Time varying fields

↳ $\nabla \cdot \vec{D} = \rho_v$

$\nabla \cdot \vec{B} = 0$

$\nabla \times \vec{E} = \boxed{-\frac{\partial \vec{B}}{\partial t}}$

$\nabla \times \vec{H} = \vec{J} \boxed{+\frac{\partial \vec{D}}{\partial t}}$ → displacement current

↳ $\vec{B} = \mu \vec{H}$

$\vec{D} = \epsilon \vec{E}$

↳ $\vec{J} = \boxed{\sigma \vec{E}} + \rho_v \vec{u}$

↑ conduction current ↖ convection current

↳ $\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}$ → continuity equation

~~~

Source-free Region

↳ $\rho_v = 0$ & $\vec{J} = 0$

$\nabla \cdot \vec{D} = 0$ $\quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\nabla \cdot \vec{B} = 0$ $\quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$

} → valid field? $\vec{G}$ → $\nabla \cdot \vec{G} = 0$ and $\nabla \times \vec{G} \neq 0$

↳ solve ~~add~~ all equations to obtain field expressions e.g. Given $\vec{E}$ → $\nabla \times \vec{E} \Rightarrow \vec{H}$

$\nabla \times \vec{H} \Rightarrow$ unknown quantity
~~~

e.g. $\vec{E}(x,t) = 20\cos(\omega t - 50x)\vec{a}_y$

$\Rightarrow \vec{H}(x,t) = \frac{1000}{\omega\mu_0}\cos(\omega t - 50x)\vec{a}_z$

$\omega = 1.5\times10^{10}$

Boundary conditions

$\vec{a}_{21}\times(\vec{E}_1-\vec{E}_2)=0$

$\vec{a}_{21}\cdot(\vec{D}_1-\vec{D}_2)=\rho_s$

$\vec{a}_{21}\cdot(\vec{B}_1-\vec{B}_2)=0$

$\vec{a}_{21}\times(\vec{H}_1-\vec{H}_2)=\vec{K}$

PEC → perfect electric conductor ($\sigma\to\infty$)

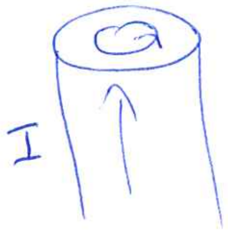
↳ $\vec{E}=0$ & $\vec{H}=0$ in PEC

$\vec{a}_{21}\times\vec{E}_1=0$

$\vec{a}_{21}\cdot\vec{D}_1=\rho_s$

$\vec{a}_{21}\cdot\vec{B}_1=0$

$\vec{a}_{21}\times\vec{H}_1=\cancel{2}\vec{K}$



$\rightarrow |\vec{J}| = \frac{I}{\pi a^2}$

Phasors

↳ time harmonic fields are often of interest

e.g. $20\cos(\omega t - 50x) \Rightarrow 20e^{-j50x}$

$\vec{A}(t) = A_0\cos(\omega t - \beta z)\vec{a}_y$

$= \mathrm{Re}\{\underbrace{\vec{A}_s}_{\text{phasor}}e^{j\omega t}\} \Rightarrow e^{j\phi} = \overset{\text{Re}}{\cos\phi} + j\sin\phi$

↳ $\vec{A}_s = A_0 e^{-j\beta z}\vec{a}_y$

$E_x =$ ① $A\cos(\omega t) \Rightarrow A \Rightarrow \mathrm{Re}\{Ae^{j\omega t}\}$

② $A\sin(\omega t+\alpha) \Rightarrow Ae^{j(\alpha-\pi/2)} = \cancel{A} - jAe^{j\alpha}$

③ $Ax^2\cos(\omega t-\beta z) \Rightarrow Ax^2e^{-j\beta z}$

④ $\frac{d}{dt}A\cos(\omega t+\alpha) \Rightarrow j\omega Ae^{j\alpha}$

$j\omega \leftarrow$

⑤ $Ae^{\alpha z}\sin(\omega t+\beta z) \Rightarrow \underbrace{-jAe^{\alpha z}e^{j\beta z}}$

$-j$

$\mathrm{Re}\{\underbrace{-j}Ae^{\alpha z}\underbrace{e^{j\beta z}e^{j\omega t}}\}$

