Solutions

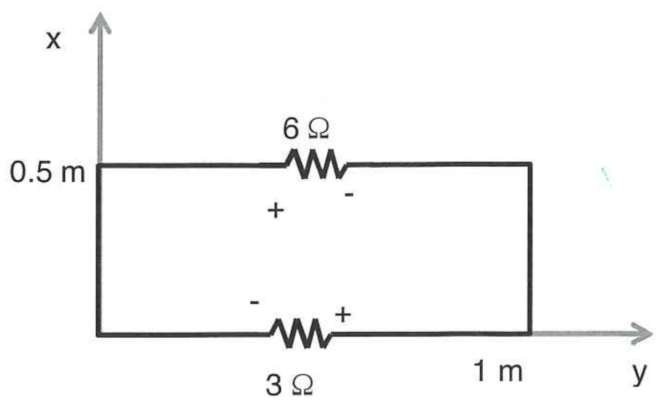
ENEL 476 – Assignment #1

Due at Monday February 6 at 5 pm

Drop boxes on 2nd floor of ICT

Question 1: The loop shown below is placed in an external magnetic flux density of

$$\mathbf{B} = -0.3e^{-t}\,\mathbf{a}_z \;\; \mathrm{Wb/m^2}$$



a) Find the total magnetic flux (ϕ) passing through the surface of the loop.
b) Find the voltage across the 6 Ω resistor
c) Find the induced current.
d) The induced current flows around the loop in the following direction (circle one):
- Clockwise
- Counterclockwise

Explain how the direction of induced current flow satisfies Lenz's law.

Question 2: An electric field in free space is given by:

$$\mathbf{E}(z,t) = (10\,\mathbf{a}_x + 4\,\mathbf{a}_y)\cos(3 \times 10^6 t - \beta z) \;\; \mathrm{V/m}$$

(2) a) Find the displacement current density, $\mathbf{J}_d(z,t)$.
(1) b) Write the electric field in phasor form, $\mathbf{E}_s(z)$.
(3.5) c) Using Faraday's law, find the magnetic field in phasor form, $\mathbf{H}_s(z)$. Leave your answer in terms of β.
(2.5) d) Using Ampere's law, find the value for β.
(1) e) Write the magnetic field in time-domain form, $\mathbf{H}(z,t)$, substituting in the value of β.

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Question 3

Consider a uniform plane wave propagating in a source-free region of free space (ε_r=1, μ_r=1, σ=0 S/m). The electric field is given by:

$$\mathbf{E}(z,t)=(10\,\mathbf{a}_x + 4\,\mathbf{a}_y)\cos(3 \times 10^6 t-\beta z)\ \text{V/m}$$

Find:

a) The phase constant, β.
b) The wavelength, λ.
c) The phase velocity, v_p or u.
d) The intrinsic impedance, η.
e) The magnetic field, $\mathbf{H}(z,t)$.

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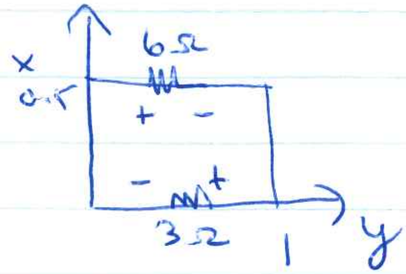
1. $\vec{B} = -0.3e^{-t}\vec{a}_z$ Wb/m²



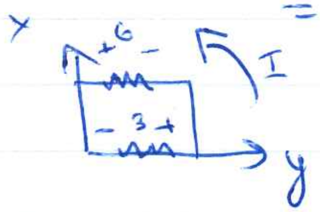
a) $\phi = \int \vec{B} \cdot d\vec{s}$

$$= \int_0^1 \int_0^{0.5} (-0.3e^{-t}\vec{a}_z) \cdot (dx\,dy\,\vec{a}_z)$$

$$= -0.3e^{-t}(0.5)(1)$$
$$= -0.15e^{-t} \text{ Wb}$$

b) $V_{emf} = -\frac{d}{dt}\phi$

$$= -\frac{d}{dt}(-0.15e^{-t})$$
$$= 0.15(-1)e^{-t}$$
$$= -0.15e^{-t}$$



$$V_{6\Omega} = \left(\frac{2}{3}\right)(-0.15e^{-t})$$
$$= -0.1e^{-t} \text{ V}$$
(wrt polarity indicated in figure)

c) $I = V_{emf}/R_{total}$

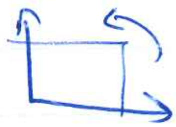
$$= -\frac{0.15}{9}e^{-t} \text{ A}$$

$I = -0.0167e^{t}$ A

d) counter-clockwise

e) Lenz's law → as time increased, Φ decreased

$\therefore$



induced current has induced flux in $-\vec{a}_z$ direction, which opposes decrease in $-\vec{a}_z$ direction of original flux

