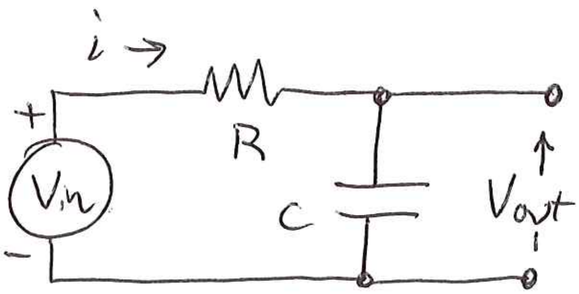
UNIT 1 NOTES

Go through LTI analysis Notes



Find DEQ of system.

Start with energy storage device ie capacitor

$$i = C \frac{dV_{out}}{dt}$$

also $$i = \frac{V_{in} - V_{out}}{R}$$

$$\frac{V_{in} - V_{out}}{RC} = \frac{dV_{out}}{dt}$$

Standard form $$\underbrace{\frac{d}{dt} V_{out}}_{\text{time derivative system variable}} = \underbrace{-\frac{1}{RC}}_{\text{constant times system variable}} V_{out} + \underbrace{\frac{1}{RC}}_{\text{constant}} \underbrace{V_{in}}_{\text{input}}$$

Convert to s domain

Laplace identity $\quad x(t) \Longleftrightarrow X(s)$

$$\mathcal{L}\left(\frac{dx(t)}{dt}\right) = s\,X(s) - \underbrace{x(0^-)}_{\text{initial condition}}$$

Assume: $V_{in}(t) = 0 \quad$ for $\quad t < 0$

$V_{out}(0^-) = 0$

$$s\,V_{out}(s) = -\frac{1}{RC}V_{out}(s) + \frac{1}{RC}V_{in}(s)$$

$$\frac{V_{out}(s)}{V_{in}(s)} = \underbrace{\frac{\frac{1}{RC}}{s + \frac{1}{RC}}}_{\substack{\text{Transfer function} \\ H(s)}}$$

$$V_{out}(s) = H(s)\,V_{in}(s)$$

Time domain solution

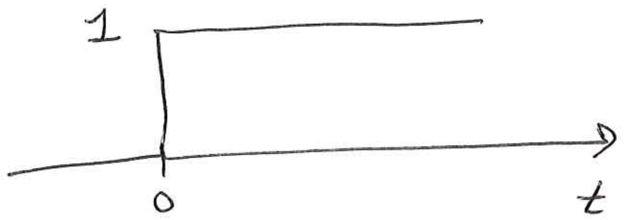
$$H(s) \iff h(t)$$

transfer function   impulse response

$$H(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}, \qquad \mathcal{L}^{-1}(H(s)) = h(t)$$

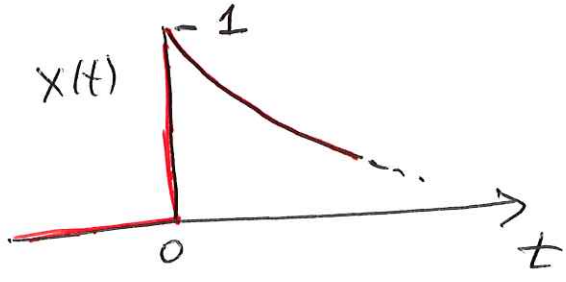
First need an identity

$$u(t) \triangleq \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



step function

let $x(t) = u(t)\, e^{-at}$



Find $X(s) \iff x(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+s)t}\, dt = -\left.\frac{e^{-(a+s)t}}{(a+s)}\right|_{0}^{\infty}$$

