Notes Feb 27

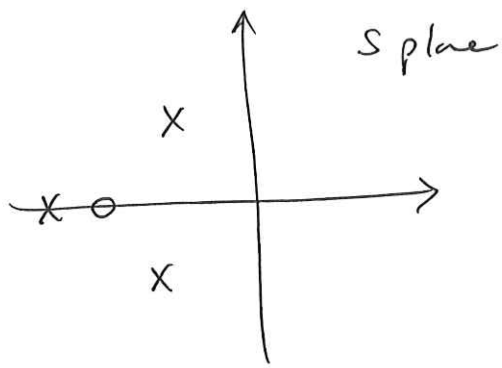
What is a pole?

$$X(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$$

$$H(s) = \frac{N(s)}{D(s)}$$

$N(s) = 0 \rightarrow$ roots are zeros
$D(s) = 0 \rightarrow$ roots are poles

Plot poles & zeros

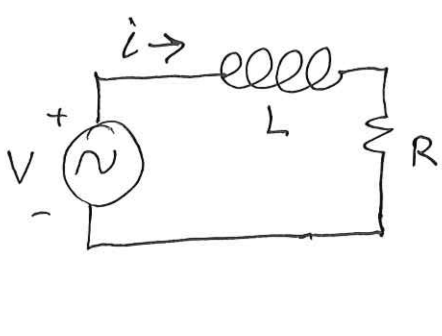


X - poles
O - zeros

Poles are singularity points in s plane where response becomes infinite magnitude or undefined.

What does that mean?

Easiest to show with example



$$L \frac{di}{dt} + Ri = V$$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{1}{L} V$$

$v(t)$ - forcing function.
Solution to $v(t)$ is particular solution

$v(t) = 0 \rightarrow \frac{di}{dt} + \frac{R}{L} i = 0$

homogenious DEQ

Solution to homogenious DEQ is $\begin{cases} \text{natural mode solution} \\ \text{eigen mode solution.} \end{cases}$

Try $i(t) = a e^{st}$ a - constant

$$a s e^{st} + \frac{R}{L} a e^{st} = 0$$

$$\underbrace{\left(s + \frac{R}{L}\right)}_{} a e^{st} = 0$$

choose $s = \frac{-R}{L}$
make zero

Hence $i(t) = a e^{-R/L t}$ natural mode solution or eigen mode of the L-R network.

Next consider the particular solution for a input of $v(t) = a e^{st}$ $\{ a, s = \text{constants}$

Need to solve

$$\frac{di}{dt} + \frac{R}{L} i = ae^{st}$$

try a solution $i = be^{st}$ $b \rightarrow$ constant

$$bse^{st} + \frac{R}{L} be^{st} = ae^{st}$$

$$\left(s + \frac{R}{L}\right) be^{st} = ae^{st}$$

$$s + \frac{R}{L} = \frac{a}{b}$$

valid for any s we can find b such that

LHS = RHS

interesting point when $s \rightarrow -R/L$

$$\left| s + \frac{R}{L} \right| = \left| \frac{a}{b} \right|$$

$s \Rightarrow -R/L \Rightarrow$ LHS $\Rightarrow 0$

RHS $\Rightarrow 0$ if $b \rightarrow \infty$

but that implies respose of $i(t) = be^{st}$ has $|b| \rightarrow \infty$

Note this particular solution is different than the

One sided solutions we are use to with Laplace

analysis.

natural mode solution

$$\frac{di(t)}{dt} + \frac{R}{L} i(t) = \frac{1}{L} \underbrace{a e^{st}}_{v(t)}$$

