Preface

I feel extremely happy to present this book, **RELATIVISTIC MECHANICS AND ASTROPHYSICS** *for the benefit of the Core Course in Physics, Sixth semester B.Sc. Programme of University of Calicut. The book is prepared in accordance with the latest syllabus of the restructured, choice-based credit and semester system. Maximum care has been taken to maintain the standard and quality of the content, precision and perfection of the treatment, and the lucidity, simplicity and clarity of the style. All efforts have also been made for the sequential arrangement of the conceptual aspects of the subject matter in a systematic format, incorporating as much relevant facts as possible to make the description clear, straight-forward and self-explanatory. A large number of solved and unsolved problems, typical examples and all types of model questions, covering almost all fundamental aspects, have been given for an elaboration and systematic analysis of the basic concepts, and a deep insight into the subject matter.*

I do not claim the originality of the subject matter, since the essential materials have been collected from authentic reference sources. Despite our careful scrutiny, errors and omissions might have occurred and we will be highly obliged to those who bring them to our notice. All suggestions for improvement will be thankfully acknowledged and considered. I hope that the book will be highly useful to cater to the needs of the students.

Calicut, **P. Sethumadhavan**
November, 2021.

UNIT ONE

1 SPECIAL RELATIVITY

Introduction

The special theory of relativity, shortly called relativity, was put forward by Albert Einstein in the year 1905 which revolutionised the concept of space, time and motion on which Newtons laws were founded. Probably no physical theory in twentieth century has been the object of more discussion amongst philosophers and scientists, and at the same time caught the imagination of the intelligent layman, than the theory of relativity. This is essentially due to the fact that the concepts underlying the theory of relativity are not only radically new but also provide a frame work which embraces practically all the branches of the physical sciences. Actually everything stemmed from the inadequacy of Newtonian mechanics.

The entire edifice of mechanics was built upon the 3+1 laws of Newton. The first 3 stands for the three laws of motion and 1 stands for the law of gravitation of Newton. With these laws, Newton explained the macroscopic world with amazing success. It lead scientists to believe that these laws were universal in their applicability. To explain laws of motion, Newton assumed that space and time are absolute. But he could not support his conviction by any scientific argument, nevertheless he clung to this on theological grounds. But with the development of the wave theory of light scientists find it necessary to endow absolute space with certain mechanical properties. For this scientists evolved a hypothetical substance called 'ether' which they decided must pervade all space. It provided a mechanical model for all known phenomena of nature and a fixed frame of reference, the absolute space, which Newtons laws are required.

Michelson-Morley-Experiment

According to Christian Huygens wave theory, light propagates through ether which is stationary with respect to earth. So velocity of light should be different for different directions. To verify this Michelson and Morley used Michelson's interferometer. it consists of a semi silvered glass plate P, a compensating plate G, and two plane mirrors M_1 and M_2 kept perpendicular to each other. The compensating plate

G is kept to make the optical path travelled the same for both rays moving in perpendicular directions. Light from a monochromatic source 'S' on falling the plate P kept at angle of 45^0 gets split into two parts. One part undergoes reflection and goes towards M_1, at the same time the other part undergoes refraction through P and goes towards M_2. The rays get reflected from M_1 and M_2 enters an eyepiece T forming interference fringes. Let v be the velocity of the earth around the sun. Therefore with respect to the ether medium the apparatus has a velocity v. The effect is the same as if the apparatus is at rest and ether is moving in the opposite direction. As far as the reference frame lab is concerned the apparatus is at rest. So ether moves in the opposite direction with respect to the lab frame. Now consider a ray which has started towards M_1. Since the ether medium tracks this wave with it to reach M_1, light wave should be travelling in the direction PA so that the resultant velocity in ether is PM_1. Let c be the velocity of light when there is no relative motion between the source of light and ether. Therefore resultant velocity along $PM_1 = \sqrt{c^2 - v^2}$. Let L be the distance from P to M_1.

$\therefore$ The time taken by the light wave to travel from P to $M_1 = \dfrac{L}{\sqrt{c^2 - v^2}}$.

Time taken by the light wave to travel from M_1 to P will also be the same. Since in this case also the light wave is crossing the ether wind.

The time taken by the light wave to travel from P to M_1 and M_1 to P

$$t_1 = \frac{2L}{\sqrt{c^2 - v^2}} = \frac{2L/c}{\sqrt{1 - v^2/c^2}} \qquad \text{...... (1)}$$

$v = 3\times10^4$ m/s (velocity of earth)

$c = 3\times10^8$ m/s

$$\therefore \qquad \frac{v^2}{c^2} = \frac{9\times10^8}{9\times10^{16}} = 10^{-8}$$

Since $\dfrac{v^2}{c^2}$ is small, using Binomial approximation equation (1) can be written as

$$\left[\begin{array}{c}(1+x)^n \simeq 1 + nx \\ \text{when } x << 1\end{array}\right]$$

$$t_1 = \frac{2L}{c}\left(1-\frac{v^2}{c^2}\right)^{-1/2} = \frac{2L}{c}\left(1+\frac{v^2}{2c^2}\right) \qquad \text{...... (2)}$$

Velocity of the light wave which travels from P to M_2 is c-v (since it is in the opposite direction of the ether wind)

Velocity of light which travels from M_2 to P = c +v.

$\therefore$ Time taken by the light wave to travel from P to M_2 and M_2 to P.

$$t_2 = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2Lc}{c^2 - v^2} = \frac{2L/c}{1-\frac{v^2}{c^2}}$$

(since $PM_2 = L$)

As before neglecting higher powers of $\frac{v^2}{c^2}$

$$t_2 = \frac{2L/c}{1-v^2/c^2} = 2L/c\left(1-\frac{v^2}{c^2}\right)^{-1} = \frac{2L}{c}\left(1+\frac{v^2}{c^2}\right) \qquad \text{....... (3)}$$

$\therefore$ The time difference between the two waves entering the telescope.

$$t_2 - t_1 = \frac{2L}{c}\left(1+\frac{v^2}{c^2}\right) - \frac{2L}{c}\left(1+\frac{v^2}{2c^2}\right)$$

$$t_2 - t_1 = \Delta t = \frac{Lv^2}{c^3} \qquad \text{....... (4)}$$

Equation 4 shows that there is a time difference between the two rays entering the telescope. A difference of time Δt causes a path difference

$$\Delta t \cdot c = \frac{Lv^2}{c^2} \qquad \text{....... (5)}$$

Because of this path difference interference fringes are formed in the telescope. Adjust the telescope and the cross wire is made to coincide with one of the fringes.

Now the apparatus is turned through 90° so that the two beams interchange their paths. i.e. in the rotated position the beam which was perpendicular to v now becomes parallel to v and viceversa, the path difference will be $= -\frac{Lv^2}{c^2}$.

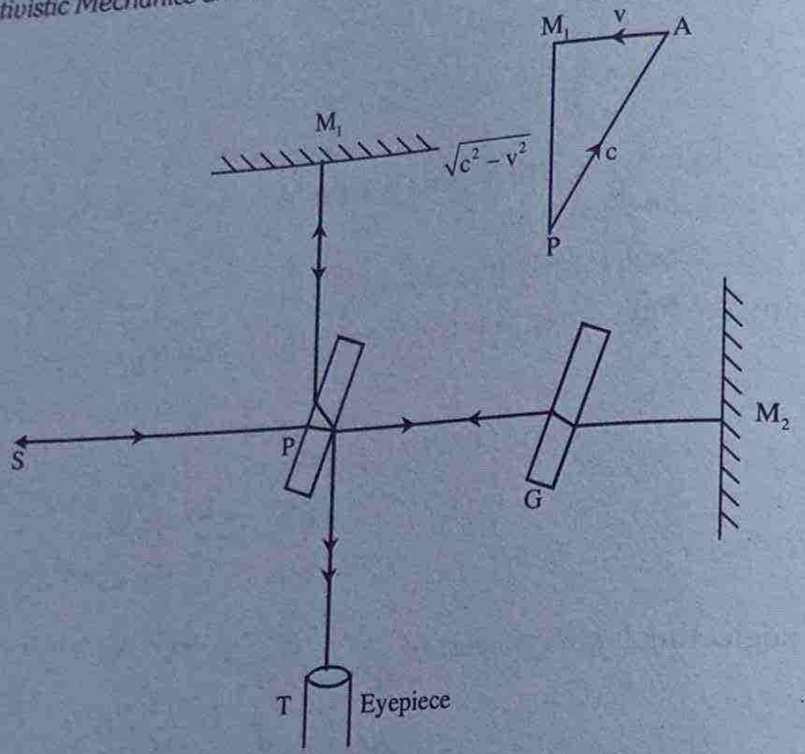


Figure 1.1

Displacement of the interference fringes

$$= \frac{Lv^2}{c^2} - - \frac{Lv^2}{c^2} = \frac{2Lv^2}{c^2}. \qquad \text{...... (6)}$$

But in the experiment no such fringe shift was observed. This shows that this experiment gives a negative result. This null result suggests that there is no ether medium which is stationary with respect to the earth and the velocity of light is a constant in all directions.

Example 1

In actual Michelson-Morley experiment the total distance from the partially silvered mirror to each of the mirrors was 10 m. The wave length of light used was 5000Å. If the orbital velocity of earth is taken as $30\,\text{km s}^{-1}$, calculate the total fringe shift when the apparatus is rotated through 90°.

Solution

$L = 10\,\text{m}$

$v = 30 \times 10^3\,\text{ms}^{-1}$

$\lambda = 5000 \times 10^{-10}\text{m}$

Fringe shift, $\delta = 2L\dfrac{v^2}{c^2} = \dfrac{2\times10\times9\times10^8}{9\times10^{16}} = 2\times10^{-7}\text{m}$

Number of expected fringe shift $= \dfrac{\delta}{\lambda} = \dfrac{2\times10^{-7}}{5\times10^{-7}} = 0.4.$

Note : It may be noted that Michelson Morley experiment was sensitive enough to detect a fringe shift of the order of 0.01 fringe.

Example 2

If the arms of a Michelson interferometer have lengths l_1 and l_2. Show that the fringe shift when the interferometer is rotated by 90° with respect to the velocity v through the ether is $\delta = \dfrac{l_1 + l_2}{\lambda}\dfrac{v^2}{c^2}$, where λ is the wavelength of right.

Solution

See figure 1.1

The time taken by the light wave to travel from P to $M_1 = \dfrac{l_1}{\sqrt{c^2 - v^2}}$

Then time taken by the light wave to travel from M_1 to P will also be the same.

∴ The time taken by the light wave to travel from P to M_1 and M_1 to P

$$t_1 = \frac{2l_1}{\sqrt{c^2 - v^2}} = 2l_1(c^2 - v^2)^{-\frac{1}{2}}$$

$$t_1 = \frac{2l_1}{c}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \frac{2l_1}{c}\left(1 + \frac{v^2}{2c^2}\right)$$

Time taken by the light wave to travel from P to M_2 and M_2 to P

$$t_2 = \frac{l_2}{c - v} + \frac{l_2}{c + v} = \frac{2l_2 c}{c^2 - v^2}$$

$$t_2 = \frac{2l_2}{c}(1 - v^2/c^2)^{-1}$$

$$t_2 = \frac{2l_2}{c}\left(1+\frac{v^2}{c^2}\right)$$

$\therefore$ The time difference between the two waves entering the telescope is

$$t_2 - t_1 = \frac{2l_2}{c}\left(1+\frac{v^2}{c^2}\right) - \frac{2l_1}{c}\left(1+\frac{v^2}{2c^2}\right)$$

If the apparatus is rotated through 90°, the arms are interchanged. The time difference is

$$t'_2 - t'_1 = \frac{2l_2}{c}\left(1+\frac{v^2}{2c^2}\right) - \frac{2l_1}{c}\left(1+\frac{v^2}{c^2}\right)$$

$\therefore$ The time shift

$$\Delta t = (t'_2 - t'_1) - (t_2 - t_1)$$

$$\Delta t = -\left(\frac{l_1 + l_2}{c}\right)\frac{v^2}{c^2}$$

$\therefore$ The fringe shift, $\delta = |\upsilon \Delta t|$

$$\delta = \frac{c}{\lambda}\frac{(l_1 + l_2)}{c}\cdot\frac{v^2}{c^2}$$

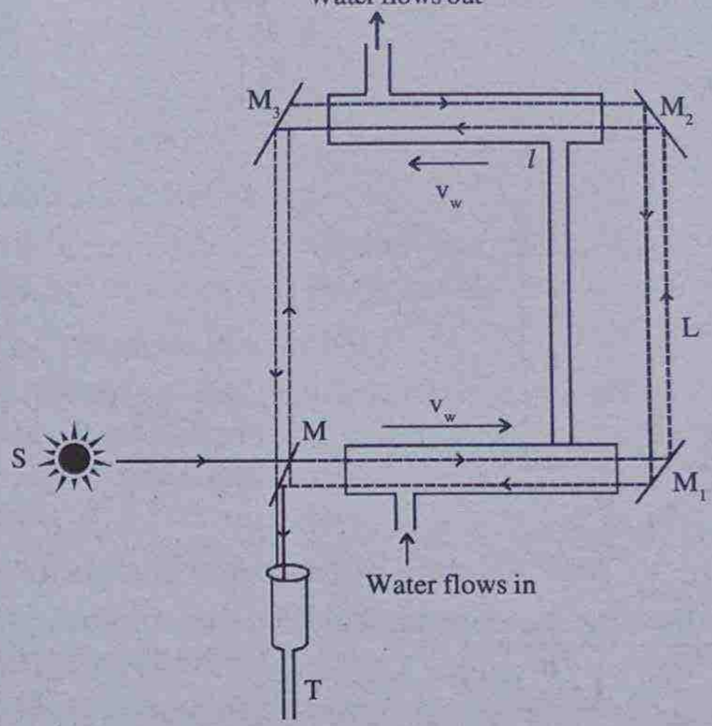
i.e.,

$$\delta = \frac{(l_1 + l_2)}{\lambda}\cdot\frac{v^2}{c^2}$$

Example 3

H. L. Fizeau investigated the velocity of light through a moving medium using the interferometer shown below.

Light of wavelength λ from a source S is split into two beams by the mirror M. The light travel around the interferometer in opposite directions and are combined at the telescope of the observer O who sees the fringe pattern. Two arms of the interferometer pass through water filled tubes of length l with flat glassend plates. The water runs through the tubes, so that one of the light beams travel down stream while the other goes upstream. Calculate the fringe shift.



M, M_1, M_2 and M_3 are mirrors at 45°

Figure 1.2

Solution

Time taken by light to go around the interferometer anticlockwise is

$$t_1 = \frac{l}{v_w + \frac{c}{n}} + \frac{L}{c} + \frac{l}{v_w + \frac{c}{n}} + \frac{L}{c}$$

$$t_1 = \frac{2l}{v_w + \frac{c}{n}} + \frac{2L}{c}$$

where v_w is the speed of water and n is the refractive index of water.

Time taken by light to go around the interferometer clockwise is

$$t_2 = \frac{2l}{\frac{c}{n} - v_w} + \frac{2L}{c}$$

∴ The time difference of light wave entering the telescope is

$$\Delta t = t_2 - t_1 = \frac{2l}{\frac{c}{n} - v_w} - \frac{2l}{v_w + \frac{c}{n}}$$

$$\Delta t = 2l\left(\frac{1}{\frac{c}{n} - v_w} - \frac{1}{v_w + \frac{c}{n}}\right)$$

$$\Delta t = 2l\left(\frac{n}{c - nv_w} - \frac{n}{nv_w + c}\right)$$

$$\Delta t = \frac{2ln}{c}\left(\frac{1}{1 - \frac{nv_w}{c}} - \frac{1}{1 + \frac{nv_w}{c}}\right)$$

$$\Delta t = \frac{2ln}{c}\left[\left(1 - \frac{nv_w}{c}\right)^{-1} - \left(1 + \frac{nv_w}{c}\right)^{-1}\right]$$

$$\Delta t = \frac{2ln}{c}\left[\left(1 + \frac{nv_w}{c}\right) - \left(1 - \frac{nv_w}{c}\right)\right]$$

$$\Delta t = \frac{2ln}{c}\frac{2nv_w}{c}$$

$$\Delta t = 4n^2 l \frac{v_w}{c^2}$$

Hence the fringe shift $= \nu\Delta t = \frac{c}{\lambda}\Delta t$

$$\delta = 4n^2 \frac{l}{\lambda c} v_w$$

Note: The actual fringe measured by Fizeau was $\delta = 4n^2 \frac{l}{\lambda c} v_w f$, where $f = 1 - \frac{1}{n^2}$ known as Fresnel drag coefficient. This was explained only after the advent of relativity.

Postulates of Relativity

The entire edifice of relativity was built upon two basic postulates. They are called (i) The principle of relativity (ii) Principle of constancy of speed of light.

I. The principle of relativity

According to this principle all laws of physics are the same in all inertial frames. It means that it is impossible to designate an inertial frame as stationary or moving by conducting experiments. We can speak of the relative motion of the two frames.

II. Principle of constancy of speed of light

According to this principle the speed of light in free space has the same value in all inertial frames.

These two postulates - one stating that all motion is relative, and the other speed of light is relative to nothing but it is an absolute constant, seem contradictory. But in the world of relativity they do not conflict. No experimental objection to Einsteins special theory of relativity has yet been found.

Galilean transformation

Suppose a physical phenomenon (event) is observed from two separate reference frames, naturally they will have two seperate sets of co-ordinates corresponding to their frames of reference.

Equations relating the two sets of co-ordinates of the event in the two frames are called transformation equations. If the two frames are inertial ones, the transformation is referred to as Galilean transformation.

*[The position and time of occurrence of a physical phenomenon taken together is called event].

Galilean transformation equations

There are three Galilean transformation equations

(i) Galilean co-ordinate transformation.

(ii) Galilean velocity transformation.

(iii) Galilean accelaration transformation.

Galilean coordinate transformation equation

Let us consider two inertial frames of reference S and S'. S be at rest and S' be moving with a uniform velocity v in the positive X-direction. Let t and t' be the time of an event measured by S and S'. Let us consider two observers one in S and other in S'. To begin with let the two frames coincide i.e. $t = t' = 0$. After a time t the two frames observe an event taking place at P. During this time t the S' frame has moved a distance vt along that direction with respect to the rest frame S. With respect to the rest frame S, the co-ordinates are (x, y, z, t) and with respect to the moving frame S', the co-ordinates are (x', y', z', t').

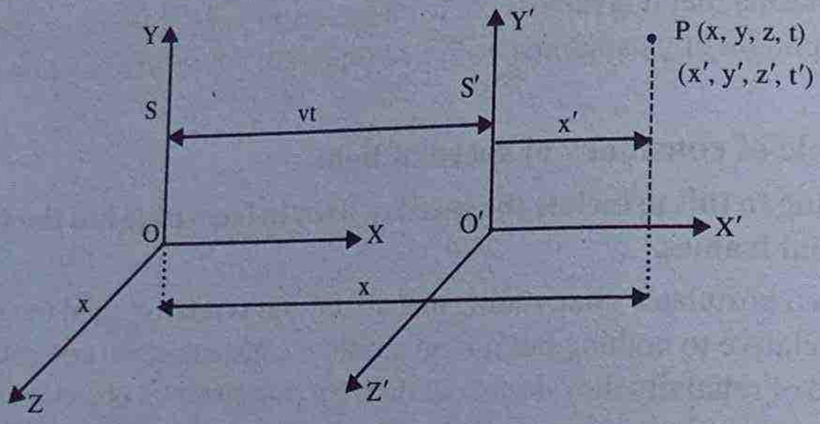


Figure 1.3

Therefore we can write (see figure 1.3)

$$x' = x - vt \qquad \text{...... (1)}$$

Since there is no motion along the Y and Z directions

$$y' = y \quad \text{and} \qquad \text{...... (2)}$$

$$z' = z \qquad \text{...... (3)}$$

Since according to classical idea the event should appear simultaneously to the two observers.

$$t' = t \qquad \text{...... (4)}$$

Equations 1, 2, 3 and 4 are known as Galilean co-ordinate transformation equations.

These transformation equations allow us to go from one inertial frame S to another inertial frame S'. If we substitute this transformation of co-ordinates into Newtons laws we can see that the laws of Newton are of the same form in two frames (see example 2). But if we transform Maxwell's equations by the substitution of above transformations their form does not remain the same. It shows that the above transformation does not give a law of physics equivalant to the two frames of references. That is Galilean transformaltion doesnot satisfy the first postulate of relativity. (see unit III)

Galilean velocity transformation equation

Differentiating equations 1, 2 and 3 with respect to t gives.

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v \quad \left(\frac{d}{dt'} = \frac{d}{dt}, \text{ since } t' = t\right)$$

$$\frac{dy'}{dt'} = \frac{dy}{dt}$$

$$\frac{dz'}{dt'} = \frac{dz}{dt}$$

using $\frac{dx'}{dt'} = u'_x$, the x-component of velocity measured with respect to S′ and $\frac{dx}{dt} = u_x$, the x-component of velocity measured with respect to S and so on. Then the above equations become

$$u'_x = u_x - v \qquad \text{...... (5)}$$

$$u'_y = u_y \qquad \text{...... (6)}$$

$$u'_z = u_z \qquad \text{...... (7)}$$

Equations 5, 6 and 7 are called Galilean velocity transformation equations or simply the classical velocity addition theorem or Galilean law of additon of velocities. Clearly in the more general cases in which velocity v has components along all three axes, we would obtain the general vector result

$$\vec{u}' = \vec{u} - \vec{v} \qquad \text{...... (8)}$$

For example the velocity of an aeroplane with respet to the air $(\vec{u}' = \vec{v}_{PA})$ equals the velocity of the plane with respect to the ground $(\vec{u} = \vec{v}_{PG})$ minus the velocity of the air with respect to the ground $(\vec{v} = \vec{v}_{AG})$

i.e., $\vec{v}_{PA} = \vec{v}_{PG} - \vec{v}_{AG}$

or $\vec{v}_{PA} + \vec{v}_{AG} = \vec{v}_{PG}$

See the same subscript symbols fusing to get the result.

If we apply these Galilean velocity transformation equations to particles which are moving very fast, we can see that this equations fail badly. For example let $u'_x = c'$ and $u_x = c$ be the velocity of light observed by the persons in S' and S respectively

i.e., $c' = c - v$

This equation shows that velocity of light is different for different inertial frames. It is contrary to experimentally observed fact that velocity of light is same for all inertial frames. That is it violates the second postulates of relativity (see unit III).

Galilean acceleration transformation equations

Differenting equations 5, 6 and 7 with respect to $t(t = t')$

we get $\dfrac{du'_x}{dt'} = \dfrac{du_x}{dt} \quad \left(\because \dfrac{dv}{dt} = 0\right)$

The second term on the R.H.S of equation 5 does not give a derivative with respect to time since $\vec{v}$ the velocity with which the frame S' is moving is a constant.

$$\frac{du'_y}{dt'} = \frac{du_y}{dt}$$

and $\dfrac{du'_z}{dt'} = \dfrac{du_z}{dt}$

$\dfrac{du'_x}{dt'} = a'_x$, the acceleartion of the event with respect to the S' frame and $\dfrac{du_x}{dt} = a_x$, the acceleration of the event with respect to the S frame and so on.

