Particle transport due to electrostatic waves

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Topics

1. Introduction
2. $E \times B$ drift
3. Two wave hamiltonian
4. Transport
5. Two wave system - Results
6. Conclusions
7. Next steps

Introduction

Introduction

- Plasmas
- Tokamaks
- Edge transport

Plasma confinement

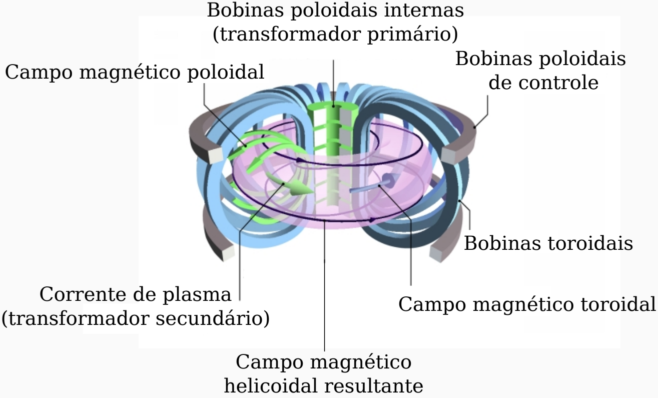


Figure 1: The magnetic fields due to the coils

$E \times B$ drift

$\vec{B} = B\hat{z}$ and $\vec{E} = 0$

$$\vec{F} = m\frac{d\vec{v}}{dt} = q(\vec{v} \times \vec{B}) \tag{1}$$

and

$$x - x_0 = r_L \sin(\omega_c t) \qquad y - y_0 = r_L \cos(\omega_c t) \tag{2}$$

x_0 e y_0 are the guiding centers of the motion.

ω_c and r_L are the gyrofrequency and the Larmor radius.

$$\omega_c = \frac{|q|B}{m} \qquad r_L = \frac{mv_\perp}{|q|B} \tag{3}$$

$\vec{B} = B\hat{z}$ and uniform $\vec{E}$

Along $\hat{z}$:

$$\dot{v}_z = \frac{q}{m} E_z \to v_z = \frac{qE_z}{m} t + v_{z0} \tag{4}$$

In $x - y$ plane:

$$m\dot{v}_x = \frac{q}{m} E_x + \omega_c v_y qBv_y \qquad m\dot{v}_y = qBv_x \tag{5}$$

$$v_x = v_\perp e^{i\omega_c t} \qquad v_y = iv_\perp e^{i\omega_c t} - \frac{E_x}{B} \tag{6}$$

$$v_\perp = \sqrt[2]{v_x^2 + v_y^2} \tag{7}$$

with drift velocity

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2} \tag{8}$$

Illustration of the $E \times B$ drift

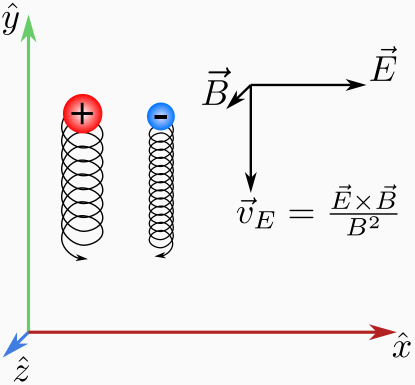


Figure 2: Electric drift for two different kinds of particles

Hamiltonian model model for the guiding centers

$$\vec{B} = B_0 \vec{z} \qquad \vec{E} = -\nabla\phi(x, y, t) \tag{9}$$

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2} \tag{10}$$

$$v_x = \frac{dx}{dt} = -\frac{1}{B_0}\frac{\partial}{\partial y}\phi(x, y, t) \qquad v_y = \frac{dy}{dt} = \frac{1}{B_0}\frac{\partial}{\partial x}\phi(x, y, t) \tag{11}$$

$$\frac{dx}{dt} = -\frac{\partial}{\partial y}H(x, y, t) \qquad \frac{dy}{dt} = \frac{\partial}{\partial x}H(x, y, t) \tag{12}$$

$$H(x, y, z) = \frac{\phi(x, y, z)}{B_0} \tag{13}$$

x and y are canonical conjugates

Potential and reference frame

$$\phi(x,y,t) = \phi_0(x) + \sum_i A_i \sin(k_{xi}x + \theta_{xi}) \sin(k_{yi}y - \omega_i t + \theta_{yi}) \qquad (14)$$

