Yani's cheap plastic review of Newton-Raphson (NR) Method

Idea: Using linear approximation of the function $f(x)$ or set of functions F(X) to solve them.

Taylor Series:

$$f(x) = f(x_0) + \sum_{k=1}^{\infty} \frac{f^k(x_0)}{k!}(x-x_0)^k \;=\; f(x_0) + f'(x_0).(x-x_0) + \frac{f''(x_0)}{2!}.(x-x_0)^2 + \cdots$$

Example: $f(x) = e^x$. Approximate e^1 and $e^{0.01}$ using Taylor Series expansion at $x_0 = 0$.

e^x is convenient since $f(x) = f'(x) = f''(x) = f^k(x) = e^x$ and they are all equal to 1 at $x = 0$ T.S. expansion gives:

$$e^x = e^0 + e^0(x-0) + \frac{e^0}{2!}(x-0)^2 + \cdots \;= 1 + x + \frac{x^2}{2} + \cdots$$

If we approximate e^x using only the first two terms: $e^x \approx 1 + x$

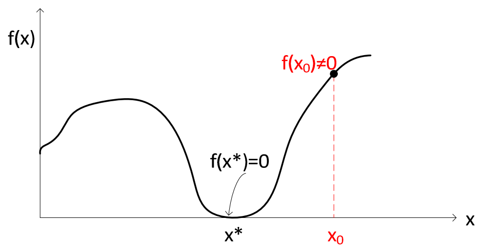
Let's calculate this for a couple of points:

x	e^x (exact value)	Taylor Series Approximation
0.01	1.0105	1.01
1	2.718	2

i.e. Taylor Series approximation is more accurate for x closer to x_0

NR for scalar non-linear function

One equation and one unknown.



Start at x_0
Want to find Δx such that $\Delta x = x^* - x_0$

Taylor Series expansion around x_0:

$$f(x) = f(x_0) + f'(x_0).(x-x_0) + H.O.T.$$

For $x = x^*$,

$$f(x^*) = f(x_0) + f'(x_0).(x^*-x_0) + \cancel{H.O.T.}$$

$$0 \approx f(x_0) + f'(x_0).\Delta x$$

Or, $\Delta x \approx -\frac{f(x_0)}{f'(x_0)}$

$$\Delta\tilde{x} = -\frac{f(x_0)}{f'(x_0)}$$

- $\Delta\tilde{x}$ can approximate how much to change x_0 to get to x^*. We will call this the update term in NR.
- If x_0 and x^* are close, $\Delta\tilde{x}$ will be more accurate. (i.e. its value will be closer to the true Δx)

Steps:

1) Choose an initial value x_0 and set $k = 0$. (k keeps track of iteration #)

2) Solve for $\Delta\tilde{x}_k = -\frac{f(x_k)}{f'(x_k)}$

3) Let $x_{k+1} = x_k + \Delta\tilde{x}_k$

4) Check for convergence, i.e. $|f(x_{k+1})| < \varepsilon$ or $|\Delta\tilde{x}_k| < \varepsilon$

5) If converged, stop. Else, k++ and go to step 2.

NR for system of non-linear equations

$$\mathrm{F(X)} = \begin{bmatrix} f_1(X) \\ f_2(X) \\ \vdots \\ f_n(X) \end{bmatrix} \quad \text{where} \quad \mathrm{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Taylor Series expansion around an (N-dimensional) point X^0:

$$F(X) = F(X^0) + J(X^0).(X - X^0) + H.O.T.$$

Jacobian: n x n matrix of partial derivatives:

$$J(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Same as before,

$$F(X^*) = F(X^0) + J(X^0).(X^* - X^0) + \cancel{H.O.T}$$

$$[0] \approx F(X^0) + J(X^0).(X^* - X^0)$$

$$J(X^0).\Delta\tilde{X} = -F(X^0)$$

Steps:

- Symbolic evaluation of the Jacobian J(X) for a generic X

1) Choose an initial value X^0 and set $k = 0$.

2) Compute the Jacobian and the mismatch vector: Plug in X^k into $J(X)$ and $F(X)$

3) Solve the linear problem $J(X^k).\Delta\tilde{X}^k = -F(X^k)$ to find $\Delta\tilde{X}^k$

4) Let $X^{k+1} = X^k + \Delta\tilde{X}^k$

5) Check for convergence, i.e. $\|F(X^{k+1})\| < \varepsilon$ or $\|\Delta\tilde{X}^k\| < \varepsilon$

6) If converged, stop. Else, k++ and go to step 2.