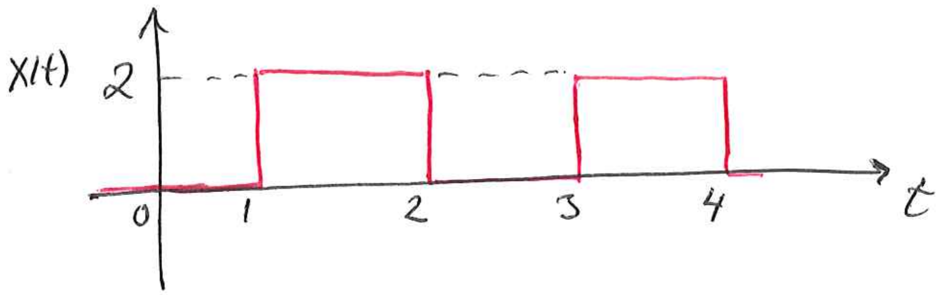
Problems for Unit 1 (Prep for Quiz 1)

P1 Find Laplace transform of $x(t)$



sol $x(t) = 2\left(u(t-1) - u(t-2) + u(t-3) - u(t-4)\right)$

$$X(s) = \frac{2}{s}\left(e^{-s} - e^{-2s} + e^{-3s} - e^{-4s}\right)$$

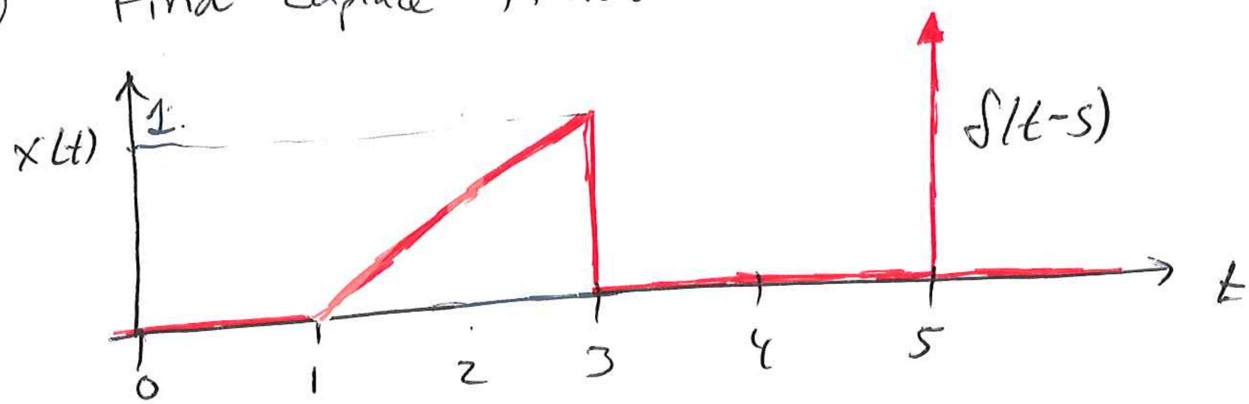
P2 Find Laplace transform of $x(t) = e^{-10t}\, t\, u(t)$

sol split up into components and operations

$$x(t) = \underbrace{e^{-10t}}_{\text{shift s by 10}} \quad \underbrace{t\,u(t)}_{\text{ramp}}$$

$$X(s) = \frac{1}{(s+10)^2}$$

3 Find Laplace Transform of $x(t)$



$$x(t) = \tfrac{1}{2}(t-1)\,u(t-1) \quad ①$$
$$-\tfrac{1}{2}(t-3)\,u(t-3) \quad ②$$
$$-\,u(t-3) \quad ③$$
$$+\,\delta(t-5) \quad ④$$

① $\frac{1}{2}\,\frac{e^{-s}}{s^2}$

③ $-\frac{e^{-3s}}{s}$

② $-\frac{1}{2}\,\frac{e^{-3s}}{s^2}$

④ e^{-5s}

$$X(s) = \frac{1}{2s^2}\left(e^{-s} - e^{-3s}\right) - \frac{e^{-3s}}{s} + e^{-5s}$$

