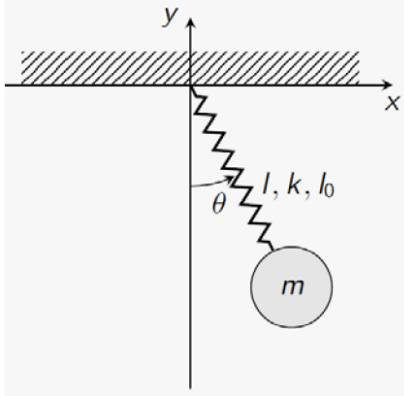
Hw1: soln:

Example: Elastic Pendulum



$$q = \begin{bmatrix} l \\ \theta \end{bmatrix}, \qquad p = l \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$$

$$\dot{p} = l\dot{\theta} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \dot{l} \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$$

$$T = \frac{1}{2} m \dot{p}^\top \dot{p} = \frac{1}{2} m (l^2\dot{\theta}^2 + \dot{l}^2), \qquad U = -mgl\cos\theta + \frac{k}{2}(l - l_0)^2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = \begin{bmatrix} m\ddot{l} \\ ml^2\ddot{\theta} + 2ml\dot{l}\dot{\theta} \end{bmatrix}$$

$$\frac{\partial L}{\partial q} = \begin{bmatrix} ml\dot{\theta}^2 + mg\cos\theta - k(l - l_0) \\ -mgl\sin\theta \end{bmatrix}$$

$$\begin{bmatrix} \ddot{l} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} l\dot{\theta}^2 + g\cos\theta - \frac{k}{m}(l - l_0) \\ \frac{g}{l}\sin\theta - \frac{2}{l}\dot{l}\dot{\theta} \end{bmatrix}$$

Equilibrium

$$\vec{x} = \begin{bmatrix} l \\ \theta \\ \dot{l} \\ \dot{\theta} \end{bmatrix} \qquad \dot{\vec{x}} = \begin{bmatrix} x_3 \\ x_4 \\ x_1 x_4^2 + g\cos x_2 - \frac{k}{m}(x_1 - l_0) \\ \frac{g}{l}\sin x_2 - \frac{2}{x_1} x_3 x_4 \end{bmatrix}$$

$f(\vec{x})$

$$f(\vec{x}_e) = 0$$

$$x_3 = x_4 = 0$$

$$\begin{cases} g\cos x_2 - \frac{k}{m}(x_1 - l_0) = 0 \\ \frac{g}{l}\sin x_2 = 0 \Rightarrow x_2 = k\pi \end{cases}$$

$k = 0, \pm 1, \pm 2 \ldots$

when $x_2 = 0$.

$$g\underbrace{\cos 0}_{1} - \frac{k}{m}(x_1 - l_0) = 0$$

$$g = \frac{k}{m}(x_1 - l_0)$$

equilibrium: $\boxed{\frac{mg}{k} + l_0 = x_1}$

stable.

when $x_2 = \pi$.

$$g\underbrace{\cos\pi}_{-1} - \frac{k}{m}(x_1 - l_0) = 0$$

$$-g - \frac{k}{m}(x_1 - l_0) = 0$$

$$x_1 = l_0 - \frac{mg}{k}$$

unstable.

Problem 2

$$\vec{z} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\dot{\vec{z}} = \begin{bmatrix} z_2 \\ 5z_2 - 10z_1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 10 & -5 \end{bmatrix}}_{A} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

equilibrium: $\vec{z} = 0$

$\text{eig}\left(\begin{bmatrix} 0 & 1 \\ 10 & -5 \end{bmatrix}\right)$ $\lambda_1 = 1.5311$ $\lambda_2 = -6.53$

Not stable