ENEL 471 – Winter 2020

Assignment 4 – Solutions

Problem 3.12

a- The output of the square-law device is given by:

$$y(t) = s^2(t) = \frac{A_c^{\,2}}{2} m^2(t)\left[1 + \cos\left(4\pi f_c t\right)\right]$$

This signal has a component around 0 Hz frequency and a component around a carrier frequency equal to 2 f_c

b- The band-pass filter removes the component around 0 Hz. It only keeps a small portion of the spectrum around 2 f_c. the maximum amplitude of that spectrum is given by:

$$Y(2f_c) = \frac{A_c^2}{2}\int_{-\infty}^{\infty} M(\lambda)M(2f_c\text{-}\lambda)d\lambda$$

$$+\frac{A_c^2}{4}\left[\int_{-\infty}^{\infty} M(\lambda)M(\text{-}\lambda)d\lambda + \int_{-\infty}^{\infty} M(\lambda)M(4f_c\text{-}\lambda)d\lambda\right]$$

Since $M(\text{-}\lambda) = M^*(\lambda)$, we may write

$$Y(2f_c) = \frac{A_c^2}{2}\int_{-\infty}^{\infty} M(\lambda)M(2f_c\text{-}\lambda)d\lambda$$

$$+\frac{A_c^2}{4}\left[\int_{-\infty}^{\infty} |M(\lambda)|^2 d\lambda + \int_{-\infty}^{\infty} M(\lambda)M(4f_c-\lambda)d\lambda\right] \qquad (1)$$

With $m(t)$ limited to $-W \leq f \leq W$ and $f_c > W$, we find that the first and third integrals reduce to zero, and so we may simplify Eq. (1) as follows

$$Y(2f_c) = \frac{A_c^2}{4}\int_{-\infty}^{\infty} |M(\lambda)|^2 d\lambda$$

$$= \frac{A_c^2 E}{4}$$

where E is the signal energy (by Rayleigh's energy theorem). Similarly, we find that

$$Y(-2f_c) = \frac{A_c^2}{4}E$$

The band-pass filter output, in the frequency domain, is therefore defined by

$$V(f) \approx \frac{A_c^2}{4}E\Delta f[(\delta f - 2f_c) + \delta(f + 2f_c)]$$

Hence,

$$v(t) \approx \frac{A_c^2}{4}E\Delta f\cos(4\pi f_c t)$$

Problem 3.20

m(t) contains {100,200,400} Hz

(a)

At the transmitter:

- The DSB signal contains: {100 kHz +/– 100 Hz, 100 kHz +/– 200, 100 kHz +/– 400} or {99.9, 100.1, 99.8, 100.2, 99.6, and 100.4} kHz
- The SSB signal can be obtained from the DSB by only retaining the upper sidebands. Therefore, this signal contains: {100.1, 100.2, 100.4} kHz

At the receiver:

- The output of the product modulator in the coherent detector contains: {100.1 +/– 100.02, 100.2 +/– 100.02, 100.4 +/– 100.02} kHz or {80 Hz, 200.12 kHz, 180 Hz, 200.22 kHz, 380 Hz, and 200.42 kHz }
- The detector output contains only the low frequency components from the output of the product modulator, which are : {80 Hz, 180 Hz, and 380 Hz }

(b)

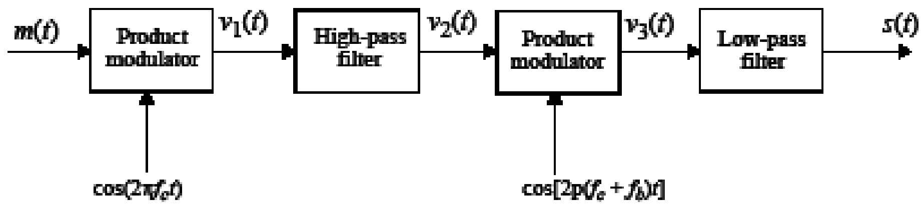
At the transmitter:

