DELTA-GAMMA HEDGING

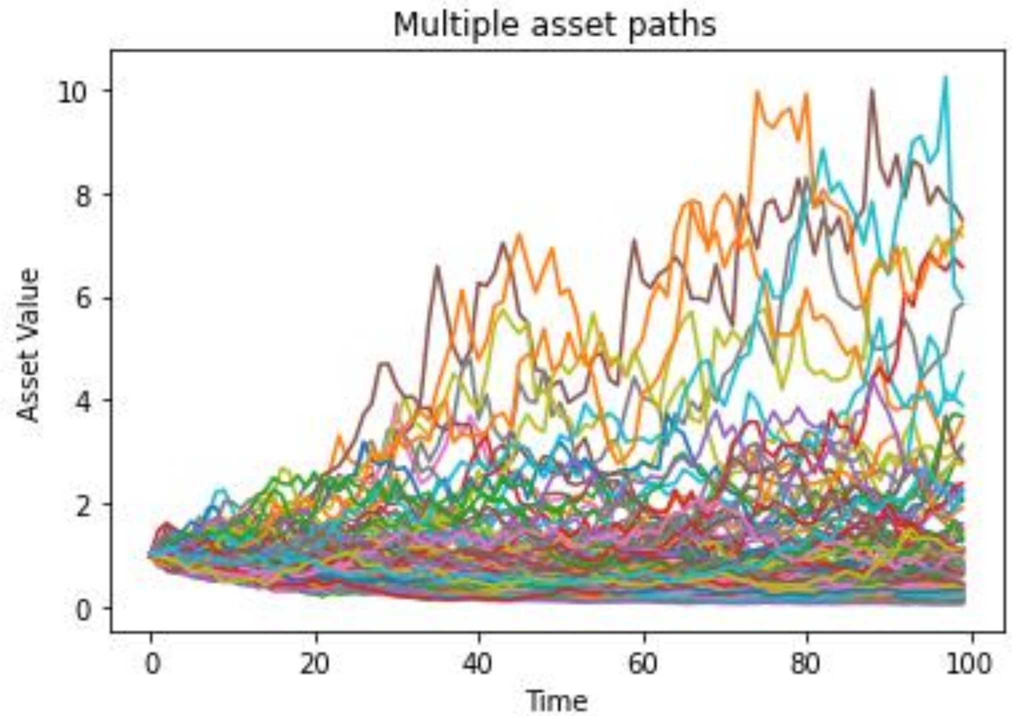
Abstract

In the following document we present the P&L graph as well as the its cumulative distribution function/probability density function, we also study the differences in various models (constant/non-constant volatility).
We also take a look at the P&L when transaction costs are considered.

Mohamed Fares Djerourou
djerouroum@cy-tech.fr

Table of Contents

Part 1: Simulation of asset path and mean/variance estimation



a) Multiple asset paths for NMC=100 and S(0)=1
b) Final asset value mean is 1.3107200467524998
   Final asset variance is 4.046108902078007
   Theoretical asset value mean is 1.2840254166877414
   Theoretical asset variance is 4.1058814053056025

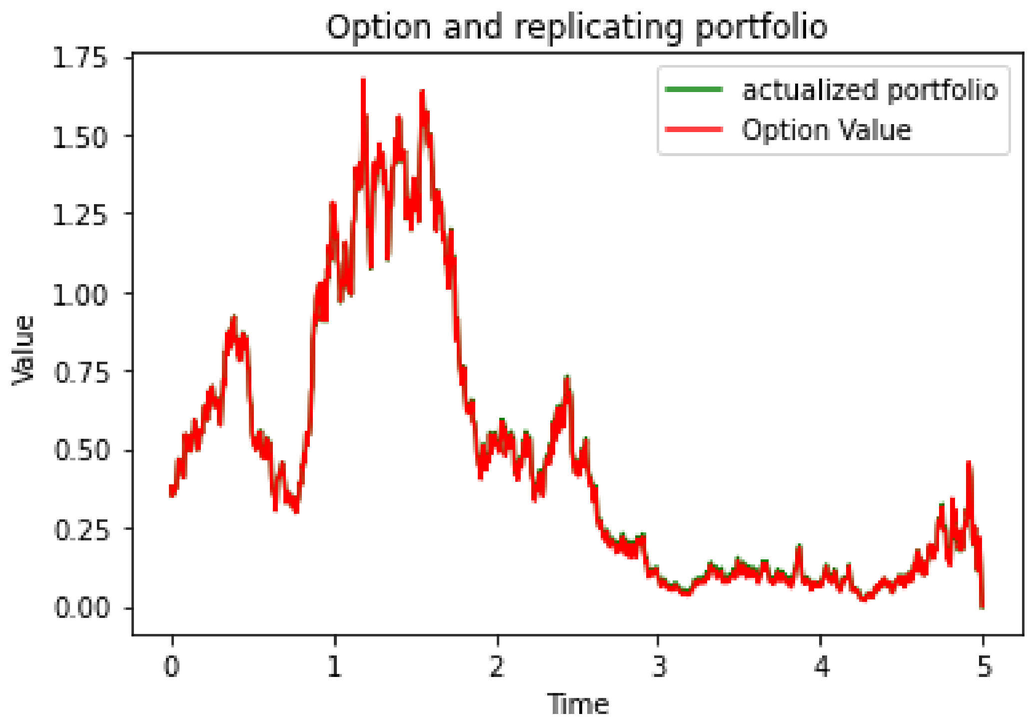
To get these values, I had to set NMC to 10000 and the the step to T/1000, to get the theoretical values I used the following formulas giving the mean variance of the exponential of a normal random variable :