$$= \underbrace{-\frac{e^{-(a+s)\infty}}{a+s}}_{=0} + \frac{1}{a+s}$$

assume $Real(a+s) > 0$

$$\therefore \quad u(t)\, e^{-at} \Longleftrightarrow \frac{1}{a+s}$$

Laplace transform is consistent such that pair is unique

ie $\mathcal{L}\left(u(t)e^{-at}\right) = \dfrac{1}{a+s}$

$$\mathcal{L}^{-1}\left(\frac{1}{a+s}\right) = u(t)e^{-at}$$

Go back to $H(s) = \dfrac{\frac{1}{RC}}{s + \frac{1}{RC}}$

$$\mathcal{L}^{-1}(H(s)) = \frac{1}{RC}\, u(t)\, e^{-t/RC}$$

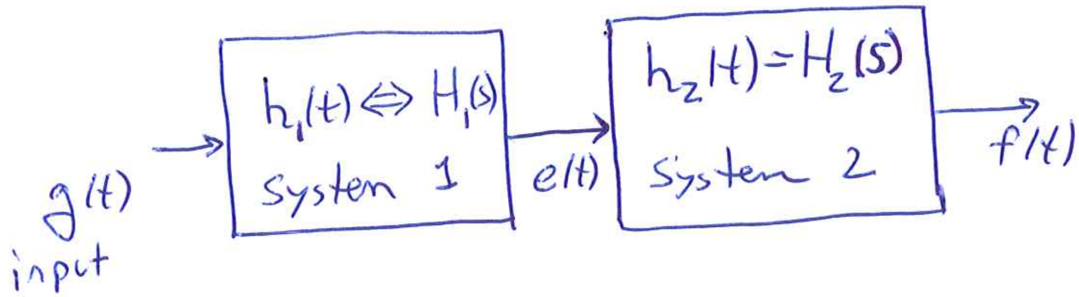
$$V_{out}(t) = h(t) * V_{in}(t)$$

$$= \int h(\tau)\, v_{in}(t-\tau)\, d\tau$$

$$= \frac{1}{RC} \int_0^{\infty} e^{-t/RC}\, V_{in}(t-\tau)\, d\tau$$

Typical Application

- cascade of two transfer functions



$$e(t) = g(t) * h_1(t)$$

$$E(s) = G(s)\, H_1(s)$$

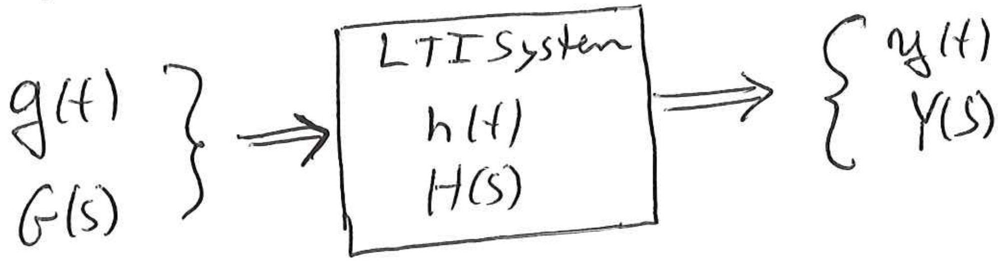
$$f(t) = e(t) * h_2(t)$$

$$= g(t) * \underbrace{h_1(t) * h_2(t)}_{\substack{\text{impulse response of} \\ H_1(s) \text{ in series with } H_2(s)}}$$

$$F(s) = E(s)\, H_2(s)$$

$$= G(s)\, \underbrace{H_1(s)\, H_2(s)}_{\substack{\text{note simple product} \\ \text{of terms.}}}$$

