Nyquist Criterion for Closed Loop stability analysis

Reading assignment Nise sections 10.3 - 10.6

If $H_{OL}(s)$ is in form of rational polynomial of s then best analysis is Root locus.

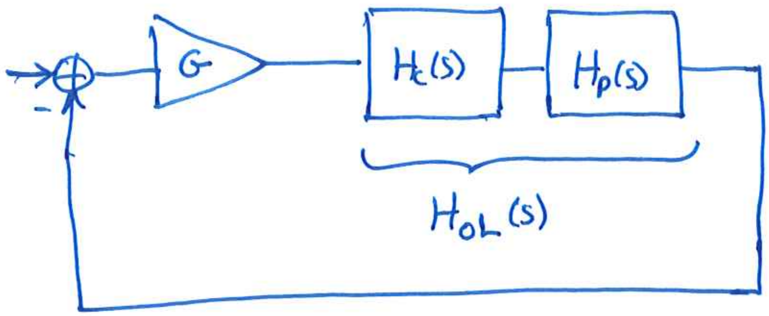
As with Bode, if only measured samples of $H_{OL}(j\omega)$ are available or $H_{OL}(j\omega)$ not in form of rational polynomial of s (or $j\omega$) then root locus not useful.

Bode simple to plot and use but has problems and analysis outcome confusing when slope of $\angle H_{OL}(j\omega)$ increases around point of $\angle H_{OL}(j\omega) = -\pi$.

More General and powerful method is Nyquist Criteria. Nyquist plot more difficult to generate but gives much more information.

Nyquist Fundamentals

Define our feedback loop as:



The closed loop response from any input to any output is:

just means some function of s in numerator that is not important here.

$$H_{CL} = \frac{\sim}{1 + G\,H_{OL}(s)}$$

Marginal stability

$$s = j\omega$$

$$1 + G\,H_{OL}(j\omega) = 0$$

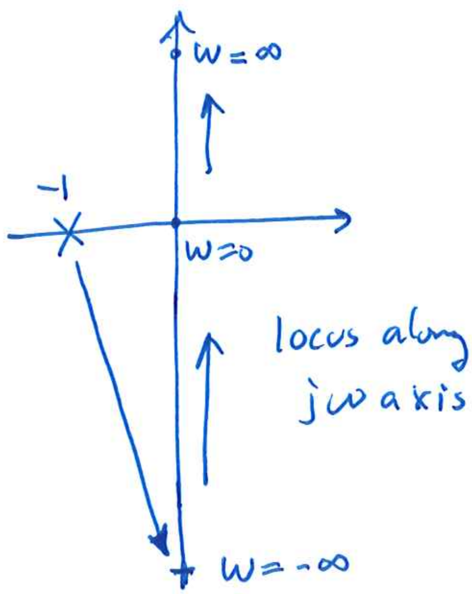
Nyquist Plot $\quad H_{OL}(j\omega)$ for $-\infty < \omega < \infty$

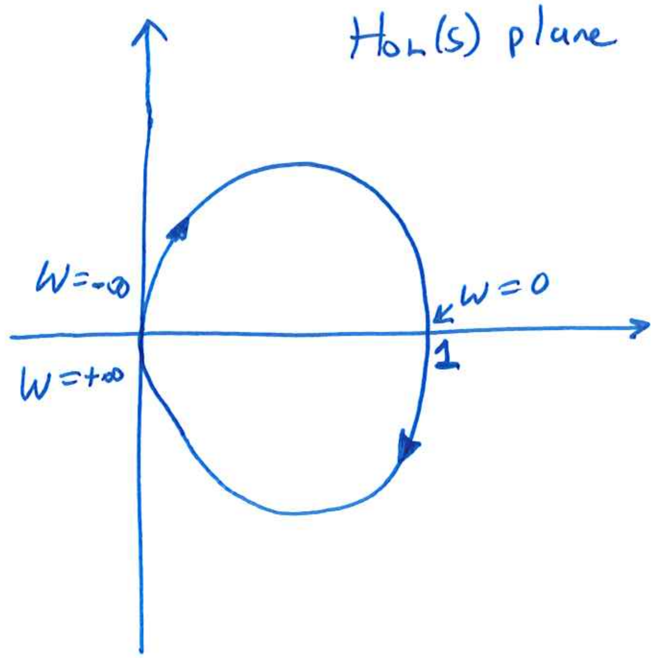
How do we use Nyquist?

find G such that $\underbrace{-\frac{1}{G} = H_{OL}(j\omega)}_{\text{condition for marginal stability.}}$

<u>Example</u>

$$H_{OL}(s) = \frac{1}{s+1}$$





Start at $s = -j\infty$ $\quad |H_{OL}| = 0$

move up $j\omega$ axis note phasor angle changes from $-90°$ towards zero as $\omega \to 0$

Hence locus starts at $+90°$ and goes towards $0°$

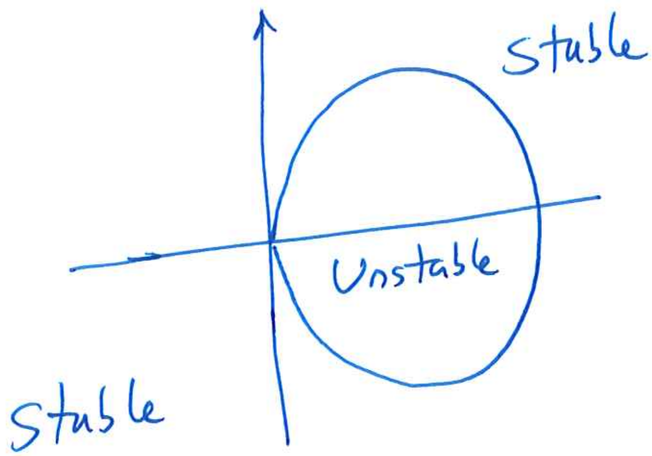
(See example 10.4 pg 555-557 in Nise.)

Plug H_{OL} into Matlab and get a Nyquist Plot.

