# Tropical Geometry and Tropical Bézout's Theorem


Amir Faris bin Mohd Kamri

Supervisor: Dr. Dmitri Panov

King's College London
Department of Mathematics


7CCM461A MSci Project

April 2021

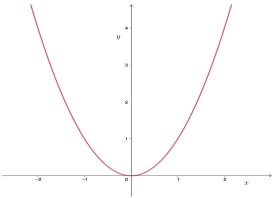
# Motivation

Why tropical geometry?

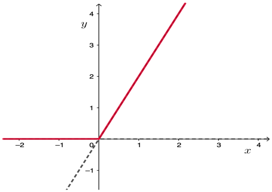
- Connects algebraic geometry and combinatorics
- Conserves a lot of definitions, statements, and properties from algebraic geometry
- Provides an easier alternative to understand complicated results in algebraic geometry
- A different approach that might lead to new methods and insights

# What is tropical geometry?

- A new variant of algebraic geometry over the tropical semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$
- redefines the rules of arithmetic with tropical addition $\oplus$ and tropical multiplication $\odot$
- Sometimes is called the combinatorial shadow of algebraic geometry
- Polynomials becomes piecewise-linear in tropical geometry



$x^2 + 0 = 0$




$x^2 \oplus 0 = 0$

# History

## Imre Simon

$14^{th}$ August 1943 - $13^{th}$ August 2009

- Hungarian-born Brazillian computer scientist
- Pioneer of "max-plus algebra", foundation of tropical geometry
- Name coined by French mathematicians because Simon is Brazillian
- Was first used in optimization
- Only recently in "mainstream" mathematics

# Tropical semiring

## Definition

The *tropical semiring* $\mathbb{T}$ is $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, with operations:

$$x \oplus y := \max(x, y) \quad \text{and} \quad x \odot y := x + y, \qquad x, y \in \mathbb{T},$$

and are called the *tropical addition* and the *tropical multiplication* respectively.

## Example

$$4 \oplus 9 = \max(4, 9) = 9, \qquad 4 \odot 9 = 4 + 9 = 13.$$

# Properties of a tropical semiring

The *tropical semiring* $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, with operations:

$$x \oplus y := \max(x, y) \quad \text{and} \quad x \odot y := x + y$$

Easy to check:

- $\oplus$ and $\odot$ are associative and commutative
- distributive law holds:
  $x \odot (y \oplus z) = x + \max(y, z) = \max(x + y, x + z) = (x \odot y) \oplus (x \odot z)$
- the identity element for $\oplus$ is $-\infty$
- the identity element for $\odot$ is 0
- the $\odot$-inverse of $x \in \mathbb{T} \setminus \{-\infty\}$ is $-x$
- no inverse for $\oplus$, solution for $x \neq -\infty$ and $y \in \mathbb{T}$ such that $x \oplus y = -\infty$ does not exist

# $\mathbb{T}$ is a semiring

### Definition(Semiring)

Semiring is similar to a ring except that each element does not need to have an additive inverse.

Note that (more properties):

- No subtraction in $\mathbb{T}$ e.g. $x \oplus 10 = 2$ has no solution for $x$
- $\oplus$ is *idempotent* i.e. $x \oplus x = x$

# Tropical monomial

## Tropical exponent

$$x^n = \underbrace{(x \odot \ldots \odot x)}_{n \text{ times}} = n \cdot x$$

We drop the notation $\odot$ when it is obvious that an equation is under tropical operations e.g. $x \oplus y \odot z = x \oplus yz$.

## Definition of tropical monomial

A *tropical monomial* is a map $m : \mathbb{T}^n \to \mathbb{T}$ of the form:

$$m(x_1, ..., x_n) = cx_1^{k_1} x_2^{k_2} \ldots x_n^{k_n},$$

where $c \in \mathbb{T}$ and $k_1, \ldots, k_n \in \mathbb{N}$.

# Tropical polynomials

## Definition

A *tropical polynomial* is a map $p : \mathbb{T}^n \to \mathbb{T}$ that is a finite linear combinations of tropical monomials of the form:

$$p(x_1, ..., x_n) = \bigoplus_{i=1}^{m} c_i x_1^{k_{i,1}} x_2^{k_{i,2}} \dots x_n^{k_{i,n}},$$

where $c_1, ..., c_m \in \mathbb{T}$ and $k_{1,1}, ..., k_{m,n} \in \mathbb{N}$.