4. Find $X(s)$ for $x(t) = (t-1)(t-2)u(t)$

$$x(t) = (t^2 - 3t + 2)\,u(t) = \underbrace{t^2 u(t) - 3t\,u(t) + 2u(t)}_{\text{split into elemental components}}$$

$$X(s) = \frac{2}{s^3} - \frac{3}{s^2} + \frac{2}{s}$$

5. Find $X(s)$ for $x(t) = (t-1)(t-2)\,u(t-1)$

Expand as $(t-1)((t-1)-1)\,u(t-1)$

$$((t-1)^2 - (t-1))\,u(t-1)$$

$$X(s) = e^{-s}\frac{2}{s^3} - e^{-s}\frac{1}{s^2}$$

6. Find $X(s)$ for $x(t) = u(t)\,u(t-1)\,u(t-2)\,u(t-3)$

$$X(s) = \mathcal{L}(u(t-3)) = \frac{e^{-3s}}{s}$$

7 Find $X(s)$ if $x(t) = (t-2)\,u(t-2)\,e^{-5t}$

$$x(t) = (t-2)\,u(t-2)\,e^{-5(t-2)}\,e^{-10}$$

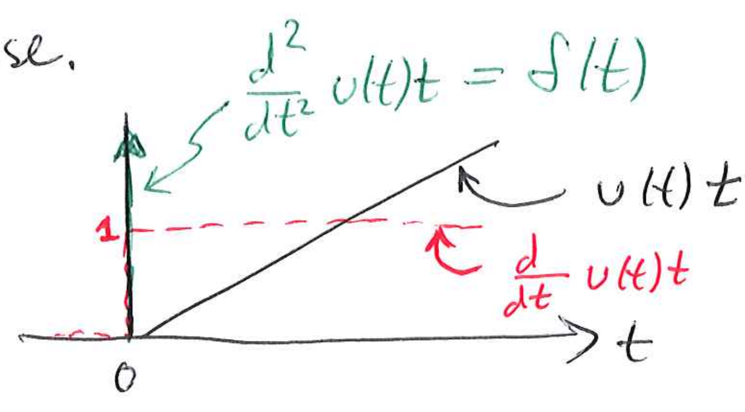
$$X(s) = e^{-10}\,e^{-2s}\,\frac{1}{(s+5)^2}$$

8 $x(t) = \frac{d^2}{dt^2}\,\underbrace{u(t)t}$, find $X(s)$

$$\Downarrow \quad \Downarrow$$

$$X(s) = s^2\,\frac{1}{s^2} = 1$$

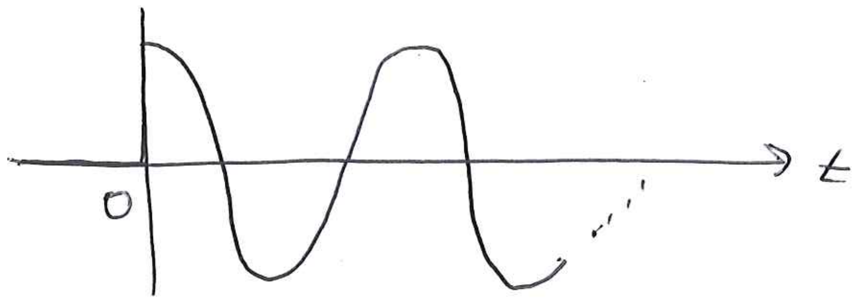
look at this graphically, see if it makes sense.



$$\delta(t) \Longleftrightarrow 1$$

(9) Find Laplace of

$$x(t) = u(t)\cos(3t)$$



Write as $$x(t) = u(t)\frac{e^{j3t}}{2} + u(t)\frac{e^{-j3t}}{2}$$

$$X(s) = \frac{1}{2}\frac{1}{s-j3} + \frac{1}{2}\frac{1}{s+j3}$$

We can simplify this as follows

$$X(s) = \frac{1}{2}\left(\frac{(s+j3)+(s-j3)}{(s-j3)(s+j3)}\right)$$

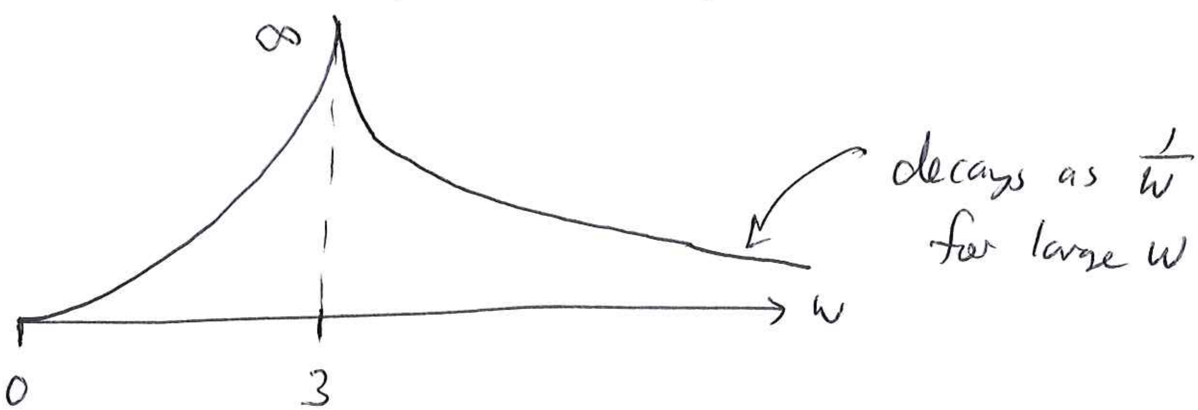
$$= \frac{1}{2}\frac{2s}{s^2+9} = \frac{s}{s^2+9}$$

