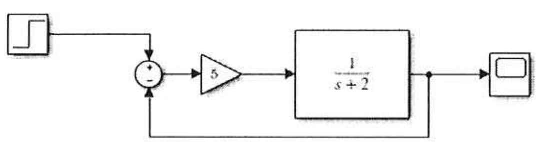
Unit 3 Feedback design - Stability and Steady State Errors

In Unit 1, Laplace frequency domain analysis was developed such that LTI systems can be easily and efficiently analyzed. Classical feedback was introduced as a means of moving poles for a more favourable transient response. In unit 2 we looked at modelling of mechatronic plants and sensors. In this unit we will go back to the details of the classical feedback and consider the desirable characteristics such as transient response and stability. This is all accomplished by moving poles with feedback. In unit 4 design tools for feedback systems will be developed such that closed loop pole can be placed for complex higher order systems.

Start with a simple example of closed loop pole placement with feedback. Suppose we have a plant of

$$H_p(s) = \frac{1}{s+2}$$

then we can move the pole from s=-2 to a more desirable pole say s=-7 (which makes the response faster) by using feedback. Consider the negative feedback loop shown below



The **open loop transfer function** of the feedback loop is

$$H_{ol}(s) = 5H_p(s) = \frac{5}{s+2}$$

The **closed loop transfer function** of the feedback loop is

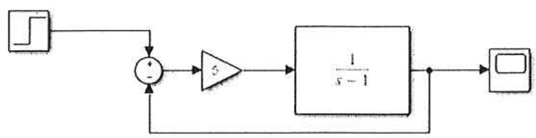
$$H_{cl}(s) = \frac{5H_p(s)}{1+5H_p(s)} = \frac{5}{s+2+5} = \frac{5}{s+7}$$

Hence with feedback the pole has moved from s=-2 to s=-7.

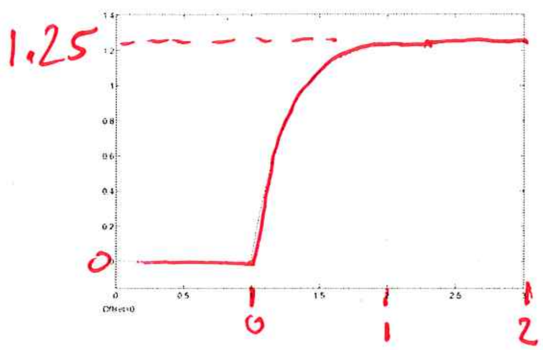
In another example suppose that the plant is unstable with a transfer function of

$$H_p(s) = \frac{1}{s-1}$$

Note that it has a pole at s=1 that will result in an unabated exponential rise of the output for an arbitrary input step or impulse excitation that can be arbitrarily small. We will move this unstable pole from s=1, which is in the RHP and unstable to a desirable stable location with negative feedback as shown below



The stable step response is shown below:



$$H_{cl}(s) = \frac{5H_p(s)}{1+5H_p(s)} = \frac{5}{s-1+5} = \frac{5}{s+4}$$

Feedback is therefore quite useful in taming unstable processing plant transfer functions.

What are other things that we would like the feedback to do?

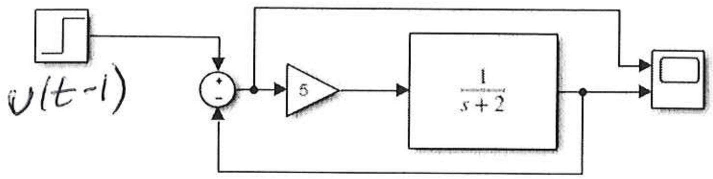
1. In many cases it would be nice if the DC gain was 1 such that the output tracked the input (after a transient). That is $H_{cl}(0)=1$. Steady state error is the persistent error or difference between the input and output of the closed loop response after the transient has died off. If $H_{cl}(0)=1$ then for a step input, the steady state error is zero.

2. Transient overshoot was minimal and with little ringing. For this we need closed loop poles with moderately large damping coefficients

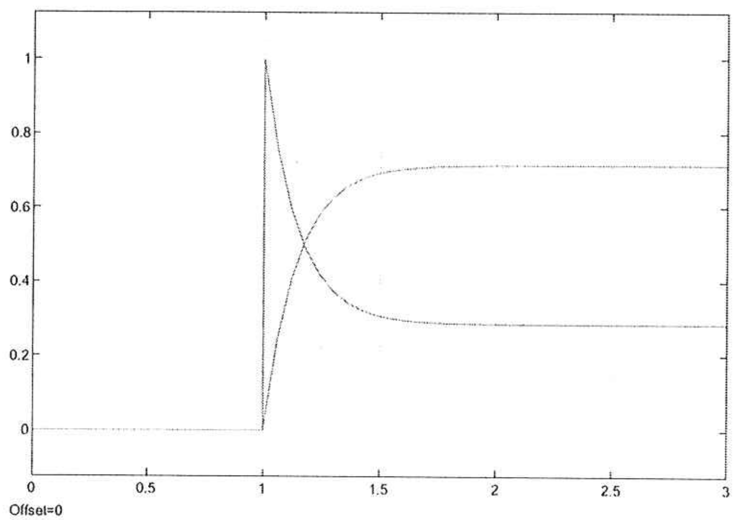
3. Transient time was short. The poles have to be away from the jw axis into the LHP.

4. Robustness to disturbances added to loop

We may also want the closed loop response to track a ramp input with zero steady state error. For instance, suppose we have a servo motor and we want to have it go to a position that is specified by a voltage input. Now if the input is a step then it is sufficient that the closed loop has a DC value of 1 such that $H_{cl}(0)=1$. Now suppose that we want the positioner to track an input that changes with time. Then it is not sufficient that $H_{cl}(0)=1$. We would likely specify that the steady state error for a ramp input is zero. As an example consider first that $H_p(s) = \dfrac{1}{s+2}$ and that we have a feedback gain of 5 as in the following Simulink model.

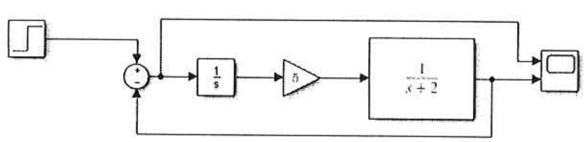


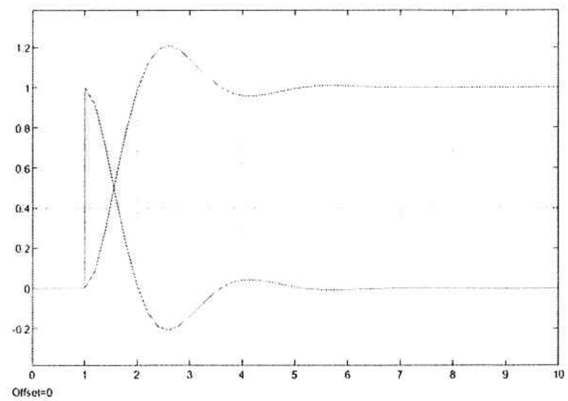
The error signal is the difference between the input reference signal (step function) and the output signal. Transient step response as shown below:



Plot of transient response to an input step function plotted as a function of time. Error in blue and output response in orange. Note the steady state error that results which is not desired.

This steady state error can be corrected by adding in an integrator. Hence we cannot have a steady state error in the loop as the integrator output would increase indefinitely. In the following Simulink model an integrator block has been added.

