1 Single Particle

To adjust: as $\tau \to 0$ it goes to dt.

Considering particle a in isolation, we have the transition matrix

$$W_a = \begin{pmatrix} -\alpha & \alpha \\ \alpha & -\alpha \end{pmatrix}, \tag{1}$$

where $w_{ij} = w_{i \leftarrow j}$. This has the decomposition

$$W_a = Q^{-1} \begin{pmatrix} 0 & 0 \\ 0 & -2\alpha \end{pmatrix} Q, \tag{2}$$

with $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then if a_t is the position of partile a at time t, we have that if $a_0 = 1$, $\mathbb{P}(a_t = 1) = \frac{1}{2}(1 + e^{-2\alpha t})$ and $\mathbb{P}(a_t = 2) = \frac{1}{2}(1 - e^{-2\alpha t})$. Then denoting $t_i = i\tau$ for small τ we can write

$$\mathbb{P}(a_{t_{i+1}} = 1 | a_{t_i} = 1) = \frac{1}{2}(1 + e^{-2\alpha\tau}) \tag{3}$$

$$\mathbb{P}(a_{t_{i+1}} = 2 | a_{t_i} = 1) = \frac{1}{2}(1 - e^{-2\alpha\tau}) \tag{4}$$

and analogues for the case where $a_{t_i} = 2$. So we have the 'stay' penalty $\frac{1}{2}(1 + e^{-2\alpha\tau})$ and the 'leave' penalty $\frac{1}{2}(1 - e^{-2\alpha\tau})$.

Now given a path of length n

$$\mathcal{P} = \{a_{t_0}, a_{t_1}, \ldots, a_{t_n}\}, \tag{5}$$

we have the transition record

$$\mathcal{T} = \underbrace{\{(a_{t_1} - a_{t_0}), (a_{t_2} - a_{t_1}), \ldots, (a_{t_n} - a_{t_{n-1}})\}}_{\text{length } n-1}. \tag{6}$$

Then the probabilty of path $\mathcal{P}$ can be expressed as

$$\mathbb{P}(\mathcal{P}) = (\frac{1}{2}(1 + e^{-2\alpha\tau}))^{n-1-l}(\frac{1}{2}(1 - e^{-2\alpha\tau}))^{l}, \tag{7}$$

where l is the number of ones in $\mathcal{T}$. It follows that

$$\log(\mathbb{P}(\mathcal{P})) = (n-1-l)\log\left(\frac{1}{2}(1 + e^{-2\alpha\tau})\right) + l\log\left(\frac{1}{2}(1 - e^{-2\alpha\tau})\right) \tag{8}$$

Taking the limit of $\tau \to 0$ is equivalent to taking $n = t/\tau$ to infinity, with t fixed. Now

$$\frac{1}{2}(1 + e^{-2\alpha\tau}) = 1 - \alpha\tau + \mathcal{O}(\tau^2) \approx 1 - \alpha t/n \tag{9}$$

and

$$\lim_{n\to\infty}(1 - \alpha t/n)^{n-1-l} = e^{-\alpha t}. \tag{10}$$

Moreover

$$\frac{1}{2}(1-e^{-2\alpha\tau}) \approx \alpha t/n$$

But taking the limit of $(\alpha/n)^l$ as $n \to \infty$ gives zero. This makes sense (?) since the measure of a given path approaches zero as n gets large. There is a special case where $l = 0$. In this case the above would suggest

$$\mathbb{P}(\mathcal{P}) = e^{-\alpha t}$$

i.e. a single path with non-zero measure. All other paths taken toghether would then have probability $1 - e^{-\alpha t}$. But we can sum over all paths with l jumps to find non-zero probability:

In general there are $\binom{n}{l}$ paths with l transitions, so summing over all such paths we have

!! change this to an integral over the l transitions times

$$\sum_{l'=l} \mathbb{P}(\mathcal{P}_{l'}) = \binom{n}{l} e^{-\alpha t}(\alpha\tau)^l. \tag{11}$$

This can be rewritten as

$$(\alpha t)^l e^{-\alpha t} \prod_{k=1}^{l+1} \frac{(1-\frac{k}{n})}{l!} \to_{n\infty} \frac{(\alpha t)^l}{l!} e^{-\alpha t}, \tag{12}$$

i.e. a Poisson distribution with parameter αt.

2 Two Particles, Four States

Considering two particles with one coupled to the other, we have the transition matrix

$$W = \begin{pmatrix} -\alpha-\beta & \beta+\gamma & 0 & \alpha \\ \beta & -\alpha-\beta-\gamma & \alpha & 0 \\ 0 & \alpha & -\alpha-\beta & \beta+\gamma \\ \alpha & 0 & \beta & -\alpha-\beta-\gamma \end{pmatrix}, \tag{13}$$

which has the decomposition

$$W = Q \begin{pmatrix} 0 & & & \\ & -2\alpha & & \\ & & -2\beta-\gamma & \\ & & & -2\alpha-2\beta-\gamma \end{pmatrix} Q^{-1}. \tag{14}$$

We have previously calculated $P_{ij\to kl}(t)$, (see Mathematica notebook).