- The DSB signal contains: {100 kHz +/– 100 Hz, 100 kHz +/– 200, 100 kHz +/– 400} or {99.9, 100.1, 99.8, 100.2, 99.6, and 100.4} kHz
- The SSB signal can be obtained from the DSB by only retaining the lower sidebands. Therefore, this signal contains: {99.9, 99.8, 99.6} kHz

At the receiver:

- The output of the product modulator in the coherent detector contains: {100.02 +/– 99.9, 100.02 +/– 99.8, 100.02 +/– 99.6} kHz or {120 Hz, 199.92 kHz, 220 Hz, 199.82 kHz, 420 Hz, and 199.62 kHz }
- The detector output contains only the low frequency components from the output of the product modulator, which are : {120 Hz, 220 Hz, and 420 Hz }

Problem 3.21



(a) The first product modulator output is

$$v_1(t) = m(t)\cos(2\pi f_c t)$$

The second product modulator output is

$$v_3(t) = v_2(t)\cos[2\pi(f_c + f_b)t]$$

The amplitude spectra of $m(t)$, $v_1(t)$, $v_2(t)$, $v_3(t)$ and $s(t)$ are illustrated on the next page:
We may express the voice signal $m(t)$ as

$$m(t) = \frac{1}{2}[m_+(t) + m_-(t)]$$

where $m_+(t)$ is the pre-envelope of $m(t)$, and $m_-(t) = m_+^*(t)$ is its complex conjugate. The Fourier transforms of $m_+(t)$ and $m_-(t)$ are defined by (See Appendix 2)

$$M_+(f) = \left\{ \begin{array}{ll} 2M(f), & f > 0 \\ 0, & f < 0 \end{array} \right\}$$

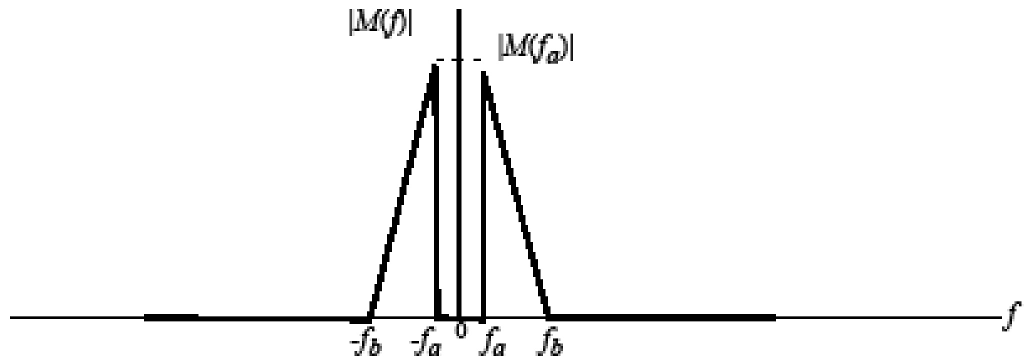
$$M_-(f) = \left\{ \begin{array}{ll} 0, & f > 0 \\ 2M(f), & f < 0 \end{array} \right\}$$

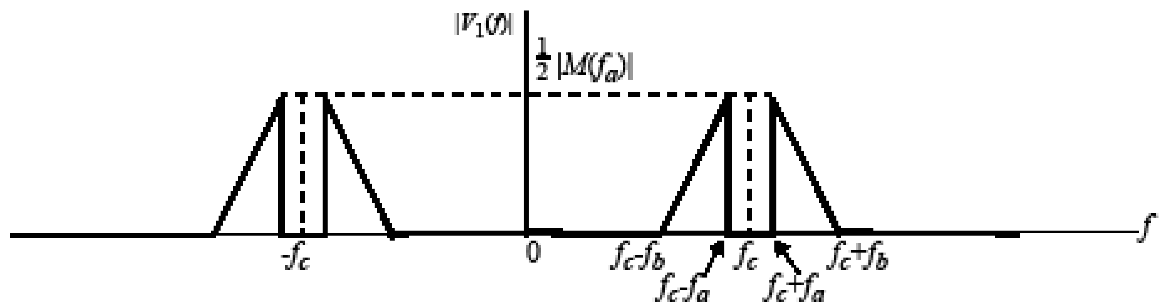
Comparing the spectrum of $s(t)$ with that of $m(t)$, we see that $s(t)$ may be expressed in terms of $m_+(t)$ and $m_-(t)$ as follows:

