ENEL 471 – Winter 2020

Assignment 6 – Solutions

Problem 4.1

For the PM case,

$$s(t) = A_c \cos[2\pi f_c t + k_p m(t)].$$

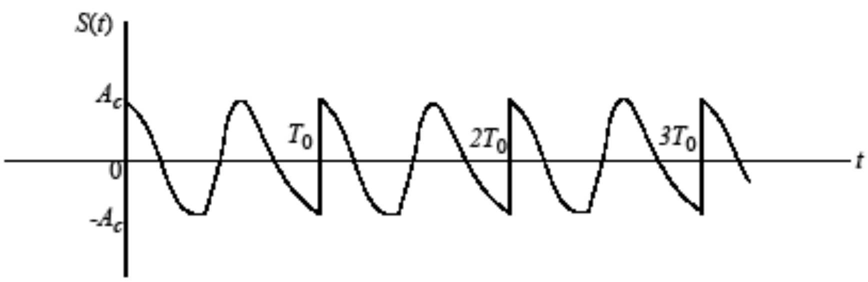
The angle equals

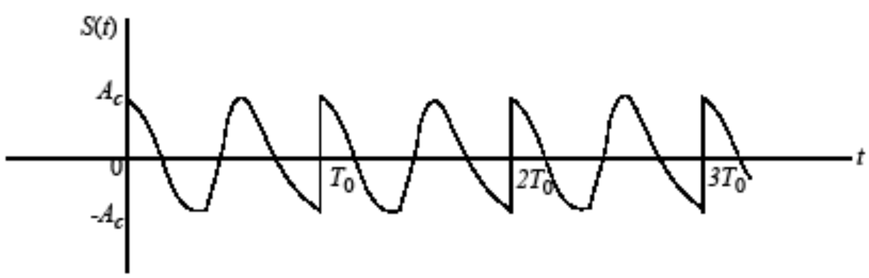
$$\theta_i(t) = 2\pi f_c t + k_p m(t).$$

The instantaneous frequency,

$$f_i(t) = f_c + \frac{Ak_p}{2\pi T_0} - \sum_n \frac{Ak_p}{2\pi}\delta(t - nT_0) \quad ,$$

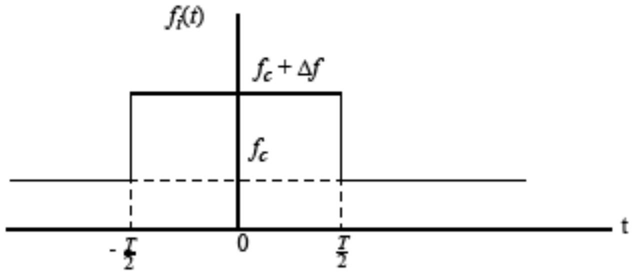
is equal to $f_c + Ak_p/2\pi T_0$ except for the instants that the message signal has discontinuities. At these instants, the phase shifts by $-k_p A/T_0$ radians.





Problem 4.3

The instantaneous frequency of the modulated wave $s(t)$ is as shown below:



We may thus express $s(t)$ as follows

$$s(t) = \begin{cases} \cos(2\pi f_c t), & t < -\frac{T}{2} \\ \cos[2\pi(f_c + \Delta f)t], & -\frac{T}{2} \le t \le \frac{T}{2} \\ \cos(2\pi f_c t), & \frac{T}{2} < t \end{cases}$$

The Fourier transform of $s(t)$ is therefore

$$\begin{aligned} S(f) &= \int_{-\infty}^{-T/2} \cos(2\pi f_c t)\exp(-j2\pi f t)dt \\ &\quad + \int_{-T/2}^{-T/2} \cos[2\pi(f_c + \Delta f)t]\exp(-j2\pi f t)dt \\ &\quad + \int_{T/2}^{-\infty} \cos(2\pi f_c t)\exp(-j2\pi f t)dt \\ &= \int_{-\infty}^{-\infty} \cos(2\pi f_c t)\exp(-j2\pi f t)dt \\ &\quad + \int_{-T/2}^{-T/2} \{\cos[2\pi(f_c + \Delta f)t - \cos(2\pi f_c t)]\}\exp(-j2\pi f t)dt \end{aligned}$$