Note that of the tweleve possible transitions, the following pairs are identical up to relabelling:

$$
\begin{aligned}
&(1,1) \to (1,1) \text{ and } (2,2) \to (2,2) && (15)\\
&(1,1) \to (1,2) \text{ and } (2,2) \to (2,1) && (16)\\
&(1,1) \to (2,1) \text{ and } (2,2) \to (1,2) && (17)\\
&(1,2) \to (1,2) \text{ and } (2,1) \to (2,1) && (18)\\
&(1,2) \to (1,1) \text{ and } (2,1) \to (2,2) && (19)\\
&(1,2) \to (2,2) \text{ and } (2,1) \to (1,1). && (20)
\end{aligned}
$$

We assume that τ is small enough such that particles a and b do not transition in the same time frame, i.e. the transitoins $(1,1) \leftrightarrow (2,2)$ and $(1,2) \leftrightarrow (2,1)$ are not allowed. We represent each state with a 3-tuple $(p(t), a(t), b(t))$, where $a(t),\ b(t) \in \{1,2\}$ and p is the parity of the state. This will take on value $p = 1$ for the odd states $(1,2)$ and $(2,1)$, and $p = 2$ for the even states $(1,1)$ and $(2,2)$. So a typical join path for particles a and b has the form

$$\mathcal{P} = \{(p_{t_1}, a_{t_1}, b_{t_1}), (p_{t_2}, a_{t_2}, b_{t_2}), \ldots, (p_{t_n}, a_{t_n}, b_{t_n})\}\,. \tag{21}$$

We now define

$$* : (i,j,k) \times (k,l,m) \mapsto \left((-1)^i(i+k), (l-j)^2, (m-k)^2\right). \tag{22}$$

Then we define the transition record to be

$$\mathcal{T} = \left\{(p_{t_2}, a_{t_2}, b_{t_2}) * (p_{t_1}, a_{t_1}, b_{t_1}), \ldots, (p_{t_n}, a_{t_n}, b_{t_n}) * (p_{t_{n-1}}, a_{t_{n-1}}, b_{t_{n-1}})\right\}. \tag{23}$$

Each possible entry in $\mathcal{T}$ has an associated penalty:

$$
\begin{aligned}
(4,0,0) &: P_{11\to11} && (24)\\
(3,0,1) &: P_{11\to12} && (25)\\
(3,1,0) &: P_{11\to21} && (26)\\
(-2,0,0) &: P_{12\to12} && (27)\\
(-3,0,1) &: P_{12\to11} && (28)\\
(-3,1,0) &: P_{12\to22} && (29)\\
& && (30)
\end{aligned}
$$

Limit of matrix powers

Theorem 2.1 *Let $A \in \mathbb{C}^{m\times m}$. Then $\lim_{n\to\infty}(\mathbb{1} + \frac{1}{n}A)^n = e^A$.*

Proof: Define $f_n(z) = (1 + z/n)^n$. Note that

$$|(1 + \frac{z}{n})^n| \le (1 + |\frac{z}{n}|)^n \tag{31}$$

so for a contour γ such that $\sup_{z\in\gamma}|z| = M$, we have the inequality

$$|(1 + \frac{z}{n})^n| \le (1 + \frac{|M|}{n})^n \uparrow e^M \tag{32}$$

Now take γ to be a simple, closed, positively oriented contour that encloses the spectrum of A. Then

$$(\mathbb{1} + \frac{1}{n}A)^n = \frac{1}{2\pi i}\oint_\gamma f_n(\zeta)(\zeta\mathbb{1} - A)^{-1}\, d\zeta. \tag{33}$$

But $f_n(z) \to e^z$ pointwise and $|f_n| \le e^M \in L^1(\gamma)$ for all n. Now take the limit of (33) as $n \to \infty$ and apply the dominated convergence theorem to find that

$$\lim_{n\to\infty}(\mathbb{1} + \frac{1}{n}A)^n = e^A \tag{34}$$

□

2.1 Markov Path Formulation

Using Bra-Ket notation, we can represent states in the two particle system as $|11\rangle, |12\rangle, |22\rangle$, and $|21\rangle$. We also define the Markov matrix

$$\begin{aligned} M &= \mathbb{1} + \tau W \quad (35) \\ &= \begin{pmatrix} 1-(\alpha+\beta)\tau & (\beta+\gamma)\tau & 0 & \alpha\tau \\ \beta\tau & 1-(\alpha+\beta+\gamma)\tau & \alpha\tau & 0 \\ 0 & \alpha\tau & 1-(\alpha+\beta+\gamma)\tau & (\beta+\gamma)\tau \\ \alpha\tau & 0 & \beta\tau & 1-(\alpha+\beta+\gamma)\tau \end{pmatrix}. \quad (36) \end{aligned}$$

Then the probability of any path of length N, $\{n_i\}_{i=1}^N$, can be written as

$$\mathbb{P}(\{n_i\}) = \langle n_0| M |n_1\rangle \langle n_1| M |n_2\rangle \langle n_2| \dots M |n_N\rangle \tag{37}$$