Thus we have

$$a'_x = a_x \quad \text{...... (9)}$$

$$a'_y = a_y \quad \text{...... (10)}$$

$$a'_z = a_z \quad \text{...... (11)}$$

It shows that both frames (or observers) measure the same acceleration. Thus according to Galilean transformation, the acceleration of a particle is the same for all observers (frames) in uniform relative motion. Since the acceleration remains invariant under Galilean transformations [passing from one inertial frame (rest) to another inertial frame (uniform motion)], acceleration is called Galilean invariant physcal quantity.

In general physical quantities which remain invariant under Galilean transformation equations are called Galilean invariant quantities.

Example 4

A rocket travelling at a speed of 500 m/s ejects the burnt gases in opposite direction from its rear. If the speed of the ejected gases relative to the ground is 1000m/s. Find the speed of the ejected gases with respect to the rocket.

Solution

Take the ground to be the rest frame (S), moving rocket to be the S' frame and the event served to be the ejected gas.

Velocity of the ejected gas with respect to ground, $u_x = -1000$ m/s

Velocity of the rocket with respect to the ground, $v = 500$ m/s

Velocity of ejected gas with respect to the rocket?

$$\begin{aligned} u'_x &= u_x - v \\ &= -1000 - 500 \\ &= -1500 \text{ ms}^{-1} \end{aligned}$$

Example 5

Show that the force acting on a particle as observed by two observers in two inertial frames of reference is the same under Galilean transformation (i.e. $v \ll c$)

Solution

Consider two inertial frames S and S'. S be at rest and S' be moving with a velocity v with respect to S along the positive x-direction. Let us consider two observers one in S and another in S'. Consider a particle of mass m moving with a velocity u_1 with respect to S. Let a force F act on the particle for a time t which changes its velocity to u_2.

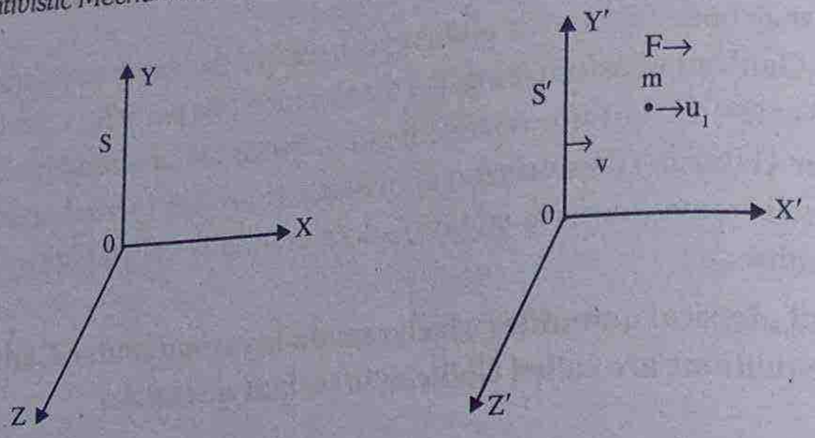


Figure : 1.4

Example 6

Two light pulses are emitted in the opposite directions from a source at rest. What is the speed of one light pulse as measured from the other.

Solution

Let the rest frame S be attached to the source and the moving frame S' be attached with the pulse moving along the x-direction. The S' frame is observing the light pulse moving along the –x direction.

Then

velocity of the light pulse with respect to S, $u_x = -c$

–ve sign comes, since the observed light pulse is moving in the –x direction

velocity of the S' frame $v = c$

$\therefore$ The velocity of the observed light pulse with respect to S',

$$u'_x = u_x - v$$

$$u'_x = -c - c = -2c$$

–ve sign shows that light pulse is moving along the –ve x-direction. It shows that the observed velocity of the light pulse is different in different frames, thus violating the second postulate of relativity. This tells us that Galilean transformation equations are not suitable to apply to fast moving particles ($v \to c$)

Example 7

Show that the length or distance between two points is invariant under Galilean transformation.

Solution

Let S and S' be two frames. S be at rest and S' be moving with a speed v along the +ve x-direction. Consider a rod of length L in the frame S with co-ordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) with respect to S′ frame the co-ordinates are (x_1', y_1', z_1') and (x_2', y_2', z_2').

The length of the rod with respect to S frame

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \qquad \text{......(1)}$$

The length of the rod with respect to S′ frame

$$L' = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2} \qquad \text{......(2)}$$

using $\quad x' = x - vt$

we have $\quad x_1' = x_1 - vt$

and $\quad x_2' = x_2 - vt$

$\therefore \quad x_2' - x_1' = x_2 - x_1$

using $\quad y' = y$

$\quad y_1' = y_1$ and $y_2' = y_2$

Then $\quad y_2' - y_1' = y_2 - y_1$

using $\quad z' = z$

$\quad z_1' = z_1$

$\quad z_2' = z_2$

$\therefore \quad z_2' - z_1' = z_2 - z_1$

Substituting these in eqn. 2, we get

$$L' = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Thus $\quad L' = L$ Hence the result

Lorentz transformation equations

We have seen that Galilean transformation equations do not obey first postulate and second postulate of relativity (see unit - I). Therefore Galiliean transformation equations must be replaced by new ones consistent with experiment. These new equations are called Lorentz transformation equations.

Consider two inertial frames S and S′. Let S be at rest and S′ be moving with a uniform velocity v in the positive x-direction. Let t and t' be the time of an event measured by S and S' respectively. To begin with let the two frames coincide i.e. $t = 0, t' = 0$. After a time t (with respect to S) an event taking place at P represented by (x, y, z, t). The same event is represented by S' as (x', y', z', t'). After a time t S frame has moved forward to a distance vt with respect to the S frame.

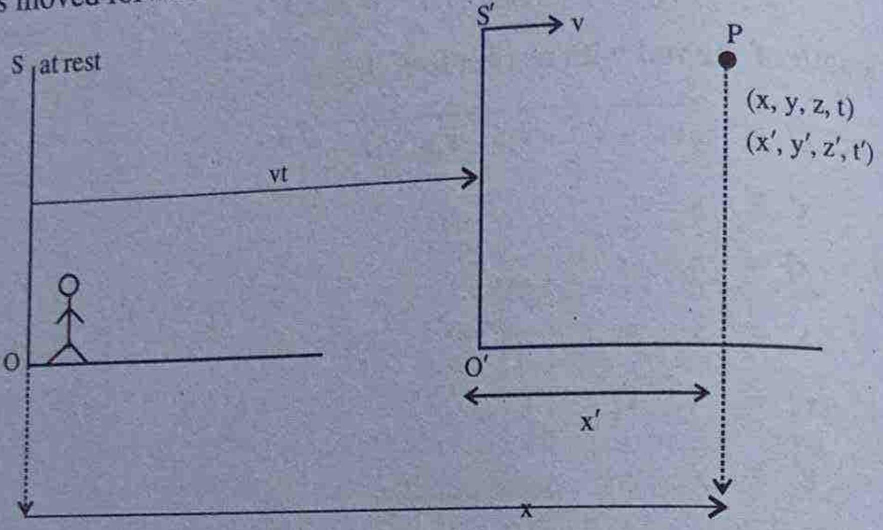


Figure 1.5

The simplest possible form of this equation can be

$$x' = k(x - vt) \qquad \text{...... (7)}$$

This is a linear equation. It should be so since a single event should appear as single to both the observers. The constant k is used in equation (7) to satisfy the two postulates of relativity. It will be a function of v, since the difference is caused by the relative motion between the two frames. According to the first postulate the observations made in the S frame must be identical to those made in S' frame except for a change in the sign of v and having the same value of k.

$$x = k(x' + vt') \qquad \text{...... (8)}$$

Here consider S' be at rest and S be moving backward with a velocity v, also the time measured by the S' frame is t'. Therefore the distance between the two frames will appear as vt'.

Since the motion is confined to x-direction only

$$y' = y \qquad \text{...... (9)}$$

$$z' = z \qquad \text{...... (10)}$$

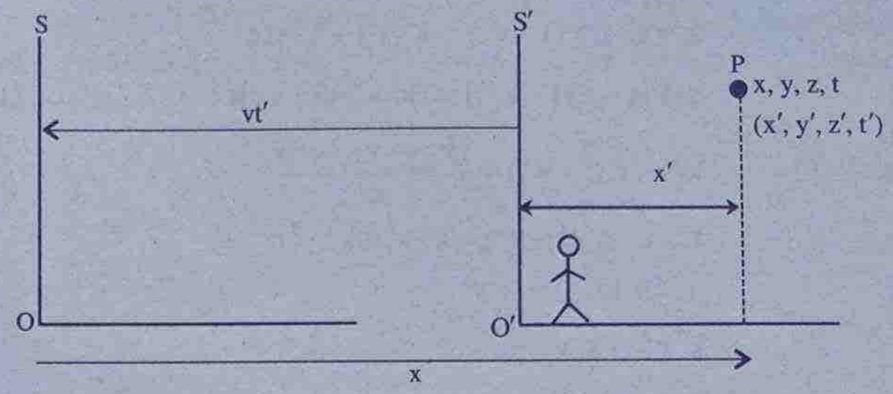


Figure 1.6

To find t'

Substitute for x' from equation (7) in equation (8)

$$x = k\{k(x - vt) + vt'\}$$

$$x = k^2x - k^2vt + kvt'$$

$$kvt' = x - k^2x + k^2vt$$

$$\therefore \quad t' = \frac{x(1-k^2)}{kv} + kt \qquad \text{...... (11)}$$

To find k

Consider again two frames S and S'. The observer in the S frame measures the time t and the observer in the S' frame measures the time t' for a flash of light. The observer in the S frame sees that the flash has moved through a distance

$$x = ct \qquad \text{...... (12)}$$

and the observer in the S' frame sees that the flash has moved through a distance

$$x' = ct' \qquad \text{...... (13)}$$

The velocity of light is assumed to be the same in both frames according to second postulate.

Substituting for x' and t' in equation (13) from equations (7) and (11) respectively, we get

$$k(x - vt) = c\left\{\frac{x(1-k^2) + k^2vt}{kv}\right\}$$

$$k^2v(x - vt) = c\,x(1-k^2) + k^2vtc$$

$$k^2vx - k^2v^2t = c\,x(1-k^2) + k^2vtc$$

$$k^2vx - c\,x(1-k^2) = k^2 v^2 t + k^2 vtc$$

$$x[k^2v - c\,(1-k^2)] = (k^2 v^2 + k^2 vc)t \qquad \text{...... (14)}$$

$\dfrac{\text{equation 14}}{\text{equation 12}}$ gives
$$k^2v - c\,(1-k^2) = \frac{(k^2 v^2 + k^2 vc)}{c}$$

$$k^2vc - c^2(1-k^2) = k^2 v^2 + k^2 v^2 c$$

$$-c^2 + k^2 c^2 = k^2v^2$$

$$k^2 c^2 - k^2v^2 = c^2$$

$$k^2(c^2 - v^2) = c^2$$

$$k^2 = \frac{c^2}{c^2 - v^2} = \frac{1}{1 - v^2/c^2}$$

$$k = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Substituting the value of k in equations (7), (9), (10) and (11), we get

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \qquad \text{...... (15)}$$

$$y' = y \qquad \text{...... (16)}$$

$$z' = z \qquad \text{...... (17)}$$

$$t' = \frac{x\left(1 - \dfrac{1}{1 - v^2/c^2}\right)}{\dfrac{1}{\sqrt{1 - v^2/c^2}}\,v} + \frac{t}{\sqrt{1 - v^2/c^2}}$$

$$t' = \frac{x\left(1 - \dfrac{v^2}{c^2} - 1\right)}{\dfrac{v}{\sqrt{1 - v^2/c^2}}\,(1 - v^2/c^2)} + \frac{t}{\sqrt{1 - v^2/c^2}}$$

$$t' = \frac{-x\,v^2/c^2}{v\sqrt{1 - v^2/c^2}} + \frac{t}{\sqrt{1 - v^2/c^2}}$$

$$t' = \frac{-x\,v/c^2}{\sqrt{1 - v^2/c^2}} + \frac{t}{\sqrt{1 - v^2/c^2}}$$

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \qquad \text{...... (18)}$$

Equations (15), (16), (17) and (18) are called Lorentz transformation equations. The above equations can be written in another form

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \qquad \text{...... (19)}$$

$$y = y' \qquad \text{...... (20)}$$

$$z = z' \qquad \text{...... (21)}$$

$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \qquad \text{...... (22)}$$

Where v of the former equations are replaced by -v and we interchanged primed and unprimed quantities. Equations 19, 20, 21 and 22 are called the inverse Lorentz transformation equations.

It is now clear that the measurement of space and time are by no means absolute but are dependant upon the relative motion between the observer and the phenomenon observed. When $\frac{v}{c} \ll 1$, the Lorentz transformation equations reduce to the Galilean transformation equations.

Example 8

At what speed v, will the Galilean and Lorentz expressions for x differ by 10%.

Solution

$$x_G = x - vt$$

$$x_L = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$\therefore \quad \frac{x_L - x_G}{x_G} = \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right)$$

But $$\frac{x_L - x_G}{x_G} = 10\% = \frac{10}{100} = \frac{1}{10} = 0 \cdot 1$$

$$0 \cdot 1 = \frac{1}{\sqrt{1 - v^2/c^2}} - 1$$

$$1 \cdot 1 = \frac{1}{\sqrt{1 - v^2/c^2}} \quad \text{squaring on both sides}$$

$$1 \cdot 21 = \frac{1}{1 - v^2/c^2}$$

$$1-\frac{v^2}{c^2}=\frac{1}{1\cdot 21}$$

$$\frac{v^2}{c^2}=1-\frac{1}{1\cdot 21}=\frac{0\cdot 21}{1\cdot 21}=\frac{21}{121}$$

$$\frac{v}{c}=0\cdot 417 \quad \text{or} \quad v=0\cdot 417c$$

See the enormous speed required to have a change by 10%.

Relativistic kinematics

We found that classical mechanics obeys the Galilean transformation whereas in relativity it obeys Lorentz transformations. This shows that in the realm of relativity all Newtonian results need modification, there we develop the kinematics appropriate to the Lorentz transformation. The Lorentz transformation equations and their inverse transformation are

$$\left.\begin{aligned} x' &= \gamma(x-vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t-\frac{vx}{c^2}\right), \text{ where } \gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned}\right] \text{Lorentz transformation}$$

$$\left.\begin{aligned} x &= \gamma(x'\times vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t'+\frac{vx'}{c^2}\right) \end{aligned}\right] \text{Inverse Lorentz transformation equations}$$

The Lorentz transformation equations show that the co-ordinates of the event(x, y, z, t) with respect to one inertial frame (S) is not equal to the co-ordinates of the same event (x',y',z',t') with respect to another inertial frame (S') (see example 8 and 9). From this we can very well conclude that space and time are relative. That is the measurements of length (space) and time depend upon the nature of the inertial frames. One more thing to be noted here is that going from Newtonian mechanics to relativity the main change is brought by the factor γ. The variation of γ with $\frac{v}{c}$ is shown

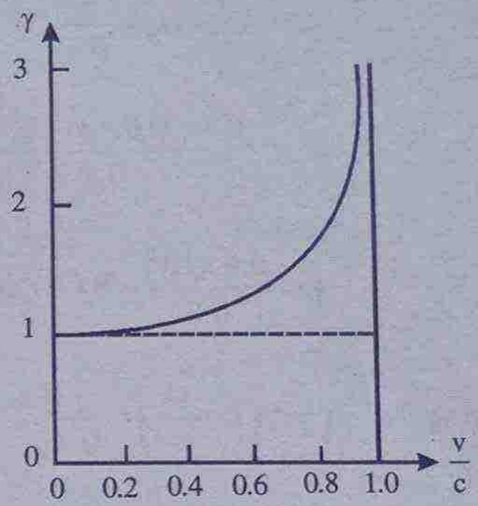


Figure 1.7

In relativity the limiting speed is velocity of light where as in Newtonian mechanics in principle the limiting speed is ∞. So if you replace c by ∞ in Lorentz transformation equations we get back Galilean transformation equations.

Example 9

Consider two inertial frames S (x, y, z, t) and $S'(x', y', z', t')$. S' is moving with speed v relative to S along the x-axis. The origins at $t = t' = 0$

Assume that $v = 0.6c$ find the co-ordinates in S' of the following.

(a) $x = 4\text{m},\ t = 0\text{ s}$ (b) $x = 10^9\text{m},\ t = 2\text{s}$

Solution

We have $\quad x' = \gamma(x - vt)$

$$\therefore \quad x' = \gamma(4 - 0) = \gamma 4$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - 0.6^2 \frac{c^2}{c^2}}} = \frac{1}{\sqrt{0.64}} = \frac{1}{0.8}$$

$$\therefore \quad x' = \frac{4}{0.8} = 5\text{m}$$

Using $\quad t' = \gamma\left(t - \frac{vx}{c^2}\right)$

$$t' = \frac{1}{0.8}\left(0 - \frac{0.6c \times 4}{c^2}\right)$$

$$t' = -\frac{0.6 \times 4}{0.8 \times c} = -\frac{2.4}{0.8 \times 3 \times 10^8}$$

$$t' = -10^{-8}\text{s}$$

b) $$x' = \gamma(x - vt) = \frac{1}{0.8}(10^9 - 0.6c \times 2)$$

$$x' = \left(\frac{10^9 - 0.6 \times 3 \times 10^8 \times 2}{0.8}\right)$$

$$x' = \frac{6.4 \times 10^8}{0.8} = 8 \times 10^8 \text{m}$$

Using $$t' = \gamma\left(t - \frac{vx}{c^2}\right) = \frac{1}{0.8}\left(2 - \frac{0.6c \times 10^9}{c^2}\right)$$

$$t' = \frac{1}{0.8}\left(2 - \frac{0.6 \times 10^9}{3 \times 10^8}\right) = \frac{1}{0.8}(2-2)$$

$$t' = 0$$

Example 10

An event occurs in S at $x = 6 \times 10^8$m and in S′ at $x' = 6 \times 10^8$m, $t' = 4$s Find the relative velocity of the systems.

Solution

$x = 6 \times 10^8$m, $x' = 6 \times 10^8$m and $t' = 4$s

Using inverse Lorentz transformation

$$x = \gamma(x' + vt')$$

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$$

$$6 \times 10^8 = \frac{6 \times 10^8 + v \times 4}{\sqrt{1 - v^2/c^2}}$$

or $$\sqrt{1 - v^2/c^2} = \frac{6 \times 10^8 + 4v}{6 \times 10^8}$$

$$\sqrt{1 - v^2/c^2} = 1 + \frac{4v}{6 \times 10^8} = 1 + 2\frac{v}{c}$$

Squaring on both sides we get

$$1-\frac{v^2}{c^2}=1+4\frac{v}{c}+\frac{4v^2}{c^2}$$

$$-\frac{v^2}{c^2}=4\frac{v}{c}+4\frac{v^2}{c^2}$$

$$-5\frac{v^2}{c^2}=4\frac{v}{c}$$

$$-5\frac{v}{c}=4$$

$$v=-\frac{4}{5}c=-\frac{4\times3\times10^8}{5}$$

$$v=-2.4\times10^8\,ms^{-1}$$

Simultaneity

We have seen that space and time are relative in its nature. One of the most important consequences of this nature is that simultaneity is relative. The events that seem to take place simultaneously to one observer may not be simultaneous to another observer in relative motion and vice versa.

Suppose that two events take place at same instant at two different positions x_1 and x_2 in the frame of reference S. An observer in the frames S' moving relative to S would measure the instants at which the two events occur as

$$t_1'=\frac{t_1-\frac{vx_1}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{...... (1)}$$

and

$$t_2'=\frac{t_2-\frac{vx_2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{...... (2)}$$

eq (2) - eq (1) gives

$$t_2' - t_1' = \frac{(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$t_2 - t_1$ is the time interval measured in the S frame. Since the events take place simultaneously with respect to S frame $t_2 - t_1 = 0$.

$$\therefore \quad t_2' - t_1' = \frac{\frac{v}{c^2}(x_1 - x_2)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This shows that $t_2' - t_1'$, which is the time intervals of two events with respect to S' frame is not zero. Thus the observer in the S' frame concludes that events are not simultaneous. i.e. The two events that are simultaneous in one frame are not simultaneous in another frame of reference.

In other words simultaneity of events is relative.

(see examples 9 and 10)

The order of events: Timelike and space like intervals

Consider two events A and B occur in space. An observer in the S frame measures the co-ordinates of A and B as (x_A, t_A) and (x_B, t_B).

With respect to the S frame, the events are separated by a distance say L

$$L = x_B - x_A$$

and the time separation of the events be

$$T = t_B - t_A$$

Assume that $x_B > x_A$ and $t_B > t_A$, so that L and T are positive.

Suppose these events A and B are observed by observer in the S' frame moving with a speed v along the positive x-axis. Let x_A', t_A' be the coordinates of event A and x_B', t_B' be the coordinates of event B with respect to S' frame.

The space separation between the two events A and B with respect to S' frame is

$$L' = x_B' - x_A'$$

Using Lorentz transformation equation

$$x' = \gamma(x - vt)$$

we get $\quad x'_B = \gamma(x_B - vt_B)$

and $\quad x'_A = \gamma(x_A - vt_A)$

$\therefore \quad L' = \gamma[(x_B - x_A) - v(t_B - t_A)]$

But $\quad x_B - x_A = L$ and $t_B - t_A = T$

Thus, $\quad L' = \gamma(L - vT) \qquad .....(1)$

Similarly the time interval between two events A and B with respect to S' frame is

$$T' = t'_B - t'_A$$

Using $\quad t' = \gamma(t - vx/c^2)$

we have $\quad t'_B = \gamma\left(t_B - v\frac{x_B}{c^2}\right)$

and $\quad t'_A = \gamma\left(t_A - v\frac{x_A}{c^2}\right)$

$\therefore \quad T' = \gamma\left[(t_B - t_{A)} - \frac{v}{c^2}(x_B - x_A)\right]$

or $\quad T' = \gamma\left(T - \frac{v}{c^2}L\right) \qquad .....(2)$

If $L > cT$, equation (1) shows that

L' is always positive

If $L > cT$ equation (2) becomes

$$T' = \gamma\left(T - \frac{v}{c^2} \times > cT\right)$$

or $\quad T' = \gamma\left(T - > \frac{v}{c}T\right)$

Depending upon the value of v, T' can be positive, zero or negative.

When the space intervals is positive and the time interval is positive, zero or negative, then the space-time interval is called space like. For this to happen L' must be greater than cT. In a space like interval it is possible to choose an inertial system in which the events are simultaneous. Events in S' frame simultaneous means $T'=0$. This gives $T=\frac{v}{c^2}L$ or $v=c^2\frac{T}{L}$.

