Chapter 9

Numerical Differentiation

In this chapter we investigate numerical techniques for estimating the derivatives of a function. Several formulas for approximating a first and second derivative by a difference quotient are given.

One important application of numerical differentiation is the development of algorithms for solving ordinary and partial differential equations.

EXAMPLE 9.1 : Velocity of a Particle

The distance s (meters) traveled by a particle moving along a coordinate line is given by the following table:

Time t (sec.)	0	2	4	6	8
s (m.)	1.00	2.72	7.38	20.08	54.59

One quantity of interest in Physics is the velocity of the particle at a given time t. It is given by $s'(t)$ and can be estimated by using numerical differentiation.

9.1 NUMERICAL DIFFERENTIATION

In many practical cases, we are faced with the problem of finding the derivative of a function whose form is either known only as a tabulation of data, or not practical to use for calculations. In this section, we will study some techniques for the numerical computation of first and second derivatives. These results will also be used to obtain the numerical solution of differential equations that will be presented in Chapters 12 and 15.

We begin with the series expansion of $f(x)$ about x. We assume that $f(x)$ has as many continuous derivatives as may be required. From Taylor's formula

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots. \tag{9.1}$$

Solving Eqn. (9.1) for $f'(x)$ yields

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2}f''(x) - \frac{h^2}{6}f'''(x) + \cdots \tag{9.2}$$

so that an approximation for $f'(x)$ may be written as

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}, \quad \text{for a small value of } h. \tag{9.3}$$

The expression for $f'(x)$ in Eqn. (9.3) is called the **forward-divided difference** approximation. Graphically, it represents the slope of the line passing through the points $(x, f(x))$ and $(x+h, f(x+h))$. Another approximation for $f'(x)$ can be obtained by replacing h by $-h$ in Eqn. (9.1). That is,

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}, \quad \text{for a small value of } h. \tag{9.4}$$

Eqn. (9.4) is called the **backward-divided difference** approximation to $f'(x)$.

Note that, from Eqn. (9.2), the **truncation error** for both the forward and backward-difference approximation to $f'(x)$ is

$$-\frac{h}{2}f''(\zeta)$$

for some ζ in the interval $(x, x+h)$, for the forward difference and for some ζ in $(x-h, x)$, for the backward difference.

One can see that for a linear function, the approximations are exact since the error term is zero in this case.

Again, if we replace h by $-h$ in Eqn. (9.1), and then subtract the resulting equation from the old one, we obtain a very popular formula for approximating $f'(x)$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{3!}f'''(x) - \frac{h^4}{5!}f^{(5)} - \cdots. \tag{9.5}$$

That is,

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}, \quad \text{for a small value of } h. \tag{9.6}$$

Formula (9.6) is known as the **central-difference** approximation of $f'(x)$. Hence, we have an approximation to $f'(x)$ with an error of the order h^2.

EXAMPLE 9.2

Given $f(x) = e^x$, approximate $f'(1.5)$ using formulas (9.6) and (9.3) with $h = 0.1$. Compare the results with the exact value $f'(x) = e^{1.5}$.

Set $x = 1.5$ and $h = 0.1$ in (9.6) and (9.3) to get

$$f'(1.5) \approx \frac{e^{1.6} - e^{1.4}}{0.2} \approx 4.489162287752$$

$$f'(1.5) \approx \frac{e^{1.6} - e^{1.5}}{0.1} \approx 4.713433540571.$$

The absolute errors are

$$|e^{1.5} - 4.489162287752| = 0.007473$$
$$|e^{1.5} - 4.713433540571| = 0.231744.$$

Observe that the central-difference formula gives a better approximation than the forward-difference formula. We will now proceed to find approximations for the second derivative of $f(x)$. Again, consider the Taylor series expansion of $f(x)$ about x

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f^{(3)}(x) + \\ &\quad + \frac{h^4}{4!}f^{(4)}(x) + \cdots \qquad (9.7) \\ f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f^{(3)}(x) + \\ &\quad + \frac{h^4}{4!}f^{(4)}(x) - \cdots . \qquad (9.8) \end{aligned}$$

Add the two equations to get

$$f(x+h) + f(x-h) = 2f(x) + \frac{2h^2}{2!}f''(x) + \frac{2h^4}{4!}f^{(4)}(x) + \cdots .$$

We now solve for $f''(x)$ to obtain

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{1}{12}h^2 f^{(4)}(x) + \cdots . \qquad (9.9)$$

Therefore, an approximation formula for $f''(x)$ is given by

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}, \quad \text{for a small value of } h. \qquad (9.10)$$

From (9.9) one can see that the truncation error is $-\frac{1}{12}h^2 f^{(4)}(\zeta)$ for some ζ in the interval $(x-h, x+h)$. Formula 9.10 is known as the **central-difference formula** for the second derivative.