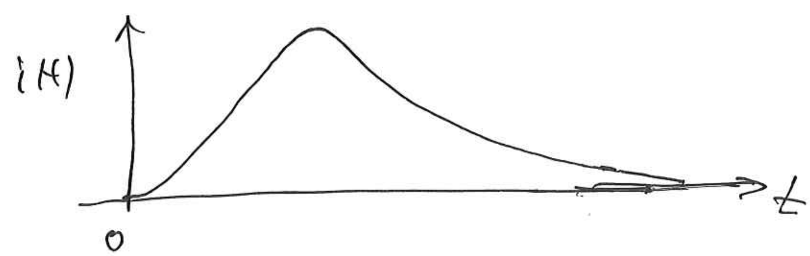
is much different than the case of particular solution

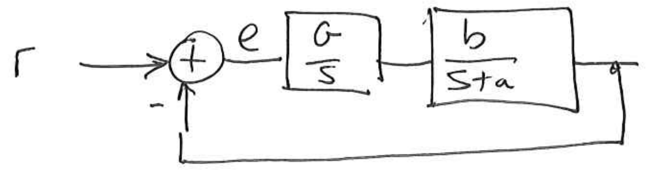
of $$\frac{di(t)}{dt} + \frac{R}{L} i(t) = \frac{1}{L} a e^{st} \underline{u(t)}$$

Solve this with Laplace for case of $s = -R/L$

$$I(s) = \frac{\frac{1}{L} \mathcal{L}\left(a e^{-R/L t} u(t)\right)}{s + R/L}$$

$$= \frac{1}{L} \frac{1}{(s + R/L)^2} = \frac{1}{L} t e^{-R/L t} u(t)$$



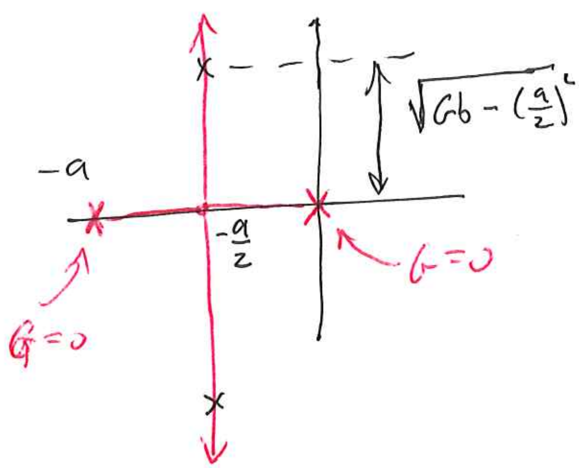


$$a = \frac{D}{J} + \frac{K_T K_b}{RJ}$$

$$b = \frac{K_T}{JR}$$

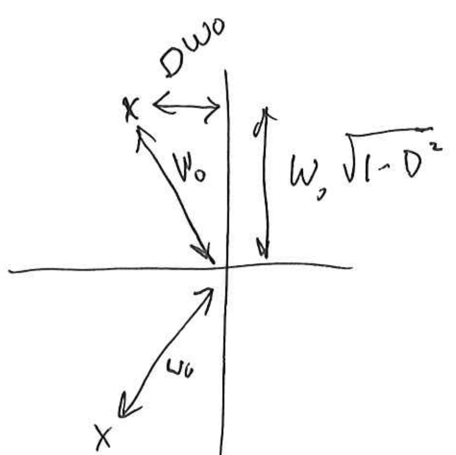
$$\frac{\Omega}{R} = H_{CL} = \frac{Gb}{s^2 + sa + Gb}$$

