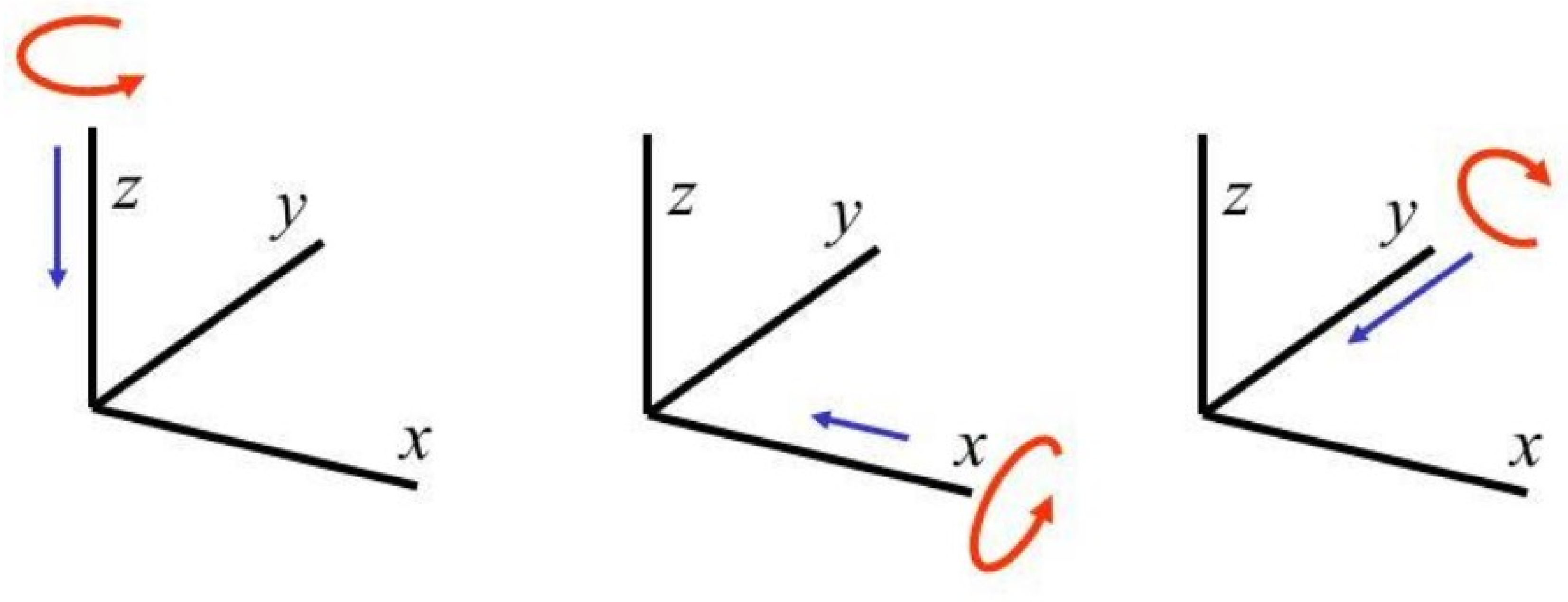
Spatial Descriptions and Transformations

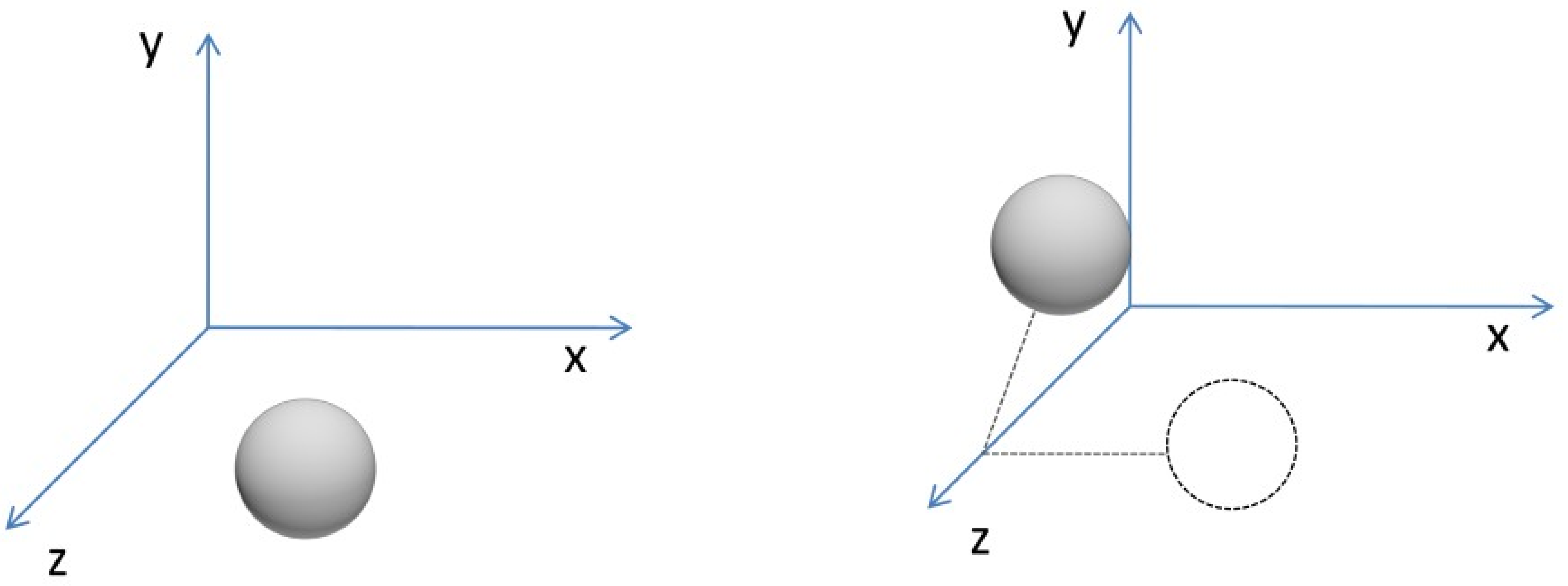
Computer Graphics (Recap)