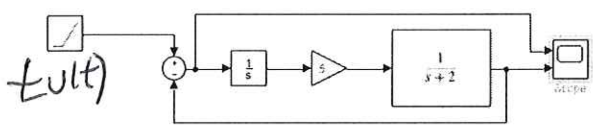


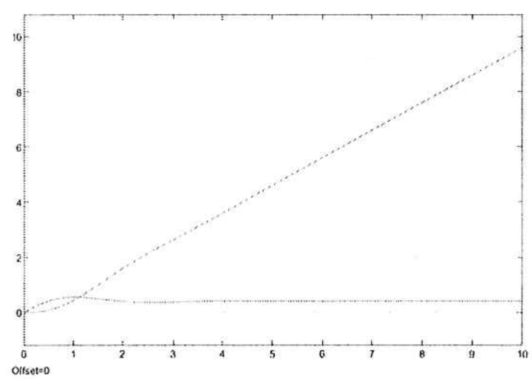


Plot of transient response to an input step function. Error in blue and output response in orange. Note that after a transient that the error settles to zero which is desired.

Now the steady state error for the closed loop is zero. However, we have messed up the transient and slowed down the circuit. Later we will learn how to fix this but for now let us focus on the steady state response.

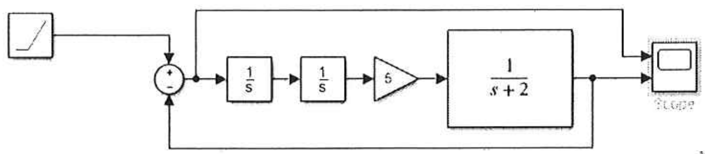
Instead of applying a step let us apply a ramp input and look at the steady state error. This is simulated in the following Simulink model.

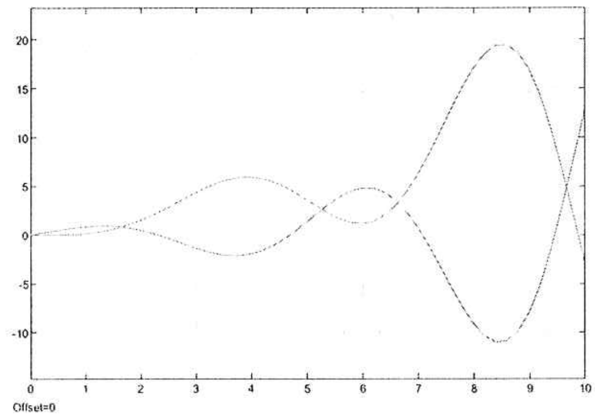




Plot of transient response to an input ramp function. Error in blue and output response in orange. Note that after a transient that the error settles to a nonzero steady state value.

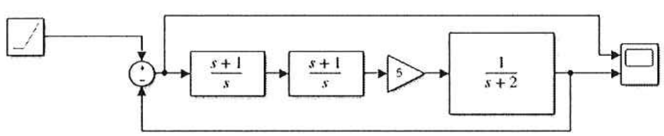
We can see from the blue trace which is the error that the steady state error for a ramp input is not zero. This is undesirable. As we will learn shortly, this can be fixed by adding another integrator so let us try that. Intuitively, if we can integrate the error such that it becomes bigger with time then it should have more ability to correct the steady state error.



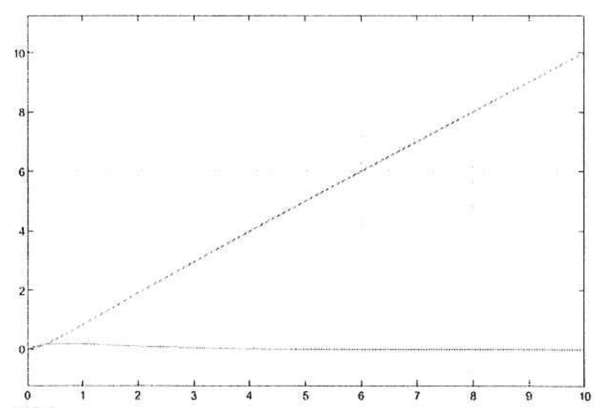


Plot of transient response to an input ramp function with two integrators. Error in blue and output response in orange. Note that the closed loop response becomes unstable.

This is bad news – the additional integrator seems to make the closed loop response unstable. Indeed if the analysis is done then we will see that the closed loop pole is in the RHP. What we will learn in the next unit is that by adding a zero to the integrator we can make the circuit stable again. Hence our feedback loop is as follows:



and the ramp response is as follows



Stable response is a result of two zeros added.

Note that this is now a stable closed loop response where the steady state error goes to zero for a ramp input. Furthermore the response is reasonably fast. Later when we look at root locus it will become evident why we chose to add two zeros specifically at the location of s=-1.

In this unit we will consider the closed loop response of the feedback loop specifically for the steady state response when a number of integrators are added. However, to make it stable is another issue. We will have to develop the root locus method before that problem can be solved.

Along the way we will introduce the 'PID controller' where:

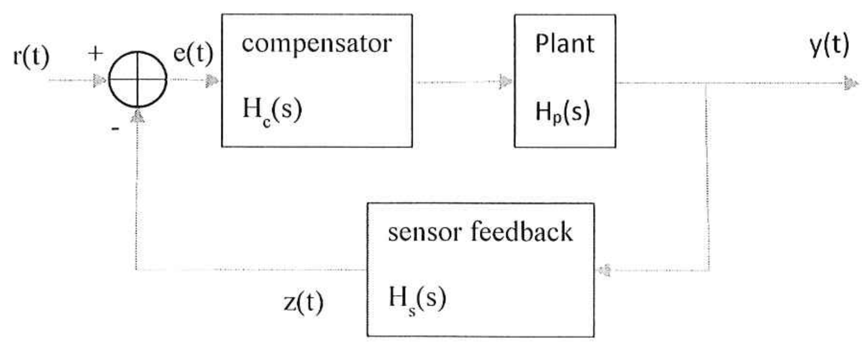
P – proportional feedback
I – integral feedback
D – derivative feedback

To get the steady state performance that is desired it is not sufficient to use a single integrator in the feedback loop and we may have to add additional integrators. We will introduce the type-1 loop with a single integrator, type-2 loop with two integrators and so on.

All this adds complexity in that the loop order is higher. Also this is exasperated by higher order plant models. We will quickly realize that our tools for designing these more complex feedback loops is inadequate. In the following unit 4 we will develop the Root Locus method and the Nyquist Stability method as necessary tools for design and analysis of negative feedback systems. As well we will introduce SISOtool as an interactive graphical tool for analysis and design of the compensator for an arbitrary type-n loop with a complex plant model.

Negative Feedback Loop

The loop configuration that we will be considering in ENEL441 is given in the figure below:



Here r(t) is the reference input with is the independent function. e(t) is the error signal that is processed by the compensator of $H_c(s)$. The plant is as usual $H_P(s)$ and the sensor for the feedback is denoted by $H_s(s)$. The output is y(t).

The objective of the feedback is to have y(t) track r(t) as closely as possible. Complicating this is that we do not observe the output directly but rather through a sensor. That is we have access to z(t) and not y(t). The sensor will have a transfer function that can be usually be ignored. That is

$$H_s(s) \approx 1$$

such that $z(t) \approx y(t)$. However, the response time of the sensor will be finite. If the response time is much faster than the time constants associated with the closed loop poles then it can be ignored. Otherwise we will have to account for additional poles of the sensor transfer function.

