Note on the induction equation

1 Generalities

We want to solve the radial induction equation at the core-mantle boundary (CMB) which reads

$$\frac{\partial B_r}{\partial t} = -\boldsymbol{\nabla}_H \cdot (\boldsymbol{u}_H B_r) \tag{1.1}$$

where B_r is the radial magnetic field, $\nabla_H \cdot$ the horizontal divergence operator and $\boldsymbol{u}_H$ the tangential flow. We can distribute the divergence operator over the whole product and get

$$\frac{\partial B_r}{\partial t} = -B_r \boldsymbol{\nabla}_H \cdot \boldsymbol{u}_H - \boldsymbol{u}_H \cdot \boldsymbol{\nabla}_H B_r \tag{1.2}$$

Generally, we assume the mantle to be insulating. As a result, no currents are circulating and $\boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{0}$ as well as $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$. Then, the magnetic field derives from a potential and one may write

$$\boldsymbol{B} = -\boldsymbol{\nabla} V \tag{1.3}$$

with V the magnetic potential.

At the CMB, the core flow is constrained radially by the mantle. As a consequence, its radial component vanishes (ie. $u_r = 0$). Likewise, as it is incompressible, it is also divergence-free (ie. $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$). Then, it admits a unique poloidal-toroidal decomposition.

$$\boldsymbol{u} = \boldsymbol{\nabla} \times (\boldsymbol{r}\,\mathcal{T}) + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\boldsymbol{r}\,\mathcal{S})) \tag{1.4}$$

where $\mathcal{T}$ and $\mathcal{S}$ are the toroidal and poloidal scalar fields, respectively. At the CMB, the expression is further simplified and the tangential flow reads

$$\boldsymbol{u}_H = \nabla\times(\boldsymbol{r}\,\mathcal{T}) + \nabla_H\, r\,\mathcal{S} \tag{1.5}$$

In this context, the condition $\boldsymbol{\nabla}\cdot\boldsymbol{u}=0$ is directly encoded into that of $\boldsymbol{u}_H$.

2 Decomposition in spherical coordinates

2.1 Expression for $\boldsymbol{u}_H$

$$\begin{aligned} \boldsymbol{u}_H &= \boldsymbol{\nabla}\times(\boldsymbol{r}\,\mathcal{T}) + \boldsymbol{\nabla}_H\, r\,\mathcal{S} \\ &= -\boldsymbol{r}\times\boldsymbol{\nabla}\mathcal{T} + r\boldsymbol{\nabla}_H\mathcal{S} \\ &= \begin{pmatrix} -\dfrac{\partial_\phi}{\sin(\theta)} \\ \partial_\theta \end{pmatrix}\mathcal{T} + \begin{pmatrix} \dfrac{\partial_\theta}{\sin(\theta)} \\ \partial_\phi \end{pmatrix}\mathcal{S} \end{aligned} \tag{2.1}$$

2.2 Expression for $\boldsymbol{\nabla}_H\cdot\boldsymbol{u}_H$

$$\begin{aligned} \boldsymbol{\nabla}_H\cdot\boldsymbol{u}_H &= \frac{1}{r\sin(\theta)}\partial_\theta(u_\theta\sin(\theta)) + \frac{1}{r\sin(\theta)}\partial_\phi(u_\phi) \\ &= \frac{1}{r\sin(\theta)}(\partial_{\theta\theta} + \partial_{\phi\phi})\mathcal{S} \end{aligned} \tag{2.2}$$

2.3 Expression for $\boldsymbol{\nabla}_H B_r$

$$\boldsymbol{\nabla}_H B_r = \begin{pmatrix} \dfrac{1}{r}\partial_\theta \\ \dfrac{1}{r\sin(\theta)}\partial_\phi \end{pmatrix} B_r \tag{2.3}$$

3 Equation in spherical coordinates

$$\begin{aligned}\frac{\partial B_r}{\partial t} &= -\frac{B_r}{r\sin(\theta)}\,(\partial_{\theta\theta}+\partial_{\phi\phi})\mathcal{S}\\ &- \frac{1}{r\sin(\theta)}[(-\partial_\phi\mathcal{T}+\partial_\theta\mathcal{S})\partial_\theta B_r+(\partial_\theta\mathcal{T}+\partial_\phi\mathcal{S})\partial_\phi B_r]\end{aligned} \tag{3.1}$$

4 Condition on the flow

We are adding a supplementary constraint on the flow, given by the relation

$$\boldsymbol{\nabla}_H\cdot(\boldsymbol{u}_H\cos(\theta))=0 \tag{4.1}$$

which in spherical coordinates reads

$$\begin{aligned}\boldsymbol{\nabla}_H\cdot(\boldsymbol{u}_H\cos(\theta)) &= \frac{1}{r\tan(\theta)}\,(\partial_{\theta\theta}+\partial_{\phi\phi})\mathcal{S}\\ &+ \frac{1}{r}(\partial_\phi\mathcal{T}-\partial_\theta\mathcal{S})\end{aligned} \tag{4.2}$$