3

WAVELIKE PROPERTIES OF PARTICLES

Intrduction

In classical mechanics particles and waves are distinct and have clear cut definitions. Properties of particles are governed by Newtonian mechanics and waves are governed by wave theories. The energy of particles are confined to a small region of space where as a wave distributes its energy throughout space in its wave fronts. In contrast to this distinct behaviour of particles and waves, when we come to microscopic world this distinction disappears and sometimes particles obey the rules of waves. The system of mechanics associated with microscopic particles is called wave mechanics, because it deals with wavelike behaviour of particles. In this chapter we deal with experimental evidences in support of wave like behaviour of microscopic particles.

De Broglie Hypothesis

In chapter 1 we dealt with particles properties of waves and conclusively proved the wave particle duality of radiation. Following the acceptance of this duality concept for radiation, it was natural to think of the converse. The initiative in this regard was taken in 1924 by the French physicist, Louis de Broglie. Louis de Broglie proposed that if radiation exhibits dual nature (properties of both waves and particles) then matter which consists of particles must also possess wave properties. This is called the de Broglie hypothesis. The waves associated with these particles are called matter waves or de Broglie waves.

De Broglie's hypothesis attributing a dual particle wave character to matter. He arrived at this from the following observations.

(i) Universe is composed of radiations and matter (basic constituents of the physical world).

(ii) Nature loves symmetry (matter and radiation should be similar in nature)

(iii) Radiation has dual nature.

De Broglie's idea thus meant that any matter like a cricket ball, bullet, earth, electrons, protons, neutrons etc. might show the characteristics of waves. It is interesting to note that de Broglie's revolutionary hypothesis would perhaps have been regarded as one of the philosophical conjectures, but for the fact that within

two years of their formulation they received striking experimental confirmation. The existence of de Broglie waves was confirmed in diffraction experiments with electron beams in 1927 and in 1929 de Broglie received the Nobel prize.

After introducing the idea of matter waves, de Broglie proceeded to calculate the wavelength to be associated with a moving particle. De Broglie further suggested that in discussing such fundamental entities, as waves and particles, certain basic physical formula should apply in both cases.

According to quantum theory of radiation, the energy of a photon is

$$E = h\nu \qquad \text{..... (1)}$$

since photon is a particle of zero rest mass $(m_o = 0)$, according to the theory of relativity

$$E = \sqrt{p^2c^2 + m_o c^4}$$

For a photon $m_o = 0$

Thus $$E = pc \qquad \text{..... (2)}$$

Comparing eqns 1 and 2, we get

$$pc = h\nu$$

or $$p = \frac{h\nu}{c} = \frac{h}{\lambda} \quad (\because \ c = \nu\lambda)$$

or $$\lambda = \frac{h}{p} \qquad \text{..... (3)}$$

It is the relation connecting between wave character of radiation (λ) and particle character (momentum) of radiation. De Broglie asserted that an equation applicable to radiation must also be applicable to all material particles as well because nature loves symmetry. Thus, the wavelength that must be associated with a material particle of momentum $(p = mv)$ is given by

$$\lambda = \frac{h}{p} = \frac{h}{mv} \qquad \text{..... (4)}$$

This is the famous de-Broglie formula. From the above equation it is obvious that since h is so very small, the wavelength would be so small for all matter except the very light elementary particles. The wave properties of matter are, therefore, likely to manifest themselves only in the domain of the elementary particles.

From the de Broglie's formula it follows that

(i) Lighter the particle, greater its de Broglie wavelength

(ii) The faster the particle moves, smaller its de Broglie wavelength

(iii) The de Broglie wave of a particle is independent of the charge or nature of the particle.

(iv) The matter waves are not electromagnetic in nature as the electromagnetic waves are produced only by charged particles.

(v) If $v = 0$, $\lambda = \frac{h}{0} = \infty$ and $v = \infty$, $\lambda = \frac{h}{\infty} = 0$. This means that the matter waves are present only if the particle is in motion.

(vi) When the particle moves with a velocity comparable to the velocity of light, the

$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$, where m is the relativistic mass of the body.

Then $\lambda = \frac{h}{mv} = \frac{h\sqrt{1 - v^2/c^2}}{m_0 v}$

(vii) Matter waves propagate in the form of wave packet with group velocity.

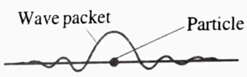


Figure 3.1: Wave associated with a particle

De Broglie wavelength in terms of potential

Consider a charged particle with charge e starts from rest and accelerated through a potential difference V. The velocity v acquired by the particle is given by

$$\frac{1}{2} mv^2 = eV$$

where m is the mass of the particle

$$\therefore \quad v = \sqrt{\frac{2eV}{m}} \qquad \text{..... (5)}$$

∴ The momentum of the particle is

$$p = mv = m\sqrt{\frac{2eV}{m}} = \sqrt{2meV}$$

∴ The de Broglie wavelength $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2meV}} = \frac{h}{\sqrt{2mE}}$ $(eV = E)$

For example in the case of electron $m = 9.1 \times 10^{-31}$ kg, $e = 1.6 \times 10^{-19}$ C and take $h = 6.62 \times 10^{-34}$ Js