If $L<cT$, equation (1) shows that L' can be positive, negative or zero. But equation (2) shows that, T' is always positive. That is when the time interval is positive and the space interval is positive, zero or negative, then the space-time interval is said to be time like. In a time like interval it is always possible to find an inertial system in which the events occur at the same point.

Lorentz length contraction

The length of a body is measured to be greatest when it is at rest relative to the observer. When it moves with a velocity relative to the observer its length is contracted in the direction of its motion where as its dimension perpendicular to the direction are unaffected.

Proof

Consider a rod lying at rest along the x'-axis of the S' frame which moves with a velocity v in the positive x- direction. Its end points are measured to be at x_2' and x_1' so that its length.

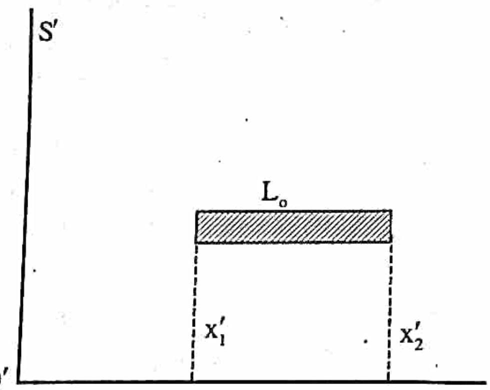


Figure 1.8

$$L_0 = x_2' - x_1' \qquad \text{...... (1)}$$

An observer in the rest frame measures the points as x_2 and x_1. The length L as measured by the observer in S is

$$L = x_2 - x_1 \qquad \text{...... (2)}$$

From the first Lorentz transformation equations we have

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$\therefore \qquad x_1' = \frac{x_1 - vt_1}{\sqrt{1-v^2/c^2}}$$

$$x_2' = \frac{x_2 - vt_2}{\sqrt{1-v^2/c^2}}$$

So that
$$x_2' - x_1' = \frac{(x_2 - x_1) - v(t_2 - t_1)}{\sqrt{1-v^2/c^2}}$$

But the measurement of length involves the simultaneous determination of the spatial coordinates of its end point whether it is with respect to an observer at rest or in motion i.e., $t_2 = t_1$.

$$\therefore \qquad x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1-v^2/c^2}}$$

Substituting for $x_2' - x_1'$ and $x_2 - x_1$ from equations (1) and (2) we have

$$L_0 = \frac{L}{\sqrt{1-v^2/c^2}}$$

or
$$L = L_0\sqrt{1-v^2/c^2}$$

i.e.,
$$L < L_0.$$

This shows that the length of a moving rod appears to contract from its rest length in the direction of its motion. This is called Lorentz - Fitzgerald contraction. This contraction is appreciable only when the velocity v is comparable to the velocity of light c. Even then it has been verified experimentally several times.

When $v = c, L = o$

This shows that it is impossible to impart a velocity equal to that of light to a body. It is due to length contraction a fastly moving ring appears to be oblate in shape (ellipse), sphere appears to be an ellipsoid, a cube appears to be a parallelopiped and so on.

Example 11

The length of a space ship is measured to be exactly half its proper length. What is the speed of the space ship relative to the observers frame.

Solution

We have

$$L = L_0 \sqrt{1 - v^2/c^2}$$

$$L = \frac{L_0}{2} \text{ given}$$

$$\therefore \quad \frac{L_0}{2} = L_0 \sqrt{1 - v^2/c^2}$$

$$\tfrac{1}{2} = \sqrt{1 - v^2/c^2} \text{ Squaring on both sides}$$

$$\tfrac{1}{4} = 1 - \frac{v^2}{c^2}$$

$$\frac{v^2}{c^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\frac{v}{c} = \frac{\sqrt{3}}{2} = 0 \cdot 866$$

$$v = 0 \cdot 866c = 2 \cdot 598 \times 10^8 \text{ms}^{-1}$$

Example 12

A circular ring in $x - y$ plane moves parallel to the x- axis. What should be its velocity so that its area appears to be half the stationary area.

Solution

Area of the ring $= \pi R_0^2, R_0$ be the radius of the ring. As it moves along x-direction, its radius along x-axis suffers contraction. As a result the ring assumes the shape of an ellipse with semi major axis R_0 and semi minor axis R

Area of the ellipse $= \pi R_0 R$

As this area appears to be half that of ring we have

$$\pi R_0 R = \frac{\pi R_0^2}{2} \qquad R = \frac{R_0}{2}$$

using $\quad L = L_0 \sqrt{1 - v^2/c^2}$ with $L_0 = R_0, L = R = \frac{R_0}{2}$

$$\frac{R_0}{2} = R_0\sqrt{1 - v^2/c^2}$$

$\frac{1}{2} = \sqrt{1 - v^2/c^2}$ squaring on both sides

$\frac{1}{4} = 1 - v^2/c^2$

$$\frac{v^2}{c^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$v = \frac{\sqrt{8}}{2}c = 0 \cdot 866\ c = 2 \cdot 598 \times 10^8\,\mathrm{ms^{-1}}$$

The orientation of a moving rod

Consider a rod of length L' lies in the S' frame making an angle θ'. The frame S' is moving with a velocity v along the positive x-direction. Here our aim is to calculate the length and orientation of the rod with respect to an observer in the S frame.

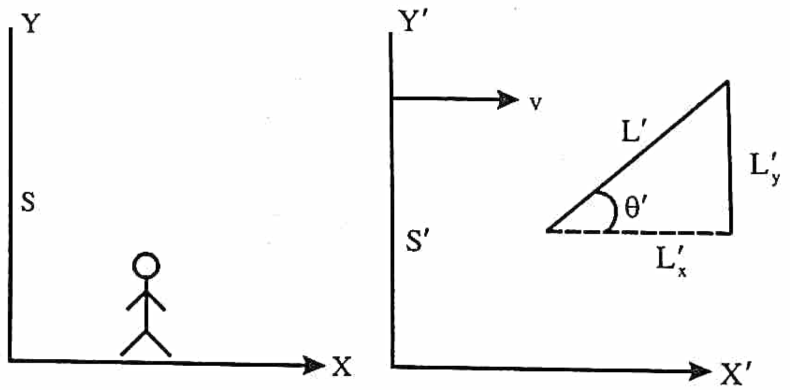


Figure 1.9

From the S' frame, we have

$$L'_x = L'\cos\theta'$$

and $$L'_y = L'\sin\theta'$$

$$\therefore \qquad \frac{L'_y}{L'_x} = \tan\theta' \qquad ......(1)$$

Let L be the length of the rod and θ be the orientation of the rod with respect to the S frame. Then

$$L_x = L\cos\theta$$

and $$L_y = L\sin\theta$$

$$\therefore \quad \frac{L_y}{L_x} = \tan\theta \qquad \text{.......(2)}$$

Using $L = L_0\sqrt{1-v^2/c^2}$

$$L_x = L'_x\sqrt{1-v^2/c^2}$$

and $L_y = L'_y$ (no contraction along y-direction)

$$\therefore \quad \frac{L_y}{L_x} = \gamma\frac{L'_y}{L'_x}$$

substituting for $\frac{L_y}{L_x}$ and $\frac{L'_y}{L'_x}$ from equations 1 and 2 we get

$$\tan\theta = \gamma\tan\theta'$$

or $$\theta = \tan(\gamma\tan\theta') \qquad \text{.....(3)}$$

To find the length of the rod, we use Pythagorus theorem

$$L^2 = L_x^2 + L_y^2$$

$$L^2 = \left(L'_x\sqrt{1-\frac{v^2}{c^2}}\right)^2 + L_y'^2$$

or $$L^2 = (L'\cos\theta'\sqrt{1-v^2/c^2})^2 + (L'\sin\theta')^2$$

$$L^2 = L'^2\cos^2\theta'\left(1-\frac{v^2}{c^2}\right) + L'^2\sin^2\theta'$$

$$L^2 = L'^2\left(1-\frac{v^2}{c^2}\cos^2\theta'\right)$$

$$L = L'\left(1-\frac{v^2}{c^2}\cos^2\theta'\right)^{\frac{1}{2}} \qquad \text{.....(4)}$$

Equation 3 says that the angle measured by an observer in the S frame appears to increase and equation 4 says that length appears to be shortened with respect to S frame observer. See examples 13 and 14.

Example 13

A space craft antenna is at an angle of $10°$ relative to the axis of the space craft. If the space craft moves away with a speed of 0.7c. What is the angle of the antenna as seen from the earth.

Solution

$$\theta' = 10°$$
$$v = 0.7c$$

Using $\theta = \tan^{-1}(\gamma \tan\theta')$

$$v = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-0.7^2}} = \frac{1}{\sqrt{.51}} = 1.4003$$

$$\theta = \tan^{-1}(1.4003 \tan 10°)$$

$$\theta = \tan^{-1}(0.2469) = 13.86°$$

Example 14

A light beam is emitted at angle θ' with respect to the x'-axis in S'. Find the angle θ the beam makes with respect to the x-axis in S

Solution

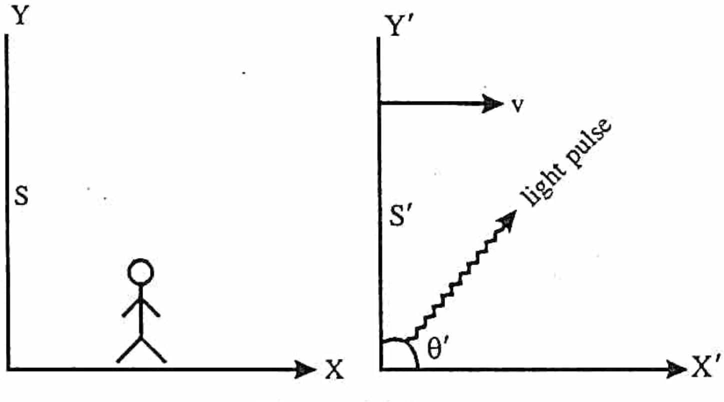


Figure 1.10

Assume that the light wave starts from the origin O'. After a time t', the x', y' coordinates of light pulse are

$$x' = ct' \cos\theta'$$

and $$y' = ct' \sin\theta'$$

with respect to the S frame the coordinates are

$$x = \gamma(x' + vt') = \gamma t'c\left(\cos\theta' + \frac{v}{c}\right)$$

$$y = y'$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

$$t = \gamma t'\left(1 + \frac{v}{c}\cos\theta'\right)$$

From the S frame we also have

$$x = ct\,\cos\theta$$

$$\therefore \qquad \cos\theta = \frac{x}{ct} = \frac{\gamma t'c\left(\cos\theta' + \frac{v}{c}\right)}{c\gamma t'\left(1 + \frac{v}{c}\cos\theta'\right)}$$

or $$\theta = \cos^{-1}\left[\frac{\left(\cos\theta' + \frac{v}{c}\right)}{\left(1 + \frac{v}{c}\cos\theta'\right)}\right]$$

Time dilation

According to relativity time intervals are affected by the relative motion between frames. Consider two inertial frames S and S'. Let S be at rest and S' be moving with a velocity v in the positive x-direction. To begin with let the two frames coincide i.e. $t = 0, t' = 0$. Then two events occur at any given point x' in frame S' at times t_1' and t_2' as noted on the clock carried by it and times t_1 and t_2 as noted on the colck carried by frame S. That is the time interval between two events as noted on the clock in the moving frame S' is $t_2' - t_1' = \Delta t_0$ and time interval between same two events as noted on the clock in the rest frame S is $t_2 - t_1 = \Delta t$.

Using Lorentz inverse transformation

$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

$$t_1 = \frac{t_1' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

$$t_2 = \frac{t_2' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

$$t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1 - v^2/c^2}}$$

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}$$

i.e., $\Delta t > \Delta t'$.

That is an interval of time observed in the rest frame is longer than in the S' frame. This effect is called time dilation. This means that a moving clock runs slow with respect to a stationary clock. Remember that smaller time intervals means moving slow.

When $v \to c$, $\Delta t' \to 0$

That is, the passage of time and also the process of aging will be stopped. This explains what has come to be known as the twin paradox.

Example 15

A particle with mean proper life time of 2×10^{-6} s moves through the laboratory with a speed of 0.99 c. Calculate its life time as measured by an observer in laboratory.

Solution

$$\Delta t' = 2 \times 10^{-6} \text{s}$$

$$v = 0 \cdot 998 c$$

Using $$\Delta t = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2\times10^{-6}}{\sqrt{1-9\cdot98^2}} \simeq 32\times10^{-6}\,\text{s}.$$

Example 16

Consider two identical twins of age 25 years. One remains on earth the other travels within a space ship with a velocity $\frac{\sqrt{3}}{2}c$. After 25 years elapsed on earth, traveller returns. Then what are their ages?

Solution

$$v = \frac{\sqrt{3}}{2}c$$

$$\Delta t = 25 \text{ years}$$

$$\Delta t' = ?$$

Using $$\Delta t = \frac{\Delta t'}{\sqrt{1-v^2/c^2}}$$

$$\Delta t' = \Delta t\sqrt{1-v^2/c^2} = 25\sqrt{1-\frac{3}{4}} = 12\cdot5\,\text{years}.$$

$\therefore$ Age of the traveller $= 25+12\cdot5 = 37\cdot5$ years.

Age of one who stayed on earth $= 25+25 = 50$ years.

Example 17

A particle with a mean proper life of $1\ \mu s$ moves through the laboratory at $2\cdot7\times10^8\,ms^{-1}$. What will be the distance travelled by it before disintegration.

Solution

$$v = 2\cdot7\times10^8\,ms^{-1}$$

$$\Delta t' = 10^{-6}s$$

$$\Delta t = \frac{\Delta t'}{\sqrt{1-v^2/c^2}} = \frac{10^{-6}}{\sqrt{1-\left(\frac{2\cdot7\times10^8}{3\times109}\right)^2}} = 2\cdot3\times10^{-6}s$$

$\therefore$ Distance travelled $x = v\Delta t = 2\cdot7\times10^8\times2\cdot3\times10^{-6} = 620\,m$

Proper frame, proper length and proper time

The inertial frame of reference in which observed body is at rest is called the proper frame of reference. **The length of a rod as measured in the inertial frame in which it is at rest is called the proper length.**

In the relation $L = L_0\sqrt{1-v^2/c^2}$

L_0 is called the proper length.

Like wise the proper time interval is the time interval recorded by a clock attached to the observed body. The relation between the proper time interval $\Delta\tau_0$ and non proper time $(\Delta\tau)$ is as follows.

$$\Delta\tau = \frac{\Delta\tau_0}{\sqrt{1-v^2/c^2}}.$$

Proper time interval is an invariant quantity in relativity.

Muon decay

The time dilation effect can be indirectly verified by the fact of μ^+ mesons reaching the ground level. These elementary particles are produced in the upper atmosphere as a result of collision of cosmic rays with the air molecules. Though they move with different velocities the faster ones among them have a velocity of 0.998 c.

These particles can travel a distance of

$$x = vt = 0\cdot998c \times 2\times10^{-6} = 600 \text{ m}$$

where $t = 2\times10^{-6}$s is the deacy time of μ mesons. These particles are produced at an altitude of more than 10km above the surface of the earth can hardly reach the surface of earth. But a large number of mesons reach the earths surface. This is because of time dilation. The decay time 2×10^{-6}s is the time with respect to their own frame of reference. i.e. with respect to a person on the earths surface this time will be dilated. Using equation

$$\Delta\tau = \frac{\Delta\tau_0}{\sqrt{1-v^2/c^2}} = \frac{2\times10^{-6}}{\sqrt{1-(0\cdot998)^2}} = 32\times10^{-6}\text{s}$$

i.e. more than 16 times than in their own frame of reference. In this much longer life time, they can travel a distance

$$x = vt = 0\cdot998 \times 3 \times 10^{8} \times 32 \times 10^{-6}$$

$$= 10\,\text{km}$$

This explains the presence of mesons near the surface of the earth and indirectly verifies the time dilation effect.

Role of time dilation in atomic clock

Now a days atomic clocks are used as standard references to measure unit of length(metre) and also unit of time (second). This is because atomic clocks have several advantages. They are easily accessible, invariable and highly precise. Here we talk about the measurement time and time dilation effect.

When an atom jump from an higher state to lower state it emits radiation with definite frequency given by $\upsilon \simeq \frac{\Delta E}{h} = \frac{1.6 \times 10^{-19}}{6.6 \times 10^{-34}} \simeq 10^{15}\,\text{Hz}$. If ΔE is of the order of electron volt, light emitted is in the optical region.

When the energy change is very small the emitted radiation is in the microwave region $(\upsilon \simeq 10^{10}\,\text{Hz})$.

These microwave radiations can be detected and amplified electronically and need as our standard reference to govern the rate of an atomic clock. Atomic clocks are highly precise. Then precision is of the order of I part 10^{13}. This means that, for example, two maser clocks run for 33,000,00 years, they commit an error of 1 second.

Each atom radiating at its natural frequency serves as a miniature clock. Atoms in a gas are in random motion so these clocks are not at rest with respect to the rest (Laboratory) frame. As a result the observed frequency as shifted due to time dilation effect. Hence we calculate the effect of time dilation.

Consider an atom emitting radiation of its natural frequency υ_0 in its frame. The observed frequency in the lab frame is υ.

Using $\quad \upsilon_0 = \frac{1}{\Delta t_0}$

and $\quad \upsilon = \frac{1}{\Delta t}$

or $\quad \frac{\upsilon}{\upsilon_0} = \frac{\Delta t_0}{\Delta t}$

Using $\Delta t = \gamma\Delta t_0$, we get

$$\frac{\upsilon}{\upsilon_0} = \frac{1}{\gamma}$$

or $$\upsilon = \frac{1}{\gamma}\upsilon_0$$

$$\upsilon = \upsilon_0\sqrt{1-\frac{v^2}{c^2}},$$ Using Binomial approximation

$$\upsilon = \upsilon_0\left(1-\frac{v^2}{c^2}\right)^{1/2} \simeq \upsilon_0\left(1-\frac{1}{2}\frac{v^2}{c^2}\right)$$

$$\therefore \quad \frac{\upsilon-\upsilon_0}{\upsilon_0} = -\frac{1}{2}\frac{v^2}{c^2}$$

L.H.S gives the fractional change in frequency $\frac{\delta\upsilon}{\upsilon_0}$.

i.e., $$\frac{\delta\upsilon}{\upsilon_0} = -\frac{1}{2}\frac{v^2}{c^2}$$

Multiply numerator an denominator by m, the mass of the atom.

$$\frac{\delta\upsilon}{\upsilon_0} = -\frac{1}{2}\frac{mv^2}{mc^2}$$

According to kinetic theory of gases $\frac{1}{2}mv^2$ is the kinetic energy of atom due to thermal motion. According to statistical mechanics

$$\frac{1}{2}mv^2 = \frac{3}{2}kT$$

where k is the Boltzmann's constant and T is the temperature in kelvin.

$$\therefore \quad \frac{\delta\upsilon}{\upsilon_0} = -\frac{3kT}{2mc^2}$$

substituting the value of $k = 1.38\times10^{-23}\,JK^{-1}$

$m = 1.67 \times 10^{-27}$ kg and $c = 3 \times 10^{8}\,ms^{-1}$

we get $\dfrac{\delta\upsilon}{\upsilon_0} = \dfrac{-3 \times 1.38 \times 10^{-23} \times T}{3 \times 1.67 \times 10^{-27} \times 9 \times 10^{16}}$

$$\frac{\delta\upsilon}{\upsilon_0} = -1.377 \times 10^{-13} T$$

This shows that in order to have an accuracy of 1 part in 10^{13}, it is necessary that the temperature measurement of the radiating atoms should have an accuracy of one kelvin. If we go for an accuracy of 1 part in 10^{15}, T should have an accuracy of 10^{-3} K. It is an herculean task to achieve this.

Relativistic transformation of velocity

Suppose a particle has a velocity u in the rest frame its components along the three coordinates are

$$u_x = \frac{dx}{dt},\ u_y = \frac{dy}{dt} \text{ and } u_z = \frac{dz}{dt}$$

and its velocity in a frame of reference S' moving with a velocity v relative to S along the positive x- direction is u'. Its components are

$$u'_x = \frac{dx'}{dt'},\ u'_y = \frac{dy'}{dt'} \text{ and } u'_z = \frac{dz'}{dt'}$$

We will see that how these component velocities in the two frames are related to each other. From the Lorentz transformation equations we have

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}},\ y' = y,\ z' = z,\ t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

Taking the differentials of above four equations, we have

$$dx' = \frac{dx - vdt}{\sqrt{1 - v^2/c^2}} \qquad \text{...... (1)}$$

$$dy' = dy \qquad \text{...... (2)}$$

$$dz' = dz \qquad \text{...... (3)}$$

$$dt' = \frac{dt - vdx/c^2}{\sqrt{1 - v^2/c^2}} \qquad \text{...... (4)}$$

$\dfrac{\text{equation 1}}{\text{equation 4}}$ gives $$\frac{dx'}{dt'}=\frac{dx-v\,dt}{dt-\dfrac{v\,dx}{c^2}}=\frac{\left(\dfrac{dx}{dt}-v\right)}{\left(1-\dfrac{v}{c^2}\dfrac{dx}{dt}\right)}$$

$$u'_x=\frac{u_x-v}{1-\dfrac{u_x v}{c^2}} \qquad \text{...... (5)}$$

$\dfrac{\text{equation 2}}{\text{equation 4}}$ gives $$\frac{dy'}{dt'}=\frac{dy\sqrt{1-v^2/c^2}}{dt\ -v\,dx/c^2}=\frac{\dfrac{dy}{dt}\sqrt{1-v^2/c^2}}{1-\dfrac{v\,dx/dt}{c^2}}$$

$$u'_y=\frac{u_y\sqrt{1-v^2/c^2}}{1-\dfrac{u_x v}{c^2}} \qquad \text{...... (6)}$$

$\dfrac{\text{equation 3}}{\text{equation 4}}$ gives $$u'_z=\frac{u_z\sqrt{1-v^2/c^2}}{1-\dfrac{u_x v}{c^2}} \qquad \text{...... (7)}$$

Equations 5, 6 and 7 are called velocity transformation equations.

The inverse velocity trasformation equations are

$$u_x=\frac{u'_x+v}{1+u'_x v/c^2} \qquad \text{...... (8)}$$

$$u_y=\frac{u'_y\sqrt{1-v^2/c^2}}{1+\dfrac{u'_x v}{c^2}} \qquad \text{...... (9)}$$

$$u_z=\frac{u'_z\sqrt{1-v^2/c^2}}{1+\dfrac{u'_x v}{c^2}} \qquad \text{...... (10)}$$

To see that velocity of light is invariant under the relativistic velocity transformations, let us consider two photons A and B approaching one another as shown in figure.