For a single wave:

$$H(x,y,t) = \phi_0(x) + A_1 \sin(k_{x1}x) \sin(k_{y1}y - \omega_1 t) \qquad (15)$$

Change the reference frame through the canonical transformation:

$$F_2(x', y) = x'(y - v_1 t) \qquad v_1 = \frac{\omega_1}{k_{y1}} \qquad (16)$$

Equations of motion for a single wave

$$H(x,y) = \phi_0(x) - v_1 x + A_1 \sin(k_{x1}x)\sin(k_{y1}y) \tag{17}$$

$$\frac{dx}{dt} = -k_{yi}A_i \sin(k_{xi}x)\cos(k_{yi}y) \tag{18}$$

$$\frac{dy}{dt} = \left[\frac{d\phi_0}{dx} - v_1\right] + k_{xi}A_i \cos(k_{xi}x)\sin(k_{yi}y) \tag{19}$$

Control parameter $U(x)$

We can also define a control profile $U(x)$:

$$U(x) = \frac{1}{A_1 k_{x1}} \left[\frac{d\phi_0}{dx} - B_0 v_1 \right] = \frac{B_0}{A_1 k_{kx1}} \left[-\frac{E(x)}{B_0} - v_1 \right] \qquad (20)$$

$$U(x) = \frac{B_0}{A_1 k_{x1}} \left[v_E(x) - v_1 \right] \qquad (21)$$

When $U = 0$, we have a resonance, where the phase velocity of the wave is the same as the electric drift.

From now on $U(x) = U = cte$

About some parameters k_x, k_y

Restrictions on k_x and k_y to satisfy space periodicity:

$$k_y = \frac{2\pi n}{L_y} \qquad k_x = \frac{2\pi m}{L_x} \qquad n, m \in \mathrm{Z} \tag{22}$$

with L_x and L_y, being the x and y period.

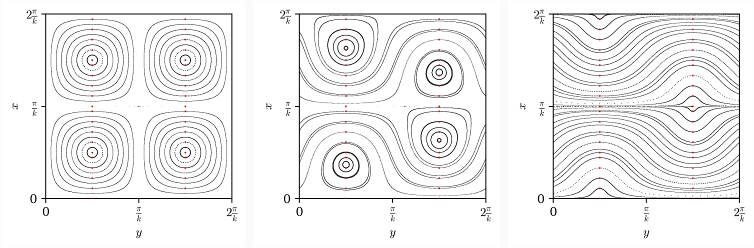
$L_{x,y} = 2\pi$ so that $k_{x,y}$ are integers

In the resonance, $U = 0$, the trajectories structure themselves with a lattice of elliptic and hyperbolic points

$$P_H = \left(\frac{(2m+1)\pi}{2k_x}; \frac{(2n+1)\pi}{2k_y} \right) \qquad P_E = \left(\frac{m\pi}{k_x}; \frac{n\pi}{k_y} \right) \tag{23}$$

Trajectories

As $|U|$ increases, the structure of the cells changes



(a) $U = 0.0$ **(b)** $U = 0.4$ **(c)** $U = 1.0$

Figure 3: Guiding centers trajectories for a single wave

Two wave hamiltonian

Hamiltonian

$$H(x,y,t) = \phi_o + A_1 \sin(k_{x1}x)\sin(k_{y1}y - \omega_1 t) + A_2 \sin(k_{x2}x + \theta_x)\sin(k_{y2}y - \omega_2 t) \tag{24}$$

With the new reference frame

$$H(x,y,t) = \phi_o - v_1 x + A_1 \sin(k_{x1}x)\sin(k_{y1}y) + A_2 \sin(k_{x2}x + \theta_x)\sin(k_{y2}(y - vt)) \tag{25}$$

$$v = \frac{\omega_2}{k_{y2}} - \frac{\omega_1}{k_{y1}} \qquad \tau = \frac{2\pi}{k_{y2}v} \tag{26}$$