- Rotation around axis
  - Counter-clockwise, w.r.t rotation axis.



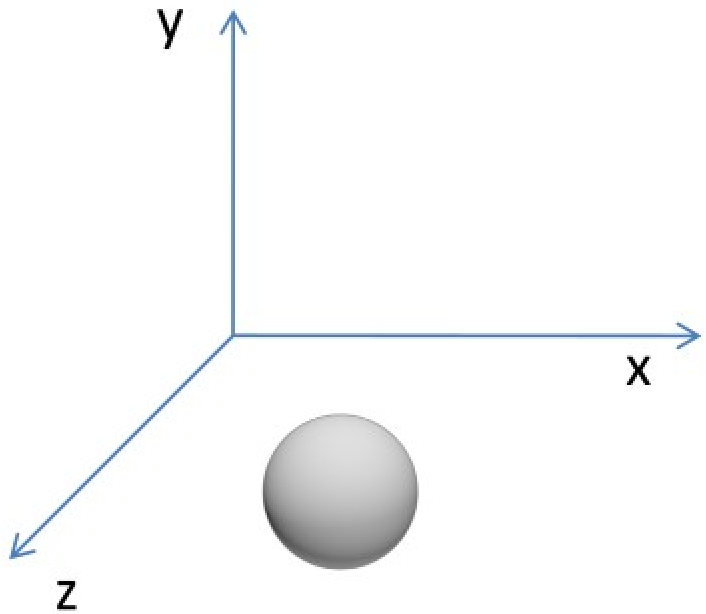
Computer Graphics (Recap)

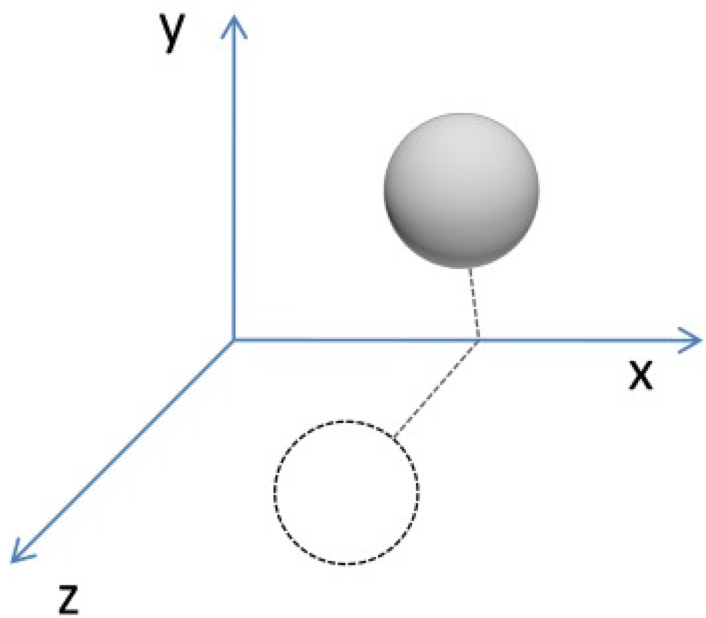
$$\text{rotate-z}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Computer Graphics (Recap)

$$\text{rotate-x}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$





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Computer Graphics (Recap)

$$\text{rotate-z}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{rotate-x}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$\text{rotate-y}(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

Computer Graphics (Recap)

$$\text{rotate-z}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{rotate-x}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$\text{rotate-y}(\phi) = \begin{bmatrix} \cos\phi & 0 & \boxed{\sin\phi} \\ 0 & 1 & 0 \\ \boxed{-\sin\phi} & 0 & \cos\phi \end{bmatrix}$$

Q: Why is it different?