Only consider LTI systems in this course



Consists of delays, poles, zeros
nothing else

Laplace Transform

$$\mathcal{L}(h(t)) = H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

t - time variable

s - complex frequency variable

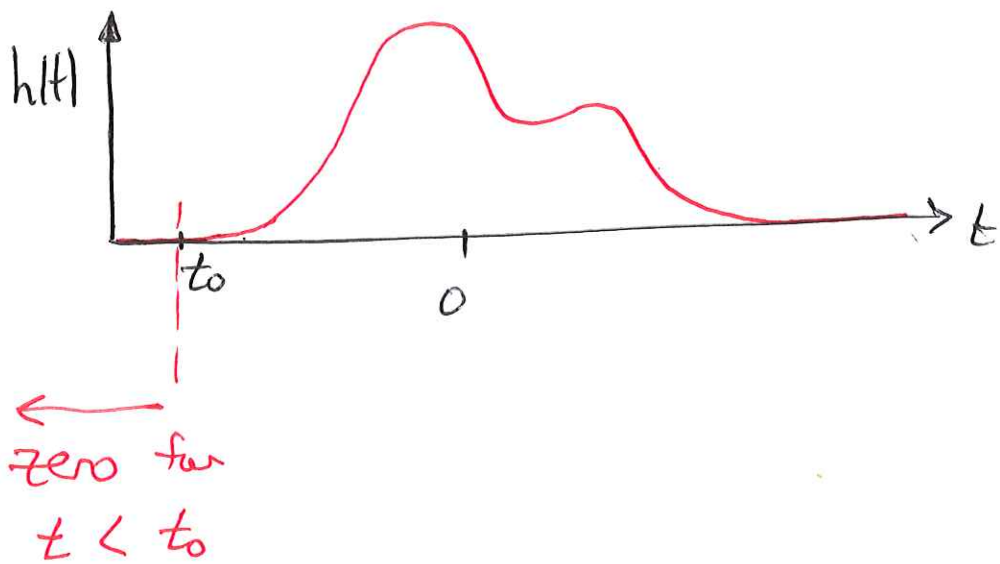
In controls usually only interested in "one sided" Laplace transforms.

Usually one sided implies

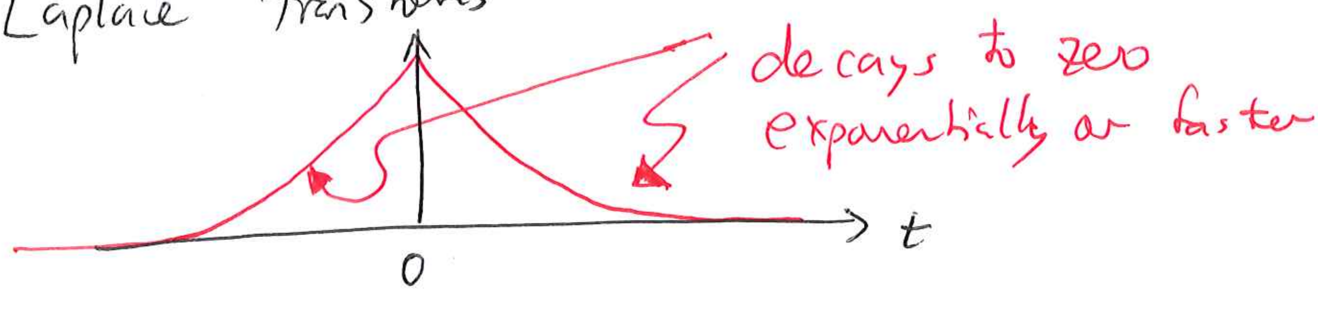
$h(t) = 0 \qquad t < 0$

$h(t)$ – any value for $t \geq 0$

We can start before $t = 0$ for a one sided function



Special cases of two sided functions having Laplace transforms



Important Laplace transform pairs or elemental functions

Impulse function

$$f(t) = \delta(t) = \begin{cases} \infty & t=0 \\ 0 & \text{otherwise} \end{cases}$$

Also $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$ necessary to complete definition of $\delta(t)$

$$\mathcal{L}(\delta(t)) = \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\, dt$$

zero every where except at $t=0$ - plug in $t=0$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-s\cdot 0}\, dt$$

$$= \underbrace{e^{-s\cdot 0}}_{1} \underbrace{\int_{-\infty}^{\infty} \delta(t)\, dt}_{1}$$

$$= 1$$

Laplace transform pair

$$\delta(t) \Longleftrightarrow 1$$

Laplace transform of a Step Function

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$U(s) = \mathcal{L}(u(t))$$

$$= \int_{-\infty}^{\infty} u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-st}\, dt = \left.\frac{e^{-st}}{-s}\right|_{0}^{\infty}$$

What is $\lim\limits_{t \to \infty} e^{-st}$?

We need to make assumption that:

$$\text{Real}(s) > 0$$

such that $\lim_{t \to \infty} e^{-st} = 0$

This is some of the weirdness of limits associated with Laplace analysis that we will not get into here.

