Поверхностные интегралы I-го рода

№1

$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = v \end{cases} \Rightarrow r(x,y,z) = r(u\cos v,\ u\sin v,\ v) \Rightarrow \begin{cases} r'_u = (\cos v,\ \sin v,\ 0) \\ r'_v = (-u\sin v,\ u\cos v,\ 1) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} E = \langle r'_u, r'_u \rangle = \cos^2 v + \sin^2 v = 1 \\ G = \langle r'_v, r'_v \rangle = u^2\sin^2 v + u^2\cos^2 v + 1 = u^2+1 \\ F = \langle r'_u, r'_v \rangle = 0 \end{cases} \Rightarrow dS = \sqrt{EG-F^2}\,du\,dv = \sqrt{u^2+1}\,du\,dv$$

ответ

№2

$$\begin{cases} x = (R+r\cos\theta)\cos\varphi \\ y = (R+r\cos\theta)\sin\varphi \\ z = r\sin\theta \end{cases} \Rightarrow r(x,y,z) = ((R+r\cos\theta)\cos\varphi,\ (R+r\cos\theta)\sin\varphi,\ r\sin\theta) \Rightarrow$$

$(\varphi\in(0,2\pi),\ \theta\in[-\pi,\pi),\ R,r>0)$

$$\Rightarrow \begin{cases} r'_\varphi = (-(R+r\cos\theta)\sin\varphi,\ (R+r\cos\theta)\cos\varphi,\ 0) \\ r'_\theta = (-r\sin\theta\cos\varphi,\ -r\sin\theta\sin\varphi,\ r\cos\theta) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} E = \langle r'_\varphi, r'_\varphi \rangle = (R+r\cos\theta)^2 \\ G = \langle r'_\theta, r'_\theta \rangle = r^2\sin^2\theta + r^2\cos^2\theta = r^2 \\ F = \langle r'_\varphi, r'_\theta \rangle = 0 \end{cases} \Rightarrow \sqrt{EG-F^2} = \sqrt{EG} = r\sqrt{(R+r\cos\theta)^2} \Rightarrow$$

$$\Rightarrow dS = \sqrt{EG-F^2}\,d\varphi\,d\theta = r|R+r\cos\theta|\,d\varphi\,d\theta$$

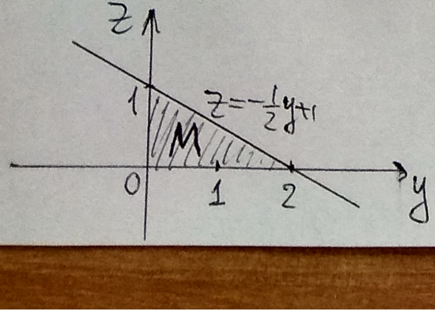
ответ

Кудрявцев (т.3, §11):

№1(1)

$$I = \iint_S (x+y+z)\,dS, \qquad S = \left\{(x,y,z)\ \middle|\ \begin{matrix} x+2y+4z=4 \\ x\geq 0,\ y\geq 0,\ z\geq 0 \end{matrix}\right\}$$

$$x = 4-2y-4z \Rightarrow dS = \sqrt{1+(x'_y)^2+(x'_z)^2}\,dy\,dz = \sqrt{1+(-2)^2+(-4)^2}\,dy\,dz = \sqrt{21}\,dy\,dz$$



Проекция S на yOz: $2y+4z=4 \Leftrightarrow z = -\frac{1}{2}y+1$

$$I = \iint_M (4-2y-4z+y+z)\sqrt{21}\,dy\,dz = \int_0^2 dy \int_0^{-\frac{1}{2}y+1} (-y-3z+4)\sqrt{21}\,dz =$$

$$= \sqrt{21}\int_0^2 dy\left(-yz-\frac{3}{2}z^2+4z\right)\Big|_0^{-\frac{1}{2}y+1} = \sqrt{21}\int_0^2\left(y\left(\frac{1}{2}y-1\right)-\frac{3}{2}\left(\frac{1}{2}y-1\right)^2-2y+4\right)dy =$$

$$= \sqrt{21}\int_0^2\left(\frac{1}{2}y^2-y-\frac{3}{2}\cdot\frac{1}{4}y^2+\frac{3}{2}y-\frac{3}{2}-2y+4\right)dy = \sqrt{21}\int_0^2\left(\frac{1}{8}y^2-\frac{3}{2}y+\frac{5}{2}\right)dy =$$

$$= \sqrt{21}\left(\frac{1}{8}\cdot\frac{8}{3}-\frac{3}{2}\cdot\frac{4}{2}+\frac{5}{2}\cdot 2\right) = \sqrt{21}\left(\frac{1}{3}-3+5\right) = \sqrt{21}\left(\frac{1}{3}+\frac{6}{3}\right) = \frac{7\sqrt{21}}{3}$$