Computer Graphics (Recap)

	2D	3D
T	$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$	$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

R	$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$	$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ $R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ $R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Basic Terminologies

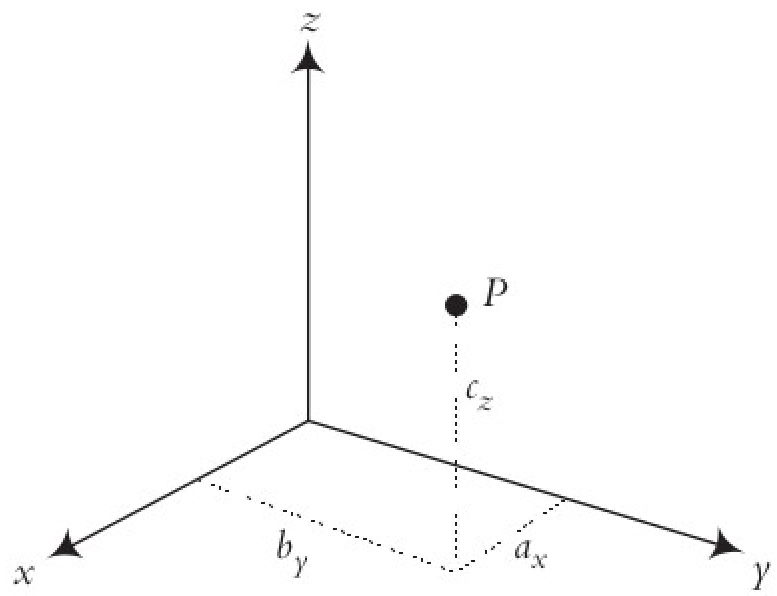
Description: A description is used to specify attributes of various objects with which a manipulation system deals.

We discuss the description of **positions**, of **orientations**, and of an entity that contains both of these descriptions: **the frame**.

Representation of a Vector in Space

A vector can be represented by three coordinates of its tail and its head. If the vector starts at point A and ends at point B, then it can be represented by

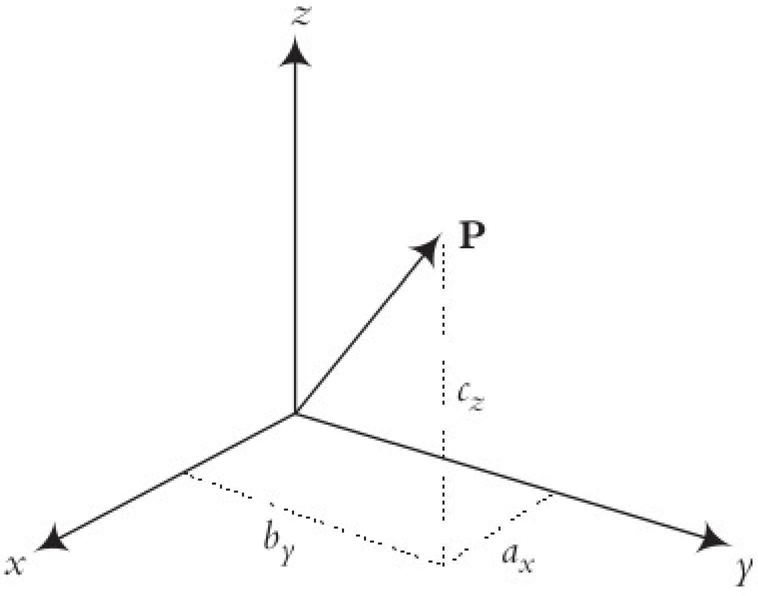
$$\mathbf{P}_{AB} = (B_x - A_x)\mathbf{i} + (B_y - A_y)\mathbf{j} + (B_z - A_z)\mathbf{k}.$$



Representation of a Vector in Space

Specifically, if the vector starts at the origin

$$\mathbf{P} = a_x\mathbf{i} + b_y\mathbf{j} + c_z\mathbf{k}$$



Representation of a Vector in Space

In Matrix Form,

$$\mathbf{P} = \begin{bmatrix} a_x \\ b_y \\ c_z \end{bmatrix}$$

This representation can be slightly modified to also include a scale factor **w**

$$\mathbf{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \\ w \end{bmatrix} \quad \text{where } a_x = \frac{P_x}{w}, b_y = \frac{P_y}{w}, \text{ etc.}$$

Representation of a Vector in Space

- **w** may be any number and, as it changes, it can change the overall size of the vector.
- This is similar to the zooming function in computer graphics.
- As the value of w changes, the size of the vector changes accordingly.
- If w is bigger than 1, all vector components enlarge.
- If w is smaller than 1, all vector components become smaller.
- However, if w = 0 it becomes a direction vector.

Example I

A vector is described as $\mathbf{P} = 3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$. Express the vector in matrix form:

(a) With a scale factor of 2.
(b) If it were to describe a direction as a unit vector.

Example I

A vector is described as $\mathbf{P} = 3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$. Express the vector in matrix form:

(a) With a scale factor of 2.
(b) If it were to describe a direction as a unit vector.