Back to problem

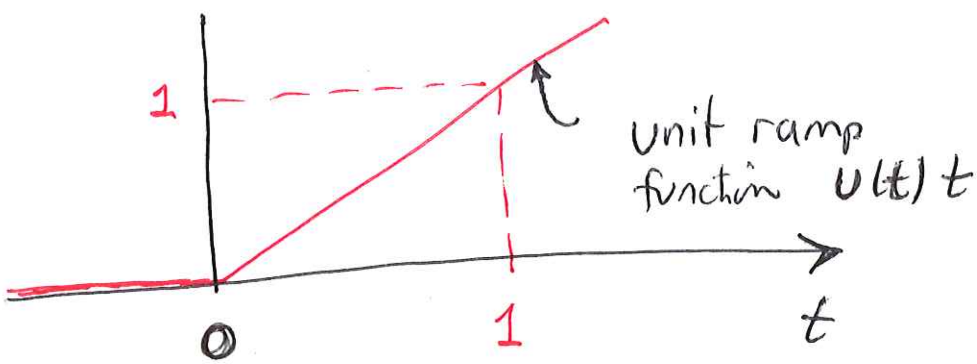
$$U(s) = \frac{e^{-st}}{-s}\Bigg|_0^\infty = \underbrace{\frac{e^{-s\infty}}{-s}}_{0} - \underbrace{\frac{e^{-s\cdot 0}}{-s}}_{-\frac{1}{s}}$$

$$U(s) = \frac{1}{s}$$

Laplace transform pair $\quad u(t) \Longleftrightarrow \frac{1}{s}$

Ramp Function

$$f(t) = u(t)\, t$$



Laplace transform of unit ramp function.

$$F(s) = \mathcal{L}(f(t)) = \mathcal{L}(u(t)t)$$

$$= \int_{-\infty}^{\infty} u(t)\, t\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} t\, e^{-st}\, dt$$

To evaluate integral - integration by parts

$$\int_{t_1}^{t_2} b\, da = ab\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} a\, db$$

$$\frac{da}{dt} = e^{-st} \qquad a = \frac{e^{-st}}{-s}$$

$$da = e^{-st} dt$$

$$b = t \qquad db = dt$$

$$F(s) = \int_0^{\infty} \underbrace{t}_{b}\, \underbrace{e^{-st} dt}_{da} = \underbrace{\frac{e^{-st}}{-s}}_{a}\, \underbrace{t}_{b}\Big|_0^{\infty} - \int_0^{\infty} \underbrace{\frac{e^{-st}}{-s}}_{a}\, \underbrace{dt}_{db}$$

$$F(s) = \lim_{t\to\infty} \frac{e^{-st}}{s} t - \lim_{t\to 0} \frac{e^{-st}}{s} t - \lim_{t\to\infty} \frac{e^{-st}}{s^2} + \lim_{t\to 0} \frac{e^{-st}}{s^2}$$

1. $$\lim_{t\to\infty} \underbrace{\frac{e^{-st}}{s}}_{\text{decays exponentially}} t \longrightarrow 0$$

increases linearly

2. $$\lim_{t\to 0} \frac{e^{-st}}{s} t \longrightarrow 0$$

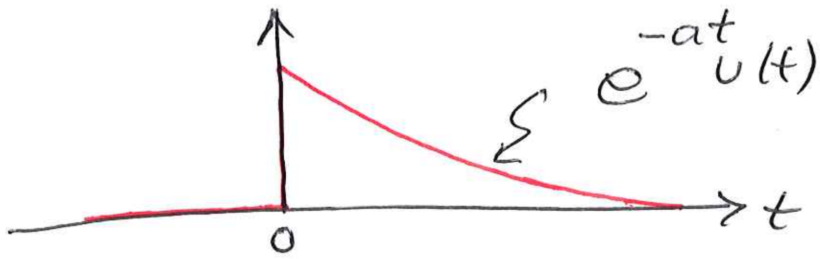
3. $$\lim_{t\to\infty} \frac{e^{-st}}{s^2} \longrightarrow 0$$ (decays exponentially)