↑ ответ

№ 1(2)

$$\begin{cases} x=\sin\theta\cos\varphi \\ y=\sin\theta\sin\varphi \qquad \theta\in[0,\pi] \\ z=\cos\theta \qquad \varphi\in[0,2\pi] \\ \qquad z\geqslant 0 \end{cases} \Rightarrow \theta\in\left[0;\frac{\pi}{2}\right]$$

$dS=\sin\theta\, d\varphi\, d\theta$ (см. семинар)

$$\iint_S (x+y+z)\,dS = \int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{2}}\left(\sin^2\theta\cos\varphi+\sin^2\theta\sin\varphi+\frac{1}{2}\sin 2\theta\right)d\theta =$$

$$= \int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{2}}\frac{\cos\varphi+\sin\varphi}{2}(1-\cos 2\theta)\,d\theta + \frac{1}{2}\int_0^{2\pi}d\varphi\int_0^{\frac{\pi}{2}}\sin 2\theta\, d\theta =$$

$$= \int_0^{2\pi}\frac{\cos\varphi+\sin\varphi}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{2}}d\varphi - \frac{1}{4}\int_0^{2\pi}\cos 2\theta\Big|_0^{\frac{\pi}{2}}d\varphi =$$

$$= \frac{\pi}{4}\int_0^{2\pi}(\cos\varphi+\sin\varphi)\,d\varphi - \frac{1}{4}\cdot 2\pi(-2) = \frac{\pi}{4}(\sin\varphi-\cos\varphi)\Big|_0^{2\pi}+\pi = \pi$$

← ответ

№ 2(2)

$I = \iint_S (x^2+y^2)\,dS$, $\quad S$ — поверхность конуса $\sqrt{x^2+y^2}\leq z\leq 1$

$$I = \underbrace{\iint_{S_1}(x^2+y^2)\,dS_1}_{I_1} + \underbrace{\iint_{S_2}(x^2+y^2)\,dS_2}_{I_2},$$

где $\begin{cases} S_1 \text{ — боковая поверхность конуса} \\ S_2 \text{ — основание конуса} \end{cases}$

$$z=\sqrt{x^2+y^2} \Rightarrow z'_x = \frac{x}{\sqrt{x^2+y^2}}, \quad z'_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$dS_1=\sqrt{(z'_x)^2+(z'_y)^2+1}\,dxdy=\sqrt{2}\,dxdy$$

$$I_1=\iint\limits_{S_1}(x^2+y^2)dS_1=\sqrt{2}\iint\limits_{M_1}(x^2+y^2)dxdy=\sqrt{2}\int\limits_0^{2\pi}d\varphi\int\limits_0^1 r^2\cdot r\,dr=\sqrt{2}\cdot 2\pi\cdot\frac{1}{4}=\frac{\pi\sqrt{2}}{2}$$

$$dS_2=\sqrt{(x'_\varphi)^2+(y'_\varphi)^2}\,d\varphi=\sqrt{((r\cos\varphi)'_\varphi)^2+((r\sin\varphi)'_\varphi)^2}\,d\varphi=$$

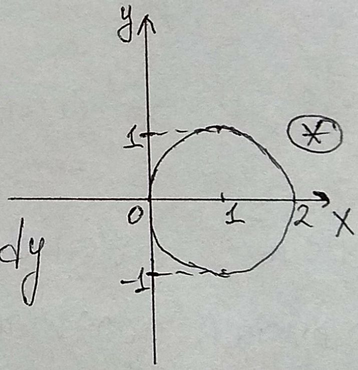
$$=\sqrt{(-r\sin\varphi)^2+(r\cos\varphi)^2}=\sqrt{r^2}=r$$

$$I_2=\iint\limits_{S_2}(x^2+y^2)dS_2=\iint\limits_{M_2}(x^2+y^2)r\,dxdy=\int\limits_0^{2\pi}d\varphi\int\limits_0^1 r^2\cdot r\,dr=2\pi\cdot\frac{1}{4}=\frac{\pi}{2}$$

$$I=I_1+I_2=\frac{\pi\sqrt{2}}{2}+\frac{\pi}{2}=\frac{\pi}{2}(\sqrt{2}+1)\leftarrow \text{ответ}$$

№7(1)