Solution: The vector can be expressed in matrix form with a scale factor of 2 as well as 0 for direction as:

$$\mathbf{P} = \begin{bmatrix} 6 \\ 10 \\ 4 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 0 \end{bmatrix}$$

Example I

A vector is described as $\mathbf{P} = 3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$. Express the vector in matrix form:

(a) With a scale factor of 2.
(b) If it were to describe a direction as a unit vector.

However, in order to make the vector into a unit vector, we normalize the length to be equal to 1. To do this, each component of the vector is divided by the square root of the sum of the squares of the three components:

$\lambda = \sqrt{P_x^2 + P_y^2 + P_z^2} = 6.16$ and $P_x = {}^3/_{6.16} = 0.487$, etc. Therefore,

$$\mathbf{P}_{unit} = \begin{bmatrix} 0.487 \\ 0.811 \\ 0.324 \\ 0 \end{bmatrix}$$

Note that $\sqrt{0.487^2 + 0.811^2 + 0.324^2} = 1$.

Example II

A vector **p** is 5 units long and is in the direction of a unit vector **q** described below. Express the vector in matrix form.

$$\mathbf{q}_{unit} = \begin{bmatrix} 0.371 \\ 0.557 \\ q_z \\ 0 \end{bmatrix}$$

Example II

A vector **p** is 5 units long and is in the direction of a unit vector **q** described below. Express the vector in matrix form.

$$\mathbf{q}_{unit} = \begin{bmatrix} 0.371 \\ 0.557 \\ q_z \\ 0 \end{bmatrix}$$

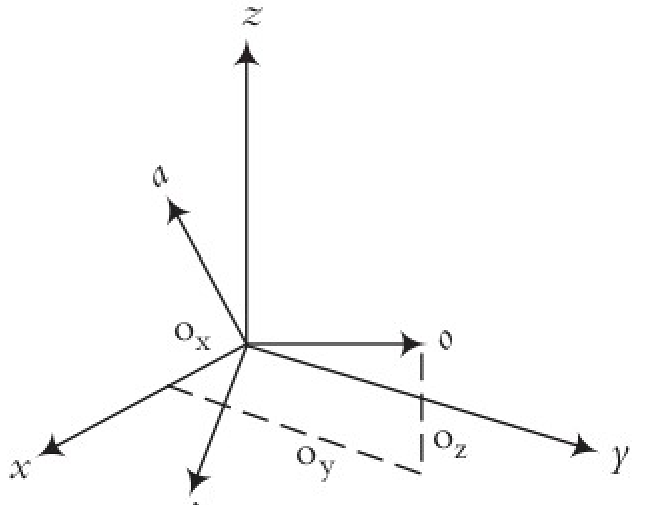
Solution: The unit vector's length must be 1. Therefore,

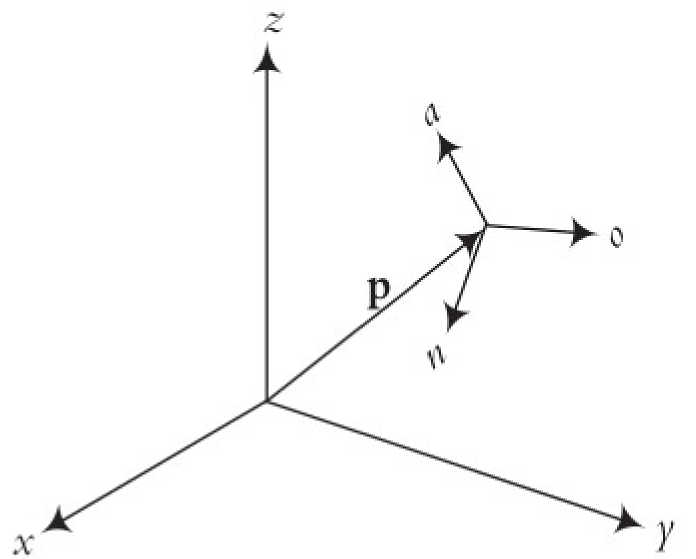
$$\lambda = \sqrt{q_x^2 + q_y^2 + q_z^2} = \sqrt{0.138 + 0.310 + q_z^2} = 1 \quad \rightarrow \quad q_z = 0.743$$

$$\mathbf{q}_{unit} = \begin{bmatrix} 0.371 \\ 0.557 \\ 0.743 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{p} = \mathbf{q}_{\,unit} \times 5 = \begin{bmatrix} 1.855 \\ 2.785 \\ 3.715 \\ 1 \end{bmatrix}$$