4. $$\lim_{t\to 0} \frac{e^{-st}}{s^2} \longrightarrow \frac{1}{s^2}$$

$\therefore$ Laplace transform pair

$$t\,u(t) \Longleftrightarrow \frac{1}{s^2}$$

Laplace transform of $f(t)=e^{-at}u(t)$, $a>0$



$\left\{ \begin{array}{l} a \in \text{Real} \\ a > 0 \end{array} \right.$

$$\mathcal{L}(f(t)) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-(a+s)t} dt$$

$$= \left. \frac{e^{-(a+s)t}}{-(a+s)} \right|_{0}^{\infty}$$

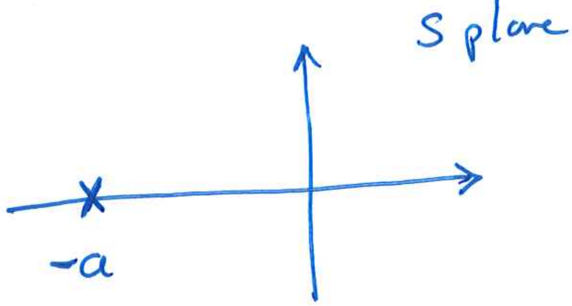
Need to make assumption that Real $(a+s) > 0$

$$\mathcal{L}(f(t)) = \frac{1}{a+s}$$

Laplace transform pair

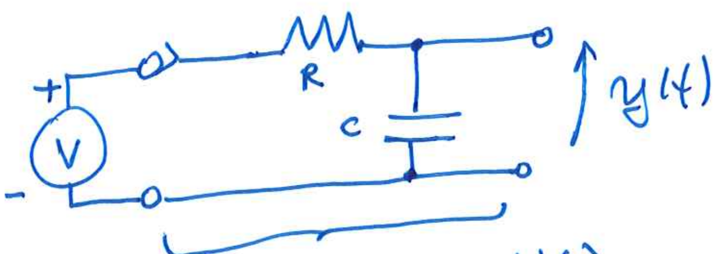
$$e^{-at}u(t) \Longleftrightarrow \frac{1}{a+s}$$

Decaying exponential has a laplace transform that is a single pole in left hand plane.



single pole in LHP - stable.

<u>Example</u>

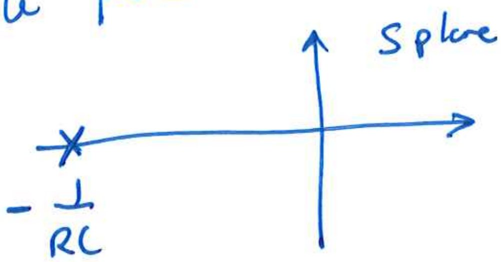


$h(t) \Longleftrightarrow H(s)$

$$\frac{Y(s)}{V(s)} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R} = \frac{1}{1 + sCR}$$

$$= \frac{1}{CR} \frac{1}{\frac{1}{CR} + s}$$

Single pole at $s = -\frac{1}{CR}$



Note

① Laplace transform is self-consistent meaning

$$V(s) = \mathcal{L}(v(t))$$

$$\mathcal{L}^{-1}(V(s)) = \mathcal{L}^{-1}(\mathcal{L}(v(t))) = v(t)$$

② Laplace transform pair is unique meaning

$$v(t) \Longleftrightarrow V(s)$$

if I have $\begin{cases} v(t) & \text{then } V(s) \text{ is unique} \\ V(s) & \text{then } v(t) \text{ is unique.} \end{cases}$

Hence in previous example if

$$H(s) = \frac{1}{CR} \frac{1}{s + \frac{1}{CR}}$$

and I know that

$$x(t) = u(t) e^{-at} \Rightarrow X(s) = \frac{1}{s+a}$$

then $h(t) = \frac{1}{CR} u(t) e^{-\frac{1}{RC} t}$

Without further calculation!

In other words we never really have to compute the inverse laplace transform directly. We can get this from identities.

What is a laplace identity?

$$u(t)\, e^{-at} \Longleftrightarrow \frac{1}{s+a}$$

So when I see a single pole at $-a$ I can infer a time domain response associated with this pole as $u(t)e^{-at}$.