Consider now a path of length two. Suppose that in this path, particle A begins in state 1 and stays there in the second step. For example,

$$|11\rangle \to |12\rangle$$

would be such a path.Considering all such paths and summing over the nuissance paths of B, we find that

$$\begin{aligned} \mathbb{P}(|1\rangle \to |1\rangle) &= \frac{1}{\sqrt{2}}(\langle 11| + \langle 12|)\, M \frac{1}{\sqrt{2}}(|11\rangle + |12\rangle) \quad (38) \\ &= \frac{1}{2}(\langle 11| M |11\rangle + \langle 11| M |12\rangle + \langle 12| M |11\rangle + \langle 12| M |12\rangle) \quad (39) \\ &= \frac{1}{2}(M_{11} + M_{12} + M_{21} + M_{22}) \quad (40) \\ &= \frac{1}{2} - (\alpha+\beta)\frac{\tau}{2} + \beta\frac{\tau}{2} + (\beta+\gamma)\frac{\tau}{2} + \frac{1}{2} - (\alpha+\beta+\gamma)\frac{\tau}{2} = 1 - \alpha\tau. \quad (41) \end{aligned}$$

On the other hand, the probability of particle A starting in state 1 and moving to state 2 in the second step is

$$
\begin{aligned}
\mathbb{P}\left(|1\rangle \to |2\rangle\right) &= \frac{1}{\sqrt{2}}\left(\langle 11| + \langle 12|\right) M \frac{1}{\sqrt{2}}\left(|21\rangle + |22\rangle\right) && (42)\\
&= \frac{1}{2}\left(\langle 11| M |21\rangle + \langle 11| M |22\rangle + \langle 12| M |21\rangle + \langle 12| M |22\rangle\right) && (43)\\
&= \frac{1}{2}\left(M_{14} + M_{13} + M_{24} + M_{23}\right) && (44)\\
&= \alpha\tau/2 + 0 + 0 + \alpha\tau/2 = \alpha\tau. && (45)
\end{aligned}
$$

From the block diagonal structure of M we can surmise that also

$$
\mathbb{P}\left(|2\rangle \to |1\rangle\right) = \alpha\tau \tag{46}
$$

and

$$
\mathbb{P}\left(|2\rangle \to |2\rangle\right) = 1 - \alpha\tau. \tag{47}
$$

In other words, for an N step path summing over the nuissance paths of B yieds

$$
\mathbb{P}(\text{path of particle A}) = (1 - \alpha\tau)^{N-l}(\alpha\tau)^{l} \to e^{-\alpha t}(\alpha\tau)^{l}, \tag{48}
$$

where l is the number of jumps made by particle A as before. Here we have recovered the behaviour of the single particle system by summing over the nuissance paths of B in the two particle system.

Consider a path, as in equation (21), restricted by the end points $A(0) = A(t) = 1$. That is, a path of the form

$$
\mathcal{P} = \left\{(p_{t_1}, 1, b_{t_1}), (p_{t_2}, a_{t_2}, b_{t_2}), \ldots, (p_{t_n}, 1, b_{t_n})\right\}. \tag{49}
$$

In the Markov path formulation, the probability of this path can be written

$$
\mathbb{P}(\mathcal{P}) = \langle 1B_1| M |A_2B_2\rangle \langle A_2B_2| M \ldots M |1B_n\rangle . \tag{50}
$$

We may then sum over the nuissance paths of paritcle B to find the probability of the path taken by particle A, denoted by $\mathcal{P}_A$ in the sequel.

$$
\mathbb{P}(\mathcal{P}_A) = \left(\langle 11| + \langle 12|\right) M \left(|A_2 1\rangle \langle A_2 1| + |A_2 2\rangle \langle A_2 2|\right) M \ldots M \left(|11\rangle + |12\rangle\right). \tag{51}
$$

Focussing on the quantity $M\left(|A_2 1\rangle \langle A_2 1| + |A_2 2\rangle \langle A_2 2|\right)$, we note that if $A_2 = 1$, then

$$
\begin{aligned}
|A_2 1\rangle \langle A_2 1| + |A_2 2\rangle \langle A_2 2| &= |11\rangle \langle 11| + |12\rangle \langle 12| && (52)\\
&= \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} && (53)\\
&= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, && (54)
\end{aligned}
$$

and likewise

$$|21\rangle\langle 21| + |22\rangle\langle 22| = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{55}$$

We then write M in block form

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}, \tag{56}$$

such that

$$M(|11\rangle\langle 11| + |12\rangle\langle 12|) = \begin{pmatrix} M_1 & 0 \\ M_3 & 0 \end{pmatrix}, \qquad M(|21\rangle\langle 21| + |22\rangle\langle 22|) = \begin{pmatrix} 0 & M_2 \\ 0 & M_4 \end{pmatrix} \tag{57}$$