$H_{OL} = tf(1, [1, 1])$

Nyquist (H_{OL})

Back to Nyquist Plot of $H_{OL}(s) = \frac{1}{s+1}$



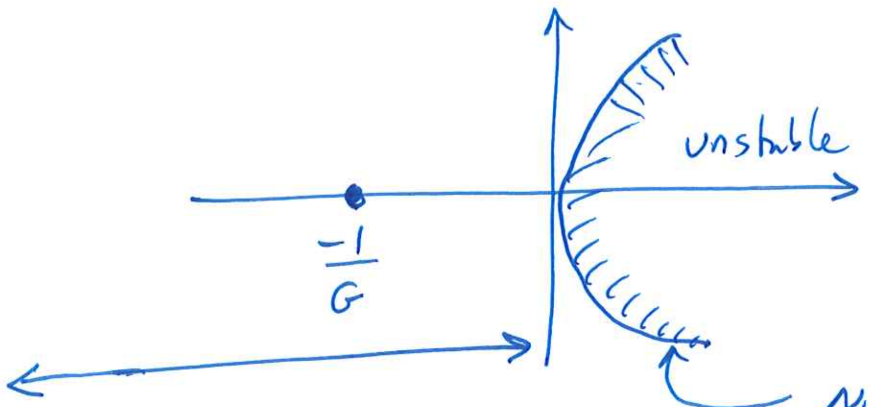
Nyquist stability condition (simplified)

The negative feedback loop is stable if the point $-1/G$ is outside any encirclement of the Nyquist plot.

Hence the labelling for stable and unstable regions in plot above.

(use simplified criteria for cases where all poles of $H_{OL}(s)$ are in LHP.)

Stability for region of $G \in$ Real, $G > 0$

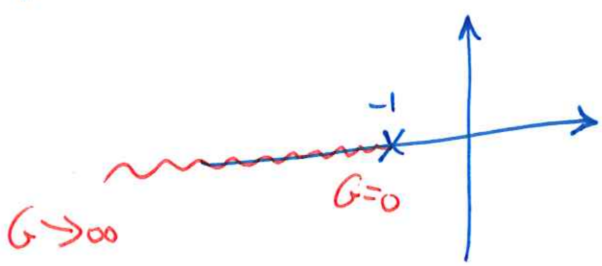


Nyquist plot for $\frac{1}{s+1}$

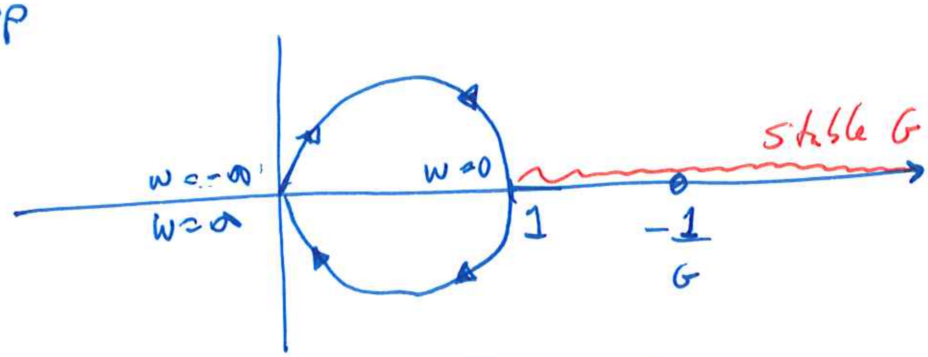
G stable in this region

$0 < G < \infty$

Note this is consistent with Root Locus for $\frac{1}{s+1}$



But there is another region where G results in a stable loop



$-1 < G < 0$

This range of $G<0$ is not covered by the Root locus we have looked at.

$G>0$ & negative feedback, 180° root locus
8 - rules
familiar with this

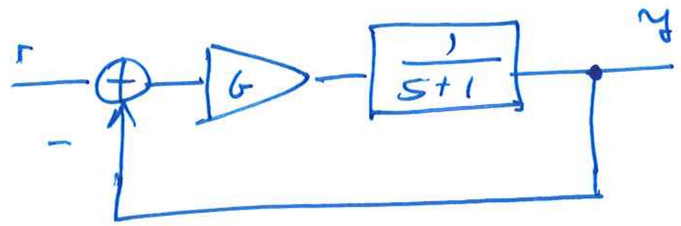
$G<0$ & negative feedback
$G>0$ & positive feedback
- 0° root locus
- new rules
- less common
- have not studied in 441.

In matlab
rlocus (H_{OL}) 180° root locus
rlocus ($-H_{OL}$) 0° root locus.

Going back to the Nyquist plot we have the two ranges of G giving stable behavior:

$G>0$
$-1<G<0$
combine these regions
$G>-1$
$G \in$ Real

Let us do a direct calculation of stability region by calculation the closed loop pole directly

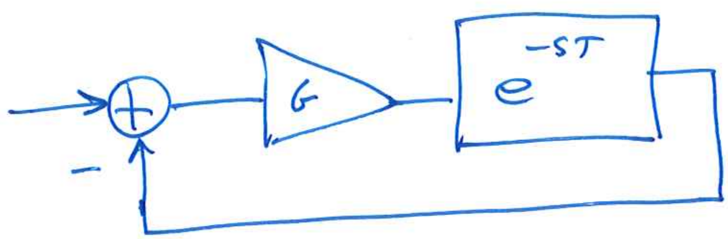


$$H_{CL}(s) = \frac{\frac{G}{s+1}}{1+\frac{G}{s+1}} = \frac{G}{s+(1+G)}$$

$G > -1$ for stability $\Rightarrow$ agrees with Nyquist plot.

Now consider $H_{OL}(s)$ to be a pure delay.

$H_{OL}(s) = e^{-sT} \Longleftrightarrow \delta(t-T)$ (delay of T)