If $v \neq 0$ the system is no longer integrable. τ is the period of the perturbation

Equations of motion

$$\frac{dx}{dt} = -k_{y1}A_1 \sin(k_{x1}x)\cos(k_{y1}y) - k_{y2}A_2 \sin(k_{x2}x + \theta_x)\cos(k_{y2}(y - vt)) \quad (27)$$

$$\frac{dy}{dt} = \left[\frac{d\phi_0}{dx} - v_1\right] + k_{x1}A_1 \cos(k_{x1}x)\sin(k_{y1}y) + k_{x2}A_2 \cos(k_{x2}x + \theta_x)\sin(k_{y2}(y - vt)) \quad (28)$$

Transport barriers with $U = 0$

If there is some value $x = x_b$ such that

$$\sin(k_{x1} x_b) = \sin(k_{x2} x_b + \theta_x) = 0 \tag{29}$$

$$x_b = \frac{n_1 \pi}{k_{x1}} \ , \ x_b = \frac{n_2 \pi}{k_{x2}} - \frac{\theta_x}{k_{x2}}, \ n_1, n_2 \in \mathbb{Z} \tag{30}$$

$$n_2 - n_1 \frac{k_{x2}}{k_{x1}} = \frac{\theta_x}{\pi} \tag{31}$$

If k_{x1} and $k_{x2} \in \mathbb{Z}$ barriers exist only if $\theta_x = n_3 \pi$, $n_3 \in \mathbb{Z}$

Transport

Transport characterization

One of many ways to characterize the regime is through the mean square displacement

$$\langle \sigma(t)^2 \rangle = \frac{1}{N} \sum_{i}^{N} (x_i(t) - x_i(t_0))^2 \approx C t^{\gamma} \tag{32}$$

subdiffusive	diffusive	superdiffusive
$\gamma < 1$	$\gamma = 1$	$\gamma > 1$

If $\gamma = 1$

$$\langle D_x(t) \rangle = \frac{1}{2tN} \sum_{i} (x_i(t) - x_i(t_0))^2 \tag{33}$$

Computational details

- C/C++
- Python: OpenCV, Scikit
- 4th order Runge-Kutta
- $\delta t = \tau \times 10^{-3}$

Two wave system - Results

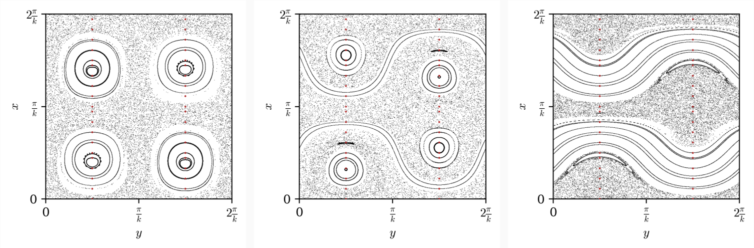
Two wave parameters

i	A	ω	k_x	k_y	v_y	θ_x
1	1	3	3	3	1	0
2	-	6	3	3	2	$\pi/2$

Table 1: Numeric parameters for the simulations with two waves.

Stroboscopic map - Control parameter U

To produce the stroboscopic map, we integrate for an initial condition, and mark the position at each period of the perturbation. Here we see the influence of the control parameter U

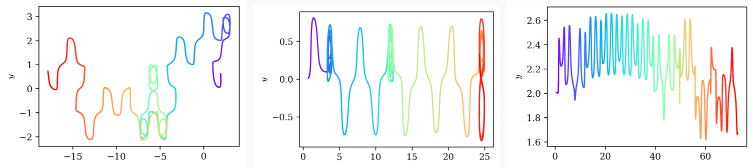


(a) $U = 0.0$ **(b)** $U = 0.4$ **(c)** $U = 1.0$

Figure 4: Stroboscopic maps for $A_2 = 0.3$. Initial conditions in red.

Individual trajectories

We see the influence of U by taking the same initial condition.