Suppose now that the path $\mathcal{P}$, composed of n total steps, is such that particle A spends m_1 steps in state 1, m_2 steps in state 2, then again m_3 steps in state 1 and so on until the path ends with particle A in state 1 for m_{l+1} steps (such that there are l transitions for particle A in total). Then we can write that path probability of $\mathcal{P}_A$ as

$$\begin{aligned} &(\langle 11| + \langle 12|)\left[M(|11\rangle\langle 11| + |12\rangle\langle 12|)\right]^{m_1-1}\left[M(|21\rangle\langle 21| + |22\rangle\langle 22|)\right]^{m_2}\dots \\ &\qquad \left[M(|11\rangle\langle 11| + |12\rangle\langle 12|)\right]^{m_{l+1}-1}(|11\rangle) + |12\rangle) \end{aligned} \tag{58}$$

$$= \left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right)^T \begin{pmatrix} M_1 & 0 \\ M_3 & 0 \end{pmatrix}^{m_1-1} \begin{pmatrix} 0 & M_2 \\ 0 & M_4 \end{pmatrix}^{m_2} \dots \begin{pmatrix} M_1 & 0 \\ M_3 & 0 \end{pmatrix}^{m_{l+1}-1} \left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right) \tag{59}$$

Now, some simple algebra allows us to write

$$\begin{pmatrix} M_1 & 0 \\ M_3 & 0 \end{pmatrix}^m = \begin{pmatrix} M_1^m & 0 \\ M_3 M_1^{m-1} & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & M_2 \\ 0 & M_4 \end{pmatrix}^m = \begin{pmatrix} 0 & M_2 M_4^{m-1} \\ 0 & M_4^m \end{pmatrix} \tag{60}$$

Hence, letting

$$\Lambda_1 = M_1^{m_1-1} M_2 M_4^{m_2-1} M_3 M_1^{m_3-1} \dots M_3 M_1^{m_{l+1}-2} \tag{61}$$

$$\Lambda_3 = M_3 M_1^{m_1-2} M_2 M_4^{m_2-1} M_3 \dots M_3 M^{m_{l+1}-2}, \tag{62}$$

we compute the matrix sandwitched in (59) to be

$$\begin{pmatrix} M_1 & 0 \\ M_3 & 0 \end{pmatrix}^{m_1-1} \begin{pmatrix} 0 & M_2 \\ 0 & M_4 \end{pmatrix}^{m_2} \dots \begin{pmatrix} M_1 & 0 \\ M_3 & 0 \end{pmatrix}^{m_{l+1}-1} = \begin{pmatrix} \Lambda_1 & 0 \\ & \\ \Lambda_3 & 0 \end{pmatrix} \tag{63}$$

We now consider the quantity Λ_1. First, note that, as $\tau \to 0$, the number of steps between any successive pair of jumps can be taken to be large, i.e. each m_i is large. We also know that $M_i = \mathbb{1} + \tau W_i$, so

$$\lim_{\tau\to 0} \Lambda_1 = e^{t_1 W_1} M_2 e^{t_2 W_4} M_3 e^{t_3 W_1} \dots M_3 e^{t_{l+1} W_1}, \tag{64}$$

where $\sum_i t_i = t$, is the total time. We take note of the commutation relations

$$\left[e^{t_i W_1}, M_2\right] = \left[e^{t_i W_4}, M_3\right] = \frac{e^{-t_i\alpha}(\alpha\tau)}{2\beta\gamma}\zeta \underbrace{\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}}_{(*)}, \tag{65}$$

where $\zeta = (e^{-t_i(2\beta+\gamma)} - 1)$. In addition, we have the equality

$$M_2 M_3 = M_3 M_2 = M_2^2 = \begin{pmatrix} 0 & \alpha\tau \\ \alpha\tau & 0 \end{pmatrix}^2 = (\alpha\tau)^2 \mathbb{1}, \tag{66}$$

and since $W_1 = W_4$ we have

$$e^{t_i W_1} e^{t_j W_4} = e^{t_i W_1 + t_j W_4} = e^{(t_i + t_j) W_1} \tag{67}$$

(that is to say, the matrix exponentials commute). Now, we can use (65)-(67) to rewrite (64) as follows:

$$\begin{aligned}
\lim_{\tau\to 0} \Lambda_1 &= e^{t_1 W_1} M_2 e^{t_2 W_4} M_3 e^{t_3 W_1} \dots M_3 e^{t_{l+1} W_1} \\
&= e^{t_1 W_1} M_2 \left(M_3 e^{t_2 W_4} + \frac{e^{-t_i\alpha}(\alpha\tau)}{2\beta\gamma}\zeta \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \right) M_3 e^{t_3 W_1} M_2 \dots M_3 e^{t_{l+1} W_1} \qquad (68) \\
&= \left(e^{t_1 W_1} M_2 M_3 e^{t_2 W_4} + \frac{e^{-t_2\alpha}(\alpha\tau)}{2\beta\gamma}\zeta e^{t_1 W_1} M_2 \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \right) \\
&\left(e^{t_3 W_1} M_2 M_3 e^{t_4 W_4} + \frac{e^{-t_4\alpha}(\alpha\tau)}{2\beta\gamma}\zeta e^{t_3 W_1} M_2 \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \right) \\
&\vdots \\
&\left(e^{t_{l-1} W_1} M_2 M_3 e^{t_l W_4} + \frac{e^{-t_l\alpha}(\alpha\tau)}{2\beta\gamma}\zeta e^{t_{l-1} W_1} M_2 \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \right) e^{t_{l+1} W_1} \qquad (69)
\end{aligned}$$