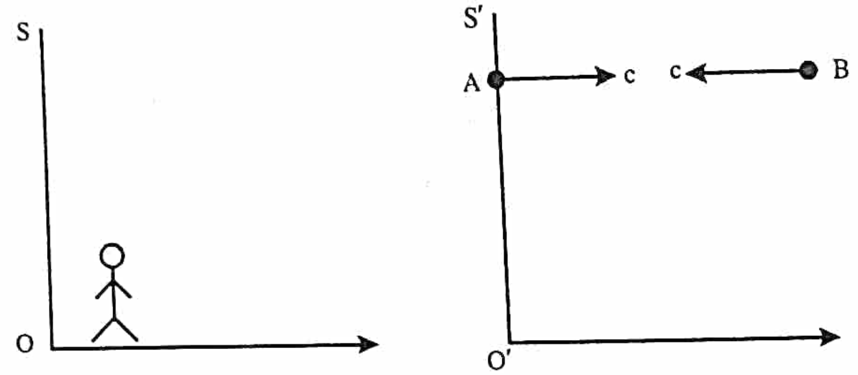


Figure 1.11

We have to calculate the velocity of photon B with respect to A. For this consider an observer in the rest frame S. Let another observer sitting on photon A which is considered as the moving frame S'. i.e. S' frame is moving with speed c. If B is a photon observed by the observer in the rest frame S. Then $u_x = -c$ and $v = c$

$\therefore$ The velocity of the photon B with respect to A

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{-c - c}{1 - \frac{c \times -c}{c^2}} = -c$$

Let us now consider two photons A and B moving in the same direction as shown in figure below.

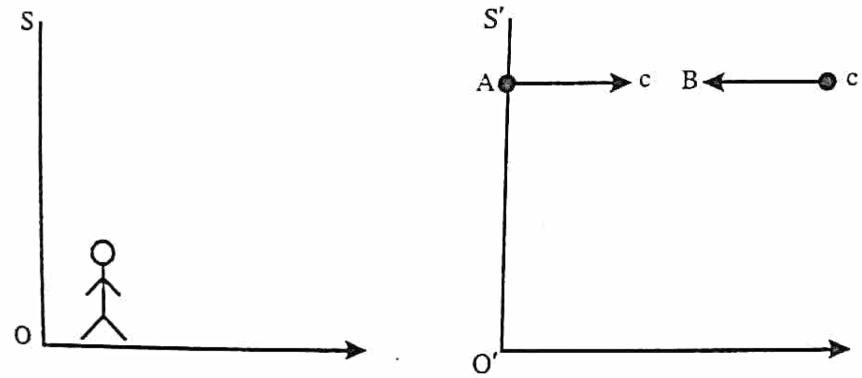


Figure 1.12

Proceeding as described above, we have

$$u_x = c, \quad v = c$$

$$\therefore \quad u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{c - c}{1 - \frac{c \cdot c}{c^2}} = \frac{0}{0}$$

This is indeterminate. To determine the value of u'_x, we take the limit of above equation with $v \to c$.

$$u'_x = \mathrm{Lt}_{v\to c} \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \mathrm{Lt}_{v\to c} \frac{c - v}{1 - \frac{v}{c}}$$

Using L'Hospital's rule, we get

$$= \left. \frac{\frac{d}{dv}(c - v)}{\frac{d}{dv}\left(1 - \frac{v}{c}\right)} \right|_{v=c} = \frac{-1}{-\frac{1}{c}} = c$$

This is in well agreement with the second postulate of special theory of relativity.

The relativistic velocity transformation equations were tested experimentally by T. Alvagar at CERN. Alvager used a beam of protons of energy 20 GeV. The protons bombarded a target to produce neutral pions (π°) of energy more than 6 GeV. π° decays into two γ-ray photons. Alvagar determined the velocity of γ-ray photon moving in the forward direction. The velocity of γ-ray photon in a frame at rest with respect to π° (i.e. S' frame) $u'_x = c$. The velocity of π° with respect to $S = 0.99975c$

$\therefore$ Velocity of γ-ray photon with respect to S frame.

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} = \frac{c + 0.99975c}{1 + \frac{0.99975c^2}{c^2}} = c$$

The experimental measurement was carried out by measuring the time taken by the γ-ray to travel between two detectors placed 0.3m apart. The experimental value was in excellent agreement with the velocity calculated using relativistic velocity transformations.

Example 18

Two electrons leave a radio active sample in opposite directions, each having a speed 0.67c with respect to the sample. Calculate the speed of one electron with respect to the other (i) classically and (ii) relativistically.

Solution

Consider one electron as the S-frame, the sample as the S' frame, and the other electron as the object whose speed in the S-frame is to be determined. Then $u' = 0.67c$, $v = 0.67c$.

Using inverse velocity transformation equation we have

$$u = \frac{u'+v}{1+\frac{u'v}{c^2}}$$

$$u = \frac{0.67c + 0.67c}{1+\frac{0.67c \times 0.67c}{c^2}}$$

$$u = \frac{1.34c}{1.45} = 0.92c$$

According to classical relation (Galilean velocity transformation equation)

$$u = u' + v$$

$$u = 0.67c + 0.67c = 1.34c$$

Example 19

Two photons approach each other. What is the velocity of one photon with respect to another.

Solution

Here $u' = c$ and $v = c$

using $$u = \frac{u'+v}{1+\frac{uv}{c^2}} = \frac{c+c}{1+\frac{c^2}{c^2}} = \frac{2c}{2} = c$$

Example 20

If a photon traverses the path in such a way that it moves in $x' - y'$ plane and makes an angle θ with the axis of the frame S', then prove for frame S,

$$u_x^2 + u_y^2 = c^2$$

Solution

and $$\left.\begin{aligned} u'_x &= c\cos\theta \\ u'_y &= c\sin\theta \end{aligned}\right] \text{ given}$$

using $$u_x = \frac{u'_x + v}{1 + \dfrac{u'_x v}{c^2}}$$

and $$u_y = \frac{u'_y\sqrt{1 - \dfrac{v^2}{c^2}}}{1 + \dfrac{u'_x v}{c^2}}$$

We get $$u_x = \frac{c\cos\theta + v}{1 + \dfrac{v}{c}\cos\theta} \qquad \text{...... (1)}$$

$$u_y = \frac{c\sin\theta\sqrt{1 \dfrac{v^2}{c^2}}}{1 + \dfrac{v}{c}\cos\theta} \qquad \text{...... (2)}$$

Squaring and adding eqs (1) and (2) we get

$$u_x^2 + u_y^2 = \frac{(c\cos\theta + v)^2}{\left(1 + \dfrac{v}{c}\cos\theta\right)^2} + \frac{c^2\sin^2\theta\left(1 - \dfrac{v^2}{c^2}\right)}{\left(1 + \dfrac{v}{c}\cos\theta\right)^2}$$

$$u_x^2 + u_y^2 = \frac{c^2\cos^2\theta + v^2 + 2vc\cos\theta + c^2\sin^2\theta - v^2\sin^2\theta}{\left(1 + \dfrac{v}{c}\cos\theta\right)^2}$$

$$u_x^2 + u_y^2 = \frac{c^2 + v^2 + 2v\cos\theta - v^2\sin^2\theta}{\left(1 + \dfrac{v}{c}\cos\theta\right)^2}$$

$$u_x^2 + u_y^2 = \frac{c^2 + v^2 + 2vc\cos\theta - v^2(1-\cos^2\theta)}{\left(1+\frac{v}{c}\cos\theta\right)^2}$$

$$u_x^2 + u_y^2 = \frac{c^2 + 2vc\cos\theta + v^2\cos^2\theta}{\left(1+\frac{v}{c}\cos\theta\right)^2}$$

$$u_x^2 + u_y^2 = \frac{c^2\left(1+\frac{2v}{c}\cos\theta + \frac{v^2}{c^2}\cos^2\theta\right)}{\left(1+\frac{v}{c}\cos\theta\right)^2}$$

$$u_x^2 + u_y^2 = \frac{c^2\left(1+\frac{v}{c}\cos\theta\right)^2}{\left(1+\frac{v}{c}\cos\theta\right)^2} = c^2$$

Thus $\quad u_x^2 + u_y^2 = c^2$

Example 21

Two velocities 0.8c each are inclined to one another at angle of $30°$. Obtain the value of the result.

Solution

Using $\quad u_x = \dfrac{u'_x + v}{1+\dfrac{u'_x v}{c^2}}$

$u'_x = 0.8c \times \cos 30$ and $v = 0.8c$

We get $\quad u_x = \dfrac{0.8c\cos 30 + 0.8c}{1+\dfrac{0.8c\cos 30 \times 0.8c}{c^2}} = 0.96c$

and $$u_y = \frac{u'_y\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{u'_x v}{c^2}} = \frac{0.8c\sin 30\sqrt{1-0.8^2}}{1+\frac{0.8c\cos 30\times 0.8c}{c^2}}$$

$$= 0.15c$$

$\therefore$ $$u = \sqrt{u_x^2+u_y^2} = \sqrt{(0.96c)^2+(0.15c)^2} = 0.97c$$

Example 22

Two rods having the same length l_0 move lengthwise towards each other parallel to a common axis with the same velocity v relative to lab frame. What is the length of each rod in the frame fixed to the other rod.

Solution

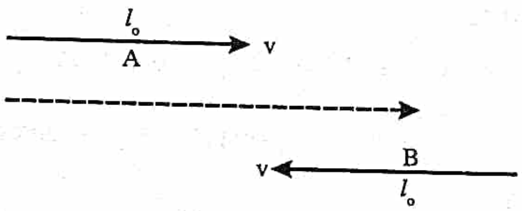


Figure 1.13

Velocity of A with respect t_0 B,

$$v_{AB} = \frac{v_A+v_B}{1+\frac{v_A v_B}{c^2}} = \frac{v+v}{1+\frac{v^2}{c^2}}$$

$$v_{AB} = \frac{2v}{1+\frac{v^2}{c^2}}$$

The length of the rod as measured by B is

Using $l = l_0\sqrt{1-\frac{v_{AB}^2}{c^2}}$

$$l = l_0\sqrt{1-\left(\frac{2v}{1+\frac{v^2}{c^2}}\right)^2\cdot\frac{1}{c^2}}$$

$$l = l_0\sqrt{1-\frac{4v^2.c^2}{(c^2+v^2)^2}}$$

$$l = l_0\sqrt{\frac{(c^2+v^2)^2-4v^2c^2}{(c^2+v^2)^2}}$$

$$\therefore \qquad l = l_0\frac{\left(1-\frac{v^2}{c^2}\right)}{1+\frac{v^2}{c^2}}$$

Speed of light in a medium

Consider a tube filled with water at rest. The velocity of light in the water with respect to laboratory frame is $\frac{c}{n}$, where n is the refractive index of water. Recall that

$$n = \frac{\text{velocity of light in vacuum}}{\text{velocity of light in medium}}$$

Suppose the water is flowing through the tube with speed v. Then what is the speed of light in moving water with respect to lab frame?

Using velocity addition formula, we have

$$u = \frac{u'+v}{1+\frac{u'v}{c^2}},$$ where , u' is the velocity of light with respect to water

$$\therefore \qquad u = \frac{\frac{c}{n}+v}{1+\frac{v}{nc}}$$

$$u = \frac{c}{n}\frac{\left(1+\frac{nv}{c}\right)}{\left(1+\frac{v}{nc}\right)}$$

$$u = \frac{c}{n}\left(1+n\frac{v}{c}\right)\left(1+\frac{v}{nc}\right)^{-1}$$

using Binomial approximation to the last term, we get

$$u = \frac{c}{n}\left(1+\frac{nv}{c}\right)\left(1-\frac{v}{nc}\right)$$

$$u = \frac{c}{n}\left(1+\frac{nv}{c}-\frac{v}{nc}-\frac{v^2}{c^2}\right)$$

Neglecting $\frac{v^2}{c^2}$, we get

$$u = \frac{c}{n}\left(1+\frac{nv}{c}-\frac{v}{nc}\right)$$

or $$u = \frac{c}{n}+v-\frac{v}{n^2}$$

or $$u = \frac{c}{n}+\left(1-\frac{1}{n^2}\right)v$$

This shows that the velocity of light in moving water with respect to lab frame is increased. In other words light appears to be dragged by the water by a factor $1-\frac{1}{n^2}$ times the speed of water. This effect is entirely due to relativity.

The Doppler effect

C.J. Doppler in 1842 observed that **whenever there is a relative motion between the source of sound and observer there is an apparent changes in frequency of the source of sound heard by the observer. This phenomenon is called Doppler effect.**

Doppler's effect is easily observed by a person standing on a platform when an engine sounding horn passes by. It is observed that the pitch(frequency) of engine appears to increase when the train approaches the person and appears to decrease when the train receeds away from the person. Though Doppler's effect is most commonly and easily observed with sound waves, all types of waves including light waves exhibit Doppler effect. Here we discuss Doppler effect in sound as well as in light waves.

Doppler shift in sound

Consider a source of sound S and an observer O. Let v and w be the velocities of the source and sound respectively. υ be the frequency of sound from the source.

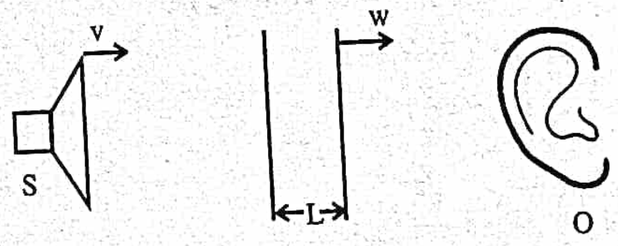


Figure 1.14

In time t the sound travels a distance of wt, consider the sound as a series of pulses separated by a time $\tau = \frac{1}{\upsilon}$.

If L be the separation of two pulses, then the number of pulses reaching the observer

$$= \frac{\text{Distance travelled}}{\text{pulse separation}}$$

$$= \frac{wt}{L}$$

$\therefore$ The number of pulses received per second $= \frac{w}{L}$

The number of pulses received per second by the observer is the frequency heard by the observer. it is denoted by υ'

i.e, $$\upsilon' = \frac{w}{L}$$

Now we can calculate L.

To determine L, consider a pulse emitted at $t = 0$ and the next pulse emitted at $t = \tau$. During this time τ first pulse travelled a distance of $w\tau$ and the source travelled a distance of $v\tau$.

$\therefore$ Then distance between the two pulses,

$$L = w\tau - v\tau = (w - v)\tau$$

$$L = \frac{(w-v)}{\upsilon} \left(\because \ \upsilon = \frac{1}{\tau}\right)$$

$$\therefore \qquad \upsilon' = \frac{w}{L} = \frac{w}{\frac{(w-v)}{\upsilon}} = \upsilon \frac{w}{w-v}$$

or

$$\upsilon' = \upsilon \frac{1}{1-\frac{v}{w}}$$

This is the expression for the apparent frequency heard by the observer when source is moving towards the observer. Obviously $\upsilon' > \upsilon$

When the source is moving away from the observer replace v by $-v$. Thus

$$\upsilon' = \upsilon \frac{1}{1+\frac{v}{w}} \text{ here } \upsilon' < \upsilon.$$

Now we consider a different situation where the source of sound is at rest and the observer is moving towards the source with a speed v.

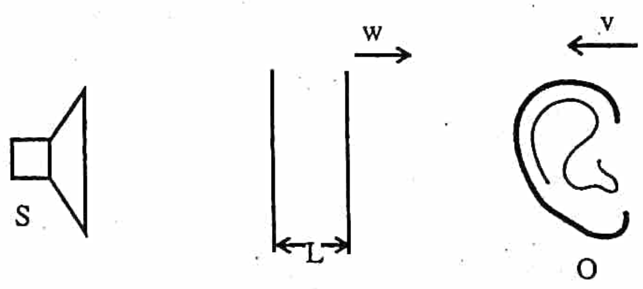


Figure 1.15

Since the source is at rest the distance between the two pulses, $L = w\tau = \frac{w}{\upsilon}$.

In time t, the number of pulses received by the observer $= \frac{(w+v)t}{L}$

$\therefore$ The number of pulses received by the observer in one second $= \frac{(w+v)}{L}$

This is nothing but the frequency heard by the observer (υ').

i.e. $$\upsilon' = \frac{(w+v)}{L}$$

Substituting for L, we get $\upsilon' = \dfrac{(w+v)}{w/\upsilon} = \upsilon\dfrac{(w+v)}{w}$

$$\upsilon' = \upsilon\left(1+\frac{v}{w}\right)$$

This is the expression for apparent frequency heard by the observer, when observer is moving towards the source.

When the observer is moving away from the source replace v by $-v$.

Thus $$\upsilon' = \upsilon\left(1-\frac{v}{w}\right)$$

The Doppler effect in sound gives us an important information. That is, knowing υ, v and w we can calculate υ' from which we can tell whether it is the observer or source is moving.

If this result is applicable to light waves (relativistic Doppler effect) then we would be able to distinguish between inertial frames which one is at rest or motion. This is contrary to the principle of relativity that it is not possible to distinguish between two inertial frames at rest and in uniform motion. To resolve this we go for relativistic Doppler effect.

Relativistic Doppler effect

Consider a light source emitting pulses of frequency υ in its rest frame $\left(\upsilon = \dfrac{1}{\tau_0}\right)$. The source is moving towards an observer with velocity.

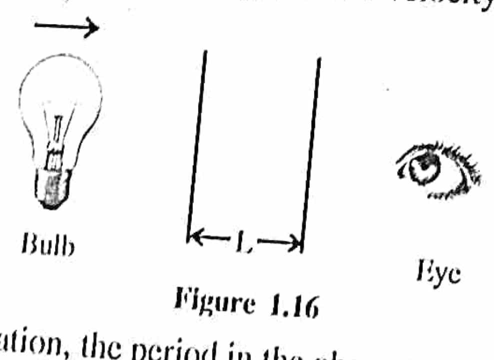


Figure 1.16

It is due to time dilation, the period in the observers rest frame is

$$\tau = \gamma\, \tau_0$$

Since the speed of light is a universal constant, the pulse arrive at the observer with velocity c

The frequency of the pulses observed by the observer is

$\upsilon' = \frac{c}{L}$, where L is the separation between the two pulses with respect to the observer.

Since the source is moving towards the observer

$$L = c\tau - v\tau \quad = (c - v)\tau$$

$$\therefore \qquad \upsilon' = \frac{c}{(c-v)\tau} = \frac{c}{(c-v)} \frac{1}{\gamma\tau_0}$$

$$\upsilon' = \upsilon\sqrt{1 - v^2/c^2} \, . \, \frac{c}{c-v}$$

or
$$\upsilon' = \upsilon \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c}}$$

$$\upsilon' = \upsilon\sqrt{\frac{1 + v/c}{1 - v/c}}$$

This is the expression for the observed fréquency. Obviously $\upsilon' > \upsilon$ as we expect since the source is moving towards the observer.

When the source is moving away from the observer replace v by $-v$.

i.e.,
$$\upsilon' = \upsilon\sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} \text{ here } \upsilon' < \upsilon$$

Note: It may be noted that the relativistic Doppler frequency is the geometric mean of two classical results.

Doppler effect for an observer off the line of motion

So far we considered the Doppler effect for a source and observer along the line of motion. It may not be always so. So we consider the general case.

Consider an observer is at angle θ from the line of motion. In this case the only parameter that get affected is the distance between the two pulses L.

When the source and observer are along the line of motion, we have

$$L = c\tau - v\tau$$

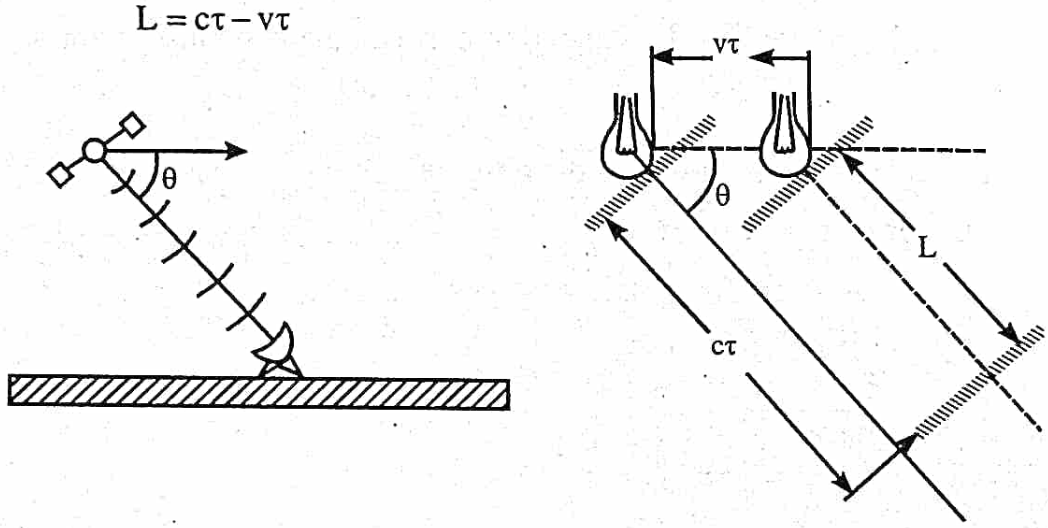


Figure 1.17

when the observer is at angle θ from the line of motion, the distance travelled by light is $c\tau$ as before since light is pervading every where. But the distance travelled by the source towards the observer is only $v\cos\theta\ \tau$. Thus

$$L = c\tau - v\cos\theta\ \tau = (c - v\cos\theta)\tau$$

Using $$\upsilon' = \frac{c}{L} = \frac{c}{(c - v\cos\theta)\gamma\tau_0}$$

$$\upsilon' = \upsilon \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c}\cos\theta}$$

It may be noted that θ is the angle measured in the rest frame of the observer.

When $\theta = 0$

$$\upsilon' = \upsilon \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

We get back our old result. This is called the longitudinal Doppler effect.

$$\theta = 90°, \ \upsilon' = \upsilon\sqrt{1-\frac{v^2}{c^2}}$$

This is called transverse Doppler effect, a phenomenon not found in Doppler effect of sound. It is solely due to time dilation. Doppler effect of light has been experimentally confirmed by Ives and Stilwell in the year 1938.

Note: Comparing Doppler effect of sound and light we can see that both formulae are the same to first order approximation in $\frac{v}{c}$. So in order to differentiate between them the effects of order $\frac{v^2}{c^2}$ has to be considered, which is a difficult task.

Application of Doppler effect – Doppler navigation

Doppler effect can be used to track a moving body such as a satellite or a fighter plane from a reference point on the earth. The precision of this tracking is fantastically high. For example the position of a satellite 10^8 m away can be determined to a fraction of a centimeter. This method was effectively used in II world war to locate enemy fighter planes.

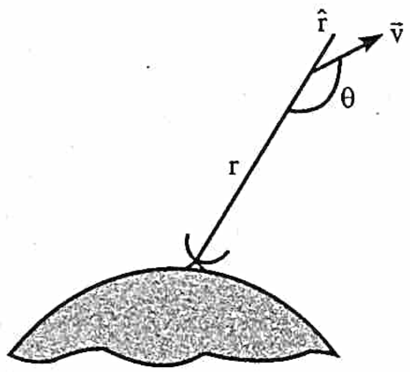


Figure 1.18

Consider a satellite moving with a velocity v at angle θ with the line of sight with respect to the ground observation point. An oscillator on the satellite broadcasts a signal with a frequency υ with respect to the satellite.