---

① $A_s = e^{-j\beta_0 x} \Rightarrow \mathrm{Re}\{e^{j\omega t}e^{-j\beta_0 x}\}$

$= \mathrm{Re}\{e^{j(\omega t-\beta_0 x)}\}$

$= \cos(\omega t-\beta_0 x)$

② $je^{jkx} \Rightarrow \mathrm{Re}\{e^{jkx}e^{j\omega t}e^{j\pi/2}\}$

③ $10\cos\alpha$

④ $\underbrace{-j}_{\to \sin}\underbrace{\cos(ky)}_{\to \text{amplitude}}e^{-j\underbrace{\alpha z}_{\text{phase}}}$

or $\mathrm{Re}\{je^{j(kx+\omega t)}\}$

$= \mathrm{Re}\{j(\cos(\omega t+kx) + j\sin(\omega t+kx))\}$

$= \mathrm{Re}\{j\cancel{\cos}(\omega t+kx) - \sin(\omega t+kx)\}$

$= -\sin(\omega t+kx)$

③ $\mathrm{Re}\{10\cos\alpha\, e^{j\omega t}\}$

$= 10\cos\alpha\cos\omega t$

④ $\cos(ky)\sin(\omega t-\alpha z)$

Source-free region · $\rho_v = 0, \vec{J} = 0$

↳ Maxwell's equations in phasor form

$$\nabla \cdot \vec{D}_s = 0 \qquad \nabla \times \vec{E}_s = -j\omega \vec{B}_s$$

$$\nabla \cdot \vec{B}_s = 0 \qquad \nabla \times \vec{H}_s = j\omega \vec{D}_s$$

Ex// $\mu = 3\times10^{-5}$ H/m

$\epsilon = 1.2\times10^{-10}$ F/m

$\sigma = 0$

$$\vec{H}(x,t) = 2\cos(\overbrace{10^{10}}^{\omega}t - \beta x)\vec{a}_z \text{ A/m}$$

a) $\vec{B}_s(x) = \mu \vec{H}_s(x)$ $\qquad \vec{H}_s(x) = 2e^{-j\beta x}\vec{a}_z$

b) $\vec{D}_s(x)$ $\qquad \vec{B}_s(x) = (3\times10^{-5})(2e^{-j\beta x}\vec{a}_z$

c) $\vec{E}(x,t)$ $\qquad \longrightarrow \underline{\nabla \times \vec{H}_s} = j\omega\vec{D}_s$

d) β $\qquad \nabla \times \vec{E}_s = -j\omega \vec{B}_s$

$$\nabla \times \vec{H}_s = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 0 & 0 & H_{sz} \end{vmatrix}$$

$$= \vec{a}_x\left(\cancelto{0}{\frac{d}{dy}H_{sz}}\right) - \vec{a}_y\left(\frac{d}{dx}H_{sz}\right) + \vec{a}_z(0)$$

$$= -\vec{a}_y\left(\frac{d}{dx}(2e^{-j\beta x})\right)$$

$$= -\vec{a}_y\left(2(-j\beta)(e^{-j\beta x})\right)$$

$$= j2\beta e^{-j\beta x}\vec{a}_y$$

$$\nabla \times \vec{H} = j\omega \vec{D}_s$$

$$\Rightarrow \quad \vec{D}_s = \frac{\cancel{j}\, 2\beta e^{-j\beta x}}{\cancel{j}\omega} \, \vec{a}_y$$

$$= \frac{2\beta}{\omega} e^{-j\beta x} \, \vec{a}_y$$

$$\vec{D}(x,t) = \frac{2\beta}{\boxed{\omega}} \cos(\boxed{\omega t} - \boxed{\beta} x) \, \vec{a}_y$$

$$\nabla \times \vec{E}_s = -j\omega \vec{B}_s \quad \Rightarrow \text{solve for } \beta$$

$$\vec{D}_s = \epsilon \vec{E}_s$$

Department of Electrical and Computer Engineering

Time-Harmonic Fields

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- Time varying electric or magnetic fields are often time-harmonic.
- **Definition:** Time-harmonic field is a field (electric or magnetic) that varies periodically or sinusoidally with time.
- Phasor representation of time-harmonic fields simplifies EM analysis provided that all of Maxwell's equations are satisfied.
- **Definition:** Phasor is a complex number that contains amplitude and phase of sinusoidal oscillation.

