Plasma waves and instabilities : from drift waves to kinetic MHD modes

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Outline

- Introduction to waves and instabilities – reactive and kinetic instabilities

- MHD instabilities – kinetic MHD

- Drift waves

- Non linear saturation processes (sketchy)

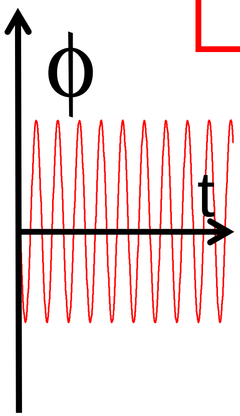
Plasma waves: basics

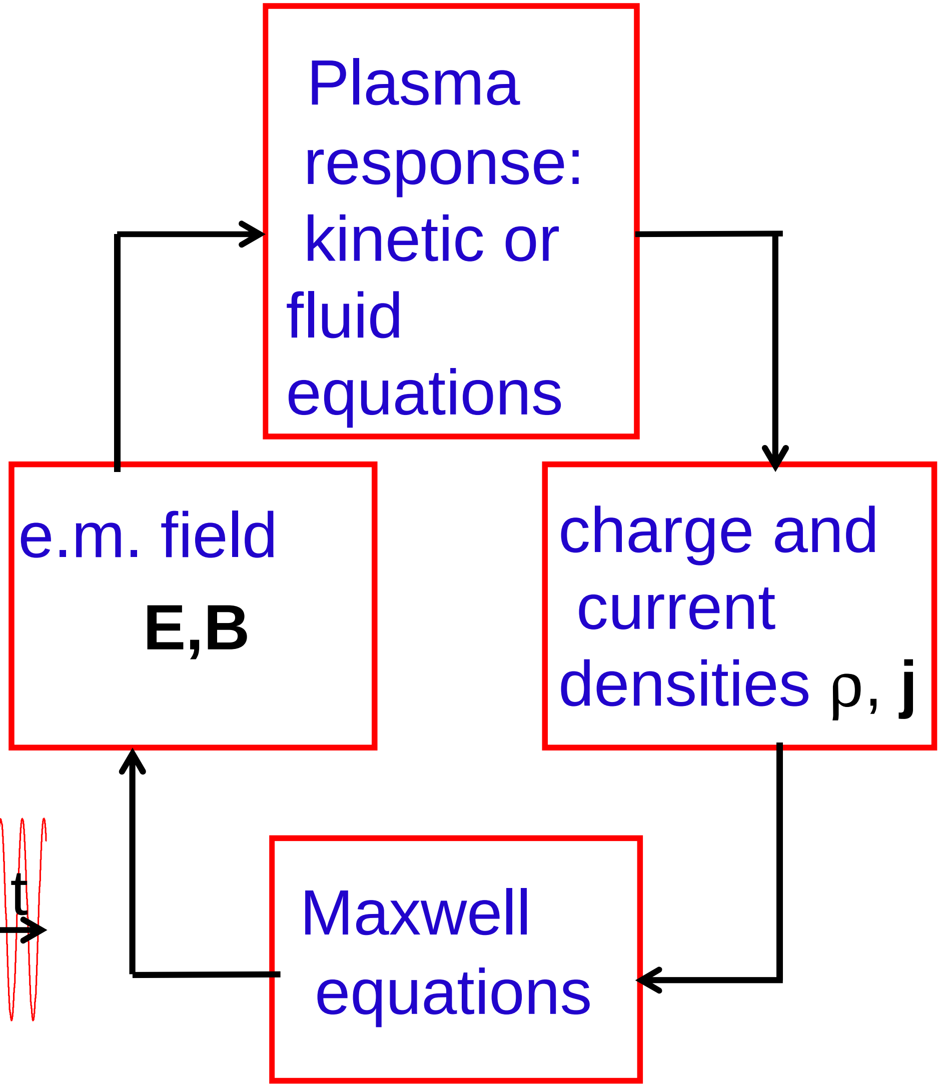
- Small sinusoidal perturbation of the electromagnetic field

$$\phi(\mathbf{x},t)=\phi_{\mathbf{k}\omega}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}+c.c.$$

- Linear response of current of charge and current densities

- Self-consistent problem





Leads to a dispersion relation

- Maxwell + plasma response leads to a dispersion relation

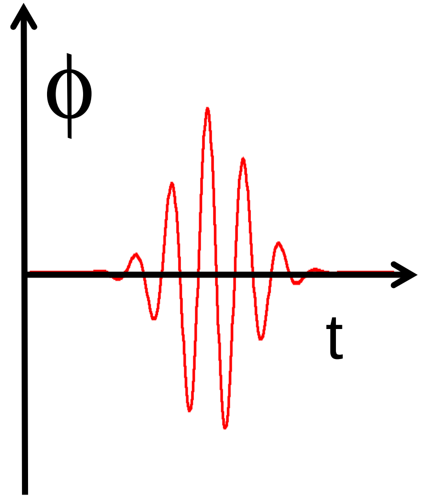
$$\epsilon(\mathbf{k},\omega) = \epsilon_r(\mathbf{k},\omega) + i\epsilon_i(\mathbf{k},\omega) = 0$$

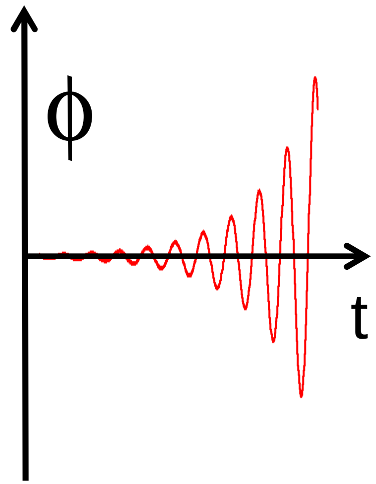
- Solution usually complex

$$\omega(\mathbf{k}) = \omega_r(\mathbf{k}) + i\gamma(\mathbf{k})$$

- Real solution $\omega_r(\mathbf{k}) \rightarrow$ wave

- Complex solution $\gamma(\mathbf{k}) > 0 \rightarrow$ instability





A bit more on the dispersion relation

- Convenient to calculate the Lagrangian of the electromagnetic field

Current density

Charge density

$$L_{\mathbf{k}\omega} = \underbrace{\epsilon_0 \mathbf{E}_{\mathbf{k}\omega} \cdot \mathbf{E}^\star_{\mathbf{k}\omega} - \frac{1}{\mu_0} \mathbf{B}_{\mathbf{k}\omega} \cdot \mathbf{B}^\star_{\mathbf{k}\omega}}_{\text{Electromagnetic field}} + \underbrace{\mathbf{J}_{\mathbf{k}\omega} \cdot \mathbf{A}^\star_{\mathbf{k}\omega} - \rho_{\mathbf{k}\omega} \phi^\star_{\mathbf{k}\omega}}_{\text{Wave-particle interaction}}$$

- Electric field $\mathbf{E}_{\mathbf{k}\omega} = i\omega \mathbf{A}_{\mathbf{k}\omega} - i\mathbf{k}\phi_{\mathbf{k}\omega}$

- Magnetic field $\mathbf{B}_{\mathbf{k}\omega} = i\mathbf{k} \times \mathbf{A}_{\mathbf{k}\omega}$

A bit more on the dispersion relation

- Maxwell equations $$\frac{dL}{d\phi^{\star}_{\mathbf{k}\omega}} = 0 \qquad \frac{dL}{d\mathbf{A}^{\star}_{\mathbf{k}\omega}} = 0$$

- Small perturbations: $\rho_{\mathbf{k}\omega}$ and $\mathbf{J}_{\mathbf{k}\omega}$ are linear functions of $\phi_{\mathbf{k}\omega}$ and $\mathbf{A}_{\mathbf{k}\omega}$.

- Dispersion relation

$$L(\mathbf{k},\omega) = \epsilon(\mathbf{k},\omega)\epsilon_0 \left|\mathbf{E}_{\mathbf{k}\omega}\right|^2$$

Energetics of an instability

- Energy exchanged between e.m. field and particles (**real** ω)

$$P(\mathbf{k},\omega) = 2\omega Im\left(L_{\mathbf{k}\omega}\right) = \mathbf{J}_{\mathbf{k}\omega}\cdot\mathbf{E}^{\star}_{\mathbf{k}\omega} + \mathbf{J}^{\star}_{\mathbf{k}\omega}\cdot\mathbf{E}_{\mathbf{k}\omega}$$

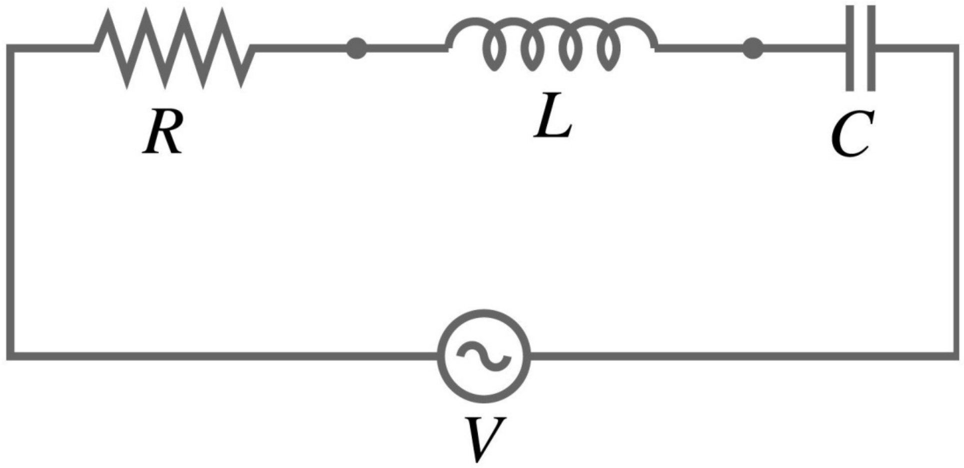
- Two types of instabilities

$P(\mathbf{k},\omega) = 0 \quad \rightarrow$ "reactive"

$P(\mathbf{k},\omega) \neq 0 \quad \rightarrow$ "kinetic"

- Marginal stability

$$P_{\mathbf{k}} = P(\mathbf{k},\omega_r(\mathbf{k}))$$



Reactive mode	R=0
Damping	R>0
"Kinetic" instability	R<0

Kinetic instability

- Situation close marginal stability $\gamma(\mathbf{k}) \ll \omega_r(\mathbf{k})$
- Taylor development of $\varepsilon(\mathbf{k},\omega)=0$
  - Lowest order $\epsilon_r(\mathbf{k},\omega_r)=0$ → pulsation $\omega_r(\mathbf{k})$
  - Next order $\gamma_{\mathbf{k}} = -\dfrac{P_{\mathbf{k}}}{W_{\mathbf{k}}}$ → growth rate $\gamma(\mathbf{k})$
- Energy density

$$W_{\mathbf{k}} = \left.\frac{d\left(\omega\epsilon_r\right)}{d\omega}\right|_{\omega=\omega_r(\mathbf{k})}$$

- Instability $\gamma(\mathbf{k})>0$ if $P_{\mathbf{k}}<0$: energy transferred from particles to wave

Reactive instability

- Since $P(\mathbf{k},\omega)=0$, if $\omega_{\mathbf{k}}$ is solution, $\omega_{\mathbf{k}}^*$ is solution too

- At threshold

$$\epsilon_r = 0 \quad \text{and} \quad \frac{d\epsilon_r}{d\omega} = 0$$

$\rightarrow$ energy $W_{\mathbf{k}}=0$

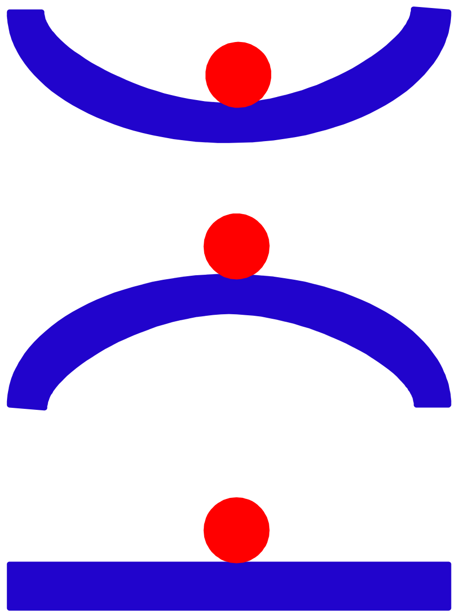
- Reactive instability sometimes called "negative energy" wave

Analogy with particle motion in a potential

Stable

Unstable

Neutral



A simple illustration of the reactive vs kinetic character of an instability

- Vlasov equation, 1D and electrostatic

$$\partial_t f + \mathbf{v} \cdot \partial_{\mathbf{x}} f + \frac{e\mathbf{E}}{m} \cdot \partial_{\mathbf{v}} f = 0$$