Agrees with Eq. 7 of table 2.1

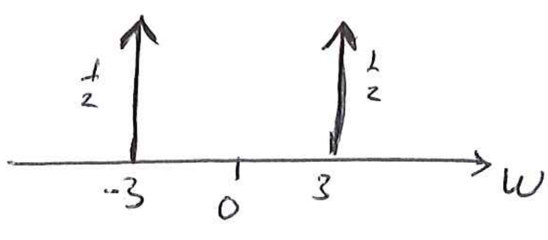
10. Evaluate magnitude of frequency spectrum of Q9.

$$X(s) = \frac{s}{s^2+9}$$

$$|X(j\omega)| = \frac{|j\omega|}{|(j\omega)^2+9|} = \frac{\omega}{|9-\omega^2|}$$



11. Why is this not the usual spectrum of a cosine ie



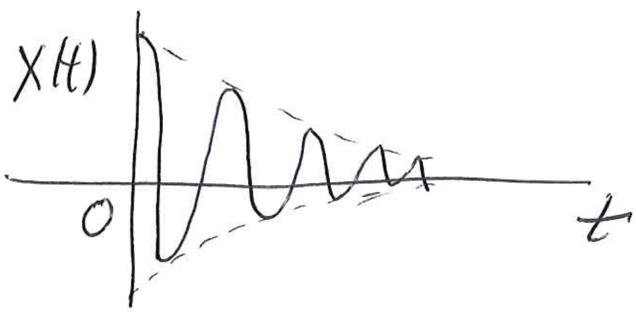
?

Because of $u(t)$ suddenly turns on at $t=0$.

(12) Find Laplace of $x(t) = u(t)e^{-7t}\cos(3t)$ (7)

We know that $u(t)\cos(3t) \Longleftrightarrow \dfrac{s}{s^2+9}$

$\therefore\ u(t)\cos(3t)e^{-7t} \Longleftrightarrow \dfrac{s+7}{(s+7)^2+9}$



(13) Find Laplace of $x(t) = \dfrac{d^2}{dt^2} u(t)e^{-7t}\cos(3t)$

The $\dfrac{d^2}{dt^2}$ contributes an s^2 to the numerator of $X(s)$. Also IC's are zero.

$$\therefore\ X(s) = \frac{s^2(s+7)}{(s+7)^2+9}$$

14 Find $X(s)$ if

$$x(t) = \int_0^t d\tau \int_0^\tau d\lambda \; \lambda\, u(\lambda)$$

We know that $t\,u(t) \Longleftrightarrow \frac{1}{s^2}$

Also the double integral contributes s^2 to the denominator of $X(s)$.

$$\therefore \quad X(s) = \frac{1}{s^4}$$

Is this correct? Work out integrals directly.

$$\left.\begin{aligned} \int_0^t \tau\, d\tau &= \frac{t^2}{2} \\ \int_0^t \frac{\tau^2}{2}\, d\tau &= \frac{t^3}{6} \end{aligned}\right\} \quad \frac{t^3}{6} u(t) \Longrightarrow \frac{3!}{s^4} \cdot \frac{1}{6} = \frac{1}{s^4}$$

(15) Show that $x(t) = \mathrm{rect}(t)$ has a Laplace transform of $X(s)\Big|_{s=j2\pi f} = \mathrm{sinc}(f)$

We know that from the Fourier transform of

$$\mathrm{rect}(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

that $\mathrm{rect}(t) \Longleftrightarrow \mathrm{sinc}(f) = \frac{\sin(\pi f)}{\pi f}$

Show with Laplace.