The right hand side in the above can be further simplified to give

$$\begin{aligned}
\lim_{\tau\to 0} \Lambda_1 = &\left((\alpha\tau)^2 e^{t_1 W_1} e^{t_2 W_4} + \frac{e^{-t_2\alpha}(\alpha\tau)}{2\beta\gamma}\zeta e^{t_1 W_1} M_2 \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \right) \\
&\left((\alpha\tau)^2 e^{t_3 W_1} e^{t_4 W_4} + \frac{e^{-t_4\alpha}(\alpha\tau)}{2\beta\gamma}\zeta e^{t_3 W_1} M_2 \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \right) \\
&\vdots \\
&\left((\alpha\tau)^2 e^{t_{l-1} W_1} e^{t_l W_4} + \frac{e^{-t_l\alpha}(\alpha\tau)}{2\beta\gamma}\zeta e^{t_{l-1} W_1} M_2 \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \right) e^{t_{l+1} W_1}
\end{aligned} \tag{70}$$

Terms in the expansion of the above that contain two instances of $(*)$ are in general of the form

$$\begin{aligned}
&\frac{(\alpha\tau)^l}{(2\beta\gamma)^2}\zeta^2 e^{-\alpha t_i} e^{-\alpha t_j} \\
&\qquad \exp\left\{ W_1 \sum_{k<j} t_k \right\} M_2 \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \exp\left\{ W_1 \sum_{j<p<i} t_p \right\} M_2 \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \exp\left\{ W_1 \sum_{q>i} t_q \right\}.
\end{aligned} \tag{71}$$

Then, using that

$$e^{t_i W_1} M_2 \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} e^{t_j W_1} = \alpha\tau e^{-\alpha t_j - (\alpha+2\beta+\gamma)t_i} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \tag{72}$$

we find that (71) evaluates to zero. Similarly we may show that any terms in the expansion that include more than two instances of the matrix $(*)$ also evaluate to zero. Finally, we are left with the follwing simple expression for the limit of Λ_1 for small τ.

$$\lim_{\tau\to 0}\Lambda_1 = (\alpha\tau)^l e^{tW_1} + \frac{(\alpha\tau)^l\zeta}{2\beta\gamma}\sum_{i=2}^{l} e^{-\alpha t_i}\exp\left\{W_1\sum_{j<i}t_j\right\}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\exp\left\{W_1\sum_{j>i}t_j\right\} \tag{73}$$

In general if we have a 4×4 matrix in block form given by

$$A = \begin{pmatrix}A_1 & A_2\\ A_3 & A_4\end{pmatrix},$$

then we will have

$$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}^T A \begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \sum_{a_{ij}\in A_1} a_{ij}.$$

In other words, the result of the inner product given in (59) depends only on Λ_1, so we need not evaluate Λ_3. Moreover, the second term on the RHS of (73) does not contribute to the inner product in (59) since

$$\exp\left\{W_1\sum_{j<i}t_j\right\}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\exp\left\{W_1\sum_{j>i}t_j\right\} = \exp\left\{-\alpha\sum_{j>i}t_j - (\alpha+2\beta+\gamma)\sum_{j<i}t_j\right\}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix} \tag{74}$$

and furthermore

$$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}^T \begin{pmatrix}1 & 1 & 0 & 0\\ -1 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = 0. \tag{75}$$

So, we can evaluate (59) in the limit of small τ to be

$$\mathbb{P}(\mathcal{P}_A) = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}^T \begin{pmatrix}M_1 & 0\\ M_3 & 0\end{pmatrix}^{m_1-1}\begin{pmatrix}0 & M_2\\ 0 & M_4\end{pmatrix}^{m_2}\cdots\begin{pmatrix}M_1 & 0\\ M_3 & 0\end{pmatrix}^{m_{l+1}-1} \tag{76}$$

$$\xrightarrow{\tau\to 0} \begin{pmatrix}1\\1\\0\\0\end{pmatrix}^T \begin{pmatrix}(\alpha\tau)^l e^{tW_1} & 0\\ \Lambda_3 & 0\end{pmatrix}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = 2e^{-\alpha t}(\alpha\tau)^l. \tag{77}$$

Hence, we have recovered the behaviour of the single particle system.

2.2 The Coupled Two Particle System

Now we seek to find the path probability of particle B by summing over the nuissance paths of particle A. This turns out to be extremely difficult to do exactly, and to proceed analytically we have to make the two assumptions that $\gamma/\alpha << 1$ and $\gamma/\beta << 1$. We will also need the following lemma.