(a) $U = 0.0$ **(b)** $U = 0.4$ **(c)** $U = 1.0$

Figure 5: Trajectories for the same initial condition. Color represents time

Mean square displacement

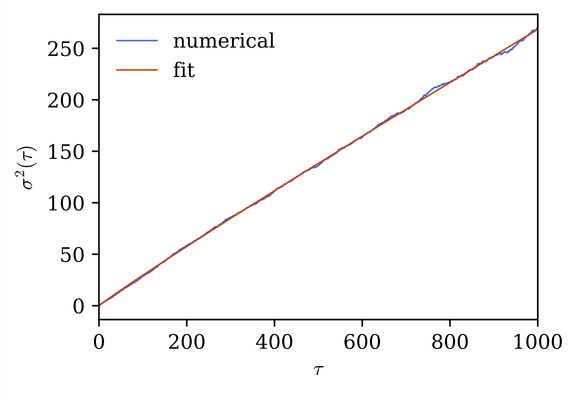


Figure 6: Mean square displacement, for $U = 0.0$, $A_2 = 0.3$, $\theta_x = \pi/4$, $\gamma = 0.968$

Diffusion in relation to U

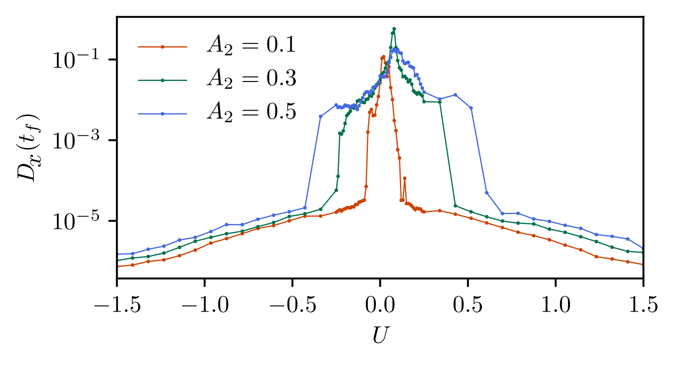


Figure 7: D_x for different values of U and A_2

Presence of anomalous transport

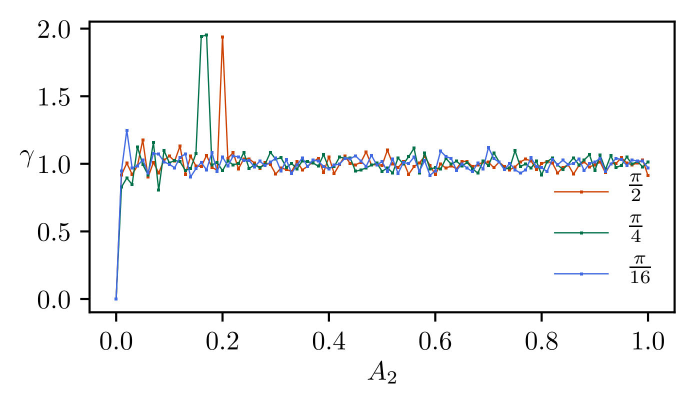
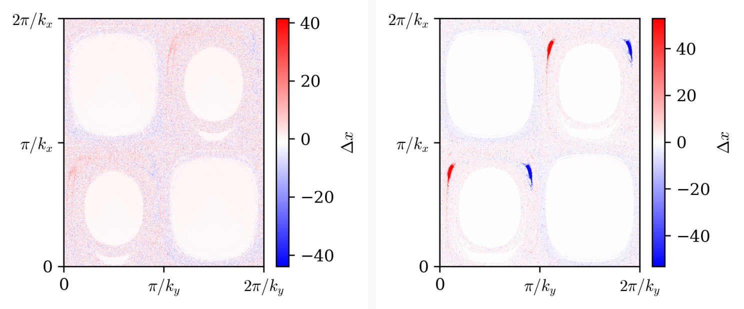


Figure 8: γ in relation to A_2; for some values superdiffusion is present

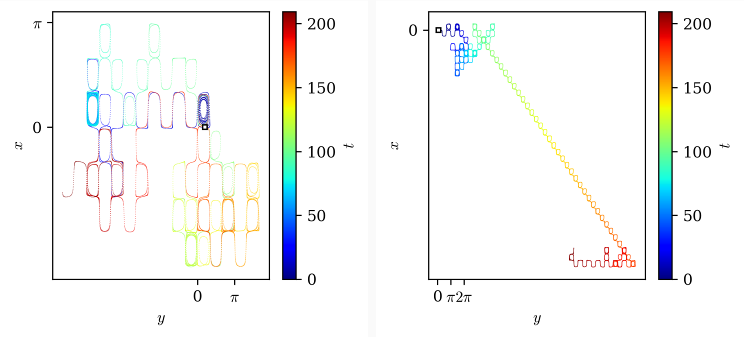
What is causing it?