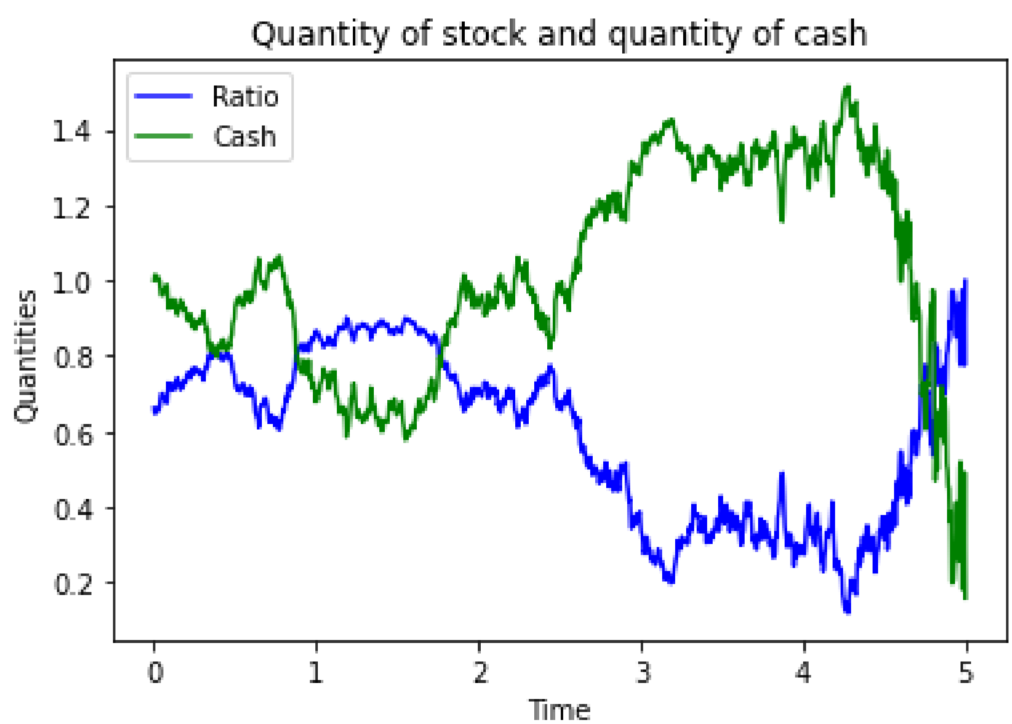
$$E[\exp(X)] = \exp(\mu + \sigma^2/2)$$

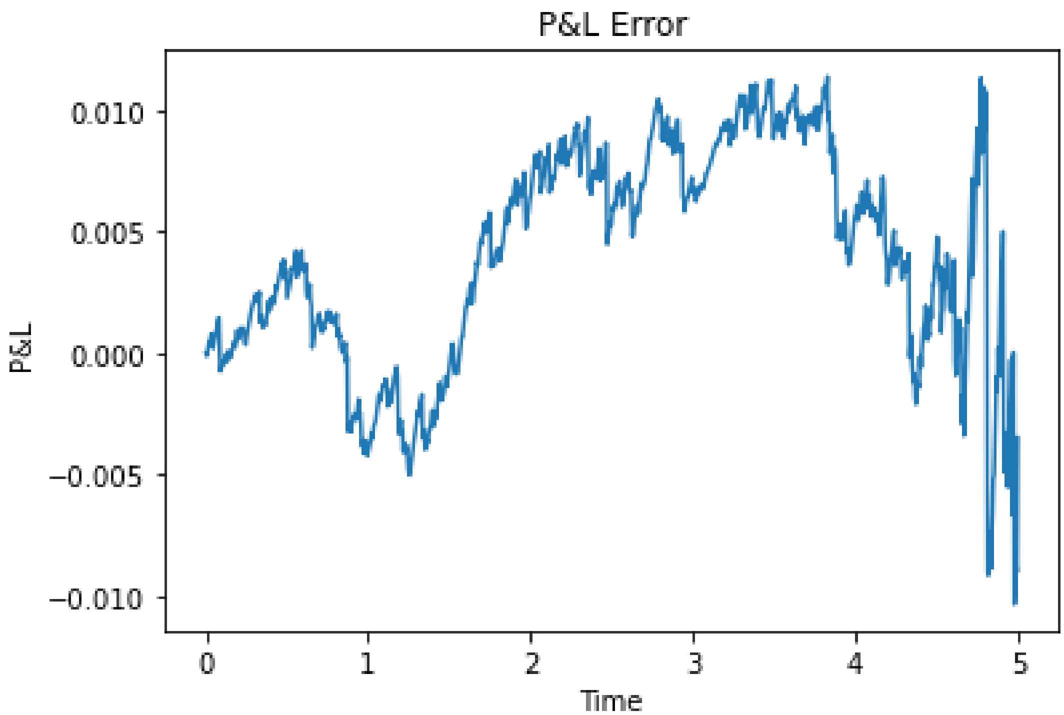
$$Var(\exp(X)) = (\exp(\sigma^2) - 1)\exp(2\mu + \sigma^2)$$

The values are close, although we have to take nmc large enough to get a good approximation, especially for the variance, which seems to be more dependent on step and number of Monte-Carlo values, intuitively it makes some sense since if we have an “abnormal” value, it will be squared in the variance estimation, and hence the variance will be further away from the theoretical variance.

Part2: Simulation of the hedging portfolio







The value of Delta if the option is in the money is 1, conversely the value of Delta if the option is out the money is 0. This makes sense because if the option is in the money, you "want" to buy the stock, because it is rising and we are talking about a call option. If the option is out the money, then Delta will be 0 at maturity, or rather tend to 0 as we approach maturity, because the stock value is falling and thus you do not want to buy it, if you bought some quantity A of it you would only be losing more money since it's falling.

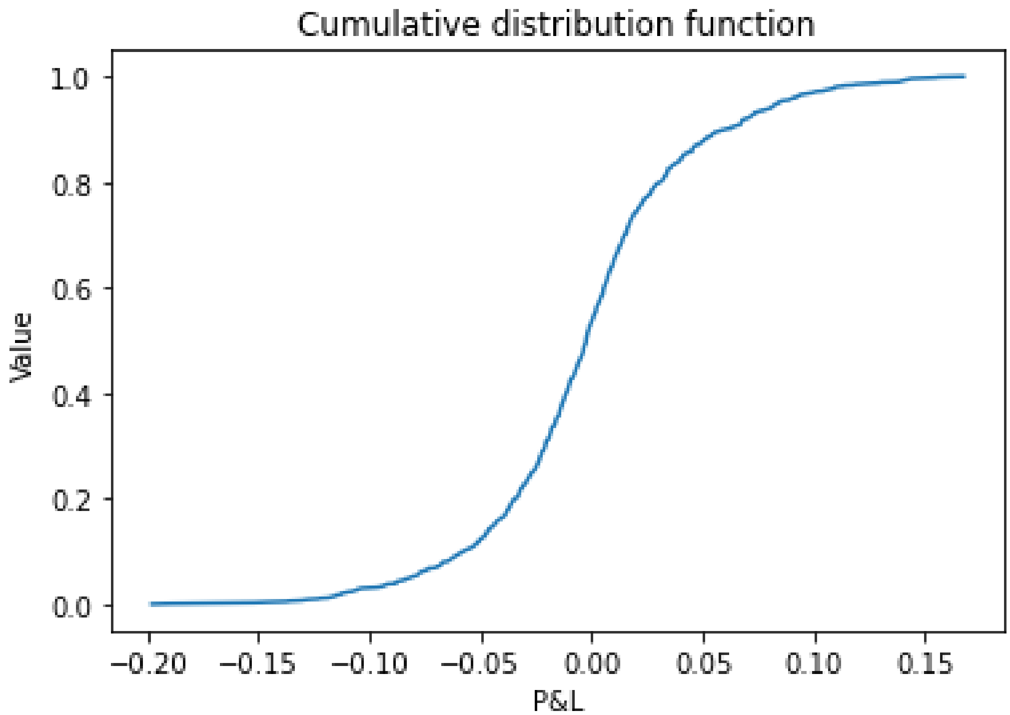
Part 3: Computation of the mean and variance of the final P&L

