On the appearance of 1/f noise in the abelian sandpile model

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October 2019

Abstract

Complex systems are those which are intrinsically difficult to model due to the large number of variables that are involved in their operation. Typically the dynamic between the constituents of such systems is not limited to simple interactions through the four fundamental forces. A wide range of areas, including neurology, finance, geophysics, sociology, and computer science are faced with the challenge of understanding such systems.
An important paradigm in the study of complex systems is that of self organized criticality. This was first proposed by Bak, Tang and Wiesenfeld in 1987, along with a sandpile model that displays self organized criticality.
Here explore the statistics of systems that obey the Bak-Tang-Wiesenfeld (abelian) sandpile model. The statistically stable mass of these systems is investigated and found to be proportional to their area. We also investigate some characteristic statistical properties of such sandpiles and find that they follow a pink noise pattern, except in the case of avalanche losses. It is also found that some properties of an abelian sandpile are scale invariant.

Introduction

In their 1987 paper, titled "Self-organized criticality: an exploration of 1/f noise", Bak, Tang and Wiesenfeld put forward the idea of self organized criticality. Systems that display self organized criticality (SOC) are those that evolve into a critical point given a wide array of initial conditions. They suggested that systems with many, spatially separated degrees of freedom often follow SOC dynamics and that this is the underlying cause of the prevalence of 1/f noise in nature.[2] This argument has since become part of the paradigm in the study of naturally arising complexity,[1] finding application in a wide range of fields, including economics, geophysics, and neurobiology.[3,4,6]
In this paper, they also described an example of such a system, known as the 'BTW sandpile model' or the 'Abelian sandpile model':[2]
Consider an $N \times N$ grid of points with a grain of sand being dropped onto it at regular intervals. The mechanism by which sand is added to the grid we will call the driving force. The site to which the grain of sand is added is picked from a uniform distribution. Once a site holds more than four grains of sand we will have a toppling event. That is, if $z(x, y)$ is the mass of sand at site $\mathbf{x} = (x, y)$, then

$$z(x,y) = 4 \rightarrow \begin{cases} z(x,y) \rightarrow 0 \\ z(x \pm 1, y) \rightarrow z(x\pm, y) + 1 \\ z(x, y \pm 1) \rightarrow z(x, y \pm 1) + 1 \end{cases} \tag{1}$$

, and the sand topples onto the neighbouring sites. If one or more of the neighbouring sites reach their own critical mass of four as a result of this toppling at $\mathbf{x}$, then they will be toppled as well. This process will continue until every site on the grid is at a stable mass. Then another grain of sand will be added. Such a collection of toppling events we will refer to as an 'Avalanche'.

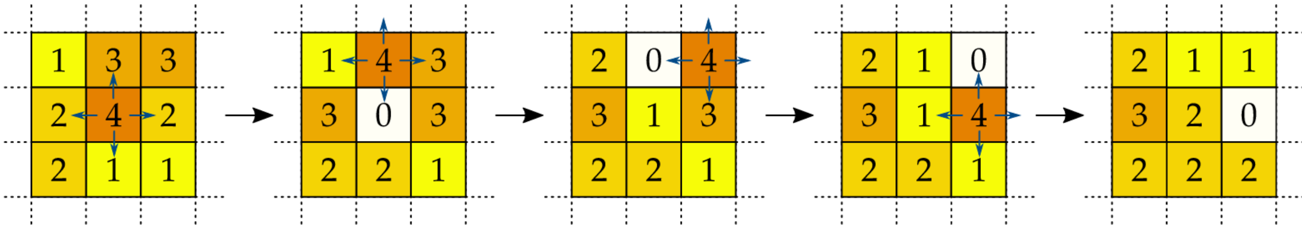

Figure 1: An avalanche in progress

Some avalanches will include toppling events at the edges of the grid. The system will lose the sand that topples off the edges in such instances. This is known as the loss of an avalanche.

Each avalanche also has a magnitude, length, and area, defined as follows:

- Magnitude - the total number of toppling events in the avalanche
- Length - the maximum distance between the origin of the avalanche and a toppled site, measured in Manhattan distance
- Area - the total number of distinct points toppled during the avalanche

These four properties (loss, magnitude, length, and area) form the basis of our statistical analysis of avalanches on $N \times N$ grids.