EXAMPLE 9.3

Let $f(x) = \sin x$. Use formulas (9.3) and (9.10) with $h = 0.1$ to approximate $f'(0.5)$ and $f''(0.5)$. Compare with the true values, $f'(0.5) = 0.87758256$ and $f''(0.5) = -0.47942554$.

Using formula (9.3) we get

$$\begin{aligned} f'(0.5) &\approx \frac{f(0.6) - f(0.5)}{0.1} \\ &\approx \frac{0.56464247 - 0.47942554}{0.1} \approx 0.8521693 \\ \text{Error} &= 0.02541326. \end{aligned}$$

Similarly, using formula (9.10), we get

$$\begin{aligned} f''(0.5) &\approx \frac{f(0.6) - 2f(0.5) + f(0.4)}{(0.1)^2} \\ &\approx \frac{0.56464247 - 0.95885108 + 0.38941834}{(0.1)^2} \approx -0.479027 \\ \text{Error} &= 0.000399. \end{aligned}$$

MATLAB's Methods

MATLAB approximates the derivative of a function $f(x)$ with the use of the built-in function diff. We will demonstrate the uses of the diff function only for vectors. It can be used on matrices and gives the same thing for each column of the matrix that it does for a vector. For a vector with elements

$\mathbf{x} = [x_1, x_2, ..., x_n]$

diff computes the consecutive differences of the values of $\mathbf{x}$, that is:

$\text{diff(x)} = [x_2 - x_1, x_3 - x_2, ..., x_n - x_{n-1}].$

If we apply for example diff to the vector $\mathbf{x}$

```
>> x=[0.1   1.0   1.2   0.5   0.8   0.9];

>> diff(x)
ans =
     0.9000   0.2000   -0.7000   0.3000   0.1000
```

The derivative estimate of a function $y = f(x)$ is obtained by the quantity

$$\text{diff(y)./diff(x)}|_{x=x_n} \approx \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}$$

which is the forward-divided difference. As an application let us approximate the derivative of the function $y = \sin(x)$ at the entries of:

```
>> x = 0 : pi/32 : pi;
```

```
>> y = sin(x);

>> dy = diff(y);

>> dx = diff(x);

>> dsinx = dy./dx;
```

The length of each of the vectors dx, dy, dsinx is:

```
>> length(dsinx)
ans =
       32
```

The entries of the vector dsinx are the estimates of the derivative of $\sin x$ at $x_1, x_2, \ldots, x_{32}$.

We will make our comparison with the exact values of the derivative $y' = \cos x$ over the same number of values of x using the command plot:

```
>> x = x(1:32);
>> plot(x,dsinx,'*',x,cos(x),'+')
>> xlabel('x')
>> text(1.5,.5,'+ Appx. with dt = pi/32')
>> text(1.5,.4,' . cos(x)')
```

The result is shown in Figure 9.1.

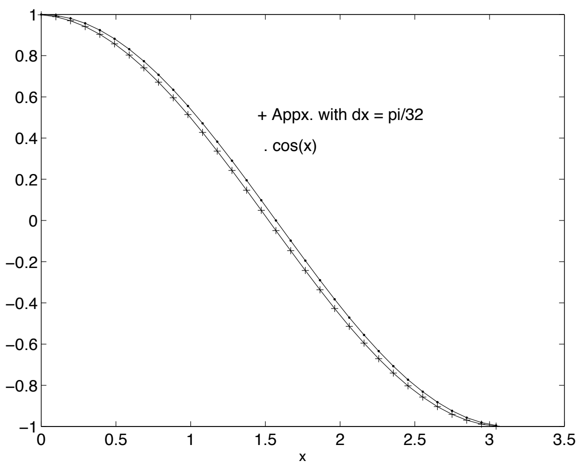


FIGURE 9.1
Numerical Approximation of the derivative of sin x.