## In usual arithmetic form:

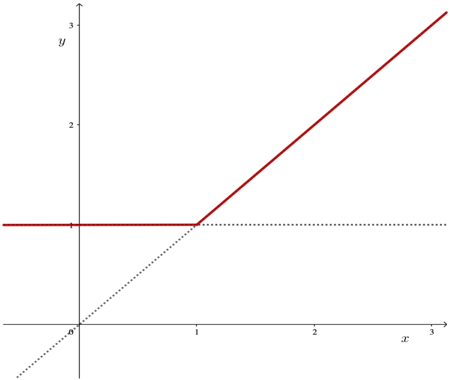
$$p(x_1, \dots, x_n) = \max(c_1 + k_{1,1} \cdot x_1 + \dots + k_{1,n} \cdot x_n,$$
$$c_2 + k_{2,1} \cdot x_1 + \dots + k_{2,n} \cdot x_n, \quad \dots \ )$$

# Tropical polynomials in one variable

**Example:**

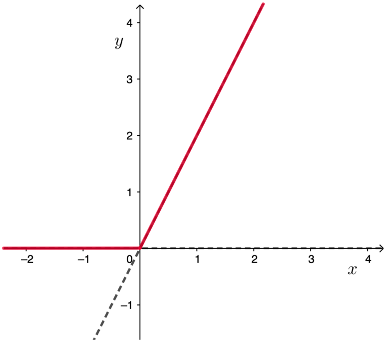
- $p_1(x) = x \oplus 1 = \max(x, 1)$,

$$p_1(x) = \begin{cases} 1 & \text{if } x \le 1 \\ x & \text{if } x \ge 1, \end{cases}$$

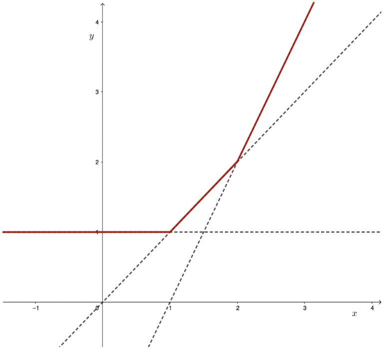


# More examples:

- $p_2(x) = x^2 \oplus 0 = \max(2x, 0)$,
- $p_3(x) = (-2)x^2 \oplus x \oplus 1 = \max(2x-2, x, 1)$.




$p_2(x) = x^2 \oplus 0$



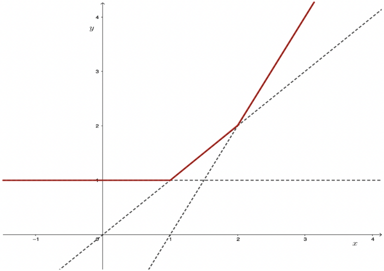

$p_3(x) = (-2)x^2 \oplus x \oplus 1$

# Properties of tropical polynomials

We can observe from the graphs in the previous slides and see the properties of tropical polynomial $p$ where:

1. $p$ is continuous;
2. $p$ is a **piecewise linear** function with the number of pieces being finite;
3. $p$ is convex.

$$p_3(x) = (-2)x^2 \oplus x \oplus 1$$

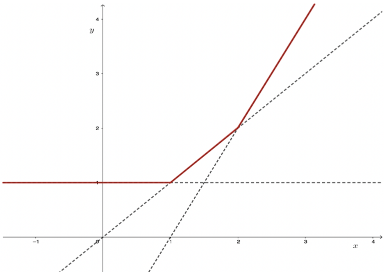


# Tropical roots

## Definition

The *roots* of a tropical polynomial $p(x_1, \ldots, x_n)$ are the points $w_1, \ldots, w_n \in \mathbb{T}$ such that the maximum are obtained at least twice i.e. the points in $p(x_1, \ldots, x_n)$ where they are non-differentiable.

$$
\begin{aligned}
p_3(x) &= (-2)x^2 \oplus x \oplus 1 \\
&= \max(2x-2, x, 1) \\
&= \begin{cases} 1 & \text{if } x \leq 1 \\ x & \text{if } 1 \leq x \leq 2 \\ 2x-2 & \text{if } x \geq 2. \end{cases}
\end{aligned}
$$



# Tropical hypersurface

## Definition

The *tropical hypersurface* of $p$, denoted $V(p)$, is the set of all $w_1, \ldots, w_n \in \mathbb{T}$ such that the maximum is obtained at least twice.
In other words, $V(p)$ is the set of all tropical roots.