According to Doppler formula the frequency υ' received by the ground station is

$$\upsilon' = \upsilon\frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}\cos\theta}$$

since $v \ll c$, $\frac{v^2}{c^2}$ can be neglected.

$$\therefore \qquad \upsilon' = \frac{\upsilon}{1-\frac{v}{c}\cos\theta} = \upsilon\left(1-\frac{v}{c}\cos\theta\right)^{-1}$$

or $$\upsilon' = \upsilon(1+\frac{v}{c}\cos\theta)$$

or $$\upsilon' - \upsilon = \upsilon\frac{v}{c}\cos\theta$$

This shift in frequency (Doppler shift) can be measured from the ground station. For this we keep an oscillator identical to the one in the satellite, and by simple electronic methods the difference in frequency $\upsilon' - \upsilon$ (beat frequency) can be measured.

Now we have to calculate distance of the satellite by knowing $\upsilon' - \upsilon$.

The velocity of the satellite is $\bar{v} = v_r\ \hat{r} + v_\theta\ \hat{\theta}$, when $v_r = \frac{dr}{dt}$ and $v_\theta = r\frac{d\theta}{dt}$.

So $$\hat{r}\cdot\bar{v} = \frac{dr}{dt}\hat{r}\,.\,\hat{r}$$

or $$\frac{dr}{dt} = \hat{r}\cdot\bar{v}$$

$$\frac{dr}{dt} = v\cos(180-\theta) = -v\cos\theta$$

But $$v\cos\theta = \frac{c}{\upsilon}(\upsilon' - \upsilon)$$

$$\therefore \qquad \frac{dr}{dt} = -\frac{c}{\upsilon}(\upsilon' - \upsilon)$$

$$\frac{dr}{dt} = -\lambda\ (\upsilon' - \upsilon)$$

Where λ is the wave length of the pulse emitted from the oscillator.

Integrating the above equation with respect to time within limits t_a and t_b, we get

$$r_b - r_a = -\lambda\int_{t_a}^{t_b}(\upsilon' - \upsilon)dt$$

The integral on the R.H.S is the number of cycles of beat frequency which occurs in the interval $t_b - t_a$.

i.e, $$r_b - r_a = \lambda N_{ba}$$

Knowing λ and the number of beats received we calculate $r_b - r_a$.

Satellite communication systems operate at a typical wavelength 10cm and since the beat signal can be measured to a fraction of a cycle, satellites can be tracked to about 1cm.

If the satellite and ground station oscillators do not match, a two way Doppler tracking system can be used in which a signal from the ground is broadcast to the satellite which then amplifies it and relays back to the ground station. This will double the Doppler effect there by increasing the resolution by a factor of two.

Example 23

One of the most prominent spectral lines of hydrogen is H_α line, a bright red line with a wavelength of 656.1×10^{-9}m. What is the expected wavelength of the H_α line from a star receding with a speed of 3000kms^{-1}.

Solution

$$\lambda = 656.1\times10^{-9}\text{m},\ v = 3\times10^{6}\text{ms}^{-1}$$

When the source is receding away, we have

$$\upsilon' = \upsilon\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

using $\upsilon' = \dfrac{c}{\lambda'}$ and $\upsilon = \dfrac{c}{\lambda}$

$$\frac{c}{\lambda'} = \frac{c}{\lambda}\sqrt{\frac{1-v/c}{1+v/c}}$$

or $$\lambda' = \lambda \sqrt{\frac{1-v/c}{1+v/c}}$$

$$\lambda' = 656.1\times10^{-9}\sqrt{\frac{1+\frac{3\times10^{6}}{3\times10^{8}}}{1-\frac{3\times10^{6}}{3\times10^{8}}}}$$

$$\lambda' = 656.1\times10^{-9}\sqrt{\frac{1+10^{-2}}{1-10^{-2}}}$$

$$\lambda' = 656.1\times10^{-9}\sqrt{\frac{1.01}{0.99}}$$

$$\lambda' = 656.1\times10^{-9}\times1.01$$

$$\lambda' = 662.7\times10^{-9}\,\text{m}$$

Example 24

The H_α line measured on earth from opposite ends of equator differ in wave length by $9\times10^{-12}\,\text{m}$. Assume that the effective is caused by rotation of the Sun, find the period of rotation.

The radius of the Sun is $696.34\times10^{6}\,\text{km}$ $\lambda = 656.1\times10^{-9}\,\text{m}$

Solution

Let λ_1' be the observed wave length of H_α line coming from A towards the observer

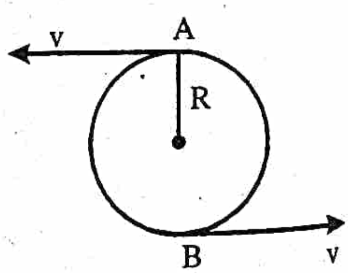


Figure 1.19

i.e., $$\lambda_1' = \lambda\sqrt{\frac{1-v/c}{1+v/c}} \simeq \lambda\sqrt{\left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right)^{-1}}$$

$$\lambda_1' \simeq \lambda\sqrt{1-\frac{2v}{c}} = \lambda\left(1-\frac{v}{c}\right).$$

Similarly λ_2' be the observed wave length of the line moving away from B, we get

$$\lambda_2' \approx \lambda\left(1+\frac{v}{c}\right)$$

$$\therefore \qquad \lambda_2' - \lambda_1' = \lambda\left(1+\frac{v}{c}\right) - \lambda\left(1-\frac{v}{c}\right)$$

$$\lambda_2' - \lambda_1' = 2\lambda\frac{v}{c}$$

$$v = \frac{c(\lambda_2' - \lambda_1')}{2\lambda}$$

$$v = \frac{3\times10^8 \times 9\times10^{-12}}{2\times656.1\times10^{-9}}$$

$$v = 2.058\text{x}10^3\,\text{ms}^{-1}.$$

Using $\quad v = R\omega$

$$\omega = \frac{v}{R} = \frac{2.058\times10^3}{696.340\times10^6}$$

$$\omega = 2.955\times10^{-6} \text{ radians}$$

$$\therefore \qquad T = \frac{2\pi}{\omega} = \frac{2\times3.14}{2.955\times10^{-6}}$$

$$T = 2.125\times10^6\,\text{s}$$

$$T = \frac{2.125\times10^6}{60\times60\times24} \text{ days}$$

$$T = 24\cdot59 \text{ days}$$

Example 25

A rocket ship is receding from the earth at a speed of 0.2c. A light in the rocket ship appears blue to passengers on the ship what colour would it appear to be to an observer on the earth. $\lambda_{\text{blue}} = 470\text{nm}$

Solution

$$v = 0.2c, \ \lambda_{\text{blue}} = 470\text{nm}$$

Using $\lambda' = \lambda \sqrt{\dfrac{1+\frac{v}{c}}{1-\frac{v}{c}}}$, $\dfrac{v}{c} = 0.2$

$$\lambda' = 470 \times 10^{-9} \sqrt{\frac{1+0.2}{1-0.2}}$$

$$\lambda' = 470 \times 10^{-9} \times 1.225$$

$\lambda' = 575.6\text{mm}$ The colour appears to be yellow.

Twin Paradox

Consider two synchronised clocks. One is kept on earth the other is taken in a fast moving space ship. If the fast moving clock is brought back to earth after a long time we can see that the time elapsed in moving clock will be less than the time elapsed in clock on earth. This is because a moving clock runs slow. Actually there is no difference between a physical clock and a biological clock accordingly heart beats of a person can be taken as clock. i.e. when a person moves with high speed, his heart beat will be slower. If his age is counted with reference to his heart beat his age will be growing slower. Here comes the twin paradox.

Consider two identical twins A and B, if A goes about at a high speed v in a rocket and B stays behind on earth. When A returns to earth he will be younger than B. In relativity situations are interchangeable. Hence we can assume that A is rest and B is moving with velocity -v. After return B will be found to be younger than A. That is each should find the other younger. A logical contradiction. i.e. a paradoxical statement. This is called twin paradox. This paradox comes because we assumed that twins situations are symmetrical and interchangeable, an assumption that is not correct. Thus there is no paradox at all.

Quantitative analysis of ageing of twins

The twin B stays on earth observes that A travels away a distance L in time $t = \dfrac{L}{v}$, where t is the taken by the twin A to travel forward with respect to B. After t, twin A rapidly reverses his motion and returns with the same velocity. The time for return trip is also t. If we neglect the time for turn around, the total time taken by A with respect to B is $t_B = 2t$

It is due to time dilation A's clock runs slow as far as B is concerned. The time measured by A for the round trip is

$$t'_A = \frac{t_B}{\gamma} = t_B\sqrt{1-\frac{v^2}{c^2}}$$

$$\therefore \qquad \frac{\text{Age of A}}{\text{Age of B}} = \frac{t'_A}{t_B} = \sqrt{1-v^2/c^2}$$

From this B concludes that A is younger.

Now we calculate the age of B as far as A is concerned. Twin A sees that B going away for distance L with velocity –Vand return. This takes time $t_A = 2t$ on 's clock and A sees time t'_B elapse on B's clock, then

$$t'_B = \frac{t_A}{\gamma} = t_A\sqrt{1-\frac{v^2}{c^2}}$$

$$\therefore \qquad \frac{\text{Age of A}}{\text{Age of B}} = \frac{t_A}{t'_B} = \frac{1}{\sqrt{1-v^2/c^2}}$$

From this A concludes that B is younger. This is the paradox. A thinks that B is younger and B thinks that A is younger.

Example 26

A young man voyages to the nearest star, α centauri, 4.3 light years away. He travels in space ship at a velocity of $\frac{c}{5}$. When he returns to earth, how much younger is he than his twin brother who stayed at home.

Solution

The time taken for the round trip $t = \dfrac{2\times 4.3\times c}{v}$ years.

$$= \frac{2\times 4.3\times c}{\frac{c}{5}} = 43 \text{ years}$$

Using time dilation $\qquad t' = t\sqrt{1-\dfrac{v^2}{c^2}} \quad = 43\sqrt{1-\dfrac{1}{25}}$

$$t' = 43\sqrt{\frac{24}{25}} = 43\times .97979$$

$$t' = 42.13 \text{ years}$$

$\therefore$ Age difference $= t - t'$

$$= 43 - 42.131$$

$$= 0.87 \text{ years}$$

$$= 0.87\times 12 \text{ months}$$

$$\simeq 10 \text{ months}$$

i.e., the voyager is 10 months younger.

Relativistic momentum

Here we develop the dynamics of special theory of relativity. There are several ways to do this. One approach is to develop formal procedure for writing the laws of physics in a form which satisfies the postulates of relativity. We start with conservation of momentum, because this is one of the basic laws of nature. We seek what modifications are required to preserve this principle in relativity. This will tell us we must modify our idea of mass to preserve conservation of momentum in relativity. Once this is achieved we can extend this to develop the entire dynamics in relativity.

Concept of mass and momentum in relativity

For this consider a glancing (oblique) elastic collision of two identical particles A and B each having mass m and move with velocity u in opposite directions as shown in figure.

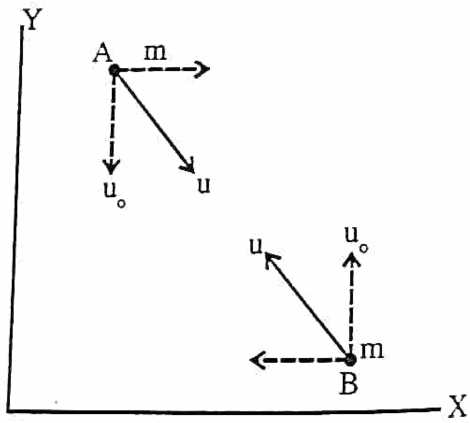


Figure 1.20

After some time particles collide and move in opposite directions. Here we are going to view the collision process (before and after) with two special frames A and B. The A-frame moves with a velocity same as that of the x-component velocity of the particle A. Thus with respect to A frame particle A has only the y-component velocity of the particle A. Let it be u_0. The B-frame moves with a velocity same as that of the x-component velocity of the particle B. With respect A-frame, the particle B has x component velocity and y component velocity.

The x component velocity of the particle B with respect to $A = v$

v is actually the relative velocity between the two frames.

From the velocity addition theorem, we can calculate the velocity of the particle B with respect to the A frame.

Using $$u_y' = \frac{u_y\sqrt{1-v^2/c^2}}{1-\frac{u_x v}{c^2}}$$

Here $u_y = u_0$, $v = v$ and $u_x = 0$

Thus $$u_y' = \frac{u_y}{\gamma}, \quad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

Since the particles are identical, their velocities are the same and the situations are symmetric, we can write down the velocities of the particles A and B with respect to B-frame. The x-component velocity of the particle B with respect to frame $A = u_0$

The x-component velocity of the particle A with respect to B-frame = v

The y-component velocity of the particle A with respect to B-frame $\frac{u_0}{\gamma}$

The velocities of particles A and B before collision with respect to A-frame and B-frame are shown in figure.

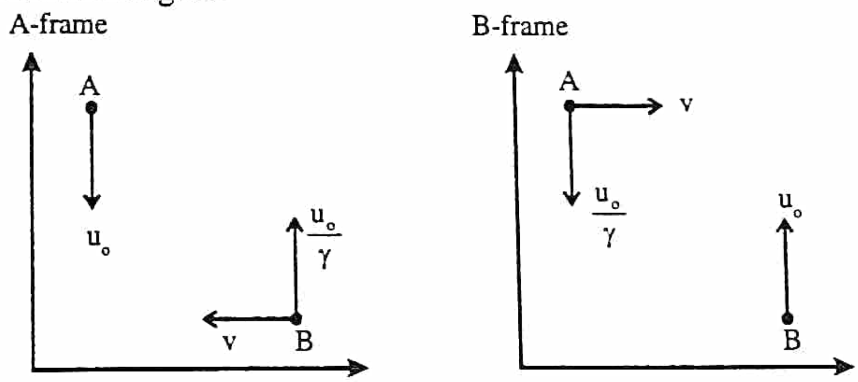


Figure 1.21: Before collision

After collision the x-component velocities remain as such and y-component velocities reversed their directions. If the y-speeds of particles in their own frame is u'

then the y-speed of other particle is $\frac{u'}{\gamma}$. The situation after collision is depicted in figure below.

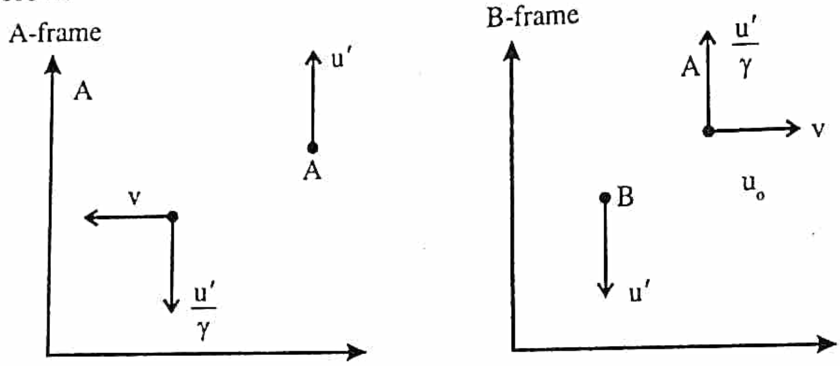


Figure 1.22: After collision

Our aim is to find a conserved quantity analogous to classical momentum. We define the momentum as the product of mass (which is a function of particles velocity w)and velocity w.

i.e $\qquad \bar{p} = m(w)\bar{w}$

where m(w) is a scalar quantity to be determined.

In A-frame the velocity of the particle before collision along x-direction is entirely due to particle B. The speed of particle B with respect to A - frame is $w = \left(v^2 + \frac{u_0^2}{\gamma^2}\right)^{\frac{1}{2}}$ before collision. After collision A is $w' = \left(v^2 + \frac{u'^2}{\gamma^2}\right)^{\frac{1}{2}}$. That is before collision m is a function of w and after collision it is a function of w' along x-direction. Applying law of conservation of momentum along x-direction in the A-frame.

Momentum before collision in the x direction in the frame A

= momentum after collision in the x-direction in the frame A

i.e., $\qquad m(w)v = m(w')v$

Thus we get $w = w'$

Applying law of conservation of momentum along the y-direction in the A-frame. i.e., Momentum before collision in the y-direction in the A-frame = Momentum after collision in the y-direction in A-frame.

i.e., $-m(u_0)+m(w)\dfrac{u_0}{\gamma}=m(u')u'-m(w')\dfrac{u'}{\gamma}$

we proved that $\quad w=w'$

$$\left(v^2+\frac{u_0^2}{\gamma^2}\right)^{\frac{1}{2}}=\left(v^2+\frac{u'^2}{\gamma^2}\right)^{\frac{1}{2}}$$

This implies that $u^0=u'$

Re-writing the above equation, we get

$$-m(u_0)u_0+m(w)\frac{u_0}{\gamma}=m(u_0)u_0-m(w)\frac{u_0}{\gamma}$$

or $$2m(w)\frac{u_0}{\gamma}=2m(u_0)u_0$$

or $$m(w)=vm(u_0)$$

In the limit, when $u_0\to 0$, $m(u_0)\to m(0)$ we call it as the rest mass of the particle denoted by m_0.

We have $w=\left(v^2+\dfrac{u_0^2}{\gamma^2}\right)^{\frac{1}{2}}$

when $u_0\to 0$, $\quad w=v$

$\therefore$ we have $m(v)=\gamma m_0$

or $$m(v)=\frac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$$

It shows that in relativity mass depends on velocity.

Now we are in a position to get an expression for momentum.

$$\bar{p}=m(u)\bar{u}$$

$$p=\frac{m_0\ \ \bar{u}}{\sqrt{1-u^2/c^2}}=\gamma m_0 u=m\bar{u}.$$

Velocity dependence of electrons mass

The effect of velocity on the electrons mass was experimentally detected by Bucherer and he verified the relativistic mass formula. For this he designed an experimental setup as shown in figure below.

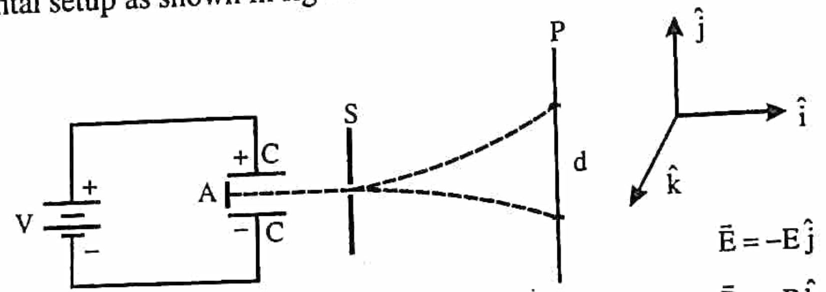


Figure 1.23

The whole apparatus is arranged in an evacuated chamber.

It consists of a source of electron A (radium salt) which emits β-rays which have broad energy spectrum of the order of 1 MeV. These electrons are passed through a velocity filter which can select a monoenergetic electrons. Velocity selector consists of two parallel metal plates C connected to a battery V. i.e, electrons are subjected to an electric field $\vec{E}(-E\,\hat{j})$. A transverse magnetic field $\vec{B}(-B\,\hat{k})$ is also applied to the electrons. Now E, B and v (velocity of electrons) are mutually perpendicular. Now the electric experience electric force qE and magnetic force $q(v \times B)$.

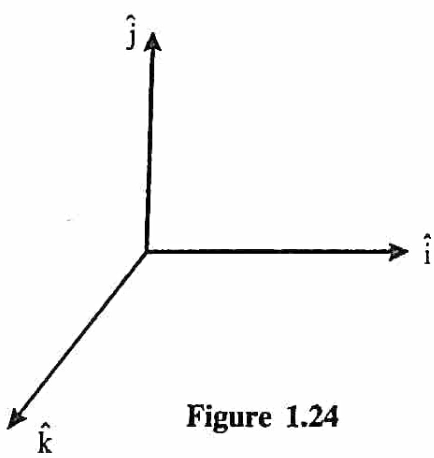


Figure 1.24

The electric force on the charge is $\vec{F}_e = qE\hat{j}$

The magnetic force on the charge

$$\vec{F}_m = qv\hat{i} \times -B\hat{k}$$

$$\vec{F}_m = +qvB\hat{j}$$

Adjust E and B such that the total force on the charge is zero. We have

$$\vec{F}_T = \vec{F}_e + \vec{F}_m$$

When $\vec{F}_T = 0$, we get

$$-\vec{F}_e = \vec{F}_m$$

or $$--qE\hat{j} = qV\hat{j}$$

or $$v = \frac{E}{B}.$$

When the electrons experience no force they pass through the slit S undeflected all having the same velocity $v = \frac{E}{B}$. This is the principle of velocity selector.

Beyond S only perpendicular magnetic field acts. When a charged particle q, moves in a perpendicular magnetic field B, it moves along a circular path. The centripetal force $\frac{mv^2}{r}$ required is supplied by magnetic Lorentz force qvB.

i.e. $$\frac{mv^2}{r} = qvB$$

or $$r = \frac{mv}{qB}$$

using $$v = \frac{E}{B}$$

Radius of curvature $r = \frac{mE}{qB^2}$

Finally the electrons are allowed to fall on a photograph plate P. By reversing $\vec{E}$ and $\vec{B}$ the sense of deflection is reversed. From the total deflection d, and using the geometry of the apparatus, we can evaluate r. By finding r for different velocities, the velocity dependence of $\frac{m}{q}$ can be studied.

From his observations he found that $\frac{q}{m}$ is not constant. $\frac{q}{m}$ was found to vary with velocity. Since charge is independent of velocity, the variation of $\frac{v}{m}$ can be attributed to variation in m alone, Bucherer replaced m by $\frac{m_0}{\sqrt{1-v^2/c^2}}$ and drawn a graph between $\frac{m}{m_0}$ and $\frac{v}{c}$ using his experimental data. On the same graph he plotted $\frac{m}{m_0}$ versus $\frac{v}{c}$ using $m = m_0\sqrt{1-v^2/c^2}$, surprisingly the graphs were one and the same.

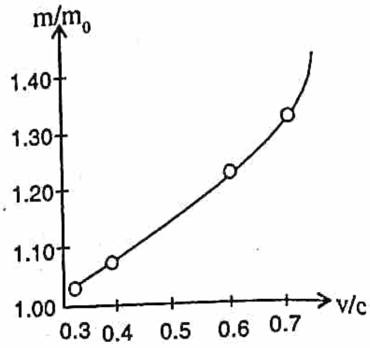


Figure 1.25

The variation of mass with velocity is shown in figure 1.25.