- Response function

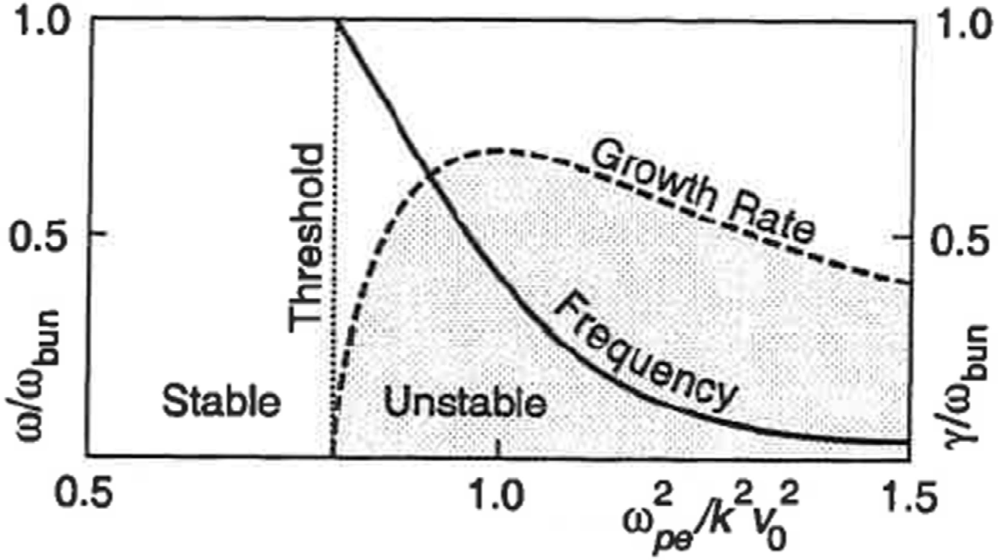
Unperturbed dist. function

$$\epsilon(k,\omega) = 1 + \sum_{species} \frac{\omega_{ps}^2}{k} \int_{-\infty}^{+\infty} dv \frac{\partial_v \hat{f}_{0s}}{\omega - kv + io^+}$$

Plasma frequency $\omega_{ps}^2 = \frac{n_s e_s^2}{m_s \epsilon_0}$

Hydrodynamic limit: Buneman instability

- Hydrodynamic limit $\omega \gg kv$ for ions and electron beam $f_0 = n_e \delta(v - v_0)$



Baumjohann & Treumann 2012

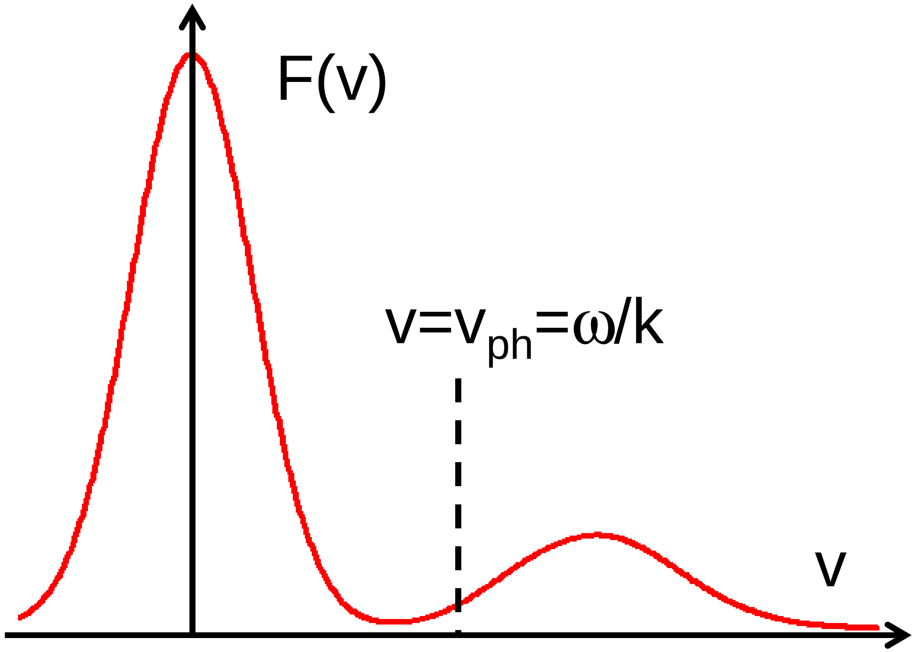
- Dispersion relation

$$\epsilon(k,\omega) = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(\omega - kv_0)^2} = 0$$

- Negative energy wave is unstable if $kv_0 \simeq \omega_{pe} \longrightarrow \omega = \frac{-1+\sqrt{3}i}{2}\left(\frac{m_e}{2m_i}\right)^{1/3}\omega_{pe}$

Kinetic limit : bump on tail

- Hydrodynamic limit for all thermal species ω>>kv
+ hot beam



- Dielectric

$$\epsilon(k,\omega) \simeq 1 - \frac{\omega_p^2}{\omega^2} - i\pi\, \partial_v \hat{f}_0\Big|_{v=v_{ph}}$$

- Kinetic instability

$$\gamma(k) = -\frac{\epsilon_i}{\partial_\omega \epsilon_r\big|_{\omega=\omega_p}} = \omega_p \frac{\pi}{2}\frac{\omega_{pb}^2}{k^2}\, \partial_v \hat{f}_0\Big|_{v=v_{ph}}$$

Single fluid ideal MHD

- MHD equations

Magnetic field

Velocity

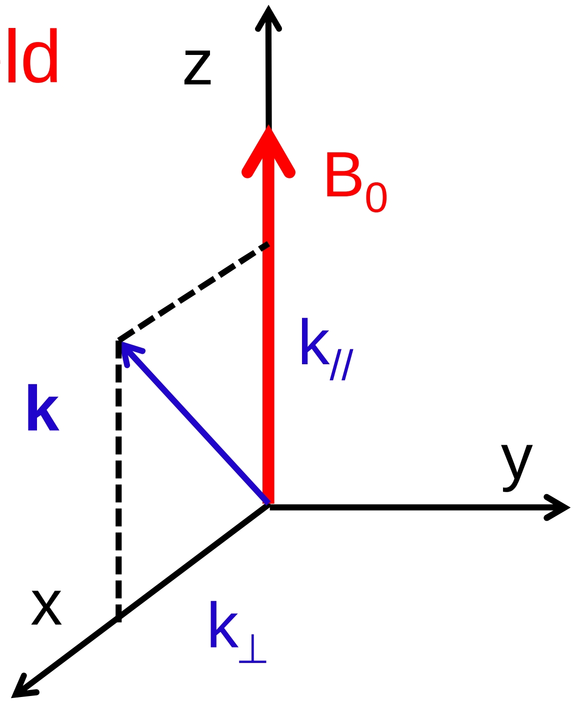
$$\rho d_t \mathbf{V} = -\nabla P + \mathbf{J} \times \mathbf{B}$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{V} \times \mathbf{B})$$

Pressure

$$d_t P + \Gamma P \nabla \cdot \mathbf{V} = 0$$

Adiabatic index



- Lagrangian derivative $d_t = \partial_t + \mathbf{V} \cdot \nabla$
- MHD displacement $\dfrac{d\boldsymbol{\xi}(\mathbf{x}, t)}{dt} = \mathbf{V}(\mathbf{x}, t)$

Alfvén waves in incompressible medium ρ=cte

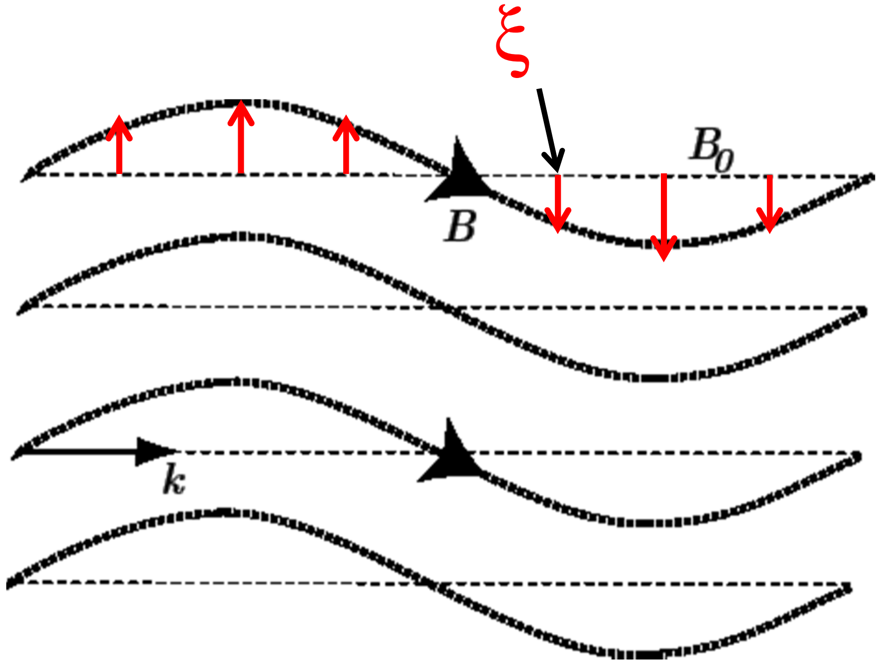
- 3 solutions: shear Alfvén wave, fast and slow magneto-acoustic waves

- Shear Alfvén wave

$$\omega = k_{\|} V_A$$

- Alfvén velocity

$$V_A = \sqrt{\frac{B^2}{\mu_0 \rho}}$$



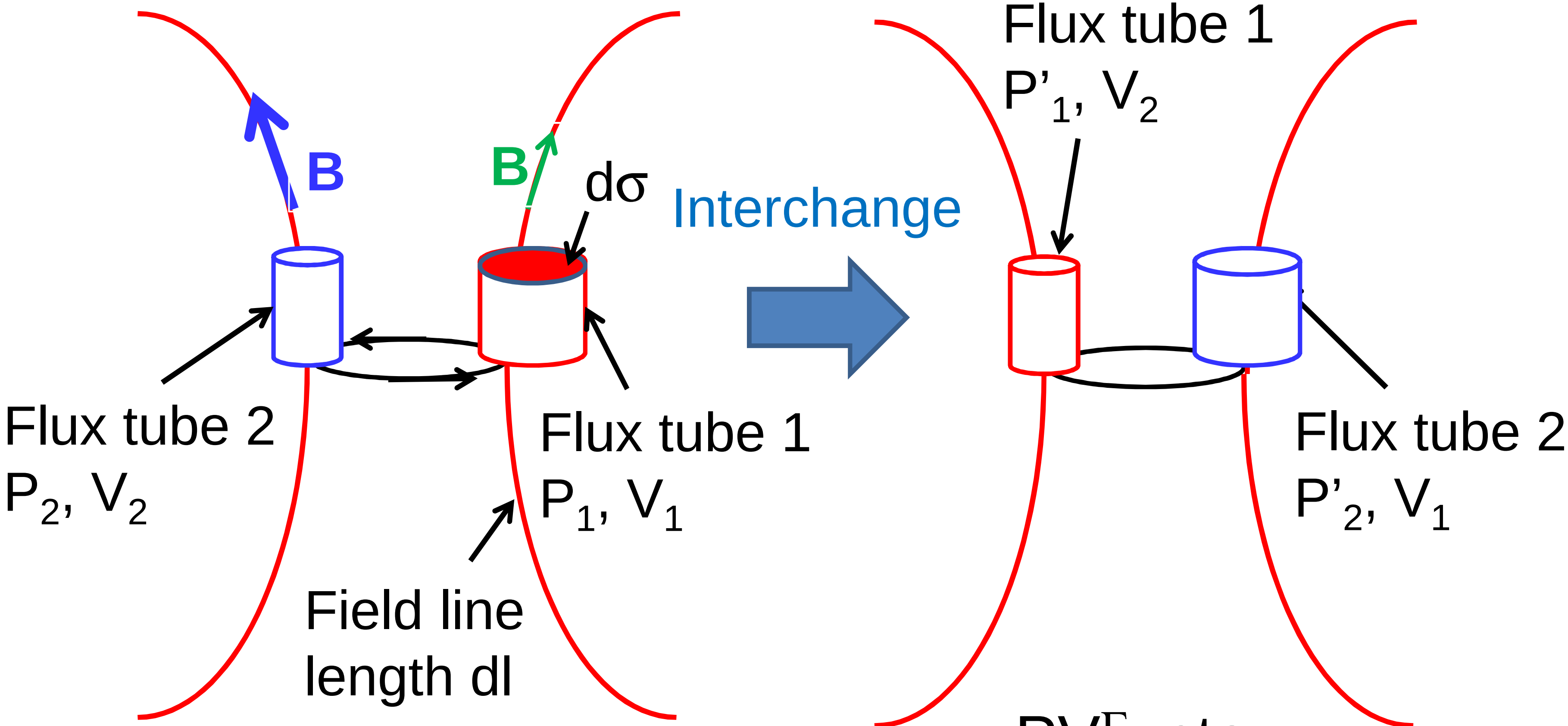
MHD instabilities

• In single fluid MHD, two main destabilizing terms:

1) Pressure gradient $\rightarrow$ interchange instability

2) Current density gradient $\rightarrow$ kink mode

MHD instabilities : interchange



PV^{Γ}=cte

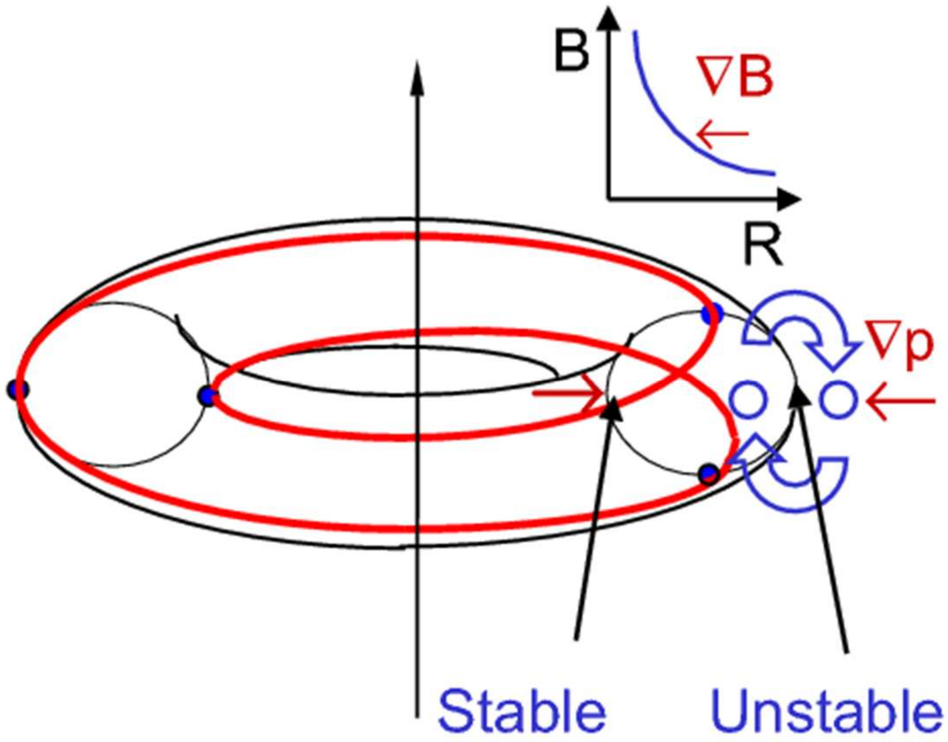
MHD instabilities : interchange (cont.)

- Exchange of two flux tubes

- Released energy

$$\delta W = \delta P \delta V \simeq \delta P \delta \int \frac{d\ell}{B}$$

- Interchange instability ∇P aligned with ∇B



Energy principle

- Force balance equation $\rho\omega^2\boldsymbol{\xi}_\omega = -\mathbf{F}_\omega = \mathbf{L}_\omega \cdot \boldsymbol{\xi}_\omega$

$$\frac{1}{2}\omega^2 \int d^3\mathbf{x}\rho \left|\boldsymbol{\xi}_\omega\right|^2 = \delta W_{MHD} = \int d^3\mathbf{x} \left(\boldsymbol{\xi}_\omega^* \cdot \mathbf{L}_\omega \cdot \boldsymbol{\xi}\right)$$

- MHD energy δW combines wave character and instability sources

$$\delta W_{MHD} = \delta W_{wave} + \delta W_{interchange} + \delta W_{kink}$$

- MHD instabilities are reactive

$$L_{MHD} = \omega^2 \int d^3\mathbf{x}\rho \left|\boldsymbol{\xi}_\omega\right|^2 - 2\delta W_{MHD}$$ real for real ω

The MHD description usually fails at low frequency

- Two reasons for breakdown of the MHD description:

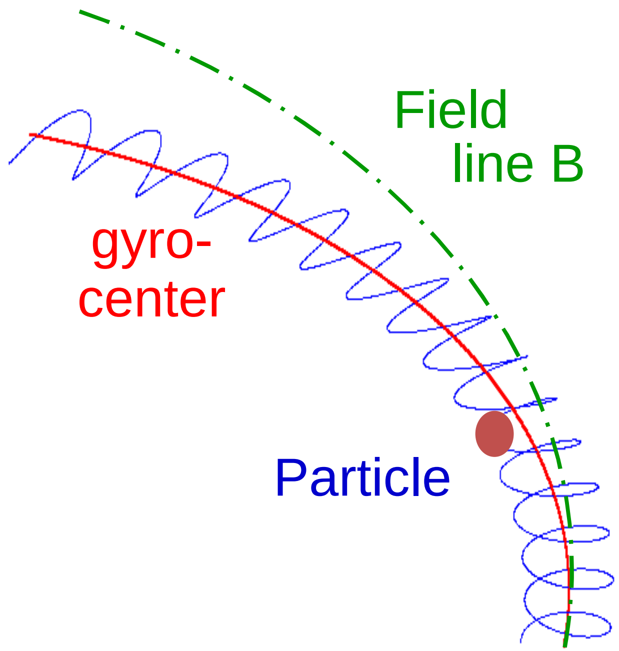
1) Landau resonances $\omega = \mathbf{k.v}$

2) Effects of finite orbit width δ

- Ideal MHD approach valid when

$$\omega >> \mathbf{k.v} \text{ and } k_\perp\delta<<1$$

- Difficulties occur at low frequencies



Low frequency limit

- Momentum equation for each species

$$\rho d_t \mathbf{V} = -\nabla P + ne\left(\mathbf{E} + \mathbf{V} \times \mathbf{B}\right) + \ \ldots$$

- Strong guide field B

Polarization drift

Parallel flow

$$\mathbf{V} = V_{\|}\mathbf{b} + \mathbf{V}_E + \mathbf{V}^* + \mathbf{V}_{pol} + \ \ldots$$

EXB velocity

Diamagnetic velocity

$$\mathbf{V}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

$$\mathbf{V}^* = \frac{\mathbf{B}}{B^2} \times \frac{\nabla P}{ne}$$

Drift wave

- Simple slab geometry, electrostatic, fluid ions

$$\partial_t n_i + \nabla \cdot (n_i \mathbf{V}_E) = 0$$

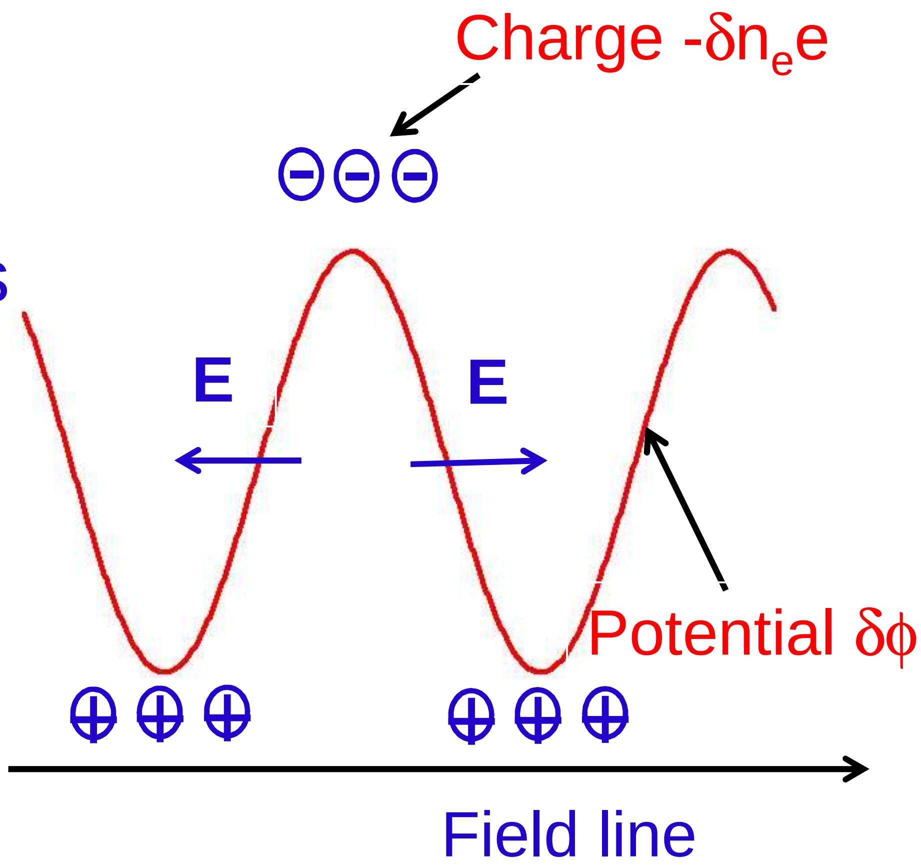
- fast motion along field lines $\rightarrow$ adiabatic electrons

$$\frac{\delta n_e}{n_{eq}} = \frac{e\delta\phi}{T_e}$$

- electro-neutrality ($k\lambda_D \ll 1$)

$$n_e = n_i$$

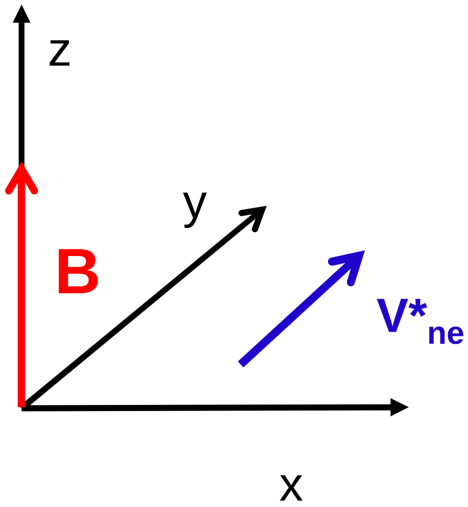
$n_e \approx n_0 \exp(e\phi/T_e)$



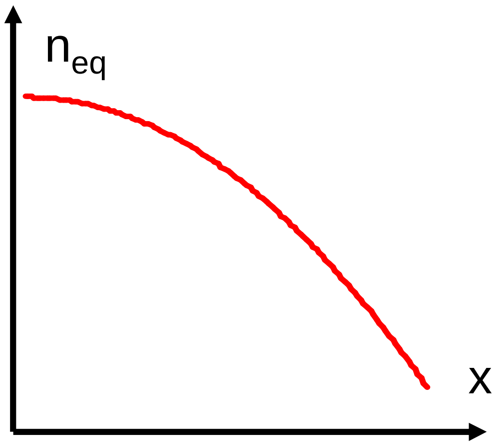
Drift wave (cont.)

- Density gradient in the x direction, uniform B, y,z periodic

- Phase velocity = electron diamagnetic velocity



$$v_{ph} = \frac{\omega}{k_y} = V^*_{ne}$$

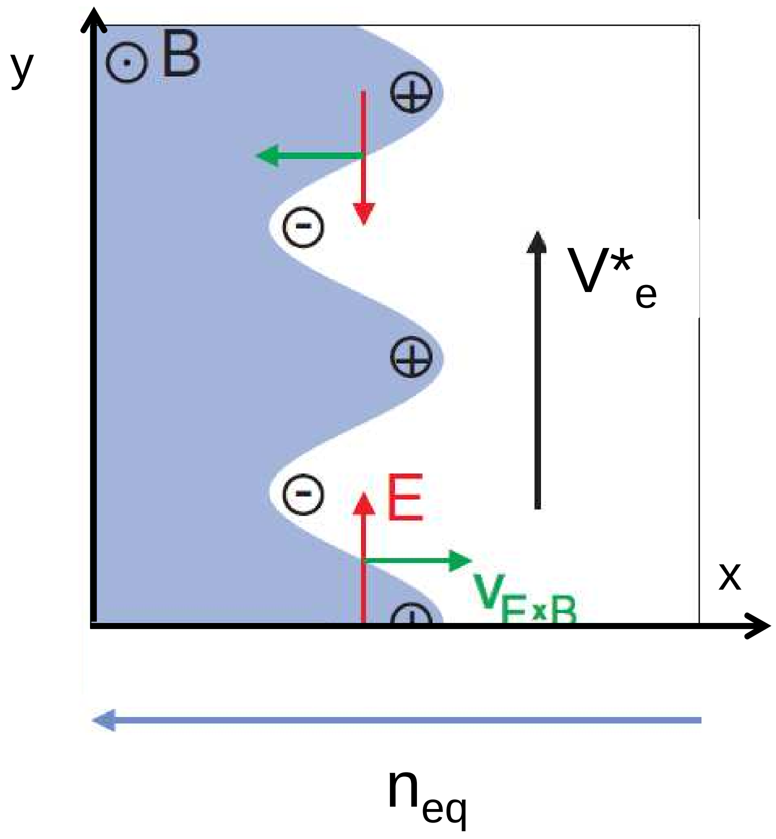


$$\mathbf{V}^*_{ne} = -\frac{T_0}{eB_0}\frac{\partial n_{eq}}{\partial x}\mathbf{e}_y$$