Lecture Outline:

- Review Phasor notation and conversion between time and frequency
- Apply Phasor representation to Maxwell's equations.
- Solve for E (or H) given H (or E) and also solve for any unknown parameter in expressions.

Review of Phasor notation

- **Phasor** is a complex number that contains amplitude and phase of sinusoidal oscillation.

② $z = x + jy = r\angle\phi$ (1)

① $z = re^{j\phi} = r(\cos\phi + j\sin\phi)$ (2)

where r is magnitude and ϕ is phase of z

③ $r = |z| = \sqrt{(x^2 + y^2)}, \phi = \tan^{-1}\frac{y}{x}$ (3)

- For time-harmonic fields ϕ consists of both time t and phase elements θ:

$$\phi = \omega t + \theta \quad (4)$$

θ can be function of time, space, or constant.

Review of Phasor notation

- Substitute (4) in (2)

$$z = re^{j\phi} = re^{j(\omega t+\theta)} \quad (5)$$

$$z = re^{j\omega t}e^{j\theta} \quad (6)$$

- The Real and Imaginary parts of (6) are given by

$$re^{j\phi} = r\cos(\omega t+\theta) + jr\sin(\omega t+\theta) \quad (7)$$

$$Re\left(re^{j\phi}\right) = r\cos(\omega t+\theta) \quad (8)$$

$$Im\left(re^{j\phi}\right) = r\sin(\omega t+\theta) \quad (9)$$

Review of Phasor notation

- Thus, a time harmonic field $\overrightarrow{A}(t)$ given by

$$\overrightarrow{A}(t) = \mathrm{A_o}\cos(\omega t+\theta) \quad (10)$$

Is equal to

$$\overrightarrow{A}(t) = Re\left(A_o e^{j\theta}e^{j\omega t}\right) \quad (11)$$

$$\overrightarrow{A}(t) = Re\left(A_s e^{j\omega t}\right) \quad (12)$$

where $A_s = A_o e^{j\theta}$

- $\overrightarrow{A}(t)$: is time-harmonic field
- A_s: is Phasor form of $\overrightarrow{A}(t)$

Review of Phasor notation

- **Example 1:** Given time-harmonic field

$$\vec{A}\,(t) = \mathrm{A_o}\cos(\omega t - \beta x)a_y$$

Determine Phasor form of $\vec{A}\,(t)$:

Solution:

$$\vec{A}\,(t) = \mathrm{A_o}\cos(\omega t - \beta x)a_y$$
$$\vec{A}\,(t) = Re\left(\mathrm{A_o}e^{-j\beta x}a_y\; e^{j\omega t}\right)$$
$$\vec{A}\,(t) = Re\left(\mathrm{A}_s\mathrm{e}^{j\omega t}\right)$$

Thus, Phasor form of $\vec{A}\,(t)$ is:

$$\boxed{\mathrm{A}_s = \mathrm{A_o}e^{-j\beta x}a_y}$$

Review of Phasor notation

- **Example 2:** Given Phasor representation

$$\mathrm{A}_s = e^{-j\beta_o x}$$

Determine time-harmonic field $\vec{A}\,(t)$

Solution:

$$\vec{A}\,(t) = Re\left(\mathrm{A}_s e^{j\omega t}\right)$$
$$\vec{A}\,(t) = Re\left(e^{-j\beta_o x}e^{j\omega t}\right)$$
$$\vec{A}\,(t) = Re\left(e^{j(\omega t-\beta_o x)}\right)$$
$$\vec{A}\,(t) = Re[\cos(\omega t - \beta_o x) + j\sin(\omega t - \beta_o x)]$$

$$\boxed{\vec{A}\,(t) = \cos(\omega t - \beta_o x)}$$

Review of Phasor notation

- **Example 3:** Given Phasor representation

$$\mathrm{A}_s = je^{jkx}$$

Determine time-harmonic field $\vec{A}\,(t)$

Solution:

$$\vec{A}\,(t) = Re\left(\mathrm{A}_s e^{j\omega t}\right) = Re\left(je^{jkx}e^{j\omega t}\right)$$

