ENEL 471 – Winter 2020

Assignment 8 – Solutions

Problem 4.11

(a) The angle of the PM wave is

$$\theta_i(t) = 2\pi f_c t + k_p m(t)$$

$$= 2\pi f_c t + k_p A_m \cos(2\pi f_m t)$$

$$= 2\pi f_c t + \beta_p \cos(2\pi f_m t)$$

where $\beta_p = k_p A_m$. The instantaneous frequency of the PM wave is therefore

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt}$$

$$= f_c - \beta_p f_m \sin(2\pi f_m t)$$

We see that the maximum frequency deviation in a PM wave varies linearly with the modulation frequency f_m.

Using Carson's rule, we find that the transmission bandwidth of the PM wave is approximately (for the case when $\beta_p >> 1$)

$$B_T \approx 2(f_m + \beta_p f_m) = 2f_m(1 + \beta_p) \approx 2f_m\beta_p.$$

This shows that B_T varies linearly with f_m.

(b) In an FM wave, the transmission bandwidth B_T is approximately equal to $2\Delta f$, if the modulation index $\beta >> 1$. Therefore, for an FM wave, B_T is effectively independent of the modulation frequency f_m.

Problem 4.13

The overall frequency multiplication ratio is

$$n = 2 \times 3 = 6$$

Assume that the instantaneous frequency of the FM wave at the input of the first frequency multiplier is

$$f_{i1}(t) = f_c + \Delta f \cos(2\pi f_m t)$$

The instantaneous frequency of the resulting FM wave at the output of the second frequency multiplier is therefore

$$f_{i2}(t) = nf_c + n\Delta f \cos(2\pi f_m t)$$

Thus, the frequency deviation of this FM wave is equal to

$$n\Delta f = 6 \times 100 = 60\text{kHz}$$

and its modulation index is equal to

$$\frac{n\Delta f}{f_m} = \frac{60}{5} = 12$$

The frequency separation of the adjacent side-frequencies of this FM wave is unchanged at $f_m = 5$ kHz.

Problem 4.14

$$v_2 = av_1^2$$

$$\begin{aligned} s(t) &= A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t)) \\ &= A_c \cos(2\pi f_c t + \beta m(t)) \end{aligned}$$

$$\begin{aligned} v_2 &= a \cdot s^2(t) \\ &= a \cdot \cos^2(2\pi f_c t + \beta m(t)) \\ &= \frac{a}{2} \cdot \cos(4\pi f_c t + 2\beta m(t)) \end{aligned}$$

The DC component can be filtered out.

The square-law device produces a new FM signal centred at $2f_c$ and with a frequency deviation of 2β. This doubles the frequency deviation.

Problem 4.18

The envelope detector input is

$$v(t) = s(t) - s(t-T)$$

$$= A_c \cos[2\pi f_c t + \phi(t)] - A_c \cos[2\pi f_c (t-T) + \phi(t-T)]$$

$$= -2A_c \sin\left[\frac{2\pi f_c (t-T) + \phi(t) + \phi(t-T)}{2}\right] \sin\left[\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}\right] \qquad (1)$$

where

$$\phi(t) = \beta \sin(2\pi f_m t)$$

The phase difference $\phi(t)$ - $\phi(t - T)$ is

$$\phi(t) - \phi(t-T) = \beta \sin(2\pi f_m t) - \beta \sin[2\pi f_m (t-T)]$$

$$= \beta[\sin(2\pi f_m t) - \beta \sin 2\pi f_m (t + \quad) \cos(2\pi f_m T) + \cos(2\pi f_c t) \sin(2\pi f_m T)]$$

$$\approx \beta[\sin(2\pi f_m t) - \sin(2\pi f_m t) + 2\pi f_m T \cos(2\pi f_m t)]$$

$$= 2\pi \Delta f T \cos(2\pi f_m t)$$

where

$$\Delta f = \beta f_m.$$

Therefore, noting that $2\pi f_c T = \pi/2$, we may write

$$\sin\left[\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}\right] \approx \sin[\pi f_c T + \pi \Delta f T \cos(2\pi f_m t)]$$

$$= \sin\left[\frac{\pi}{4} + \pi \Delta f T \cos(2\pi f_m t)\right]$$

$$= \sqrt{2} \cos[\pi \Delta f T \cos(2\pi f_m t)] + \sqrt{2} \sin[\pi \Delta f T \cos(2\pi f_m t)]$$

$$= \sqrt{2} + \sqrt{2}\pi \Delta f T \cos(2\pi f_m t)$$

where we have made use of the fact that $\pi\Delta f T << 1$. We may therefore rewrite Eq. (1) as

$$v(t) \approx -2\sqrt{2}A_c[1+\pi\Delta f T\cos(2\pi f_m t)]\sin\left[\pi f_c(2t-T)+\frac{\phi(t)+\phi(t-T)}{2}\right]$$

Accordingly, the envelope detector output is

$$a(t) \approx 2\sqrt{2}A_c[1+\pi\Delta f T\cos(2\pi f_m t)]$$

which, except for a bias term, is proportional to the modulating wave.