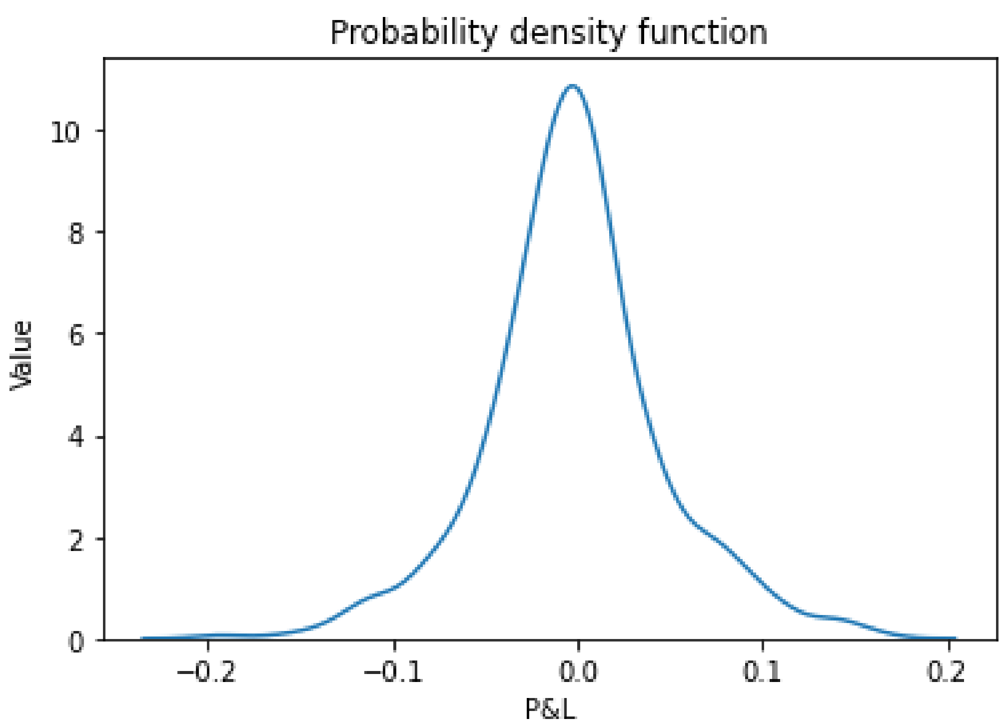
These values were computed with Nmc=1000 and N=100, of course the mean was computed as the expectation of the difference between the portfolio and the option's final values divided by the number of simulations.

The variance was computed as the difference of the mean of the square of the random variable minus the square of the mean.

P&L mean -0.001735598791153703

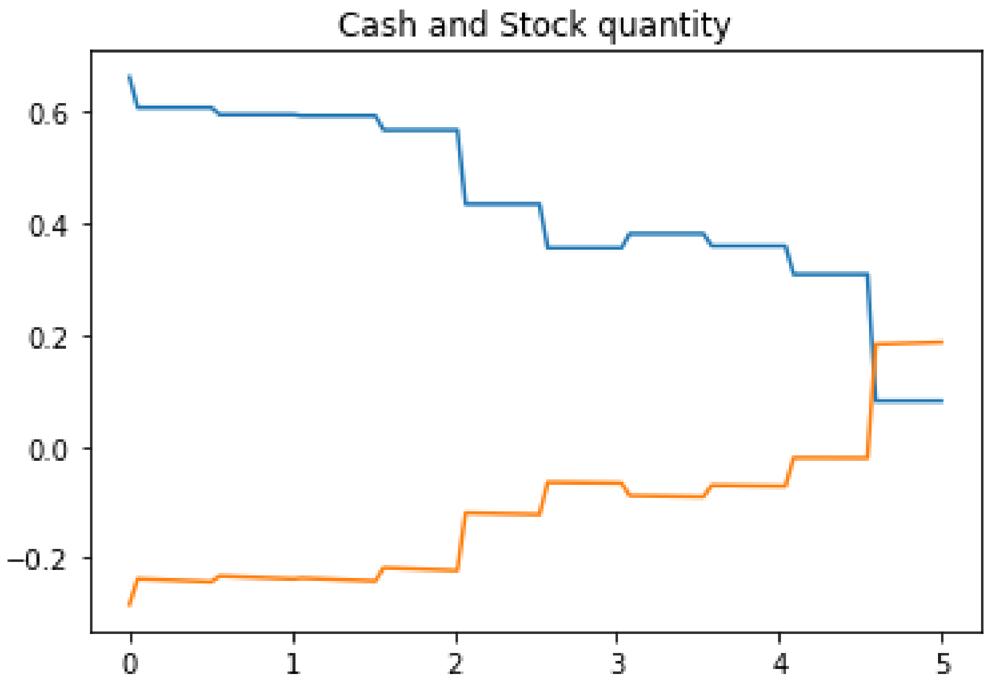
P&L Variance 0.002360928202544324



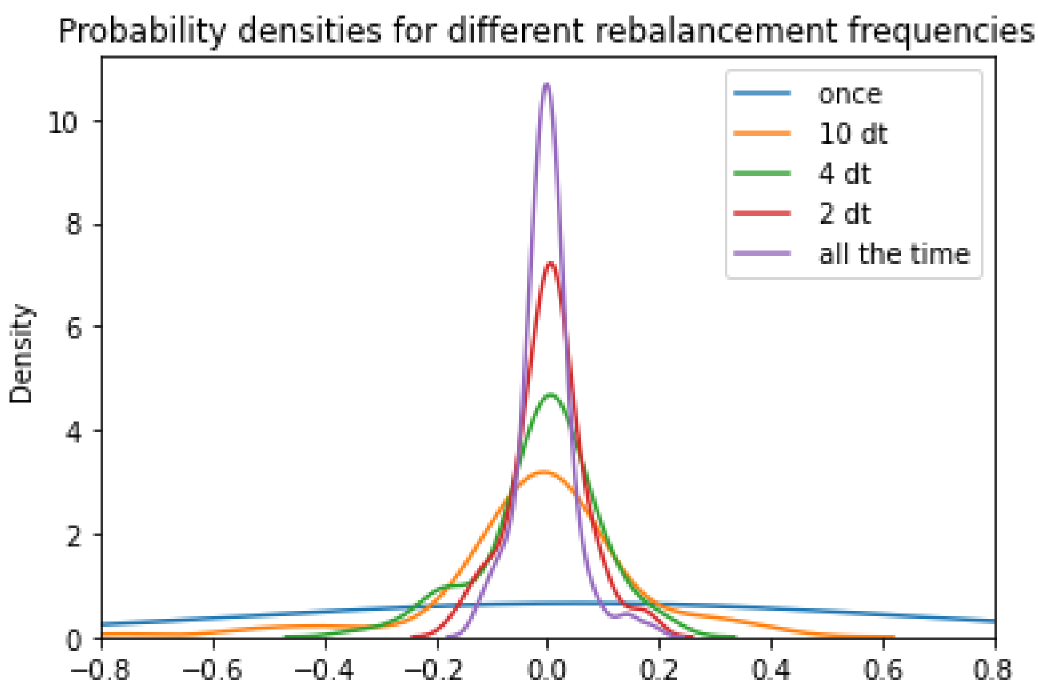


Part 4: The influence of trading frequency on the P&L Variance

The following is the stock ratio and cash plot when we rebalance once out of 10 occurrences