Example 27

Find the velocity at which the mass of a particle is double its rest mass.

Solution

$$m = 2m_0 \text{ (given)}$$

using
$$m = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

$$2m_0 = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

$$2 = \frac{1}{\sqrt{1-v^2/c^2}} \text{ squaring on both sides}$$

$$4 = \frac{1}{1-v^2/c^2}$$

$$1-\frac{v^2}{c^2} = \frac{1}{4}$$

$$1-\frac{1}{4} = \frac{v^2}{c^2}$$

$$\frac{3}{4} = \frac{v^2}{c^2}$$

$$\frac{\sqrt{3}}{2} = \frac{v}{c}$$

$$v = \frac{\sqrt{3}}{2}c = 0\cdot866c = 0\cdot866 \times 3 \times 10^8$$

$$= 2\cdot598 \times 10^8 \, ms^{-1}$$

Example 28

A man weighing 60 kg on the ground. When he is in a space ship in motion his mass is 65 kg measured by an observer on the ground. What is the speed of the space ship.

Solution

$$m_0 = 60\,kg$$

$$m = 65\,kg$$

Using $$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

$$\sqrt{1 - \frac{v^2}{c^2}} = \frac{60}{65} = \frac{12}{13} \text{ squaring}$$

$$1 - \frac{v^2}{c^2} = \frac{144}{169}$$

$$1 - \frac{144}{169} = \frac{v^2}{c^2}$$

$$\frac{25}{169} = \frac{v^2}{c^2}$$

$$\frac{5}{13} = \frac{v}{c}$$

$$v = \frac{5}{13}c = \frac{5}{13} \times 3 \times 10^8 = \frac{15}{13} \times 10^8$$

$$v = 1\cdot154 \times 10^8 \, ms^{-1}.$$

Mass - Energy relation

The kinetic energy K of a particle is defined as the work done by an external force in increasing the speed of the particle from zero to some value v.

i.e., $$K = \int_0^v \vec{F}\cdot d\vec{x}$$

But $$F = \frac{dp}{dt} = \frac{d(mv)}{dt}$$

$$K = \int_0^v \frac{d}{dt}(mv)\cdot dx = \int_0^v d(mv)\cdot\frac{dx}{dt} = \int_0^v d(mv)\cdot v$$

$$K = \int_0^v (m\,dv + v\,dm)\cdot v = \int_0^v (mv\,dv + v^2dm) \quad \text{......(1)}$$

(Here m and v are variables)

Using $m = \dfrac{m_0}{\sqrt{1-v^2/c^2}}$ squaring and rearranging, we have

$$m^2(c^2 - v^2) = m_0^2c^2$$

Taking the differenitials on both sides, we get

$$2m\,dm\,c^2 - 2mv^2dm - 2m^2v\,dv = 0$$

$$c^2dm - v^2dm - mv\,dv = 0$$

$$mv\,dv + v^2dm = c^2dm$$

Using this, equation (1) becomes

$$K = \int_{m_0}^{m} c^2dm$$

where m_0 is the mass when velocity is zero and m is the mass when velocity is v

$$K = c^2[m]_{m_0}^{m} = mc^2 - m_0c^2$$

or $$K = (m - m_0)c^2.$$

It shows that change in mass is related to kinetic energy. i.e., Kinetic energy is defined as the product of increase in its mass and c^2. When the body is at rest $K = 0$.

i.e., $$mc^2 - m_0c^2 = 0.$$

This shows that each term on L.H.S. should represent energy according to the principle of homogeneity of dimensions. Since m_0c^2 does not involve velocity of the body it should give energy associated with the particle at rest. Therefore m_0c^2 should be the internal energy of particle.

$\therefore$ Total energy of the particle $E = K + m_0c^2$

i.e.,
$$E = mc^2 - m_0c^2 + m_0c^2$$
$$E = mc^2$$

This shows that mass and energy are interconvertible and proves to be universal. The best examples of conversion of mass into energy are the nuclear fusion, nuclear reaction processes and phenomenon of pair annihilation. The best example of conversion of energy into mass is the phenomenon of pair production.

Example 29

Calculate the energy equivalent of 1kg of coal.

Solution

$$m = 1\text{kg}$$

using $E = mc^2 = 1\times(3\times10^8)^2 = 9\times10^{16}\,J$

Example 30

Prove that when $\frac{v}{c} \ll 1$, the relativistic kinetic energy becomes the classical one.

Solution

We have relativistic kinetic energy

$$K = mc^2 - m_0c^2, \text{ using } m = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

$$K = \frac{m_0}{\sqrt{1-v^2/c^2}}c^2 - m_0c^2$$

$$K = m_0c^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)$$

$$K = m_0c^2\left\{\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1\right\}, \text{ when } \frac{v}{c} << 1$$

$$\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} = 1+\frac{v^2}{2c^2}$$

$$\therefore \quad K = m_0c^2\left(1+\frac{v^2}{2c^2}-1\right)$$

$$K = m_0c^2\cdot\frac{v^2}{2c^2} = \frac{1}{2}m_0v^2$$

This is classical kinetic energy.

Example 31

Find the velocity of a proton having kinetic energy 900 MeV

$$m_0 = 1\cdot 67\times 10^{-27}\,\text{kg}.$$

Solution

Kinetic energy,

$$K = 900\text{MeV} = 900\times 10^6\text{eV} = 9\times 10^8\times 1\cdot 6\times 10^{-19}\text{J}$$

$$= 9\times 1\cdot 6\times 10^{-11}\text{J} = 14.4\times 10^{-11}\text{J}$$

using $K = (m - m_0)c^2$

$$14\cdot 4\times 10^{-11} = (m - m_0)\,(3\times 10^8)^2$$

or $m - m_0 = \dfrac{14.4\times 10^{-11}}{9\times 10^{16}} = 1\cdot 6\times 10^{-27}\,\text{kg}.$

or $m = m_0 + 1\cdot 6\times 10^{-27} = 1\cdot 67\times 10^{-27} + 1\cdot 6\times 10^{-27} = 3\cdot 27\times 10^{-27}.$

Using $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$

$$3\cdot 27\times 10^{-27} = \frac{1\cdot 67\times 10^{-27}}{\sqrt{1-v^2/c^2}}$$

$$\sqrt{1-v^2/c^2} = \frac{1\cdot 67}{3\cdot 27} = 0\cdot 51$$

Squaring, $$1-\frac{v^2}{c^2} = 0\cdot 26$$

$$\frac{v^2}{c^2} = 1-0\cdot 26 = 0.74$$

or $$\frac{v}{c} = 0\cdot 86$$

$$v = 0\cdot 86c = 0\cdot 86\times 3\times 10^8$$

$$v = 2\cdot 58\times 10^8\,ms^{-1}\cdot$$

Example 32

Calculate the increase in mass when 1kg is moving with a velocity of 0.9 c. Hence find the kinetic energy.

Solution

$$m_0 = 1kg;\ v = 0\cdot 9c$$

using $$m = \frac{m_0}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-0\cdot 9^2}} = \frac{1}{\sqrt{1-0\cdot 81}}$$

$$m = \frac{1}{\sqrt{0\cdot 19}} = 2\cdot 294\,kg$$

$\therefore$ Increase in mass $= m - m_0$

$$= 2\cdot 294 - 1 = 1\cdot 294\,kg$$

$\therefore$ Kinetic energy $K = (m - m_0)c^2$

$$= 1\cdot 294\times (3\times 10^8)^2 = 1\cdot 294\times 9\times 10^{16}$$

$$= 1\cdot 165\times 10^{17}\,J\cdot$$

Relativistic energy and momentum in an inelastic collision

Inelastic collision in Newtonian mechanics

Consider two identical particle each of mass m, moving with same velocity in opposite directions. After collision they stick together.

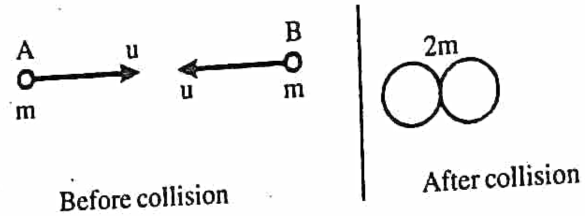


Figure 1.26

According to law of conservation of momentum

Momentum before collision = momentum after collision

$$mu - mu = 2mv$$
$$0 = 2mv$$

This implies that v = 0. i.e., after collision particles stick together and comes to rest

Total kinetic energy before collision

$$= \frac{1}{2}mu^2 + \frac{1}{2}mu^2 = mu^2$$

Total kinetic energy after collision

$$= \frac{1}{2}\cdot 2m0^2 = 0$$

This shows that during an inelastic collision energy is lost in the form of heat. This will not occur in relativity. We shall see to it.

Inelastic collision in relativity

Consider two identical particles each of rest mass m_0, moving with same velocity u in opposite directions. After collision particles stick together. In relativity we have two inertial frames S and S'. S is at rest and S' is moving uniformly. We analyse the collision with respect to S and S' separately.

In S-frame

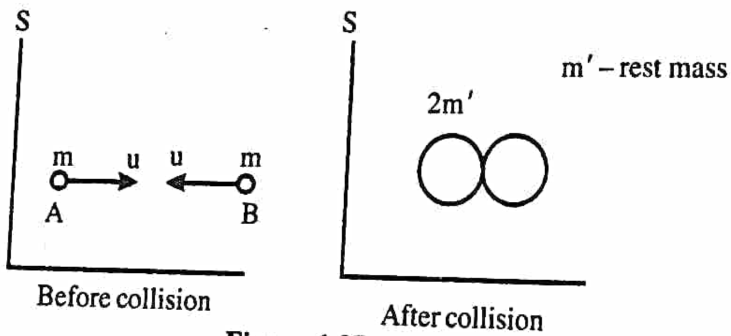


Figure 1.27

The total energy before collision = $mc^2 + mc^2$

$$= 2mc^2 = \frac{2m_0c^2}{\sqrt{1-\frac{u^2}{c^2}}}$$

The total energy after collision $= 2m'c^2$

As no external work has been done on the particles

Total energy before collision = Total energy after collision

i.e.,
$$\frac{2m_0c^2}{\sqrt{1-\frac{u^2}{c^2}}} = 2m'c^2 \qquad .....(1)$$

or
$$m' = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$$

i.e.,
$$m' > m_0$$

This shows that final rest mass (m') is greater than the initial rest mass. Unlike in Newtonian mechanics, in relativity the energy is not lost as heat but used to increase the mass according to mass-energy relation. In other words we can say that in relativity total energy is always conserved.

In S' - frame

Let u be the velocity with which S' frame is moving along positive x-direction.

$\therefore$ The velocity of the particle A with respect to S' frame = 0

The velocity of the particle B with respect to S' frame

$$v = \frac{u+u}{1+\frac{uu}{c^2}} = \frac{2u}{1+\frac{u^2}{c^2}}$$

After collision the particles stick together and move with velocity u with respect to S' - frame. Situations are shown in figure below.

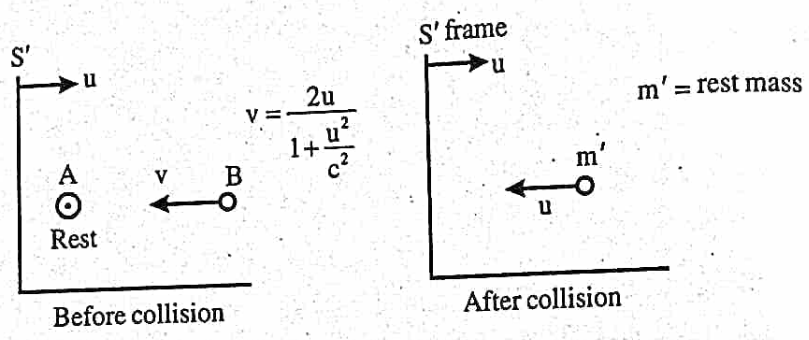


Figure 1.28

According to conservation of momentum

Momentum before collision = Momentum after collision

$$mv = \frac{2m'u}{\sqrt{1-u^2/c^2}}$$

or

$$\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2m'u}{\sqrt{1-\frac{u^2}{c^2}}} \qquad \text{..... (2)}$$

$$1-\frac{v^2}{c^2} = 1-\frac{(2u)^2}{\left(1+\frac{u^2}{c^2}\right)^2 c^2} = \frac{\left(1+\frac{u^2}{c^2}\right)^2 c^2 - 4u^2}{\left(1+\frac{u^2}{c^2}\right)^2 c^2}$$

$$= \frac{\left((1+2\frac{u^2}{c^2}+\frac{u^4}{c^4}\right)c^2 - 4u^2}{\left(1+\frac{u^2}{c^2}\right)^2 c^2}$$

$$= c^2 \frac{\left(1+\frac{2u^2}{c^2}+\frac{u^4}{c^4}-\frac{4u^2}{c^2}\right)}{\left(1+\frac{u^2}{c^2}\right)^2 c^2}$$

$$1-\frac{v^2}{c^2}=\frac{1-2\frac{u^2}{c^2}+\frac{u^4}{c^4}}{\left(1+\frac{u^2}{c^2}\right)^2}=\frac{\left(1-\frac{u^2}{c^2}\right)^2}{\left(1+\frac{u^2}{c^2}\right)^2} \qquad \text{..... (3)}$$

$$\therefore \qquad \frac{v}{\sqrt{1-\frac{v^2}{c^2}}}=\left(\frac{2u}{1+\frac{u^2}{c^2}}\right)\frac{\left(1+\frac{u^2}{c^2}\right)}{\left(1-\frac{u^2}{c^2}\right)}=\frac{2u}{1-\frac{u^2}{c^2}}$$

put this in equation (2), we get

$$m_0\frac{2u}{1-\frac{u^2}{c^2}}=\frac{2m'u}{\sqrt{1-\frac{u^2}{c^2}}}$$

or

$$\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}=m' \qquad \text{..... (4)}$$

we get back the same result in the S-frame. This shows that S and S' frames are identical.

Now we apply law of conservation of energy.

Total energy before collision = Total energy after collision

i.e.,

$$m_0c^2+\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{2m'c^2}{\sqrt{1-\frac{u^2}{c^2}}}$$

or

$$m_0+\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{2m'}{\sqrt{1-\frac{u^2}{c^2}}}$$

substituting for $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ from equation (3) we get

$$m_0 + \frac{m_0\left(1+\frac{u^2}{c^2}\right)}{\left(1-\frac{u^2}{c^2}\right)} = \frac{2m'}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\frac{m_0\left(1-\frac{u^2}{c^2}\right)+m_0\left(1+\frac{u^2}{c^2}\right)}{1-\frac{u^2}{c^2}} = \frac{2m'}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\frac{2m_0}{\sqrt{1-\frac{u^2}{c^2}}} = 2m'$$

or $$m' = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$, we got back again the same result. Thus establishes the fact that S and S' frame are identical and they yield the same results.

Example 33

Two particles of rest mass m_0 approach each other with equal and opposite velocity u in the laboratory frame. What is the total energy of one particle as measured in the rest frame of the other $u = \frac{c}{\sqrt{2}}$

Solution

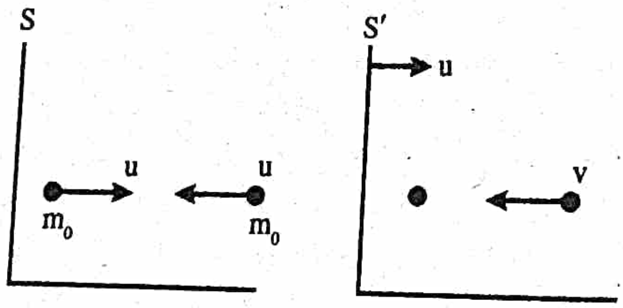


Figure 1.29

From the figure $v = \dfrac{u+u}{1+\dfrac{uu}{c^2}} = \dfrac{2u}{1+\dfrac{u^2}{c^2}}$ .....(1)

$$E' = mc^2 = \gamma m_0 c^2$$

Where $\gamma = \dfrac{1}{\sqrt{1-v^2/c^2}}$

$$\frac{u}{c} = \frac{1}{\sqrt{2}} \text{ given } \therefore \frac{u^2}{c^2} = \frac{1}{2}$$

Put this in equation (1) we get

$$v = \frac{2\cdot c}{\sqrt{2}\left(1+\frac{1}{2}\right)} = 2\frac{\sqrt{2}}{3}c$$

$$1-\frac{v^2}{c^2} = 1-\left(2\frac{\sqrt{2}}{3}\right)^2 = 1-\frac{8}{9} = \frac{1}{9}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{\frac{1}{9}}} = 3$$

So $E' = 3m_0c^2$

Example 34

A particle of rest mass m and speed v collides and sticks to a stationary particle of mass M. What is the final speed of the composite system.

Solution

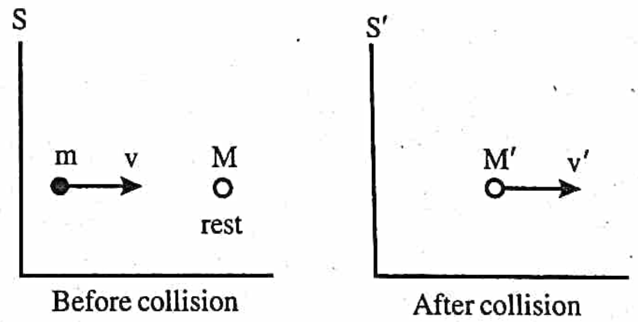


Figure 1.30

From momentum conservation

$$\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{M' v'}{\sqrt{1-\frac{v'^2}{c^2}}} \qquad \text{.....(1)}$$

From energy conservation

$$\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} + Mc^2 = \frac{M' c^2}{\sqrt{1-\frac{v'^2}{c^2}}} \qquad \text{.....(2)}$$

$\frac{Eq(1)}{Eq(2)}$ gives
$$\frac{\frac{mv}{\sqrt{1-v^2/c^2}}}{\frac{mc^2}{\sqrt{1-v^2/c^2}}+Mc^2} = \frac{v'}{c^2}$$

Take $\frac{1}{\sqrt{1-v^2/c^2}} = \gamma \qquad \frac{\gamma mv}{\gamma m + M} = v'$

Example 35

A particle of rest mass m_0 and kinetic energy of $6m_0c^2$ strikes and sticks to an identical particle at rest. What is the rest mass of the resultant particle.

Solution

We have $\quad K = (m - m_0)c^2,\ K = 6m_0c^2$ gives

$$6m_0c^2 = (m - m_0)c^2$$

$$7m_0 = m \qquad \text{.....(0)}$$

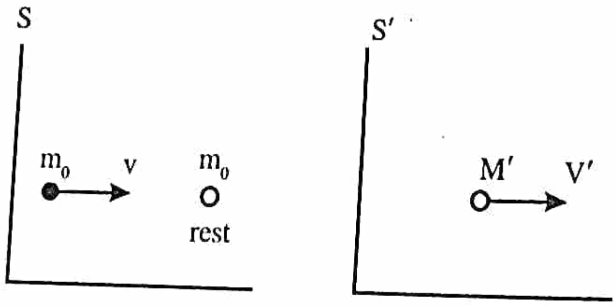


Figure 1.31

From momentum conservation

$$7m_0v = \frac{M'\,v'}{\sqrt{1-\frac{v'^2}{c^2}}} \qquad .....(1)$$

From energy conservation

$$7m_0c^2 + m_0c^2 = \frac{M'\,c^2}{\sqrt{1-\frac{v'^2}{c^2}}} \qquad .....(2)$$

$\frac{\text{eq 1}}{\text{eq 2}}$ gives $\quad \frac{7}{8}v = v' \qquad .....(3)$

From eq (0) we have $\quad 7m_0 = \frac{m_0}{\sqrt{1-v^2/c^2}}$

or $\quad \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{7}$ squaring

$$1-\frac{v^2}{c^2} = \frac{1}{49}$$

$$\frac{v^2}{c^2} = 1-\frac{1}{49} = \frac{48}{49}$$

$$\frac{v}{c} = \frac{\sqrt{48}}{7}$$

$$\therefore \quad v' = \frac{7}{8}\frac{\sqrt{48}}{7}c = \frac{\sqrt{48}}{8}c$$

$$\frac{v'}{c} = \frac{\sqrt{48}}{8}$$

$$\frac{v'^2}{c^2} = \frac{48}{64}$$

$$1-\frac{v'^2}{c^2}=1-\frac{48}{64}=\frac{16}{64}$$

$$\sqrt{1-\frac{v'^2}{c^2}}=\frac{4}{8}=\frac{1}{2}$$

Put the value of v' and $\sqrt{1-\frac{v'^2}{c^2}}$ in equation (1), we get

$$7m_0v=M'\frac{7}{8}v$$

$$\therefore \quad M'=4m_0.$$

The equivalence of mass and energy

The relativistic mass energy relation was experimentally confirmed by Cockcroft and Walton in 1932 by designing a high energy proton accelerator. The accelerator consists of mainly 4 parts.

1. A source of protons

Protons are produced by supplying an electrical discharge to hydrogen gas.

2. A power supply and a mechanism to boost the voltage of power supply

They used a power supply with a voltage of 150 kV. By using a simple electrical circuit consisting of capacitors and rectifiers, they boosted the voltage of the power supply step by step to 600 kV.

3. A target

They used ^7Li as the target

4. A zinc sulphide fluorescent screen construction and working

Protons produced were accelerated in vacuum step by step by the applied high voltage. When they attain maximum energy, they are allowed fall on the lithium target kept at 45°. The fluorescent screen is kept in front of the target as shown in figure 1.32.

They found that the fluorescent screen emitted occasional flashes. By various tests they determined that the flushes were due to α - particles. They interpreted this as follows. The ^7Li captures a proton and the resulting nucleus of mass 8 immediately disintegrates into alpha particles.

i.e.,

$$^1H+{}^7Li\rightarrow {}^4He+{}^4He$$

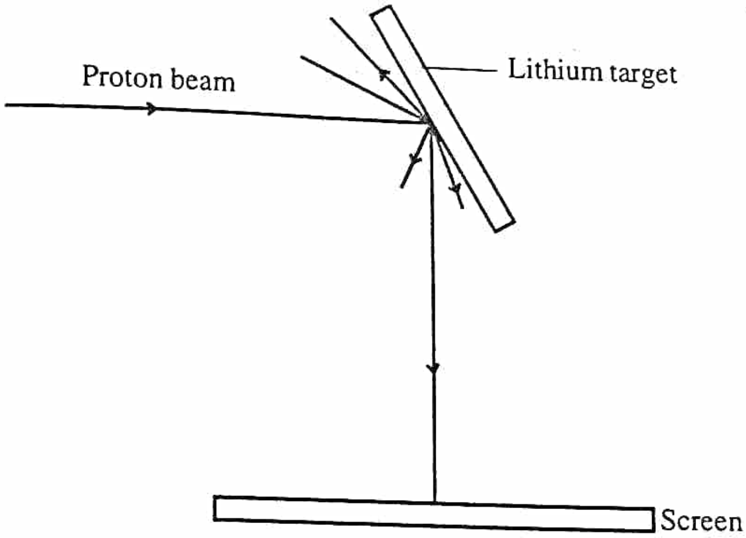


Figure 1.32

Writing mass-energy equation for the above reaction, we get

Kinetic energy of ^1H + Rest energy of ^1H and ^7L_i

= Kinetic energy of $2\,^4He$ + rest energy of $2\,^4He$

i.e $K(^1H) + M_pc^2 + M_{Li}c^2 = 2K(^4He) + 2M_\alpha c^2$

Rewriting this equation as

$(M_P + M_{Li} - 2M_\alpha)c^2 = 2K(^4He) - K(^1H)$

symbolically, we can write

$$\Delta Mc^2 = K \qquad \text{Mass – energy equation}$$

Cockcroft and Walton obtained the value $K = 17.2\,\text{MeV}$ from their experiment.