Representation of a Frame Relative to a Fixed Reference Frame

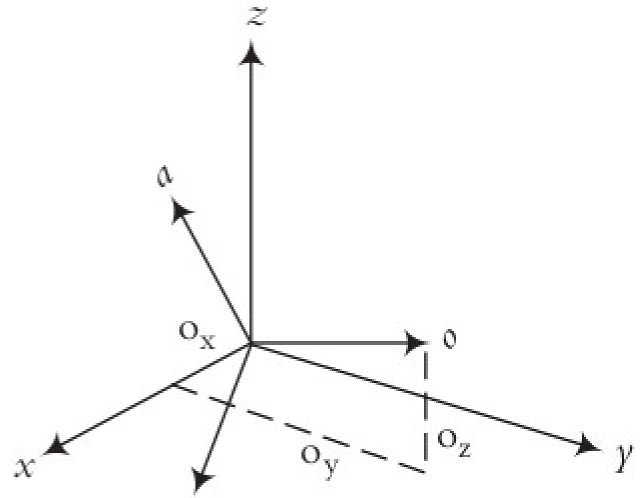
To fully describe a frame relative to another frame, both the location of its origin and the directions of its axes must be specified. If a frame is not at the origin (or, in fact, even if it is at the origin) of the reference frame, its location relative to the reference frame is described by a **vector** between the origin of the frame and the origin of the reference frame

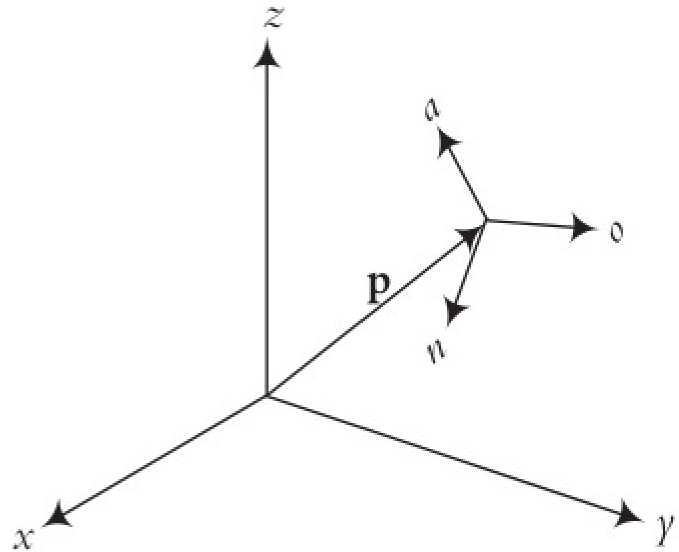




Representation of a Frame Relative to a Fixed Reference Frame

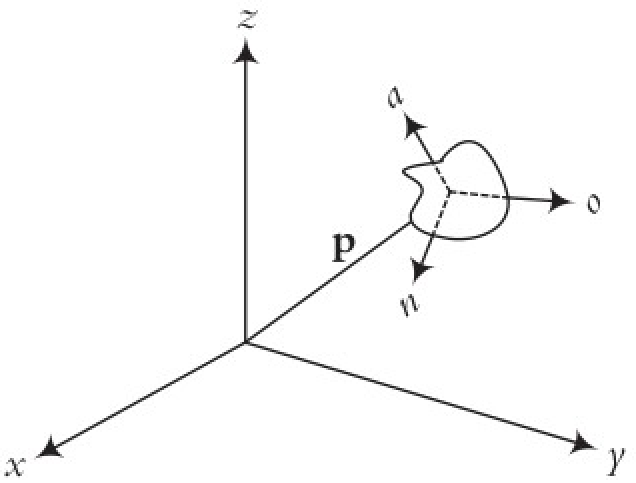
$$F = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





Representation of a Rigid Body

$$F_{object} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

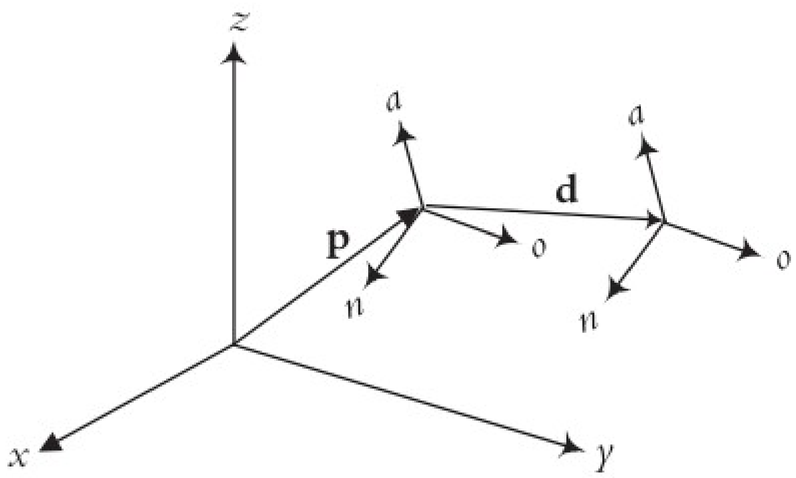


Representation of Transformations

A transformation may be in one of the following forms:

- A pure translation
- A pure rotation about an axis
- A combination of translations and/or rotations

Representation of a Pure Translation



$$T = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where dx , dy , and dz are the three components of a pure translation vector d relative to the x-, y-, and z-axes of the reference frame. The new location of the frame is,

$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x + d_x \\ n_y & o_y & a_y & p_y + d_y \\ n_z & o_z & a_z & p_z + d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example III

A frame F has been moved 10 units along the y-axis and 5 units along the z-axis of the reference frame. Find the new location of the frame.