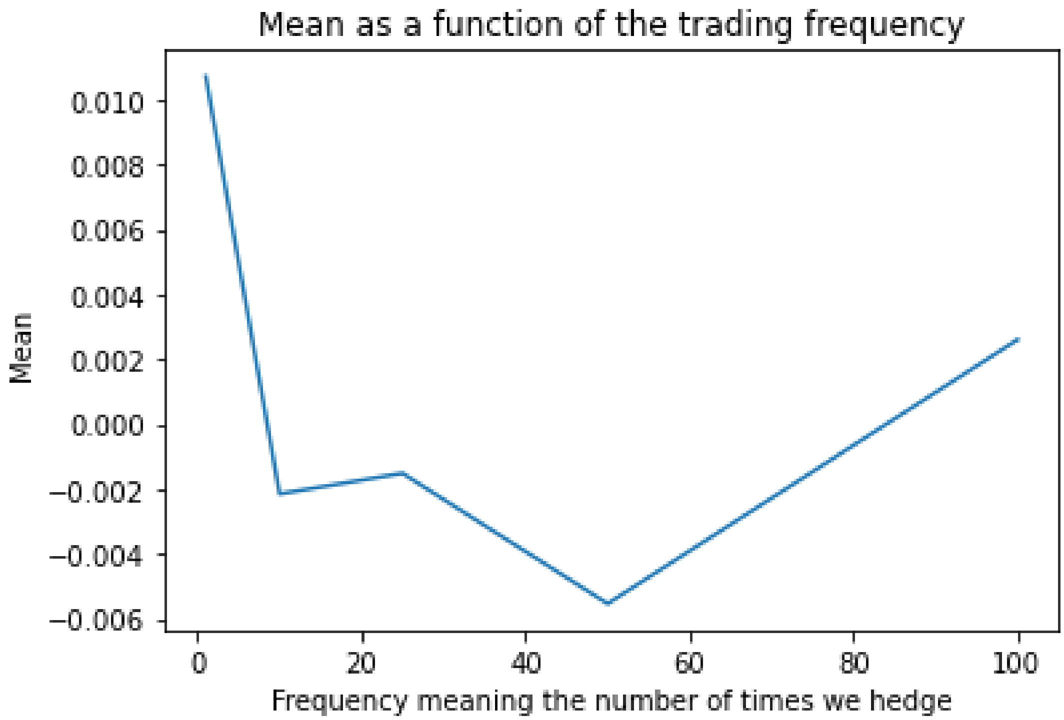


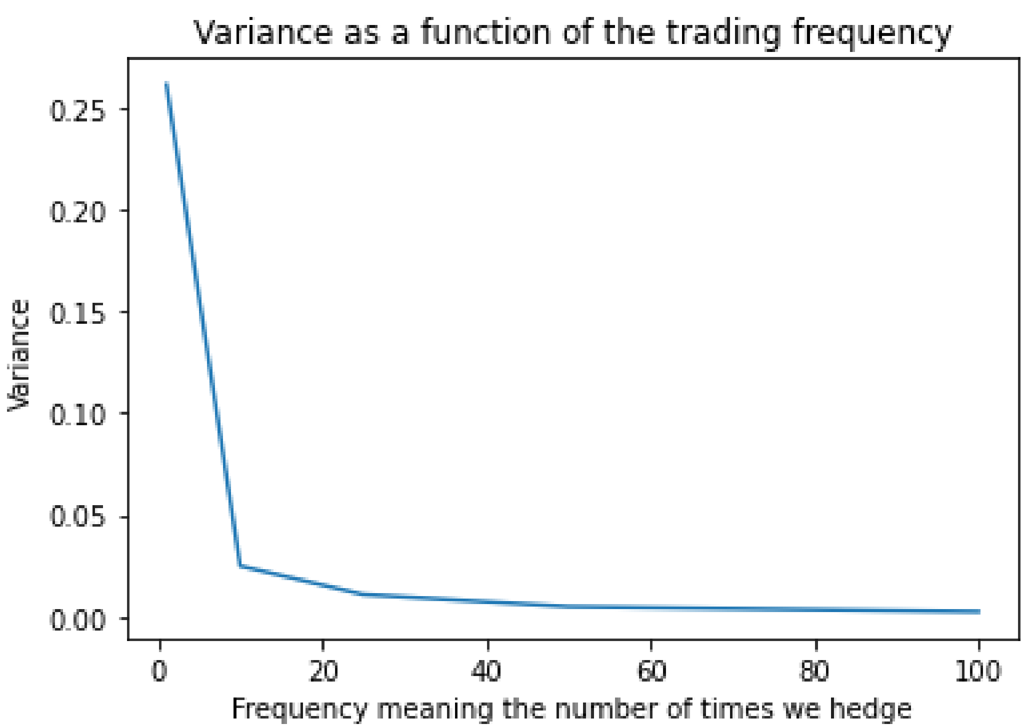
Next, we plot together the probability density functions for different trading frequencies



This makes sense, the volatility of the distribution decreases as we hedge more frequently, and the probability that the P&L is 0, or close to 0, is also higher in that case, as we can see through the probability densities.

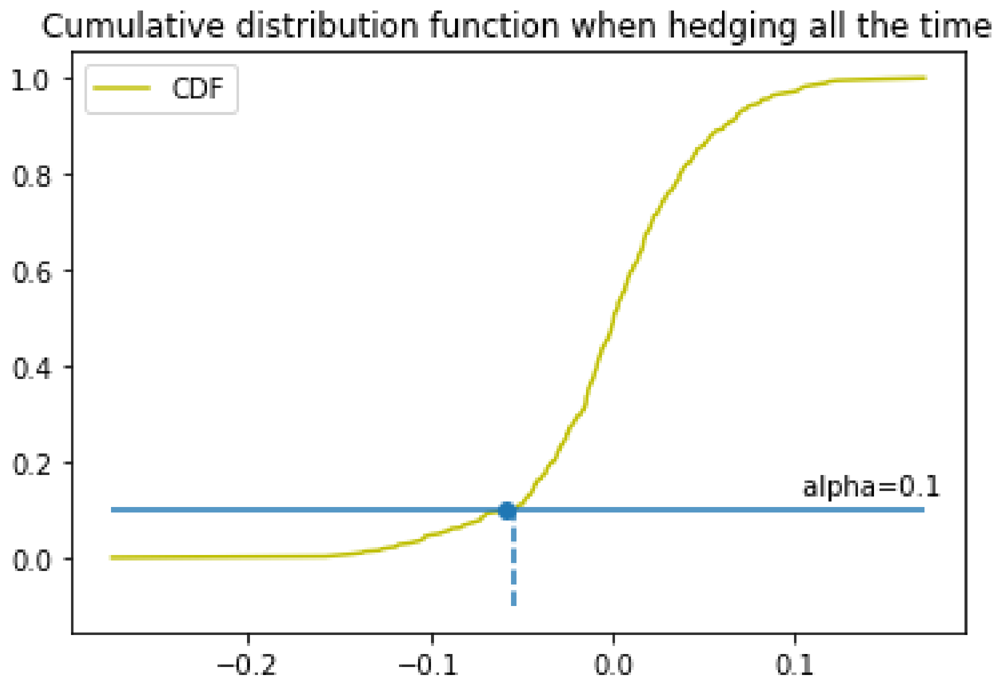
Let’s take a look at the mean and variance of the final P&L:



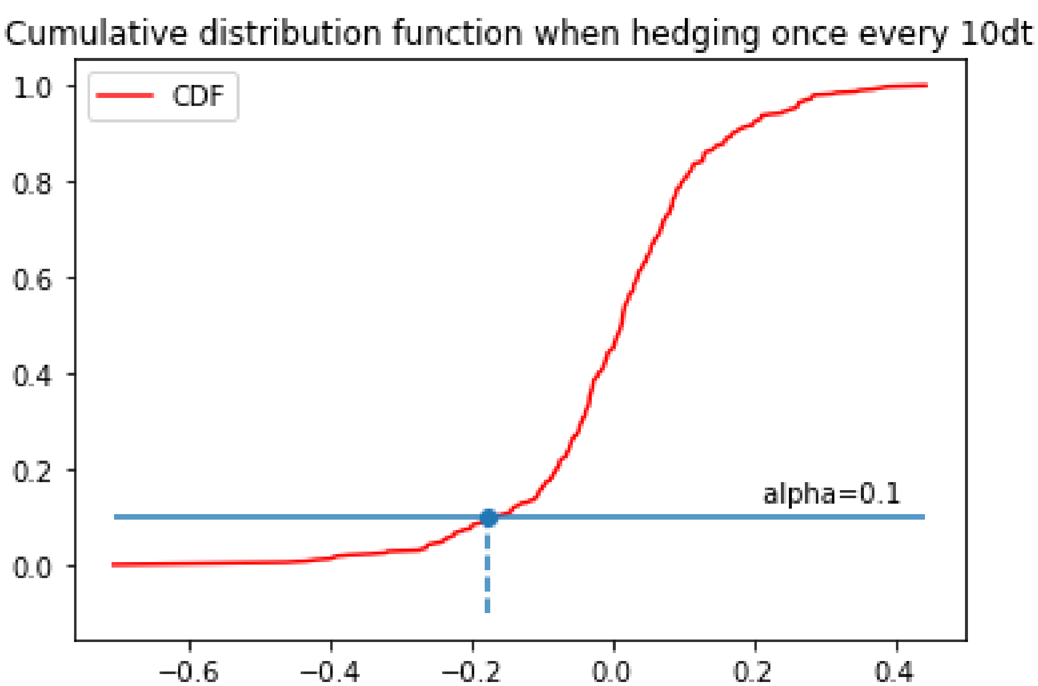


These results seem reasonable, the variance of the final P&L decreases as we hedge more frequently, and so does the mean if we account for the randomness and the sign of the mean.

Part 5: Value At Risk



This is how we can get the value at risk visually, using the cumulative distribution function, the above plot is the CDF when we hedge every dt, down below is the CDF when we rebalance once every ten times.



To get the value at risk, we can simply sort our data array and uniformly plot it to a [0,1] array of the same length, it will give us the distribution we want as every point will be associated with its natural order in the array as a percentage proportion, which is exactly the cumulative distribution function.

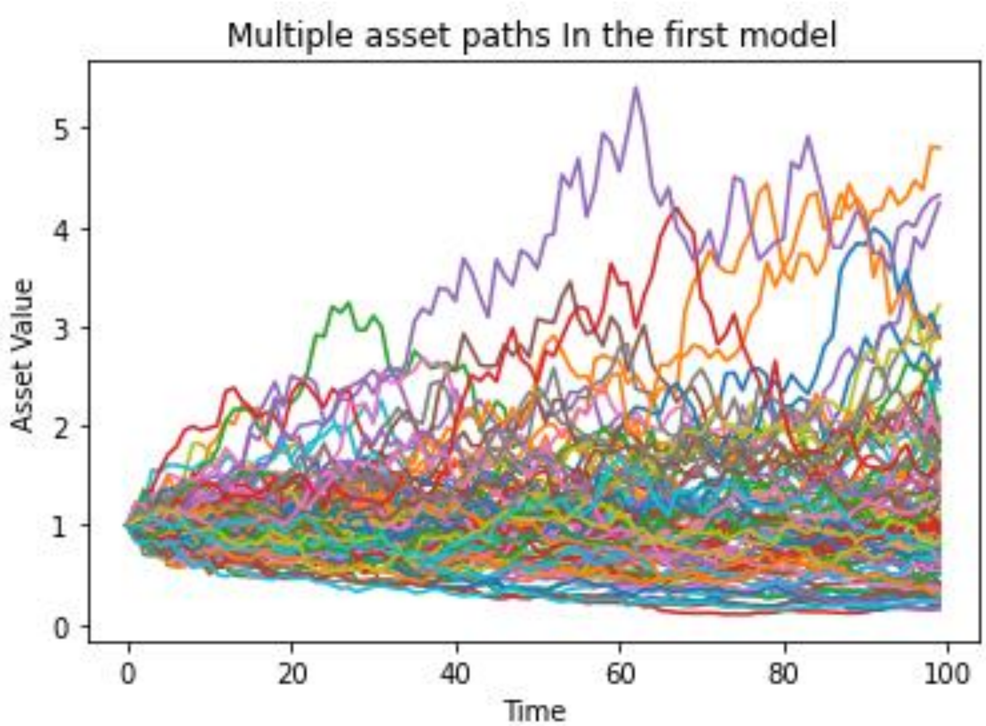
Doing so, we get the following values respectively if we rebalance all the time, and if we rebalance once every ten times:

The value at risk if we rebalance every dt is **-0.053591402510657415**

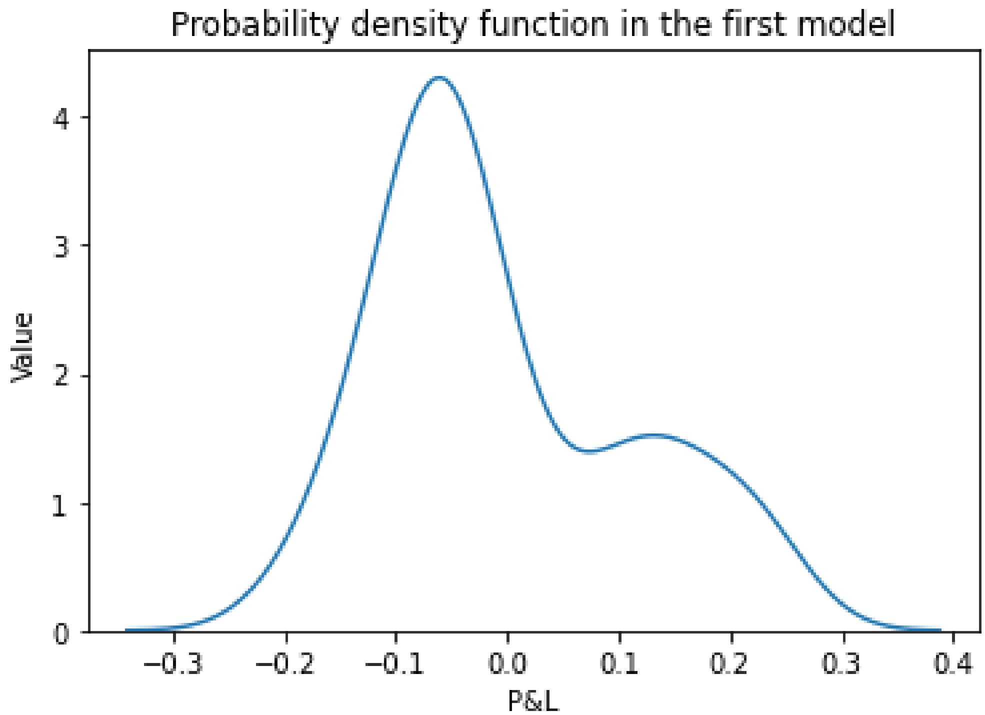
The value at risk if we rebalance once every 10*dt is **-0.17579565350910742**

These values fit well with the visual approximations.