To get ΔMc^2, they substituted the values of M_p, M_{Li} and M_α

$$M_p = 1.0072 \text{ amu}$$

$$M_{Li} = 7.0104 \pm 0.0030 \text{ amu}$$

$$M_\alpha = 4.0011 \text{ amu}$$

$$\therefore \quad \Delta M = (1.0072 + 7.0104 \pm 0.0030) - 2 \times 4.0011)$$

$$= (0.0154 \pm 0.0030)$$

Using 1 amu $= 931\,\text{MeV}$

$$\Delta Mc^2 = (14.34 \pm 2.79)\,\text{MeV}$$

The difference between K and ΔMc^2 is $= (17.2 - 14.34)$ MeV $= 2.86$ MeV. This is only slightly larger than the allowed experimental uncertainty 2.79. From this they almost confirmed the equivalence of mass and energy.

Momentum - Energy relation

If a body of mass m moving with a velocity v its momentum

$$p = mv \qquad \text{..... (1)}$$

and its energy

$$E = mc^2 \qquad \text{..... (2)}$$

Using $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$ squaring on both sides and rearranging we get

$$m^2 = \frac{m_0^2}{1 - v^2/c^2}$$

$$m^2\left(1 - \frac{v^2}{c^2}\right) = m_0^2$$

$$m^2\left(\frac{c^2 - v^2}{c^2}\right) = m_0^2$$

$$m^2(c^2 - v^2) = m_0^2 c^2$$

$$m^2 c^2 - m^2 v^2 = m_0^2 c^2$$

$$m^2 c^2 = m^2 v^2 + m_0^2 c^2$$

Multiplying throughout by c^2, we get

$$m^2 c^4 = m^2 v^2 c^2 + m_0^2 c^4$$

Using equations (1) and (2) we get

$$E^2 = p^2 c^2 + m_0^2 c^4$$

$$E = \sqrt{p^2 c^2 + m_0^2 c^4}$$

This is the energy-momentum relation.

Massless particles

In the case of a photon or neutrino $m_0 = 0$

$$\therefore \qquad E = pc$$

using equations (1) and (2), we have

$$mc^2 = mvc$$

$$v = c$$

This shows that the velocity of a particle of zero rest mass is c.

Remember that only rest mass of the photon is zero. Using $E = mc^2$ and $E = h\upsilon$

We have $$mc^2 = h\upsilon$$

or $$m = \frac{h\upsilon}{c^2}$$

This is the mass equivalent of photon.

It shows that in relativity a photon of energy E has a mass $\frac{h\upsilon}{c^2}$. Since this massless particle interacts with charged particles like electrons, protons etc., they can be easily detectable. Similarly the particles neutrino and graviton are massless particles though they interact with matter weakly they cannot be detected easily. So far neutrinos have been detected but not gravitons.

Tachyons

In nature we find particles like electrons, protons etc. for which $m_0 > 0$. From this an interesting proposal has been made by E.C.G. Sudarsan (an Indian scientist) and others that it in nature particles with $m_0 > 0$ and $m_0 = 0$ exist then particles with $m_0 < 0$ should also exist. Such particles can have velocity greater than c.

Using $$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

If $v > c, \sqrt{1 - v^2/c^2}$ is imaginary. If m_0 is taken to be imaginary. The imaginary in the numerator and the imaginary in the denominator will cutoff there by making m a real quantity. The rest mass imaginary means that this particle cannot have a rest position. The rest mass m_0 imaginary does not matter provided the experimentally observable entities E and p are real and positive. This condition is satisfied provided $p^2c^2 > m_0^2c^4$ (From $E^2 = p^2c^2 + m_0^2c^4$). Also since $E = \gamma mc^2$ and $p = \gamma m_0 v$, E and p can be real only if γ is imaginary. Physically this means that particle is moving with a velocity greater than c. These particles are called Tachyons. This name is derived from Greek word Tachyons meaning swift. Several attempts have been made to observe these particles but so far none of them has been successful.

Example 36

Show that the particles move with velocity equal to that of light have zero rest mass

Solution

From Energy-momentum relation, we have

$$E = \sqrt{p^2 c^2 + m_0^2 c^4}$$

or $\quad E^2 = p^2 c^2 + m_0^2 c^4$ using $E = mc^2$ and $p = mv$

$$m^2c^4 = m^2 v^2 c^2 + m_0^2 c^4$$

But $\quad v = c$ given, then

$$m^2c^4 = m^2 c^4 + m_0^2 c^4$$

or $\quad m_0^2 c^4 = 0 \Rightarrow m_0 = 0.$

Example 37

Calculate the rest mass of a particle whose momentum is $130\text{MeV}/c$ when its kinetic energy is 50MeV.

Solution

$$p = 130\frac{\text{MeV}}{c} = \frac{130\times10^6\times1.6\times10^{-19}}{3\times10^8}\text{kgms}^{-1}$$

$$K = 50\text{ MeV} = 50\times10^6\times1.6\times10^{-19}\text{J}$$

The kinetic energy K and the total energy are related by

$$E = K + m_0c^2$$

and also we have

$$E = \sqrt{p^2c^2 + m_0^2c^4}$$

equating the two equations, we get

$K + m_0c^2 = \sqrt{p^2c^2 + m_0^2c^4}$ squaring on both sides.

$$K^2 + m_0^2c^4 + 2Km_0c^2 = p^2c^2 + m_0^2c^4$$

$$2Km_0c^2 = p^2c^2 - K^2$$

$$m_0 = \frac{p^2c^2 - K^2}{2Kc^2}$$

$$= \frac{(130\times10^6\times1\cdot6\times10^{-19})^2\,(3\times10^8)^2 - (50\times10^6\times1\cdot6\times10^{-19})^2}{2\times50\times10^6\times1\cdot6\times10^{-19}\,(3\times10^8)^2}$$

$$= 2\cdot56\times10^{-28}\,\text{kg}.$$

The photoelectric effect

The phenomenon of photoelectric effect had been studied extensively in the quantum mechanics paper. Photoelectric effect confirmed that light possesses particle nature. But in relativity Michelson interferometer experiment confirmed that light has wave nature. Einstein's energy relation $mc^2 = h\upsilon$ provides a link between the particle and wave nature of light.

Radiation pressure of light

According to Maxwell's electro magnetic theory, light wave carries momentum. This momentum will be transferred to the surface when it is absorbed or reflected. It is due to the change in moment, when light falls on a surface, the surface experiences a pressure called radiation pressure. The calculation of radiation pressure on the basis of wave theory is complicated, but with photos picture it is very simple.

Calculation of radiation pressure on the basis of photon picture

Consider a stream of photons striking a perfectly reflecting mirror normally. After reflection photons move with the same speed.

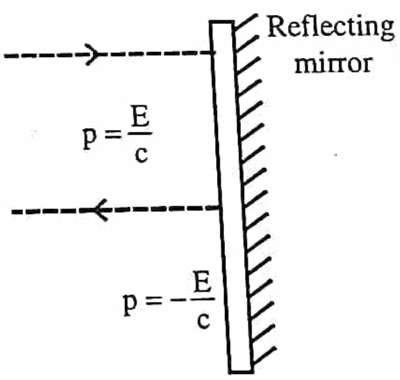


Figure 1.33

The initial momentum of each photon $p = \frac{E}{c}$

Final momentum (after reflection) of each photon,

$$p = -\frac{E}{c}$$

$\therefore$ Change in momentum of the photon $= \frac{E}{c} - \frac{-E}{c} = \frac{2E}{c}$

If there are n photons stricking the surface per unit area in unit time

The total change in momentum in unit time per unit area $= \frac{2nE}{c}$

The change in momentum in unit time is called force F.

Thus Force per unit area $= \frac{2nE}{c}$

The force per unit area is called pressure P

$$\therefore \qquad P = \frac{2nE}{c}$$

nE is the total energy per unit area per second is called the intensity of light I

$$\text{Thus } P = \frac{2I}{c}$$

This is the expression for radiation pressure when light reflects from a reflector.

Similarly the radiation pressure on an absorbing surface $P = \frac{I}{c}$

The average intensity of sunlight falling on the earth surface at normal incidence is calculated to be $1400\,Wm^{-2}$.

$\therefore$ The radiation pressure on a mirror due to sunlight $= \frac{2I}{c} = \frac{2 \times 1400}{3 \times 10^8}$

$$= 9.33 \times 10^{-6}\,Nm^{-2}$$

This pressure is very small when comparing to atmospheric pressure $10^5\,Nm^{-2}$. However on the cosmic scale the radiation pressure is very large, it helps keep stars from collapsing under their own gravitation forces. The gravitational force of a star is acting inward and the radiation pressure is acting outward. So long as when these two forces balance each other the star lives as a youngster one.

Another important aspect of photons is that unlike classical particles photons can be created and destroyed. In other words photon number is not conserved in nature. The absorption of light by matter leads to destruction of photon where as emission of radiation creates photons. Nevertheless, the law of conservation of momentum and energy are generalised with in the frame work of relativity. This will provide us a very powerful tool to treat processes involving photon without a detailed knowledge of interaction. One illustrative example is given below.

The photon picture of the Doppler effect

We already discussed the Doppler effect of light by considering light as a wave. This time we are going to discuss Doppler effect of light by considering light as a particle (photon).

Consider an atom with rest mass m_0 at rest. If the atom emits a photon of energy $h\upsilon$ the mass of the atom becomes m_0'. From the conservation of energy we have

$$m_0c^2 = m_0'c^2 + h\upsilon$$

i.e.,

$$m_0'c^2 = m_0c^2 - h\upsilon \qquad \text{.......(1)}$$

Now suppose that the atom moves freely with a velocity v, before emitting the photon.

Before emitting the photon

Energy of the atom, $E = \dfrac{m_0c^2}{\sqrt{1-v^2/c^2}}$

Momentum of the atom $p = \dfrac{m_0v}{\sqrt{1-v^2/c^2}}$

After emitting the photon of energy $h\upsilon'$, the atom has velocity v', rest mass m_0', energy E' and momentum p'.

According to conservation momentum, we have

$$p = p' + \frac{h\upsilon'}{c}$$

or

$$pc - h\upsilon' = p'c \qquad \text{.....(2)}$$

According to conservation of energy, we have

$$E = E' + h\upsilon'$$

or

$$E - h\upsilon' = E' \qquad \text{......(3)}$$

squaring equations(2) and (3) and subtracting we get

$$(E-h\upsilon')^2-(pc-h\upsilon')^2=E'^2-(p'c)^2$$

From the energy-momentum relation, we have

$$E^2=p^2c^2+m_0^2c^4$$

so

$$E'^2=p'^2c^2+m_0'^2c^4$$

or

$$E'^2-p'^2c^2=m_0'^2c^4$$

Then above equation becomes

$$(E-h\upsilon')^2-(pc-h\upsilon')^2=m_0'^2c^4$$

substituting for $m_0'^2c^2$ from eq. (1), yields

$$(E-h\upsilon')^2-(pc-h\upsilon')^2=(m_0c^2-h\upsilon)^2$$

$$E^2-2Eh\upsilon'+h^2\upsilon'^2-p^2c^2+2pch\upsilon'-h^2\upsilon'^2$$

$$=m_0^2c^4-2m_0c^2h\upsilon+h^2\upsilon^2$$

substituting for $E^2-p^2c^2=m_0^2c^4$ on the L.H.S and cut likes terms on both sides, we get $-2Eh\upsilon'+2pch\upsilon'=-2m_0c^2h\upsilon+h^2\upsilon^2$

$$2h\upsilon'(-E+pc)=-h\upsilon(2m_0c^2-h\upsilon)$$

or

$$\upsilon'=\upsilon\frac{(2m_0c^2-h\upsilon)}{2(E-pc)}$$

Now we evaluate $E-pc$

$$E-pc=\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}-\frac{m_0vc}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$E-pc=\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}\left(1-\frac{v}{c}\right)=m_0c^2\sqrt{\frac{\left(1-\frac{v}{c}\right)}{\left(1+\frac{v}{c}\right)}}$$

$$\therefore \quad \upsilon' = \upsilon\, 2m_0c^2\left(1-\frac{h\upsilon}{2m_0c^2}\right)\cdot\frac{1}{2m_0c^2}\sqrt{\frac{\left(1+\frac{v}{c}\right)}{\left(1-\frac{v}{c}\right)}}$$

$$\upsilon' = \upsilon\left(1-\frac{h\upsilon}{2m_0c^2}\right)\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

For massive sources the term $\frac{h\upsilon}{2m_0c^2}$ is negligible.

$$\therefore \quad \upsilon' = \upsilon\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

A result which is exactly in agreement with our old result. This shows that the wave nature and the particle nature of light predict the same results.

Does light travel at the velocity of light

The question does light travel at the velocity of light may seem to be rhetorical. But answering this question is not as simple as we think of. We know that the entire edifice of relativity is built upon on two postulates where velocity plays a crucial role. In relativity a universal character is attributed to the velocity of light i.e, the velocity of light in vacuum is a universal constant and is the same for the observer in all inertial frames. We can very well prove that there is only one such universal velocity in relativity.

To prove this we assume that there is a second universal velocity c^* emitted by a system or due to some phenomenon other than light. For example the velocity of gravitons or neutrinos or tachyons. We call this phenomenon as Γ.

Consider a light and a Γ - pulse emitted along the x-axis from the origin of a coordinate system (S frame) at $t = 0$

The x-coordinates of the pulses after a time t are

$$x_l = ct$$

$$x_r = c^* t$$

The relative velocity between the two pulses is given by

$$u = \frac{d}{dt}(x_r - x_l) = \frac{d}{dt}(c^* t - ct)$$

$$u = c^* - c$$

Now consider the same pulses in the S' frame moving with velocity v along the positive x-axis.

The x- coordinates of the pulses with respect to S' frame are

$$x'_l = ct'$$

$$x'_r = c^* t'$$

∴ The relative velocity of the two pulses is given by

$$u' = \frac{d}{dt'}(x'_r - x'_l) = \frac{d}{dt}(c^* t' - ct')$$

$$u' = c^* - c$$

From the velocity addition theorem

we have $$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

For light pulse $$c' = \frac{c - v}{1 - \frac{cv}{c^2}} = \frac{c - v}{1 - \frac{v}{c}} = c$$

For Γ - pulse $$(c^*)' = \frac{c^* - v}{1 - \frac{c^* v}{c^2}}$$

$c' = c$ implies that the velocity of light in S frame and S' frame is the same there by attaining a universal character in accordance with relativity. We also proved that

$(c^*)' \neq c^*$ implies that Γ - pulse is not universal in its character. Moreover using Lorentz transformation we can very well write

$$u' = (c^*)' - c$$

if c^* and c are universal.

or
$$u' = \frac{c^* - v}{1 - \frac{c^* v}{c^2}} - c$$

This result disagrees with our former result.

$$u' = c^* - c$$

For these results to be agreeable only when $c^* = c$

In this case $u = c^* - c = 0$

and
$$u' = \frac{c - v}{1 - \frac{cv}{c^2}} - c = 0$$

$c = c^*$ implies that there is only one universal velocity i.e, the velocity of light. We conclude that light travels at the universal velocity. We could also see that when we try to incorporate a second universal velocity the entire edifice of relativity collapsed. In other words special theory of relativity cannot accommodate more than one universal velocity.

Scientific community is still investigating a second universal velocity moving faster than light. If it is found we have to modify special theory of relativity in order to accommodate the newly discovered.

The rest mass of the photon

We found that within the frame work of relativity the rest mass of the photon is zero and it moved with velocity of light of the photon has a non-zero rest mass, its velocity would be less than c.

If m_p is the rest mass of the photon, then its energy

$$E = \gamma m_p c^2$$

If we assume that $E = h\upsilon$ is valid, the above equation can be written as

$$h\upsilon = \frac{m_p c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

squaring on both sides, we get

$$(h\upsilon)^2 = \frac{(m_p c^2)^2}{1-\frac{v^2}{c^2}}$$

If υ_0 be the characteristic frequency of photon, then $m_p c^2 = h\upsilon_0$

$$\therefore \quad (h\upsilon)^2 = \frac{(h\upsilon_0)^2}{1-\frac{v^2}{c^2}}$$

$$\upsilon^2 = \frac{\upsilon_0^{\ 2}}{1-\frac{v^2}{c^2}}$$

or

$$\upsilon^2\left(1-\frac{v^2}{c^2}\right) = \upsilon_0^{\ 2}$$

$$1-\frac{v^2}{c^2} = \frac{\upsilon_0^{\ 2}}{\upsilon^2}$$

$$\therefore \quad \frac{v^2}{c^2} = 1-\frac{\upsilon_0^{\ 2}}{\upsilon^2}$$

If $\upsilon_0 = 0$, then $v = c$, otherwise v depends on frequency. This happen when light passes through dispersive medium such as water, glass etc. This phenomenon is called dispersion. Our experimental challenge is to test whether or not vacuum (empty place) exhibits dispersion. If it is proved we will be forced to put a limit on the rest mass of the photon. So far the frequency dependence of light in vacuum is not confirmed. So the rest mass of the photon can be taken to be zero.

Light from a pulsar

Pulsar is the abbreviated form of a pulsating star. Pulsars emit light signals at regular intervals of time. The time intervals vary from milliseconds to 10 seconds. Depending upon the nature of light signals emitted by pulsars we have several types of pulsars such as radio pulsars, x-ray pulsars, gamma ray pulsars etc. The first pulsar was detected by Joselyn Bell and Anthony Hewish in the year 1967. It is named as PSR1919+21. The time interval of signals of this pulsar was 1.337s. For this discovery Anthony Hewish was awarded the Nobel prize in physics for the year 1974. The peculiar properties of light signals emitted by pulsars attracted the scientific community. As a result of this more than 1000 pulsars have been detected so far. The most interesting pulsar is the one that we found in the crab nebula called crab pulsar.

It emits signals of time intervals 0.033s. It is due to high precision of time interval of emission of signals, crab pulsar is considered as the superclock of the universe. Crab pulsar emits signals in the optical, x-ray and radio frequency regions. As the pulses are quite sharp their arrival time can be measured to an accuracy of micro seconds. it is known that light from the pulsars at different optical wavelengths arrives simultaneously within the experimental resolving time. We can use these facts to put a limit on the rest mass of the photon.

Signal from the crab pulsar takes 5000 years, to reach on earth. Suppose that signals at two different frequencies travel with a small difference in velocity Δv and obviously arrive at slightly different times t and $t+\Delta t$ on earth.

Using, time $= \dfrac{\text{distance}}{\text{velocity}}$

$$t = \frac{l}{v}$$

Where l is the distance between the crab pulsar and the earth.

or $$v = \frac{l}{t} \qquad \text{..... (1)}$$

$\therefore$ $$\Delta v = -\frac{l}{t^2}\Delta t \qquad \text{..... (2)}$$

$\dfrac{\text{eq2}}{\text{eq1}}$ gives $$\frac{\Delta v}{v} = -\frac{\Delta t}{t}$$

Actually no such velocity difference has been observed. But by estimating the sensitivity of the experiment we can set an upper limit to $\Delta v \cdot \Delta t$ can be measured to an accuracy of about 2 milliseconds.

i.e., $$\Delta t = 2\times10^{-3}\text{s}$$

we know that $$t = 5000 \text{ years}$$

$$t = 5000\times3.15\times10^{7}\text{s}$$

(1 year $\simeq 3.15\times10^{7}$s) $t \simeq 1.58\times10^{11}$s

$$\therefore \quad \left|\frac{\Delta v}{v}\right| = \left|\frac{\Delta t}{t}\right| \le \frac{2\times10^{-3}}{1.5\times10^{11}}$$

or
$$\left|\frac{\Delta v}{v}\right| \simeq 10^{-14}$$

If we take $v \simeq c$

$$\left|\frac{\Delta v}{c}\right| \simeq 10^{-14}$$

Recall our earlier result

$$\frac{v^2}{c^2} = 1 - \frac{\upsilon_0^2}{\upsilon^2}$$

For the two signal frequencies υ_1 and υ_2 and the velocities v_1 and v_2, we get

$$\frac{v_1^2}{c^2} = 1 - \frac{\upsilon_0^2}{\upsilon_1^2}$$

and
$$\frac{v_2^2}{c^2} = 1 - \frac{\upsilon_0^2}{\upsilon_2^2}$$

subtracting, we get

$$\frac{v_1^2 - v_2^2}{c^2} = \upsilon_0^2\left(\frac{1}{\upsilon_2^2} - \frac{1}{\upsilon_1^2}\right)$$

or
$$\frac{(v_1+v_2)(v_1-v_2)}{c^2} = \upsilon_0^2\left(\frac{1}{\upsilon_2^2} - \frac{1}{\upsilon_1^2}\right)$$

Take $v_1 + v_2 \simeq 2c$ and $v_1 - v_2 = \Delta v$, we have

$$2\frac{\Delta v}{c} = \upsilon_0^2\left(\frac{1}{\upsilon_2^2} - \frac{1}{\upsilon_1^2}\right)$$

Take $\upsilon_1 = 8\times10^{14}$Hz (blue light)

and $\upsilon_2 = 5\times10^{14}$Hz (red light)

$$2\frac{\Delta v}{c} = \frac{\upsilon_0^2}{10^{28}}\left(\frac{1}{5^2} - \frac{1}{8^2}\right) = 2.44 \times 10^{-30}\upsilon_0^2$$

Using the limit $\frac{\Delta v}{c} = 10^{-14}$, we have

$$2 \times 10^{-14} = 2.44 \times 10^{-30}\upsilon_0^2$$

or $$\upsilon_0^2 = \frac{2 \times 10^{-14}}{2.4 \times 10^{-30}} = 0.833 \times 10^{16}$$

$$\upsilon_0^2 < 10^{16}$$

$$\upsilon_0 < 10^8$$

Putting this value in the expression for rest mass of the photon

$$m_p = \frac{h\upsilon_0}{c^2} = \frac{6.6 \times 10^{-34} \times < 10^8}{9 \times 10^{16}}$$

$$m_p < 0.733 \times 10^{-42}$$

That is the limiting mass of the photon is 10^{-42} kg. If we increase the experimental accuracy $\frac{\Delta v}{c}$, the limiting rest mass of the photon will also increase.