(a) $A_2 = 0.20$ **(b)** $A_2 = 0.16$

Figure 9: displacement on the phase space after 50 iterations, in color. $\theta_x = \frac{\pi}{4}$

What is causing it?



(a) $A_2 = 0.20$

(b) $A_2 = 0.16$

Figure 10: Individual trajectories. $\theta_x = \frac{\pi}{4}$

Segment and test - Why and how to do it?

- Diffusion is expensive
- Ballistic modes are sufficient condition for anomalous transport
- Adaptive

	Dsplacement	γ evaluation	Segment and test
Parameters	$N_x \times N_y \times N_{it}$	$N \times N_{it}$	$N \times N_{it}$
Order	$10^3 \times 10^3 \times 10^2$	$10^3 \times 10^4$	$10^2 \times 10^4$
Order	10^8	10^7	10^6

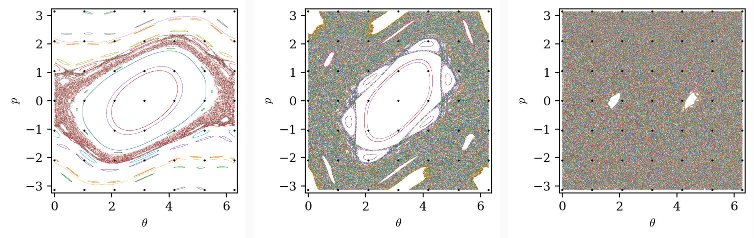
Table 2: Approximated iterations of some methods to identify anomalous transport

General idea

1. Pick some initial conditions ≈ 100
2. Create the stroboscopic map
3. Separate regions with morphology
4. Test the regime in each region

Standard map as a benchmark

$$p_{n+1} = p_n + K\sin(\theta_n) \quad , \quad \theta_{n+1} = \theta_n + p_{n+1} \mod(2\pi) \tag{34}$$



(a) $K = 0.9$ **(b)** $K = 1.5$ **(c)** $K = 6.5$

Figure 11: Phase space for some values of K.