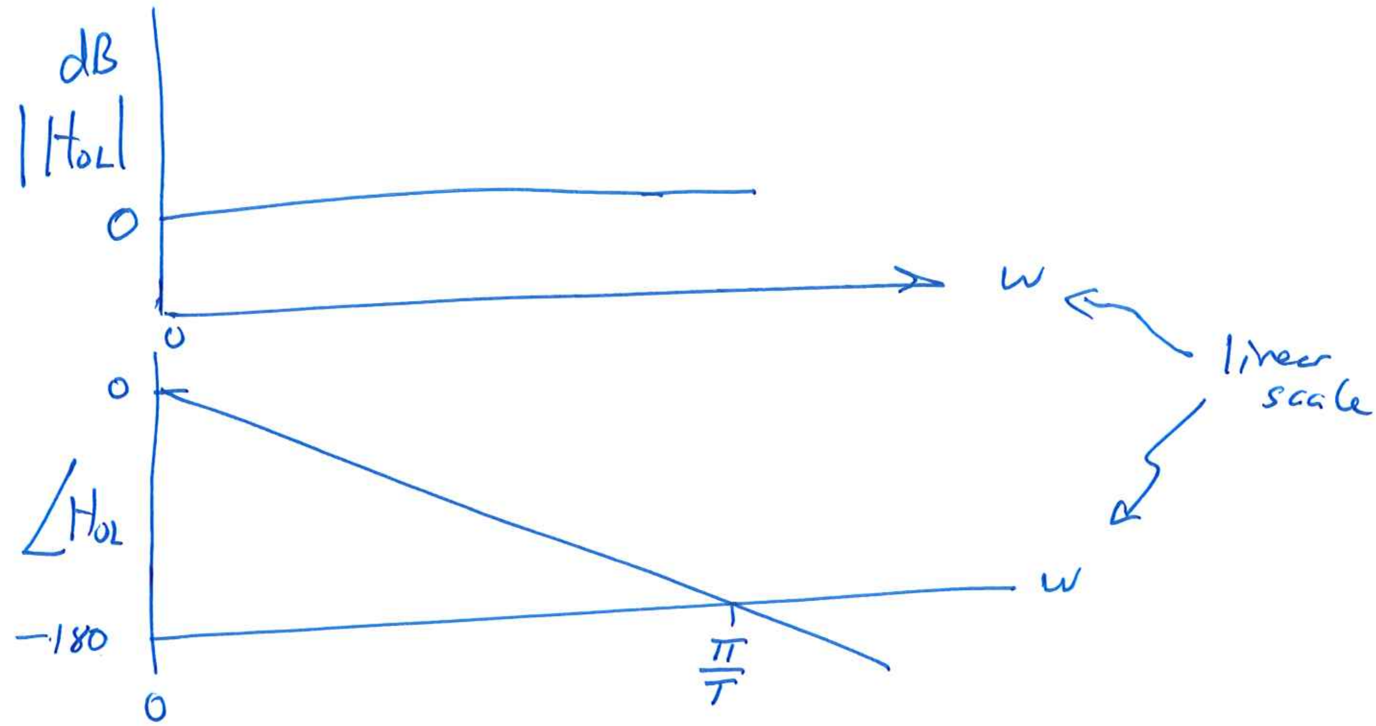


Nyquist plot of $e^{-j\omega T}$

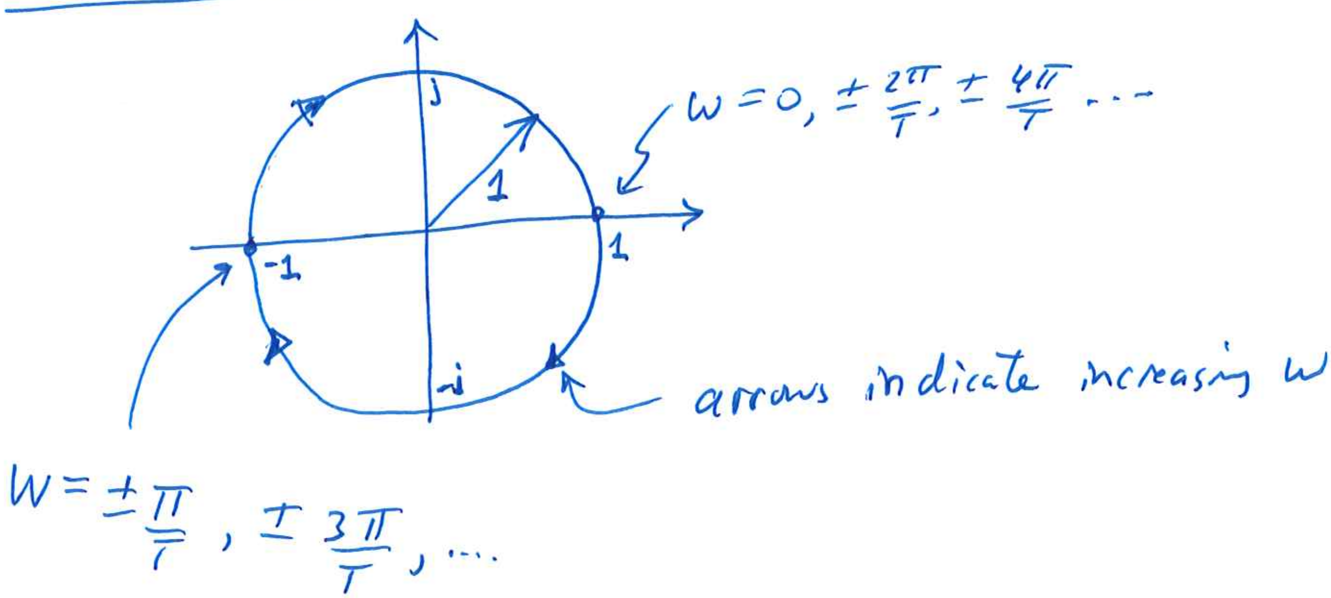
$\left| e^{-j\omega T} \right| = 1$ for all ω

$\angle e^{-j\omega T} = -\omega T$

Bode Plot of $H_{OL}(j\omega) = e^{-j\omega T}$



Nyquist Plot of $H_{OL}(j\omega) = e^{-j\omega T}$ (Pure Delay)



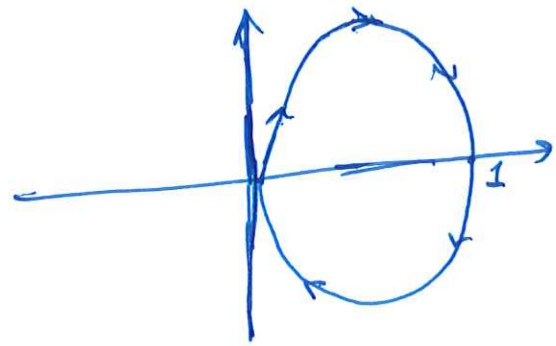
Again outside of region that is encircled is stable. Hence $|G| < 1$ for stability as point $-\frac{1}{G}$ will be outside encirclement of Nyquist plot.

An advantage with Nyquist over root locus is that a pole-zero model of $H_{OL}(s)$ is not required.

