Chapter IV – Angle Modulation

ENEL 471 – Introduction to Communications Systems and Networks

Chapter Objectives

- At the end of this chapter, you will be able to:
  - Define angle modulation and distinguish between phase modulation and frequency modulation
  - Analyze the time domain and frequency domain representations of frequency modulated signals
  - Perform and analyze the frequency demodulation of signals in the absence of noise
  - Perform and analyze the frequency demodulation of signals in the presence of channel noise

Outline

- Angle Modulation
  - Definition of angle modulation
  - Phase modulation and frequency modulation
  - Properties of angle modulation
- Frequency Modulation (FM)
  - Narrowband and wideband frequency modulation
  - Transmission bandwidth of FM signals
  - Generation of FM signals
- Frequency Demodulation
  - Demodulation of FM signals in the absence of noise
  - Demodulation of FM signals in the presence of noise

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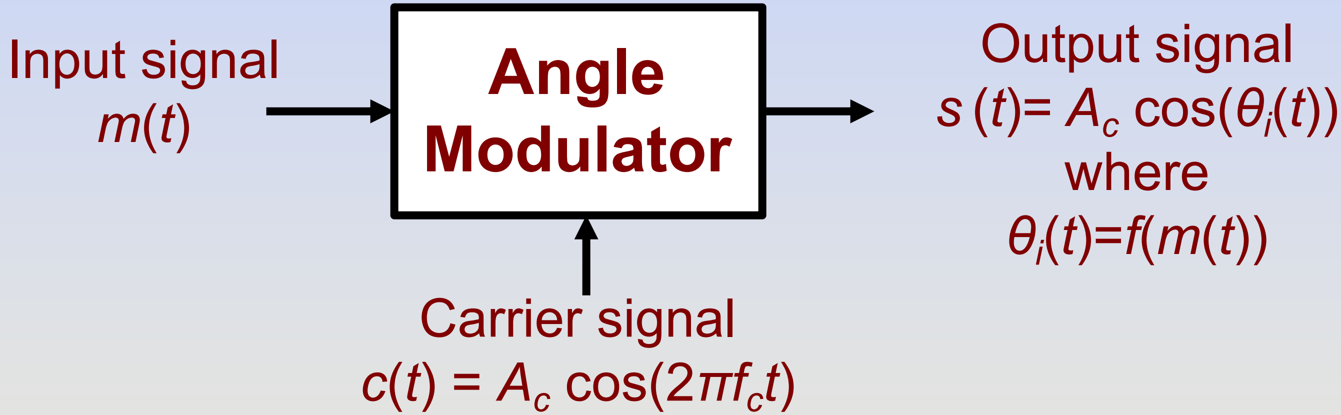
Definition of Angle Modulation



- For Angle modulation :
  - The angle of the carrier is the characteristic of the carrier that is varied in accordance with the modulating wave (signal)

$$s(t) = A_c \cdot \cos(\theta_i(t))$$

→ The information in the input signal is encoded in the angle of the carrier signal, that is: $\theta_i(t) = f(m(t))$

Instantaneous Frequency

- The angle modulation signal is given by:

$$s(t) = A_c \cdot \cos(\theta_i(t))$$

- In general, the angle can have the form:

$$\theta_i(t) = 2\pi f_i(t)t + \phi_i(t)$$

- The message can be encoded either in the phase (phase modulation – constant frequency) or frequency (frequency modulation – constant phase).
- If the message is encoded in the frequency, the angle is given by:

$$\theta_i(t) = 2\pi f_i(t)t + \phi_o$$

- *The instantaneous frequency* of the modulated signal is then given by:

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt}$$

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Phase Modulation (PM)

- In the case of phase modulation, the information is encoded in the phase. The frequency is constant. The angle is then given by:

$$\theta_i(t) = 2\pi f_c t + k_p m(t)$$

- The phase modulated signal is then given by:

$$s(t) = A_c \cos\left(2\pi f_c t + k_p m(t)\right)$$

- k_p is a constant called phase sensitivity used to scale the message signal.
- The instantaneous frequency of the phase modulated signal is:

$$f_i(t) = f_c + \frac{k_p}{2\pi}\frac{dm(t)}{dt}$$

it varies as a function of the derivative of the message signal

Frequency Modulation (FM)

- In the case of frequency modulation, the information is encoded in the frequency. The phase is zero. The instantaneous frequency is then given by:

$$f_i(t) = f_c + k_f m(t) = \frac{1}{2\pi} \frac{d\theta_i(t)}{dt}$$

  it varies as a function of the message signal

- The angle of the frequency modulated signal is then:

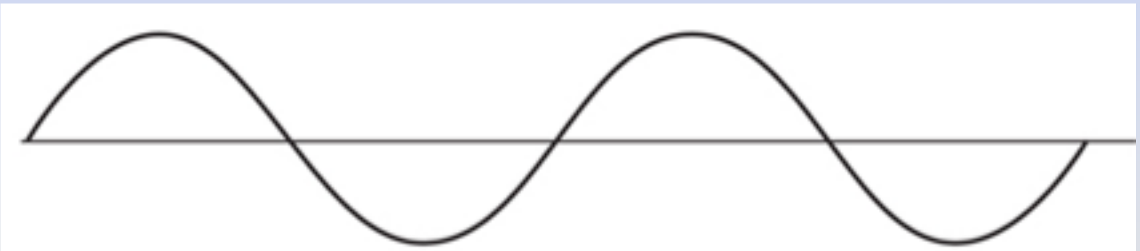
$$\theta_i(t) = 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau$$

- The phase modulated signal is then given by:

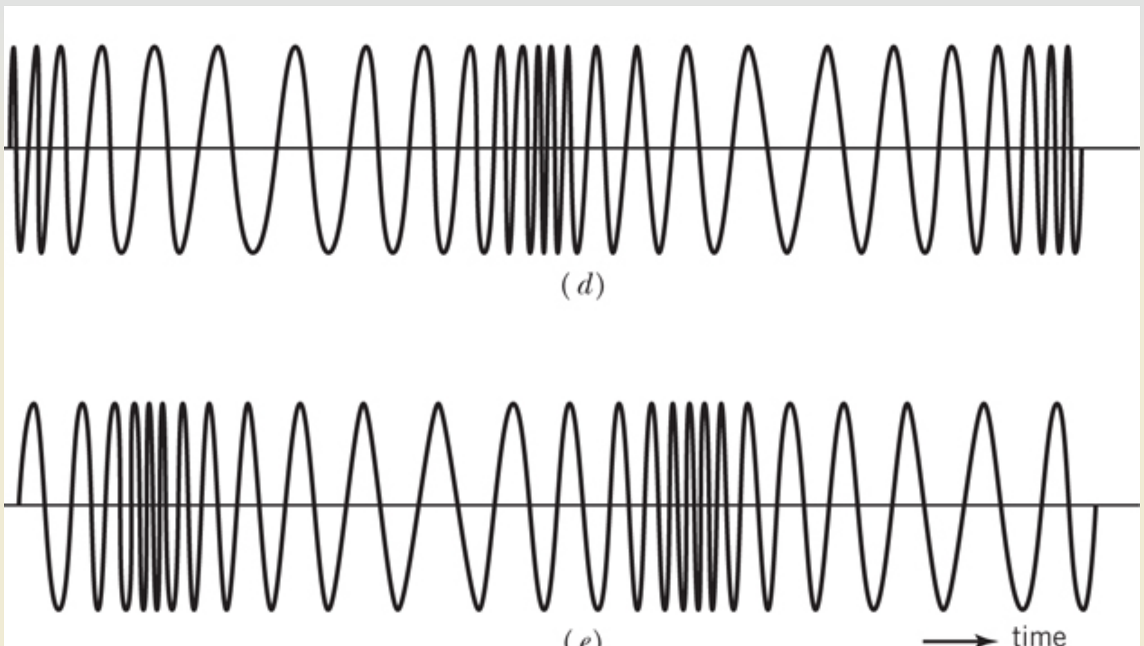
$$s(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau\right)$$

- k_f is a constant called frequency sensitivity used to scale the message signal.

Time Domain Analysis of FM and PM



Message signal
m(*t*)



PM signal
the instantaneous frequency
is a linear function of the
derivative of *m*(*t*)

FM signal
the instantaneous frequency
is a linear function of *m*(*t*)

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Transmitted Power in Angle Modulations

- For both, phase and frequency modulation, the amplitude of the modulated signal is a constant values equal to the amplitude of the carrier signal, A_c
- The amplitude is independent from the phase and frequency sensitivities

→ The average transmitted power in angle modulation signals is constant and is given by:

$$P_{av}(t) = \frac{A_c^{\,2}}{2}$$

Nonlinearity of the Modulation Process

- Contrary to amplitude modulation, angle modulation (phase of frequency) is a nonlinear operation
- If a message signal is composed by the summation of two message signals:

$$m(t) = m_1(t) + m_2(t)$$

the modulated signal (for example for phase modulation)

$$s(t) = A_c \cos\left(2\pi f_c t + k_p m(t)\right)$$

is different from the summation of the modulated signals of m_1 and m_2

If $\quad s_1(t) = A_c \cos\left(2\pi f_c t + k_p m_1(t)\right)$

and $\quad s_2(t) = A_c \cos\left(2\pi f_c t + k_p m_2(t)\right)$

we have $s(t) \neq s_1(t) + s_2(t)$

Irregularity of Zero-Crossing

- Zero-crossings: are defined as the instants of time at which a waveform changes the signal of its amplitude
- The angle modulation makes the carrier instantaneous frequency changes as a function of the message signal (linear function of the message for frequency modulation, and linear function of the derivative of the message for phase modulation)

→ The zero crossings in angle modulation are irregular and they include the information of the message signal.