Lemma 2.2 *If X is a traceless, 2×2 matrix, then*

$$e^X = \cos\sqrt{\det X}\,\mathbb{1} + \frac{\sin\sqrt{\det X}}{\sqrt{\det X}} X.$$

Proof: See *'Lie Groups: An Introduction Through Linear Groups'* by W. Rossmann, Chapter 1.2, Example 9. □

We now re-number our basis of states so that the algebra of the problem can be simplified into one involving block matricies. Throughout this subsection we will have the basis ordering

$$|11\rangle = e_1,\ |21\rangle = e_2,\ |12\rangle = e_3,\ |22\rangle = e_4.$$

This leads to the updated transition matrix Ω given by

$$\Omega = \begin{pmatrix} -\alpha-\beta & \alpha & \beta+\gamma & 0 \\ \alpha & -\alpha-\beta-\gamma & 0 & \beta \\ \beta & 0 & -\alpha-\beta-\gamma & \alpha \\ 0 & \beta+\gamma & \alpha & -\alpha-\beta \end{pmatrix}. \tag{78}$$

Let us also define $\Psi = \mathbb{1} + \tau\Omega$, the object analogous to the matrix M in the previous section. Then, given a path $\mathcal{P}$ and corresponding B-path $\mathcal{P}_B$ we can write

$$\mathbb{P}(\mathcal{P}_B) = \frac{1}{\sqrt{2}} \left(\langle 1B(\tau)| + \langle 2B(\tau)|\right) \Psi \left(|1B(2\tau)\rangle \langle 1B(2\tau)| + |2B(2\tau)\rangle \langle 2B(2\tau)|\right) \ldots \tag{79}$$

$$\Psi \left(|1B((n-1)\tau)\rangle \langle 1B((n-1)\tau)| + |2B((n-1)\tau)\rangle \langle 2B((n-1)\tau)|\right) \Psi \frac{1}{\sqrt{2}} \left(|1B(n\tau)\rangle + |2B(n\tau)\rangle\right).$$

Notice that due to the change of basis ordering, the algebra resembles that in the previous section, in the sense that

$$\Psi(|11\rangle\langle 11| + |21\rangle\langle 21|) = \begin{pmatrix} \Psi_1 & 0 \\ \Psi_3 & 0 \end{pmatrix} \text{ (particle B in state 1)} \tag{80}$$

and

$$\Psi(|12\rangle\langle 12| + |22\rangle\langle 22|) = \begin{pmatrix} 0 & \Psi_2 \\ 0 & \Psi_4 \end{pmatrix} \text{ (particle B in state 2).} \tag{81}$$

Let us now assume that $\mathcal{P}_B$ begins and ends with particle B in state 1. Then proceeding as before to evaluate the sandwitched matrix in (79), we find that

$$\mathbb{P}(\mathcal{P}_B) = \frac{1}{2}(\langle 11| + \langle 21|) \begin{pmatrix} \psi_1 & 0 \\ \psi_3 & 0 \end{pmatrix} (|11\rangle + |21\rangle), \tag{82}$$

where

$$\psi_1 = \Psi_1^{m_1-1}\Psi_2\Psi_4^{m_2-1}\Psi_3\Psi_1^{m_3-1}\dots\Psi_3\Psi_1^{m_{l+1}-2}, \tag{83}$$

$$\psi_3 = \Psi_3\Psi_1^{m_1-2}\Psi_2\Psi_4^{m_2-1}\Psi_3\Psi_1^{m_3-1}\dots\Psi_3\Psi_1^{m_{l+1}-2}. \tag{84}$$

As before, we are only concerned with evaluating ψ_1 as τ goes to zero.

$$\lim_{\tau\to 0}\psi_1 = e^{t_1\Omega_1}\Psi_2 e^{t_2\Omega_4}\Psi_3\dots\Psi_3 e^{t_{l+1}\Omega_1} \tag{85}$$

We now note the commutator relation:

$$\begin{aligned}
e^{t_i\Omega_4}\Psi_3 &= \Psi_3 e^{t_i\Omega_4} + [e^{t_i\Omega_4}, \Psi_3] \\
&= \Psi_3 e^{t_i\Omega_4} + (\alpha\gamma\tau)\frac{2\sinh\left(\frac{t_i}{2}\sqrt{4\alpha^2+\gamma^2}\right)}{\sqrt{4\alpha^2+\gamma^2}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \\
&= \Psi_3 e^{t_i\Omega_4} + \gamma\tau\sinh(\alpha t_i)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} + \mathcal{O}(\gamma^2/\alpha^2),
\end{aligned} \tag{86}$$

So (85) can be rewritten

I don't think this is technically correct, the approximation works, but only if we also assume $\alpha t_i \approx 1$, which should be true for physical systems

$$\begin{aligned}
\lim_{\tau\to 0}\psi_1 &= \left(\prod_{i=1}^{l/2}\left(e^{t_{2i-1}\Omega_1}\Psi_2\Psi_3 e^{t_{2i}\Omega_4} + \gamma\tau\sinh(at_{2i})e^{t_{2i-1}\Omega_1}\Psi_2\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)\right)e^{t_{l+1}\Omega_1} \\
&= \left(\prod_{i=1}^{l/2}\left(\beta(\beta+\gamma/2)\tau^2 e^{t_{2i-1}\Omega_1}e^{t_{2i}\Omega_4} + \gamma\tau\sinh(at_{2i})e^{t_{2i-1}\Omega_1}\Psi_2\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)\right)e^{t_{l+1}\Omega_1}
\end{aligned} \tag{87}$$

If we were to expand the product above, every term would have a scalar coefficient proportional to

$$\beta^{l/2-k}(\beta+\gamma/2)^{l/2-k}\gamma^k = \mathcal{O}(\gamma^k/\beta^k), \quad k = 0, 1, \dots, l/2. \tag{88}$$

Hence we can ignore the terms in the expansion where $k \geq 2$.