$$\vec{A}\,(t) = Re\left(e^{j\frac{\pi}{2}}e^{jkx}e^{j\omega t}\right)$$

where $e^{j\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right) = j$

$$\vec{A}\,(t) = Re\left(e^{j\left(\omega t + kx + \frac{\pi}{2}\right)}\right)$$

$$\vec{A}\,(t) = Re\left[\cos\left(\omega t + kx + \frac{\pi}{2}\right) + j\sin\left(\omega t + kx + \frac{\pi}{2}\right)\right]$$

$$\vec{A}\,(t) = \cos\left(\omega t + kx + \frac{\pi}{2}\right)$$

$$\boxed{\vec{A}\,(t) = -\sin(\omega t + kx)}$$

Review of Phasor notation

- **Example 4:** Given Phasor representation

$$\mathrm{A}_s = 10\cos(\alpha)$$

Determine time-harmonic field $\vec{A}\,(t)$

Solution:

$$\vec{A}\,(t) = Re\left(\mathrm{A}_s e^{j\omega t}\right) = Re\left(10\cos(\alpha)\, e^{j\omega t}\right)$$

$$\vec{A}\,(t) = 10\cos(\alpha)\, Re\left(e^{j\omega t}\right)$$

$$\vec{A}\,(t) = 10\cos(\alpha)\, Re[\cos(\omega t) + j\sin(\omega t)]$$

$$\boxed{\vec{A}\,(t) = 10\cos(\alpha)\cos(\omega t)}$$

Review of Phasor notation

- **Example 5:** Given Phasor representation

$$A_s = -j\cosh(y)e^{-j\alpha z}$$

Determine time-harmonic field $\vec{A}(t)$

Solution:

$$\vec{A}(t) = Re\left(A_s e^{j\omega t}\right) = Re\left(-j\cosh(y)e^{-j\alpha z}\, e^{j\omega t}\right)$$

$$\vec{A}(t) = \cosh(y)\, Re\left(e^{-\frac{j\pi}{2}}e^{-j\alpha z}e^{j\omega t}\right)$$

$$\vec{A}(t) = \cosh(y)\, Re\left(e^{j\left(\omega t-\alpha z-\frac{\pi}{2}\right)}\right)$$

$$\vec{A}(t) = \cosh(y)\cos\left(\omega t - \alpha z - \frac{\pi}{2}\right)$$

$$\boxed{\vec{A}(t) = \cosh(y)\sin(\omega t - \alpha z)}$$

Review of Phasor notation

- **Example 6:** Given Phasor representation

$$A_s = 5e^{\alpha z}\cos(z)$$

Determine time-harmonic field $\vec{A}(t)$

Solution:

$$\vec{A}(t) = Re\left(A_s e^{j\omega t}\right) = Re\left(5e^{\alpha z}\cos(z)\, e^{j\omega t}\right)$$

$$\vec{A}(t) = 5e^{\alpha z}\cos(z)\, Re\left(e^{j\omega t}\right)$$

$$\boxed{\vec{A}(t) = 5e^{\alpha z}\cos(z)\cos(\omega t)}$$

Note that $e^{\alpha z}$ is not complex, thus it is not expanded

Review of Phasor notation

- **Example 7:** Given Phasor representation

$$\mathrm{A}_s = -j\sin(\alpha z)\, e^{j\beta z}$$

Determine time-harmonic field $\overrightarrow{A}(t)$

Solution:

$$\overrightarrow{A}(t) = Re\left(\mathrm{A}_s e^{j\omega t}\right) = Re\left(-je^{j\beta z}e^{j\omega t}\right)$$

$$\overrightarrow{A}(t) = \sin(\alpha z)\, Re\left(e^{-\frac{j\pi}{2}}e^{j\beta z}e^{j\omega t}\right)$$

$$\overrightarrow{A}(t) = \sin(\alpha z)\, Re\left(e^{j\left(\omega\mathrm{t}+\beta z-\frac{\pi}{2}\right)}\right)$$

$$\overrightarrow{A}(t) = \sin(\alpha z)\cos\left(\omega\mathrm{t}+\beta z-\frac{\pi}{2}\right)$$

$$\boxed{\overrightarrow{A}(t) = \sin(\alpha z)\sin(\omega\mathrm{t}+\beta z)}$$

Review of Phasor notation

- To convert time-harmonic expression into its equivalent Phasor notation, use following steps:
  - Re-write the time-harmonic expression as:

$$\overrightarrow{A}(t) = Re\left[A_o e^{j\alpha}e^{j\omega t}\right] \quad (13)$$