We define also the toppling matrix:

$$\Delta_{\mathbf{x},\mathbf{x}'} = \begin{cases} 4 & \mathbf{x} = \mathbf{x}' \\ -1 & \mathbf{x}' = \mathbf{x} \pm \mathbf{i} \ \text{ OR } \ \mathbf{x}' = \mathbf{x} \pm \mathbf{j} \\ 0 & \textit{otherwise} \end{cases} \tag{2}$$

, where $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$. This means that the toppling operation of the point $\mathbf{x}$ is given by:

$$T_{\mathbf{x}}(z(\mathbf{y})) = \begin{cases} z(\mathbf{y}) - \Delta_{\mathbf{x},\mathbf{y}}, & z(\mathbf{x}) \geq \Delta_{\mathbf{x},\mathbf{x}} \\ z(\mathbf{y}), & \textit{otherwise} \end{cases} \tag{3}$$

One can easily show that

$$(T_{\mathbf{x}} \circ T_{\mathbf{x}'}) z(\mathbf{y}) = (T_{\mathbf{x}'} \circ T_{\mathbf{x}}) z(\mathbf{y}) \tag{4}$$

This shows that the final state of the grid after an avalanche is independent of the order in which topplings events are resolved. A less trivial fact is that the number of toppling events occurring at a given site during an avalanche is also independent of the order of operation.This is due to the fact that under this model, adding\removing sand to\from vertex $\mathbf{y}$ and then toppling vertex $\mathbf{x}$ is equivalent to toppling $\mathbf{x}$ and then adding\removing sand to\from $\mathbf{y}$

Another important quality of these systems is that once their mass reaches a certain threshold, the loss incurred by avalanches will on average cancel out the addition of sand by the driving force. At this point the total mass of the system will begin to undergo oscillations bounded above and below. This is the Statistically Stable State (SSS) of the system.

This stable mass will not be at the minimally stable state of the system (i.e. $z(\mathbf{x}) = 3$ for all $\mathbf{x}$), since, as discussed in the original 1987 paper, the grid will evolve towards a configuration where the network of minimally stable sites has been broken down such that perturbations cannot be proliferated through infinite distances. [2]

What we are interested in is the distributions of the length, area, loss, and magnitude of avalanches that occur in the SSS of the system.

Methods

Working with a computer simulation of the abelian sandpile model, to find the SSS of a grid, we allowed the system to evolve, recording the mass at each time step. Then, we checked if the maximum mass in the mass history of the grid had increased. If it had not increased in some k steps, with k a parameter, then the simulation was terminated. The statistically stable mass was then deduced from a plot of the mass history.

After finding the statistically stable value of the mass, the grid was initiated into a random microstate with this mass. Subsequently, any avalanches encoded in this initial state were resolved, resulting in a stable state at approximately the desired mass.

A grain of sand was then added to the system, avalanches always being resolved in between, until 10^5 avalanches had occurred. This process was repeated for grids of size 5×5, 10×10, 50×50, and 100×100. The magnitude, length, area, and loss of every avalanche was recorded.