The second term of Eq. (1) is recognized as the difference between the Fourier transforms of two RF pulses of unit amplitude, one having a frequency equal to $f_c + \Delta f$ and the other having a frequency equal to f_c. Hence, assuming that $f_c T >> 1$, we may express $S(f)$ as follows:

$$S(f) \approx \begin{cases} \frac{1}{2}\delta(f - f_c) + \frac{T}{2}\text{sinc}\,[T(f - f_c - \Delta f)] - \frac{T}{2}\text{sinc}\,[T(f - f_c)], & f > 0 \\ \frac{1}{2}\delta(f + f_c) + \frac{T}{2}\text{sinc}\,[T(f + f_c + \Delta f)] - \frac{T}{2}\text{sinc}\,[T(f + f_c)], & f < 0 \end{cases}$$

Problem 4.5

(a) The phase-modulated wave is

$$s(t) = A_c \cos[2\pi f_c t + k_p A_m \cos(2\pi f_m t)]$$

$$= A_c \cos[2\pi f_c t + \beta_p \cos(2\pi f_m t)]$$

$$= A_c \cos(2\pi f_c t)\cos[\beta_p \cos(2\pi f_m t)] - A_c \sin(2\pi f_c t)\sin[\beta_p \cos(2\pi f_m t)] \qquad (1)$$

If $\beta_p \leq 0.5$, then

$$\cos[\beta_p \cos(2\pi f_m t)] \approx 1$$

$$\sin[\beta_p \cos(2\pi f_m t)] \approx \beta_p \cos(2\pi f_m t)$$

Hence, we may rewrite Eq. (1) as

$$s(t) \approx A_c \cos(2\pi f_c t) - \beta_p A_c \sin(2\pi f_c t)\cos(2\pi f_m t)$$

$$= A_c \cos(2\pi f_c t) - \frac{1}{2}\beta_p A_c \sin[2\pi(f_c + f_m)t]$$

$$-\frac{1}{2}\beta_p A_c \sin[2\pi(f_c - f_m)t] \qquad (2)$$

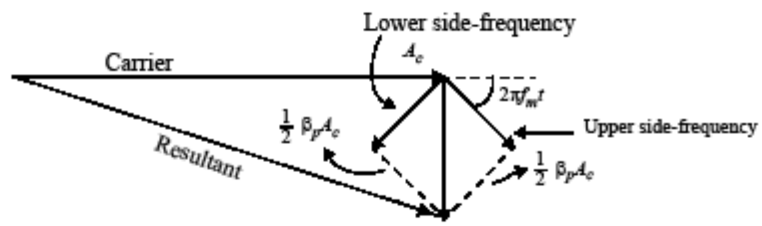
The spectrum of $s(t)$ is therefore

$$S(f) \approx \frac{1}{2}A_c[\delta(f - f_c) + \delta(f + f_c)]$$

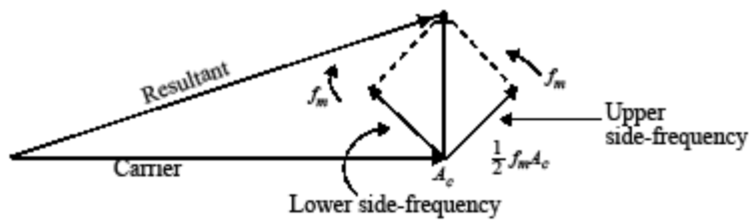
$$-\frac{1}{4j}\beta_p A_c[\delta(f - f_c - f_m) - \delta(f + f_c + f_m)]$$

$$-\frac{1}{4j}\beta_p A_c[\delta(f - f_c + f_m) - \delta(f + f_c - f_m)]$$

(b) The phasor diagram for $s(t)$ is deduced from Eq. (2) to be as follows:



The corresponding phasor diagram for the narrow-band FM wave is as follows:



Comparing these two phasor diagrams, we see that, except for a phase difference, the narrow-band PM and FM waves are of exactly the same form.