EXAMPLE 9.4

Given $f(x) = e^x$, use the MATLAB function diff to approximate $f'(1.5)$ with $h = 0.1$.

The approximation can be obtained in the following way:

```
>> x = [1.5    1.5+0.1];
>> y = exp(x);
>> dx = diff(x);
>> dy = diff(y);
>> format long
>> dy./dx
ans =
        4.71343354057050
```

EXERCISE SET 9.1

1. Suppose a polynomial p_2 interpolates f at x_i, $i = 0, 1, 2$, where $x_i = a + ih$. By differentiating p_2, show that

$$p_2''(x) = \frac{f(x_1 + h) - f(x_1 - h)}{h^2}.$$

2. Derive the following approximate formulas:

   (a) $f'(x) \approx \frac{1}{4h} [f(x + 2h) - f(x - 2h)]$,

   (b) $f'(x) \approx \frac{1}{2h} [4f(x + h) - 3f(x) - f(x + 2h)]$.

3. Using Taylor's series, derive the following approximation for the third derivative of f

$$f'''(x) \approx \frac{1}{2h^3} [f(x + 2h) - 2f(x + h) + 2f(x - h) - f(x - 2h)].$$

4. Using Taylor's series, determine the error term for the approximate formula

$$f'(x) \approx \frac{1}{2h} [f(x + 2h) - f(x)].$$

5. Given $f(x) = e^x$, approximate $f'(1)$ using the central-difference formula with $h = 0.1$.

6. Using Taylor's series, determine the error term for the approximate formula

$$f'(x) \approx \frac{1}{5h} [f(x + 4h) - f(x - h)].$$

7. Use Taylor's series to derive the following approximation formula for the third derivative of f.

$$f'''(x) \approx \frac{1}{h^3}\left[-f(x) + 3f(x+h) - 3f(x+2h) + f(x+3h)\right].$$

8. Use Taylor's series to derive the following approximation formula for the first derivative of f.

$$f'(x) \approx \frac{2f(x+3h) - 9f(x+2h) + 18f(x+h) - 11f(x)}{6h}.$$

9. Show that the approximation formula in Exercise 8 has an error of $O(h^3)$.

10. What does the difference scheme approximates. Give its error order?

$$\frac{1}{2h}[f(x+3h) + f(x-h) - 2f(x)].$$

11. Derive a difference formula for $f''(x_0)$ through $f(x_0)$, $f(x_0-h)$, and $f(x_0+2h)$ and find the leading error term.

12. Show the approximation

$$f'(x) = \frac{8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h)}{12h}.$$

13. Compute the derivative of $f(x) = \sin x$ at $\pi/3$ using

   (a) The forward-difference formula with $h = 10^{-3}$,

   (b) The central-difference formula with $h = 10^{-3}$,

   (c) The formula

$$f'(x_0) \approx \frac{f(x_0 + \alpha h) + (\alpha^2 - 1)f(x_0) - \alpha^2 f(x_0 - h)}{\alpha(1+\alpha)h}$$

   with $h = 10^{-3}$ and $\alpha = 0.5$.

14. Using the following data find $f'(6.0)$ with error $= O(h)$, and $f''(6.3)$, with error $= O(h^2)$

x	6.0	6.1	6.2	6.3	6.4
$f(x)$	0.1750	−0.1998	−0.2223	−0.2422	−0.2596

15. Define

$$S(h) = \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h}.$$

(a) Show that

$$f'(x) - S(h) = c_1h^2 + c_2h^3 + c_3h^4 + ...$$

and state c_1.

(b) Compute $f'(0.398)$ using $S(h)$ and the table

x	0.398	0.399	0.400	0.401	0.402
$f(x)$	0.408591	0.409671	0.410752	0.411834	0.412915

16. The following table contains values of $f(x)$ in $[-6, -1]$

x	-6	-5	-4	-3	-2	-1
$f(x)$	932	487	225	89	28	6

Approximate $f''(-4)$ and $f''(-3)$ using the central-difference formula.