Poles at $-\frac{a}{2} \pm j\sqrt{Gb - \left(\frac{a}{2}\right)^2}$



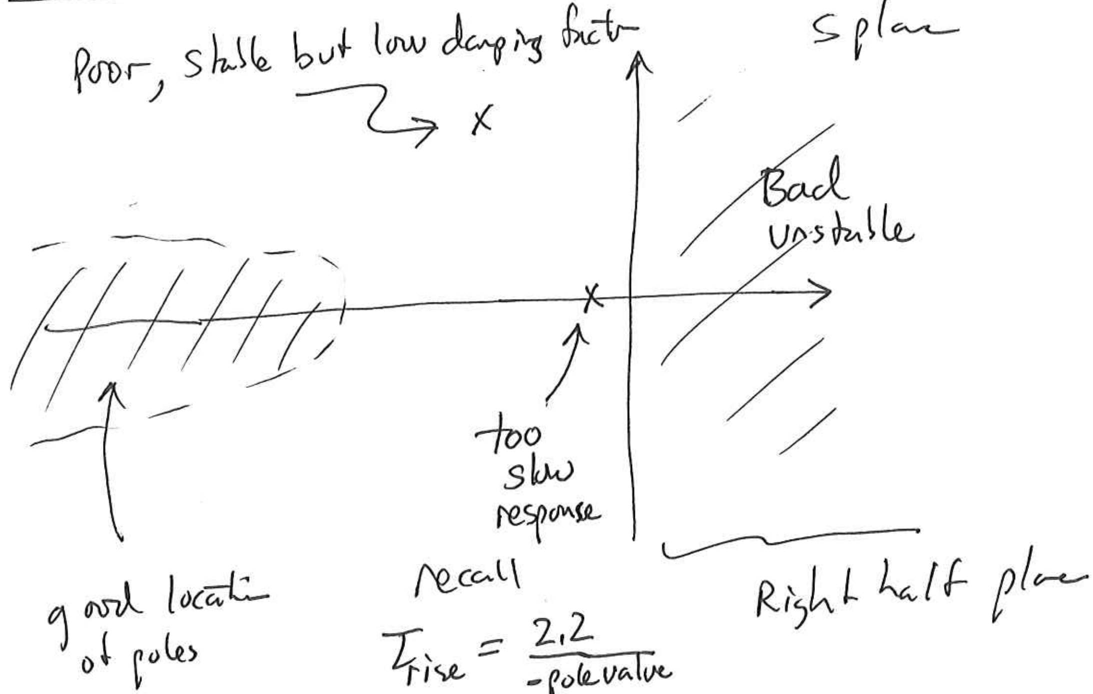
<u>Damping coefficient</u> $\quad 2D\omega_0 = a \qquad \omega_0 = \sqrt{Gb}$

$$D = \frac{a}{2\sqrt{Gb}} \qquad G\uparrow \quad D \to 0$$



Note Damping coefficient decreases with increase in G.

Good Pole Locations for Closed Loop Poles



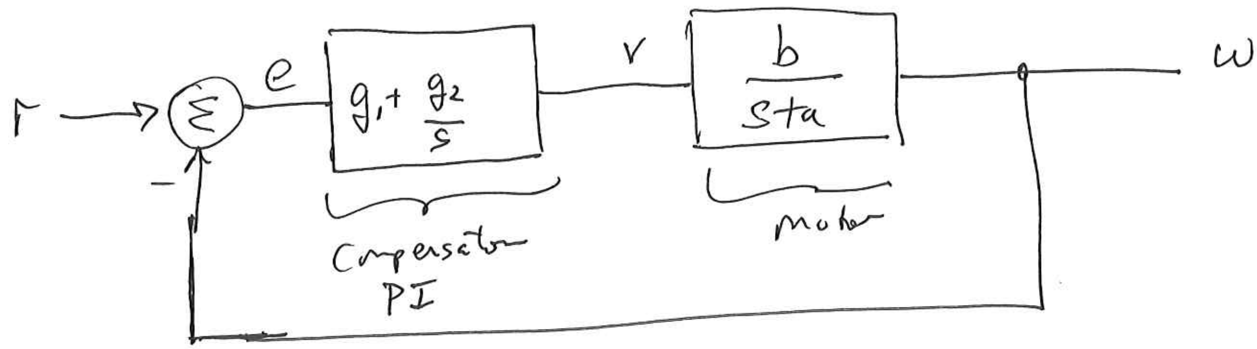
recall

$$T_{rise} = \frac{2.2}{-\text{pole value}}$$

Damping coeff
close to 1

For the motor control can we have good speed and have steady state error of zero?