Solutions to additional problems

Problem 1

1) The modulation index is

$$\beta = \frac{k_f \max[|m(t)|]}{f_m} = \frac{\Delta f_{\max}}{f_m} = \frac{20\times 10^3}{10^4} = 2$$

The modulated signal $u(t)$ has the form

$$\begin{aligned} u(t) &= \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c+nf_m)t+\phi_n) \\ &= \sum_{n=-\infty}^{\infty} 100 J_n(2)\cos(2\pi(10^8+n10^4)t+\phi_n) \end{aligned}$$

The power of the unmodulated carrier signal is $P = \frac{100^2}{2} = 5000$. The power in the frequency component $f = f_c + k10^4$ is

$$P_{f_c+kf_m} = \frac{100^2 J_k^2(2)}{2}$$

The next table shows the values of $J_k(2)$, the frequency $f_c + kf_m$, the amplitude $100J_k(2)$ and the power $P_{f_c+kf_m}$ for various values of k.

Index k	$J_k(2)$	Frequency Hz	Amplitude $100J_k(2)$	Power $P_{f_c+kf_m}$
0	.2239	10^8	22.39	250.63
1	.5767	10^8+10^4	57.67	1663.1
2	.3528	$10^8+2\times 10^4$	35.28	622.46
3	.1289	$10^8+3\times 10^4$	12.89	83.13
4	.0340	$10^8+4\times 10^4$	3.40	5.7785

As it is observed from the table the signal components that have a power level greater than 500 (= 10% of the power of the unmodulated signal) are those with frequencies $10^8 + 10^4$ and $10^8 + 2 \times 10^4$. Since $J_n^2(\beta) = J_{-n}^2(\beta)$ it is conceivable that the signal components with frequency $10^8 - 10^4$ and $10^8 - 2 \times 10^4$ will satisfy the condition of minimum power level. Hence, there are four signal components that have a power of at least 10% of the power of the unmodulated signal. The components with frequencies $10^8 + 10^4$, $10^8 - 10^4$ have an amplitude equal to 57.67, whereas the signal components with frequencies $10^8 + 2 \times 10^4$, $10^8 - 2 \times 10^4$ have an amplitude equal to 35.28.

2) Using Carson's rule, the approximate bandwidth of the FM signal is

$$B_c = 2(\beta + 1)f_m = 2(2 + 1)10^4 = 6 \times 10^4 \text{ Hz}$$

Problem 2

1)

$$\begin{aligned}\beta_p &= k_p \max[|m(t)|] = 1.5 \times 2 = 3 \\ \beta_f &= \frac{k_f \max[|m(t)|]}{f_m} = \frac{3000 \times 2}{1000} = 6\end{aligned}$$

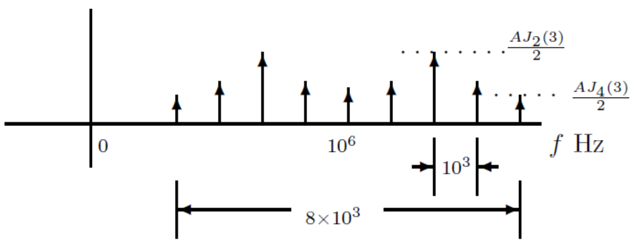
2) Using Carson's rule we obtain

$$\begin{aligned}B_{\mathrm{PM}} &= 2(\beta_p + 1)f_m = 8 \times 1000 = 8000 \\ B_{\mathrm{FM}} &= 2(\beta_f + 1)f_m = 14 \times 1000 = 14000\end{aligned}$$

3) The PM modulated signal can be written as

$$u(t) = \sum_{n=-\infty}^{\infty} AJ_n(\beta_p)\cos(2\pi(10^6 + n10^3)t)$$

The next figure shows the amplitude of the spectrum for positive frequencies and for these components whose frequencies lie in the interval $[10^6 - 4 \times 10^3, 10^6 + 4 \times 10^3]$. Note that $J_0(3) = -.2601$, $J_1(3) = 0.3391$, $J_2(3) = 0.4861$, $J_3(3) = 0.3091$ and $J_4(3) = 0.1320$.