Example – PM and FM for a Ramp Message

- Consider a message signal defined by:

$$m(t) = \begin{cases} t & ; t > 0 \\ 0 & ; t < 0 \end{cases}$$

- This message signal is modulated with angle modulation using a carrier of amplitude 1 and frequency of ¼ Hz.
- Analyze, in time-domain, the phase modulated signal with sensitivity pi/2
- Analyze, in time-domain, the frequency modulated signal with sensitivity 1
- What can you conclude about the zero crossings?

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Frequency Modulation

- Frequency modulation time-domain expression

$$s(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\, d\tau\right)$$

- Instantaneous frequency is given by:

$$f_i(t) = f_c + k_f m(t)$$

- The instantaneous frequency is varying as a function of the amplitude of the signal $m(t)$.
- The bandwidth of the FM signal may be different from the bandwidth of $m(t)$
- The spectrum is not simply the spectrum of $m(t)$ translated in frequency. It is more difficult to analyze.
- To simplify the analysis, we start with the simplest case: $m(t)$ is a cosine wave.

Frequency Modulation for a Cosine-wave

- The message is a cosine wave:

$$m(t) = A_m \cos(2\pi f_m t)$$

- Instantaneous frequency is given by:

$$f_i(t) = f_c + \underbrace{k_f A_m}_{=\Delta f} \cos(2\pi f_m t) = f_c + \Delta f \cdot \cos(2\pi f_m t)$$

- The instantaneous frequency will vary between $f_c - \Delta f$ and $f_c + \Delta f$
- Δf is called the **frequency deviation**
- The instantaneous phase is given by:

$$\theta_i(t) = 2\pi \int_0^t f_i(\tau) d\tau = 2\pi f_c t + \underbrace{\frac{\Delta f}{f_m}}_{=\beta} \sin(2\pi f_m t)$$

- β is called the **modulation index** or **phase deviation**
- It is the ratio between the maximum frequency deviation of the FM signal and the bandwidth of the message signal

Types of Frequency Modulation

- The frequency modulation can be categorized as a function of the value of the modulation index in two different categories:

$$\beta < 1$$

  - <u>Narrowband frequency modulation</u>: $\beta < 1$

  the frequency deviation is smaller than the bandwidth of the message signal

  - <u>Wideband frequency modulation</u>: $\beta > 1$

  The frequency deviation is greater than the bandwidth of the message signal

Narrow-Band Frequency Modulation

- In general, for a cosinusoidal message signal

$$m(t) = A_m \cos(2\pi f_m t)$$

the modulated signal is given by:

$$s(t) = A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_m t)\right)$$

- Using trigonometric identities, the modulated signal can be written as:

$$s(t) = A_c \cos(2\pi f_c t)\cos\left(\beta \sin(2\pi f_m t)\right) - A_c \sin(2\pi f_c t)\sin\left(\beta \sin(2\pi f_m t)\right)$$

- For narrowband frequency modulation ($\beta < 1$), we can make these approximations:

$$\cos\left(\beta \sin(2\pi f_m t)\right) \approx 1 \qquad \textit{and} \qquad \sin\left(\beta \sin(2\pi f_m t)\right) \approx \beta \sin(2\pi f_m t)$$

- Then the modulated signal can be expressed as:

$$s(t) \approx A_c \cos(2\pi f_c t) - A_c \beta \sin(2\pi f_c t)\sin(2\pi f_m t)$$

or $$s(t) \approx A_c \cos(2\pi f_c t) + \frac{A_c \beta}{2}\left[\cos\left(2\pi (f_c + f_m) t\right) - \cos\left(2\pi (f_c - f_m) t\right)\right]$$

Narrow-Band Frequency Modulation

$$s(t) \approx A_c \cos(2\pi f_c t) + \frac{A_c \beta}{2}\left[\cos(2\pi(f_c + f_m)t) - \cos(2\pi(f_c - f_m)t)\right]$$

- The time-domain representation above of the FM modulated signal in the case of narrow-band frequency modulation shows that:
  - The FM signal expression is similar to the conventional AM expression
  - The only difference is that the sign of the upper sideband and lower sideband are opposite in the case of narrow-band FM modulation
  - The bandwidth of the modulated narrow-band FM signal is equal to the message signal bandwidth and is independent from the frequency deviation

Example - Narrow-Band Frequency Modulation

- A sinusoidal modulating wave

$$m(t) = 2\cos(2000\pi t)$$

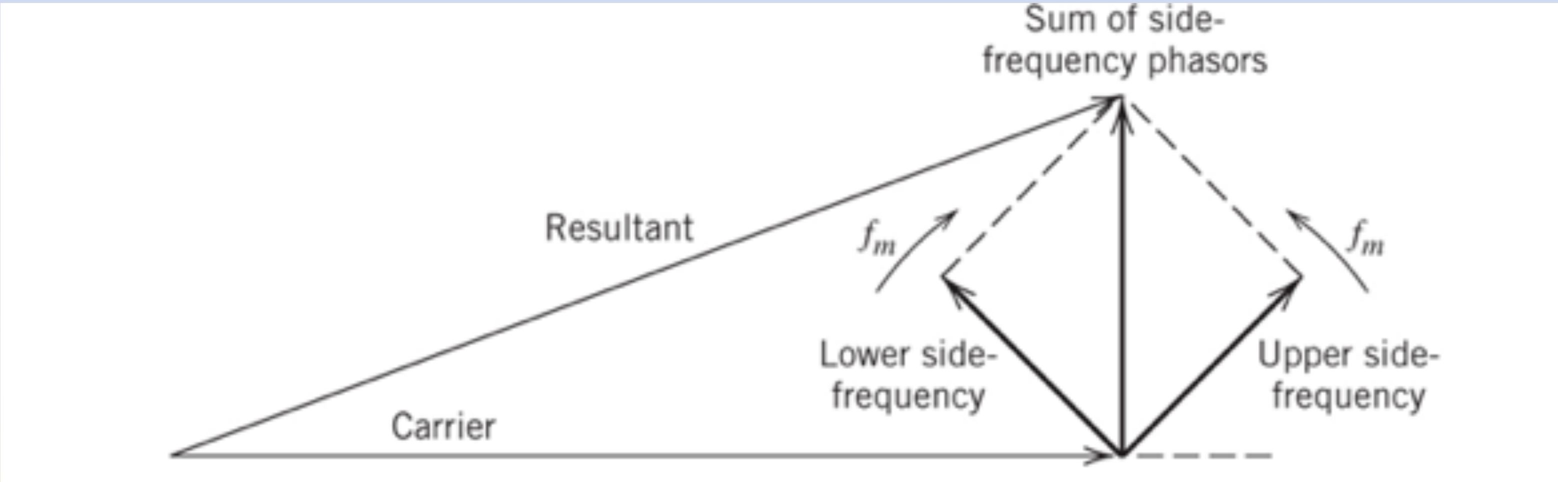
Is applied to a frequency modulator with frequency sensitivity k_f and. The modulating carrier wav has a frequency fc = 100 kHz and amplitude Ac = 1.

(a) Determine the spectrum of the resulting frequency-modulated signal, assuming that the maximum phase deviation is 0.1 radians

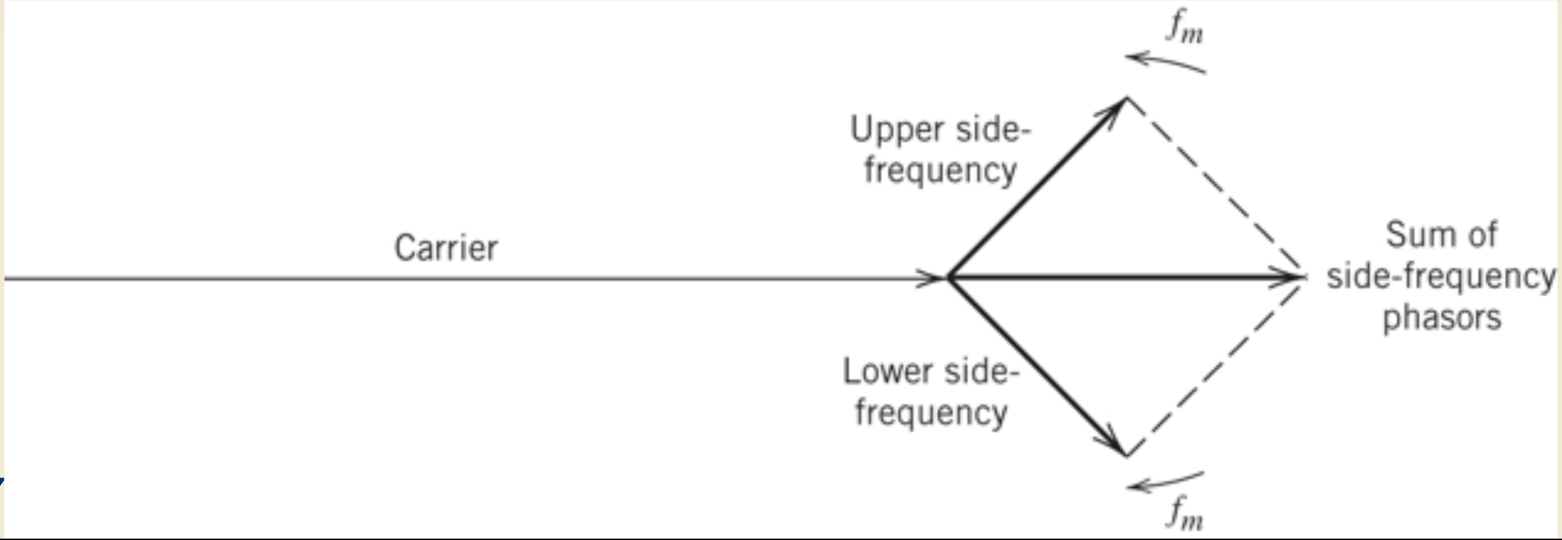
(b) Construct a phasor diagram for this modulated signal