Try to go to Proportional + integral compensator control.
PI - control



$$H_c(s) = g_1 + \frac{g_2}{s} = \frac{g_1 s + g_2}{s} = \frac{g_1\left(s + g_2/g_1\right)}{s}$$

$$H_c(s) = \frac{G\,(s+I)}{s} \quad \text{notation } \{G, I\}$$

$$H_{CL}(s) = \frac{\Omega(s)}{R(s)} = \frac{G(s+I)/s \quad b/(s+a)}{1 + \frac{G(s+I)}{s}\,\frac{b}{s+a}}$$

$$H_{CL}(s) = \frac{Gb(s+I)}{s^2 + sa + Gs + GIb}$$

Denominator $\quad D(s) = s^2 + s(a+G) + GIb$

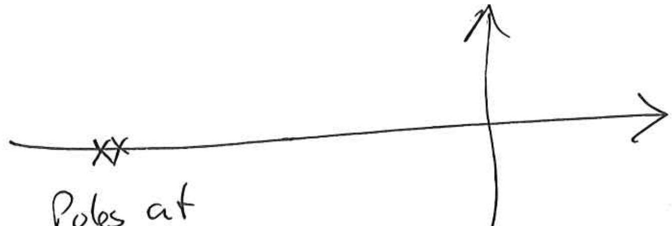
Poles $\Rightarrow -\frac{a+G}{2} \pm j\sqrt{GIb - \left(\frac{a+G}{2}\right)^2}$

Given a, b set real part of pole
ie set desired speed of control loop

$-\frac{a+G}{2} \rightarrow$ desired real part.

Then set I for a damping value of 1, ie both poles on real axis

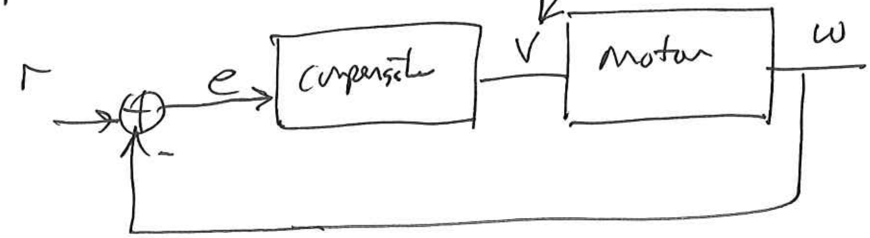
$$I = \left(\frac{a+G}{2}\right)^2 \frac{1}{Gb}$$



Poles at $-\frac{a+G}{2}$

What is price to be paid for a fast closed loop pole?

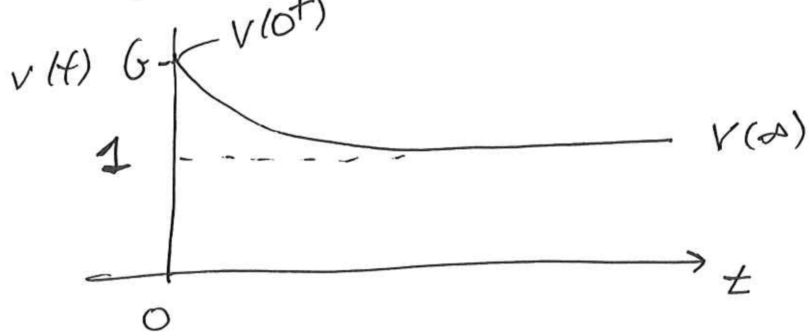
v(t) gets much larger when poles are moved.



$$H_{vr}(s) = \frac{V(s)}{R(s)} = \frac{\frac{G(s+I)}{s}}{1 + \frac{G(s+I)}{s}\left(\frac{b}{s+a}\right)}$$

$$H_{vr}(s) = \frac{G(s+I)(s+a)}{s^2 + sa + Gs + GIb}$$

Instead of trying to determine step response of v(t) directly use approximation.



$$v(0^+) = \lim_{s \to \infty} s\, H_{vr}(s) \underbrace{\frac{1}{s}}_{R(s)} = \frac{G}{1}$$

$$V(\infty) = \lim_{s \to 0} s\, H_{vr}(s) \frac{1}{s} = \frac{GIa}{GIa} = 1$$

To get a faster motor response we have to increase G. When G is increased so is the initial drive voltage to the motor.