9.2 RICHARDSON'S FORMULA

In this section, we shall develop a technique for approximating derivatives of a function f that will enable us to reduce the truncation error. This technique is known as the **Richardson extrapolation** and has special utilities in computer programs for differentiation and integration of arbitrary functions.

To illustrate the method, let us consider Eqn. (9.5) in the form

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + c_2h^2 + c_4h^4 + \cdots \tag{9.11}$$

where $c_2, c_4, \ldots$ depend on f and x.

For x fixed, we define the function

$$a(h) = \frac{f(x+h) - f(x-h)}{2h}. \tag{9.12}$$

Now, we rewrite (9.11) as

$$f'(x) = a(h) + c_2h^2 + c_4h^4 + \cdots. \tag{9.13}$$

The idea for improving the truncation error is to try to eliminate the term c_2h^2 in Eqn. (9.13). To do so, we replace h by $h/2$ in (9.13) to get

$$f'(x) = a\left(\frac{h}{2}\right) + c_2\left(\frac{h}{2}\right)^2 + c_4\left(\frac{h}{2}\right)^4 + \cdots. \tag{9.14}$$

We can now eliminate the terms in h^2 by multiplying Eqn. (9.14) by four and subtracting Eqn. (9.13) from it. The result is

$$f'(x) = a\left(\frac{h}{2}\right) + \frac{1}{3}\left[a\left(\frac{h}{2}\right) - a(h)\right] - \frac{1}{4}c_4 h^4 + \cdots. \tag{9.15}$$

One can see that we have improved the accuracy of our estimate of the derivative by reducing the error to an order h^4. The new approximation for $f'(x)$ is given by

$$f'(x) \approx a\left(\frac{h}{2}\right) + \frac{1}{3}\left[a\left(\frac{h}{2}\right) - a(h)\right]. \tag{9.16}$$

This extrapolation method can be extended since an estimate of order h^4 can be shown to have an error of order h^6 and so on.

EXAMPLE 9.5

Given $f(x) = e^x$, approximate $f'(1.5)$ using formula (9.16) with $h = 0.1$. Compare with the true value $e^{1.5}$.

Using formula (9.12) with $x = 1.5$ and $h = 0.1$, we get

$$a(h) = a(0.1) = \frac{f(1.6) - f(1.4)}{0.2} \approx 4.489162287752$$
$$a(h/2) = a(0.05) = \frac{f(1.55) - f(1.45)}{0.1} \approx 4.483556674219.$$

In view of (9.16), we obtain

$$\begin{aligned} f'(1.5) &\approx 4.483556674219 + \frac{1}{3}(4.483556674219 - 4.489162287752) \\ &\approx 4.481688136375. \end{aligned}$$

The error is 9.3×10^{-7}. As we can see, formula (9.16) gives a better approximation to $f'(1.5)$ than formula (9.6) for $h = 0.1$.

In general, given an approximation $a(h)$ and having computed the values

$$D_{n,1} = a\left(\frac{h}{2^{n-1}}\right), \quad n = 1, 2, \ldots$$

for an appropriate $h > 0$, the process can be extended to m columns. Such columns and approximations are generated recursively by the formula:

$$D_{n,m+1} = \frac{4^m}{4^m - 1}D_{n,m} - \frac{1}{4^m - 1}D_{n-1,m}. \tag{9.17}$$

The truncation error associated with the entry $D_{n,m+1}$ is of order $O(h^{2m+2})$.

The procedure is best illustrated by arranging the quantities in a table of the form shown in Table 9.1.

$$
\begin{array}{lllll}
D_{1,1} & & & & \\
D_{2,1} & D_{2,2} & & & \\
D_{3,1} & D_{3,2} & D_{3,3} & & \\
\vdots & \vdots & \vdots & \ddots & \\
D_{N,1} & D_{N,2} & D_{N,3} & \ldots & D_{N,N}
\end{array}
$$

Table 9.1 Two-dimensional triangular array $D_{N,N}$.

Observe that, since round-off error affects calculation by computers, computing $D_{N,N}$ for large values of N does not result, in general, to better accuracy of $D_{1,1}$. Therefore, one should put a limit on the number of repetitions of the process. The answer to the question, when one should stop the process in order to get the best approximation, is not known. On one hand, we want h small for accuracy, but on the other hand, we want h large for stability. One can empirically check whether the truncation error formula is being maintained from one level to the next. When this fails, the extrapolations should be broken off.