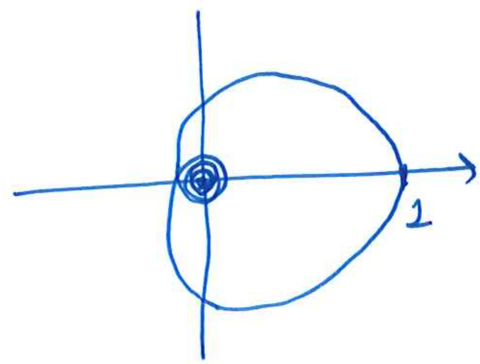
Can accomodate delays directly in Nyquist.

eg. $H_{OL}(s) = \dfrac{e^{-sT}}{s+1}$

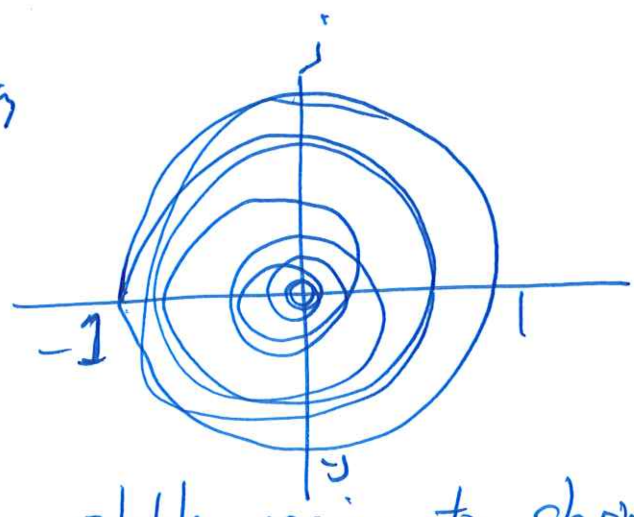
$H_{OL}(j\omega)$ for no delay



$H_{OL}(j\omega)$ for small delay
$T \ll 1$



$H_{OL}(j\omega$ for very large delay
$T \gg 1$



Note how delay causes stable region to shrink and unstable region to grow.

You can generate these plots in Matlab using:

H_{OL} = tf(1, [1, 1], 'iodelay', T)

nyquist(H_{OL})

As with Bode plots you can enter a specific frequency range as

w = logspace(w_{Low}, w_{high}, #samples)

nyquist(H_{OL}, w)

We use simplified Nyquist criteria where we have assumed no poles of $H_{OL}(s)$ in right hand plane. But what about

$$H_{OL}(s) = \frac{1}{s} \frac{1}{s+1}$$

marginal stable pole on $j\omega$ axis

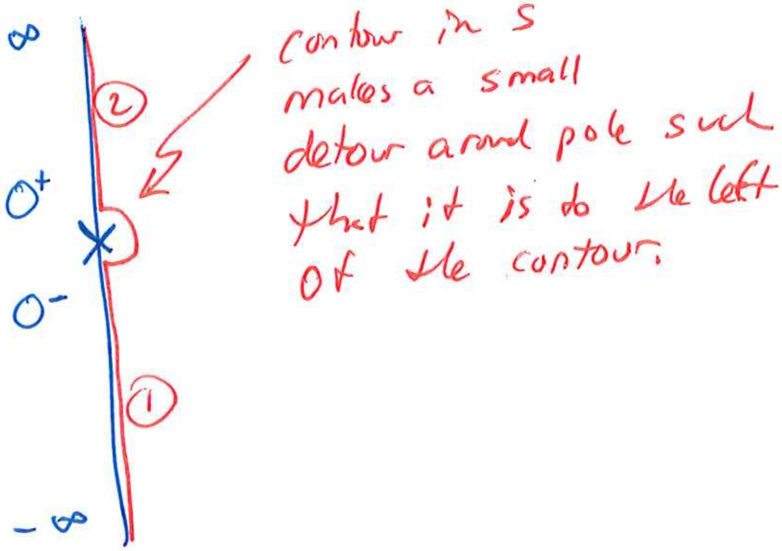
Or how to use for unstable problems such as inverted pendulum.

$$H_p(s) = \frac{1}{s^2 - 1}$$

Issue with poles on the $j\omega$ axis

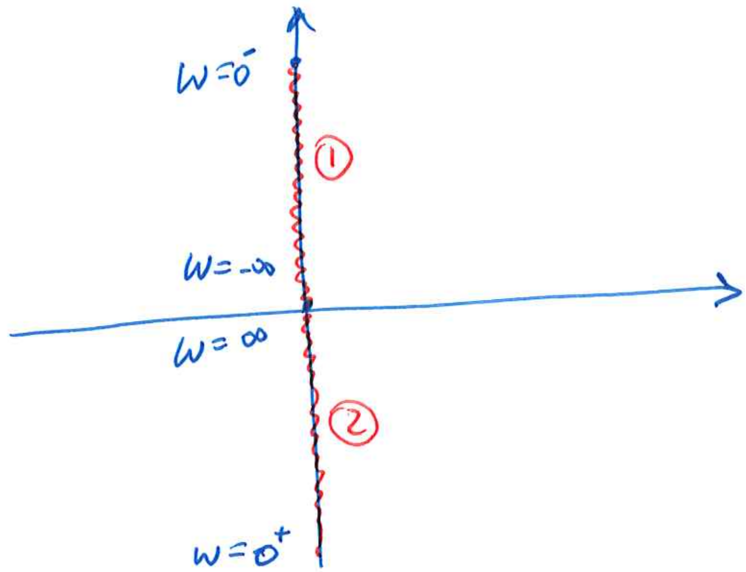
Consider $H_{OL}(s) = \frac{1}{s}$

A problem with application of Nyquist is that the contour in ω has to be on the right side of any open loop poles. In this case we have a pole at $s=0$

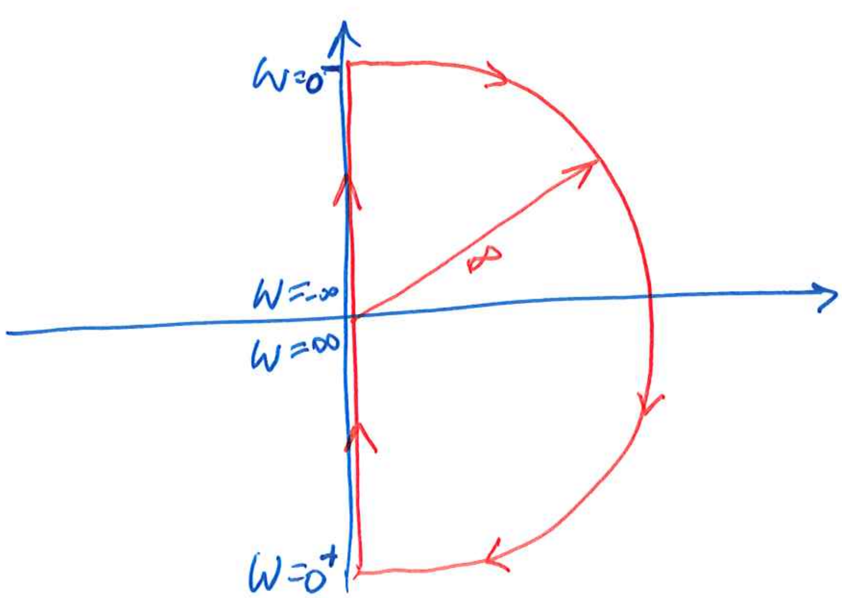


Draw Nyquist of segment $\omega = -\infty$ to $\omega = 0^-$ and the segment of $\omega = 0^+$ to $\omega = \infty$.