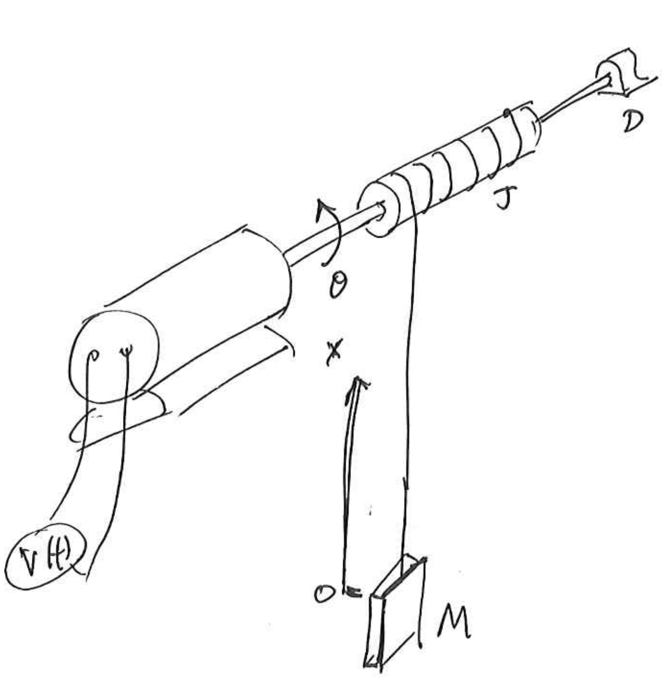
motor has to handle larger $v(t)$

compensator has to deliver larger current.

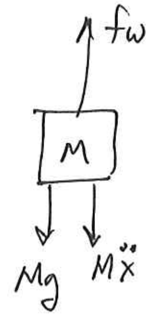
This is typical of control systems

To move the closed loop poles more drive signal is needed.

In this example drive signal is $v(t)$.



control loop for elevator



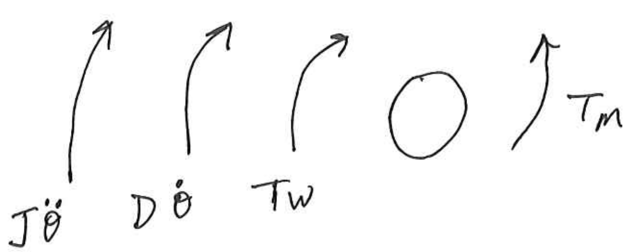
$$f_w = Mg + M\ddot{x}$$

$$T_w = r f_w$$

$$= rMg + Mr\ddot{x}$$

$$r\theta = x$$

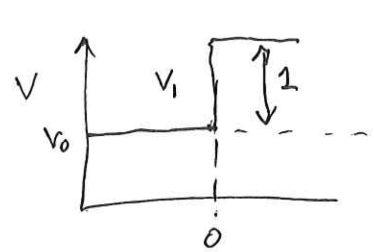
$$T_w = rMg + Mr^2\ddot{\theta}$$



$$T_M = K_T \frac{V}{R} - \frac{K_T K_b}{R} \omega$$

$$\underline{\frac{K_T}{R} V} - \frac{K_T k_b}{R}\dot{\theta} = J\ddot{\theta} + D\dot{\theta} + \underline{rMg} + Mr^2\ddot{\theta}$$