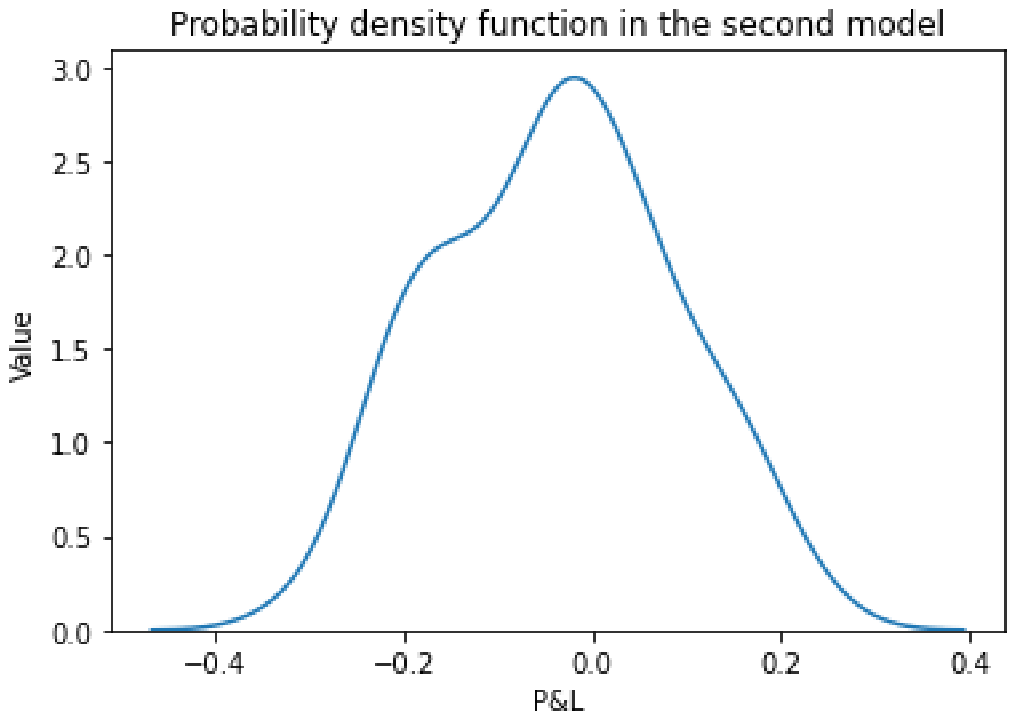
Part 6 : Stochastic volatility models



Since the volatility changes, the asset paths follow cycles of different volatilities, and in this case if we compare it with the previous asset paths, since the volatility decreases to 0.3 from 0.5 and has a relatively higher probability of staying at 0.3, the asset paths' maximum value is overall smaller, obviously it's random, but randomness tends to "vanish" with the number of simulations.



This strange “hump” is telling us that when the volatility is in the exact scenario of model1, it spikes once at around zero, and another time at some positive value. And in this case the option is not as well-hedged as in the constant volatility model. As for the interpretation, I suspect this is due to the fact that the analytical solution of the BS-Call equation we have chosen is the solution for a constant volatility, since we are changing the volatility randomly, the equation does not exactly hold anymore, we would have to “derive” a new equation and hence another solution with a variable volatility to get a “better” hedging.

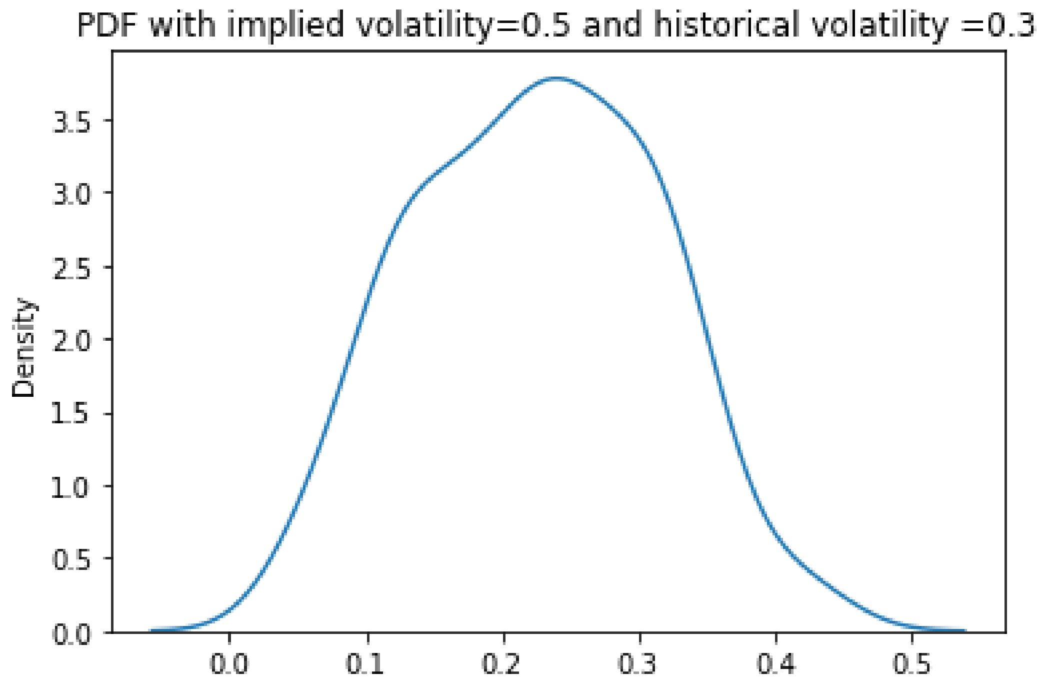


Here I had some strange results, we were given the transition probability, but we weren't given the starting sigma, meaning the sigma we should start the transitions with for each Monte-Carlo path simulation, so I assumed that the starting sigma was picked following a simple Bernoulli random variable with probability one, nevertheless the observations are similar to the previous model.