$I=\iint\limits_S(xy+yz+zx)dS$, S — часть конуса $z=\sqrt{x^2+y^2}$ внутри цилиндра $x^2+y^2=2x$



$$x^2+y^2=2x \Longleftrightarrow (x-1)^2+y^2=1$$

$$\left.\begin{aligned} z'_x&=\frac{x}{\sqrt{x^2+y^2}}\\ z'_y&=\frac{y}{\sqrt{x^2+y^2}}\end{aligned}\right\}\Rightarrow dS=\sqrt{(z'_x)^2+(z'_y)^2+1}\,dxdy=\sqrt{2}\,dxdy$$

$I=\sqrt{2}\iint\limits_M\left(xy+(y+x)\sqrt{x^2+y^2}\right)dxdy$; Пусть $\begin{cases}x=r\cos\varphi\\ y=r\sin\varphi\end{cases}$; Тогда:

$$(r\cos\varphi-1)^2+(r\sin\varphi)^2\le 1 \Longleftrightarrow r^2\cos^2\varphi-2r\cos\varphi+1+r^2\sin^2\varphi\le 1 \Longleftrightarrow$$

$\Longleftrightarrow r^2-2r\cos\varphi\le 0 \Longleftrightarrow r\le 2\cos\varphi$, причём $-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$ *

$$I=\sqrt{2}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi\int\limits_0^{2\cos\varphi}\left(r^2\cos\varphi\sin\varphi+r(\cos\varphi+\sin\varphi)\cdot r\right)r\,dr=\sqrt{2}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos\varphi\sin\varphi+\cos\varphi+\sin\varphi)d\varphi\int\limits_0^{2\cos\varphi}r^3dr=$$

$$=\sqrt{2}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\tfrac{1}{2}\sin 2\varphi+\cos\varphi+\sin\varphi\right)\cdot\frac{r^4}{4}\Big|_0^{2\cos\varphi}d\varphi=4\sqrt{2}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\tfrac{1}{2}\sin 2\varphi+\cos\varphi+\sin\varphi\right)\cos^4\varphi\,d\varphi=$$

$$=4\sqrt{2}\Bigg(\underbrace{\frac{1}{2}\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin 2\varphi\cos^4\varphi\,d\varphi}_{0}+\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^5\varphi\,d\varphi+\underbrace{\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\varphi\cos^4\varphi\,d\varphi}_{0}\Bigg)=8\sqrt{2}\int\limits_0^{\frac{\pi}{2}}\cos^5\varphi\,d\varphi=$$

$$=8\sqrt{2}\int\limits_0^{\frac{\pi}{2}}\cos^4\varphi\,d\,\overset{\sin}{\cancel{\cos}}\,\varphi=$$

$$=8\sqrt{2}\int\limits_0^{\frac{\pi}{2}}\left(1-\sin^2\varphi\right)^2d\sin\varphi=8\sqrt{2}\int\limits_0^{\frac{\pi}{2}}\left(1-2\sin^2\varphi+\sin^4\varphi\right)d\sin\varphi=$$

ответ

$$=8\sqrt{2}\left(\sin\varphi-\frac{2}{3}\sin^3\varphi+\frac{1}{5}\sin^5\varphi\right)\Bigg|_0^{\frac{\pi}{2}}=8\sqrt{2}\left(1-\frac{2}{3}+\frac{1}{5}\right)=8\sqrt{2}\left(\frac{15-10+3}{15}\right)=\frac{64\sqrt{2}}{15}$$

№10

$I=\iint\limits_S z^2\,dS$, $\quad S$ – часть конич. пов-ти $\begin{cases}x=u\cos v\sin\alpha\\ y=u\sin v\sin\alpha\\ z=u\cos\alpha\end{cases}$ $\quad\alpha=\text{const}$, $0<\alpha<\frac{\pi}{2}$, $u\in[0;1]$, $v\in[0;2\pi]$

$r'_u=(\cos v\sin\alpha,\ \sin v\sin\alpha,\ \cos\alpha)$

$r'_v=(-u\sin v\sin\alpha,\ u\cos v\sin\alpha,\ 0)$

$E=\langle r'_u,r'_u\rangle=1;\quad G=\langle r'_v,r'_v\rangle=u^2\sin^2\alpha;\quad F=\langle r'_u,r'_v\rangle=0$

$dS=\sqrt{EG-F^2}\,du\,dv=u\sin\alpha\,du\,dv$

$$I=\iint\limits_{D(u,v)}u^2\cos^2\alpha\,u\sin\alpha\,du\,dv=\int\limits_0^{2\pi}dv\int\limits_0^1u^3\cos^2\alpha\sin\alpha\,du=2\pi\cdot\cos^2\alpha\sin\alpha\cdot\frac{1}{4}=$$

$$=\frac{\pi\sin\alpha\cos^2\alpha}{2}$$ ← ответ

№8(1)

$$I=\iint\limits_S f(x;y;z)dS, \qquad f=\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}, \quad S-\text{эллипсоид } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\begin{cases} x=a\sin u\cos v \\ y=b\sin u\sin v \\ z=c\cos u \end{cases} \Rightarrow \begin{cases} r'_u=(a\cos u\cos v,\ b\cos u\sin v,\ -c\sin u) \\ r'_v=(-a\sin u\sin v,\ b\sin u\cos v,\ 0) \end{cases}$$