$$V(t) = V_0 + V_1(t)$$



$$\frac{rMgR}{K_T}$$

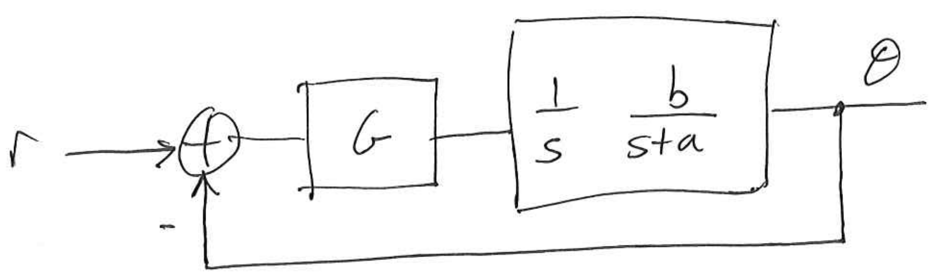
$$\frac{K_T}{R} V_1(t) = (J + Mr^2)\ddot{\theta} + \left(D + \frac{K_T k_b}{R}\right)\dot{\theta}$$

$$\frac{\Theta}{V_1} = \frac{\frac{K_T}{R}}{(Mr^2+J)s^2 + \left(D+\frac{K_T K_b}{R}\right)s}$$

$$a = \frac{D+\frac{K_T K_b}{R}}{Mr^2+J}, \quad b = \frac{K_T/R}{Mr^2+J}$$

$$\frac{\Theta}{V_1} = \frac{b}{s(s+a)}$$

Position Control loop

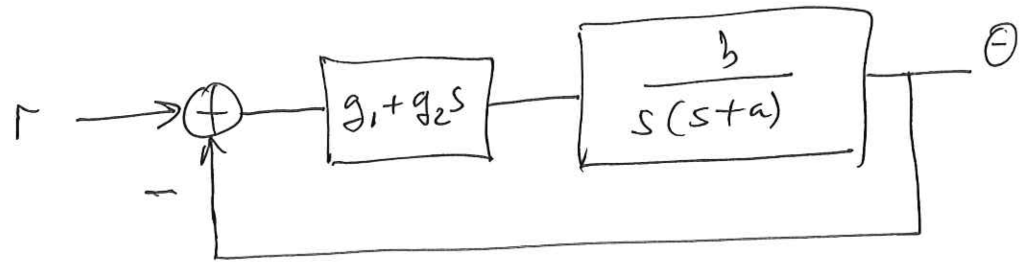


$$\frac{\Theta}{R} = H_{CL} = \frac{\frac{Gb}{s(s+a)}}{1+\frac{Gb}{s(s+a)}} = \frac{Gb}{s^2+as+Gb}$$

$$H_{CL}(0) = 0$$

Poles $\quad -\frac{a}{2} \pm j\sqrt{ab - \left(\frac{a}{2}\right)^2}$

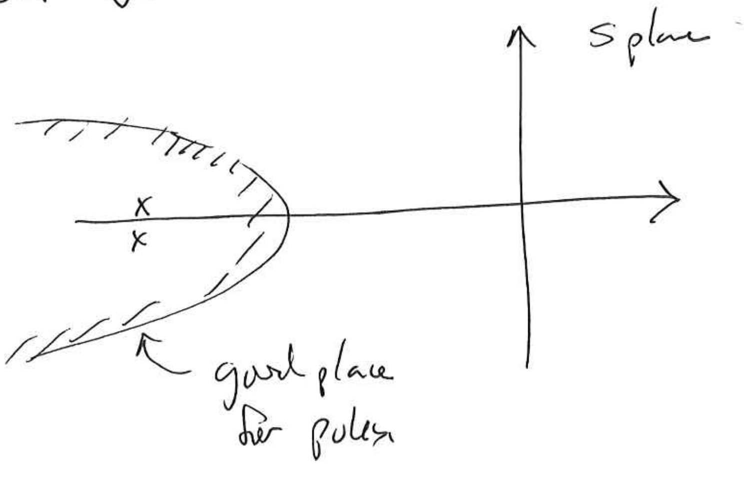
How to speed it up?