Drift wave (cont.) – schematic picture

- Start with a density $n_i = n_e$ corrugation
- Fast electron response along field lines → potential adjusts → electric field E
- ExB drifts shifts the perturbation along V^*_e

Grulke&Klinger 02



The ion inertia plays a crucial for non linear saturation

- Drawbacks of the previous model: no instability, infinity of non linear solutions
- Add the polarization drift (ion inertia)

$$\mathbf{V} = \mathbf{V}_E + \mathbf{V}^* + \mathbf{V}_{pol} + \ \ldots$$

Divergence of polarization current $\nabla \cdot (ne\mathbf{V}_{pol}) \simeq -\frac{n_{eq}m_i}{B^2} d_t \nabla_\perp^2 \phi$

Lagrangian derivative $d_t = \partial_t + \mathbf{V}_E \cdot \nabla$

A paradigm : the CHM equation

- Same assumptions+ polarization drift

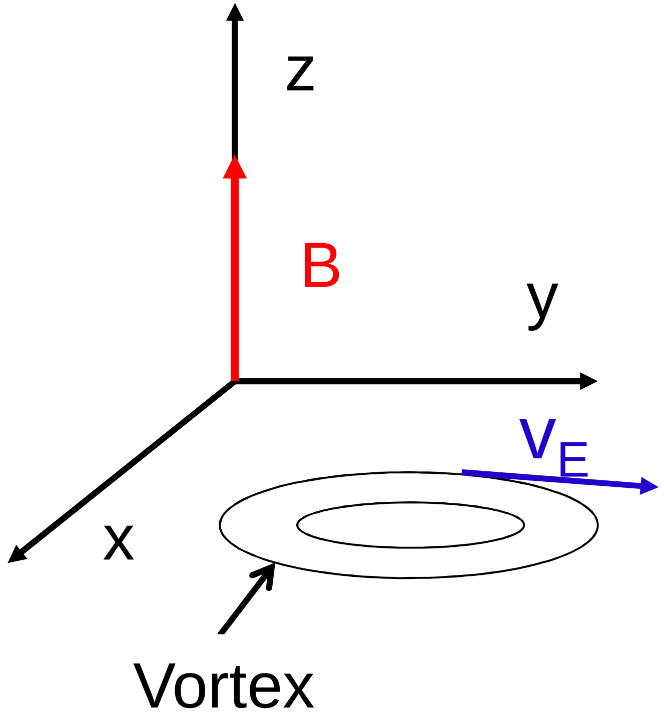
- Charney-Hasegawa-Mima (CHM) equation

$$d_t\left(\phi - \rho_s^2 \nabla_\perp^2 \phi\right) + \mathbf{V}_{ne}^* \cdot \nabla \phi = 0$$

Ion gyroradius $\rho_s = \dfrac{\sqrt{m_s T_e}}{eB}$

- Dispersion relation

$$\omega = \frac{k_y V_{ne}^*}{1 + k_\perp^2 \rho_s^2}$$



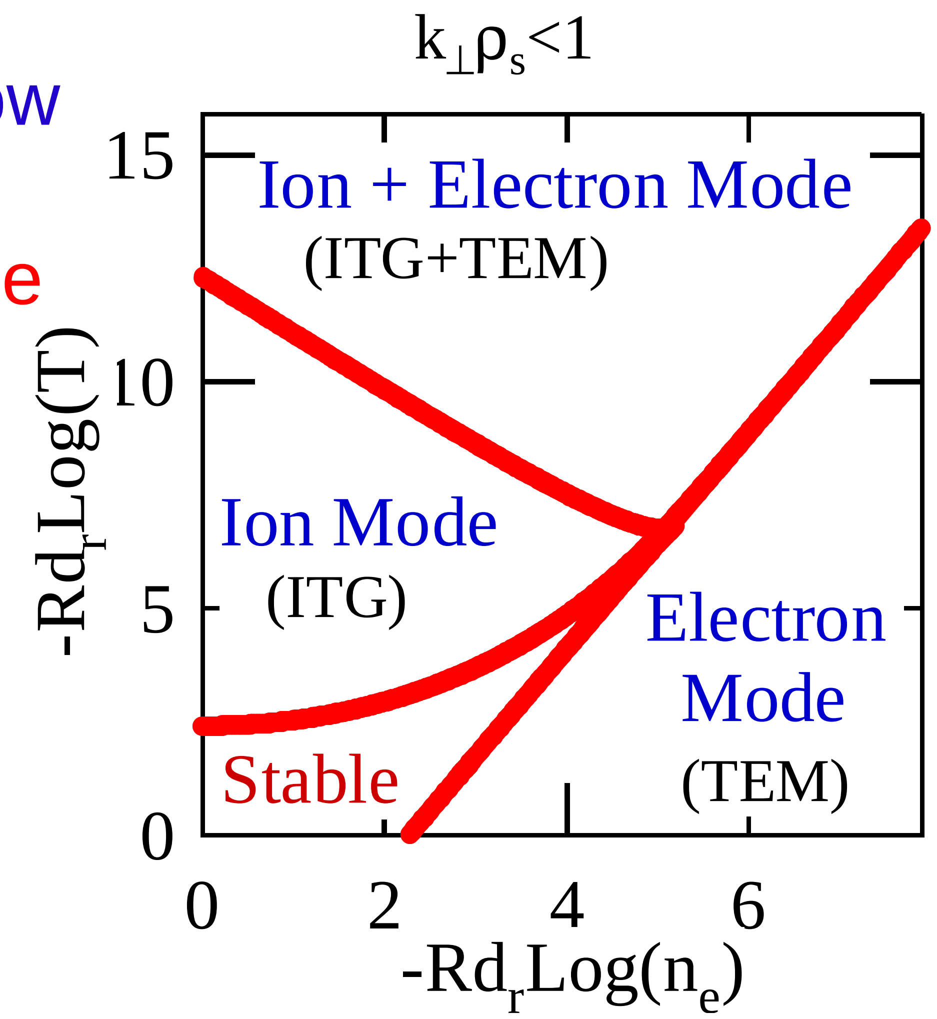
$$\mathbf{v_E} = \frac{\mathbf{B}}{B^2} \times \nabla\phi$$

Example of a tokamak: electrostatic modes

- Dominant instabilities at low frequency : drift waves mainly driven by interchange

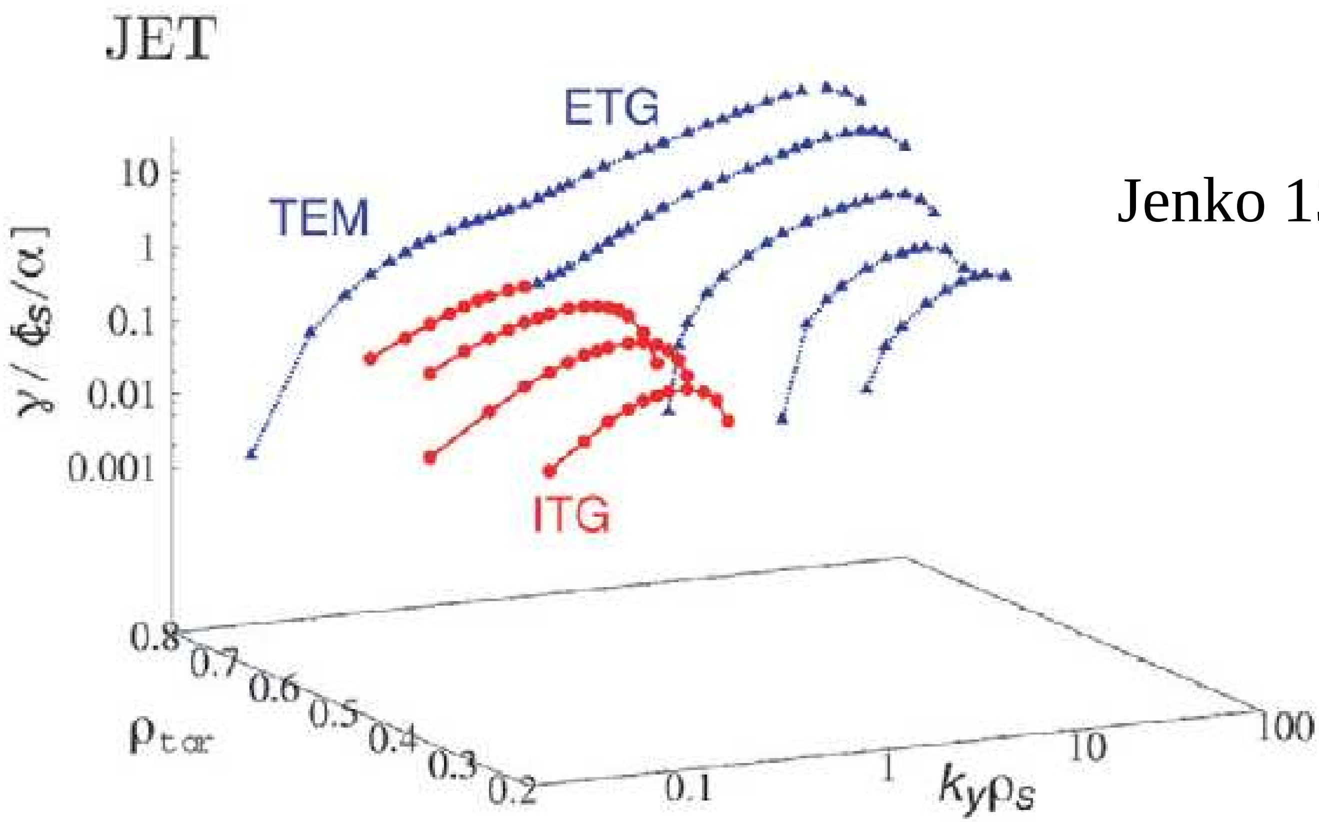
- Kinetic type: driven via resonances by electrons or ions.

- Threshold in temperature and density gradients



Growth rates

- Several branches may co-exist.
- Electron branch at $k_{\perp}\rho_s<1$

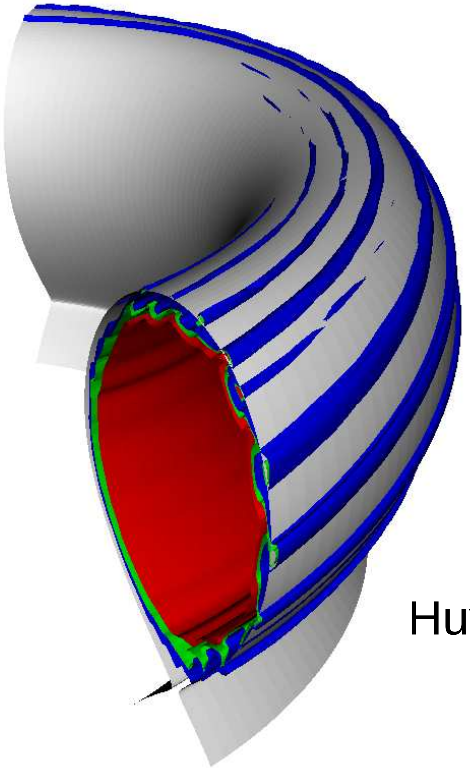


Jenko 13

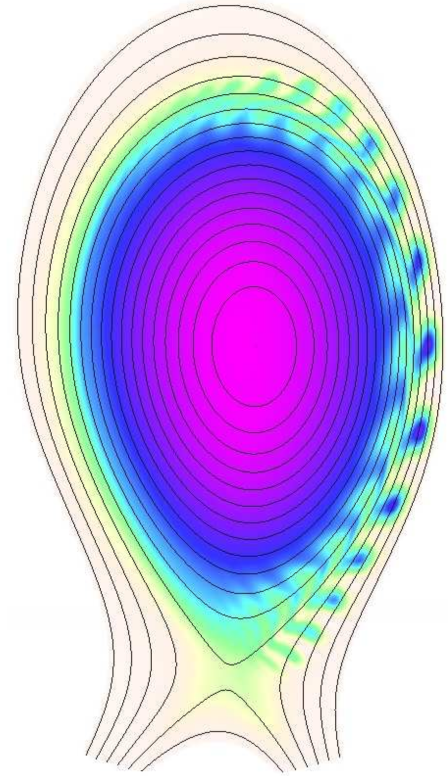
Example of a tokamak: electromagnetic modes

