1) $m\ddot{z} + \gamma \dot{z} + kz = 0$

write down the state space form:

$$X = \begin{bmatrix} z \\ \dot{z} \end{bmatrix}, \dot{X} = \begin{bmatrix} \dot{z} \\ \ddot{z} \end{bmatrix} ; \dot{X} = AX + Bu \Rightarrow \dot{X} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix} u}_{\text{no input}}$$

$$\dot{X} = \begin{bmatrix} X_2 \\ -\frac{\gamma}{m}X_2 - \frac{k}{m}X_1 \end{bmatrix}, m\dot{X}_2 = -\gamma\dot{z} - kz \rightarrow \ddot{z} = \dot{X}_2 = -\frac{\gamma}{m}X_2 - \frac{k}{m}X_1$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2) write down the equation of motion & put it in a state space form:

equation of motion: $m\ddot{z} + \gamma\dot{z} + kz = F$

state space form: $X = \begin{bmatrix} z \\ \dot{z} \end{bmatrix}, \dot{X} = \begin{bmatrix} \dot{z} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} X_2 \\ \frac{F}{m} - \frac{\gamma}{m}X_2 - \frac{k}{m}X_1 \end{bmatrix}$

$m\ddot{z} = F - \gamma\dot{z} - kz$

$m\dot{X}_2 = F - \gamma X_2 - kX_1 \xRightarrow{\div m} \ddot{z} = \dot{X}_2 = \frac{F}{m} - \frac{\gamma}{m}X_2 - \frac{k}{m}X_1$

$$\dot{X} = AX + Bu \Rightarrow \begin{bmatrix} X_2 \\ \frac{F}{m} - \frac{\gamma}{m}X_2 - \frac{k}{m}X_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + [0]u$$

Question 3: Given, $k=2$, $m=5$, $\gamma=1$ is the system Controllable?

Substituting the Constants in the A & B matrices $\Rightarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{2}{5} & -\frac{1}{5} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{5} \end{bmatrix}$

$M_c = [B \quad AB]$

$$AB = \begin{bmatrix} 0 & 1 \\ -\frac{2}{5} & -\frac{1}{5} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ -\frac{1}{25} \end{bmatrix} = \begin{bmatrix} 0.2 \\ -0.04 \end{bmatrix} ; M_c = \begin{bmatrix} 0 & 0.2 \\ 0.2 & -0.04 \end{bmatrix}$$ M_c is full rank

Therefore is Controlable. ✓

4) Define the output, $y = Cx$ with $C = [1 \;\; 0]$, that is the output is the position of the mass. Let the origin $\vec{X} = 0$ be the equilibrium of the system when no external force $F = 0$ is applied. Design a set point controller so that the system stabilizes to $y_r = 5$. with zero velocity.

For gain matrix K we have to place the poles of the closed loop system at a location that ensures stability. This means that the eigen values of closed loop system must have negative real parts. We assume that we would want our poles to be located at $(-1, -2.5)$. Thus, we have: $(\lambda+1)(\lambda+2.5) \rightarrow$ desired pole locations.

$$A - BK = \begin{bmatrix} 0 & 1 \\ -0.4 & -0.2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} [K_1 \;\; K_2] = \begin{bmatrix} 0 & 1 \\ -\frac{K_1}{5} - \frac{4}{10} & -\frac{2K_2}{10} - \frac{2}{10} \end{bmatrix}$$

$\text{EigenV}(A - BK) = (\lambda+1)(\lambda+2.5) \rightarrow$ finding the values for K_1, K_2

$\rightarrow K_1 = 10.5,\ K_2 = 16.5 \rightarrow K = [10.5 \;\; 16.5]$ now we have the desired gain which places the closed loop poles @ $-1, -2.5$.

now we calculate K_r using the following formula:

$$K_r = -(C(A - BK)^{-1}B)^{-1} =$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -\frac{4}{10} & -\frac{2}{10} \end{bmatrix} - \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} [10.5 \;\; 16.5] = \begin{bmatrix} 0 & 1 \\ -2.5 & -3.5 \end{bmatrix} \rightarrow (A - BK)^{-1} = \begin{bmatrix} -1.4 & -0.4 \\ 1 & 0 \end{bmatrix}$$

$$K_r = -\left([1 \;\; 0] \begin{bmatrix} -1.4 & -0.4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} \right)^{-1} = \underline{12.5}$$

$$u = K_r r - KX \rightarrow u = 12.5 * 5 - [10.5 \;\; 16.5] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} =$$

$$u = 62.5 - 10.5 X_1 - 16.5 X_2$$

$$\dot{X} = AX + Bu \rightarrow (A - BK)X + BK_r r \rightarrow \begin{bmatrix} 0 & 1 \\ -2.5 & -3.5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} * 12.5 * 5 = \dot{X}$$

$$y_r = [5; 0]' = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$