The objective of having y(t) track r(t) (through the sensor) is achieved by careful design and implementation of the compensator $H_c(s)$ with the objective that

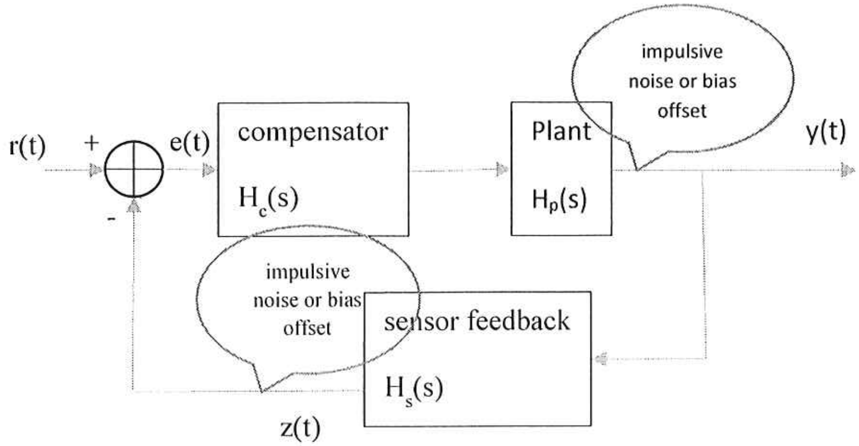
$$e(t) = r(t) - z(t) \approx 0$$

for typical reference inputs of r(t). r(t) is typically a step or ramp input. In the real world it is typically a random process of a given power spectral density. However, for the sake of ENEL441 we will assume that r(t) can be represented by a low order polynomial. That is

$$r(t) = u(t)\left(a + bt + ct^2 + dt^3\right)$$

where {a,b,c,d} are real coefficients and u(t) is the unit step function.

Additionally we can have disturbances that occur within the loop and sensor that the feedback loop quickly has to recover from without instability. The disturbances are illustrated below:



Noise and bias offset is typical in the plant output. There is also the possibility of noise in the sensor. More troublesome is an offset or bias in the sensor transfer function that is incorrectly modelled. Such a distortion can be construed as a disturbance to the feedback system.

In this unit we will consider the overall design of the feedback loop such that the objectives of close tracking of the input and robustness to distortions and disturbances are handled for a variety of plant models. We will be less concerned with the transient response, frequency response and stability of the closed loop. We will solve this in the next section.

Zero steady state error for a DC input

Initially assume that $H_s(s)=1$, an ideal sensor transfer function such that the open loop transfer function is

$$H_{OL}(s)=H_c(s)H_p(s)=\frac{N(s)}{D(s)}$$

where N(s) is the numerator polynomial and D(s) is the denominator polynomial. Then the closed loop transfer function from the reference to the error is given as

$$H_e(s)=\frac{1}{1+H_{OL}(s)}=\frac{1}{1+\dfrac{N(s)}{D(s)}}=\frac{D(s)}{D(s)+N(s)}$$

To have a steady state error of zero we can also say that the DC response of $H_e(s)$ has to be zero. That is

$H_e(0) = 0$

Assuming that $D(0) + N(0) \neq 0$ which is reasonable (why?) then the condition is that

$D(0) = 0$

This means that $D(s)$ has a root at s=0 and since it is a polynomial we can then write

$D(s) = s \times (\text{some polynomial in s}) = sG(s)$

where G(s) is a polynomial with no roots at s=0.

Another condition has to be that $D(s) + N(s)$ has no roots in the RHP. Roots in the RHP would imply that the feedback loop is unstable

Another way of looking at this is that for a unit step input we have

$$E(s) = H_e(s)R(s) = \frac{\left(\frac{1}{s}\right)D(s)}{D(s)+N(s)} = \frac{G(s)}{D(s)+N(s)}$$

To proceed we need the final value theorem (FVT) which states

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

which we will prove later in this unit. The FVT is valid provided that E(s) has no poles in the RHP.

Applying this we get that the steady state error is zero provided that $H_e(s)$ is stable (no poles in RHP or pole at s=0) and that

$$\lim_{s\to 0} sG(s) = \lim_{s\to 0} D(s) = 0$$

This implies that $D(s)$ must have at least one root at s=0. Hence $H_e(s)$ must have at least one integrator.

Let us consider another case where the input is a ramp such that

$$E(s) = \frac{1}{s^2} H_e(s) = \frac{\left(\frac{1}{s^2}\right)D(s)}{D(s)+N(s)}$$

Using the FVT directly we have

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{\frac{D(s)}{s}}{D(s)+N(s)}$$

For $e(\infty)=0$ we must have the condition that D(s) has at least two roots at s=0 which implies that $H_{OL}(s)$ must have at least two integrators.

In general for an input of $r(t)=t^n u(n)$, for zero steady state error with the condition $e(\infty)=0$, we must have that $H_e(s)$ have no poles in the RHP nor at s=0 and that have at $H_{OL}(s)$ have at least n+1 integrators.

Loop Type $\equiv$ number of integrators in $H_{OL}(s)$

Proof of Final Value Theorem (FVT)

The FVT is of importance in our development of feedback systems so that we will go through the proof here. Start with the Laplace transform of the time derivative of a function as

$$L\{\dot{f}(t)\}=\int_0^\infty \dot{f}(t)e^{-st}dt$$

Using the product rule we have

$$L\{\dot{f}(t)\}=fe^{-st}\Big|_0^\infty+\int_0^\infty sf(t)e^{-st}dt$$

Assuming that f(t) does not increase faster with time that a polynomial of t and that the real part of s is slightly positive then

$$fe^{-st}\Big|_0^\infty=0-f(0)$$

This really means that f(t) cannot increase exponentially which is equivalent to stating that we cannot have f(t) with any poles in the RHP. However, for instance if $f(t)=t^4u(t)$, that is OK as this only increases in a polynomial fashion with time. Hence with this we have

$$\begin{aligned}L\{\dot{f}(t)\}&=-f(0)+s\int_0^\infty f(t)e^{-st}dt\\&=sF(s)-f(0)\end{aligned} \qquad \text{A}$$

Next go back to the beginning but this time take the limit of $s\to 0$ as

$$\lim_{s\to 0} L\{\dot{f}(t)\} = \lim_{s\to 0} \int_0^\infty \frac{d}{dt} f(t)\, e^{-st}\, dt$$

Take limit inside of integral

$$= \int_0^\infty \frac{d}{dt} f(t) \lim_{s\to 0} e^{-st}\, dt$$

$$= \int_0^\infty \frac{d}{dt} f(t)\, dt$$

$$\lim_{s\to 0} L\{\dot{f}(t)\} = f(\infty) - f(0) \quad \text{(B)}$$

Combining (A) and (B)

$$f(\infty) = \lim_{s\to 0} \left(s F(s) \right)$$

This is the final value theorem.