- Ballooning mode: shear Alfvén wave coupled to interchange instability

- Low frequency limit: diamagnetic drifts matter



Huysmans 2009



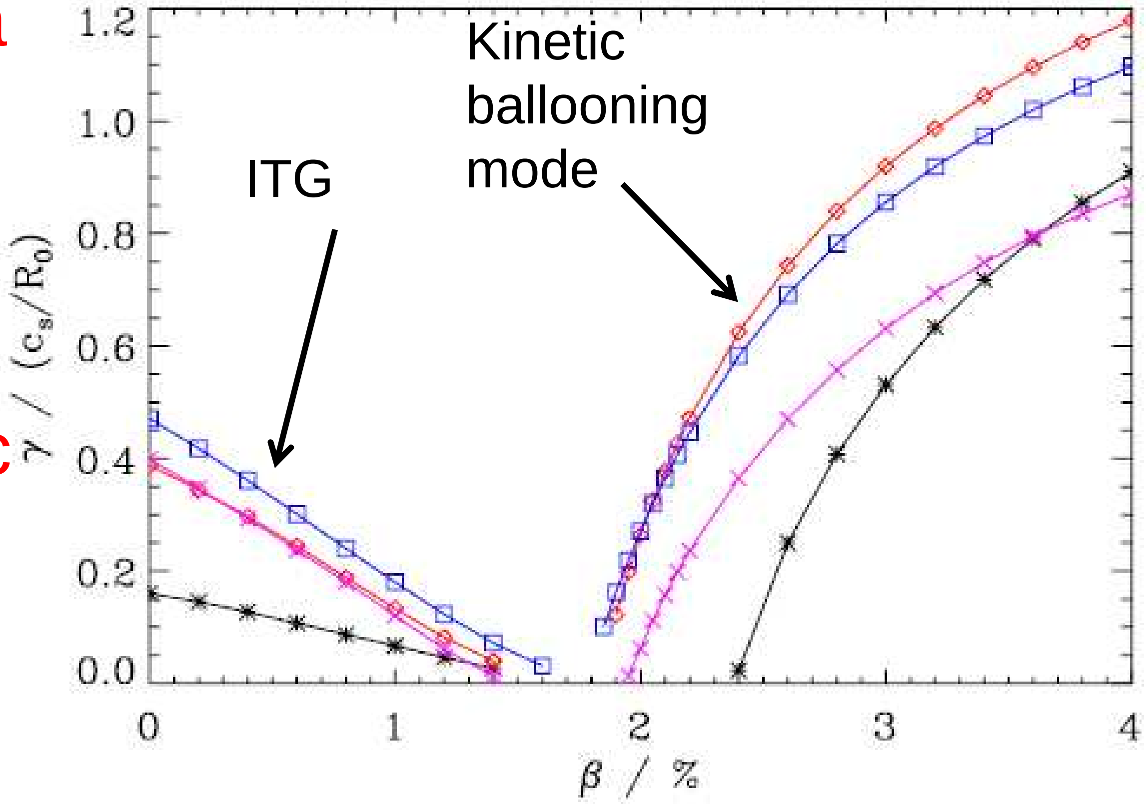
Connection with MHD

Pueschel 2010

- Drift waves dominate at low beta

$$\beta = \frac{2P}{B^2/2\mu_0}$$

- At high beta, kinetic ballooning modes become unstable



Hybrid formulation for kinetic MHD modes

Two options:

1) Solve the full kinetic problem (high k)

2) Hybrid formulation (low k)

$$\rho d_t \mathbf{V} = -\nabla P + \mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbf{\Pi}_{hot}$$

Non thermal particle stress tensor

$$\Pi_{ij} = m \int d^3\mathbf{v} f v_i v_j$$

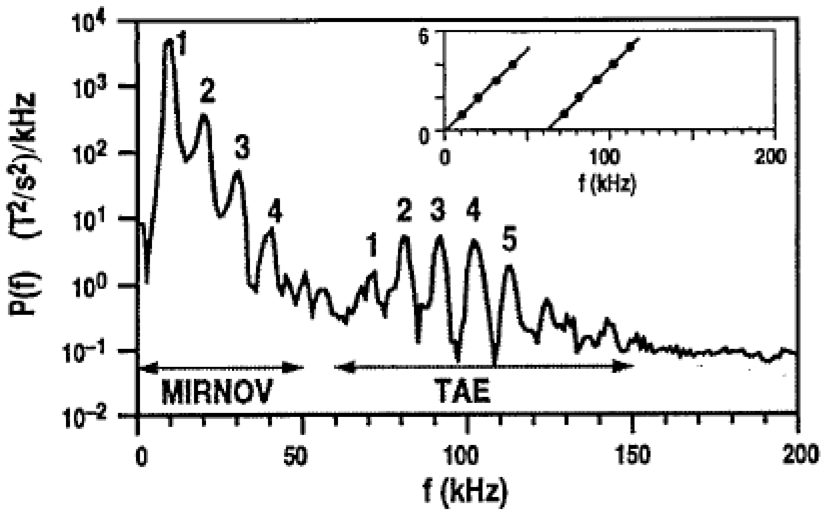
→ imaginary part of δW

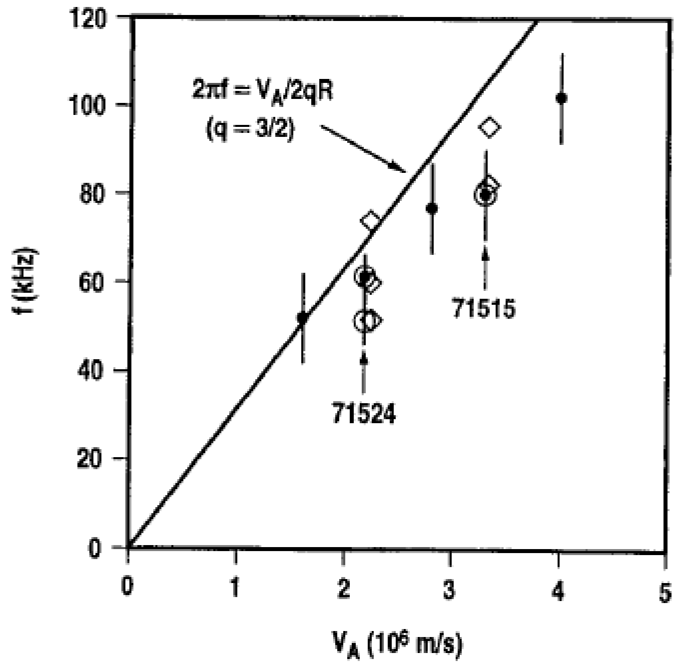
→ kinetic instabilities

Alfvén waves driven by fast particles

- Driving mechanism similar to bump on tail
- Quasi-coherent modes
- In some cases, frequency chirping

Turnbull 1993





Non linear saturation

- Variety of non linear dynamics:

1) Few modes : steady saturated state, relaxation oscillations, explosive behavior, …

2) Many coupled modes : usually evolve towards turbulence. Turbulent state is different if waves are involved.

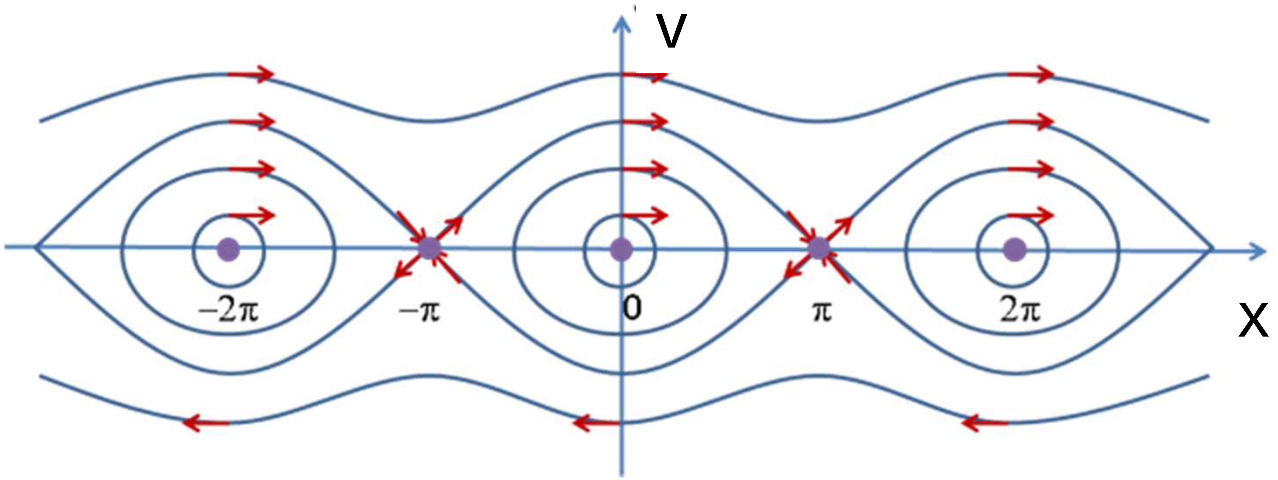
- Bump on tail instability is the testbed here

One mode only: particle trapping

- Particle energy in the wave frame of reference

$$E = \frac{1}{2}mv^2 + e\phi\cos(kx)$$

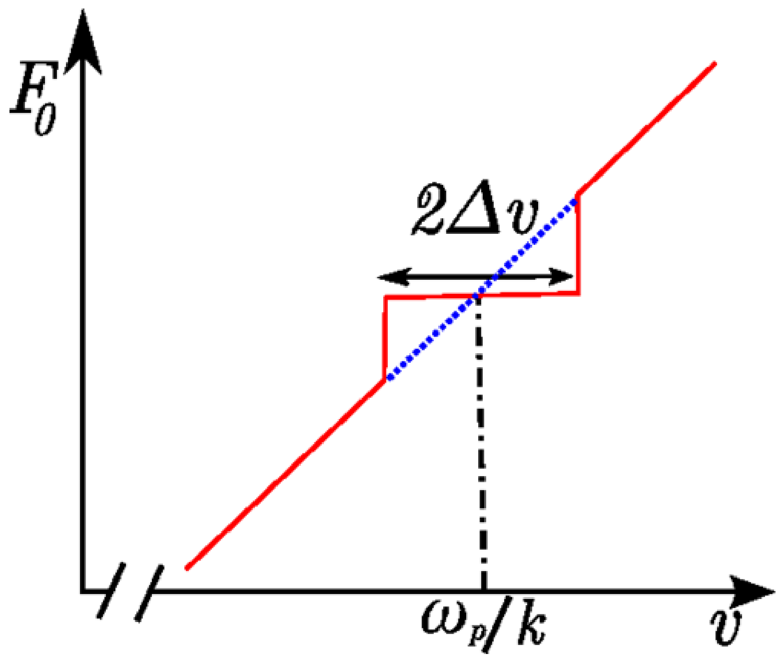
- Similar to a pendulum – trapping time $\tau_b^{-1} = \sqrt{\frac{ek\mathbf{E}_{\mathbf{k}\omega}}{m}}$



Plateau effect

- F(E) is solution of the Vlasov equation

- Flattening of the distribution in the unstable region → stabilisation Berk & Breizman 97