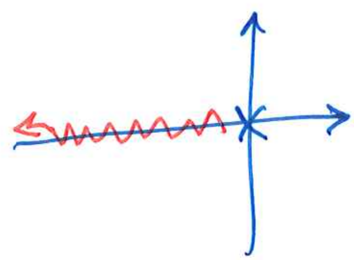
Then consider the portion of the contour from $\omega = 0^-$ to $\omega = 0^+$.



Then consider the contour from 0^- to 0^+ and note the pole is the only contribution to the Nyquist plot. We go in an infintesimal semi circle. for $180°$ ccw Therefore the Nyquist plot should do the opposite. Have an infinite radius and $180°$ cw. Then we can draw the complete locus as below:



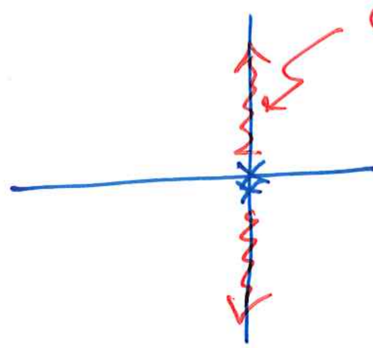
Now consider the stability based on the Nyquist plot. It reveals that $-\infty < -\frac{1}{G} < 0$ or $0 < G < \infty$. Note this agrees with Root locus,



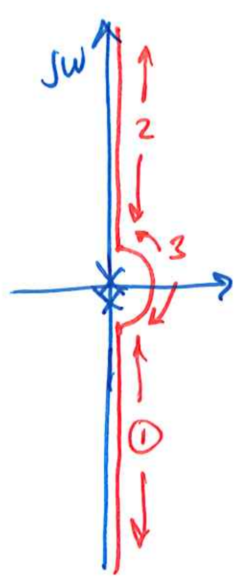
<u>Example</u> $H_{OL}(s) = \frac{1}{s^2}$

From root locus we have that this is marginally stable closed loop for all positive values of G. What does Nyquist say?

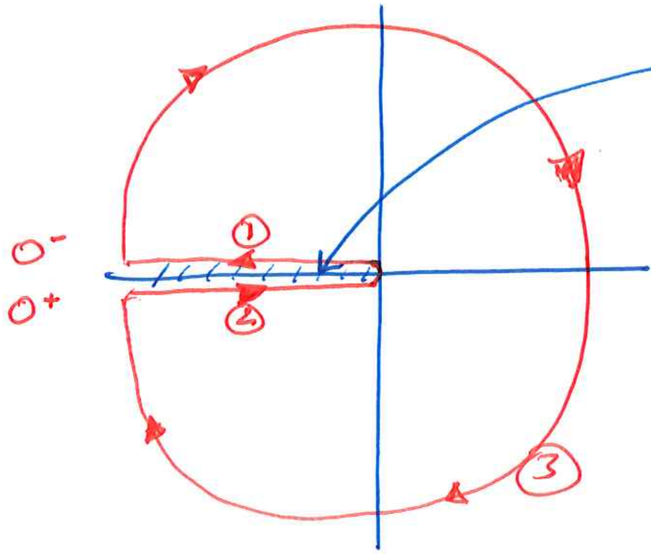
First root locus



<u>Nyquist</u>



Contour segments	
1	$-\infty < \omega < 0^-$
2	$0^+ < \omega < \infty$
3	0^- to 0^+



infintesimal sliver of a region of stability

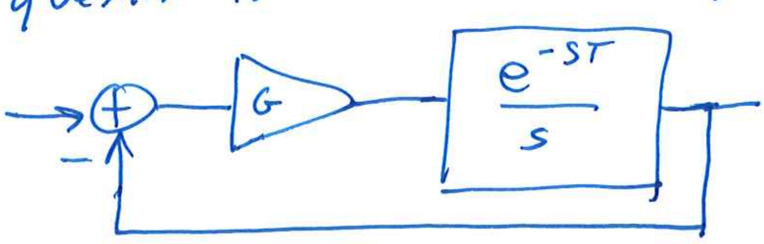
Only a sliver of infintesimal width is not enclosed and can be considered stable.

However as the region is infintesimal it is considered marginally stable.

Note we detour around two poles at $s=0$. Arc in splane is 180° but times two poles is 360°. Hence for the 180° ccw contour we do 360° cw in Nyquist.

Phase Margin

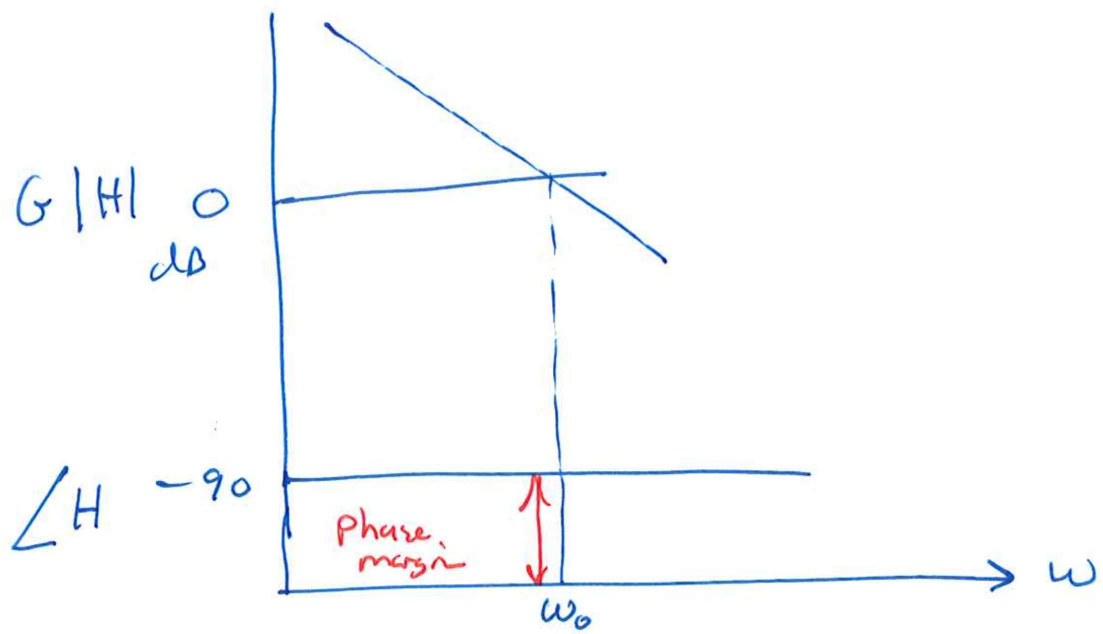
Consider the problem of an integrator with a delay. The question is how much delay can we tolerate?