EXAMPLE 9.6
Given $f(x) = e^x$ and $h = 0.25$, compute $D_{6,6}$ to approximate $f'(1)$.

We have

$$
\begin{aligned}
D_{1,1} &= a(h) = \frac{e^{1.25} - e^{0.75}}{0.5} \approx 2.7466858816 \\
D_{2,1} &= a(h/2) = \frac{e^{1.125} - e^{0.875}}{0.25} \approx 2.7253662198.
\end{aligned}
$$

Using formula (9.17), we get

$$
D_{2,2} = \frac{4}{4-1} D_{2,1} - \frac{1}{4-1} D_{1,1} \approx 2.7182596658.
$$

Continuing in this manner leads to the values in Table 9.2.
The MATLAB function f2 used in this table is defined as follows:

```
function f=f2(x)
 f=exp(x);
```

EXERCISE SET 9.2

1. Let $f(x) = \ln x$. Approximate $f'(1)$ for $h = 0.1$ and $h = 0.01$.

```
» derive('f2',0.25,1,6)
   Derivative table
```

I	h	Di,1	Di,2	Di,3	....		
1	0.250000	2.746686					
2	0.125000	2.725366	2.718260				
3	0.062500	2.720052	2.718280	2.718282			
4	0.031250	2.718724	2.718282	2.718282	2.718282		
5	0.015625	2.718392	2.718282	2.718282	2.718282	2.718282	
6	0.007813	2.718309	2.718282	2.718282	2.718282	2.718282	2.718282

Table 9.2 Approximation of the derivative of $f(x) = e^x$ at $x = 1$ using Richardson's formula.

2. Let $a > 1$ be a given number. We consider the function

$$f(x) = a^x.$$

Using the fact that $f'(0) = \ln a$, find an approximation to $\ln 6$ by using the derivative table in Example 9.6.

3. Repeat the previous exercise to approximate $\ln 3$ to five decimal digits.

4. Given the approximation formula

$$K(x) = \frac{f(x+3h) - f(x-h)}{4h}$$

show that

$$f'(x) - K(x) = c_1 h + c_2 h^2 + \cdots$$

and determine c_1.

5. Prove that the approximation formula

$$f'(x) \approx \frac{4f(x+h) - 3f(x) - f(x+2h)}{2h}$$

has an error that can be written as

$$f'(x) - \frac{4f(x+h) - 3f(x) - f(x+2h)}{2h} = c_1 h^2 + c_2 h^3 + \cdots.$$

Determine c_1 and c_2.

6. Using the following data, find $f'(6.0)$, error $= O(h)$, and $f''(6.3)$, error $= O(h^2)$.

x	6.0	6.1	6.2	6.3	6.4
$f(x)$	-0.1750	-0.1998	-0.2223	-0.2422	-0.2596

M-function 9.2

The following MATLAB function **derive.m** approximate the derivative of a function at a given point using formula 9.17 and the central difference approximation. INPUTS are a function *f*; a value of *h*; a specific point *a*; the number of rows *n*. The input function *f(x)* should be defined as an M-file.

```
function derive(f,h,a,n)
% Approximate the derivative of a function at x = a.
disp('      Derivative table')
disp('______________________________________________')
disp('   i          h          Di,1          Di,2          Di,3       ... ')
disp('______________________________________________')
D(1,1)=(feval(f,a+h)-feval(f,a-h))/(2*h);
fprintf('%2.0f %8.4f %12.4f\n',1,h,D(1,1));
for i=1:n-1
  h=h/2;
  D(i+1,1)=(feval(f,a+h)-feval(f,a-h))/(2*h);
  fprintf('%2.0f %8.4f %12.4f',i+1,h,D(i+1,1));
  for k=1:i
    D(i+1,k+1)=D(i+1,k)+(D(i+1,k)-D(i,k))/((4^k)-1);
    fprintf('%12.4f',D(i+1,k+1));
  end
  fprintf('\n');
end
```

7. Define

$$G(h) = \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h}$$

   (a) Show that

   $$f'(x) - G(h) = c_1 h^2 + c_2 h^3 + c_3 h^4 + \dots$$

   and state c_1.