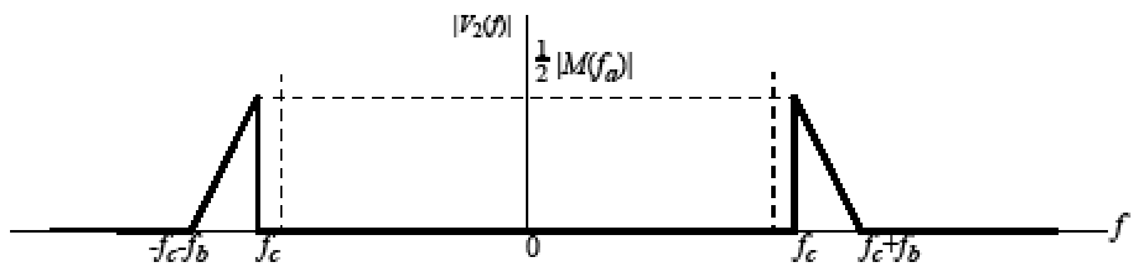
$$\begin{aligned} s(t) &= \frac{1}{8}m_+(t)\exp(-j2\pi f_b t) + \frac{1}{8}m_-(t)\exp(j2\pi f_b t) \\ &= \frac{1}{8}[m(t) + j\hat{m}(t)]\exp(-j2\pi f_b t) + \frac{1}{8}[m(t) - j\hat{m}(t)]\exp(j2\pi f_b t) \\ &= \frac{1}{4}m(t)\cos(2\pi f_b t) + \frac{1}{4}\hat{m}(t)\sin(2\pi f_b t) \end{aligned}$$

(b) With $s(t)$ as input, the first product modulator output is

$$v_1(t) = s(t)\cos(2\pi f_c t)$$


|M(f)|
|M(fa)|
-fb
-fa
0
fa
fb
f


|V1(f)|
1/2 |M(fa)|
-fc
0
fc-fb
fc-fa
fc
fc+fa
fc+fb
f


|V2(f)|
1/2 |M(fa)|
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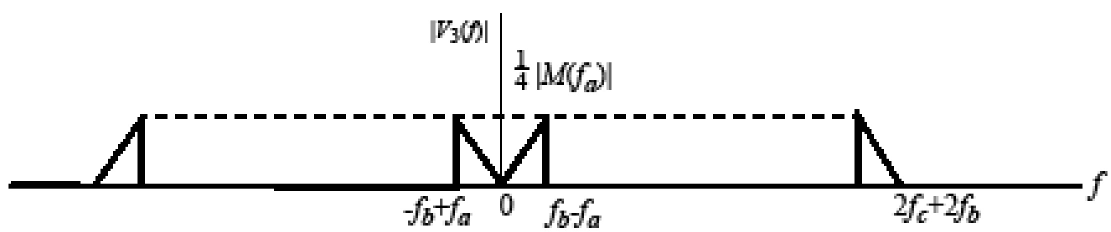
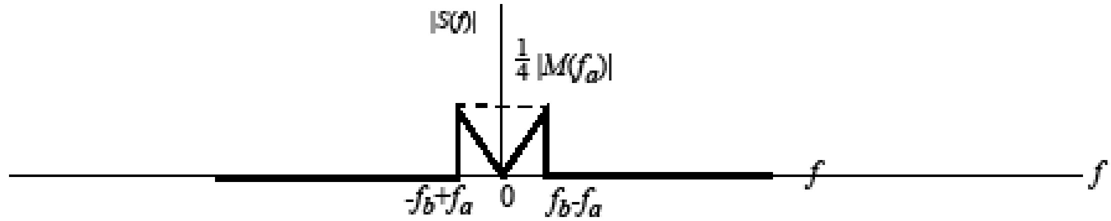
fc
0
fc
fc+fb
f