We group the delay with the gain as

$$\underbrace{G e^{-j\omega T}}_{G_0} \cdot \underbrace{\frac{1}{s}}_{H(s)}$$

Consider Bode plot for $T=0$

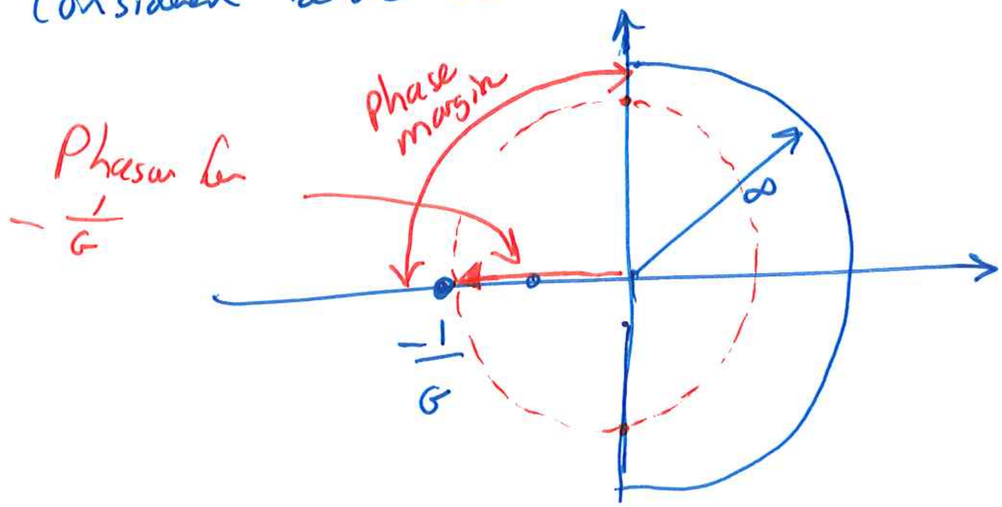


We have a phase margin of 90° as the open loop gain drops to 0 dB at ω_0

Hence we can tolerate a delay of $T\omega_0 = \frac{\pi}{2}$ before instability sets in.

What does Nyquist say?

The Nyquist plot for $H_{OL}(s) = \frac{1}{s}$ was considered before as below:



now we have a complex gain G. The phase that can be applied to G before instability is shown in the Nyquist plot by the rotation of the phasor at $-\frac{1}{G}$.

The phase margin is the angle of rotation that needs to be applied to this phasor such that it intercepts the Nyquist plot encirclement.

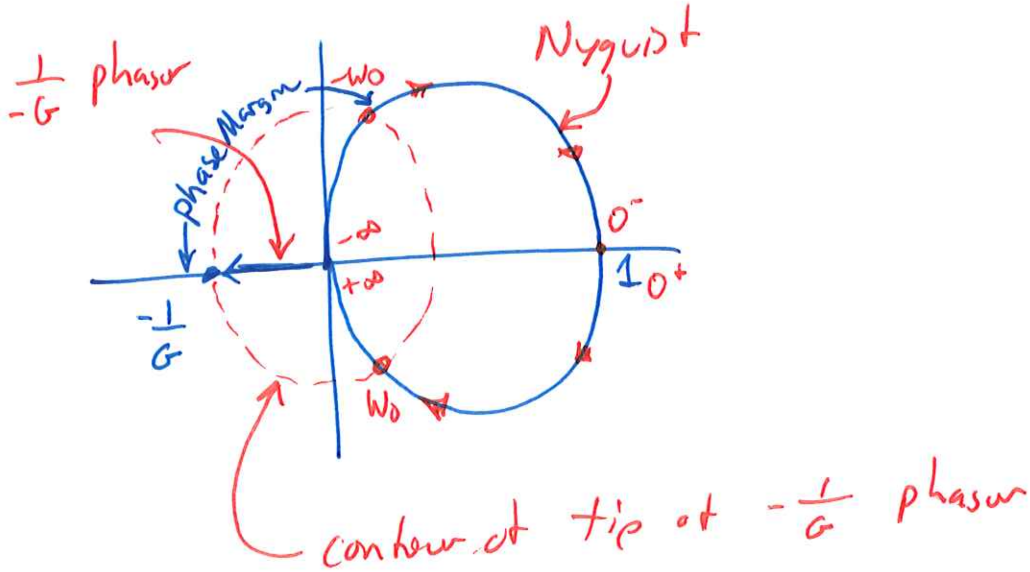
This phase margin is indicated in the diagram above.

Note in this example the phase margin is 90 degrees as for the bode plot.

Another example on phase margin

Suppose we have $H_{OL}(s) = \dfrac{e^{-sT}}{s+1}$

Sketch the Nyquist and show the phase margin.



① Select G, $G \in$ Real $G > 0$

② phasor of $-\frac{1}{G}$ rotated to form circle.

③ find points of interception of Nyquist plot. This gives W_0 the frequency corresponding to the intercept point.

④ Determine Phase Margin as in diagram

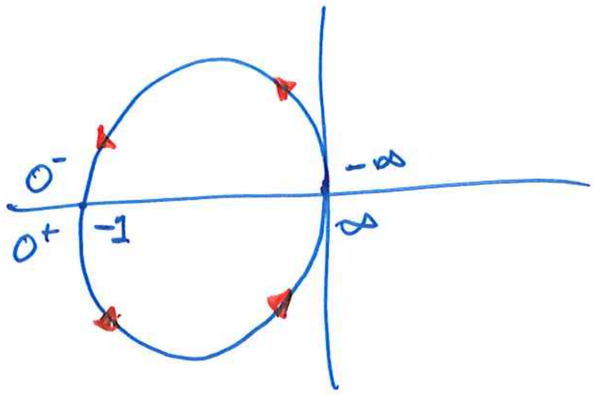
⑤ Determine maximum delay that can be tolerated as

phase margin $= W_0 \cdot$ delay that can be tolerated

Example — Unstable open loop pole.