   (b) Compute $f'(0.398)$ using the table below and $G(h)$.

x	0.398	0.399	0.400	0.401	0.402
$f(x)$	0.40859	0.40967	0.41075	0.41183	0.41292

8. Given the difference quotient known as the central-difference formula

$$D_h f = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

show that

$$\frac{4D_h f - D_{2h} f}{3}$$

is a fourth-order approximation to $f'(x)$ using Taylor's series. Assume that f is six times continuously differentiable.

9. Given the continuous smooth function $g(x)$ for which $g(0) = 8$, $g(1) = 5$, $g(2) = 3$, $g(3) = 2$, and $g(4) = 3$
   (a) Use a central-difference scheme to approximate $g''(2)$.
   (b) Use the Richardson extrapolation to improve this result.

10. The following data gives approximations to the integral $I = \int_a^b f(x)dx$ for a scheme with error terms $E = K_1h + K_2h^3 + K_3h^5 + \ldots$

$$I(h) = 2.3965,\ I(h/3) = 2.9263,\ I(h/9) = 2.9795.$$

    Construct an extrapolation table to obtain a better approximation.

11. Consider the table

x	1	2	3	4	5
$f(x)$	0.01	0.68	1.11	1.38	1.61

   (a) Approximate $f'(3)$ using the central-difference formula with $h = 2$.
   (b) Approximate $f''(3)$ using the central-difference formula with $h = 1$.
   (c) Use the Richardson extrapolation to improve the results in (a).

12. Consider the data in the table

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
$f(x)$	1.544	1.667	1.811	1.972	2.152	2.351	2.576	2.828	3.107

   (a) Approximate $f'(1.4)$ using the forward-difference formula with $h = 0.1, 0.2$.
   (b) Use the Richardson extrapolation to improve the results.

13. Compute the derivative of $f(x) = \sin x$ at $x = \pi/4$ using the Richardson extrapolation. Start with $h = 1$ and find the number of rows in the Richardson table required to estimate the derivative with six significant decimal digits.

COMPUTER PROBLEM SET 9.2

1. Write a computer program in a language of your choice that finds the first derivative of a function f at a given value α of x, using the central-difference approximation and Richardson's formula. Input data to the program should be a function f, α, the value of h, and the number of rows n.

   Test your program to find the derivative of $f(x) = e^{2x} \sin x$ at $x = 0.1, 0.5, 1, 1.5$. Compare the result with the exact answer.

2. Use the MATLAB function `derive` to approximate the derivative of the following functions:

(a) $f(x) = 3x^3 - 5x^2$ with $h = 0.2$, $N = 4$, at $x = 2.6$,

(b) $f(x) = \sqrt{x^2+1}$ with $h = 0.5$, $N = 4$, at $x = 1.3$,

(c) $f(x) = x\cos x - 3x$ with $h = 0.2$, $N = 4$, at $x = 1.4$,

(d) $f(x) = \sin x + x$ with $h = 0.3$, $N = 4$, at $x = \pi/4$,

(e) $f(x) = xe^x - x^2$ with $h = 0.2$, $N = 4$, at $x = 2.3$.

3. Use the MATLAB function derive to estimate the derivative of $f(x) = \arctan x$ at $x = 1$.

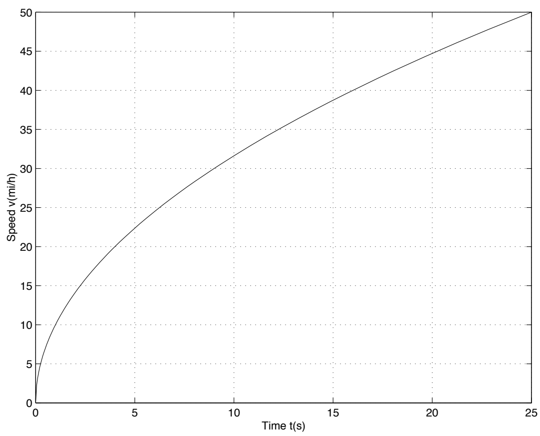
4. Use the MATLAB function derive to estimate the derivative of $f(x) = e^{x^2+1}$ at $x = 1$, $x = 2$, and $x = 3$. Compare with the exact values.