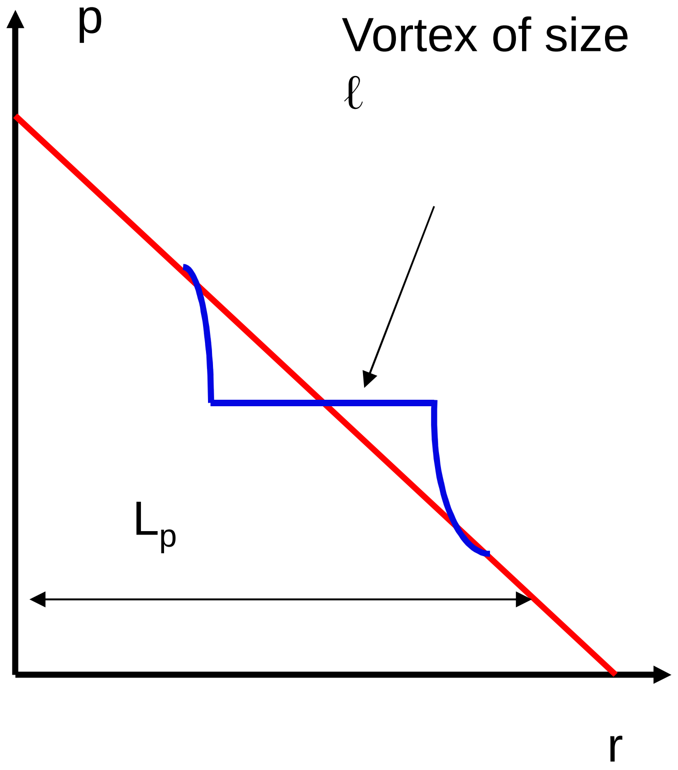


Analogy with vortex mixing : mixing-length estimate

Mixing of the pressure profile by vortex of size ℓ

$\rightarrow$ "mixing length estimate" of the fluctuation level

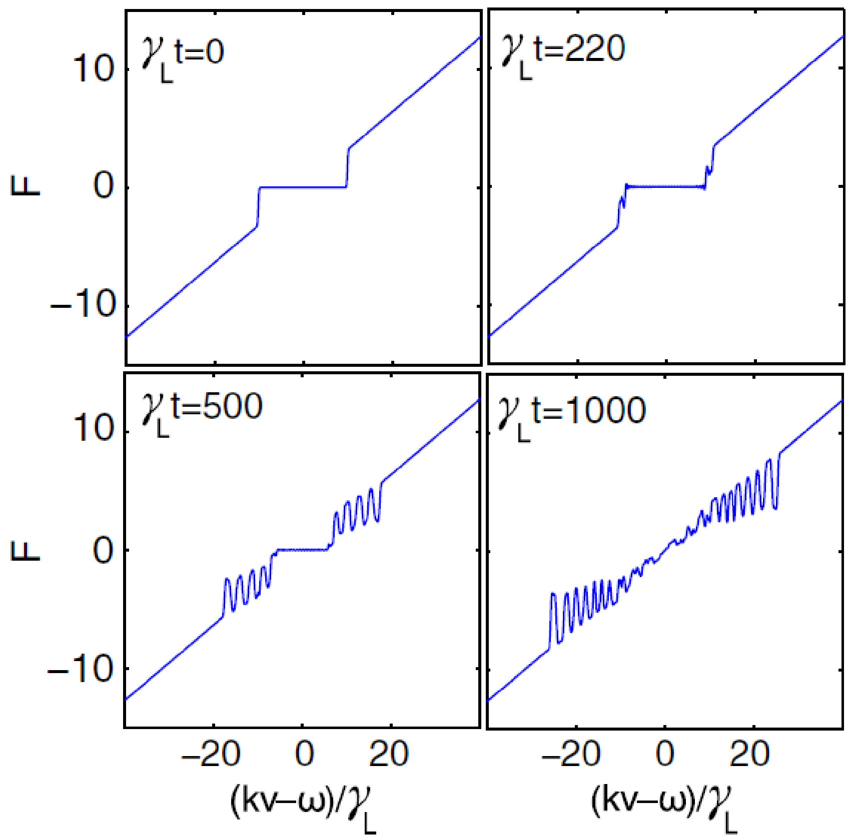
$$\frac{\delta p}{p} \approx \frac{\ell}{L_p}$$



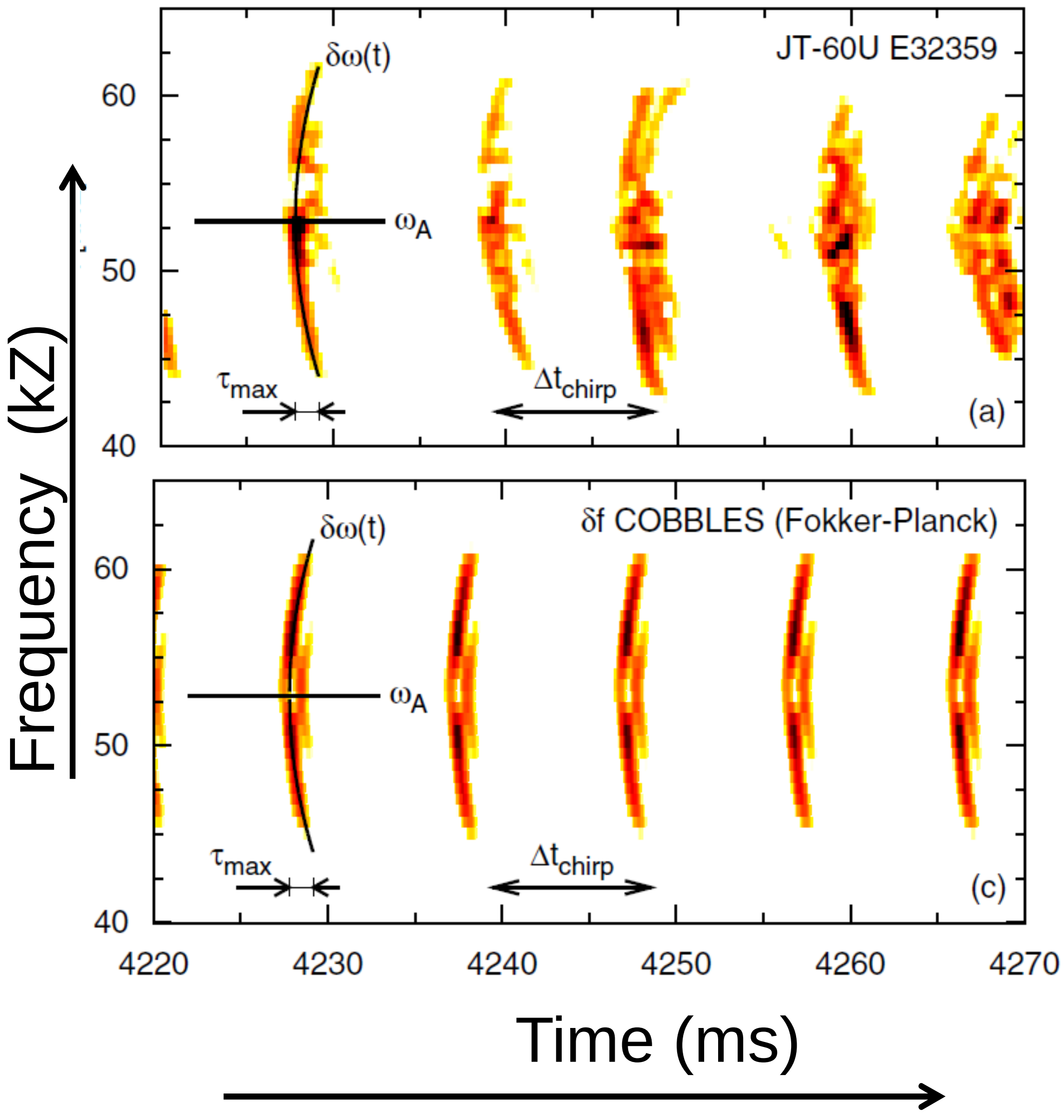
Plateau can generate a secondary instability

Lilley 15

- Edges of plateau can be unstable Lilley 15
- Plateau splits in holes an clumps
- Motion of holes/clumps ir phase space → chirping
  Berk&Breizman 97



Frequency chirping observed both in experiments and simulations



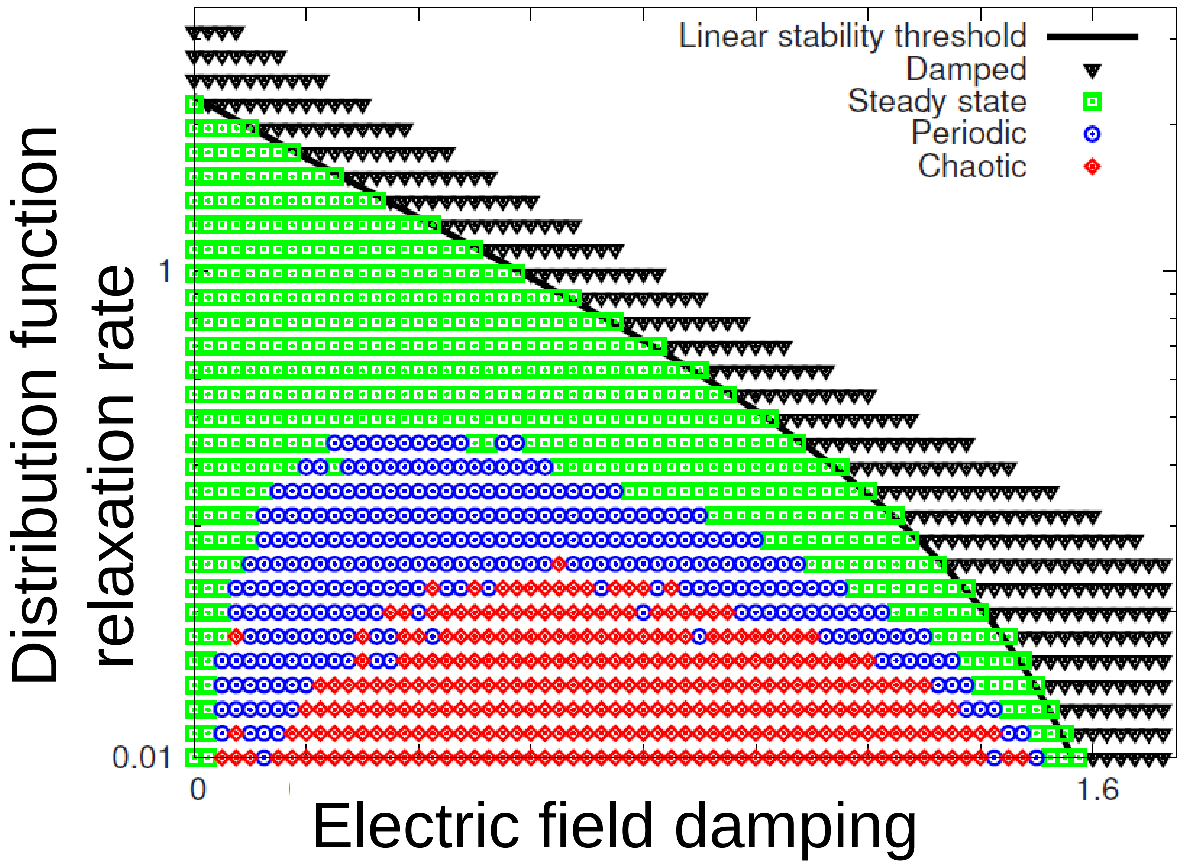
Lesur 12

Several pathways towards relaxation

Vann 03, Lesur 09

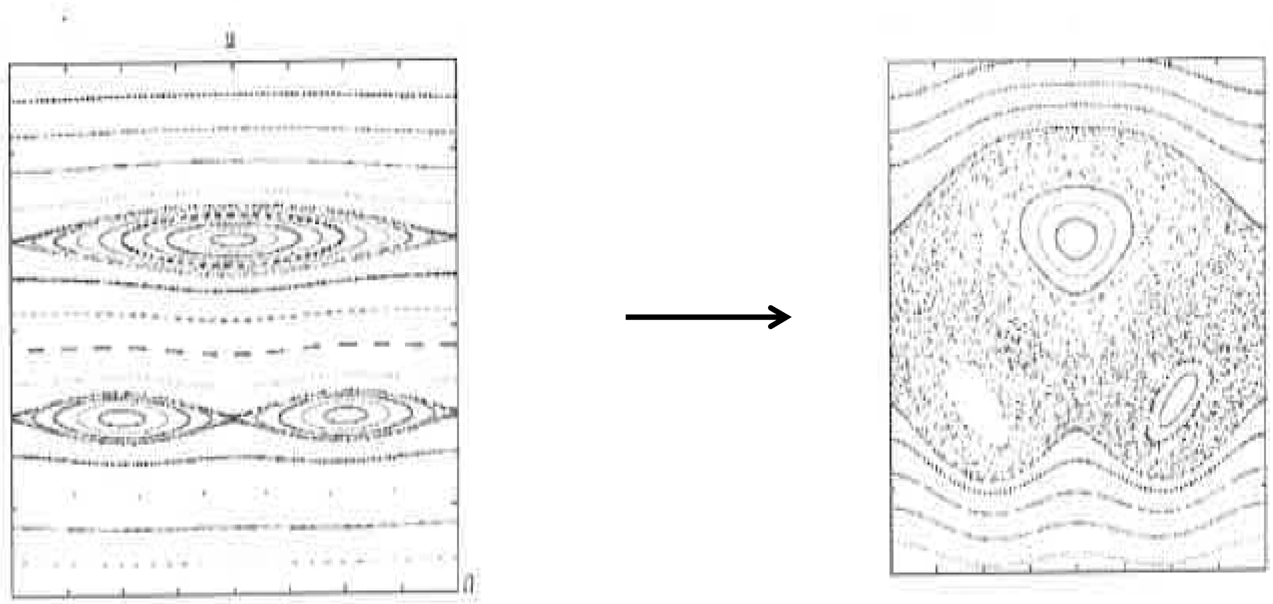
- Bump on tail: variety of dynamics depending on dissipation and drive

- Limit of strong drive still to be explored Zonca 15



Stochasticity

- Multiple modes: islands localized around $v=\omega_p/k$
- Trajectory becomes stochastic → ergodization → flattening Chirikov 59, see Lichtenberg &Liberman,1983



- Linear solution Vlasov

$$f_{\mathbf{k}\omega} = -i\frac{e\mathbf{E}_{\mathbf{k}\omega}}{m}\frac{\partial_v f_0}{\omega - kv}$$

- Evolution equation of f_0 in velocity space

$$\partial_t f_0 + \partial_v \Gamma = 0$$

Average distribution function

Flux in phase space

$$f_0(v,t) = \int \frac{dx}{2\pi} f(x,v,t)$$

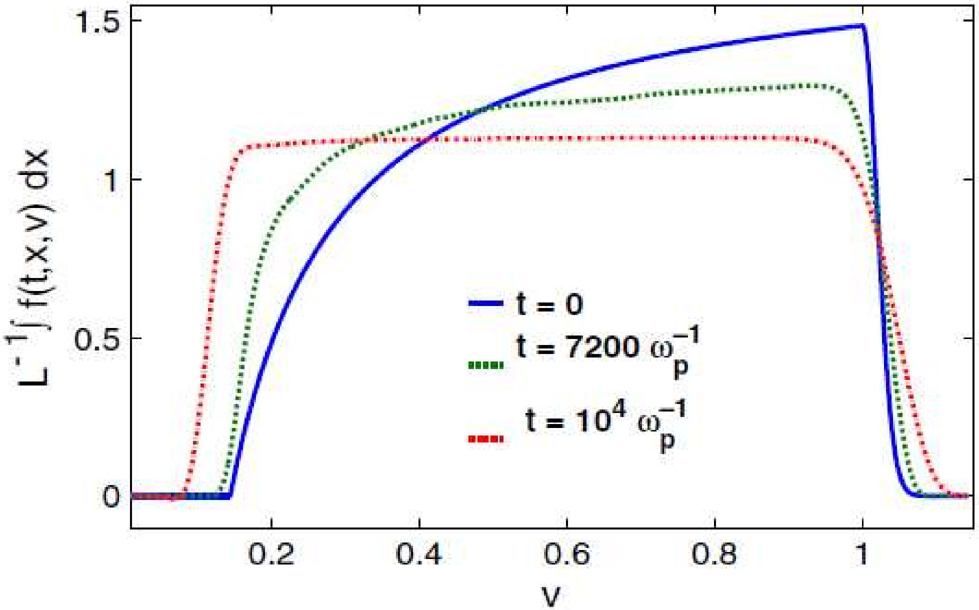
$$\Gamma = \sum_{\mathbf{k},\omega} \frac{e\mathbf{E}^\star_{\mathbf{k}\omega}}{m} f_{\mathbf{k}\omega}$$

Quasi-Linear Theory (cont.)