Narrow-Band Frequency Modulation

- Phasor Representation of a narrow-band FM signal



- Phasor representation of a conventional AM signal



Wide-Band Frequency Modulation

- In general, for a cosinusoidal message signal

$$m(t) = A_m \cos(2\pi f_m t)$$

the modulated signal is given by:

$$s(t) = A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_m t)\right) = \mathrm{Re}\left[A_c\, e^{j\left(2\pi f_c t + \beta \sin(2\pi f_m t)\right)}\right]$$

or :

$$s(t) = \mathrm{Re}\Big[\underbrace{e^{j2\pi f_c t}}_{\text{carrier}}\ \underbrace{A_c\, e^{j\beta \sin(2\pi f_m t)}}_{\text{complex envelope, } \tilde{s}(t)}\Big]$$

- $\tilde{s}(t) = A_c\, e^{j\beta \sin(2\pi f_m t)}$ is periodic with period $1/f_m$
- Using the Fourier Series representation, we can write:

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{j2\pi n f_m t}$$

where: $C_n = f_m \int_{\frac{-1}{2f_m}}^{\frac{1}{2f_m}} \tilde{s}(t)\, e^{-j2\pi n f_m t}\, dt$

Wide-Band Frequency Modulation

- The Fourier series coefficients are given by:

$$C_n = f_m \int_{\frac{-1}{2f_m}}^{\frac{1}{2f_m}} s(t)\, e^{-j2\pi n f_m t}\, dt = f_m \int_{\frac{-1}{2f_m}}^{\frac{1}{2f_m}} A_c\, e^{j\beta \sin(2\pi f_m t)}\, e^{-j2\pi n f_m t}\, dt$$

- Let $x = 2\pi f_m t$, the Fourier series coefficients can then be written as:

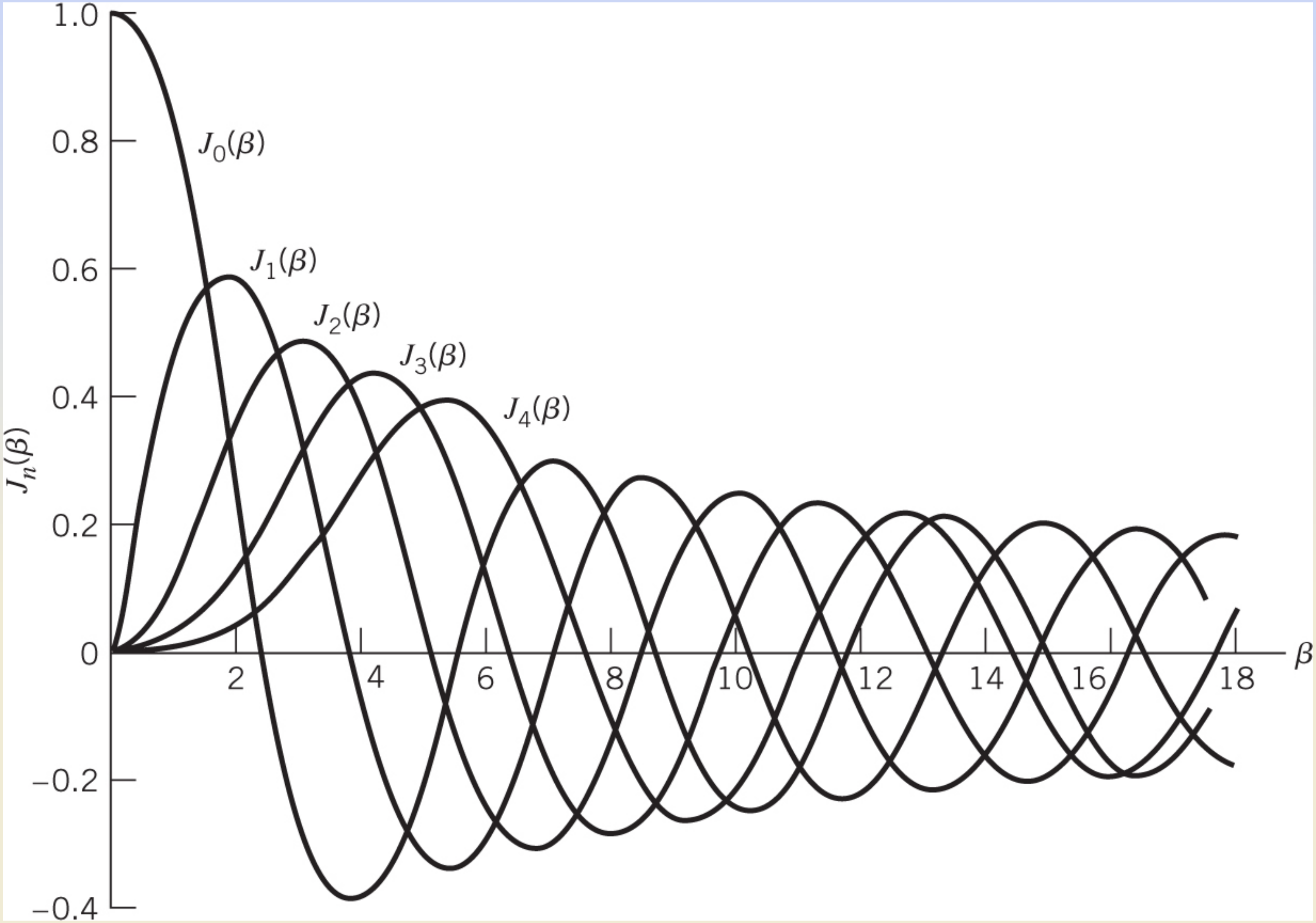
$$C_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin(x) - nx)}\, dx = A_c \cdot J_n(\beta)$$

where:

$$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin(x) - nx)}\, dx$$

is called the nth order Bessel function of the first kind, which can be proven to have real values.

Bessel Functions of the First Kind



Wide-Band Frequency Modulation

- The complex envelope can then be expressed as:

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta) e^{j2\pi n f_m t}$$

- Finally, one can write the FM modulated signal as:

$$s(t) = \mathrm{Re}\left[e^{j2\pi f_c t} \sum_{n=-\infty}^{\infty} A_c J_n(\beta) e^{j2\pi n f_m t} \right] = \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} A_c J_n(\beta) e^{j2\pi (f_c + n f_m) t} \right]$$

or simply:

$$s(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta) \cos\left(2\pi (f_c + n f_m) t\right)$$

- The spectrum of the wideband FM signal :
  - Has infinite bandwidth
  - Has impulse functions at $\pm(f_c + n f_m)$ with varying amplitudes

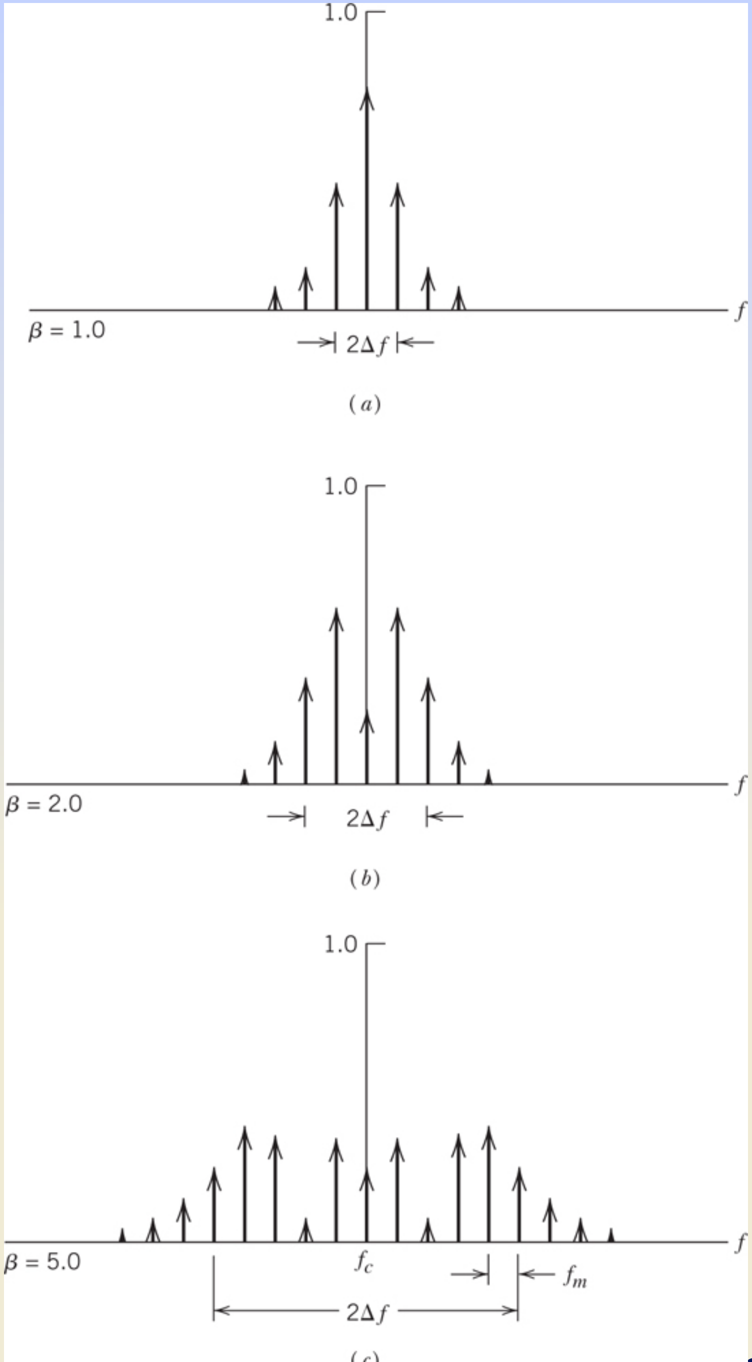
$$S(f) = \sum_{n=-\infty}^{\infty} \frac{A_c}{2} J_n(\beta) \left[\delta\left(f + (f_c + n f_m)\right) + \delta\left(f - (f_c + n f_m)\right) \right]$$