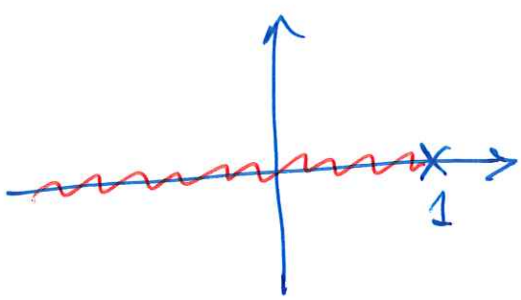
$$H_{OL}(s) = \frac{1}{s-1}$$

Nyquist plot is shown below



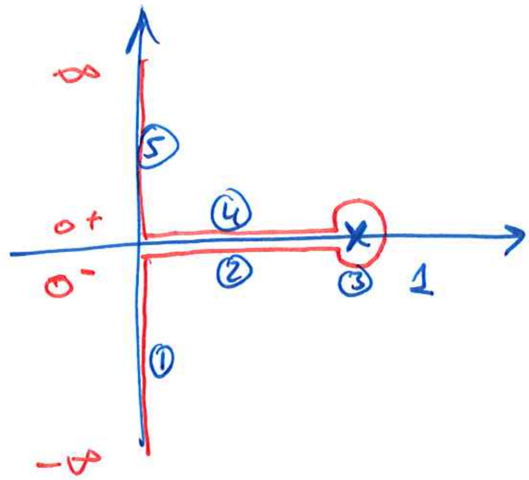
This would indicate that the closed loop is stable for $-\infty < G < 0$ and $0 < G < 1$

But this is not valid as per root locus

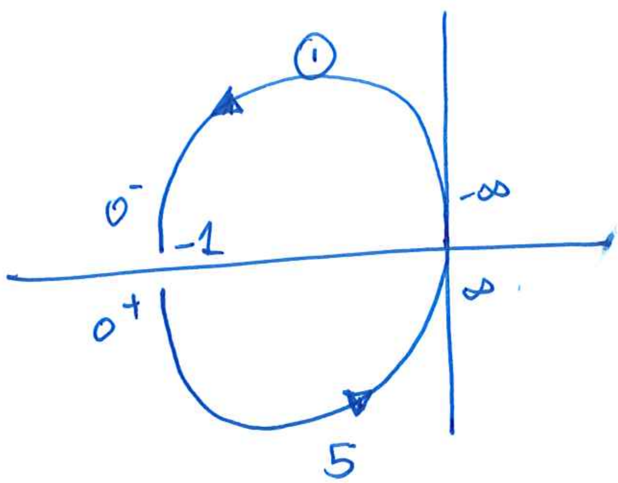


Stable for
$1 < G < \infty$
Just the opposite.

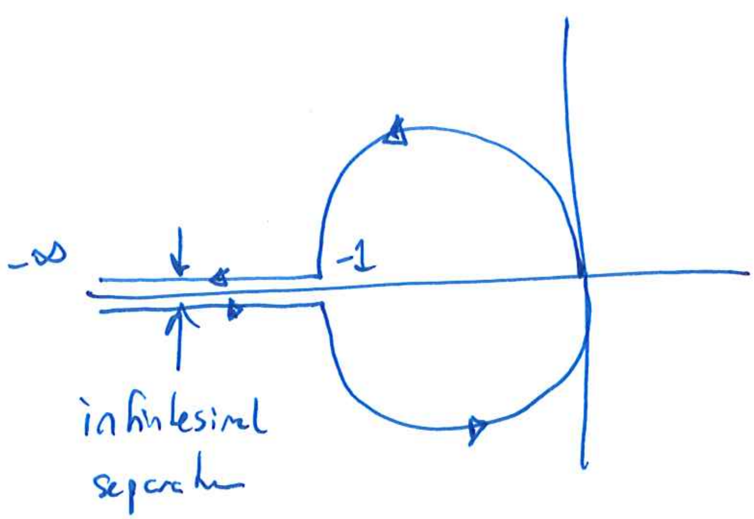
But we have violated the rule that in order to apply Nyquist we must have contour on Right side of all poles.



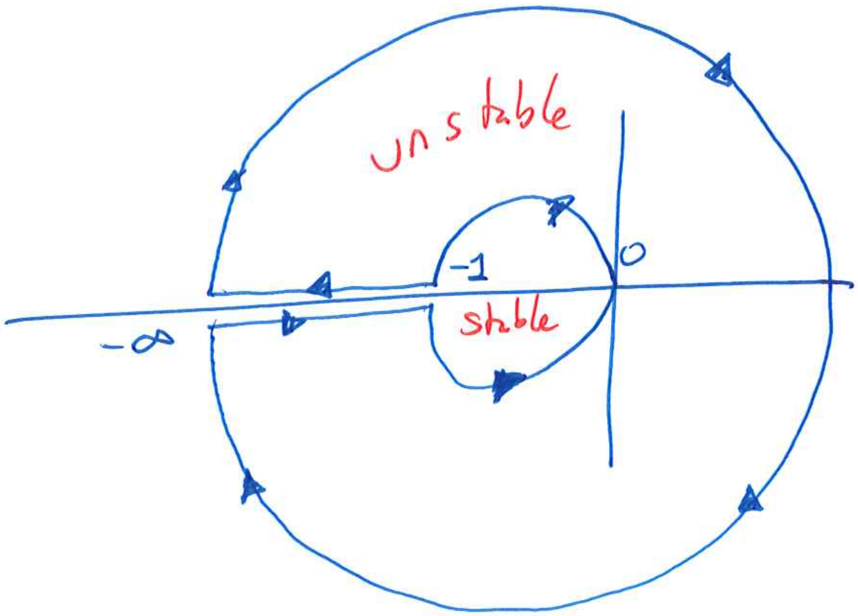
Go back an draw the contours corresponding to 1 and 5