$$x(t) = u(t + \tfrac{1}{2}) - u(t - \tfrac{1}{2})$$

$$X(s) = \frac{1}{s} e^{s/2} - \frac{1}{s} e^{-s/2}$$

$$X(s)\Big|_{s=j2\pi f} = \frac{1}{j2\pi f} e^{j\pi f} - \frac{1}{j2\pi f} e^{-j\pi f}$$

$$= \frac{1}{\pi f} \underbrace{\left(\frac{e^{j\pi f} - e^{-j\pi f}}{2j} \right)}_{\sin(\pi f)}$$

$$= \frac{\sin \pi f}{\pi f} = \mathrm{sinc}(f)$$

16 Find $x(t)$ if $X(s) = \frac{(s-5)^2}{s^3}$

(sol) expand $X(s) = \frac{s^2}{s^3} - \frac{10s}{s^3} + \frac{25}{s^3}$

$$X(s) = \underbrace{\frac{1}{s}}_{u(t)} - \underbrace{\frac{10}{s^2}}_{10tu(t)} + \underbrace{\frac{25}{s^3}}_{\frac{25}{2}t^2u(t)}$$

$$x(t) = u(t) - 10tu(t) + 12.5t^2u(t)$$

17 Find $x(t)$ if $X(s) = \frac{1}{s^2+2s+1}$

see if we can factor denominator

$$X(s) = \frac{1}{(s+1)^2}$$

$$x(t) = e^{-t}\,t\,u(t)$$

18 Find $x(t)$ if $X(s) = \frac{1}{s^2+2s+3}$

roots of denominator $\frac{-2 \pm \sqrt{4-12}}{2} = -1 \pm j\sqrt{2}$

$$X(s) = \frac{A}{s+1+j\sqrt{2}} + \frac{B}{s+1-j\sqrt{2}}$$

A, B coefficients to be determined (Partial fraction expansion)

So we have $\underbrace{1}_{\text{initial numerator}} = As + A(1-j\sqrt{2}) + Bs + B(1+j\sqrt{2})$

$A = -B$ so the s coefficient disapears

$$A\left(1-j\sqrt{2} - 1 - j\sqrt{2}\right) = 1$$

$$\therefore \quad A = \frac{1}{-2j\sqrt{2}} = j\frac{1}{2\sqrt{2}}$$

$$B = -j\frac{1}{2\sqrt{2}}$$

$$X(s) = \frac{j}{2\sqrt{2}}\left(\frac{1}{s+1+j\sqrt{2}} - \frac{1}{s+1-j\sqrt{2}}\right)$$

$$x(t) = \frac{j}{2\sqrt{2}}\left(e^{-(1+j\sqrt{2})t} - e^{-(1-j\sqrt{2})t}\right)$$

Simplify further using $\sin(b) = \dfrac{e^{jb} - e^{-jb}}{2j}$

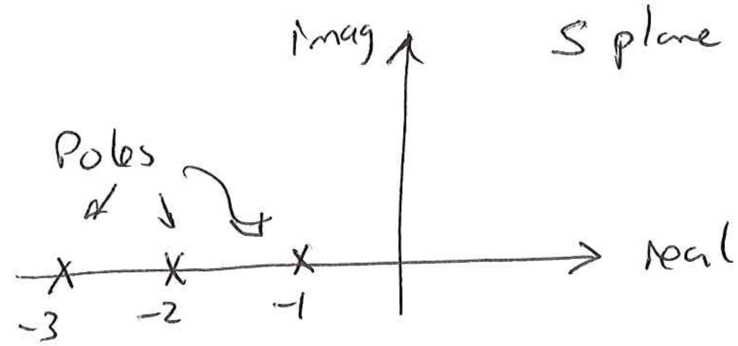
$$x(t) = \frac{1}{\sqrt{2}} \left(\frac{-1}{2j}\right) e^{-t} \left(e^{-j\sqrt{2}t} - e^{j\sqrt{2}t}\right)$$

$$= \frac{e^{-t}}{\sqrt{2}} \sin\left(\sqrt{2}\,t\right)$$

19) A transfer function of a system is

$$H(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

Sketch a pole zero map.



20) Determine impulse response of H(S) in 19.