Results and Analysis

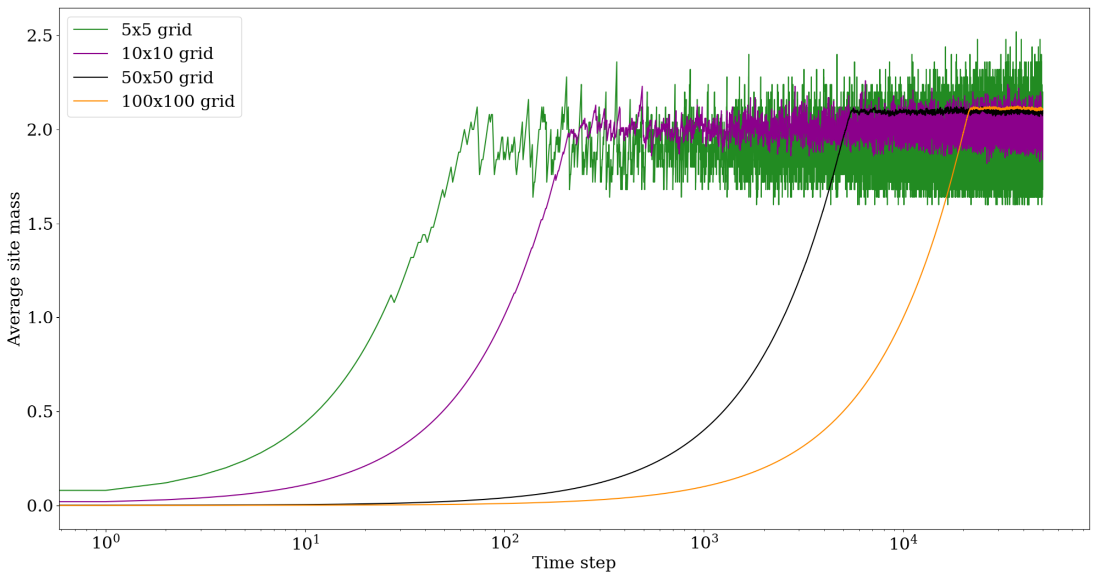

Figure 2: Average mass of sites on different grid sizes as they evolve towards their SSS

The above figure 2 is a plot of the average mass of sites on the four grids as they move towards their SSS in one particular run of the simulation. In each case, the stable mass is approximately twice the area of the grid; i.e. in the SSS of the system,

$$\frac{1}{N^2}\sum_{i,j} z(i,j) \approx 2.$$

We determine, from this plot, the stable masses for the grids to be 50, 200, 5000, and 20000, in order of their size.

Consider a microstate where the average mass of the grid sites is m, with $0 < m \leq 3/2$. Then, since the sand is being added to the grid through a uniform distribution, we are very unlikely to witness an avalanche. In such a state, avalanches that do occur will be short (in length), since longer avalanches require an extended network of minimally stable sites, which would be exceedingly unlikely in this state. It is also the case that the probability of a given avalanche having originated at the edges of the lattice as opposed to a central site goes approximately as $4/N$, which will be very small for large N. This observation, the improbability of a long avalanche or one that originates at the edges, leads us to expect that

$$3/2 < \lim_{t\to\infty} \frac{1}{N^2}\sum_{i,j} z(i,j,t) < 3$$

where the second strict inequality is derived from the fact that the minimally stable state of the grid cannot be the SSS of the system.

Notice that the average site mass oscillates much more heavily on the smaller grids. This is most likely due to the larger perimeter to area ratio on these grids, as well as the shorter average distance to the perimeter. That is, there will be more mass concentrated on the edges, and avalanches are more likely to cause toppling events at these sites.

Note that the magnitude of oscillations at the SSS for the two smaller grids is generally constant and is not in fact growing, as one might be falsely led to believe from figure 2.

Figure 3 shows the statistics collected from 10^5 avalanches in the SSS of each grid on a log-log plot. It is immediately apparent that of the four statistics collected, three of them -magnitude, area, and length- display the linear log-log behavior characteristic of 1/f noise.

The dashed lines are fits to the distributions. For the magnitude, area, and length fits we used weighted linear regression with weight function $w(f) = f^{1/2}$, where f is the relative frequency. For the losses we again used a least squares method with the same weight function,