2. $\vec{E}(z,t) = (10\vec{a}_x + 4\vec{a}_y)\cos(3\times10^6 t - \beta z)$ V/m

↳ free space

a) $\vec{J}_d(z,t) = \frac{d}{dt}\vec{D}(z,t)$

$\vec{D}(z,t) = \epsilon_0 \vec{E}(z,t)$

$$\left.\begin{aligned}\beta &= \omega\sqrt{\mu_0\epsilon_0} \\ &= \frac{3\times10^6}{3\times10^8} \\ &= 0.01\end{aligned}\right] \text{UPW} \Rightarrow \text{P3}$$

$$\vec{J}_d(z,t) = \frac{d}{dt}\left[(\epsilon_0)(10\vec{a}_x + 4\vec{a}_y)\cos(3\times10^6 t - 0.01z)\right]$$

$$= -\epsilon_0(10\vec{a}_x + 4\vec{a}_y)(3\times10^6)\sin(3\times10^6 t - 0.01z)$$

(3)

$$\vec{J}_d(z,t) = -3\times 10^6 \epsilon_0 (10\vec{a}_x + 4\vec{a}_y)\sin(3\times 10^6 t - 0.01z) \text{ A/m}^2$$

b) $$\vec{E}_s(z) = (10\vec{a}_x + 4\vec{a}_y)e^{-j0.01z} \text{ V/m}$$

c) $$\nabla\times\vec{E}_s = -j\omega\mu_0\vec{H}_s$$

$$\nabla\times\vec{E}_s = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_{sx} & E_{sy} & 0 \end{vmatrix}$$

$$= \vec{a}_x\left(-\frac{\partial}{\partial z}E_{sy}\right) - \vec{a}_y\left(-\frac{\partial}{\partial z}E_{sx}\right) + \vec{a}_z\left(\frac{\partial}{\partial x}E_{sy} - \frac{\partial}{\partial y}E_{sx}\right)^{\nearrow 0}$$

$$= -\frac{\partial}{\partial z}\left(4e^{-j0.01z}\right)\vec{a}_x + \frac{\partial}{\partial z}\left(10e^{-j0.01z}\right)\vec{a}_y$$

$$= +0.04e^{-j0.01z}\vec{a}_x - j0.1e^{-j0.01z}\vec{a}_y$$

$$\vec{J}_d(z,t) = -3\times10^6\,\epsilon_0(10\vec{a}_x + 4\vec{a}_y)\sin(3\times10^6 t - \beta z)$$

b) $\vec{E}_s(z) = (10\vec{a}_x + 4\vec{a}_y)\,e^{-j\beta z}$ V/m

c) $\nabla\times\vec{E}_s = -j\omega\mu_0\vec{H}_s$

$$\nabla\times\vec{E}_s = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{sx} & E_{sy} & E_{sz} \end{vmatrix}$$

$$= \vec{a}_x\left(\frac{-\partial}{\partial z}E_{sy}\right) + \left(\frac{\partial}{\partial z}E_{sx}\right)\vec{a}_y$$

$$= \frac{-\partial}{\partial z}(4e^{-j\beta z})\vec{a}_x + \frac{\partial}{\partial z}(10e^{-j\beta z})\vec{a}_y$$

$$= j4\beta e^{-j\beta z}\vec{a}_x - j10\beta e^{-j\beta z}\vec{a}_y$$

$$-j\omega\mu_0\vec{H}_s = j4\beta e^{-j\beta z}\vec{a}_x - j10\beta e^{-j\beta z}\vec{a}_y$$

$$\vec{H}_s = \frac{-4\beta}{\omega\mu_0}e^{-j\beta z}\vec{a}_x + \frac{10\beta}{\omega\mu_0}e^{-j\beta z}\vec{a}_y$$