example $f(t) = u(t) e^{-t}$ $\quad F(s) = \frac{1}{s+1}$

$$f(\infty) = \lim_{s\to 0} \frac{s}{s+1} = \frac{0}{0+1} = 0$$

example $f(t) = u(t)\, t^2 e^{-t}$, $\quad F(s) = \frac{2}{(s+1)^3}$

$$f(\infty) = \lim_{s\to 0} \frac{2s}{(s+1)^3} = \frac{0}{1^3} = 0$$

example $f(t) = t\,u(t)$ $F(s) = \frac{1}{s^2}$

$$f(\infty) = \lim_{s \to 0} s \cdot \frac{1}{s^2} = \infty$$

example

$f(t) = \exp(2t)$ $F(s) = \frac{1}{s-2}$

$$f(\infty) = \lim_{s \to 0} \frac{s}{s-2} = 0$$ incorrect

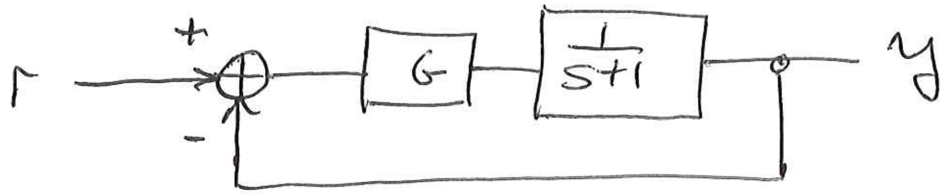
Pole in RHP so FVT cannot be used

$\boxed{\text{Annoying Limitation of FVT - must know that there are no poles in RHP to apply it.}}$

Loop Types and Steady state errors

Type 0 no integrator (no pole at $s=0$) for open loop response

Example

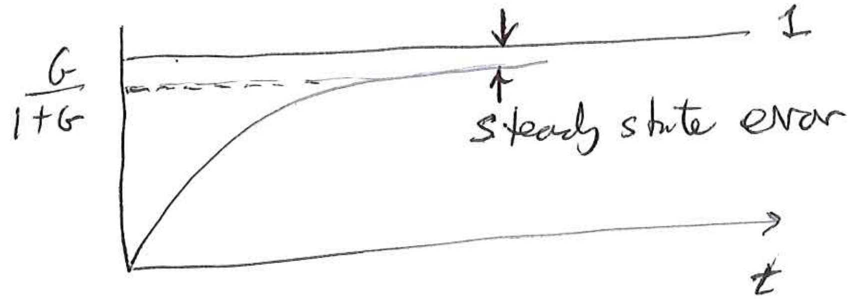


$$H_{OL}(s) = \frac{G}{s+1} \qquad \text{(no pole at zero)}$$

Response for step input $r(t) = u(t)$

$$Y(s) = \frac{\frac{G}{s+1}}{1 + \frac{G}{s+1}} \frac{1}{s} = \frac{1}{s} \frac{G}{s+1+G}$$

$$y(\infty) = \lim_{s \to 0} \left(s\, Y(s) \right) = \lim_{s \to 0} \left(\frac{G}{s+1+G} \right) = \frac{G}{1+G}$$

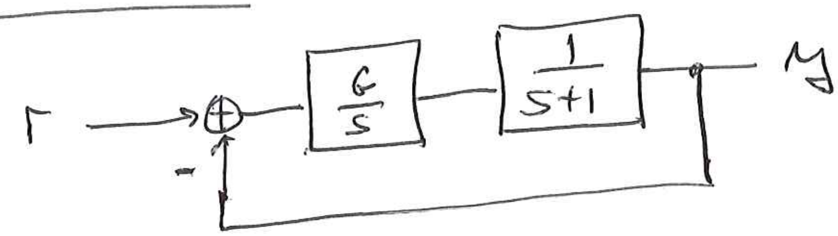


$$e_r(t) \equiv y(t) - r(t)$$

$$E_r(s) = \frac{1}{s} \frac{G}{s+1+G} - \frac{1}{s} = \frac{1}{s} \left(\frac{G}{s+1+G} - 1 \right)$$

$$e_r(\infty) = \lim_{s \to 0} \left(\frac{G}{s+1+G} - 1 \right) = \frac{-1}{1+G}$$

Add integrator



$$H_{OL}(s) = \frac{G}{s(s+1)}$$

1 pole @ $s=0$ Type-1 loop

$y(\infty)$ for $r(t) = u(t)$

$$y(\infty) = \lim_{s\to 0} s \frac{1}{s} \frac{\frac{G}{s(s+1)}}{1 + \frac{G}{s(s+1)}}$$

$$= \lim_{s\to 0} \frac{G}{s(s+1)+G} = 1$$

error signal $e(t) = r(t) - y(t)$

$$\frac{E(s)}{R(s)} = \frac{1}{1+\frac{G}{s(s+1)}} = \frac{s(s+1)}{s(s+1)+G}$$

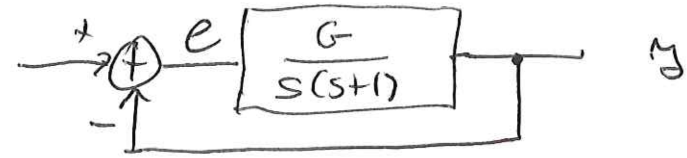
for $R(s) = \frac{1}{s}$, $r(t) = u(t)$

$$e(\infty) = \lim_{s\to 0} s \frac{1}{s} \frac{s(s+1)}{s(s+1)+G} = 0$$

Steady state error in this loop with a step input is zero.

Suppose we have a ramp input

$r(t) = t\,u(t)$



$$\frac{E(s)}{R(s)} = \frac{1}{1 + \frac{G}{s(s+1)}} = \frac{s(s+1)}{s(s+1)+G}$$

$$e(\infty) = \lim_{s \to 0} s \frac{1}{s^2} \frac{s(s+1)}{s(s+1)+G} = \frac{1}{G}$$

Finite error for type 1 loop and ramp input.

Note: for $e(\infty) = 0$ need for $r(t) = t^n u(t)$ a type $(n+1)$ loop. A type (n) loop gives a finite error.

For the previous example consider $r(t) = t^2 u(t)$

$$e(\infty) = \lim_{s \to 0} s \frac{2}{s^3} \frac{s(s+1)}{s(s+1)+G} = \infty$$

Summary $e(\infty)$

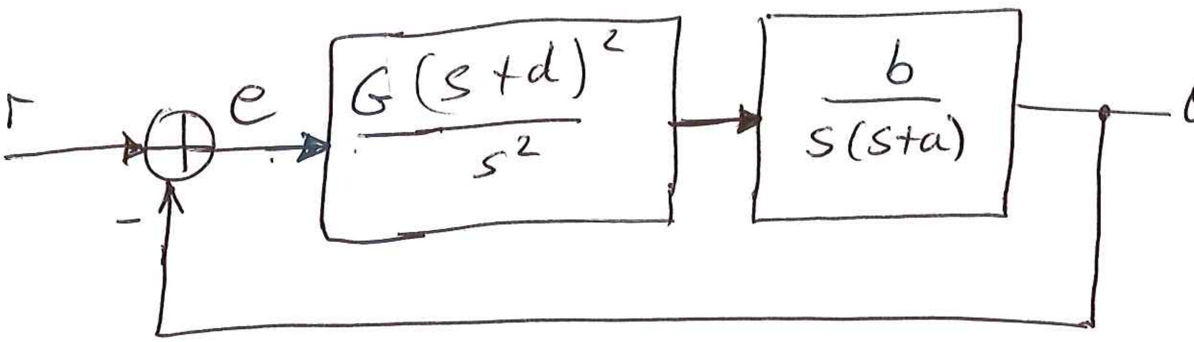
$r(t) \rightarrow$	$u(t)$	$t\,u(t)$	$t^2 u(t)$	$t^3 u(t)$	$t^4 u(t)$
Type 0	finite	∞	∞	∞	∞
Type 1	0	finite	∞	∞	∞
Type 2	0	0	finite	∞	∞
Type 3	0	0	0	finite	∞
Type 4	0	0	0	0	finite

So FVT has annoying limitation on application, ie must ensure no RHS poles

When we do root locus analysis we can get an idea if roots are all in LHP.