By hand (Partial fraction expansion)

$$H(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$= \frac{1}{(-1+2)(-1+3)} \frac{1}{s+1}$$

$$+ \frac{1}{(-2+1)(-2+3)} \frac{1}{s+2}$$

$$+ \frac{1}{(-3+1)(-3+2)} \frac{1}{s+3}$$

$$= \frac{1}{2} \frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2} \frac{1}{s+3}$$

$$h(t) = \frac{1}{2} e^{-t} u(t) - e^{-2t} u(t) + \frac{1}{2} e^{-3t} u(t)$$

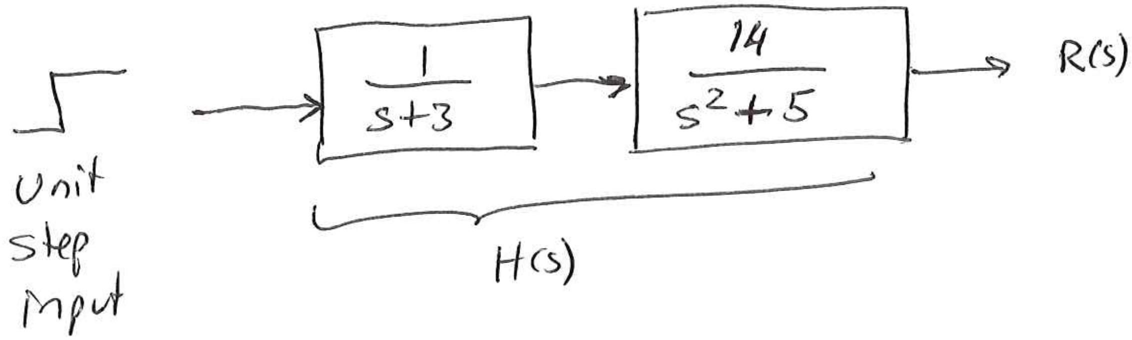
$$= \left(\frac{1}{2} e^{-t} - e^{-2t} + \frac{1}{2} e^{-3t} \right) u(t)$$

By symbolic Matlab symbolic toolbox ilaplace( )

h = ilaplace(1/((1+s)*(2+s)*(3+s)))

Same result.

21/ Find the step response of the following system.



overall response is $R(s) = \frac{1}{s} \frac{1}{s+3} \frac{14}{s^2+5}$

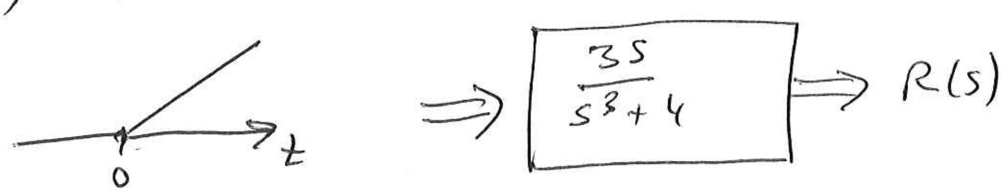
Convert to time domain by going through partial fractions as

$$R(s) = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-j\sqrt{5}} + \frac{D}{s+j\sqrt{5}}$$

Painful or use ilaplace

r = ilaplace (1/ (s * (s+3) * (s^2 +5)))

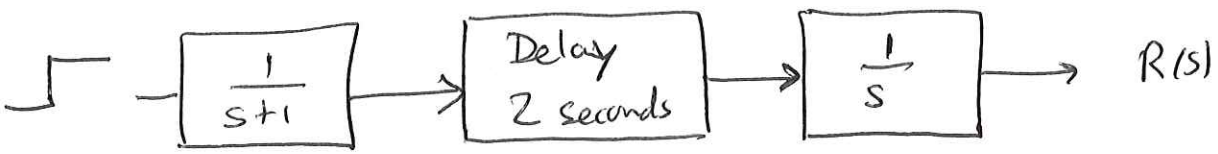
22) What is unit ramp response of $H(s) = \frac{3s}{s^3+4}$



$\frac{1}{s^2}$

$$R(s) = \frac{1}{s} \frac{3}{s^3+4}$$

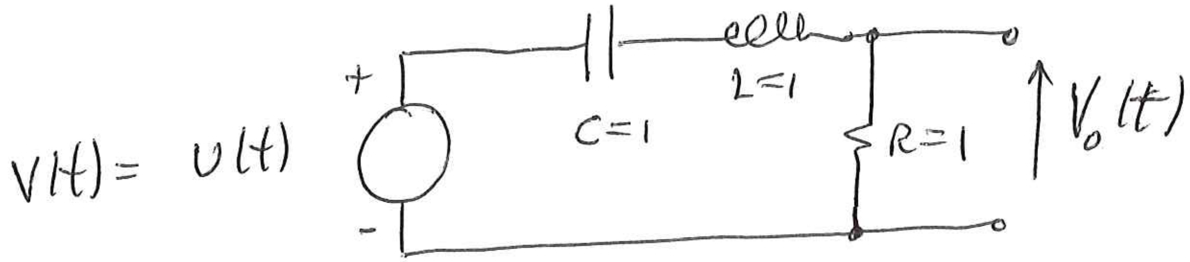
23/ Find step response of system.