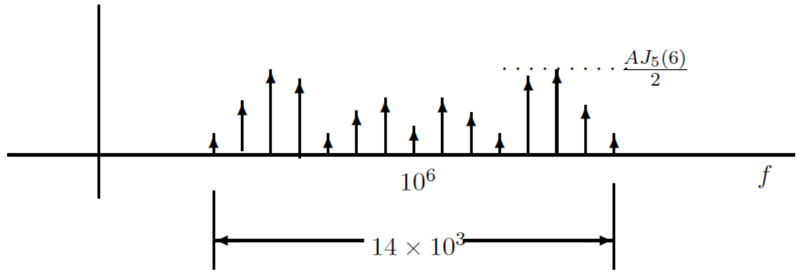


In the case of the FM modulated signal

$$\begin{aligned}u(t) &= A\cos(2\pi f_c t + \beta_f \sin(2000\pi t)) \\ &= \sum_{n=-\infty}^{\infty} AJ_n(6)\cos(2\pi(10^6 + n10^3)t + \phi_n)\end{aligned}$$

The next figure shows the amplitude of the spectrum for positive frequencies and for these components whose frequencies lie in the interval $[10^6 - 7 \times 10^3, 10^6 - 7 \times 10^3]$. The values of $J_n(6)$ for $n = 0, \ldots, 7$ are given in the following table.

n	0	1	2	3	4	5	6	7
$J_n(6)$	.1506	-.2767	-.2429	.1148	.3578	.3621	.2458	.1296



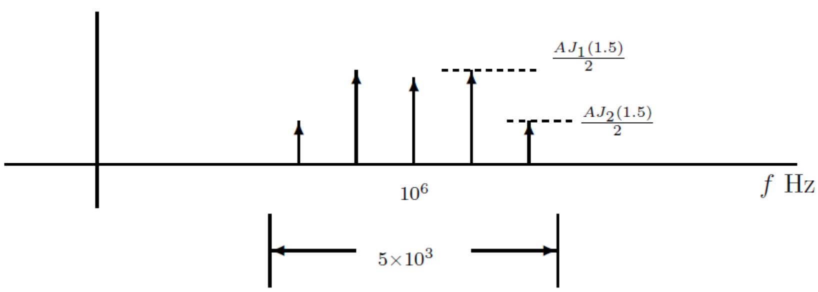
4) If the amplitude of $m(t)$ is decreased by a factor of two, then $m(t) = \cos(2\pi 10^3 t)$ and

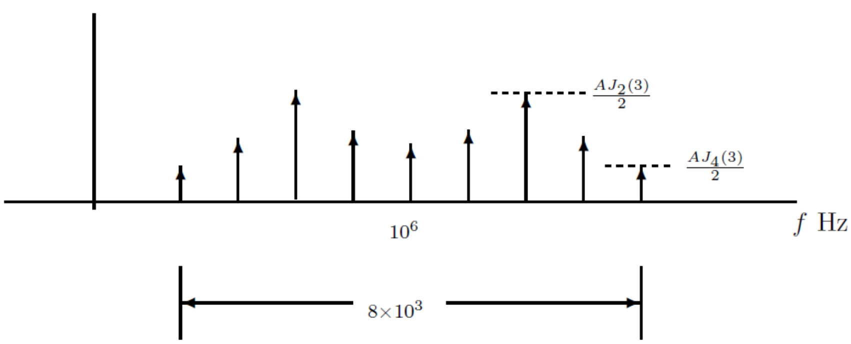
$$\begin{aligned}\beta_p &= k_p \max[|m(t)|] = 1.5 \\ \beta_f &= \frac{k_f \max[|m(t)|]}{f_m} = \frac{3000}{1000} = 3\end{aligned}$$

The bandwidth is determined using Carson's rule as

$$\begin{aligned}B_{\text{PM}} &= 2(\beta_p + 1)f_m = 5 \times 1000 = 5000 \\ B_{\text{FM}} &= 2(\beta_f + 1)f_m = 8 \times 1000 = 8000\end{aligned}$$

The amplitude spectrum of the PM and FM modulated signals is plotted in the next figure for positive frequencies. Only those frequency components lying in the previous derived bandwidth are plotted. Note that $J_0(1.5) = .5118$, $J_1(1.5) = .5579$ and $J_2(1.5) = .2321$.