In fact, we find that

there's about 23 steps missing here

$$\lim_{\tau\to 0}\psi_1 = \beta^{l/2}(\beta+\gamma)^{l/2}\tau^l e^{t_1\Omega_1}e^{t_2\Omega_2}e^{t_3\Omega_4}\dots e^{t_l\Omega_4}e^{t_{l+1}\Omega_1} + \text{nasty term} + \mathcal{O}(\gamma^2/\beta^2). \tag{89}$$

Notice also that

$$\Omega_1 = \begin{pmatrix}\gamma/2 & \alpha\\ \alpha & -\gamma/2\end{pmatrix} - (\alpha+\beta+\gamma/2)\mathbb{1}, \text{ and } \Omega_4 = \begin{pmatrix}-\gamma/2 & \alpha\\ \alpha & \gamma/2\end{pmatrix} - (\alpha+\beta+\gamma/2)\mathbb{1}, \tag{90}$$

where in each case the first term on the RHS is traceless. Using the above and lemma 2.2 we write

$$
\begin{aligned}
e^{t_i\Omega_1} &= \exp\left\{t_i\left[\begin{pmatrix}\gamma/2 & \alpha\\ \alpha & -\gamma/2\end{pmatrix} - (\alpha+\beta+\gamma/2)\mathbb{1}\right]\right\}\\
&= e^{-(\alpha+\beta+\gamma/2)t_i}\exp\left\{t_i\begin{pmatrix}\gamma/2 & \alpha\\ \alpha & -\gamma/2\end{pmatrix}\right\}\\
&= e^{-(\alpha+\beta+\gamma/2)t_i}\left[\cosh\left(\frac{t_i}{2}\sqrt{\gamma^2+4\alpha^2}\right)\mathbb{1} + \frac{2\sinh\left(\frac{t_i}{2}\sqrt{\gamma^2+4\alpha^2}\right)}{\sqrt{\gamma^2+4\alpha^2}}\begin{pmatrix}\gamma/2 & \alpha\\ \alpha & -\gamma/2\end{pmatrix}\right]\\
&= e^{-(\alpha+\beta+\gamma/2)t_i}\left[\cosh(\alpha t_i)\,\mathbb{1} + \sinh(\alpha t_i)\begin{pmatrix}\gamma/2\alpha & 1\\ 1 & -\gamma/2\alpha\end{pmatrix}\right] + \mathcal{O}(\gamma^2/\alpha^2), \qquad (91)
\end{aligned}
$$

and likewise

$$
e^{t_i\Omega_4} = e^{-(\alpha+\beta+\gamma/2)t_i}\left[\cosh(\alpha t_i)\,\mathbb{1} + \sinh(\alpha t_i)\begin{pmatrix}-\gamma/2\alpha & 1\\ 1 & \gamma/2\alpha\end{pmatrix}\right] + \mathcal{O}(\gamma^2/\alpha^2). \qquad (92)
$$

Take now the following basis of $sl(2)$

$$
H = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \ X_+ = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \ X_- = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \qquad (93)
$$

and further define the two families of indexed operators given by

$$
a_m = \left(\frac{m\gamma}{2\alpha}H + X_+\right),\ \ b_m = \left(\mathbb{1} + \frac{m\gamma}{2\alpha}X_-\right),\ \ m\in\mathbb{Z} \qquad (94)
$$

Notice that

$$
\begin{pmatrix}\gamma/2\alpha & 1\\ 1 & -\gamma/2\alpha\end{pmatrix} = a_1,\ \ \text{and}\ \ \begin{pmatrix}-\gamma/2\alpha & 1\\ 1 & \gamma/2\alpha\end{pmatrix} = a_{-1}.
$$

Furthemore, up to $\mathcal{O}(\gamma^2/\alpha^2)$, the following operator algebra holds

$$
\begin{aligned}
a_m a_n &= b_{n-m} \qquad (95)\\
a_m b_n &= a_{m+n} \qquad (96)\\
b_n a_m &= a_{m-n} \qquad (97)\\
b_n b_m &= b_{m+n} \qquad (98)
\end{aligned}
$$

dd

dd

In particular, we find that, $e^{t_i\Omega_1} = e^{t_i\Omega_4} + \frac{\gamma \sinh{(\alpha t_i)}}{\alpha} e^{-(\alpha+\beta+\gamma/2)t_i}\mathbb{1} + \mathcal{O}(\gamma^2/\alpha^2)$. Using this result we re-write the RHS of (89) as below. The first term on the RHS can be re-written as