next unit

Example Consider a type III loop positioner with compensator at $G\frac{(s+d)^2}{s^2}$



Based on root locus for some values of G, d, b and a, the closed loop poles are all in LHP.

use FVT to determine steady state errors

$$\frac{E(s)}{R(s)} = \frac{s^3(s+a)}{s^3(s+a) + G(s+d)^2 b}$$

When $R(s) = \frac{1}{s}$ step input

$$E(s) = \frac{s^2(s+a)}{s^3(s+a) + G(s+d)^2 b}$$

$$e(\infty) = \lim_{s \to 0} s\,E(s) = 0$$

Ramp input $R(s) = \frac{1}{s^2}$

$$E(s) = \frac{s(s+a)}{s^3(s+a) + G(s+d)^2 b}$$

$$e(\infty) = \lim_{s \to 0} s\,E(s) = 0$$

Parabolic $R(s) = \frac{2}{s^3}$

$$E(s) = \frac{s+a}{s^3(s+a) + G(s+d)^2 b}$$

$$e(\infty) = \lim_{s \to 0} s\,E(s) = 0$$

Cubic input

$$r(t) = t^3 u(t)$$

$$R(s) = \frac{6}{s^4}$$

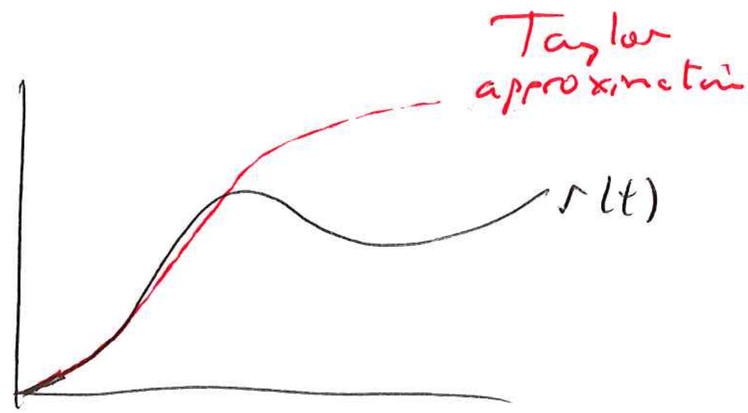
$$s E(s) = \frac{s+a}{s^3(s+a) + 6(s+d)^2 b}$$

$$e(\infty) = \lim_{s \to 0} s E(s) = \frac{a}{6 d^2 b} \Rightarrow \text{finite.}$$

for $r(t) = t^4 u(t)$ $\quad e(\infty) \to \infty$.

General input

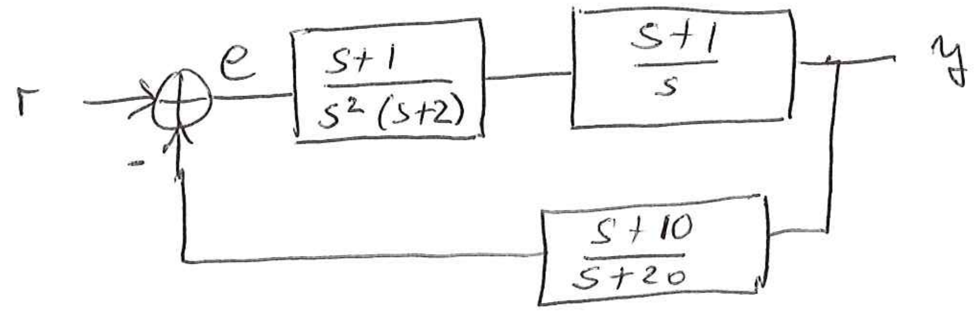
$$r(t) \simeq a_0 + a_1 t + a_2 t^2 + a_3 t^3$$



If we can approximate the input $r(t)$ as a Taylor series then determine error contribution of each term. Gives indication of RMS error.

Problems Unit 3

1) What is the loop type of the feedback loop



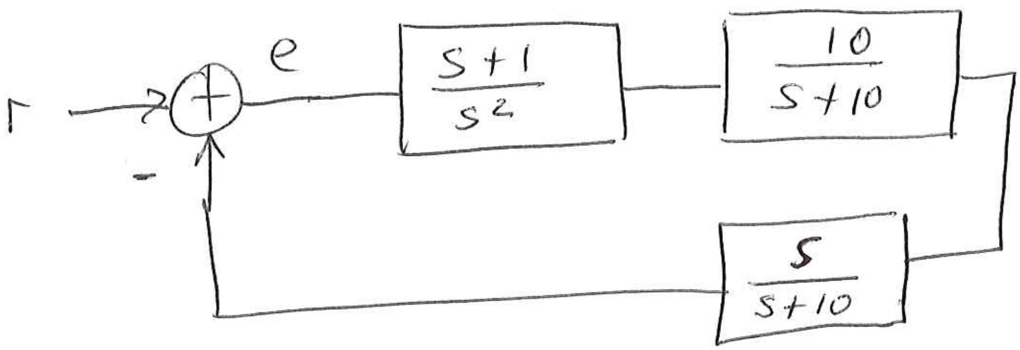
Sol

$$H_{OL}(s) = \frac{(s+1)^2(s+10)}{s^3(s+2)(s+20)}$$

$H_{OL}(s)$ has 3 poles at $s=0$
i.e. 3 integrators

Hence it is of type-3

2) What is loop type

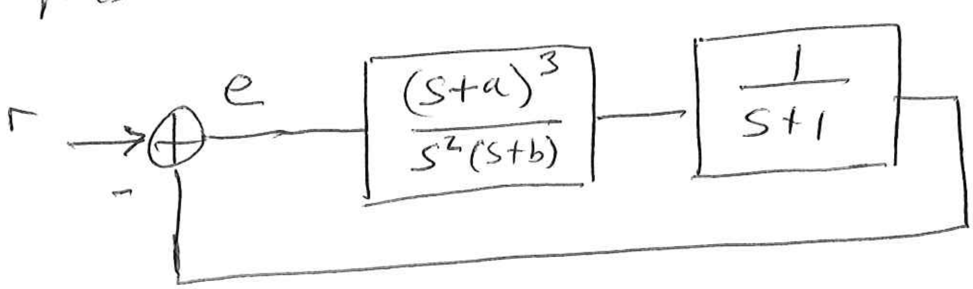


Sol loop type $= 1$

beware of cancellation in $H_{OL}(s)$

$$H_{OL}(s) = \frac{s+1}{s^{\cancel{2}}} \quad \frac{10}{s+10} \quad \frac{\cancel{s}}{s+10}$$

3) Assume a feed back loop with 'a' and 'b' selected such that there are no closed loop poles in RHP



What is $e(\infty)$ for $r = t^2 u(t)$?

Sol

$$\frac{E(s)}{R(s)} = \frac{1}{1 + \frac{(s+a)^3}{s^2(s+b)}\frac{1}{s+1}} = \frac{s^2(s+b)(s+1)}{s^2(s+b)(s+1)+(s+a)^3}$$

$e(\infty)$ for $R(s) = \frac{2}{s^3}$ is

$$e(\infty) = \lim_{s\to 0} s\,E(s)$$

(fvt valid as there are no poles in RHP of $E(s)$)

$$e(\infty) = \lim_{s\to 0} \frac{2}{\cancel{s^3}} \frac{\cancel{s^3}(s+b)(s+1)}{s^2(s+b)(s+1)+(s+a)^3}$$

$$= \frac{2\cdot b\cdot 1}{a^3} = \frac{2b}{a^3}$$

4) Redo problem 3 for $r(t) = (1+3t+4t^2)u(t)$

Sol
Use superposition and that loop is type-II

hence $r(t) = (1+3t)u(t)$ gives $e(\infty)=0$

$r(t) = 4t^2 u(t)$ is $\frac{2b}{a^3}\cdot 4$

Hence for $r(t) = (1 + 3t + 4t^2)\, u(t)$

$$e(\infty) = \frac{8b}{a^3}$$

5) What if $r(t) = u(t) e^{-t}$ find $e(\infty)$ in problem 3.