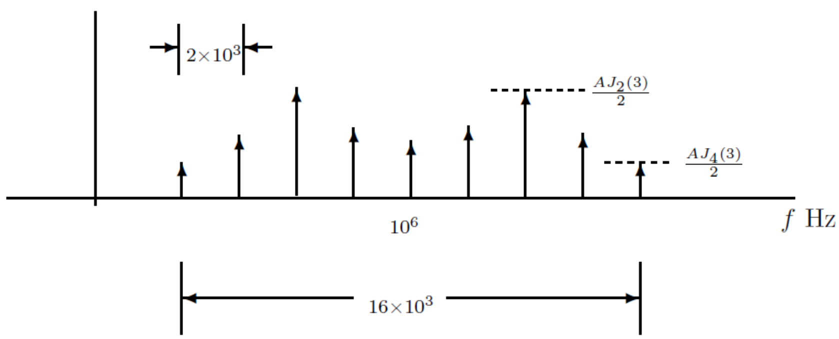
5) If the frequency of $m(t)$ is increased by a factor of two, then $m(t) = 2\cos(2\pi 2 \times 10^3 t)$ and

$$\begin{aligned}
\beta_p &= k_p \max[|m(t)|] = 1.5 \times 2 = 3 \\
\beta_f &= \frac{k_f \max[|m(t)|]}{f_m} = \frac{3000 \times 2}{2000} = 3
\end{aligned}$$

The bandwidth is determined using Carson's rule as

$$\begin{aligned}
B_{\mathrm{PM}} &= 2(\beta_p + 1) f_m = 8 \times 2000 = 16000 \\
B_{\mathrm{FM}} &= 2(\beta_f + 1) f_m = 8 \times 2000 = 16000
\end{aligned}$$

The amplitude spectrum of the PM and FM modulated signals is plotted in the next figure for positive frequencies. Only those frequency components lying in the previous derived bandwidth are plotted. Note that doubling the frequency has no effect on the number of harmonics in the bandwidth of the PM signal, whereas it decreases the number of harmonics in the bandwidth of the FM signal from 14 to 8.



Problem 3

1) The PM modulated signal is

$$
\begin{aligned}
u(t) &= 100\cos(2\pi f_c t + \frac{\pi}{2}\cos(2\pi 1000t)) \\
&= \sum_{n=-\infty}^{\infty} 100 J_n(\frac{\pi}{2})\cos(2\pi(10^8 + n10^3)t)
\end{aligned}
$$

The next table tabulates $J_n(\beta)$ for $\beta = \frac{\pi}{2}$ and $n = 0, \ldots, 4$.

n	0	1	2	3	4
$J_n(\beta)$	.4720	.5668	.2497	.0690	.0140

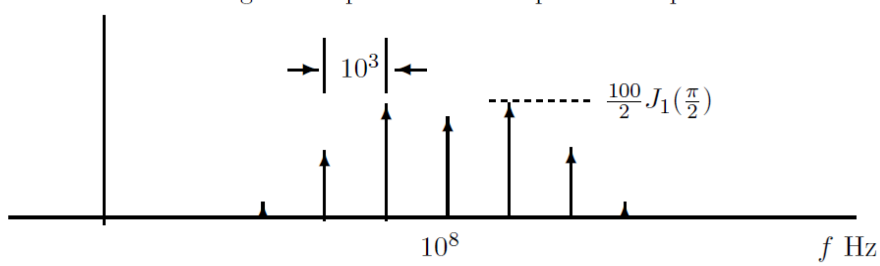
The total power of the modulated signal is $P_{\text{tot}} = \frac{100^2}{2} = 5000$. To find the effective bandwidth of the signal we calculate the index k such that

$$
\sum_{n=-k}^{k} \frac{100^2}{2} J_n^2(\frac{\pi}{2}) \geq 0.99 \times 5000 \Longrightarrow \sum_{n=-k}^{k} J_n^2(\frac{\pi}{2}) \geq 0.99
$$

By trial end error we find that the smallest index k is 2. Hence the effective bandwidth is

$$
B_{\text{eff}} = 4 \times 10^3 = 4000
$$

In the the next figure we sketch the magnitude spectrum for the positive frequencies.



2) Using Carson's rule, the approximate bandwidth of the PM signal is

$$
B_{\text{PM}} = 2(\beta_p + 1)f_m = 2(\frac{\pi}{2} + 1)1000 = 5141.6
$$

As it is observed, Carson's rule overestimates the effective bandwidth allowing in this way some margin for the missing harmonics.