|V3(f)|
1/4 |M(fa)|
-fb+fa
0
fb-fa
2fc+2fb
f


|S(f)|
1/4 |M(fa)|
-fb+fa
0
fb-fa
f
f

Problem 3.22

$$f_1 = f_c - \Delta f - W$$
$$f_2 = f_c + \Delta f$$

$$v_1(t)v_2(t) = A_1A_2\cos(2\pi f_1 t + \phi_1)\cos(2\pi f_2 t + \phi_2)$$
$$= \frac{A_1A_2}{2}[\cos(2\pi(f_1 - f_2)t + \phi_1 - \phi_2) + \cos(2\pi(f_1 + f_2)t + \phi_1 + \phi_2)]$$

The low-pass filter will only pass the first term.

$$\therefore LFP(v_1(t)v_2(t)) = \frac{1}{2}A_1A_2[\cos(-2\pi(W + 2\Delta f)t + \phi_1 - \phi_2)]$$

Let $v_0(t)$ be the final output, before band-pass filtering.

$$v_o(t) = \frac{1}{2}A_1A_2[\cos(-2\pi\left(\frac{W + 2\Delta f}{W/\Delta f + 2}\right)t + \frac{\phi_1 - \phi_2}{W/\Delta f + 2}) \cdot A_2\cos(2\pi f_2 t + \phi_2)]$$
$$= \frac{1}{2}A_1A_2^2[\cos(-2\pi\Delta f t + \frac{\phi_1 - \phi_2}{n+2} - \phi_2) \cdot \cos(2\pi f_2 t + \frac{\phi_1 - \phi_2}{n+2} + \phi_2)]$$
$$= \frac{1}{4}A_1A_2^2[\cos(-2\pi(f_c + 2\Delta f) + \frac{\phi_1 - \phi_2}{n+2} - \phi_2) + \cos(-2\pi f_c t + \frac{\phi_1 - \phi_2}{n+2} + \phi_2)]$$

After band-pass filtering, retain only the second term.

$$\therefore v_o(t) = \frac{1}{4}A_1A_2^2[\cos(-2\pi f_c t + \frac{\phi_1 - \phi_2}{n+2} + \phi_2)$$

$$\frac{\phi_1}{n+2} - \frac{\phi_2}{n+2} + \phi_2 = 0$$

rearranging and solving for ϕ_2 :

$$\phi_2 = -\frac{\phi_1}{n+1}$$

(b) At the second multiplier, replace $v_2(t)$ with $v_1(t)$. This results in the following expression for the phase:

$$\frac{\phi_1}{n+2} - \frac{\phi_2}{n+2} + \phi_1 = 0$$

$$\phi_1 = \frac{\phi_2}{n+3}$$

Additional Problems

Problem 1

$$u(t) = m(t)c(t) = A(\text{sinc}(t) + \text{sinc}^2(t))\cos(2\pi f_c t)$$

Taking the Fourier transform of both sides, we obtain

$$\begin{aligned} U(f) &= \frac{A}{2}[\Pi(f) + \Lambda(f)] \star (\delta(f - f_c) + \delta(f + f_c)) \\ &= \frac{A}{2}[\Pi(f - f_c) + \Lambda(f - f_c) + \Pi(f + f_c) + \Lambda(f + f_c)] \end{aligned}$$

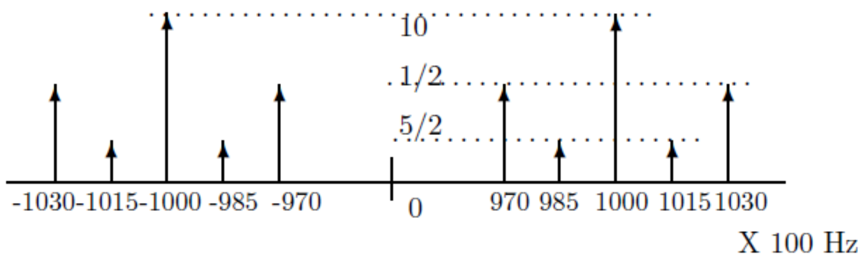
$\Pi(f - f_c) \neq 0$ for $|f - f_c| < \frac{1}{2}$, whereas $\Lambda(f - f_c) \neq 0$ for $|f - f_c| < 1$. Hence, the bandwidth of the bandpass filter is 2.