$$\frac{\Theta}{R} = \frac{(g_1+g_2 s)\, b}{s^2+sa + g_1 b + g_2 bs} = \frac{(g_1+g_2 s)\, b}{s^2+s(a+g_2 b)+g_1 b}$$

$$\text{Poles} = \frac{a+g_2}{2} \pm j\sqrt{g_1 b - \left(\frac{a+g_2 b}{2}\right)^2}$$

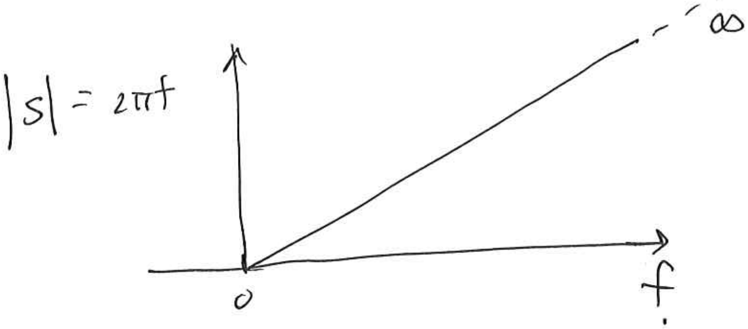
Set g_2 increase speed.
set g_1 for nice pole position



One small problem

Can not realize $g_1 + g_2 s$

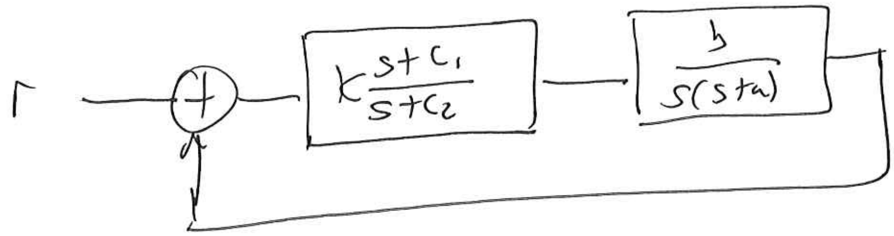
$P + D$

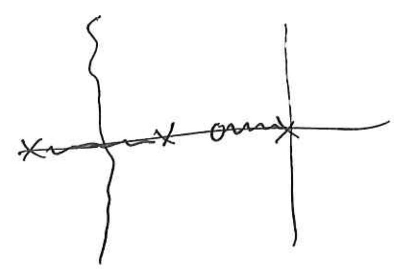


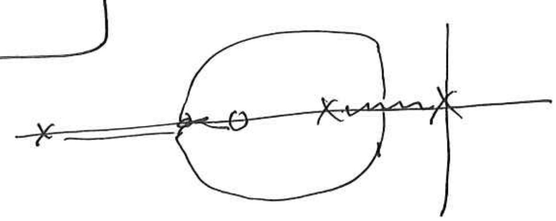
reality $\quad g_1 + \dfrac{g_2 s}{g_3 s + 1} \approx g_1 + g_2 s \qquad |s| \ll \dfrac{1}{g_3}$

$$\frac{g_1 g_3 s + g_1 + g_2 s}{g_3 s + 1} = K\left(\frac{s + c_1}{s + c_2}\right)$$

