9}$

$n^{\log_b^a} = n^{1.9} < f(n) = n^2$

$f(n) = \Omega(n^{\log_3 8 + 0.1})$

for $\varepsilon = 0.1$

$\boxed{\therefore T(n) = \Theta(n^2)}$

Case 3 applied

2) $T(n) = 10T(n/3) + n^2$

$a = 10,\ b = 3,\ f(n) = n^2$

$\log_b a = \log_3 10 = 2.1$

$n^{\log_3 10} = n^{2.1} > f(n) = n^2$

$f(n) = \Theta(n^{\log_b a - \varepsilon})$ for $\varepsilon = 0.1$

Case 1 applies

$$\boxed{\therefore T(n) = \Theta(n^{2.1})}$$

3) $T(n) = 16T(n/4) + n^2 \log^3 n$

Case 2: $f(n) = \Theta(n^{\log_b a} \lg^k n)$, where $k \geq 0$. ***Solution:*** $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

Simple case: $k = 0 \Rightarrow f(n) = \Theta(n^{\log_b a}) \Rightarrow T(n) = \Theta(n^{\log_b a} \lg n)$

$a = 16,\ b = 4,\ f(n) = n^2 \log^3 n$

$\log_b a = \log_4 16 = 2$

$f(n) = \Theta(n^{\log_4 16} \log^k n)$ for $k = 3$

$f(n) = \Theta(n^2 \log^3 n) = f(n)$

Case 2 applied

$$\boxed{\therefore T(n) = \Theta(n^2 \log^4 n)}$$

$$T(n) = 9T(n/3) + n^3$$

$a = 9$, $b = 3$, $f(n) = n^3$

$\log_b^a = \log_3^9 = 2$, $n^{\log_b^a} = n^{\log_3^9} = n^2$

$f(n) = \Omega(n^{\log_3^9 + 1})$ for $\varepsilon = 1$, $f(n) = \Omega(n^{\log_3^{10}})$

$af(n/b) \leq cf(n)$ for some $c < 1$

$9(n/3)^3 \leq cn^3$

$9(\frac{1}{3}n)^3 \leq cn^3$

$\frac{\frac{9}{27}\cancel{n^3}}{\cancel{n^3}} \leq \frac{c\cancel{n^3}}{\cancel{n^3}}$

$\boxed{c \geq 1/3}$

Case 3 applied

$\boxed{\therefore T(n) = \Theta(n^3)}$