$$F = \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example III

A frame F has been moved 10 units along the y-axis and 5 units along the z-axis of the reference frame. Find the new location of the frame.

$$F = \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: Using Equation (2.15) or (2.16), we get:

$$F_{new} = Trans(d_x, d_y, d_z) \times F_{old} = Trans(0, 10, 5) \times F_{old}$$

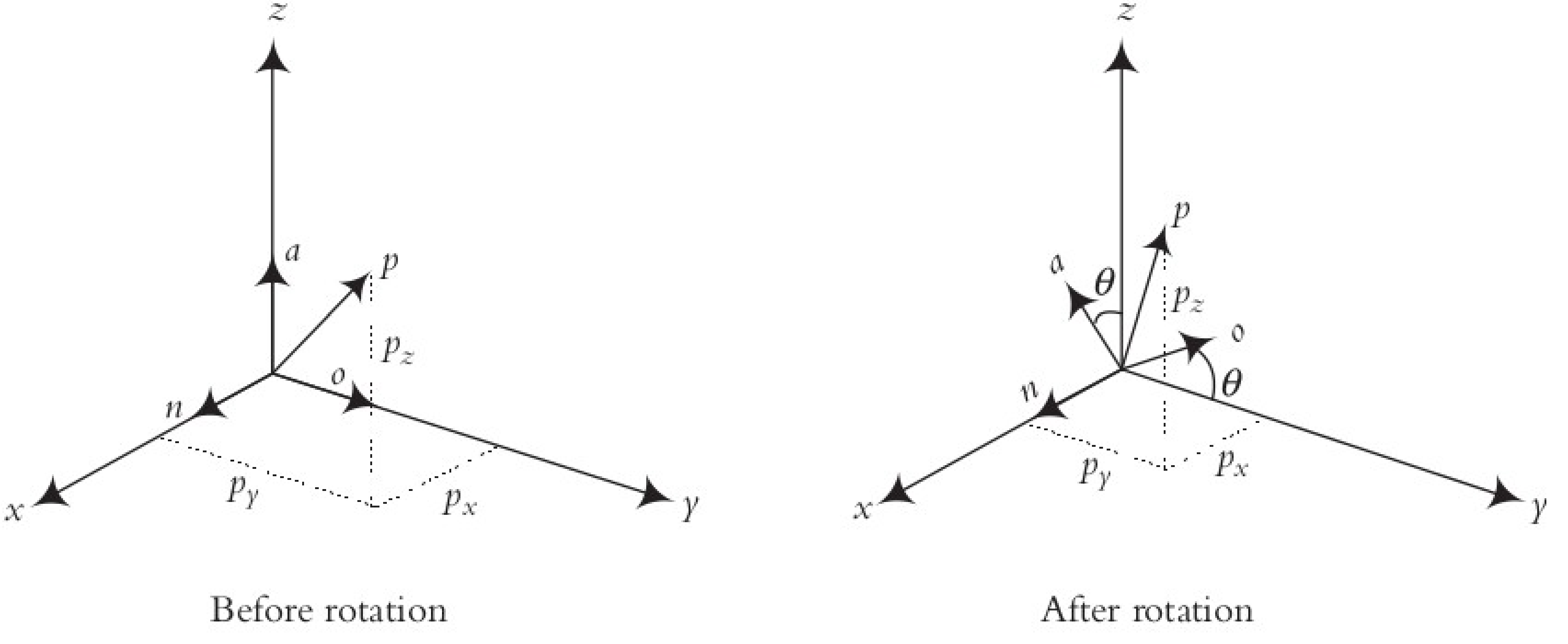
and

$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 13 \\ -0.766 & 0 & 0.643 & 13 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Representation of a Pure Rotation about an Axis

Rotation along X-axis



Representation of a Pure Rotation about an Axis

Rotation along X-axis

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_n \\ p_o \\ p_a \end{bmatrix} \qquad Rot(x, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}$$

Similarly

$$Rot(y, \theta) = \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix} \quad \text{and} \quad Rot(z, \theta) = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example IV