- Linear solution of Vlasov equation → flux

$$\Gamma = -D_{QL}\partial_v f_0$$

- Quasi-linear diffusion coefficient



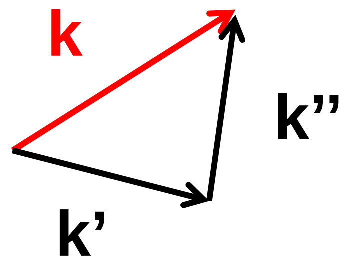
$$D_{QL} = \sum_{\mathbf{k},\omega} \left| \frac{e\mathbf{E}_{\mathbf{k}\omega}}{m} \right|^2 \frac{\gamma(k)}{\left(\omega_r(k) - kv\right)^2 + \gamma^2(k)}$$

Besse 2011

- Often effective beyond validity conditions cf lecture Gürcan

Mode coupling is needed to compute the turbulence intensity

- Generic form of a non linear equation in Fourier space

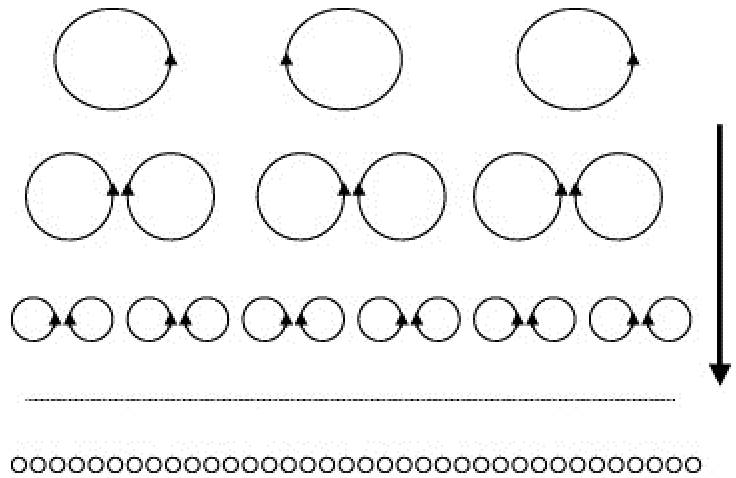


$$\phi(\mathbf{x},t) = \sum_{\mathbf{k}} \phi_{\mathbf{k}}(t) \exp\{i\mathbf{k}\cdot\mathbf{x}\}$$

$$\partial_t \phi_{\mathbf{k}}(t) = -i\omega(\mathbf{k})\phi_{\mathbf{k}}(t) + \sum_{\mathbf{k}'\mathbf{k}''} \Lambda_{\mathbf{k}'\mathbf{k}''}\phi_{\mathbf{k}'}(t)\phi_{\mathbf{k}''}(t)$$

- Triad $\mathbf{k}' + \mathbf{k}'' = \mathbf{k}$

- Coupling leads to energy transfer between waves – if coupling is ''local'' $\rightarrow$ cascade



Case of drift waves

- Coupling term for CHM equation

$$\Lambda_{\mathbf{k}'\mathbf{k}''} = -\frac{1}{2}\frac{\rho_s^2}{1+k_\perp^2\rho_s^2}\mathbf{e}_z\cdot\left(\mathbf{k}'\times\mathbf{k}''\right)\left(k_\perp''^2 - k_\perp'^2\right)$$

- Conserves energy and enstrophy

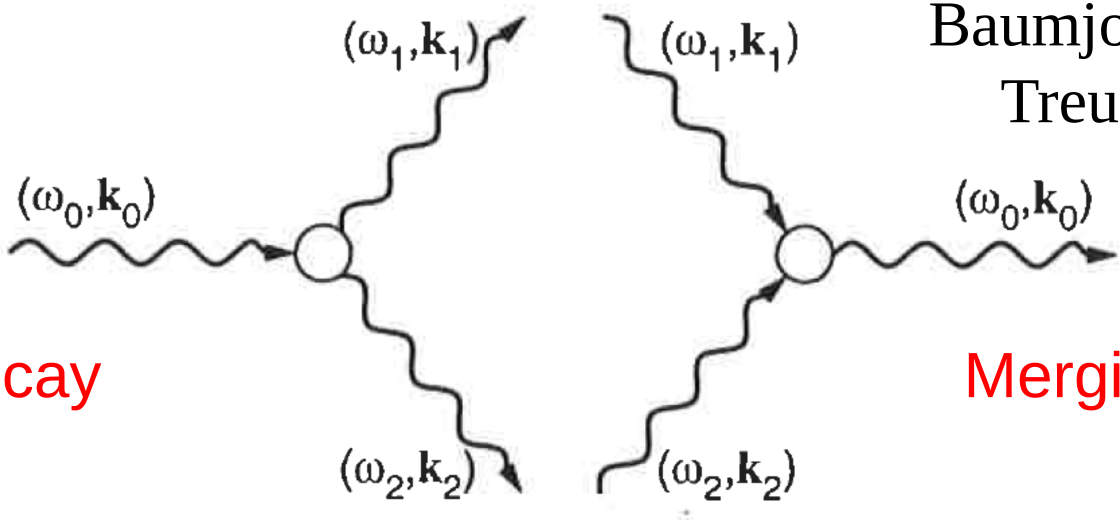
$$E = \frac{1}{2}\sum_{\mathbf{k}}\left(1+\rho_s^2 k_\perp^2\right)\left|\phi_{\mathbf{k}}(t)\right|^2$$

$$\Omega = \frac{1}{2}\sum_{\mathbf{k}}\left(1+\rho_s^2 k_\perp^2\right)k_\perp^2\left|\phi_{\mathbf{k}}(t)\right|^2$$

3 modes only : decay instability

- Decay of a « pump » mode ($\mathbf{k}_0,\omega_0$) into two « daughter » waves ($\mathbf{k}_1,\omega_1$), ($\mathbf{k}_2,\omega_2$)

- **Constraint** $\mathbf{k}_0 = \mathbf{k}_1 + \mathbf{k}_2$; $\omega_0 = \omega_1 + \omega_2$



Baumjohann & Treumann 2012

Decay

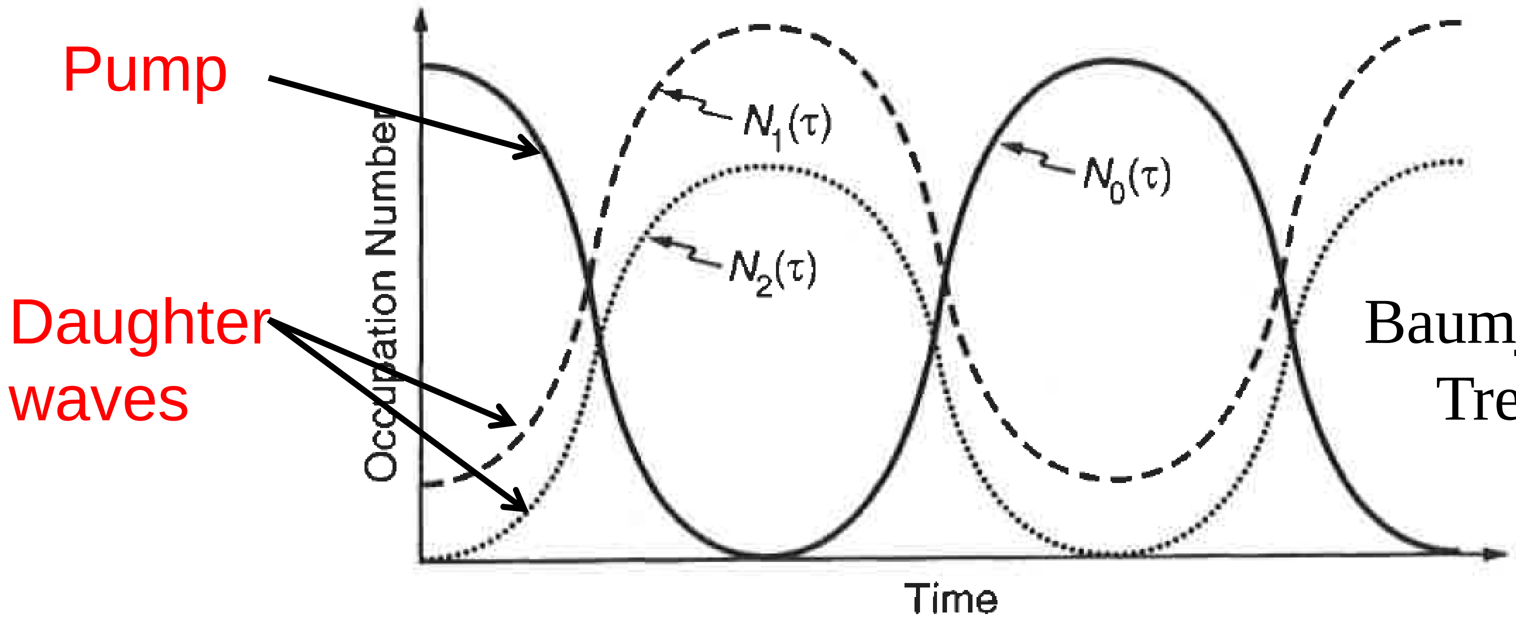
Merging

- Drift waves: decay possible if $k_1^2 < k_0^2 < k_2^2$

Decay instability (cont.)