  - If time-harmonic expression contains sine term, then express the sine term as phase-shifted cosine term
    - Recall that cosine function leads sine by $\frac{\pi}{2}$, or sine function lags cos by $\frac{\pi}{2}$
  - Compare (13) with (14) to determine A_s

$$\overrightarrow{A}(t) = Re\left[A_s e^{j\omega t}\right] \qquad (14)$$

Review of Phasor notation

- **Example 8:** Given time-harmonic expression

$$\vec{A}(t) = A\cos\omega t$$

Determine its Phasor representation A_s

Solution:

$$\vec{A}(t) = \mathrm{Re}\left(\mathrm{A}e^{j\omega t}\right) \quad (8.1)$$

Compare (8.1) with $Re\left[A_s e^{j\omega t}\right]$

$$\boxed{A_s = A}$$

Review of Phasor notation

- **Example 9:** Given time-harmonic expression

$$\vec{A}(t) = A\sin(\omega t + \alpha)$$

Determine its Phasor representation A_s

Solution:

The time-harmonic expression contains sine term, then express the sine term as phase-shifted cosine term:

$$\vec{A}(t) = A\cos\left(\omega t + \alpha - \frac{\pi}{2}\right)(9.1)$$

$$\vec{A}(t) = \mathrm{Re}\left(Ae^{j\left(\omega t + \alpha - \frac{\pi}{2}\right)}\right) \quad (9.2)$$

$$\vec{A}(t) = \mathrm{Re}\left(Ae^{-j\frac{\pi}{2}}e^{j\alpha}e^{j\omega t}\right)(9.3)$$

$$\vec{A}(t) = \mathrm{Re}\left(-jAe^{j\alpha}e^{j\omega t}\right) \quad (9.4)$$

Compare (9.4) with $Re\left[A_s e^{j\omega t}\right]$

$$\boxed{A_s = -jAe^{j\alpha}}$$

Review of Phasor notation

- **Example 10:** Given time-harmonic expression

$$\vec{A}\,(t) = Ax^2\cos(\omega t - \beta \mathrm{z})$$

Determine its Phasor representation A_s

Solution:

$$\vec{A}\,(t) = \mathrm{Re}\left(Ax^2e^{j(\omega t-\beta z)}\right) \quad (10.1)$$

$$\vec{A}\,(t) = \mathrm{Re}\left(Ax^2e^{-j\beta z}e^{j\omega t}\right) \quad (10.2)$$

Compare (10.2) with $Re\left[A_s e^{j\omega t}\right]$

$$\boxed{A_s = Ax^2e^{-j\beta z}}$$

Review of Phasor notation

- **Example 11:** Given time-harmonic expression

$$\vec{A}\,(t) = \frac{d}{dt}A\cos(\omega t)$$

Determine its Phasor representation A_s

Solution:

$$\vec{A}\,(t) = \mathrm{Re}\left(\frac{d}{dt}Ae^{j\omega t}\right) \quad (11.1)$$

$$\vec{A}\,(t) = \mathrm{Re}\left(A\frac{d}{dt}e^{j\omega t}\right) \quad (11.2)$$

$$\vec{A}\,(t) = \mathrm{Re}\left\{A(j\omega)e^{j\omega t}\right\}(11.3)$$

Compare (11.3) with $Re\left[A_s e^{j\omega t}\right]$

$$\boxed{A_s = j\omega A}$$

Review of Phasor notation

- Comparing the answers of examples 8 and 11, we observe:

$$A\cos(\omega t) \rightarrow A$$

$$\frac{d}{dt}A\cos(\omega t) \rightarrow j\omega A$$

This implies that phase notation of $\frac{d}{dt}$ is:

$$\frac{d}{dt} \rightarrow j\omega$$

Review of Phasor notation

- **Example 12:** Given time-harmonic expression

$$\vec{A}(t) = \frac{d^2}{dt^2}A\cos(\omega t - \alpha)$$

Determine its Phasor representation A_s

Solution:

$$\vec{A}(t) = \mathrm{Re}\left(\frac{d^2}{dt^2}Ae^{-j\alpha}e^{j\omega t}\right) \quad (12.1)$$

$$\vec{A}(t) = \mathrm{Re}\{(j\omega)^2 Ae^{-j\alpha}e^{j\omega t}\} \quad (12.2)$$

where $j^2 = \left(\sqrt{-1}\right)^2 = -1$

$$\vec{A}(t) = \mathrm{Re}\{-\omega^2 Ae^{-j\alpha}e^{j\omega t}\} \quad (12.3)$$

Compare (12.3) with $Re\left[A_s e^{j\omega t}\right]$

$$\boxed{A_s = -\omega^2 Ae^{-j\alpha}}$$

Review of Phasor notation

- **Example 13:** Given time-harmonic expression

$$\vec{A}(t) = A\sin\alpha\cos(\omega t - \alpha)$$

Determine its Phasor representation A_s

Solution:

$$\vec{A}(t) = \mathrm{Re}\left(A\sin\alpha\, e^{-j\alpha}e^{j\omega t}\right) \quad (13.1)$$

Compare (13.1) with $Re\left[A_s e^{j\omega t}\right]$

$$\boxed{A_s = A\sin\alpha\, e^{-j\alpha}}$$

Review of Phasor notation

- **Example 14:** Given time-harmonic expression

$$\vec{A}(t) = Ae^{\alpha z}\sin(\omega t + \beta z)$$

Determine its Phasor representation A_s

Solution:

The time-harmonic expression contains sine term, then express the sine term as phase-shifted cosine term:

$$\vec{A}(t) = Ae^{\alpha z}\cos\left(\omega t + \beta z - \frac{\pi}{2}\right) \quad (14.1)$$

$$\vec{A}(t) = \mathrm{Re}\left\{Ae^{\alpha z}e^{j\left(\omega t + \beta z - \frac{\pi}{2}\right)}\right\} \quad (14.2)$$

$$\vec{A}(t) = \mathrm{Re}\left(Ae^{\alpha z}e^{-j\frac{\pi}{2}}e^{\beta z}e^{j\omega t}\right) \quad (14.3)$$

$$\vec{A}(t) = \mathrm{Re}\left(-jAe^{\alpha z}e^{\beta z}e^{j\omega t}\right) \quad (14.4)$$

Compare (14.4) with $Re\left[A_s e^{j\omega t}\right]$

$$\boxed{A_s = -jAe^{j\alpha}e^{\beta z}}$$

Apply Phasor representation to Maxwell's equations

- Typically, time varying fields have same time variation.
- Phasor form provides convenient way to solve problems related to time varying electric and magnetic fields:
    - Write $\vec{\mathrm{E}}(t)$ and $\vec{\mathrm{H}}(t)$ in Phasor form
    - Solve Maxwell's equations for quantity of interest
    - Convert back to time

Apply Phasor representation to Maxwell's equations

- Recall that:

$$\vec{\mathrm{J}}_d = \frac{\partial}{\partial t}\vec{\mathrm{D}} = \frac{\partial}{\partial t}\epsilon\vec{\mathrm{E}} \rightarrow j\omega\epsilon\vec{\mathrm{E}} \qquad (14)$$

Then Maxwell's equation and their equivalent phase forms can be written as:

$$\nabla.\vec{B}_s = 0 \qquad (15)$$

$$\nabla.\vec{E}_s = \frac{\rho_v}{\epsilon} \qquad (16)$$

$$\nabla \times \vec{\mathrm{H}}_s = \vec{\mathrm{J}}_s + \frac{\partial}{\partial t}\vec{\mathrm{D}} \rightarrow \vec{\mathrm{J}}_s + j\omega\epsilon\vec{\mathrm{E}} \qquad (17)$$

$$\nabla \times \vec{\mathrm{E}}_s = -\frac{\partial}{\partial t}\vec{\mathrm{B}}_s \rightarrow -j\omega\vec{B}_s \qquad (18)$$