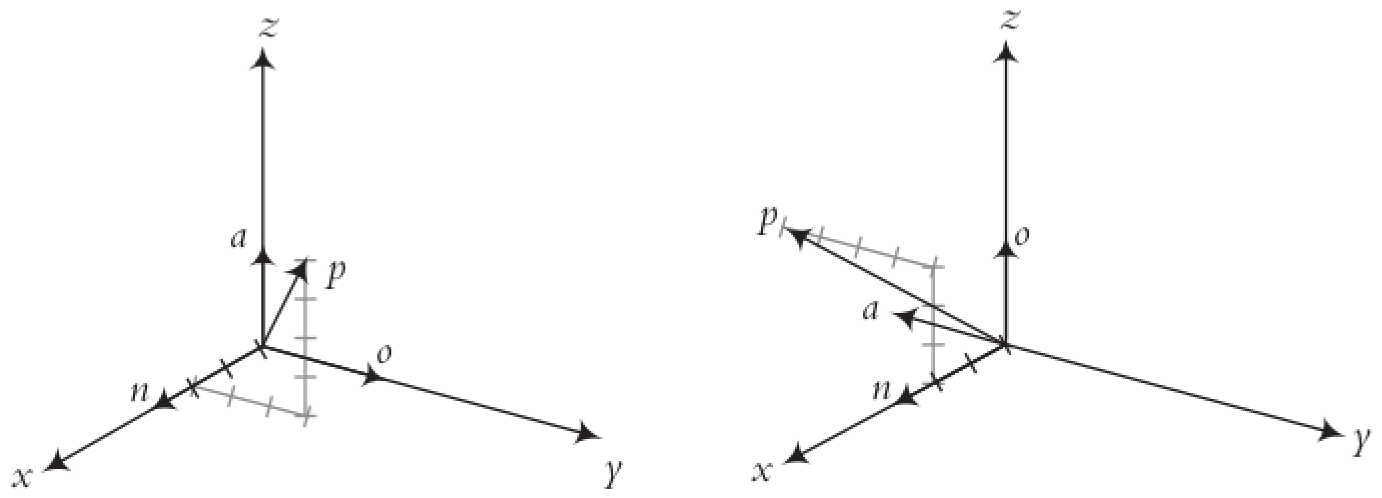
A point $p(2,3,4)^T$ is attached to a rotating frame. The frame rotates 90° about the x-axis of the reference frame. Find the coordinates of the point relative to the reference frame after the rotation, and verify the result graphically.

Example IV

Solution: Of course, since the point is attached to the rotating frame, the coordinates of the point relative to the rotating frame remain the same after the rotation. The coordinates of the point relative to the reference frame will be:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} \times \begin{bmatrix} p_n \\ p_o \\ p_a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$$

As shown in Figure 2.14, the coordinates of point p relative to the reference frame after rotation are 2, −4, 3, as obtained by the above transformation.



Representation of Combined Transformations

To see how combined transformations are handled, let's assume that a frame F_{noa} is subjected to the following three successive transformations relative to the reference frame F_{xyz}:

1. Rotation of α degrees about the x-axis,
2. Followed by a translation of $[l_1, l_2, l_3]$ (relative to the x-, y-, and z-axes respectively),
3. Followed by a rotation of β degrees about the y-axis.

$$p_{1,xyz} = Rot(x, \alpha) \times p_{noa}$$

$$p_{2,xyz} = Trans(l_1, l_2, l_3) \times p_{1,xyz} = Trans(l_1, l_2, l_3) \times Rot(x, \alpha) \times p_{noa}$$

$$p_{xyz} = p_{3,xyz} = Rot(y, \beta) \times p_{2,xyz} = Rot(y, \beta) \times Trans(l_1, l_2, l_3) \times Rot(x, \alpha) \times p_{noa}$$

Example V

A point $p(7,3,1)^T$ is attached to a frame F_{noa} and is subjected to the following transformations. Find the coordinates of the point relative to the reference frame at the conclusion of transformations.

1. Rotation of 90° about the z-axis,
2. Followed by a rotation of 90° about the y-axis,
3. Followed by a translation of $[4,-3,7]$.

Example V

A point $p(7,3,1)^T$ is attached to a frame F_{noa} and is subjected to the following transformations. Find the coordinates of the point relative to the reference frame at the conclusion of transformations.

1. Rotation of 90° about the z-axis,
2. Followed by a rotation of 90° about the y-axis,
3. Followed by a translation of [4,−3,7].

Solution: The matrix equation representing the transformation is:

$$p_{xyz} = Trans(4,-3,7)Rot(y,90)Rot(z,90)p_{noa}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 10 \\ 1 \end{bmatrix}$$

Example VI

In this case, assume the same point $p(7,3,1)^T$, attached to F_{noa}, is subjected to the same transformations, but the transformations are performed in a different order, as shown. Find the coordinates of the point relative to the reference frame at the conclusion of transformations.

1. A rotation of 90° about the z-axis,
2. Followed by a translation of $[4,-3,7]$,
3. Followed by a rotation of 90° about the y-axis.

Example VI

In this case, assume the same point $p(7,3,1)^T$, attached to F_{noa}, is subjected to the same transformations, but the transformations are performed in a different order, as shown. Find the coordinates of the point relative to the reference frame at the conclusion of transformations.

1. A rotation of 90° about the z-axis,
2. Followed by a translation of [4,−3,7],
3. Followed by a rotation of 90° about the y-axis.