(sol) Clearly this should be $e(\infty) = 0$

since $r(t) \leq u(t)$ and if $r(t) = u(t)$ then $e(\infty)$ is zero.

6) What if $r(t) = u(t) e^{t}$?

Can't use fvt as now

$$E(s) = \underbrace{\frac{s^2(s+b)(s+1)}{s^2(s+b)(s+1) + (s+a)^3}}_{\text{made assumption that no poles in RHP}} \cdot \underbrace{\frac{1}{s-1}}_{\substack{R(s) \\ \text{has pole in} \\ \text{RHP}}}$$

We cannot use FVT.

We have to do a partial fraction expansion.

$$E(s) = \frac{A}{s-1} + \underbrace{\frac{\sim}{s^2(s+b)(s+1)+(s+a)^3}}_{\text{Can be expanded but not necessary.}}$$

$$A = \lim_{s \to 1} E(s)(s-1)$$

So unless A has a zero exactly at $s = 1$ then $A \neq 0$ and $|e(\infty)| \to \infty$, as expected.

7 For problem 3 consider $r(t) \Leftrightarrow R(s)$

$$R(s) = \frac{2}{s^3} e^{-10s}$$

So we have delay of 10 but this does not matter as we have $\lim_{t\to\infty} e(t) = e(\infty)$

$\infty + 10$ is still ∞.

Hence we still have, as in question 3, e

$$e(\infty) = \frac{2b}{a^3}$$

Quiz 8 2015

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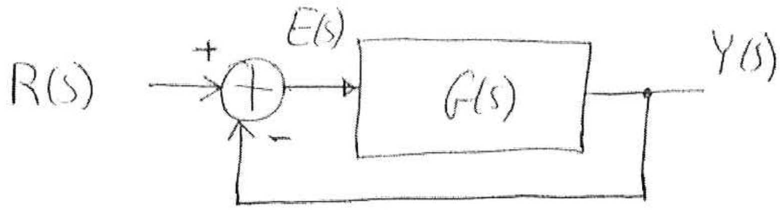
ENEL441 QUIZ 8 Name______________________________ UCID______________________________

Consider the feedback system shown in the diagram below:



Q1. Determine the steady state value of $e(\infty)$ for the transfer function of

$$\frac{E(s)}{R(s)} = \frac{(s+3)^3 s(s+1)}{(s+4)(s+2)}$$

for a ramp input signal $r(t) = tu(t) \Leftrightarrow R(s)$ and $e(t) \Leftrightarrow E(s)$.

(ans)

$$E(s) = \frac{1}{s^2}\frac{(s+3)^3 s(s+1)}{(s+4)(s+2)}$$

$$e(\infty) = \lim_{s\to 0} s\frac{1}{s^2}\frac{(s+3)^3 s(s+1)}{(s+4)(s+2)} = \frac{(3)^3(1)}{(4)(2)} = \frac{27}{8}$$

Q2. Determine if the feedback system is of type 0, type 1, type 2 or type 3. Explain your answer.

(ans) type 1 as it has a finite error for a ramp input.

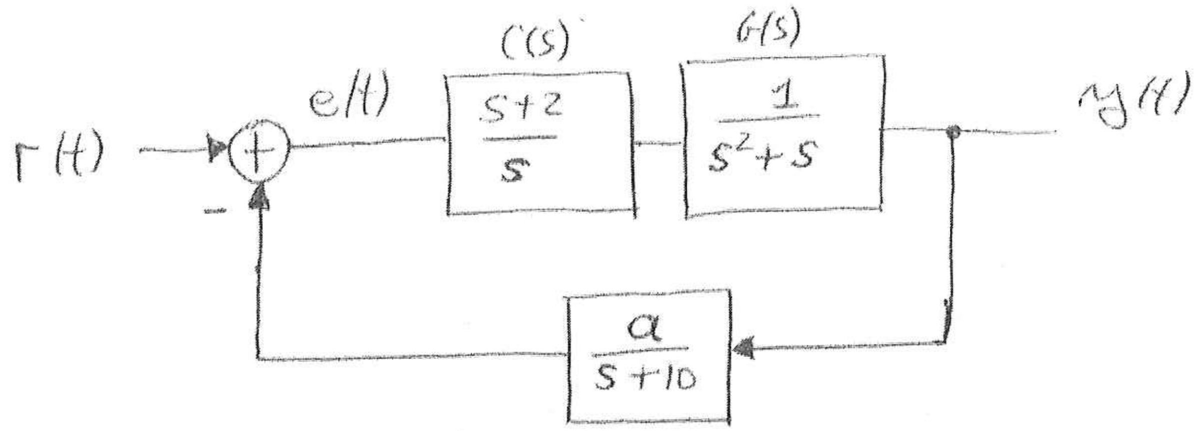
Q3. Assuming the feedback system is represented by the diagram above then determine an expression for G(s). You do not have to simplify your answer.

(ans) Let G=N/D such that E/R=1/(1+N/D) = D/(D+N). Then

$$D = (s+3)^3 s(s+1)$$
$$N = (s+4)(s+2) - (s+3)^3 s(s+1)$$
$$G = \frac{(s+4)(s+2) - (s+3)^3 s(s+1)}{(s+3)^3 s(s+1)} = \frac{1}{s}\frac{(s+4)(s+2) - (s+3)^3 s(s+1)}{(s+3)^3 (s+1)}$$

ENEL441 QUIZ 7 Name 2016 **UCID**

Consider the feedback system given below where a is a constant parameter.



Q1. Determine the transfer function $H(s)=\frac{E(s)}{R(s)}$. You do not have to simplify your answer.

$$H(s)=\frac{1}{1+\frac{s+2}{s}\frac{1}{s^2+s}\frac{a}{s+10}}$$

$$=\frac{\left(s^3+s^2\right)(s+10)}{\left(s^3+s^2\right)(s+10)+a(s+2)}$$

Q2. Is the loop of type 1, type 2 or type 3? Explain your answer.

type 2 as there are two integrators

Q3. Determine $e(\infty)$ for $r(t)=t^2u(t)$

$$e(\infty)=\lim_{s\to 0} s\frac{2}{s^3}\frac{\left(s^3+s^2\right)(s+10)}{\left(s^3+s^2\right)(s+10)+a(s+2)}$$

$$=\frac{2(s+1)(s+10)}{\left(s^3+s^2\right)(s+10)+a(s+2)}=\frac{20}{a2}$$

Q4. Without doing any calculations, what value of a is required such that $y(\infty)=r(\infty)$ for $r(t)=u(t)$

Explain your reasoning.

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(23)

a=10 such that the DC gain of the feedback is equal to unity and then as $e(\infty)=0$ it follows that $y(\infty)=r(\infty)$.