## Example

For $p_3 = (-2)x^2 \oplus x \oplus 1, \text{ we have } V(p_3) = \{1, 2\}$.

# Tropical curves in $\mathbb{R}^2$

## Definition

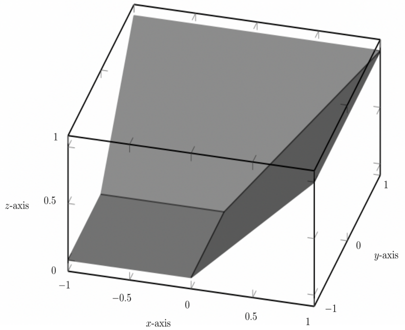
Let $p : \mathbb{R}^2 \to \mathbb{R}$. A *tropical curve* or *plane tropical curve* is the set of all roots or zeros of $p$ i.e. a tropical curve $V(p)$ is a set of all points where the maximum is obtained twice.

$$p(x, y) = \bigoplus_{(i,j)} c_{i,j} \odot x^i \odot y^j$$

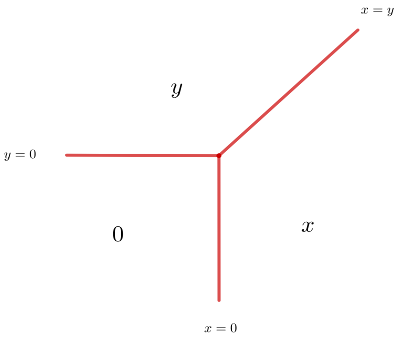
## Tropical curve $V(p)$

$$V(p) = \{(x_0, y_0) \in \mathbb{R}^2 | \ \exists (i, j) \neq (k, l), \ p(x_0, y_0) = c_{i,j} x_0^i y_0^j = c_{k,l} x_0^k y_0^l\}$$

# Examples of tropical curves



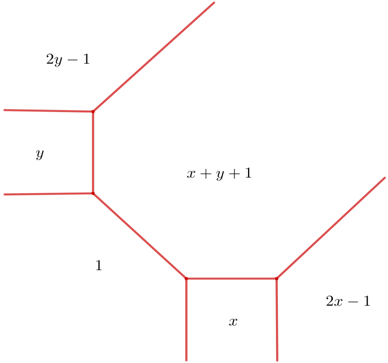

Graph of $p_4(x, y) = x \oplus y \oplus 0$



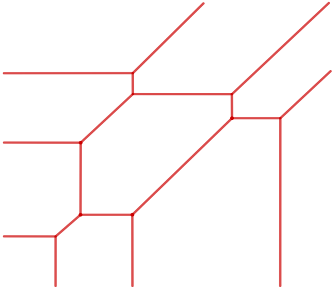

Tropical line $V(p_4)$

## Definition

A tropical curve $V(p)$ is said to have *degree k* if $V(p) \subset \mathbb{R}^2$ and $p$ is of degree $k$




Tropical conic $V(p_5)$ where $p_5 = (-1)x^2 \oplus 1xy \oplus (-1)y^2 \oplus x \oplus y \oplus 1$



General tropical cubic

# Weight

Recall that roots of $p(x, y)$ are all points $(x_0, y_0) \in \mathbb{R}^2$ such that there exists pairs $(i, j) \neq (k, l)$ satisfying $p(x_0, y_0) = c_{i,j} x_0^i y_0^j = c_{k,l} x_0^k y_0^l \}$.