Examples of Spectra of FM Signals

- $m(t) = A_m \cos(2\pi f_m t)$,

  therefore: $$\beta = \frac{k_f A_m}{f_m}$$

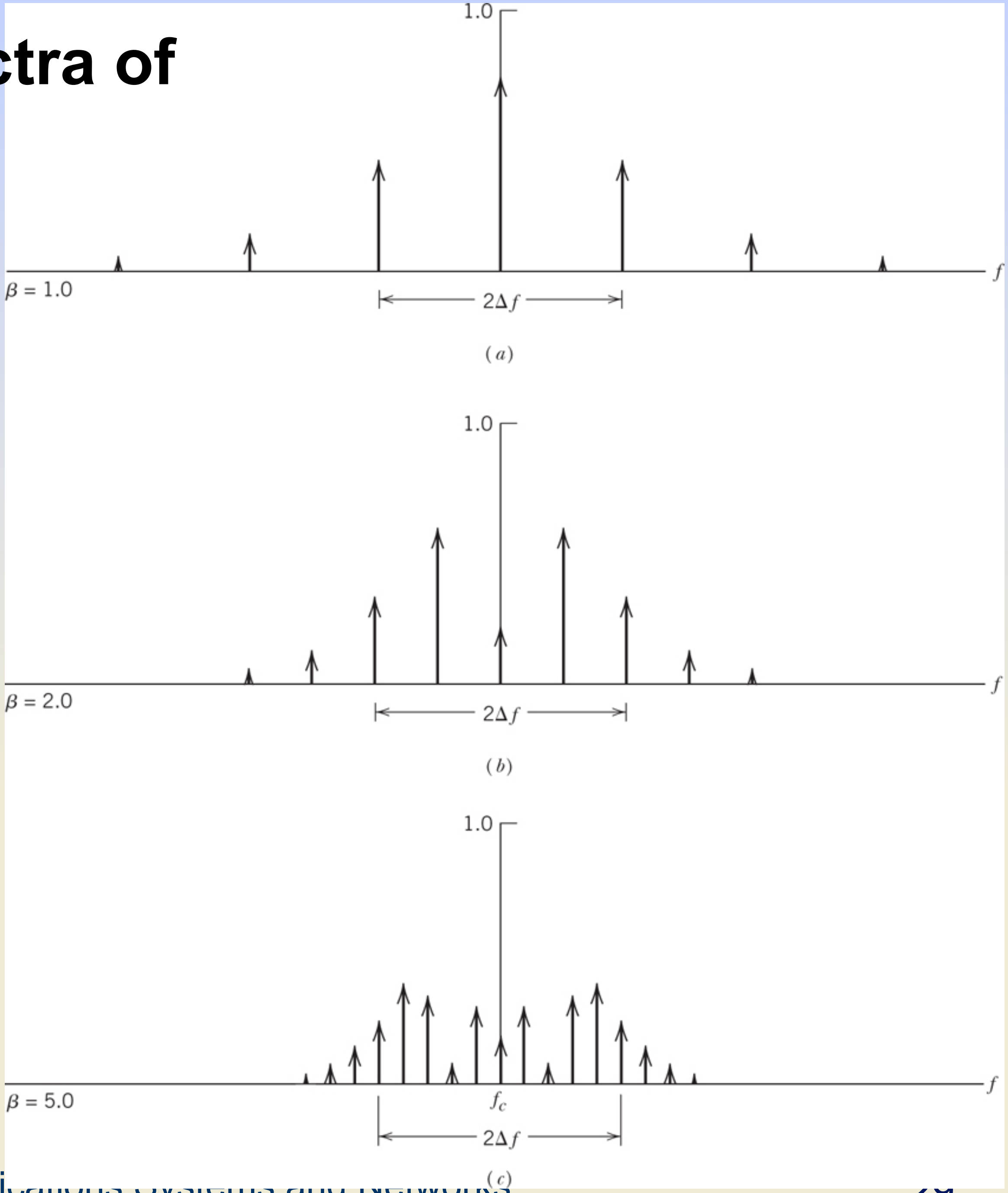
- **Case 1**: β is varied by changing A_m and keeping f_m constant



Examples of Spectra of FM Signals

- $m(t) = A_m \cos(2\pi f_m t)$, therefore: $\beta = \dfrac{k_f A_m}{f_m}$

- **Case 2**: β is varied by changing f_m and keeping A_m constant



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Carson's Rule

- If β is small, then the transmission bandwidth is:

$$B_T = 2f_m$$

- If β is large, then the transmission bandwidth can be approximated by:

$$B_T \approx 2f_m + 2\Delta f = 2\Delta f\left(1 + \frac{1}{\beta}\right)$$

Carson's rule

It is also valid for small values of β

Carson's Rule – Example

- An FM signal is generated using a sinewave modulated signal given by:

$$m(t) = 2\cos(2\pi f_m t)$$

where f_m = 10 kHz and the frequency sensitivity k_f = 10000

a- Calculate the maximum frequency deviation of the FM signal.

b- Calculate the modulation index

c- Use Carson's rule to estimate the transmission bandwidth of the FM signal.

d- Determine the amplitude spectrum of the modulated signal

1 Percent Bandwidth

- Carson's rule provide an approximation of the bandwidth without précising any condition on how to calculate it.
- One can define the transmission bandwidth of an FM wave as :

  The separation between the two frequencies beyond which none of the side frequencies is greater than 1 percent of the carrier amplitude obtained when the modulation is removed.

- When there is no modulation ($\beta = 0$) , the amplitude of the carrier is provided by:

$$A_c \cdot J_0(0) = A_c \cdot 1 = A_c$$

- If we define :

$$n_{\max} = \max_{|J_n(\beta)| > 0.01} \{n\}$$

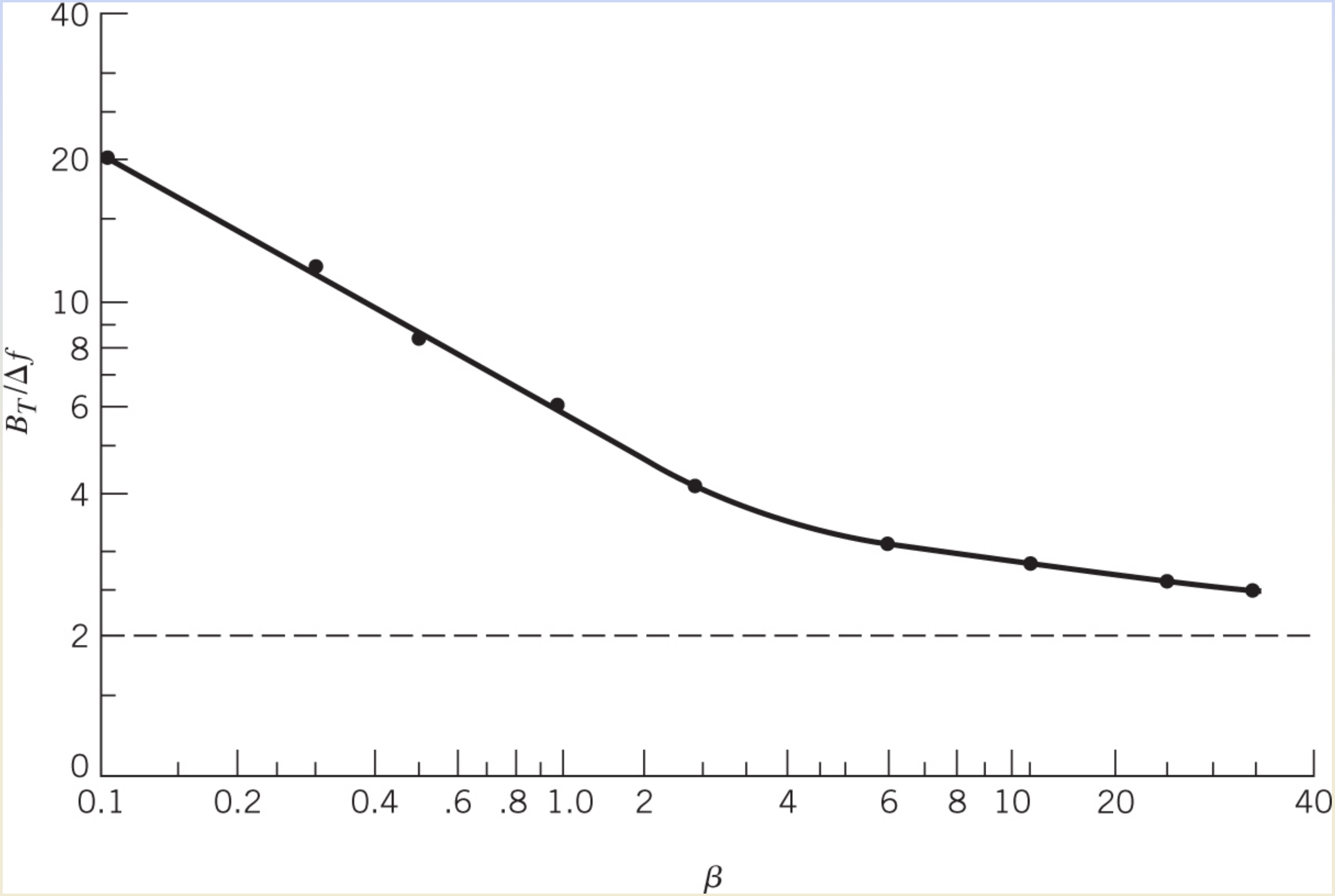
  then the transmission bandwidth is given by:

$$B_T = 2 n_{\max} f_m$$

Examples of 1 Percent Bandwidth versus β

Modulation index β	Number of significant side frequencies 2 n_{max}
0.1	2
0.3	4
0.5	4
1	6
2	8
5	16
10	28
20	50
30	70

Universal Curve for evaluating the 1 Percent Bandwidth



1 Percent Bandwidth – Example

- In the previous example:

An FM signal is generated using a sinewave modulated signal given by:

$$m(t) = 2\cos(2\pi f_m t)$$

where f_m = 10 kHz and the frequency sensitivity k_f = 10000

e- Use the 1 percent bandwidth to estimate the transmission bandwidth of the FM signal.

f- Compare this result to the result obtained with Carson's rule.

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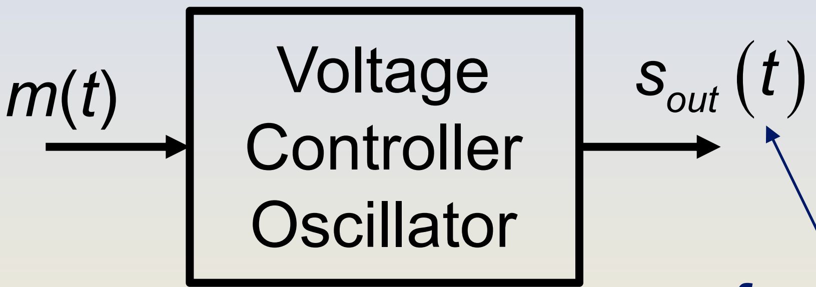
Voltage Controlled Oscillator (VCO)

- The voltage controlled oscillator is a system, where the output has a frequency that is a function of the applied voltage at the input



frequency ($s_{out}(t)$) varies linearily with the value of $m(t)$

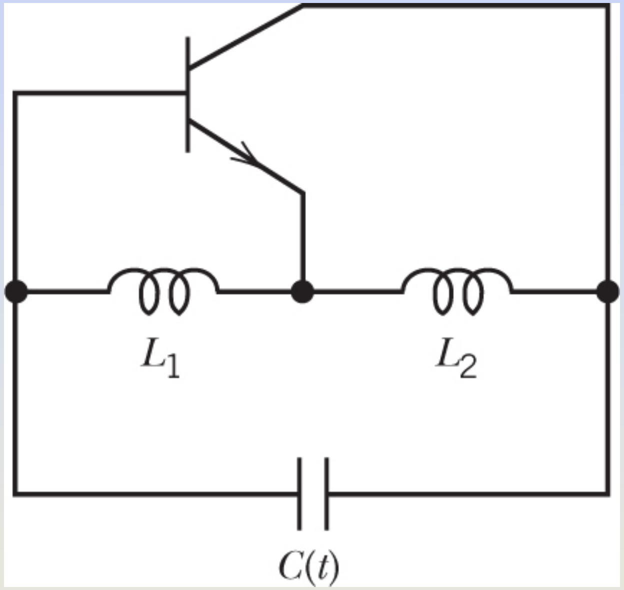
- If the input voltage is the message signal $m(t)$, then the output signal has an instantaneous frequency given by:

$$f_i(t) = f_o + \Delta f \cdot m(t)$$

Voltage Controlled Oscillator (VCO)

- Example of VCOs:

  the Hartley Oscillator →



- Using a varactor or varicap diode (varying capacitor as a function of the applied voltage to its electrodes)
- If the varying capacitor has the form:

$$C(t) = C_0 + \Delta C \cdot \cos(2\pi f_m t)$$

- It can be shown that the instantaneous frequency will have the form:

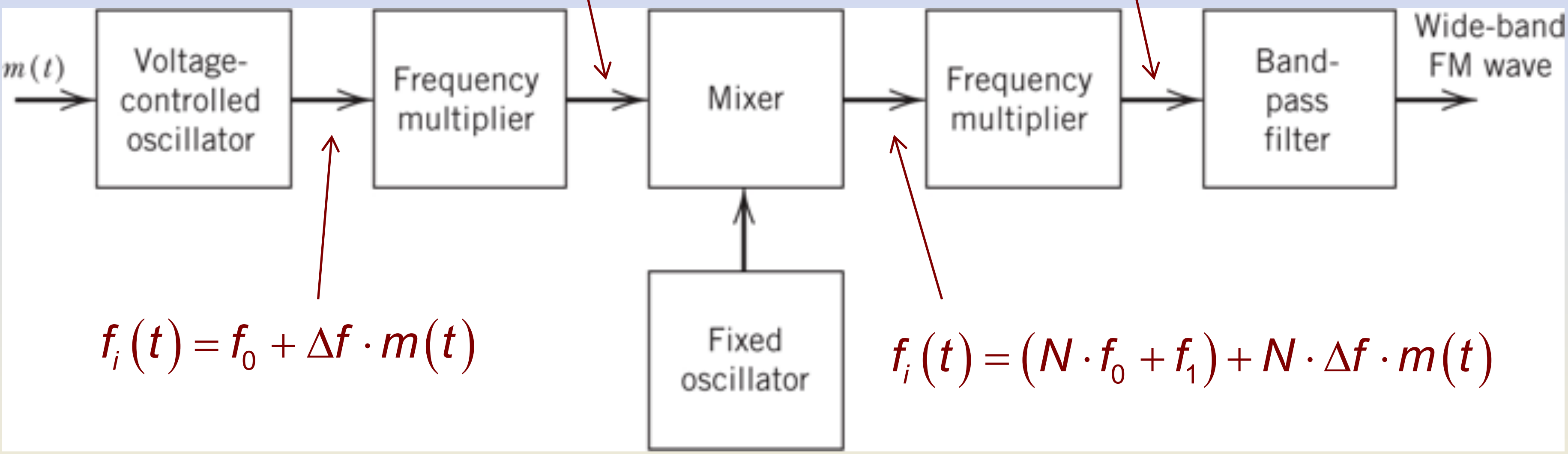
$$f_i(t) \approx f_0 + \Delta f \cdot \cos(2\pi f_m t)$$

if : $\dfrac{\Delta f}{f_0} = -\dfrac{\Delta C}{2C_0} << 1$ → **<u>approximation valid for narrowband signals</u>**

Wideband Frequency Modulation

$$f_i(t) = N \cdot f_0 + N \cdot \Delta f \cdot m(t)$$

$$f_i(t) = M \cdot (N \cdot f_0 + f_1) + N \cdot M \cdot \Delta f \cdot m(t)$$



$$f_i(t) = f_0 + \Delta f \cdot m(t)$$

$$f_i(t) = (N \cdot f_0 + f_1) + N \cdot \Delta f \cdot m(t)$$

- Higher modulation index can be achieved with the frequency multipliers
- Any carrier frequency can be obtained with the frequency multipliers and the mixer