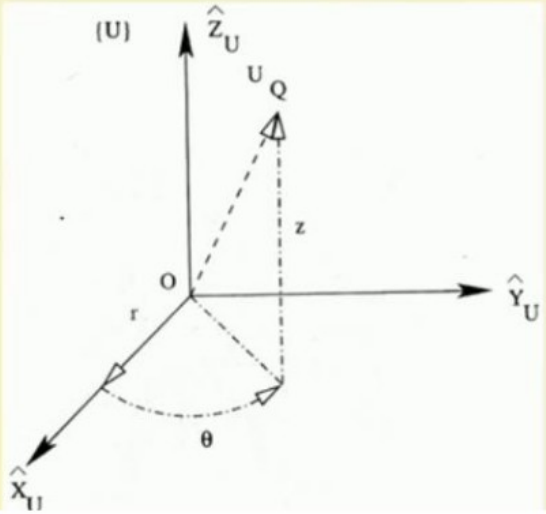
Solution: The matrix equation representing the transformation is:

$$p_{xyz} = Rot(y,90)\,Trans(4,-3,7)\,Rot(z,90)\,p_{noa}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ -1 \\ 1 \end{bmatrix}$$

Example VII

Cylindrical Coordinate System



Steps:

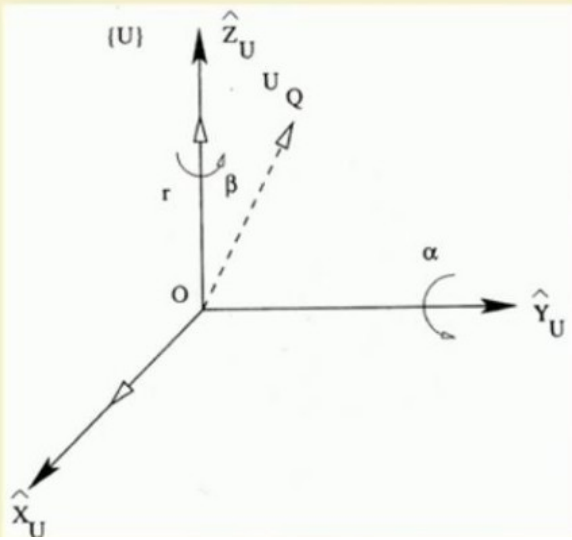
1. Starting from the origin O, translate by r units along $\hat{X}_U$ axis
2. Rotate in anti-clockwise sense about $\hat{Z}_U$ axis by an angle θ
3. Translate along $\hat{Z}_U$ axis by z units

$$[T]_{composite} = TRANS(\hat{Z}_U, z)ROT(\hat{Z}_U, \theta)TRANS(\hat{X}_U, r)$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & r\cos\theta \\ \sin\theta & \cos\theta & 0 & r\sin\theta \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example VIII

Spherical Coordinate System



Steps:

1. Starting from the origin O, translate along $\hat{Z}_U$ axis by r units
2. Rotate in anti-clockwise sense about $\hat{Y}_U$ axis by an angle α
3. Rotate in anti-clockwise sense about $\hat{Z}_U$ axis by an angle β

$$[T]_{composite} = ROT(\hat{Z}_U, \beta)ROT(\hat{Y}_U, \alpha)TRANS(\hat{Z}_U, r)$$

$$= \begin{bmatrix} \cos\alpha\cos\beta & -\sin\beta & \sin\alpha\cos\beta & r\sin\alpha\cos\beta \\ \cos\alpha\sin\beta & \cos\beta & \sin\alpha\sin\beta & r\sin\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha & r\cos\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse of Transformation Matrices

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} n_x & n_y & n_z & -\mathbf{p}\cdot\mathbf{n} \\ o_x & o_y & o_z & -\mathbf{p}\cdot\mathbf{o} \\ a_x & a_y & a_z & -\mathbf{p}\cdot\mathbf{a} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example IX

Calculate the matrix representing $Rot(x, 40°)^{-1}$.

Solution: The matrix representing a 40° rotation about the x-axis is:

$$Rot(x, 40°) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.766 & -0.643 & 0 \\ 0 & 0.643 & 0.766 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The inverse of this matrix is:

$$Rot(x, 40°)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.766 & 0.643 & 0 \\ 0 & -0.643 & 0.766 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As you can see, since the position vector of the matrix is zero, its dot product with the **n**-, **o**-, and **a**-vectors is also zero. ■

Example X

Calculate the inverse of the given transformation matrix:

$$T = \begin{bmatrix} 0.5 & 0 & 0.866 & 3 \\ 0.866 & 0 & -0.5 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example X

Calculate the inverse of the given transformation matrix:

$$T = \begin{bmatrix} 0.5 & 0 & 0.866 & 3 \\ 0.866 & 0 & -0.5 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: Based on the above, the inverse of the transformation will be:

$$T^{-1} = \begin{bmatrix} 0.5 & 0.866 & 0 & -(3 \times 0.5 + 2 \times 0.866 + 5 \times 0) \\ 0 & 0 & 1 & -(3 \times 0 + 2 \times 0 + 5 \times 1) \\ 0.866 & -0.5 & 0 & -(3 \times 0.866 + 2 \times -0.5 + 5 \times 0) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.866 & 0 & -3.23 \\ 0 & 0 & 1 & -5 \\ 0.866 & -0.5 & 0 & -1.598 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

References

Saeed B. Niku - Introduction to Robotics Analysis, Control, Applications Wiley (2010)