Sol

$$R(s) = \underbrace{\frac{1}{s}}_{\text{step input}} \frac{e^{-2s}}{(s+1)s} = \frac{e^{-2s}}{s^2(s+1)}$$

24/ Find Laplace transform of output voltage



Sol

Transfer function of circuit

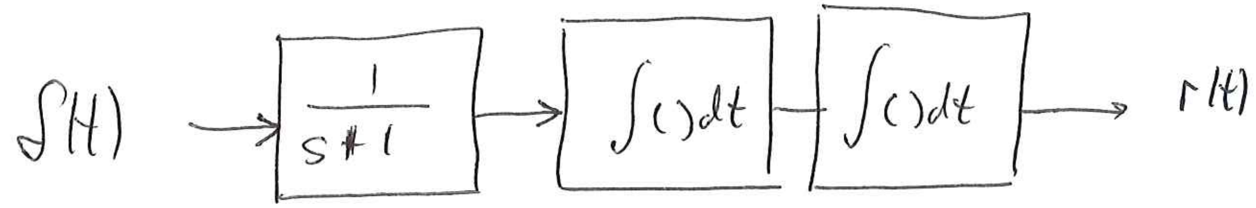
$$H = \frac{R}{R + sL + \frac{1}{sC}} = \frac{1 \cdot s}{s + s^2 + 1} = \frac{s}{s^2+s+1}$$

$$V_o(s) = \frac{1}{s} H(s) = \frac{1}{s^2+s+1}$$

$$V_o(t) = \text{ilaplace}\left(1/(s^2+s+1) \right)$$

$$V_o(t) = e^{-0.5t}\, u(t) \left(e^{j.866t}(-0.577i) + e^{-j.866t}(0.577i) \right)$$

25/ Find impulse response of system.



assume initial conditions of integrators is zero

sol

$$r(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2(s+1)} \right) = t + \exp(-t) - 1$$

due to integrators

or

$$r(t) = \int_0^t \int_0^{t_1} e^{-\lambda}\, d\lambda\, dt_1$$

$$= \int_0^t (1 - e^{-t_1})\, dt_1$$

$$= t + e^{-t} - 1$$

26/ A DEQ represents a system that is given as

$$3\frac{d^3y}{dt^3} + 5\frac{dy}{dt} + \frac{dx}{dt} + 3x = 0$$

If $x(t)$ is the independent stimulus signal and $y(t)$ is the dependent output signal find the transfer function relating $Y(s)$ to $X(s)$. Assume IC's are zero

(Sol)

$$(3s^3 + 5s)\,Y(s) = (-s-3)\,X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{-(s+3)}{3s^3+5s}$$

27) In Q26 if $y(t)$ is the independent input signal and $x(t)$ is the output find the transfer function.

(Sol)

$$H(s) = \frac{X(s)}{Y(s)} = -\frac{3s^3+5s}{s+3}$$

28) Determine $x(t)$ assuming the DEQ

$$\frac{d^2x(t)}{dt^2} + 5x(t) = u(t)$$

where $u(t)$ is the unit step function

also $x(0^-) = 0$ $\quad \frac{dx}{dt}(0^-) = 0$

$$(s^2+5)\,X(s) = \frac{1}{s}$$

$$X(s) = \frac{1}{s\,(s^2+5)}$$

Now $\dfrac{\sqrt{5}}{s^2+5} \Longleftrightarrow \sin(\sqrt{5}\,t)\,u(t)$

$$x(t) = \frac{1}{\sqrt{5}} \int_0^t \sin(\sqrt{5}\,\tau)\,d\tau$$

$$= \frac{1}{5} - \frac{1}{5}\cos(\sqrt{5}\,t)$$

Q29) Find $x(t)$ if $X(s) = \dfrac{(s-5)^2}{s^3}$

(ans) $X(s) = \dfrac{s^2 - 10s + 25}{s^3} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s^3}$

$C = X(s)\,s^3\big|_{s=0} = 25$, $B = \dfrac{d}{ds}\left(X(s)s^3\right)\Big|_{s=0} = -10$