now add in the trajectory segments 2 and 4

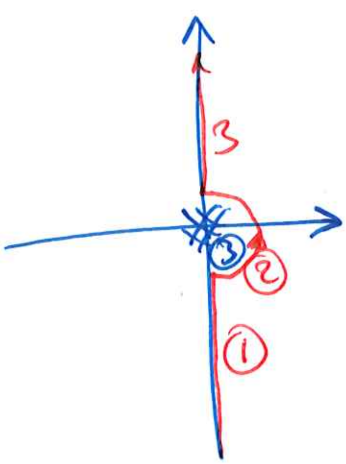


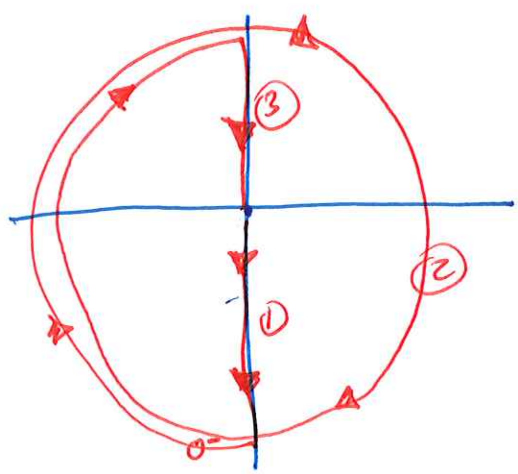
Finally add in the contour for 3 which is counter clockwise so that we need a clockwise rotation in Nyquist at 360°



Now the remarkable observation is that the disk between 0 and -1 is the only region of stability. Everywhere else it is enclosed by a loop and is unstable.

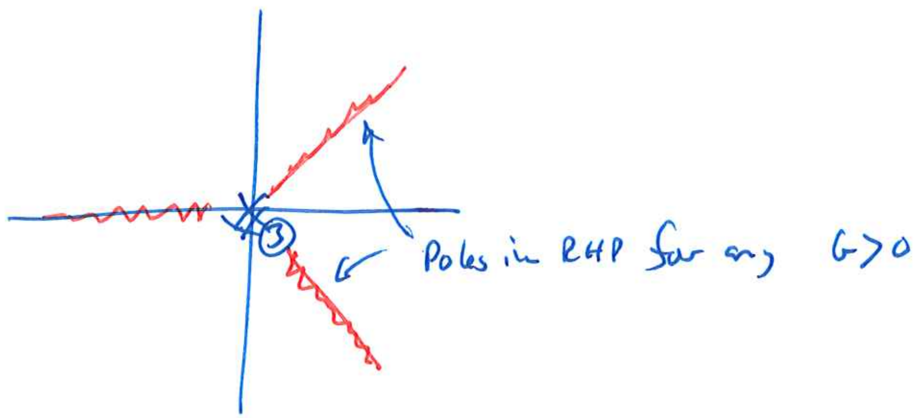
<u>Example</u> $H_{OL}(s) = \frac{1}{s^3}$



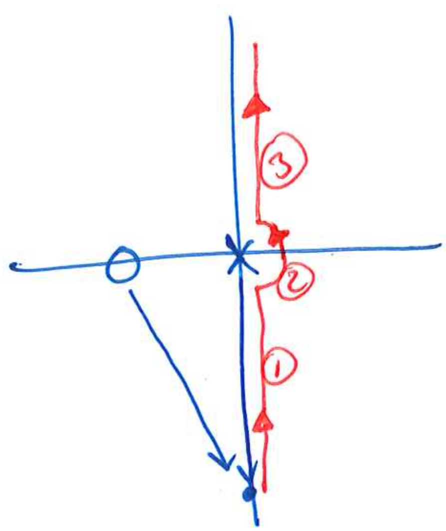


As we see the Nyquist contour encircles the entire complex plane and there for no value of G is in a stable region.

This is consistent with root locus



<u>Example</u> $$H_{OL}(s) = \frac{s+1}{s}$$



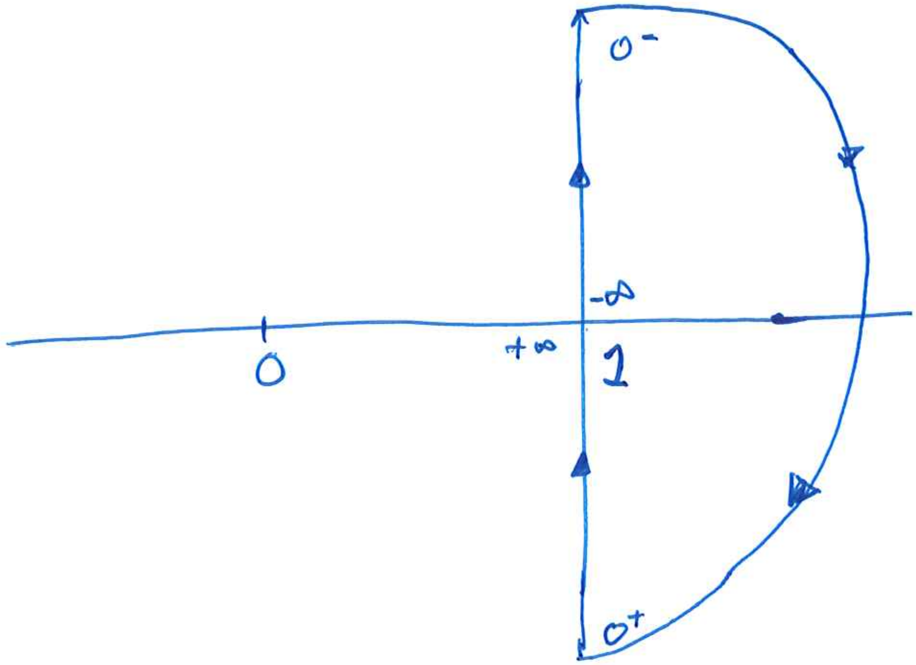
Just like in Bode plots we have to determine the phasors of multiple poles and zeros and determine what they do.

Write as

$$H_{OL}(j\omega) = \frac{j\omega}{j\omega} + \frac{1}{j\omega}$$

$$= 1 + \underbrace{\frac{1}{j\omega}}_{}$$

only variation along imaginary axis at offset of 1.



Nyquist stability

$\overleftrightarrow{\;0 < G < \infty\;}$ | $\overleftrightarrow{\;-1 < G < 0\;}$

∴ stable for $-1 < G < \infty$

Show this directly.

closed loop $\dfrac{G\,\frac{s+1}{s}}{1 + G\,\frac{s+1}{s}} = \dfrac{G(s+1)}{s(1+G)+1}$

Stable if $\dfrac{1}{1+G} > 0$ $\quad G > -1$