Problem 2

1) The spectrum of $u(t)$ is

$$
\begin{aligned}
U(f) &= \frac{20}{2}\left[\delta(f-f_c)+\delta(f+f_c)\right] \\
&+\frac{2}{4}\left[\delta(f-f_c-1500)+\delta(f-f_c+1500)\right. \\
&\left.+\delta(f+f_c-1500)+\delta(f+f_c+1500)\right] \\
&+\frac{10}{4}\left[\delta(f-f_c-3000)+\delta(f-f_c+3000)\right. \\
&\left.+\delta(f+f_c-3000)+\delta(f+f_c+3000)\right]
\end{aligned}
$$

The next figure depicts the spectrum of $u(t)$.



2) The square of the modulated signal is

$$
\begin{aligned}
u^2(t) &= 400\cos^2(2\pi f_c t)+\cos^2(2\pi(f_c-1500)t)+\cos^2(2\pi(f_c+1500)t) \\
&+25\cos^2(2\pi(f_c-3000)t)+25\cos^2(2\pi(f_c+3000)t) \\
&+\text{ terms that are multiples of cosines}
\end{aligned}
$$

If we integrate $u^2(t)$ from $-\frac{T}{2}$ to $\frac{T}{2}$, normalize the integral by $\frac{1}{T}$ and take the limit as $T \to \infty$, then all the terms involving cosines tend to zero, whereas the squares of the cosines give a value of $\frac{1}{2}$. Hence, the power content at the frequency $f_c = 10^5$ Hz is $P_{f_c} = \frac{400}{2} = 200$, the power content at the frequency P_{f_c+1500} is the same as the power content at the frequency P_{f_c-1500} and equal to $\frac{1}{2}$, whereas $P_{f_c+3000} = P_{f_c-3000} = \frac{25}{2}$.

3)

$$\begin{aligned} u(t) &= (20 + 2\cos(2\pi 1500 t) + 10\cos(2\pi 3000 t))\cos(2\pi f_c t) \\ &= 20(1 + \frac{1}{10}\cos(2\pi 1500 t) + \frac{1}{2}\cos(2\pi 3000 t))\cos(2\pi f_c t) \end{aligned}$$

This is the form of a conventional AM signal with message signal

$$\begin{aligned} m(t) &= \frac{1}{10}\cos(2\pi 1500 t) + \frac{1}{2}\cos(2\pi 3000 t) \\ &= \cos^2(2\pi 1500 t) + \frac{1}{10}\cos(2\pi 1500 t) - \frac{1}{2} \end{aligned}$$

The minimum of $g(z) = z^2 + \frac{1}{10}z - \frac{1}{2}$ is achieved for $z = -\frac{1}{20}$ and it is $\min(g(z)) = -\frac{201}{400}$. Since $z = -\frac{1}{20}$ is in the range of $\cos(2\pi 1500 t)$, we conclude that the minimum value of $m(t)$ is $-\frac{201}{400}$. Hence, the modulation index is

$$\alpha = -\frac{201}{400}$$

4)

$$\begin{aligned} u(t) &= 20\cos(2\pi f_c t) + \cos(2\pi(f_c - 1500)t) + \cos(2\pi(f_c - 1500)t) \\ &= 5\cos(2\pi(f_c - 3000)t) + 5\cos(2\pi(f_c + 3000)t) \end{aligned}$$

The power in the sidebands is

$$P_{\text{sidebands}} = \frac{1}{2} + \frac{1}{2} + \frac{25}{2} + \frac{25}{2} = 26$$

The total power is $P_{\text{total}} = P_{\text{carrier}} + P_{\text{sidebands}} = 200 + 26 = 226$. The ratio of the sidebands power to the total power is

$$\frac{P_{\text{sidebands}}}{P_{\text{total}}} = \frac{26}{226}$$