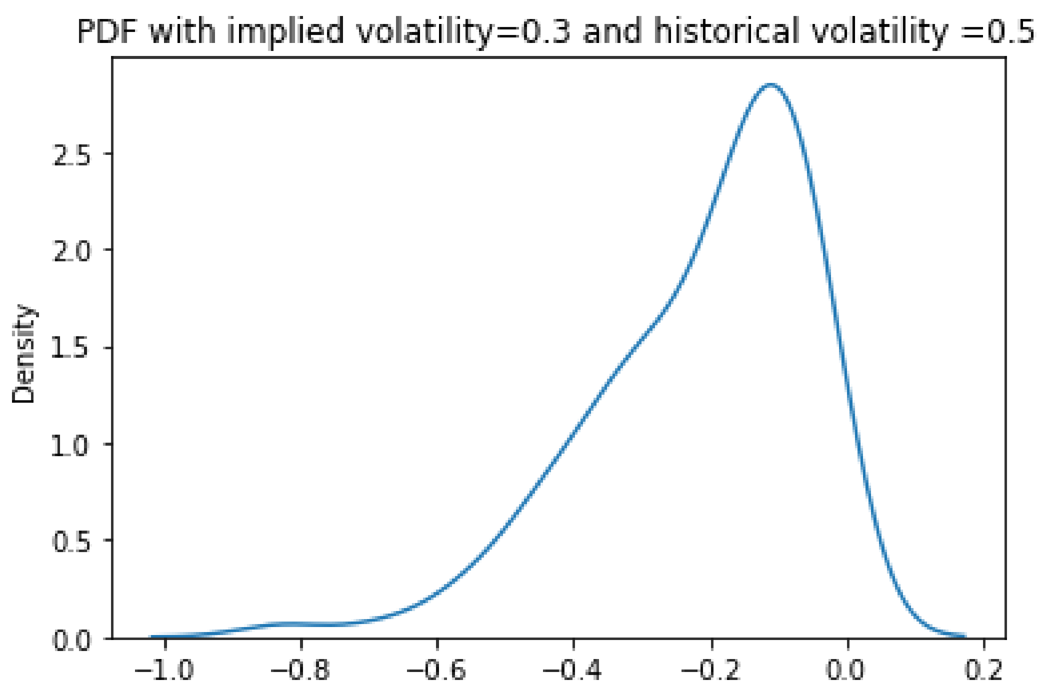
The codes are in the attached file.

<u>Part 7: Delta-hedging with implied volatility</u>

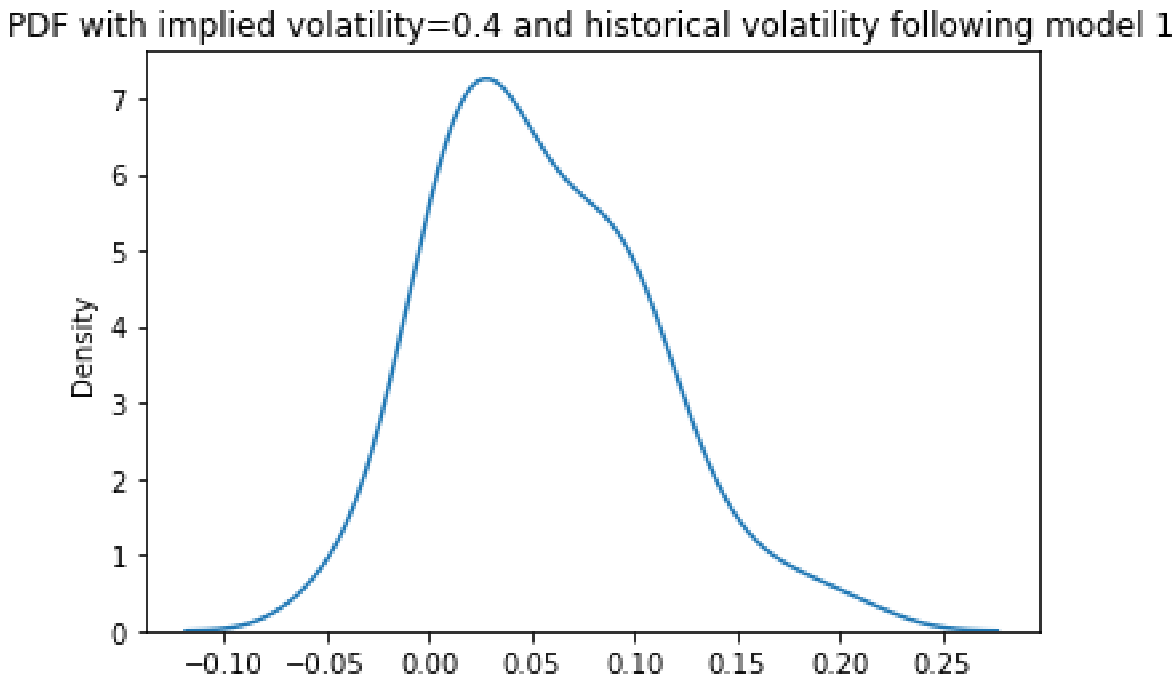
The following plots correspond to hedging using the estimated implied volatility while the asset follows a geometric Brownian motion with a set historical volatility, there are three cases to look into :



If the implied volatility is higher than the historical one, our P&L is centered around a positive value, meaning our actualized portfolio "overperforms" the option.



Conversely if it is lower than the historical volatility, it "centered" around a negative value, meaning the value of the actualized portfolio will be a bit lower than that of the option

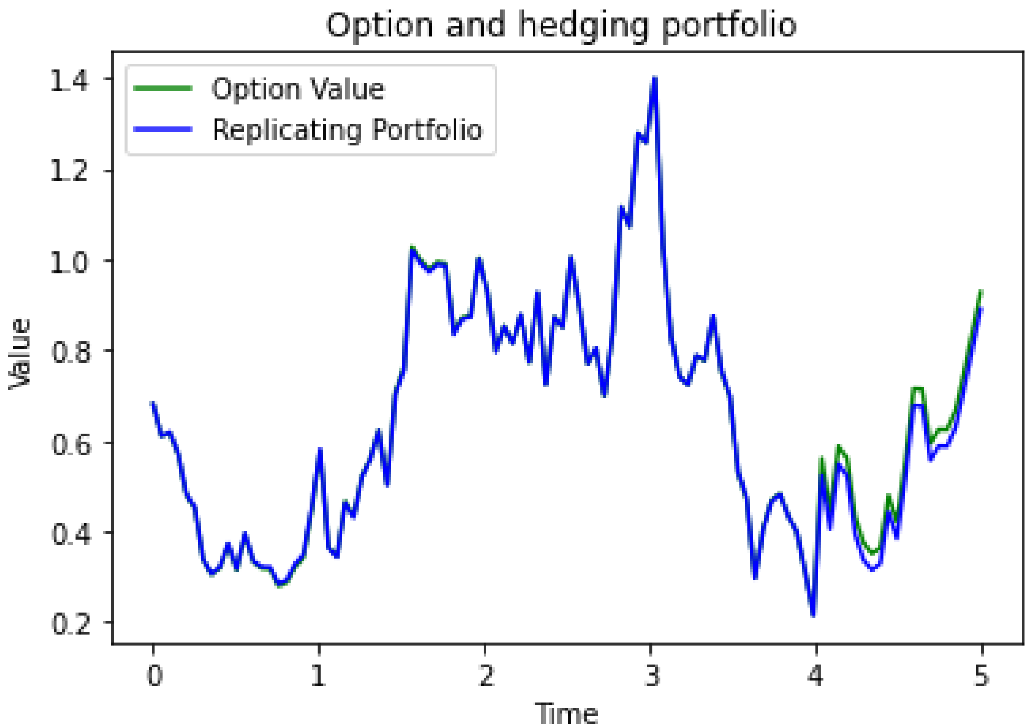


Taking the previous observations into consideration, this makes sense because here our historical volatility will be 0.3 more often than 0.5 and hence, it will be less than the 0.4 implied volatility more often that it will be bigger than it, which gives us likelier positive values.