Aid Sheet

One sided Laplace Transform $F(s)=\int_0^\infty f(t)e^{-st}dt$, $f(t)=\frac{1}{2\pi j}\oint F(s)e^{st}ds$

$f(t)$	$F(s)$	$f(t)$	$F(s)$
1	$\delta(s)$	$\dot{f}$	$sF(s)-f(0^-)$
$\delta(t)$	1	$\ddot{f}$	$s^2F(s)-sf(0^-)-\dot{f}(0^-)$
$u(t)$	$1/s$	$\dddot{f}$	$s^3F(s)-s^2f(0^-)-s\dot{f}(0^-)-\ddot{f}(0^-)$
$t^m u(t)$	$m!/s^{m+1}$	$\int f(t)dt$	$F(s)/s$
$e^{-at}u(t)$	$1/(s+a)$	$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0} sF(s)$
$\frac{1}{(m-1)!}t^{m-1}e^{-at}u(t)$	$1/(s+a)^m$	$\sin(at)u(t)$	$\frac{a}{s^2+a^2}$
$f(t-T)$	$F(s)e^{-sT}$	$\cos(at)u(t)$	$\frac{s}{s^2+a^2}$
$tf(t)$	$-\frac{d}{ds}F(s)$	$x(t)*y(t)$	$X(s)Y(s)$

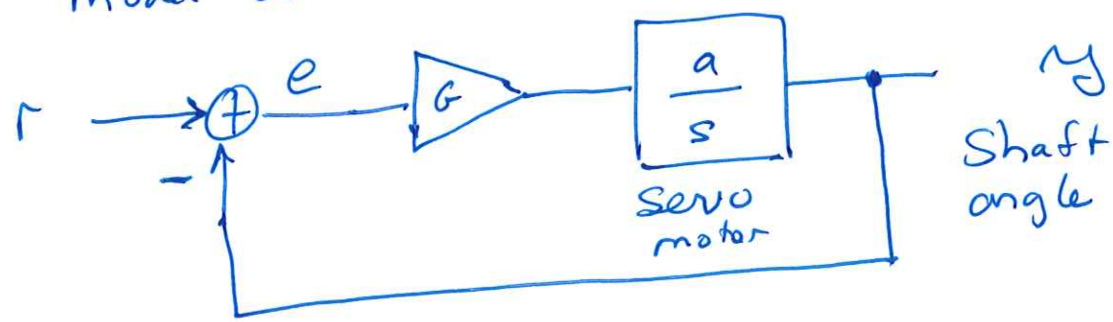
$*$ implies convolution as $x(t)*y(t)=\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau$

Examples and Quiz Prep Problems

Servo Positioner

A simple model of the motor is a simple integrator

A model of a servo control is then as follows



How well does $y(t)$ track $r(t)$ for step, ramp and parabolic?

As the open loop $H_{OL}(s) = \frac{Ga}{s}$ has one integrator then it is a type I loop which has

$$e(\infty) = \begin{cases} 0 & \text{step} \quad r(t) = u(t) \\ \text{finite} & \text{ramp} \quad r(t) = t\,u(t) \\ \infty & \text{parabolic.} \quad r(t) = t^2 u(t) \end{cases}$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+\frac{Ga}{s}} = \frac{s}{s+Ga}$$

$$e(\infty)\Big|_{r(t)=u(t)} = \lim_{s\to 0} s \cdot \frac{1}{s} \cdot \frac{s}{s+Ga} = 0$$

Zero steady state error

$$e(\infty)\Big|_{r(t)=t\,u(t)} = \lim_{s\to 0} s \cdot \frac{1}{s^2} \cdot \frac{s}{s+Ga} = \frac{1}{Ga}$$

finite steady state error

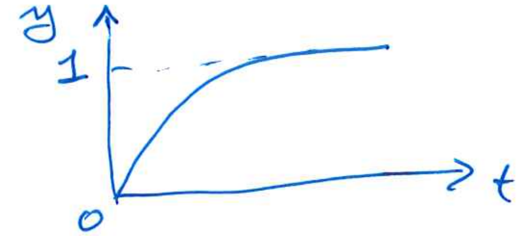
$$e(\infty)\Big|_{r(t)=t^2 u(t)} = \lim_{s\to 0} s \cdot \frac{2}{s^3} \cdot \frac{s}{s+Ga} = \infty$$

In previous example calculate $y(t)$ for step input

$$\frac{Y(s)}{R(s)} = \frac{\frac{Ga}{s}}{1+\frac{Ga}{s}} = \frac{Ga}{s+Ga}$$

$$Y(s) = \frac{1}{s}\,\frac{Ga}{s+Ga}$$

$$y(t) = 1 - e^{-Gat}$$



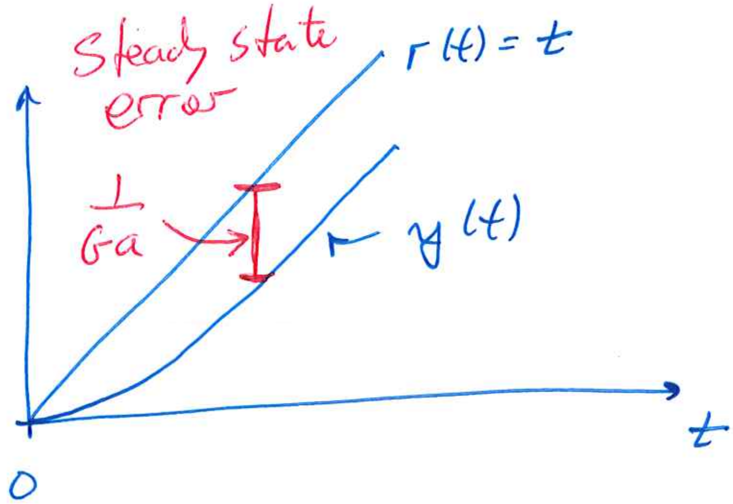
asymptotically approaches $r(t)$, ie $e(\infty) = 0$

Calculate $y(t)$ for ramp input

$$Y(s) = \frac{1}{s^2} \frac{Ga}{s+Ga}$$

$$y(t) = \underbrace{t\,u(t)}_{r(t)} - \underbrace{\frac{1}{Ga}\left(1 - e^{-Gat}\right)}_{\text{error term or transient}}$$