→ Wideband modulation and high frequency carrier are possible

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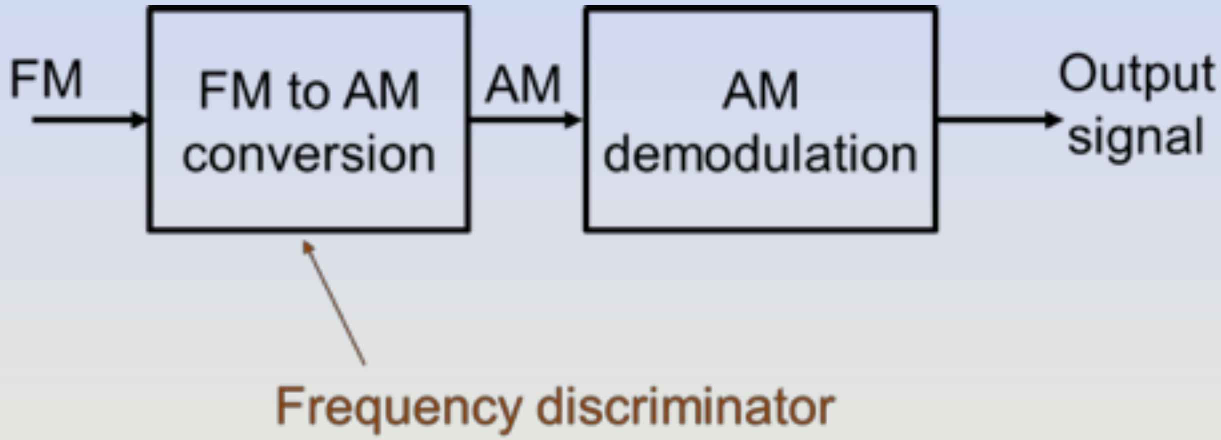
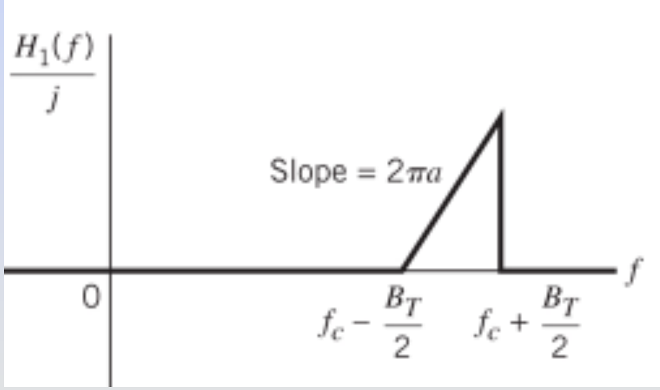
Demodulation of FM Signals



- The frequency discriminator transforms the FM signal to an AM signal whose amplitude is proportional to the instantaneous frequency of the FM signal
- What type of transfer function a frequency discriminator should have?
- How can this system be implemented?

Frequency Discriminator



- If an LTI system has a frequency response, whose amplitude is a linear function of frequency in the FM band:

$$H_1(f) = j2\pi a\left(f - f_c + \frac{B_T}{2}\right) \quad \text{for } |f - f_c| < \frac{B_T}{2}$$

then if the input is an FM signal: $s(t) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\,d\tau\right)$

the output signal of this system is given by:

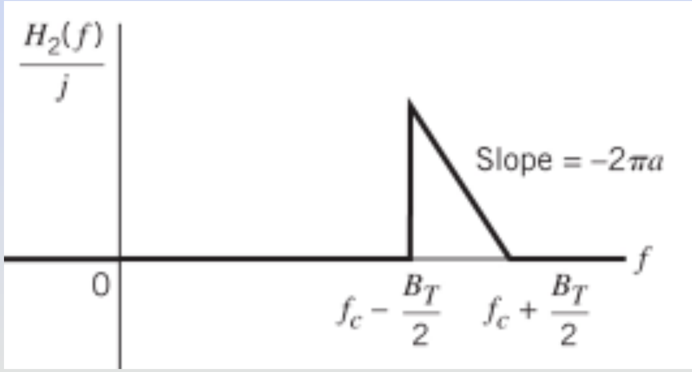
$$y_1(t) = A_c a\pi B_T\left(1 + \frac{2k_f}{B_T}\cdot m(t)\right)\cdot\cos\left(2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\,d\tau + \frac{\pi}{2}\right)$$

- If we have $\left|\frac{2k_f}{B_T} m(t)\right| < 1$ for all t, then we can use an envelope detector to detect the envelope. The output is then:

$$s_1(t) = A_c a\pi B_T\left(1 + \frac{2k_f}{B_T}\cdot m(t)\right)$$

Frequency Discriminator

- Similarly, if an LTI system has a frequency response, whose amplitude is a linear function of frequency with negative slope in the FM band:



$$H_2(f) = -j2\pi a\left(f + f_c - \frac{B_T}{2}\right) \quad \text{for } |f - f_c| < \frac{B_T}{2}$$

then we can write: $H_2(f) = H_1(-f) \quad \text{for } |f - f_c| < \frac{B_T}{2}$

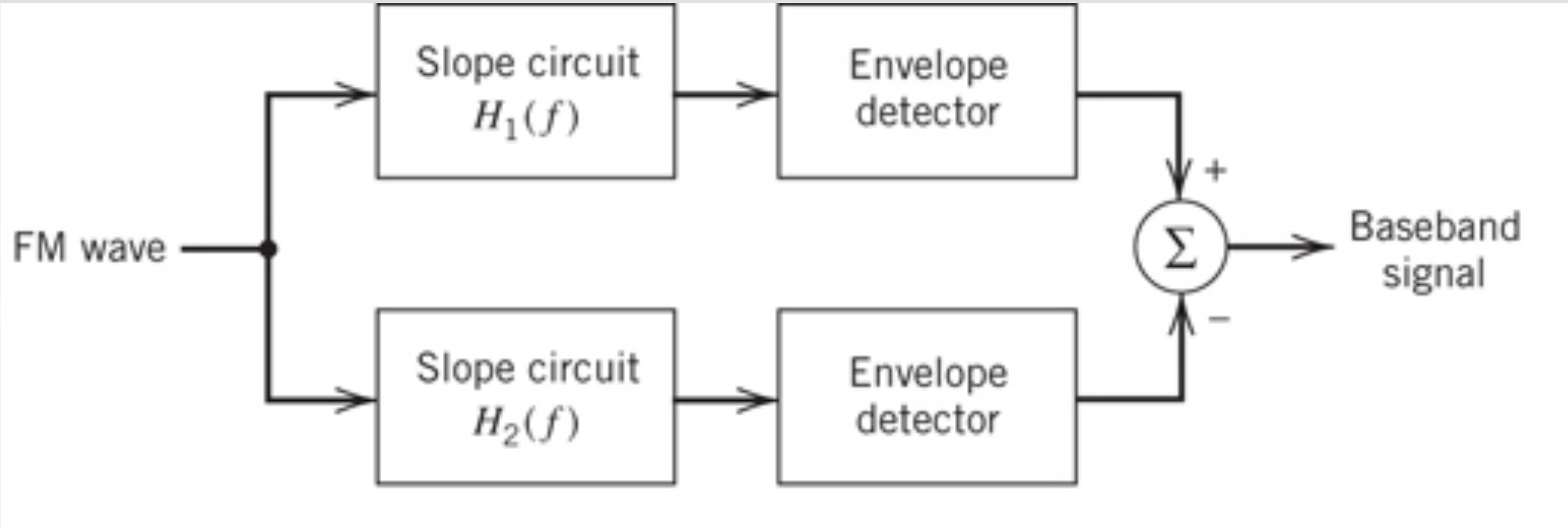
- If we have $\left|\frac{2k_f}{B_T} m(t)\right| < 1$ for all t, then we can use an envelope detector to detect the envelope. The output is then:

$$s_2(t) = A_c a\pi B_T\left(1 - \frac{2k_f}{B_T} \cdot m(t)\right)$$

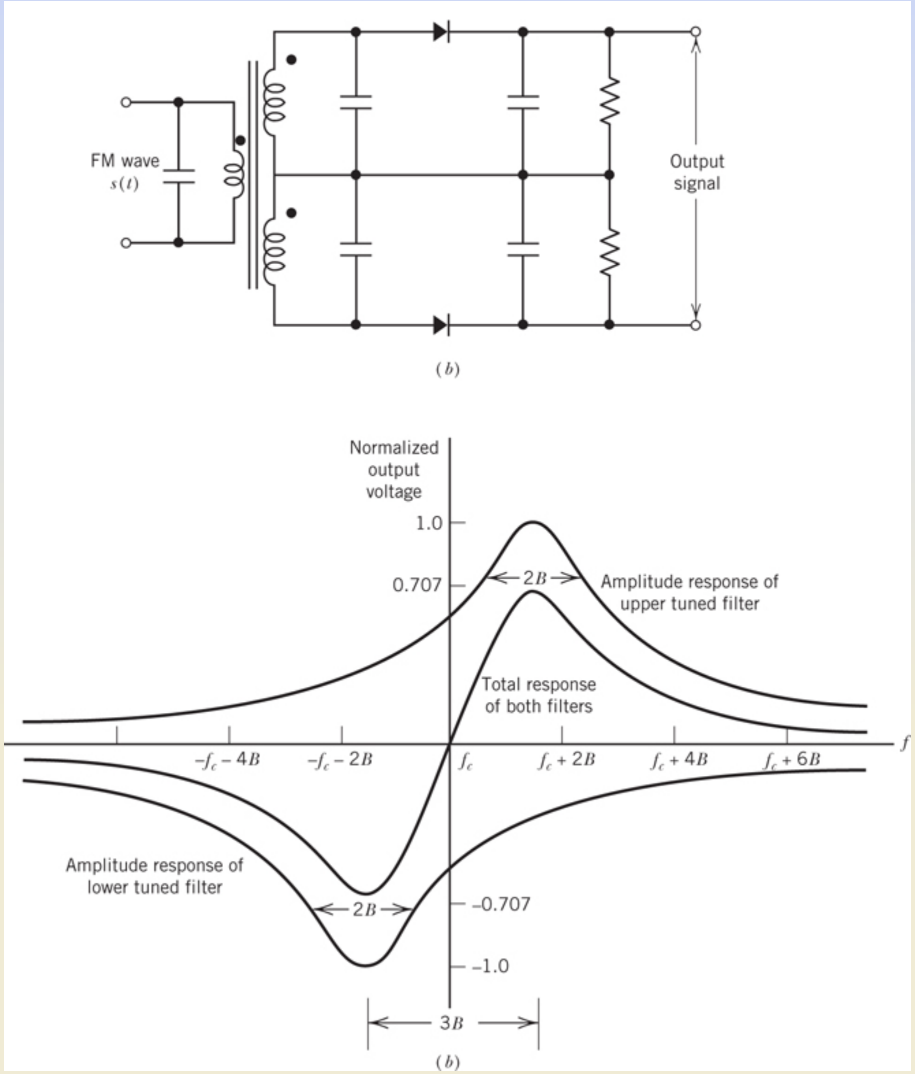
Frequency Discriminator

- If we substruct the two signals we obtain

$$s_{out}(t) = s_1(t) - s_2(t) = 4\pi A_c a k_f m(t)$$



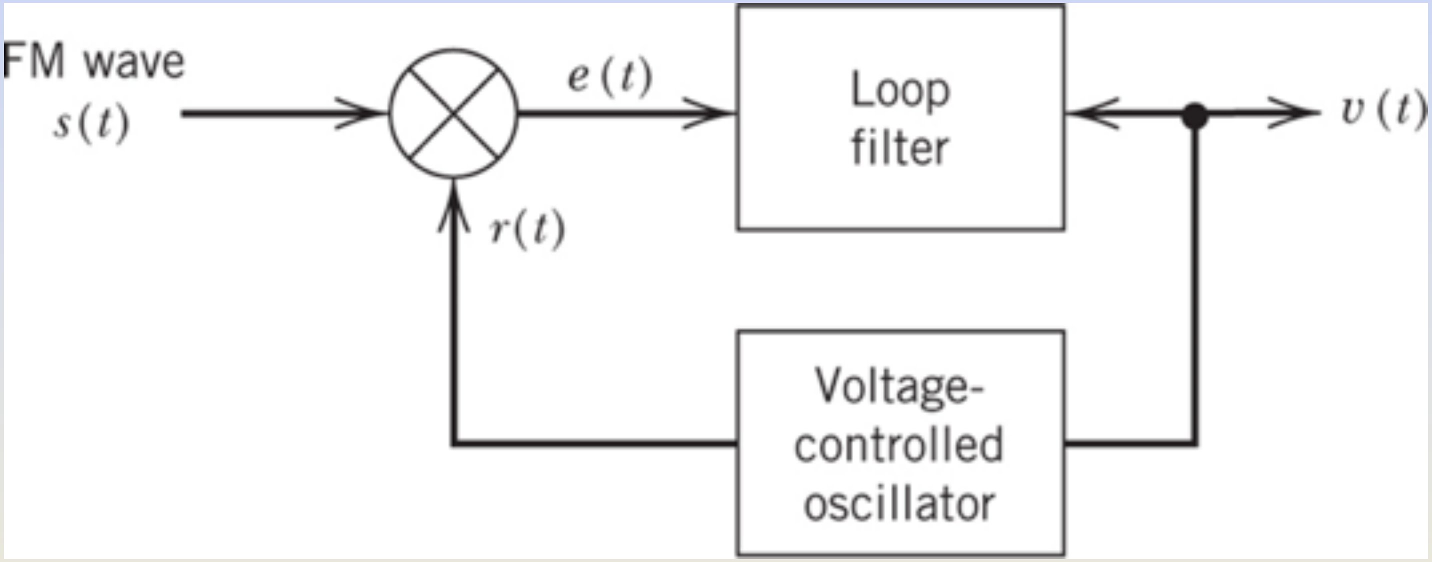
Frequency Discriminator – Practical implementation



Outline

- Angle Modulation
  - Definition of angle modulation
  - Phase modulation and frequency modulation
  - Properties of angle modulation
- Frequency Modulation (FM)
  - Narrowband and wideband frequency modulation
  - Transmission bandwidth of FM signals
  - Generation of FM signals
- Frequency Demodulation
  - Demodulation of FM signals in the absence of noise
    - Phase locked loop
  - Demodulation of FM signals in the presence of noise

Phase Locked Loop (PLL)



- The PLL is a negative feedback system composed of three elements
  - VCO: generates a sinusoidal signal with frequency determined by its input voltage
  - Mixer (or multiplier), which multiplies the FM signal with the output of the VCO
  - A loop filter, which consists of a low-pass filter that removes the high frequency components of the mixer.

Phase Locked Loop (PLL)

- We assume that:
  1. The frequency of the VCO is precisely set at the unmodulated carrier frequency when the control voltage is set to zero
  2. The VCO output has a 90-degree phase-shift with respect to the unmodulated carrier frequency
- Objective: Calculate the output of the PLL
- The input signal is :

$$s(t) = A_c \sin\left(2\pi f_c t + \phi_1(t)\right)$$

where $\phi_1(t) = 2\pi k_f \int_0^t m(\tau)\,d\tau$

- Let the output of the VCO be:

$$r(t) = A_v \cos\left(2\pi f_c t + \phi_2(t)\right)$$

where $\phi_2(t) = 2\pi k_v \int_0^t v(\tau)\,d\tau$

Nonlinear Model of the Phase Locked Loop

- At the output of the multiplier, the signal is composed of two components:
  1. A high frequency component, represented by the sum-frequency term
     $$0.5A_cA_v\sin\left(4\pi f_c t+\phi_1(t)+\phi_2(t)\right)$$
  2. A low frequency component, represented by the different-frequency term
     $$0.5A_cA_v\sin\left(\phi_1(t)-\phi_2(t)\right)$$
- The loop filter is a low pass filter that removes the high frequency component. Only the low frequency component is maintained at the input of the loop filter, which is given by:
  $$0.5A_cA_v\sin\left(\phi_e(t)\right)$$

  where: $\phi_e(t)=\phi_1(t)-\phi_2(t)=\phi_1(t)-2\pi k_v\int_0^t v(\tau)\,d\tau$

Nonlinear Model of the Phase Locked Loop

- The output of the loop filter is the convolution integral of the input and the filter impulse response. It is given by:

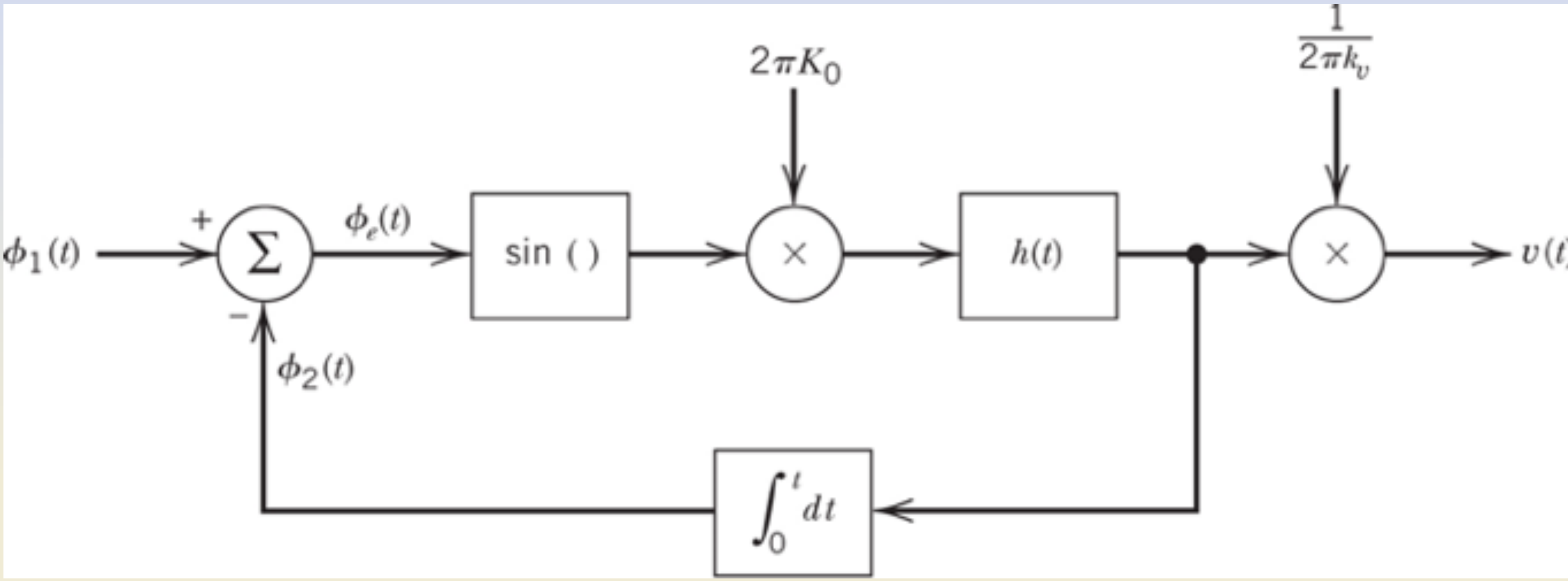
$$v(t) = \int_{-\infty}^{\infty} e(\tau) h(t-\tau) d\tau$$