## Definition

The *weight* of an edge $e$ of $V(p)$ is the maximum of the greatest common divisor of the numbers $|i - k|$ and $|j - l|$ for all pairs $(i, j) \neq (k, l)$ which corresponds to the edge such that:

$$w(e) = \max_{\mu(e)}(\gcd(|i - k|, |j - l|)),$$

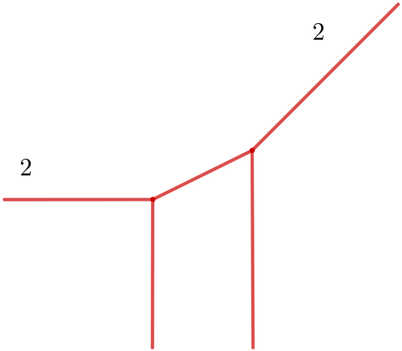
where

$$\mu(e) = \{(i, j), (k, l) | \ \forall x_0 \in e, \ p(x_0, y_0) = c_{i,j} x_0^i y_0^j = c_{k,l} x_0^k y_0^l\}$$

## Example

$p_4(x, y) = x \oplus y \oplus 0$, with $w(e) = 1$ at all edges

# Examples of weight




Tropical conic

# Newton polytopes

Let $S \subset \mathbb{R}^n$ be a finite set of points

## Definition of Convex Hull

The *convex hull* of $S$, denoted $\text{Conv}(S)$, is the unique smallest convex polygon with vertices in $S$ that contains all points of $S$. $\text{Conv}(S)$ can also be called a *polytope*.

Let $p$ be a polynomial in two variables and $c_{i,j}x^iy^j$ be the monomials of $p$. $S_{i,j} = \{(i,j) \in \mathbb{R}^2 | \ c_{i,j}x^iy^j \text{ and } c_{i,j} \neq -\infty\}$.

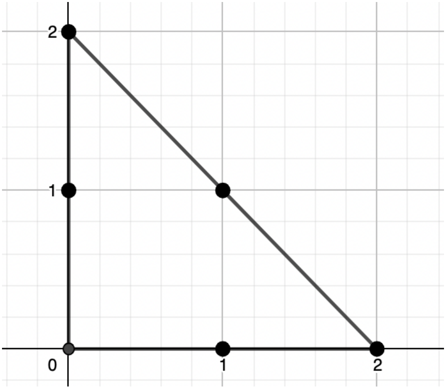
## Definition of Newton Polytope

The *Newton polytope* corresponding to $p$, denoted $\text{Newt}(p)$, is the convex hull of all $S_{i,j}$ of $p$ where:

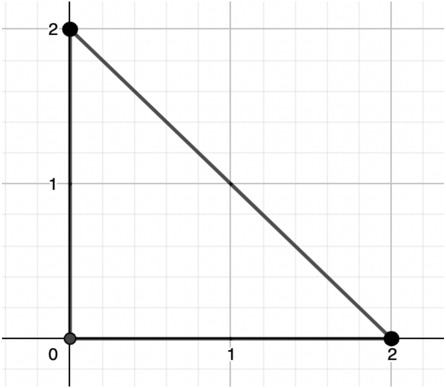
$$\mathrm{Newt}(p) = \mathrm{Conv}(S_{i,j})$$

## Example

Consider $p_8(x,y) = 1x^2 \oplus xy \oplus 1y^2 \oplus x \oplus y \oplus 1$.
$\mathrm{Conv}(p_8) = \{(0,0),(1,0),(2,0),(0,1),(0,2),(1,1)\}$. Thus,
$\mathrm{Newt}(p_8) = \{(2,0),(0,0),(0,2)\}$




$\mathrm{Conv}(p_8)$




$\mathrm{Newt}(p_8)$

The *degree* $k$ of $p(x, y) = \bigoplus_{(i,j)} c_{i,j} x^i y^j$ is $k = \max_{i,j}(i + j)$ for all coefficients $c_{i,j} \neq -\infty$.

We will assume that all polynomials of degree $k$ satisfy $c_{0,0} \neq -\infty$, $c_{k,0} \neq -\infty$, and $c_{0,k} \neq -\infty$.

## Definition A

The set $S_{i,j}$ is contained in the triangle with vertices $(0, 0)$, $(k, 0)$ and $(0, k)$. The triangle is denoted as $\Delta_k$.

## Proposition

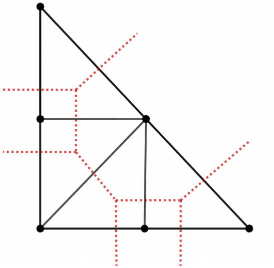
The triangle $\Delta_k$ is precisely the convex hull of the set $S_{i,j}$.