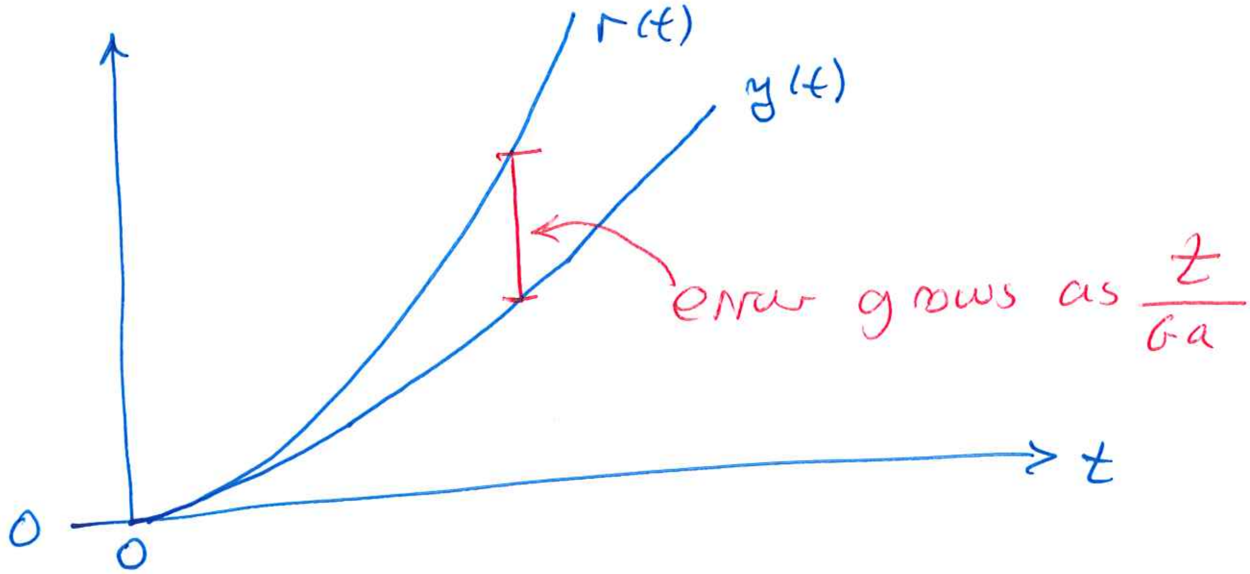
reaches steady state



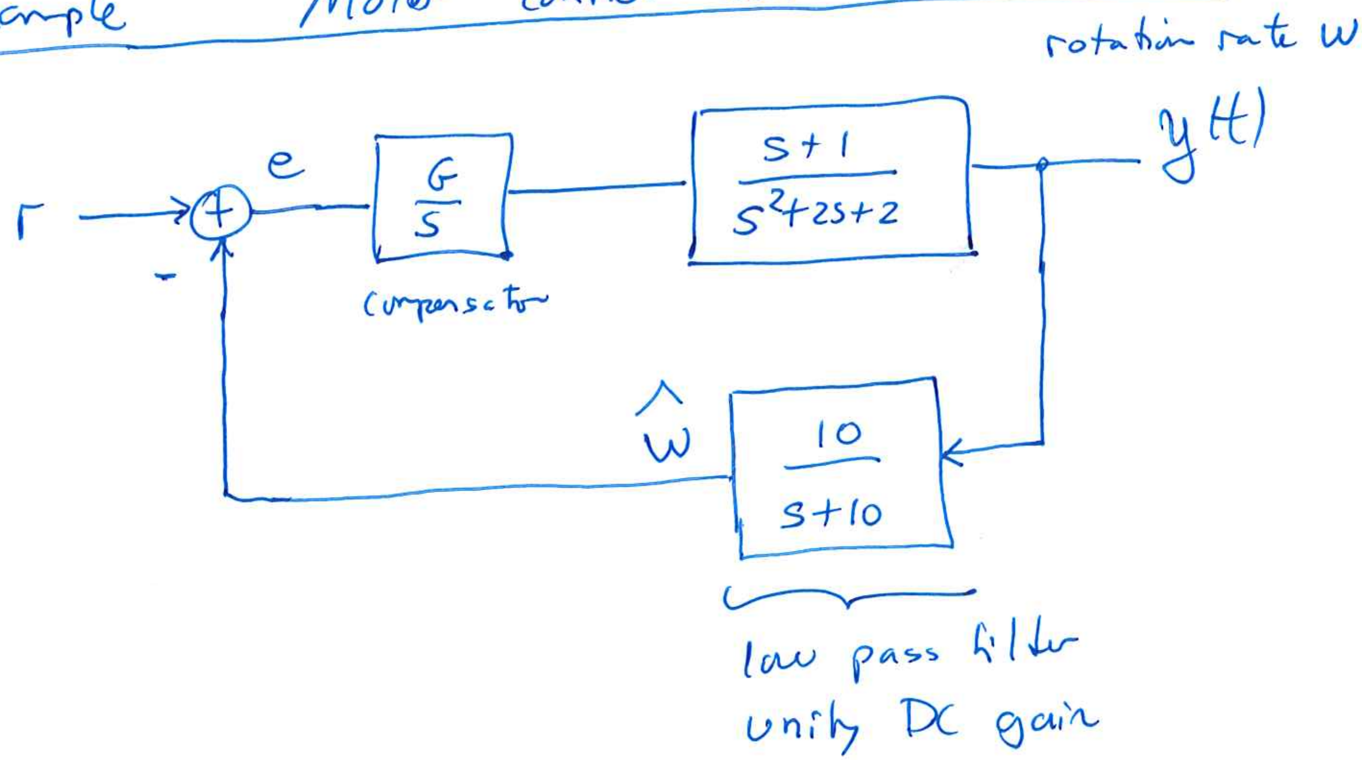
Calculate $y(t)$ for ramp input $r(t) = t^2/2$

$$Y(s) = \frac{1}{s^3} \frac{Ga}{s + Ga}$$

$$y(t) = \underbrace{\frac{t^2}{2}}_{r(t)} + \underbrace{\frac{1}{Ga^2}\left(1-e^{-Gat}\right)}_{\substack{\text{error component}\\ \text{reaches finite}\\ \text{value}}} - \underbrace{\frac{t}{Ga}}_{\substack{\text{error component}\\ \text{increases to } \infty}}$$



Example Motor control with Tachometer



Find $e(\infty)$ for a ramp input.

$$E(s) = R(s) \frac{1}{1 + \frac{G}{s} \frac{s+1}{s^2+2s+2} \frac{10}{s+10}}$$

$$e(\infty) = \lim_{s \to 0} s \cdot \frac{1}{s^2} \frac{s(s^2+2s+2)(s+10)}{s(s^2+2s+2)(s+10) + G(s+1)10}$$

$$= \frac{2 \cdot 10}{G \cdot 1 \cdot 10} = \frac{2}{G}$$

Note though this is not the true error of $r(t) - y(t)$ as tach has a LPF.

Instead calculate

$$err(\infty) = \lim_{s \to 0} s\left(R(s) - Y(s)\right)$$

$$\frac{Y(s)}{R(s)} = \frac{G(s+1)/s(s^2+2s+2)}{1 + G(s+1)\,10/s(s^2+2s+2)(s+10)}$$

$$\frac{Y(s)}{R(s)} = \frac{G(s+1)(s+10)}{s(s^2+2s+2)(s+10) + G(s+1)\,10}$$

$$\frac{R(s)-Y(s)}{R(s)} = 1 - \frac{Y(s)}{R(s)} = \underbrace{\frac{s(s^2+2s+2)(s+10) + G(s+1)10 - G(s+1)(s+10)}{s(s^2+2s+2)(s+10) + G(s+1)\,10}}$$

$$err(\infty) = \lim_{s \to 0} s \cdot \frac{1}{s^2} \cdot (\quad)$$

$$= \lim_{s \to 0} \frac{(s^2+2s+2)(s+10) - G(s+1)}{G(s+1)\,10}$$

$$= \frac{2 \cdot 10 - G}{G \cdot 10} = \frac{2}{G} - \underbrace{\frac{1}{10}}_{\text{new part.}}$$

Which is slightly different answer than before.

Why do you think that $err(\infty)$ is less than $e(\infty)$?

In the previous example, how can we tell that the loop is still a type I?

Open loop transfer function

$$H_{OL}(s) = \frac{G}{s} \cdot \frac{s+1}{s^2+2s+2} \cdot \frac{10}{s+10}$$

Still has a pole at $s=0$ $\Longrightarrow$ Type I

Suppose in the previous example that the transfer function of the tachometer is $\frac{s}{s+10}$ instead of $\frac{10}{s+10}$. Is it still type I?

$$H_{OL}(s) = \frac{G}{s} \; \frac{s+1}{s^2+2s+2} \; \frac{s}{s+1}$$

Note the zero of the tachometer at $s=0$ cancels the integrator pole of the compensator at $s=0$

Hence no longer type I.

Is this example realistic?