- Finally, the dynamic behaviour of the PLL can be described by this integro-differential equation:

$$\frac{d\phi_e(t)}{dt} = \frac{d\phi_1(t)}{dt} - 2\pi K_0 \int_{-\infty}^{\infty} \sin\left[\phi_e(\tau)\right] h(t-\tau) d\tau$$

where $K_0 = 0.5 k_v A_c A_v$

Nonlinear Model of the Phase Locked Loop

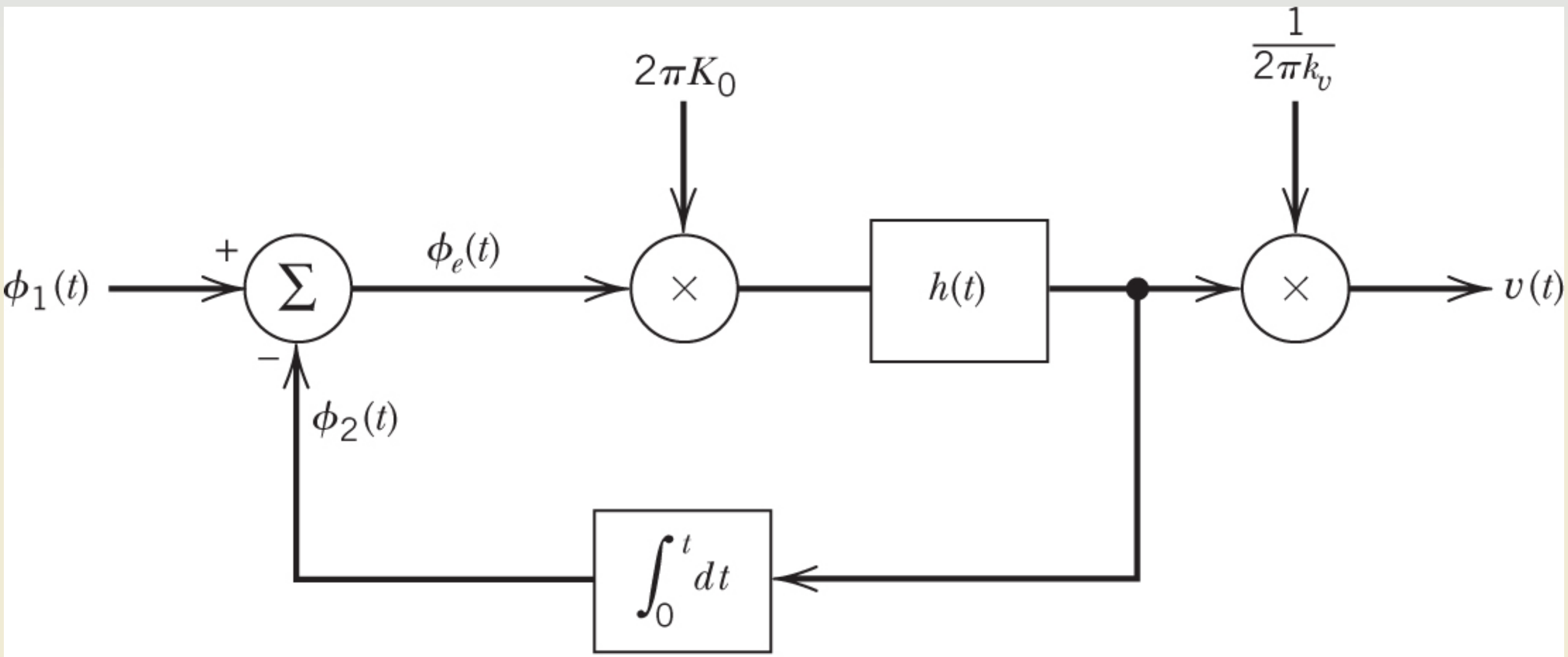


Linear Model of the Phase Locked Loop

- When the phase error $\phi_e(t)$ is zero, the loop is in phase lock
- When the phase error $\phi_e(t)$ is small, the loop is near phase lock. In this case, the following approximation holds

$$\sin(\phi_e(t)) \approx \phi_e(t)$$

- The PLL can then be described by this linear model:



Linear Model of the Phase Locked Loop

- The integro-differential equation become:

$$\frac{d\phi_e(t)}{dt} + 2\pi K_0 \int_{-\infty}^{\infty} \phi_e(\tau) h(t-\tau) d\tau = \frac{d\phi_1(t)}{dt}$$

- This equation can be represented in frequency domain as:

$$j2\pi f \cdot \Phi_e(f) + 2\pi K_0 \cdot H(f) \cdot \Phi_e(f) = j2\pi f \cdot \Phi_1(f)$$

which we can write as:

$$\Phi_e(f) = \frac{1}{1+L(f)} \cdot \Phi_1(f)$$

where $L(f) = \frac{K_0}{jf} \cdot H(f)$

- Since $v(t) = \frac{K_0}{k_v} \int_{-\infty}^{\infty} \phi_e(\tau) h(t-\tau) d\tau$, in frequency domain this becomes:

$$V(f) = H(f) \frac{K_0}{k_v} \Phi_e(f) = \frac{jf \cdot L(f)}{k_v} \Phi_e(f)$$

Simplified Linear Model of the PLL

- The output voltage can then be expressed as a function of the input phase as:

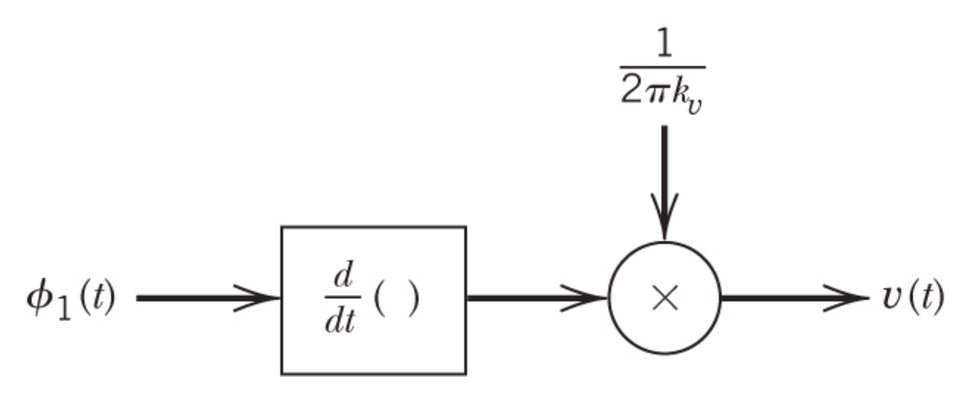
$$V(f) = \frac{jf}{k_v} \frac{L(f)}{1+L(f)} \Phi_1(f)$$

- If the PLL is implemented so that the loop gain $L(f)$ is very large: $|L(f)| >> 1$ then the relationship between input and output become:

$$V(f) \approx \frac{jf}{k_v} \Phi_1(f)$$

- Going back to time domain this relationship is:

$$v(t) \approx \frac{1}{2\pi k_v} \frac{d\varphi_1(t)}{dt} = \frac{k_f}{k_v} m(t)$$



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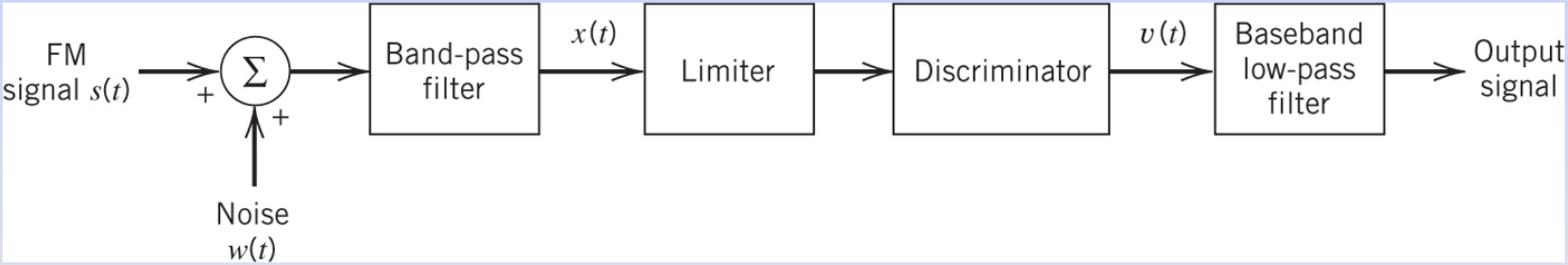
Demodulation of FM Signals in the Presence of Noise



- It can be shown that the signal-to-noise ratio at the input of an FM receiver is:

$$SNR_{in} = CNR = \frac{A_c^{\,2}}{2WN_0}$$

- The signal-to-noise ratio at the output of an FM receiver is:

$$SNR_{out} = \frac{3A_c^{\,2}k_f^{\,2}P}{2N_0W^3}$$

- Therefore, the figure of merit for FM modulation is:

$$FOM_{FM} = \frac{SNR_{out}}{SNR_{in}} = \frac{3k_f^{\,2}P}{W^2}$$

- If the message is sinusoidal, then the figure of merit becomes:

$$FOM_{FM} = \frac{SNR_{out}}{SNR_{in}} = \frac{3}{2}\beta^2$$