$K, c_1, c_2,$ selected to make approximate PD

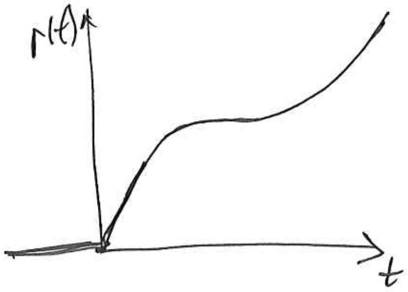




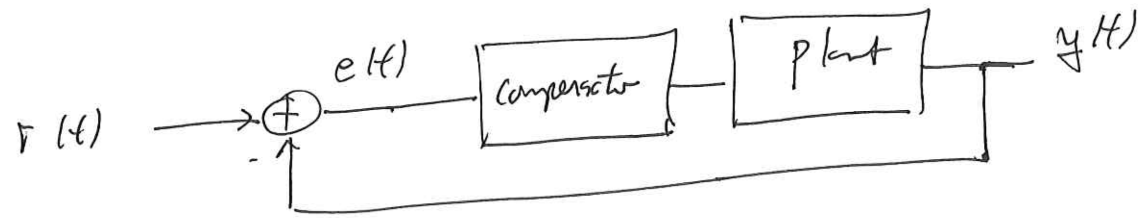


Need better tools

Steady state errors



$$r(t) = (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots) u(t)$$



How well can $y(t)$ track $r(t)$.

Consider each component of $r(t)$ separately.

Note when $e(\infty) = 0$

ie

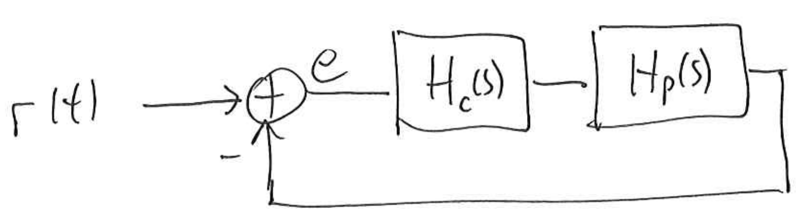
$r(t) = u(t)$ $\quad e(\infty) = 0$

$r(t) = t u(t)$ $\quad e(\infty) = 0$

$r(t) = t^2 u(t)$ $\quad e(\infty) = \text{finite}$

$r(t) = t^3 u(t)$ $\quad e(\infty) = \infty$

$\downarrow$ higher order of $r(t)$



$$H_{CL} = \frac{H_c H_p}{1 + H_c H_p}$$

$$H_{OL} = H_c H_p$$

$$H_e = \frac{1}{1 + H_c H_p}$$

$$H_{OL} = \frac{N(s)}{D(s)}$$

$$H_e = \frac{1}{1+\frac{N}{D}} = \frac{D}{D+N}$$

$$e(\infty) = \lim_{s \to 0} s \frac{D(s)}{D(s)+N(s)} R(s)$$

How many roots of $D(s)$ at $s=0$,

Loop order = no of integrators in $H_{OL}(s)$

$$H_{OL} = \frac{b}{s+a}, \quad R(s) = \frac{1}{s}$$

$$e(\infty) = \lim_{s \to 0} s \frac{1}{s} \frac{s+a}{s+a+b} = \frac{a}{b}$$ finite steady state error

loop type is 0 as there are no roots at $s=0$

$$H_{OL} = \frac{b}{s+a} \quad R(s) = \frac{1}{s^2}$$

$$e(\infty) = \lim_{s \to 0} \frac{1}{s} \frac{s+a}{s+a+b} = \infty$$

Loop type = 1 $\quad H_{OL} = \frac{b}{s(s+a)} \quad H_e = \frac{s(s+a)}{s(s+a)+b}$

$r = u(t) \quad e(\infty) = 0$

$r = t\,u(t)$ $\quad R = \frac{1}{s^2}$

$$e(\infty) = \lim_{s \to 0} s \frac{s(s+a)}{s(s+a)+b} \frac{1}{s^2} = \frac{a}{b} \quad \text{finite}$$

$r = t^2 u(t)$ $\quad R = \frac{2}{s^3}$

$$\lim_{s \to 0} s \frac{s(s+a)}{s(s+a)+b} \frac{2}{s^3} = \infty$$