$A = \dfrac{d^2 X(s)\,s^3}{2ds^2} = 1$

$$x(t) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - 10\,\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + 25\,\mathcal{L}^{-1}\left(\frac{1}{s^3}\right)$$

$$= u(t) - 10t\,u(t) + \frac{25}{2}t^2 u(t)$$

Q30) Determin $X(S)$ for

$$\frac{d^2 x(t)}{dt^2} + 5x(t) = u(t) \qquad \begin{cases} u(t) = \text{step function} \\ x(0^-) = 2 \\ \dot{x}(0^-) = 4 \end{cases}$$

$$s^2 X(S) - s\,x(0^-) - \dot{x}(0^-) + 5X(s) = \frac{1}{s}$$

$$X(s)\left(s^2 + 5\right)s = 1 + s^2 x(0^-) + s\dot{x}(0^-)$$

$$X(S) = \frac{1}{s(s^2+5)} + \frac{s\,2}{s^2+5} + \frac{4}{s^2+5}$$

2015 Quiz 1

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1.(3) Find the Laplace transform of the following function

$$f(t) = 3\delta(t-1) + 7t\,u(t)$$

$$\boxed{F(s) = 3\exp(-s) + 7\frac{1}{s^2}}$$

2.(3) Find the Laplace transform of the following function

$$f(t) = (t-1)e^{-t}\,u(t)$$

$$\boxed{\begin{aligned} f(t) &= te^{-t}\,u(t) - e^{-t}\,u(t) \\ F(s) &= \frac{1}{(s+1)^2} - \frac{1}{s+1} \end{aligned}}$$

3.(4) Find the Laplace transform of the following function

$$f(t) = \begin{cases} 1 & 1 \le t < 3 \\ 0 & otherwise \end{cases}$$

$$\boxed{\begin{aligned} f(t) &= u(t-1) - u(t-3) \\ F(s) &= \frac{\exp(-s)}{s} - \frac{\exp(-3s)}{s} \end{aligned}}$$

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1.(3) Find the Laplace transform of the following function

$$f(t) = 4\delta(t) + 7t^2\, u(t) + 4$$

$$F(s) = 4 + \frac{14}{s^3} + 4\delta(s)$$

2.(3) Find the Laplace transform of the following function

$$f(t) = (t-1)\, u(t-1)$$

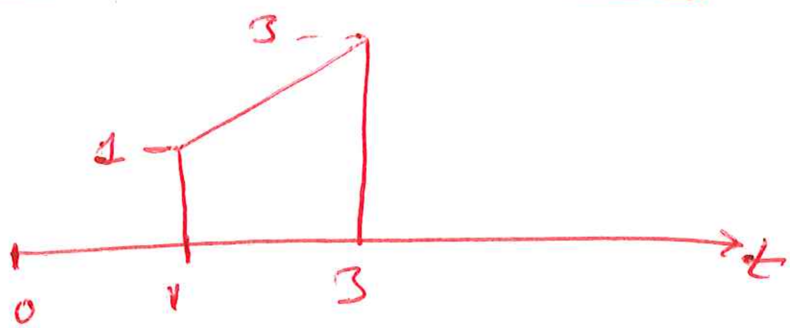
$$F(s) = e^{-s}\frac{1}{s^2}$$

3.(4) Find the Laplace transform of the following function

$$f(t) = \begin{cases} t & 1 \le t < 3 \\ 0 & otherwise \end{cases}$$

$$f(t) = u(t-1) + (t-1)u(t-1) - (t-3)u(t-3) - 3u(t-3)$$

$$F(s) = \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s^2} - \frac{3e^{-3s}}{s}$$



ENEL441 QUIZ 1 September 24, 2018 Laplace transform and transfer functions

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35 minutes, 20 marks total

1.(5) Find the Laplace transform of the following function

$$f(t) = t^2\, u(t-2)$$

$$= (t-2+2)^2\, u(t-2)$$

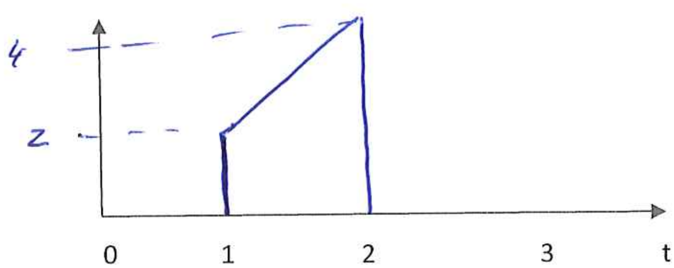
$$= (t-2)^2 u(t-2) + 4(t-2)\, u(t-2) + 4\, u(t-2)$$