APPLIED PROBLEMS FOR CHAPTER 9

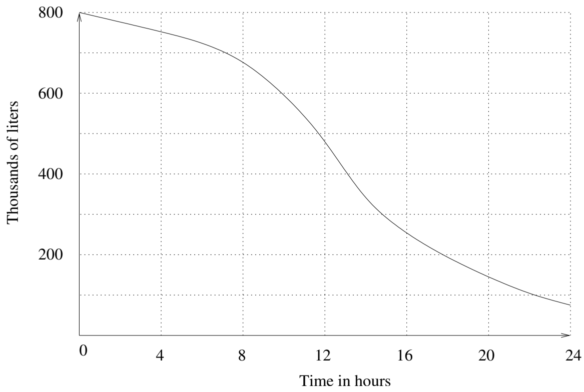
1. The accompanying table gives the distance $d(t)$ of a bullet at various points from the muzzle of a rifle. Use these values to approximate the speed of the bullet at $t = 60, 90$, and 120 using the central-difference formula.

$t(s)$	0	30	60	90	120	150	180
$d(m)$	0	100	220	350	490	600	750

2. A graph of the speed versus time t for a test run of an automobile is shown in the accompanying figure. Estimate the acceleration a at $t = 10, 15$, and 20 (Hint: $a(t) = v'(t)$).



3. The graph below shows the amount of water in a city water tank during one day when no water was pumped into the tank. Approximate the rate of change of water usage at $t = 8, 12$ and 16.

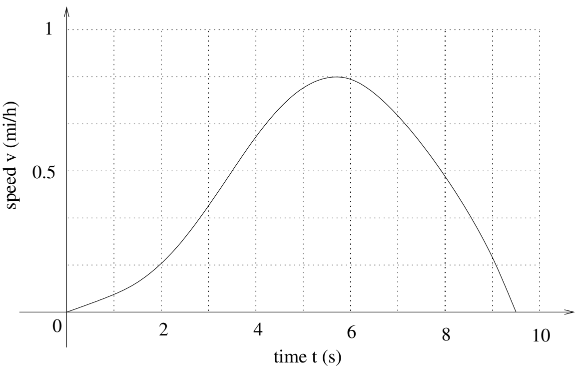


4. The following table gives the normal high temperature T for a city as a function of time (measured in days since January 1).

Days of the year	1	30	61	91	121	151	181	211	241	271
T	38	39	50	62	73	80	88	90	80	75

What is the approximate rate of change in the normal high temperature on March 2 (day number 61) and July 30 (day number 211).

5. The accompanying figure gives the speed, in miles per second, at various times of a test rocket that was fired upward from the surface of the Earth. Use the graph to approximate the acceleration of the rocket at $t = 2, 4$, and 6.



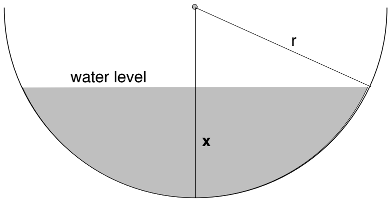
6. Water is flowing from a reservoir shaped like a hemisphere bowl of radius 12 m (see figure below). The volume of water is given by

$$V(x) = \frac{\pi}{3}x^3(3r - x).$$

The rate of change of the volume of water is

$$dV/dt = (dV/dx)(dx/dt).$$

Assume that $dx/dt = 1$ and approximate $dV/dt = dV/dx$ when $x = 3$ using the central-difference formula with $h = 0.1, 0.01$, and 0.001.



7. Let $f(x) = \sqrt{x}$ be given by the following table that has five-decimal digits of accuracy.

x	1.2	1.4	1.6	1.8	2.0
$\sqrt{x}$	1.09545	1.18322	1.26491	1.34164	1.41421
x	2.2	2.4	2.6	2.8	
$\sqrt{x}$	1.48324	1.54919	1.61245	1.673320	

Approximate $f'(x)$ at $x = 1.4, 1.8, 2.2,$ and 2.4 using the forward-difference formula with $h = 0.2$. Compare your results with the values of the exact ones.