<u>What is alternative to using identities?</u>

Laplace transform pair
$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$x(t) = \frac{1}{2\pi j} \underbrace{\oint}_{\text{complex contour integration around the s-plane}} X(s)\, e^{st}\, ds$$

Very tedious to do.

$$x(t) = \frac{1}{2\pi j} \oint \frac{1}{CR} \frac{1}{s+\frac{1}{CR}}\, e^{st}\, ds \Longleftarrow \text{difficult!}$$

Delay Identity

$x(t) \longrightarrow \boxed{\text{delay } T} \longrightarrow y(t) = x(t-T)$

$Y(s) = ?$

$$
\begin{aligned}
Y(s) &= \int y(t)\, e^{-st}\, dt \\
&= \int x(t-T)\, e^{-st}\, dt \\
&= \int x(t-T)\, e^{-s(t-T)}\, e^{-sT}\, dt \\
&= e^{-sT} \int x(t-T)\, e^{-s(t-T)}\, d(t-T)
\end{aligned}
$$

change of integration variable

$$= e^{-sT}\, X(s)$$

Hence: $\delta(t-T) \Longleftrightarrow e^{-sT}$

Useful Laplace identity.

Consider an example

$x(t) = u(t)\, e^{-at} \longrightarrow \boxed{h(t) = \delta(t-T)} \longrightarrow y(t) = ?$

$Y(s) = ?$

$$y(t) = u(t-T)\, e^{-a(t-T)}$$

$h(t)$ delays $x(t)$ by T so where ever you see "t" replace by "$t-T$".

$$Y(s) = X(s)\,H(s) = \frac{1}{s+a} \cdot e^{-sT}$$

So we have a transform pair

$$u(t-T)\, e^{-a(t-T)} \Longleftrightarrow \frac{e^{-sT}}{s+a}$$

Suppose you were given $G(s) = \dfrac{e^{-sT}}{s+a}$, find $g(t)$ directly.

One simple way is to express as operators.

$$\underbrace{\underset{T}{\text{Delay by}} \left\{ \underbrace{\mathcal{L}^{-1}\left(\frac{1}{s+a}\right)}_{u(t)\, e^{-at}} \right\}}_{u(t-T)\, e^{-a(t-T)}}$$

Laplace identity for integration

$$x(t) \longrightarrow \boxed{\int_{-\infty}^{t} (\;)\,dt'} \longrightarrow y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

$$Y(s) = \int \int_{-\infty}^{t} x(\tau)\,d\tau \; e^{-st}\,dt$$

Integration by parts $\quad \int a\,db = ab - \int b\,da$

let $a = \int_{-\infty}^{t} x(\tau)\,d\tau \qquad da = x(t)\,dt$

$db = e^{-st}\,dt \qquad b = \dfrac{-e^{-st}}{s}$

$$Y(s) = \underbrace{\int_{-\infty}^{t} x(\tau)\,dt \left(\frac{-e^{-st}}{s}\right)\Bigg|_{0}^{\infty}}_{0} - \int \frac{-e^{-st}}{s}\,x(t)\,dt$$

$$Y(s) = \frac{1}{s}\int e^{-st}\,x(t)\,dt = \frac{X(s)}{s}$$

identity $\qquad \displaystyle\int_{-\infty}^{t} x(\tau)\,d\tau \iff \frac{X(s)}{s}$

A pole at $s=0$ implies integration in the time domain.

Example

$$Y(s) = \frac{1}{s} \qquad y(t) = ?$$

let $x(t) = \delta(t)$ ie $X(s) = 1$

write $Y(s) = \frac{X(s)}{s} \iff \int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{t} \delta(\tau)\,d\tau$

$$= \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$= u(t)$$

ie $y(t) = u(t)$

Example

$$Y(s) = \frac{1}{s^2} = \frac{1}{s}\,\frac{1}{s}$$

let $x(t) = u(t)$, $X(s) = \frac{1}{s}$

$Y(s) = \frac{X(s)}{s} \iff \int_{-\infty}^{t} u(\tau)\,d\tau = \begin{cases} t & t > 0 \\ 0 & t \leq 0 \end{cases}$

