1 Linear Algebra

1.1 Derivative (14 points)

Assume that x is a vector and A is a square matrix. Show that:

- $$\frac{\partial x^T A x}{\partial x} = 2x^T A$$
- $$\frac{\partial trace(x^T A x)}{\partial x} = x^T(A + A^T)$$

1.2 Eigenvalue and Eigenvectors (14 points)

Suppose $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of matrix A. Prove:

- $\lambda_1 + \lambda_2 + ... + \lambda_n = trace(A)$
- $\lambda_1 \lambda_2 ... \lambda_n = det(A)$

1.3 Estimation (18 points)

Suppose X and Y are independent normal random variables with mean μ and variance 1, where $\mu \sim Uni(0, 1)$.

$$f_\mu(t) = \begin{cases} 1, & \text{t} \in [0, 1] \\ 0, & \text{elsewhere} \end{cases}$$

- Find the joint distribution of μ,X,Y.($f_{\mu,X,Y}(t.x.y)$).
- Find the MAP estimate of μ.

1.4 Norms (12 points)

Calculate the L1 norm, the Euclidean (L2) norm and the Maximum(L infinity) norm of the following matrix :

$$A = \begin{bmatrix} 5 & -4 & 2 \\ -1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix}$$

2 Probability

2.1 Expectation (18 points)

Let the joint probability density function of random variables X and Y be :

$$f_(x) = \begin{cases} 2x - 2, & 1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Find $\mathbb{E}(X^3 + 2X - 7)$.

2.2 Function of random variables (14 points)

Suppose $X_1, X_2, ..., X_n$ are iid random variables. Find the probability density function of $Y_1 = max[X_1, X_2, ..., X_n]$, $Y_2 = min[X_1, X_2, ..., X_n]$.

2.3 Rank (10 points)

Prove that if P is a full rank matrix, matrices M and $P^{-1}MP$ have the same set of eigenvalues.