*Proof:* Definition of a convex hull and Definition A.

# Dual subdivision of tropical curves

## Definition

Let $p(x, y)$ be a tropical polynomial with degree $k$ and $V(p)$ be its tropical curve. The *dual subdivision* of $V(p)$ is the union of triangles $\Delta_v$ in $\Delta_k$ for each vertex $v$ of $V(p)$.

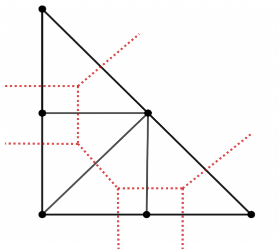


Tropical cubic and its dual subdivision

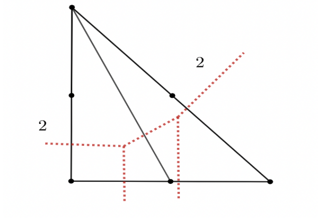
# Weights in dual subdivision

## Remark

The weight of an edge $e$ of a tropical curve $V(p)$ can be read off directly from its dual subdivision.



Dual subdivision of a tropical cubic

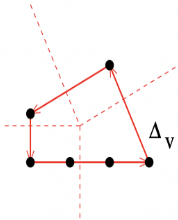



Dual subdivision of a tropical conic

# Balanced graph

## Definition B

A graph in $\mathbb{R}^2$ whose edges have rational slopes and have weights $w_i \in \mathbb{Z}^+$ is a *balanced graph* if it satisfies the balancing condition at every vertices.




## Corollary

*V(p) is a tropical curve if and only if it is a balanced graph.*

*Proof:* From Definition B.

# Classical Bézout's theorem

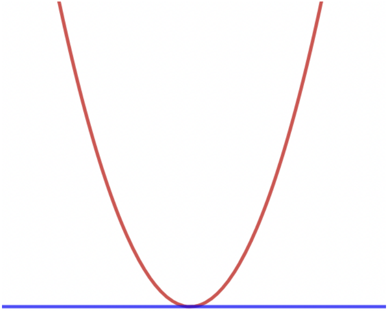
**Bézout's theorem**

Let $F$ and $G$ be curves representing the polynomials $f(x, y)$ and $g(x, y)$ without common factors over algebraically closed field $K$ . Let $d_1 = \deg(F)$ and $d_2 = \deg(G)$. Then, $F$ and $G$ intersect exactly at $d_1 \cdot d_2$ number of points if:

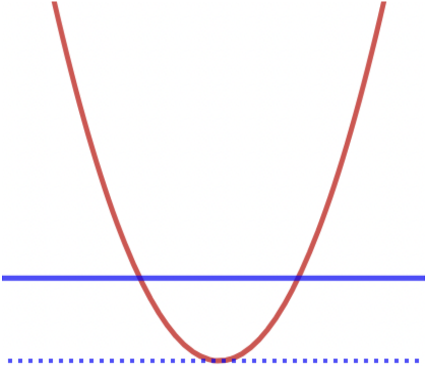
1. the intersections are considered in the projective plane $\mathbb{P}_K$;
2. the intersections are counted with multiplicities.

Let's focus on condition 2.

# Examples of the condition 2. on Bézout's theorem

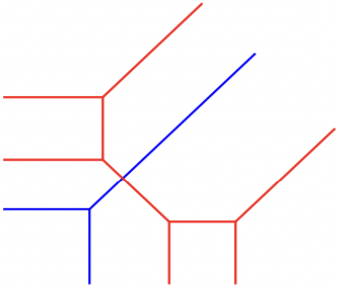


Intersection of $F = x^2 - y$ and $G = y$,

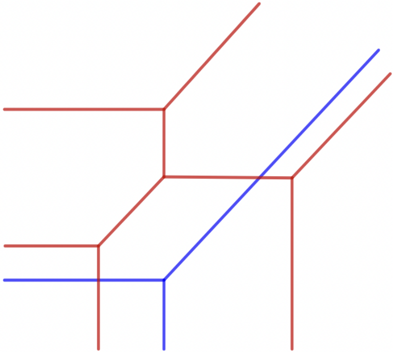


Intersection of $F = x^2 - y$ and $G = y$ with deviation.