IMPORTANT FORMULAE

1. Michelson interferometer:

   a) Displacement of interference fringes, $\delta = \frac{2Lv^2}{c^2}$

   (b) Number of fringe shifts = $\frac{\delta}{\lambda}$

   (c) Fringe shift in water = $\frac{4n^2 l}{\lambda c} v_w f$, where $f = 1 - \frac{1}{n^2}$

2. Galilean transformations:

   a) Coordinate transformation equations $x' = x - vt$, $y' = y$ and $t' = t$

b) Velocity transformation equations $u'_x = u_x - v$, $u'_y = u_y$ and $u'_z = u_z$

c) Acceleration transformation equations $a'_x = a_x$, $a'_y = a_y$, and $a'_z = a_z$

3. Lorentz transformation equations:

$$x' = \gamma(x - vt) \qquad x = \gamma(x' + vt')$$

$$y' = y \qquad y = y'$$

$$z' = z \qquad z = z'$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \qquad t = \gamma\left(t' + \frac{vx'}{c^2}\right) \quad \text{where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

4. Space like and time like intervals.

$$L' = \gamma(L - vT)$$

and $T' = \gamma\left(T - \frac{v}{c^2}L\right)$

For space like interval

$L > cT$, L' is positive and T' can be positive, zero or negative.

For time like interval

$L < cT$, T' is positive and L' can be positive, negative or zero.

5. Lorentz length contraction formula:

$$L = L_0\sqrt{1 - v^2/c^2}, \quad L < L_0.$$

6. The orientation of a moving rod:

$$L = L'\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\cos\theta'\right)^{\frac{1}{2}}$$

$$\theta = \tan^{-1}(\gamma \tan\theta'), \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$L < L', \qquad \theta > \theta'$$

7. Atomic clock correction:

$$\frac{\delta\upsilon}{\upsilon_0} = -\frac{3kT}{2mc^2}$$

8. Time dilation

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}} \qquad \Delta t > \Delta t'$$

9. Transformation of velocities:

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \qquad u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}$$

$$u'_y = \frac{u_y\sqrt{1 - v^2/c^2}}{1 - \frac{u_x v}{c^2}} \qquad u_y = \frac{u'_y\sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}}$$

$$u'_z = \frac{u_z\sqrt{1 - v^2/c^2}}{1 - \frac{u_x v}{c^2}} \qquad u_z = \frac{u'_z\sqrt{1 - v^2/c^2}}{1 + \frac{u'_x v}{c^2}}$$

10. The velocity of light in moving water with respect to lab frame.

$$u = \frac{c}{n} + \left(1 - \frac{1}{n^2}\right)v$$

11. Doppler effect in sound:

$$\upsilon' = \upsilon\frac{1}{1 - \frac{v}{w}}$$ – Source is moving towards the observer

$$\upsilon' = \upsilon\left(1 + \frac{v}{w}\right)$$ – Observer is moving towards the source

12. Relativistic Doppler effect

$$\upsilon' = \upsilon\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$ – Source is moving towards the observer

or $$\lambda' = \lambda\sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}$$

13. Doppler effect for an observer off the line motion

$$\upsilon' = \upsilon\frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c}\cos\theta}$$ Source is moving towards the observer

14. Doppler navigation formula

$$r_b - r_a = -\lambda N_{ab}$$

15. Relativistic momentum

$$\vec{p} = m\vec{u} = \frac{m_0\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}$$

16. Variation of mass with velocity

$$m(v) = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

17. Mass-energy relation:

$$E = mc^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}, \quad E = K + m_0c^2$$

18. Expression for relativistic kinetic energy

$$K = mc^2 - m_0c^2 = m_0c^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)$$

19. Energy-momentum relation

$$E = \sqrt{p^2c^2 + m_0^2c^4}$$

20. Expression for radiation pressure

$P = \frac{2nE}{c} = \frac{2I}{c}$ - Light reflected from a mirror

$P = \frac{I}{c}$ - For an absorbing surface

21. Rest mass of the photon, $m_0 = 0$

Mass of the photon $m = \frac{h\upsilon}{c^2}$

22. Orientation of a light pulse with respect to S frame

$$\cos\theta = \frac{\cos\theta' + \frac{v}{c}}{1 + \frac{v}{c}\cos\theta'}$$

UNIVERSITY MODEL QUESTIONS

Section A

(*Answer questions in about* **two** *or* **three** *sentences*)

Short answer type questions

1. What was the aim of the Michelson-Morley experiment?
2. Draw a labelled diagram of Michelson-Morley experiment
3. Explain the negative result of Michelson - Morley experiment.
4. What is the use of compensating plate in Michelson - Morley experiment?
5. Explain the significance of Michelson - Morley experiment.
6. What are tht postulates of special theory of relativity?
7. Write down the Galilean transformation equations and explain the symbols used.
8. Obtain the Galilean transformation equations for velocity of a particle moving in space.
9. Obtain the Galilean acceleration transformation equations.
10. What is a Galilean invariant quantity?
11. Write down the Lorentz transformation equations and explain the symbols used.
12. What is meant by simultaneity?
13. What is meant by time like interval?
14. What is meant by space like interval?
15. Draw the graphical variation of $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$ with $\frac{v}{c}$.
16. What is Lorentz contraction in relativity?
17. What is meant by time dilation in relativity?
18. Define the following terms

    a) Proper frame b) Proper length c) Proper time
19. Write down the velocity transformation equations in relativity.
20. What is the effect on speed of light in a moving medium?
21. Define Doppler effect.
22. Write down the relativistic Doppler formula in terms of frequency and explain the symbols used.
23. Write down two applications of Doppler effect.

24. What is twin paradox?
25. Define relativistic momentum and energy.
26. How does mass vary with velocity
27. Write down the mass energy relation and explain the symbols used.
28. State the expressions for rest energy, kinetic energy and total energy of a relativistic particle.
29. Suppose the speed of light were infinite what would happen to the relativistic predictions of length contraction, time dilation and mass variation.
30. What is meant by Doppler navigation?
31. Write down the energy momentum relation and explain the symbols used.
32. Draw the graphical variation of mass with velocity.
33. Distinguish between elastic and inelastic collision in relativity.
34. Massless particles travel with speed of light. Justify.
35. What is meant by radiation pressure of light?
36. Write down expression for radiation pressure and explain the symbols used.
37. If a photon travels with a speed other than that of light, then what would be the rest mass of the photon
38. What is a pulsar? Name two of them.
39. What is the effect of radiation pressure in stars?
40. Why pulsars are considered as super clocks of the universe?

Section B

(Answer questions in about half a page to one page)

Paragraph / Problem

1. Show that in the non-relativistic limit the Lorentz transformations reduce to Galilean transformations.
2. Explain how time dilation was verified experimentally.
3. Show that when $\frac{v}{c} \ll 1$, , the relativistic kinetic energy becomes the classical one.
4. Show that addition of velocity to the velocity of light gives velocity of light.
5. Events that are simultaneous in one frame are not simultaneous in another reference frame, prove it.
6. Explain how velocity transformation was verified experimentally.
7. Explain how mass variation was verified experimentally.
8. Arrive at space like and time like intervals.
9. A rod of length L' in S' frame moving with a speed v. Find out its orientation in the lab frame.
10. What is the role of time dilation in atomic clocks?
11. Arrive at velocity addition theorem in relativity.

12. Deduce an expression for velocity of light in moving water with respect to lab frame.
13. How will you track a moving object by Doppler effect?
14. Explain the ageing of twins quantitatively.
15. Derive the relation $E = mc^2$
16. Derive the energy-momentum relation from mass variation with velocity.
17. Derive an expression for radiation pressure of light.

Problems

18. What will be the fringe shift in Michelson Morley experiment, if the effective length of each part is 6 m and wavelength of light used is 6000 Å ? Velocity of earth $3\times10^4\,ms^{-1}$.

[$\frac{1}{5}$of a fringe]

19. In Michelson-Morley experiment the length of the paths of the two beams is 11 metres each. The wavelength of light used is 6000Å. If the expected fringe shift is 0.4 fringes, calculate the velocity of earth relative to ether [$v = 3\times10^4\,ms^{-1}$]
20. Show that for $v \ll c$, the Lorentz transformations will become Galilean transformations.
21. At what speed v, will the Galilean and Lorentz expressions for x differ by

a) 1% b) 50% [$42\times10^8\,ms^{-1}$, $2\cdot235\times10^8\,ms^{-1}$.]

22. The length of a space craft is measured to be exactly $\frac{3}{4}$ of its proper length. What is its speed with respect to earth. Also find the dilation of spacecrafts unit time.

[(a) $v = 0.66c = 1.98\times10^8\,ms^{-1}$ b) $\Delta t_0 = 0.75s$]

23. Let a meter scale be moving with a velocity half of that of light. What will be its length measured by a stationary observer. [L = 0.866 m]
24. A rod of proper length 3m moves with a velocity 0.86 c in a direction making an angle 60° with its length. Find the apparent length. [2.7495 m]
25. Find the speed of the space ship of every day spent on it may correspond to 2 days on the earth surface. [$v = 2.598\times10^8\,ms^{-1}$]
26. At what speed should a clock be moved so that it may appear to lose 1 minute in each hour [$v = 5.4\times10^7\,ms^{-1}$]
27. Calculate the energy equivalent of the rest mass of an electron. [$E = 8.19\times10^{-14}\,J$]
28. Calculate the velocity of an electron having kinetic energy 1 MeV. $m_0 = 9\times10^{-31}\,kg$.

[$v = 2\cdot62\times10^8\,ms^{-1}$]

29. Calculate the energy released in MeV when a neutron decays into a proton and an electron. $M_n = 1\cdot6747\times10^{-27}\,kg$, $M_e = 9\times10^{-31}\,kg$. [E = 0.7303 MeV]

30. An electron and a positron practically at rest come together and annihilate into other, producing two protons of equal energy. Find the energy and equivalent mass of each proton. $m_e = 9\times10^{-31}$ kg.

[energy of each proton $= 81\times10^{-15}$ J mass equivalent of each proton 9×10^{-31} kg]

31. Show that the rest mass of a particle of momentum p and kinetic energy K is $m_0 = \dfrac{p^2c^2 - K^2}{2Kc^2}$.

32. A particle with rest mass m_0 is moving with a velocity $\dfrac{c}{\sqrt{2}}$. Find out the momentum, kinetic energy and total energy of this particle. $\left[m_0c, (\sqrt{2}-1)\, m_0c^2, \sqrt{2}\, m_0c^2\right]$

33. The rest mass of an electron is 9.1×10^{-31} kg what will be its mass if it were moving with $\frac{4}{5}$th of the speed of light. [1.5×10^{-30} kg]

34. Suppose that the total mass of 1kg is transformed into energy, how large is this energy in kilowatt hours [2.5×10^{10}]

35. A certain quantity of ice at $0°$ C melts into water at ice at $0°$ C, in doing so gains 1 kg of mass. What was the initial mass. [2.68×10^{11} kg]

36. Show that $x^2 + y^2 + z^2 - c^2t^2$ is invariant under Lorentz transformation.

37. Derive $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$ starting from $E = mc^2$

38. Derive the mass transformation equation $m = \gamma m'\left(1 + \dfrac{u'_x v}{c^2}\right)$

39. Show that the differential operator

$$\Box^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$ is invariant under Lorentz transformation.

40. Show that the proper time $d\tau = dt\sqrt{1 - \dfrac{v^2}{c^2}}$ is an invariant quantity under Lorentz transformation.

41. What is the speed of π-mesons whose observed mean life is 2.5×10^{-7} s. The proper mean life of these π-mesons is 2.5×10^{-8} s. [$v = 0.995c$]

42. A beam of pions has velocity $v = 0.6c$. It has a half life of 1.8×10^{-8} s. Estimate the time taken by pions to decay to half their initial number. [2.25×10^{-8} s]

43. Find the shape of a circle at rest in a frame S when viewed from a frame S', when S' is moving with speed v along x-axis with respect to S. [ellipse]

44. Frame S' moves with velocity v relative to frame S. A bullet in frame S is fired with velocity u at an angle θ with respect to forward direction of motion. What is this angle as measured in S'.
$$\left[\tan\theta' = \tan\theta\sqrt{1-\frac{v^2}{c^2}}\left(1-\frac{v}{u\cos\theta}\right)\right]$$

45. Two β-particles A and B travel in opposite directions each with a velocity 0.98 c. What is their relative velocity as observed by a stationary observer. [0.99c]

46. A particle moves with velocity represented by a vector $u' = 3\hat{i}+4\hat{j}+12\hat{k}\,ms^{-1}$ in S' frame. Find the velocity of the particle in frame S. S' moves with velocity 0.8 c relative to S along positive x-direction. $[u = 2.4\times10^{8}\hat{i}+2.4\hat{j}+7.2\hat{k}]$

47. S and S' are two inertial frames. S be rest and S' be moving with a uniform speed v. Find the coordinates of S' of the following.
$v = 0.6c$
a) $x = 4m$, $t = 1s$ [a) $x' = 2.5\times10^{8}\,m$, $t' \simeq 1.25s$
b) $x = 1.8\times10^{8}m$, $t = 1s$ b) $x' = 0$, $t' = 0.8s$]

48. A rod of length l_0 oriented parallel to the x-axis moves with speed u along the x-axis in S. What is the length measured by an observer in S'.
$$\left[l = l_0\frac{[(c^2-v^2)(c^2-u^2)]^{\frac{1}{2}}}{c^2-uv}\right]$$

49. A photon of energy E_0 collides with a free particle of mass m_0 at rest. If the scattered photon flies at an angle θ, what is the scattering angle of the particle ϕ.
$$\left[\cot\phi = \left(1+\frac{E_0}{m_0c^2}\right)\tan\frac{\theta}{2}\right]$$

50. An electron of energy 10 MeV moving at right angles to a uniform field 2T. Calculate the radius of the circular path a) classically (b) relativistically
[(a) 0.53cm (b) 1.8 cm]

Section C

(*Answer questions in about* **one** *or* **two** *pages*)

Long answer type questions (Essays)

1. Describe the Michelson-Morley experiment and explain the null result obtained.
2. Derive the Lorentz transformation equations.
3. Explain the consequences of Lorentz transformation equations.

4. Derive velocity transformation equations from Lorentz transformation equations. How it is verified experimentally.
5. Derive the relativistic Doppler formula
6. How does the variation of mass with velocity has been verified.
7. By considering inelastic collision in relativity, establish that the inertial frames S and S′ are identical.
8. Explain the experimental verification of the equivalence of mass and energy.
9. Derive the Doppler formula on the basis of photon picture of light.
10. Explain how does light from a pulsar sets an upper limit to photon's rest mass.

Hints to problems

18. Number of expected fringe shift $= \dfrac{2Lv^2}{\lambda c^2}$

21. See example 2

22. $L = L_0\sqrt{1-v^2/c^2}$, $L = \frac{3}{4}L_0$ $\Delta t = \dfrac{\Delta t_0}{\sqrt{1-v^2/c^2}}$, $\Delta t = 1$ second

One second in spacecraft will appear to be 0.75 second for an observer on earth.

23. $L_0 = 1\text{m}$ $\quad L = L_0\sqrt{1-v^2/c^2}$

24. See example 5

$$L' = 3\text{m} \qquad L'_x = L'\cos 60 = \tfrac{3}{2}\text{m}$$

$$L'_y = L'\sin 60 = \frac{3\sqrt{3}}{2} \quad L_x = L'_x\sqrt{1-v^2/c^2} = 0.9\text{m}$$

$$L_y = L'_y = \frac{3\sqrt{3}}{2}$$

$$\therefore \quad L = (L_x^2 + L_y^2)^{\frac{1}{2}} = 2.7495\text{m}$$

25. $\Delta t_0 = 1\text{day}$

$\Delta t = 2\text{days}$ use $\Delta t = \dfrac{\Delta t_0}{\sqrt{1-v^2/c^2}}$.

26. 60 minutes in rest clock appears to be 59 minutes in the moving clock. Then

$\Delta t_0 = 59$ minutes, $\Delta t = 60$ minutes use $\Delta t = \dfrac{\Delta t_0}{\sqrt{1-v^2/c^2}}$.

27. $m = 9.1\times10^{-31}$ kg use $E = mc^2$.

28. See example 14

29. $M_N \rightarrow M_p + M_e$

loss of mass $m = M_N - (M_p + M_e)$

This loss of mass is converted into energy use $E = mc^2$ in joules

To convert joule into MeV divide by 1.6×10^{-13}.]

30. $m_e = m_p = 9 \times 10^{-31}$ kg $\qquad m_e c^2 + m_p c^2 = 2mc^2$

energy of each proton $= mc^2$

31. See example 17

32. $p = mv = m\dfrac{c}{\sqrt{2}}$, where $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}} = \sqrt{2}\, m_0$

$K = (m - m_0)c^2 \qquad T.E = K + m_0 c^2$

33. $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$ where $v = \dfrac{4}{5}c$

34. $\begin{bmatrix} E = mc^2, m = 1\text{kg} \\ 1\text{KWH} = 36 \times 10^5 \text{ joules.} \end{bmatrix}$

35. Let M be the initial mass.

Energy absorbed during melting $= M\,L$, $L = 336 \times 10^3\,J\,kg^{-1}$

This is equal to $mc^2 = 1 \times c^2$, $M\,L = mc^2$.

36. Show that $x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$ using

$x' = \gamma(x - vt)$, $y' = y$, $z' = z$ and $t' = \gamma(t - vx/c^2)$

37. Use $E^2 = (p^2c^2 + m_0^2c^4)$ then replace p by mv, which is $\gamma m_0 v$.

38. $E = mc^2$, $m = m_0/\sqrt{1 - u^2/c^2}$, $E' = m'c^2$

$m' = m_0/\sqrt{1 - u'^2/c^2}$. Use $E' = \dfrac{E - vp_x}{\sqrt{1 - v^2/c^2}}$, $p_x = mu_x$

Substitute for E', E and p_x

39. Use $\quad x' = \gamma(x - vt)$, $\quad y' = y$, $z' = z$ and $t' = \gamma(t - vx/c^2)$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial}{\partial z'}\frac{\partial z'}{\partial x} + \frac{\partial}{\partial t'}\frac{\partial t'}{\partial x}$$

$$\frac{\partial}{\partial x'} = \gamma, \ \frac{\partial y'}{\partial x} = 0, \ \frac{\partial z'}{\partial x} = 0, \ \frac{\partial t'}{\partial x} = \frac{\gamma v}{c^2}$$

$$\therefore \quad \frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t}\right)$$

Similary $\frac{\partial}{\partial y}=\frac{\partial}{\partial y'}$ and $\frac{\partial}{\partial z}=\frac{\partial}{\partial z'}$, $\frac{\partial}{\partial t}=\gamma\left(\frac{\partial}{\partial t'}-\frac{v\partial}{\partial x'}\right)$

Find $\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}$ and $\frac{\partial^2}{\partial t^2}$. All these are put in given equation it will be

$$\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}.$$

40. Put $v^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2$

41. $\Delta t_0 = 2.5\times10^{-8}$s, $\Delta t = 2.5\times10^{-7}$s

Using $\Delta t=\frac{\Delta t_o}{\sqrt{1-v^2/c^2}}$ find v.

42. $\Delta t_0 = 1.8\times10^{-8}$s, $v=0.6c$ find Δt using $\Delta t=\frac{\Delta t_0}{\sqrt{1-v^2/c^2}}$

43. For a circle $x^2+y^2=R^2$ use $x=\gamma(x'+vt')$ and $y=y'$

44. $u'_x=\frac{u_x-v}{1-\frac{u_x v}{c^2}}$ and $u'_y\frac{u_y\sqrt{1-v^2/c^2}}{1-\frac{u_x v}{c^2}}$

$u_x=u\cos\theta$ $\quad u_y=u\sin\theta$ and $\frac{u'_y}{u'_x}=\tan\theta'$

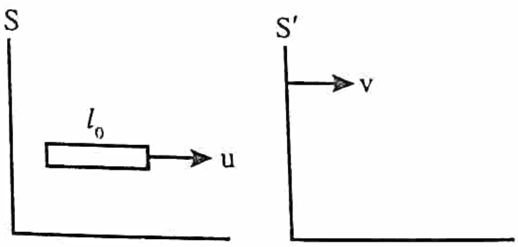
45. $u'_x=\frac{u_x-v}{1-\frac{u_x v}{c^2}}$, $u_x=.98c$, $v=0.98c$

46. $u'_x=3$ using $\quad u_x=\frac{u'_x+v}{1+\frac{u'_x v}{c^2}}$ where $v=0.8c$

$u'_y=4$ using $\quad u_y=\frac{u'_y\sqrt{1-v^2/c^2}}{1+\frac{u'_x v}{c^2}}$ $u'_z=12$ using $u_z=\frac{u'_z\sqrt{1-v^2/c^2}}{1+\frac{u'_x v}{c^2}}$

47. See example 9

48. Speed of the rod w.r. to S′, $V = \dfrac{u - v}{1 - \dfrac{uv}{c^2}}$

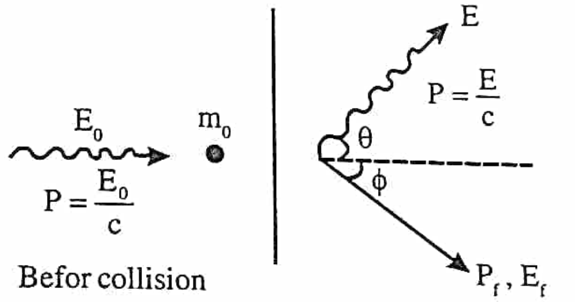


Using $L = L_0\sqrt{1 - v^2/c^2}$ .... (1)

Here $L = l'$, $L_0 = l$, $v = V = \dfrac{u - v}{1 - \dfrac{uv}{c^2}}$

substituting in eq (1), we get the result.

49.



From conservation of momentum along x-direction

$$\frac{E_0}{c} = \frac{E}{c}\cos\theta + p_f\cos\phi \qquad ..... (1)$$

along y-direction

$$0 = \frac{E}{c}\sin\theta + p_f\sin\phi \qquad ..... (2)$$

From conservation of energy, we have

$$E_0 + m_0c^2 = E + E_f \qquad ..... (3)$$

Where $E_f = \sqrt{p_f^2c^2 + m_0^2c^4}$ solving eqs (1), (2) and (3) collect ϕ.

50. a) $r = \dfrac{m_0v}{qB} = \dfrac{p}{qB} = \sqrt{\dfrac{2m_0K}{qB}}$

$m_0 = 9.1\times10^{-31}$ kg, $K = 10$ MeV $q = 1.6\times10^{-19}$ c, $B = 2$T we get r = 0.53 cm

b) $r = \dfrac{mv}{qB} = \dfrac{p}{qB}$ ...... (1)

$E = \sqrt{p^2c^2 + m_0^2c^4}$ from this calculate p and put in eq(1), we get r=1.8cm

Where $E = m_0c^2 + K$