d) $\nabla\times\vec{H}_s = j\omega\epsilon_0\vec{E}_s$

$$\nabla\times\vec{H}_s = \frac{-\partial}{\partial z}(H_{sy})\vec{a}_x + \left(\frac{\partial}{\partial z}H_{sx}\right)\vec{a}_y$$

$$= \frac{-\partial}{\partial z}\left(\frac{10\beta}{\omega\mu_0}e^{-j\beta z}\right)\vec{a}_x + \frac{\partial}{\partial z}\left(\frac{-4\beta}{\omega\mu_0}e^{-j\beta z}\right)\vec{a}_y$$

$$\nabla \times \vec{H}_s = \frac{j10\beta^2}{\omega\mu_0} e^{-j\beta z}\vec{a}_x + \frac{j4\beta^2}{\omega\mu_0} e^{-j\beta z}\vec{a}_y$$

$$\vec{E}_s = \frac{j10\beta^2}{j\omega^2\epsilon_0\mu_0} e^{-j\beta z}\vec{a}_x + \frac{j4\beta^2}{j\omega^2\mu_0\epsilon_0} e^{-j\beta z}\vec{a}_y$$

$$\cancel{10e^{-j\beta z}} = \frac{\cancel{10}\beta^2}{\omega^2\epsilon_0\mu_0} e^{-j\beta z} \qquad \frac{\cancel{4}\beta^2}{\omega^2\mu_0\epsilon_0} e^{-j\beta z} = \cancel{4e^{-j\beta z}}$$

$$\beta^2 = \omega^2\epsilon_0\mu_0 \qquad \beta^2 = \omega^2\mu_0\epsilon_0$$

$$\beta = \omega\sqrt{\epsilon_0\mu_0} \qquad \Rightarrow \beta = 0.01 \text{ rad/m}$$

$$= \frac{3\times10^6}{3\times10^8}$$

$$= 0.01 \text{ rad/m}$$

e) $$\vec{H}(z,t) = \frac{-4\beta}{\omega\mu_0}\cos(3\times10^6 t - 0.01z)\vec{a}_x + \frac{10\beta}{\omega\mu_0}\cos(3\times10^6 t - 0.01z)\vec{a}_y$$

$$\frac{\beta}{\omega\mu_0} = \frac{\omega\sqrt{\epsilon_0\mu_0}}{\omega\mu_0} = \sqrt{\frac{\epsilon_0}{\mu_0}} = \frac{1}{120\pi}$$

$$\vec{H}(z,t) = \frac{-4}{120\pi}\cos(3\times10^6 t - 0.01z)\vec{a}_x + \frac{10}{120\pi}\cos(3\times10^6 t - 0.01z)\vec{a}_y$$

$$= \frac{-1}{30\pi}\cos(3\times10^6 t - 0.01z)\vec{a}_x + \frac{1}{12\pi}\cos(3\times10^6 t - 0.01z)\vec{a}_y$$

A/m

3. $\vec{E}(z,t) = (10\vec{a}_x + 4\vec{a}_y)\cos(3\times10^6 t - \beta z)$

a) $\beta = \omega\sqrt{\mu_0\epsilon_0}$

$= \omega / c$

$= \dfrac{3\times10^6}{3\times10^8}$

$\beta = 0.01$ rad/m

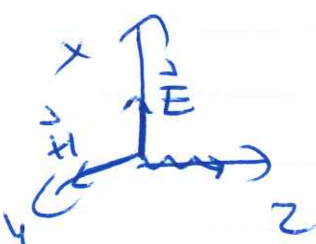
b) $\lambda = \dfrac{2\pi}{\beta}$

$= 200\pi$ m

c) $v_p = \dfrac{1}{\sqrt{\mu_0\epsilon_0}}$

$v_p = 3\times10^8$ m/s

d) $\eta = 120\pi\ \Omega$

e) 

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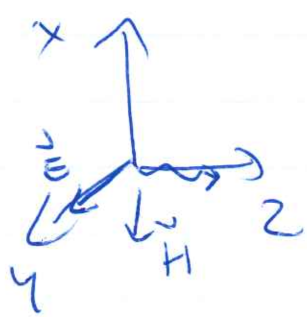

$\vec{H}(z,t) = \dfrac{-4}{120\pi}\cos(3\times10^6 t - 0.01z)\vec{a}_x + \dfrac{10}{120\pi}\cos(3\times10^6 t - 0.01z)\vec{a}_y$ A/m