- If energy is strictly conserved : pump recovers

→ dissipative processes make it irreversible

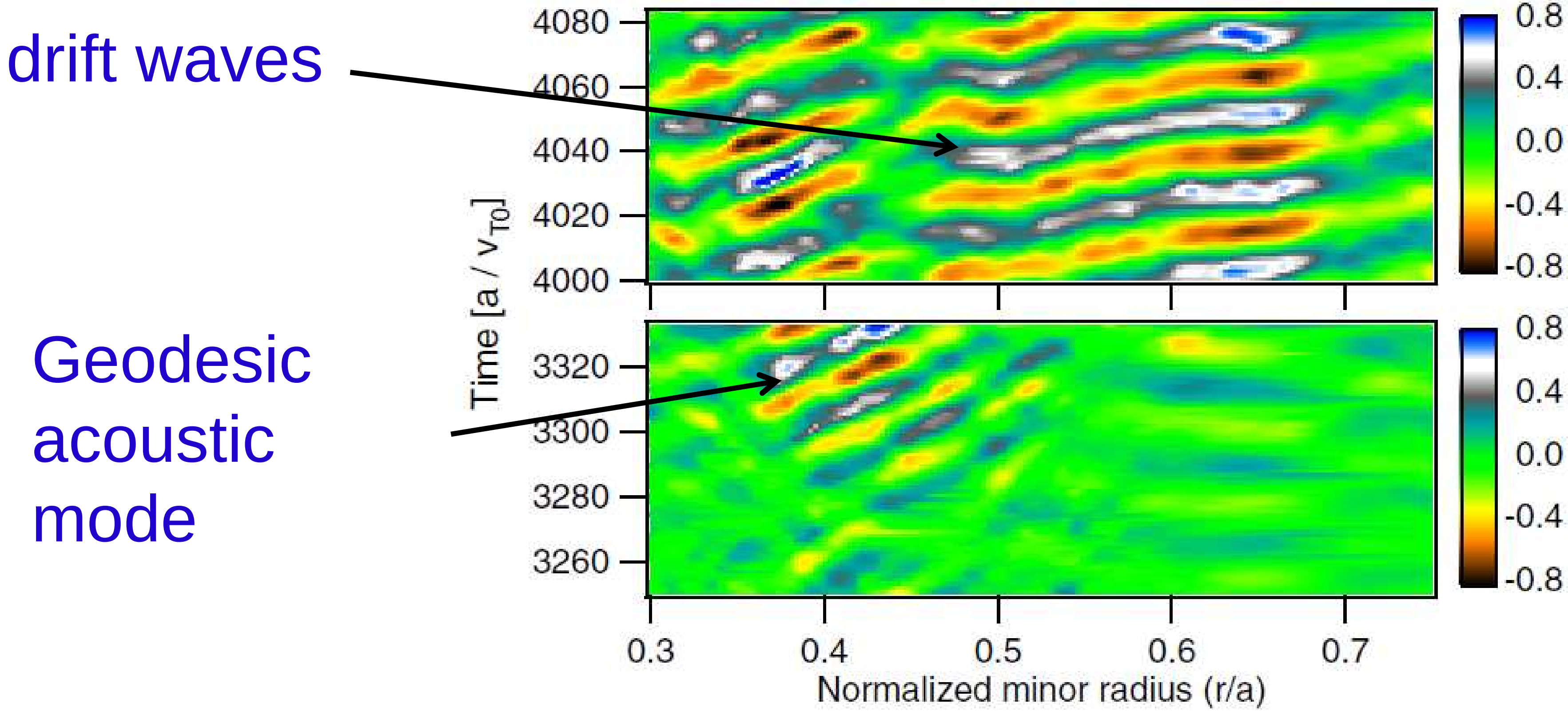


Baumjohann & Treumann 2012

Example of a parametric decay of an acoustic wave into two drift waves

- Geodesic acoustic mode driven by energetic ions
- Parametric decay into drift (ITG) waves

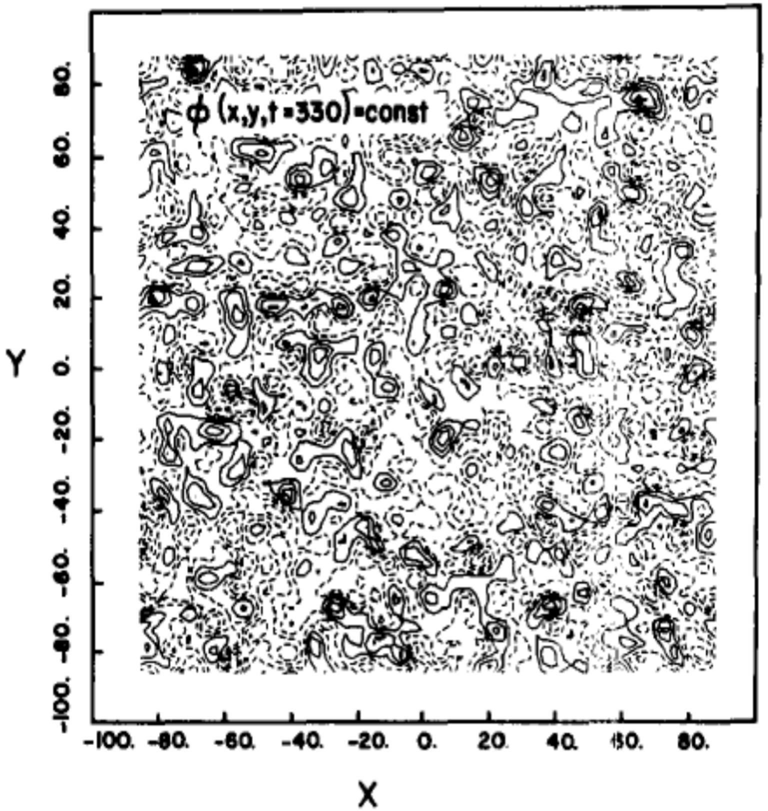
Zarzoso 13



Many unstable modes: turbulence

- If many modes are unstable : the system evolves towards a turbulent state Waltz 83, Horton 86
- Can be seen as a strange attractor in the phase space $\{\phi_k\}$

Horton 86

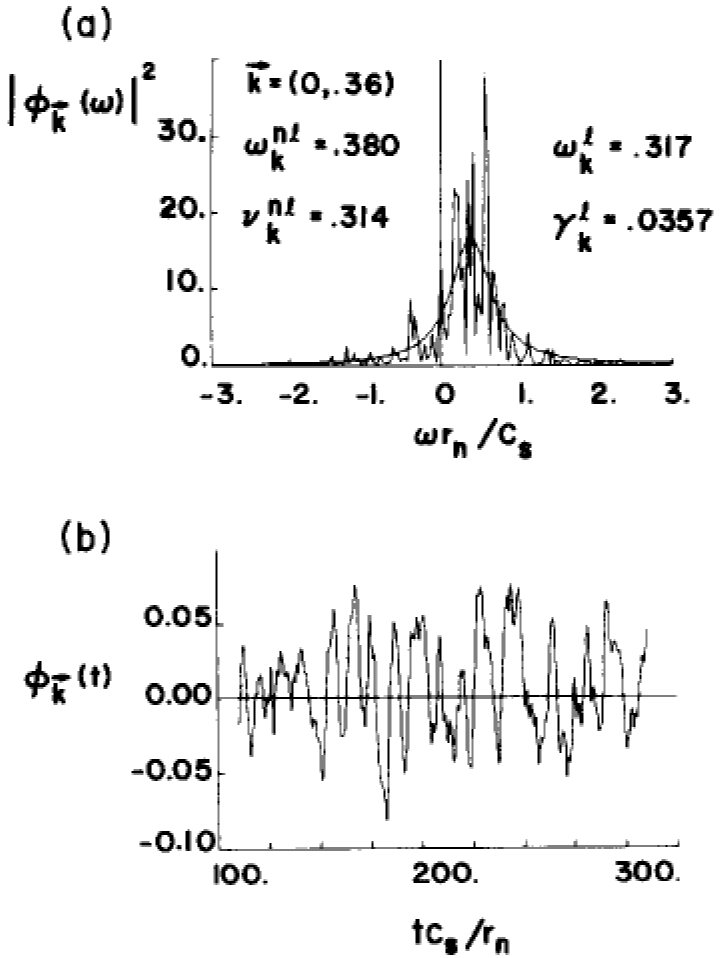


Frequency spectra

Horton 86

- Frequency spectra are broad

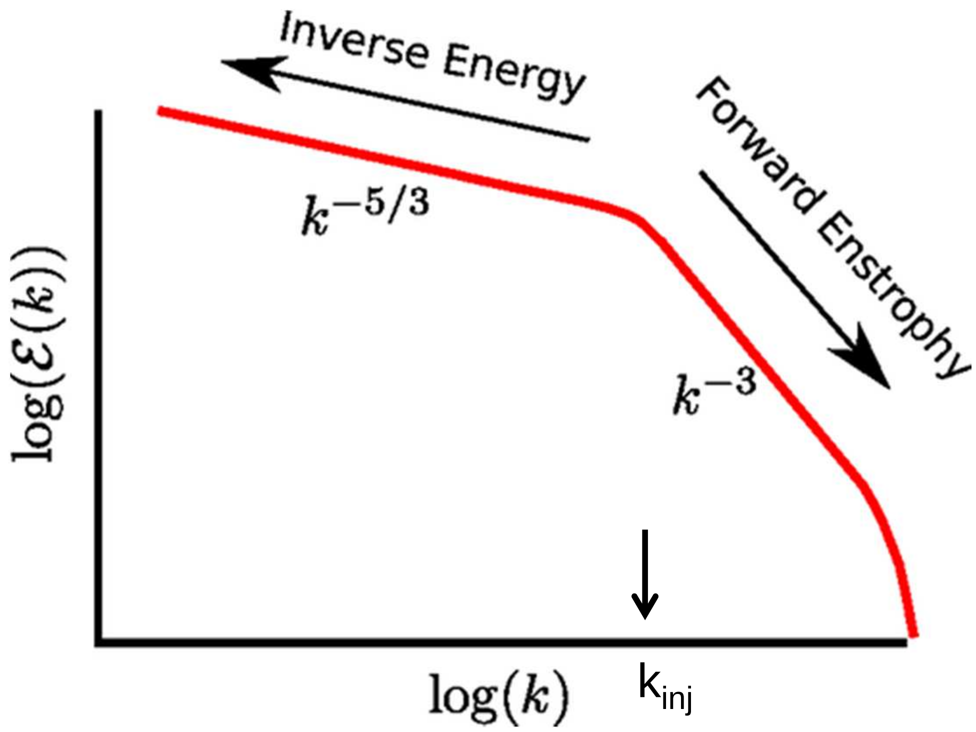
- Predictive model of the frequency spectrum shape still an open question



A few remarks on spectra and transport

- Quasi-linear theory works well for drift waves

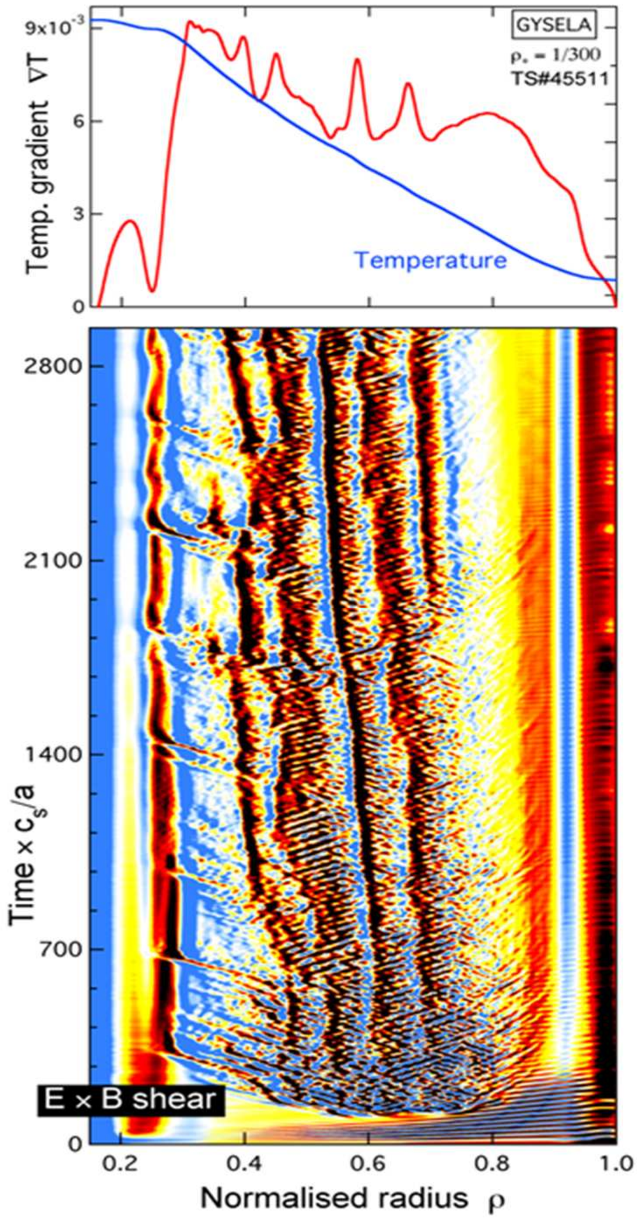
- A dual cascade is expected CHM
- Does not fit observation in tokamaks (see lecture Vermare) – many reasons:
- no inertial range
- coupling to large scale flows
- kinetic effects



A few words on wave turbulence

- Waves change the nature of turbulence see lecture Nazarenko
- Drift or Rossby waves:
- not isotropic
- generation of large scale shear flows → self-regulation

Dif-Pradalier 15



Conclusions

- Linear stability is well documented – not that simple though …
- Non linear dynamics much more complex – No simple recipe !
- For turbulent states and wave/particle interaction via Landau resonances: quasi-linear theory often works well
- Predicting a level of fluctuations is trickier
- For quasi-coherent modes: variety of non linear dynamics – parameter dependent.

Some useful textbooks

- D. B. Melrose “Instabilities in Space and Laboratory Plasmas” Cambridge UP 1986

- W. Baumjohann and R.A. Treumann “Basic Space Plasma Physics” Imperial UP 2012 vol I and II

- P. H. Diamond , S.-I. Itoh, K. Itoh, “Physical Kinetics of Turbulent Plasmas” Cambridge UP 2010

- D. Biskamp “Nonlinear Magnetohydrodynamics », 1997 Cambridge UP

- A.J. Lichtenberg and M.A. Liberman, “Regular and stochastic motion” Springer 1983

- Y. Elskens and D. Escande “Microscopic dynamics of plasmas and chaos”, IOP 2003