Part 8: Replicating portfolio

There is not much to say here except we are choosing the starting value of Cash to be the leftovers from the option's premium minus the cost of first ratio of the stock we buy, giving us the exact replicating portfolio.

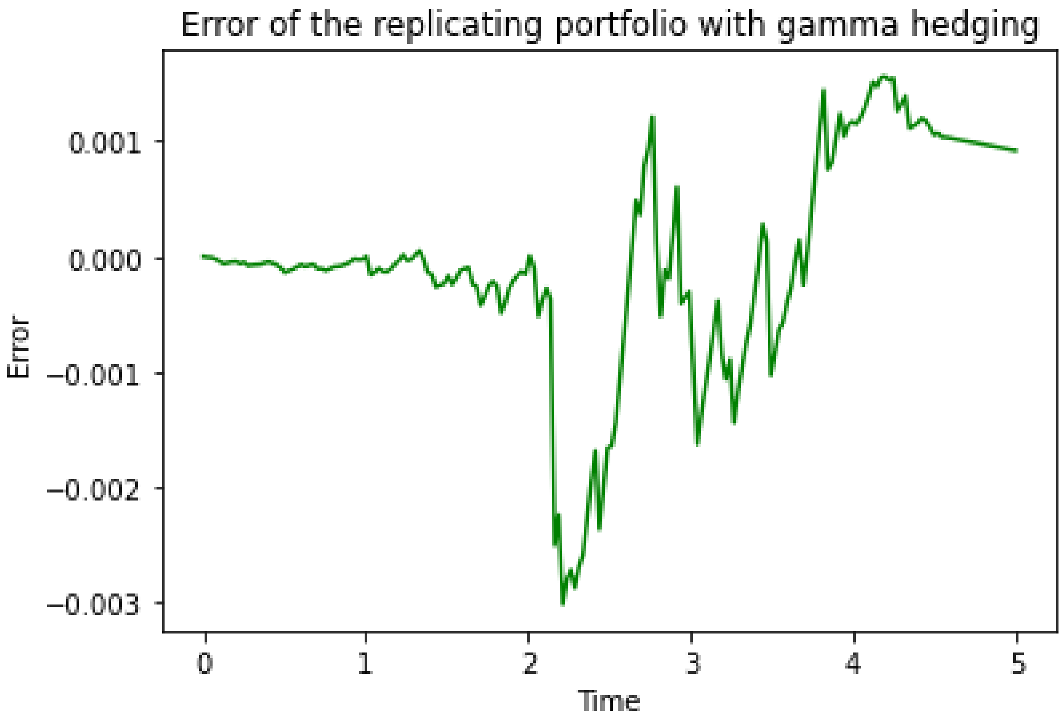
As we are hedging "as often as we can" we maintain the equality between the option and the portfolio because our portfolio is self-financed, as we can see it with the help of simulations.



As we can see the replicating portfolio perfectly tracks our option's value.

Part 9: Gamma-Hedging

The idea here is that we will use another call with a bigger time to maturity in addition to investing in the stock and in the bank to hedge our initial call, it can be shown that this provides a better approximation, meaning it reduces the error since we are fundamentally trying to reduce the risk associated with the change in delta position. (Code is attached to the file)



Part 10: Hedging with transaction costs

An assumption, we took for granted, is that we could buy and sell freely without having to pay for it, in reality there is what we call a transaction cost associated with almost any transaction someone makes in the market, here we will use a transaction cost which is proportional to the “amount” of stock we buy:

$$Cost=k*\Delta S$$

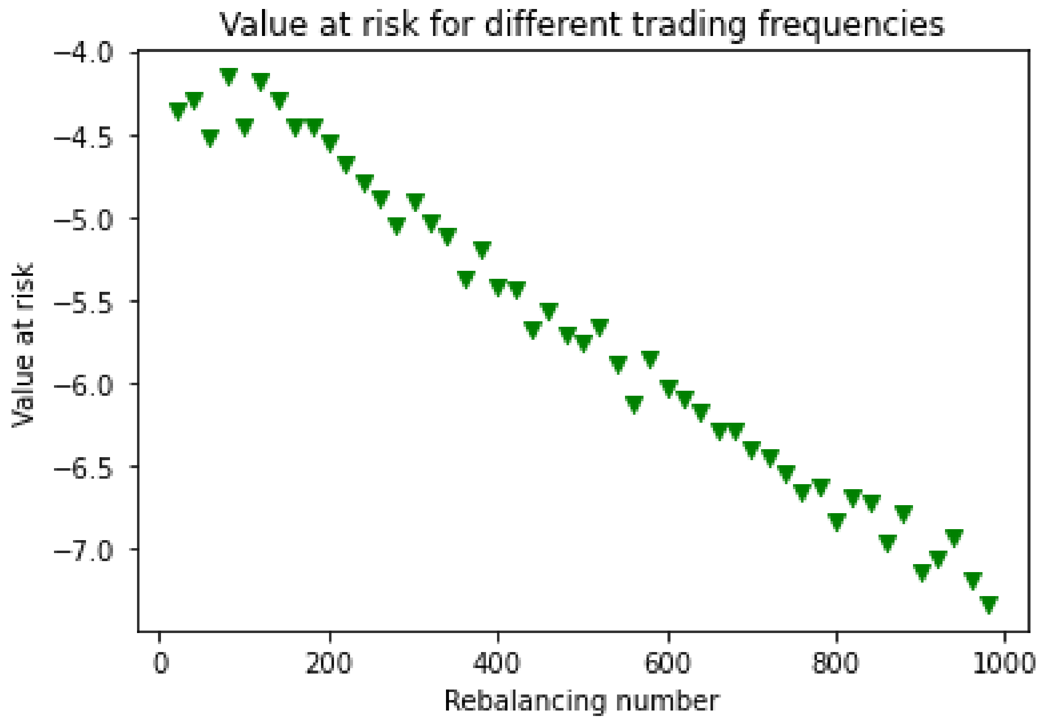
Of course, since we include transaction costs into our hedging simulation the BS model does not hold anymore, so we will use an augmented sigma value to try and reduce the chaos brought by the inclusion of transaction costs.

The code is attached to the file under the name “DeltaHedgingWithTransactionCost.py”.

We will consider different trading frequencies and look for the optimal one in terms of P&L, our measure for that will be the value at risk for alpha set to 0.1.

The following plot gives the different values at risk, from twenty rebalancing to a thousand rebalancing.

Strike=100 and S[0]=100, the transaction cost factor k is 0.01.



The minimum value (absolutely) of value at risk obtained here is for a number of rebalancing equal to 80.