$$
\begin{aligned}
&\beta^{l/2}(\beta+\gamma)^{l/2}\tau^l e^{t_1\Omega_1} e^{t_2\Omega_2} e^{t_3\Omega_4} \dots e^{t_l\Omega_4} e^{t_{l+1}\Omega_1} \\
&\qquad = \beta^{l/2}(\beta+\gamma)^{l/2}\tau^l (e^{t_1\Omega_4} + \frac{\gamma \sinh{(\alpha t_1)}}{\alpha} e^{-(\alpha+\beta+\gamma/2)t_1}\mathbb{1}) e^{t_2\Omega_4} (e^{t_3\Omega_4} + \frac{\gamma \sinh{(\alpha t_3)}}{\alpha} e^{-(\alpha+\beta+\gamma/2)t_3}\mathbb{1}) \\
&\qquad \dots e^{t_l\Omega_4} (e^{t_{l+1}\Omega_4} + \frac{\gamma \sinh{(\alpha t_{l+1})}}{\alpha} e^{-(\alpha+\beta+\gamma/2)t_{l+1}}\mathbb{1}) + \mathcal{O}(\gamma^2/\alpha^2) \\
&\qquad = \beta^{l/2}(\beta+\gamma)^{l/2}\tau^l \left(e^{t\Omega_4} + \frac{\gamma}{\alpha} \sum_{i=0}^{l/2} \sinh{(\alpha t_{2i+1})} e^{-(\alpha+\beta+\gamma/2)t_{2i+1}} e^{(t-t_{2i+1})\Omega_4} \right) + \mathcal{O}(\gamma^2/\alpha^2),
\end{aligned}
\tag{99}
$$

and for the second term we have

recall $\eta_i \approx \sinh(\alpha t_i)/\alpha$

$$
\begin{aligned}
&\beta^{l/2}(\beta+\gamma)^{l/2-1}(\gamma/\beta)\tau^l \\
&\sum_{k=1}^{l/2}\left[\left(\prod_{1\le i<k} e^{t_{2i-1}\Omega_1}e^{t_{2i}\Omega_4}\right)\left(\sinh(\alpha t_{2k})e^{t_{2k-1}\Omega_1}\Psi_2\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)\left(\prod_{k\le i\le l/2} e^{t_{2i-1}\Omega_1}e^{t_{2i}\Omega_4}\right)\right] \\
&=\beta^{l/2}(\beta+\gamma)^{l/2-1}(\gamma/\beta)\tau^l \\
&\sum_{k=1}^{l/2}\left[\left(\prod_{1\le i<k}(e^{t_{2i-1}\Omega_4}+\mathcal{O}(\gamma/\alpha))e^{t_{2i}\Omega_4}\right)\left(\sinh(\alpha t_{2k})(e^{t_{2k-1}\Omega_4}+\mathcal{O}(\gamma/\alpha))\Psi_2\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)\left(\prod_{k\le i\le l/2}(e^{t_{2i-1}\Omega_4}+\mathcal{O}(\gamma/\alpha))e^{t_{2i}\Omega_4}\right.\right. \\
&=\beta^{l/2}(\beta+\gamma)^{l/2-1}(\gamma/\beta)\tau^l \\
&\sum_{k=1}^{l/2}\left(\exp\left\{\Omega_4\sum_{1\le i<2(k-1)}t_i\right\}+\mathcal{O}(\gamma/\alpha)\right)\left(\sinh(\alpha t_{2k})(e^{t_{2k-1}\Omega_4}+\mathcal{O}(\gamma/\alpha))\Psi_2\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)\left(\exp\left\{\Omega_4\sum_{2k<i\le l+1}t_i\right\}+\mathcal{O}(\gamma/\alpha)\right. \\
&=\beta^{l/2}(\beta+\gamma)^{l/2-1}(\gamma/\beta)\tau^l \\
&\sum_{k=1}^{l/2}\left[\exp\left\{\Omega_4\sum_{1\le i\le 2k-1}t_i\right\}\sinh(\alpha t_{2k})\Psi_2\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\exp\left\{\Omega_4\sum_{2k+1\le i\le l}t_i\right\}\right]+\mathcal{O}(\gamma^2/\alpha\beta,\gamma^2/\alpha^2). \qquad (100)
\end{aligned}
$$

We now (carefully) pass (100) into the inner product (82). After some lengthy arithmetic we find that this gives contributions that are at most $\mathcal{O}(\gamma/\alpha\beta,\gamma/\beta^2)$. Hence the second term in (89) does not contribute to the inner product (82) up to order $\mathcal{O}(\gamma/\alpha\beta,\gamma/\beta^2)$. We can now evaluate (82) up to order $\mathcal{O}(\gamma/\alpha\beta,\gamma/\beta^2,\gamma/\alpha^2)$ by calclating the inner product $\begin{pmatrix}1\\1\end{pmatrix}^T$ (93) $\begin{pmatrix}1\\1\end{pmatrix}$, which gives

$$
\begin{aligned}
\mathbb{P}(\mathcal{P}_{\mathcal{B}}) &= \frac{1}{2}(\langle 11| + \langle 21|)\begin{pmatrix}\psi_1 & 0\\ \psi_3 & 0\end{pmatrix}(|11\rangle + |21\rangle) \\
&= \beta^l(1+\gamma/\beta)^{l/2}e^{-(\beta+\gamma/2)t}\tau^l\left(1+\frac{\gamma}{2\alpha}\left(\frac{l}{2}-\sum_{k=0}^{l/2}e^{-2\alpha t_{2k+1}}\right)\right) \qquad (101)
\end{aligned}
$$

Next step is to calculate $\int dt^l\mathbb{P}(\mathcal{P}_B)$.