$$F(s) = \frac{2}{s^3} e^{-2s} + 4\frac{1}{s^2} e^{-2s} + \frac{4}{s} e^{-2s}$$

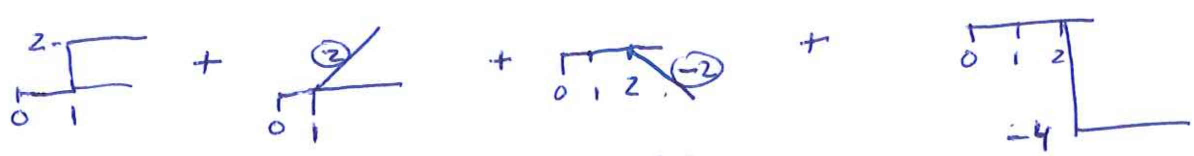
2.(5) A function is given as

$$f(t) = \begin{cases} 2t & 1 \le t < 2 \\ 0 & otherwise \end{cases}$$

a. (1) Sketch the function



b.(4) Find the Laplace transform of f(t)



$$\frac{2e^{-s}}{s} + \frac{2e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} - \frac{4}{s} e^{-2s}$$

3.(5) Determine the inverse Laplace transform of

$$F(s)=\frac{1}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$A = F(s)(s+1)\Big|_{s=-1} = \frac{1}{1\cdot 2} = \frac{1}{2}$$

$$B = F(s)(s+2)\Big|_{s=-2} = \frac{1}{(-1)(1)} = -1$$

$$C = F(s)(s+3)\Big|_{s=-3} = \frac{1}{(-2)(-1)} = \frac{1}{2}$$

$$F(s) = \frac{1}{2}\frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+3}$$

$$f(t) = \left(\frac{e^{-t}}{2} - e^{-2t} + \frac{e^{-3t}}{2}\right)u(t)$$

4.(5) An LTI system is described by the following DEQ

$$4\frac{d^2y}{dt^2} = y - x$$

where x(t) is the input excitation and y(t) is the output response. Determine Y(s) for the excitation of $x(t) = te^{-t}u(t)$. Assume the initial conditions are $y(0^-)=0 \;\; and \;\; \frac{dy(0^-)}{dt}=0$. You do not have to simplify the expression of Y(s).

$$4s^2 Y(s) = Y(s) - X(s)$$

$$(4s^2-1)Y(s) = \frac{-1}{(s+1)^2}$$

$$Y(s) = \frac{-1}{(s+1)^2(4s^2-1)}$$

Quiz 2 2015

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1.(5) Find the inverse Laplace transform of the following spectral function

$$F(s) = \frac{\exp(-2s)}{(s+1)^2}$$

$$\frac{1}{s^2} \Rightarrow tu(t)$$

$$\frac{1}{(s+1)^2} \Rightarrow \exp(-t)tu(t)$$

$$\frac{\exp(-2s)}{(s+1)^2} \Rightarrow \exp(-(t-2))(t-2)u(t-2)$$

$$f(t) = (t-2)\exp(-(t-2))u(t-2)$$

2.(5) Given that $f(t) \Leftrightarrow F(s)$ determine F(s) that satisfies the following DEQ assuming all zero initial conditions at t-.

$$\frac{df}{dt} + 3\frac{d^2 f}{dt^2} = f + u(t)$$

$$\left(3s^2 + s - 1\right)F(s) = \frac{1}{s}$$

$$F(s) = \frac{1}{3s^3 + s^2 - s}$$

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1.(5) Find the inverse Laplace transform of the following spectral function

$$F(s) = \frac{\exp(-2)s}{(s+1)^2}$$

$$\frac{1}{s^2} \Rightarrow tu(t)$$

$$\frac{1}{(s+1)^2} \Rightarrow \exp(-t)tu(t)$$

$$\frac{s}{(s+1)^2} \Rightarrow -\exp(-t)tu(t) + \exp(-t)u(t) = \exp(-t)u(t)(1-t)$$

$$f(t) = e^{-2}\exp(-t)u(t)(1-t)$$

2.(5) Given that $f(t) \Leftrightarrow F(s)$ determine F(s) that satisfies the following DEQ assuming all zero initial conditions at t=0-.

$$\frac{df}{dt} + 3\frac{d^2 f}{dt^2} = f + u(t)\exp(-t) + 5\frac{d^3 f}{dt^3} + 3\delta(t)$$

$$\left(-5s^3 + 3s^2 + s - 1\right)F(s) = \frac{1}{s+1} + 3$$

$$F(s) = \frac{1}{\left(-5s^3 + 3s^2 + s - 1\right)(s+1)} + \frac{3}{-5s^3 + 3s^2 + s - 1}$$