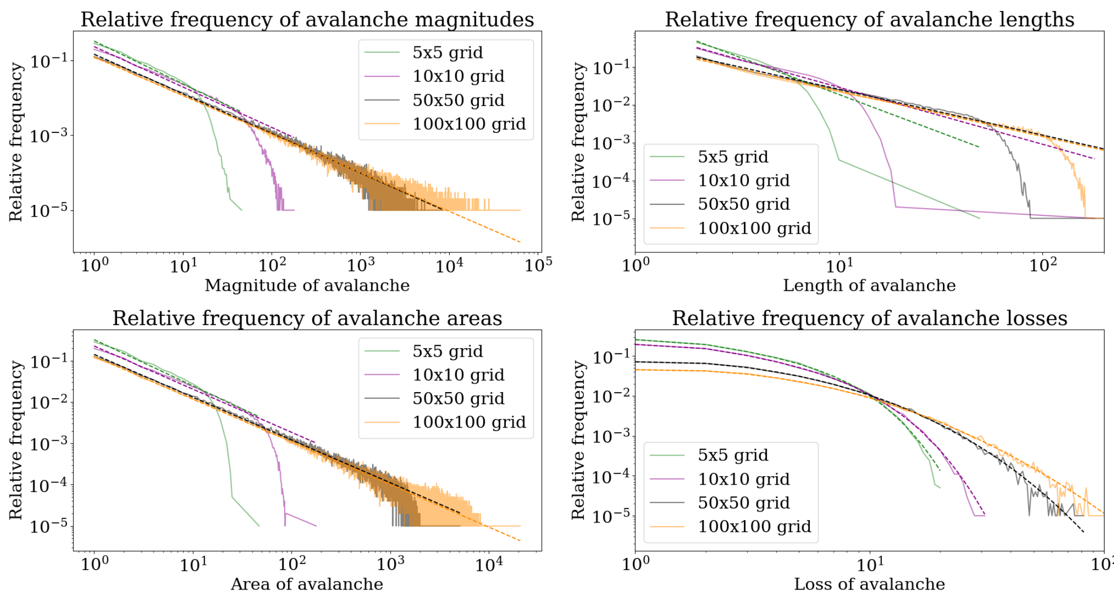

Figure 3: Statistics of avalanches occurring in the SSS of grids of different dimensions. Transparent solid lines show the data, dashed opaque lines are for the fits.

however this time fitting to a polynomial of degree four. It must be noted at this point that the erratic oscillations seen in the distributions for the larger grids are most likely artifacts of the relatively crude methods we employed: due to time and computational restraints, we only recorded 10^5 avalanches for each grid. What this leads to is that if n is a value that is relatively less likely in one of the distributions, then the values in its close neighbourhood, for example $n+1$ or $n-1$, may have never occurred in 10^5 avalanches.

The effects of finiteness can be seen across all distributions. The plots follow, except in the case of the loss, a linear path until this behavior is abruptly terminated due to the finite grid sizes.*

In acoustics, 1/f (or pink) noise is defined to be "Noise in which there is equal power per octave".[5] More generally, if $\log(y) = -\log(x) + c$, then we will have:

$$\int_a^{sa} y \mathrm{d}x = \log s, \quad s, a > 0$$

In particular, this implies that in a given sample of avalanches, there will be roughly equal numbers of topplings occurring in log bundles of equal size.

The strong correlation between the linear section of the data as well as that of the fits suggests the notion of scale invariance in abelian sandpile models. Square grids display the same distribution of magnitude and area regardless of their size, modulo finiteness effects. This is also true of the length for large grid sizes. The average mass of sites on the grid may also be such a scale invariant property.

The loss distributions do not follow a simple 1/f pattern. Still, the choice to model the log-log behaviour as a fourth order polynomial (i.e. $y \sim e^{-c \log^4 x}$) can seem somewhat contrived. In this light, it is worth mentioning that attempts to fit the distributions to lower order polynomials or an exponential function were all far less successful.

The loss distributions do support our previous explanation for the heavier oscillations about the average mass that the two smaller grids display. The 5×5 and 10×10 grids experience heavier losses in proportion to their SSS mass than their larger counterparts.

A surprising consequence of the loss distributions is that the relationship between the area (or magnitude) of an avalanche and its loss is not linear. Since the probability of a toppling event occurring at an edge vertex is fixed for a given grid size, we might naively expect that the loss associated with an avalanche will, on average, be linearly related to its magnitude.

The issue here may lie with our assumption that the probability of an edge vertex toppling is simply a function of grid dimensions. But a moment's thought will convince us that in reality the distribution of mass on the grid is likely to be highly non-uniform, and so the above is a poor assumption.

For example, consider a large grid with a roughly uniform mass distribution and allow it to evolve as usual.

*See for example the area distribution for a 10×10 grid, which terminates quickly just below 10^2

For such a large grid, the driving force will add almost no sand to the perimeter, however, avalanches will tend to redistribute the mass towards the edges. In such a setting, the only way to achieve large losses is for this redistribution of mass to occur isotropically, such that many perimeter sites can be minimally stable at once.

conclusion

The average mass of vertices on grids following the BTW sandpile model converges to a value of two regardless of the size of the grid. However, oscillations about this mean value are greater in magnitude for smaller grids.
The magnitude, length, and area of avalanches follow a pink noise pattern and, given a square lattice, are independent of the size of the grid, mudolo finiteness.
We also find that the relationship between the magnitude of an avalanche and the loss associated with it is highly non-linear.
Further work must be done to determine whether these properties hold on grids of non square shape, or those of higher dimensions, as well as infinite and semi-infinite lattices.

References

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