ie $\frac{1}{s^2} \iff t\,u(t)$

Example

$$Y(s) = \frac{e^{-s}}{s+1} + \frac{4}{s^2} + 1$$

Use linearity to deal with each term separately

$$\frac{e^{-s}}{s+1} \Longleftrightarrow \underset{1}{\text{delay}}\left\{ e^{-t} u(t) \right\} = e^{-(t-1)} u(t-1)$$

$$\frac{4}{s^2} \Longleftrightarrow 4t\,u(t)$$

$$1 \Longleftrightarrow \delta(t)$$

$$y(t) = e^{-(t-1)} u(t-1) + 4t\,u(t) + \delta(t)$$

Identity with derivative

$$f(t) \Longleftrightarrow F(s)$$

$$\frac{df(t)}{dt} \Longleftrightarrow ?$$

$$\mathcal{L}\left(\frac{d}{dt}f\right) = \int_0^\infty \frac{df}{dt} e^{-st}\, dt$$

combine

integrate by parts

$$da = \frac{df}{dt}\,dt \qquad a = f$$

$$b = e^{-st} \qquad db = -se^{-st}\,dt$$

$$\int_{-\infty}^{\infty} b\, da = ab\Big|_0^\infty - \int_0^\infty a\, db$$

$$\mathcal{L}\left(\frac{d}{dt}f\right) = \underbrace{f(t)e^{-st}\Big|_0^\infty}_{} - \underbrace{\int_0^\infty f(t)\left(-se^{-st}\,dt\right)}_{-s\,F(s)}$$

Assume $f(t)$ not exponentially increasing with t

$$\therefore \lim_{t\to\infty} f(t)e^{-st} \to 0$$

$$\lim_{t\to 0} f(t)e^{-st} = f(0)$$

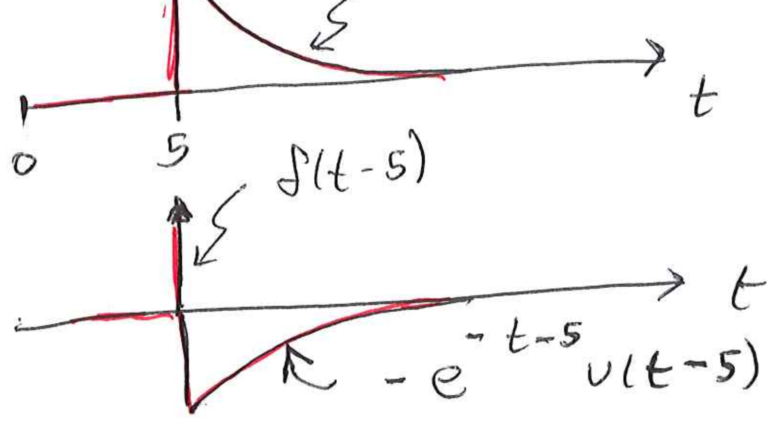
$$\mathcal{L}\left(\frac{d}{dt} f\right) = s\,F(s) - f(0)$$

<u>Example</u> Inverse Laplace of $H(s) = \dfrac{s\,e^{-5s}}{s+1}$

$$H(s) = \underbrace{\{s\}}_{\substack{\text{derivative} \\ \text{operator}}} \underbrace{\{e^{-5s}\}}_{\substack{\text{delay} \\ \text{operator}}} \underbrace{\left\{\frac{1}{s+1}\right\}}_{\substack{\text{from identity} \\ u(t)\,e^{-t}}}$$

$$\underbrace{\{e^{-5s}\}\left\{\frac{1}{s+1}\right\}}_{u(t-5)\,e^{-(t-5)}}$$

$$\underbrace{\{s\}\{e^{-5s}\}\left\{\frac{1}{s+1}\right\}}_{\frac{d}{dt}\,u(t-5)\,e^{-(t-5)}}$$



directly

$$\frac{d}{dt} u(t-s)\, e^{-(t-s)}$$

$$= \left(\frac{d}{dt} u(t-s)\right)\left(e^{-(t-s)}\right) + u(t-s)\left(\frac{d}{dt} e^{-(t-s)}\right)$$