ST - Standard map - Segmentation

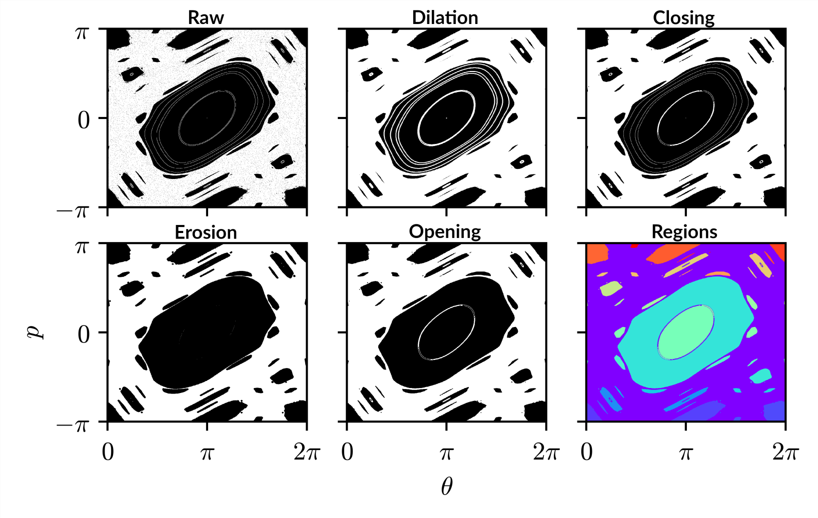


Figure 12: Morphological steps for segmentation

ST - Standard map - Categorization

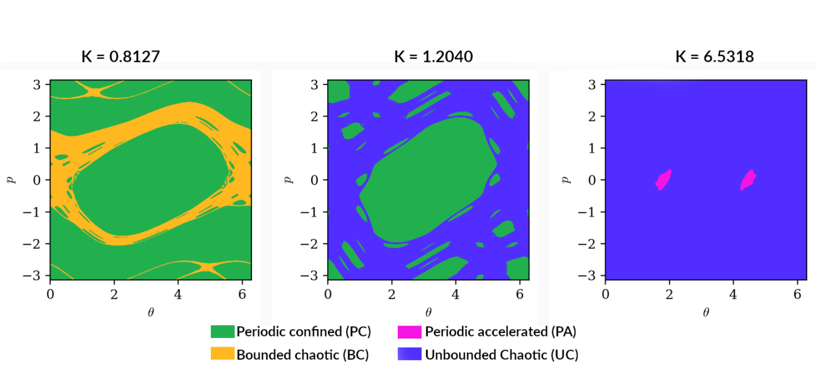


Figure 13: Categorized regions

ST - Standard map - Anomalous transport

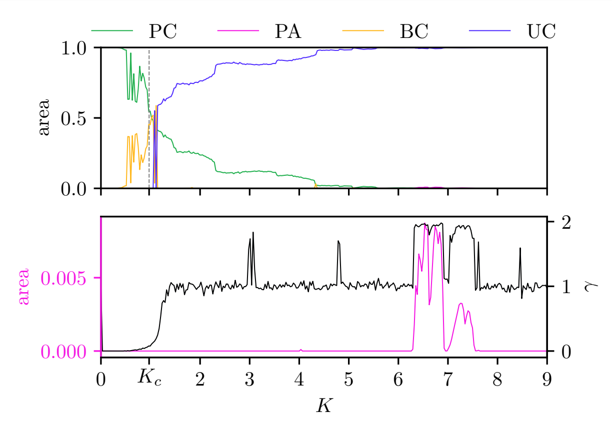
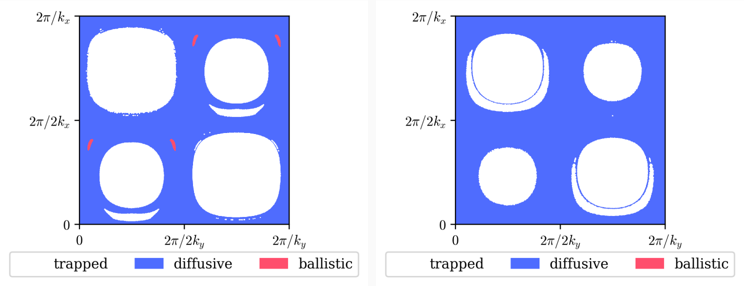


Figure 14: Existence of anomalous transport when accelerator modes are present



(a) $A_2 = 0.16$ **(b)** $A_2 = 0.2$

Figure 15: Categorization on the two wave system. $\theta_x = \frac{\pi}{4}$.

ST - Two wave system

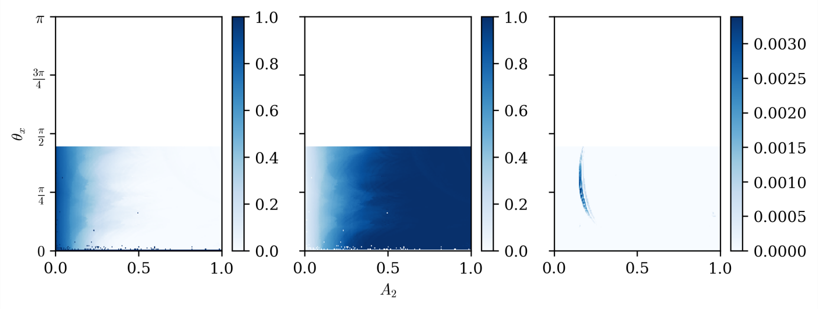


Figure 16: Categorized regions; Estimated total wall clock simulation time: 20 Days

Conclusions

Conclusions

- Investigation of the two wave system
- Identification of ballistic modes
- Development and application of ST approach

Next steps

Next Steps

- Explore the control parameter U
- Explore the influence of k_x and k_y
- System with more waves
- Can anomalous transport happen for $U \neq 0$?

Thanks :)