$(0\le u\le\pi,\ 0\le v<2\pi)$

$$\begin{vmatrix} i & j & k \\ a\cos u\cos v & b\cos u\sin v & -c\sin u \\ -a\sin u\sin v & b\sin u\cos v & 0 \end{vmatrix} =$$

$$= kab\sin u\cos u\cos^2 v+jac\sin^2 u\sin v+kab\sin u\cos u\sin^2 v+ibc\sin^2 u\cos v=$$

$$=abc\sin(u)\cdot\left(\frac{k}{c}\cos u\cos^2 v+\frac{k}{c}\cos u\sin^2 v+\frac{j}{b}\sin u\sin v+\frac{i}{a}\sin u\cos v\right)$$

$$\|r'_u\times r'_v\|=\left[(abc\sin(u))^2\left(\left(\frac{\cos u}{c}\cos^2 v+\frac{\cos u}{c}\sin^2 v\right)^2+\left(\frac{\sin u\sin v}{b}\right)^2+\left(\frac{\sin u\cos v}{a}\right)^2\right)\right]^{\frac{1}{2}}=$$

$$=abc\sin(u)\left(\frac{\cos^2 u}{c^2}+\frac{\sin^2 u\sin^2 v}{b^2}+\frac{\sin^2 u\cos^2 v}{a^2}\right)^{\frac{1}{2}}=$$

$$=abc\sin(u)\cdot\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$$

$$dS=\|r'_u\times r'_v\|du\,dv=abc\sin(u)\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$$

Следовательно, $I=\int\limits_0^{2\pi}dv\int\limits_0^{\pi}abc\sin(u)\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)du$

пусть

$$J=\int\limits_0^{\pi}\sin(u)\cdot\left(\frac{\sin^2 u\cos^2 v}{a^2}+\frac{\sin^2 u\sin^2 v}{b^2}+\frac{\cos^2 u}{c^2}\right)du=$$

$$=\frac{\cos^2 v}{a^2}\int\limits_0^{\pi}\sin^3 u\,du+\frac{\sin^2 v}{b^2}\int\limits_0^{\pi}\sin^3 u\,du+\frac{1}{c^2}\int\limits_0^{\pi}\sin u\cos^2 u\,du$$

$$\int_0^{\pi}\sin^3 u\,du = -\int_0^{\pi}\sin^2 u\,d\cos u = \int_0^{\pi}(\cos^2 u - 1)\,d\cos u = \left(\frac{\cos^3 u}{3} - \cos u\right)\Big|_0^{\pi} =$$

$$= \frac{-2}{3} + 2 = \frac{4}{3}$$

$$\int_0^{\pi}\sin u\cos^2 u\,du = -\int_0^{\pi}\cos^2 u\,d\cos u = \frac{-\cos^3 u}{3}\Big|_0^{\pi} = \frac{2}{3}$$

$$J = \frac{4}{3}\left(\frac{\cos^2 v}{a^2} + \frac{\sin^2 v}{b^2}\right) + \frac{2}{3c^2}$$

Значит, $I = abc\int_0^{2\pi}\left(\frac{4\cos^2 v}{3a^2} + \frac{4\sin^2 v}{3b^2} + \frac{2}{3c^2}\right)dv =$

$$= abc\left(\frac{4}{3a^2}\int_0^{2\pi}\cos^2 v\,dv + \frac{4}{3b^2}\int_0^{2\pi}\sin^2 v\,dv + \frac{2}{3c^2}\cdot 2\pi\right)$$

$$\int_0^{2\pi}\cos^2 v\,dv = \frac{1}{2}\int_0^{2\pi}(1+\cos 2v)\,dv = \frac{1}{2}\cdot 2\pi = \pi$$

$$\int_0^{2\pi}\sin^2 v\,dv = \frac{1}{2}\int_0^{2\pi}(1-\cos 2v)\,dv = \frac{1}{2}\cdot 2\pi = \pi$$

Окончательно получаем $I = abc\left(\frac{4\pi}{3a^2} + \frac{4\pi}{3b^2} + \frac{4\pi}{3c^2}\right) =$

$$= \frac{4\pi abc}{3}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \leftarrow \text{ответ}$$