$$= \delta(t-s)\, e^{-(t-s)} + u(t-s)\left(-e^{-(t-s)}\right)$$

$$= \left(\delta(t-s) - u(t-s)\right) e^{-(t-s)}$$

Extend the derivative identity to nth order

$$\mathcal{L}\left[\frac{d^n}{dt^n} f(t)\right] = s^n F(s) \Leftarrow \left\{ \begin{array}{l} \text{ignore the} \\ \text{IC's here} \end{array} \right.$$

Can show this identity by recursive application of integration by parts.

Last identity is the integration

$$\mathcal{L}\left(\int_{-\infty}^{t} f(t)\,dt\right) = ?$$

$$\mathcal{L}\left(\int_{-\infty}^{t} f(\tau)\,d\tau\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{t} f(\tau)\,d\tau\, e^{-st}\,dt$$

$$a(t) = \int_{-\infty}^{t} f(\tau)\,d\tau \qquad db = e^{-st}\,dt$$

$$da = f(t)\,dt \qquad b = \frac{e^{-st}}{-s}$$

$$\mathcal{L}\left[\int_{-\infty}^{t} f(\tau)\,d\tau\right] = \int_{-\infty}^{t} f(\tau)\,d\tau\,\frac{e^{-st}}{-s}\Bigg|_{-\infty}^{\infty} - \int \frac{e^{-st}}{-s} f(t)\,dt$$

$$\mathcal{L}\left[\int_{-\infty}^{t} f(\tau)\,d\tau\right] = \frac{1}{s}F(s)$$

Sumarize the Laplace identities of importance from table 2.2.

Table 2.1 Pg 36 ed. 7

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$t\,u(t)$	$\frac{1}{s^2}$
$e^{-at}\,u(t)$	$\frac{1}{a+s}$
$t^n\,u(t)$	$\frac{n!}{s^{n+1}}$
$u(t)\sin(wt)$	$\frac{w}{s^2+w^2}$
$u(t)\cos(wt)$	$\frac{s}{s^2+w^2}$

These identities are fundamental to ENEL 441
(They will be on quiz aid sheet)

Summary of identities

① Linearity $\mathcal{L}[K_1 f_1(t) + K_2 f_2(t)] = K_1 F_1(s) + K_2 F_2(s)$

K_1, K_2 constants

$f_1 \Leftrightarrow F_1$

$f_2 \Leftrightarrow F_2$

② $\mathcal{L}[e^{-at} f(t)] = F(s+a)$

③ $\mathcal{L}[f(t-T)] = e^{-sT} F(s)$

④ $\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$

⑤ $\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - \sum_{K=1}^{n} s^{n-K} f^{K-1}(0)$

⑥ $\mathcal{L}\left[\int_{-\infty}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$

Partial Fraction Expansion

Suppose we have

$$x(t) = u(t)e^{-t}$$

$$h(t) = u(t)e^{-2t}$$

$$y(t) = x(t) * h(t)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$X(s) = \frac{1}{s+1}$$

$$H(s) = \frac{1}{s+2}$$

$$Y(s) = \frac{1}{(s+1)(s+2)}$$

use partial fraction expansion

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$y(t) = u(t)\left(e^{-t} - e^{-2t}\right)$$

Example of partial fraction with multiple poles

$$F(s) = \frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}$$

$$C = F(s)(s+1)\Big|_{s=-1} = \frac{1}{(-1)^2} = 1$$

$$B = F(s)\, s^2\Big|_{s=0} = \frac{1}{1} = 1$$

$$A = \frac{d}{ds}\left(F(s)s^2\right)\Big|_{s=0} = \frac{d}{ds}\left(\frac{1}{s+1}\right)\Big|_{s=0} = \frac{-1}{(s+1)^2}\Big|_{s\to 0} = -1$$

$$F(s) = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$$

$$f(t) = -u(t) + t\,u(t) + u(t)\,e^{-t}$$

$$= u(t)\left(t + e^{-t} - 1\right)$$