# Tropical intersection theory



Intersection of tropical conic and tropical line at one point



Intersection of tropical conic and tropical line at two points

# Very important definition

## Proposition

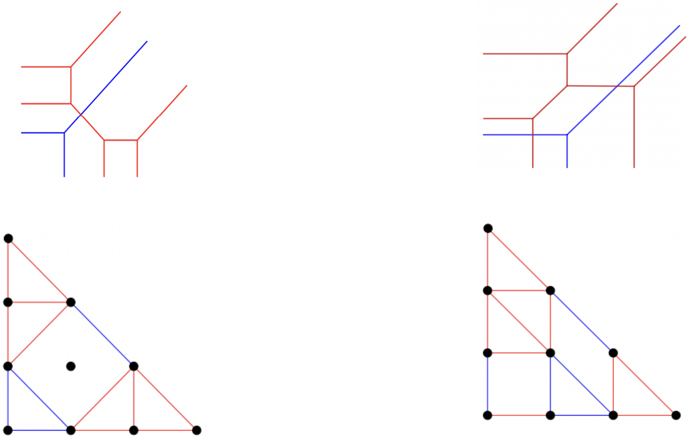
*Let $V_1(p)$ and $V_2(p)$ be tropical curves.$V_1(p) \cup V_2(p)$ is again a tropical curve.*

*Proof:* $V(p)$ is a balanced graph. A union of two balanced graphs is again a balanced graph because any sum of balanced graph would be a balanced graph.

## Important properties of $V_1(p)$ and $V_2(p)$

The dual polytope to the vertex of $V_1(p) \cup V_2(p)$ made from the points of intersection $V_1(p)$ and $V_2(p)$ is a paralleolgram.

# Dual subdivision corresponding to the tropical curves



# Tropical Bézout's theorem

## Definition of tropical multiplicity

Let $V_1(p)$ and $V_2(p)$ be two tropical curves that intersect in a finite number of points and away from the vertices of the curves. Let $q$ be the point of intersection of $V_1(p)$ and $V_2(p)$. The *tropical multiplicity* of $q$ as an intersection point of $V_1(p)$ and $V_2(p)$ is the area of parallelogram dual to $q$ in the dual subdivision $V_1(p) \cup V_2(p)$.

## Tropical Bézout's theorem (Sturmfels)

*Let $V_1(p)$ and $V_2(p)$ be two tropical curves that intersect in a finite number of points away from the vertices of the two curves. Let $d_1$ be the degree of $V_1(p)$ and $d_2$ the degree of $V_2(p)$. Then the sum of the tropical multiplicities of all points $V_1(p) \cap V_2(p)$ equal to $d_1 \cdot d_2$.*

## Proof of tropical Bézout's theorem

Define $M$ as the sum of the tropical multiplicities of all points $V_1(p) \cap V_2(p)$. Notice that there are three different types of polytopes in the dual subdivision corresponding to the tropical curve $V_1(p) \cup V_2(p)$:

1. The polytopes that are dual to a vertex of $V_1(p)$. The sum of the areas of said polytopes is equal to the area of $\Delta_{d_1}$ i.e. equal to $\frac{d_1^2}{2}$,
2. The polytopes that are dual to a vertex of $V_2(p)$. The sum of the areas of said polytopes is equal to the area of $\Delta_{d_2}$ i.e. equal to $\frac{d_2^2}{2}$,
3. The polytopes that are dual to a vertex of $V_1(p) \cap V_2(p)$. The sum of the areas of said polytopes is $M$.

Since $V_1(p) \cup V_2(p)$ is of degree $d_1 + d_2$, the sum of the area of all the polytopes is equal to the area of $\Delta_{d_1+d_2} = \frac{(d_1+d_2)^2}{2}$. We then obtain

$$M = \frac{(d_1 + d_2)^2 - d_1^2 - d_2^2}{2} = d_1 \cdot d_2 \quad \square$$

# Conclusion

There are many many more proofs that tropical geometry can provide us to understand theorems that are otherwise hard to understand. It is then an important field for us to explore as it can give us better insights and new methods to solve things.

*The End.*

*Thank you!*