∴ The above formula becomes

$$\lambda = \frac{12.27Å}{\sqrt{V}} \qquad \text{..... (6)}$$

From the above equation it is clear that $\lambda \propto \frac{1}{\sqrt{V}}$

The potential difference required to bring an electron of wavelength λÅ to rest is given by

$$V = \frac{150.6}{\lambda^2}$$

If material particles are in thermal equilibrium at temperature T, then $E = \frac{3}{2} kT$, where $k = 1.38 \times 10^{-23}\ JK^{-1}$, de Broglie. Wavelength $\lambda = \frac{h}{\sqrt{3mkT}}$

Note:

(i) For a proton $\lambda = \frac{0.286Å}{\sqrt{V}}$

(ii) For a dentron $\lambda = \frac{0.202Å}{\sqrt{V}}$

(iii) For an α-particle $\lambda = \frac{0.101Å}{\sqrt{V}}$

(iv) For a relativistic particle having rest mass m_0, kinetic energy K.E., the momentum is given by

$$E = \sqrt{p^2c^2 + m_0^2c^4}$$

$$E^2 = p^2c^2 + m_0^2c^4$$

$$(m_0c^2 + K.E)^2 = p^2c^2 + m_0^2c^4$$

so $p = \sqrt{2m_0 K.E + \frac{K.E^2}{c^2}}$

∴ corresponding Debroglie wavelength is given by

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m_0 K.E + \frac{K.E^2}{c^2}}}$$

or $$\lambda = \frac{hc}{\sqrt{K.E(K.E + 2m_0c^2)}}$$

This shows that relativistic effects reduce the value of de-Broglie wave-length.

Characteristics of Matter Waves

1. Matter waves cannot be observed. It is a wave model to describe and study matter.
2. Matter waves travel both even in vacuum, hence they are not mechanical waves.
3. Matter waves are associated with moving particles such as charged and neutral particles.
4. Matter waves are probabilistic waves because waves represent the probability of finding a particle in space.
5. The phase velocity of matter waves can be greater than that of light.
6. Waves associated with microscopic particles such as electrons can be measured.
7. Waves associated with macroscopic particles cannot be measured.
8. The phase velocity is different for different matter waves, depending on mass and velocity of the particle. It is inversely proportional to their wavelength.
9. Matter waves are not electromagnetic waves in nature.
10. The velocity of matter waves depends on the velocity of the material particle.

Example 1

Calculate the de Broglie wavelength of (i) a rifle bullet of mass 2g moving with a speed of 400m/s and (ii) a 2000kg car moving along the highway at $30ms^{-1}$.

Solution

We have $$\lambda = \frac{h}{mv}$$

(i) $m = 2g = 2\times10^{-3}kg, \quad v = 400ms^{-1}$

$$\therefore \quad \lambda = \frac{6.62\times10^{-34}}{2\times10^{-3}\times400} = 8.275\times10^{-34}m$$

(ii) $m = 2000kg, \quad v = 30ms^{-1}$

$$\therefore \quad \lambda = \frac{6.62\times10^{-34}}{2000\times30} = 1.1\times10^{-38}m.$$

See the so small wavelength associated with macroscopic particles.

Example 2

Calculate the de Broglie wavelength associated with a proton moving with a velocity equal to $\frac{1}{20}^{th}$ of the velocity of light.

Solution

$$m = 1.67\times10^{-27}kg, \quad v = \frac{c}{20} = \frac{3\times10^8}{20} = 1.5\times10^7 ms^{-1}.$$

Using $$\lambda = \frac{h}{mv} = \frac{6.62\times10^{-34}}{1.67\times10^{-27}\times1.5\times10^7} = 2.64\times10^{-14}m.$$

Example 3

Find the de Broglie wavelength associated with an electron moving with a velocity 0.5 c. $m_o = 9.1\times10^{-31}kg$

Solution

$m_o = 9.1\times10^{-31}kg, \quad v = 0.5c = 1.5\times10^8 ms^{-1}$

Since the particle's velocity is comparable to the velocity of light, mass of the particle

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\therefore \quad \lambda = \frac{h}{mv} = \frac{h\sqrt{1-\frac{v^2}{c^2}}}{m_0 v}$$

$$\lambda = \frac{6.62\times10^{-34}\sqrt{1-\frac{c^2}{4c^2}}}{9.1\times10^{-31}\times1.5\times10^{8}}$$

$$\lambda = \frac{6.62\times10^{-34}\times\sqrt{3}}{9.1\times10^{-31}\times1.5\times10^{8}\times2}$$

$$\lambda = 4.2\times10^{-12}\,\text{m}$$

Example 4

What is the energy of the gamma ray photon having wavelength of 1Å.

Solution

$$\lambda = 1\text{Å} = 10^{-10}\,\text{m}$$

Using $\lambda = \frac{h}{p}$ $\quad\therefore\quad p = \frac{h}{\lambda} = \frac{6.62\times10^{-34}}{10^{-10}}$

$$p = 6.62\times10^{-24}$$

Energy of gamma photon, $E = pc = 6.62\times10^{-24}\times3\times10^{8}$

$$E = 19.86\times10^{-16}\,\text{J}$$

or

$$E = \frac{19.86\times10^{-16}}{1.6\times10^{-19}}\,\text{eV}$$

$$E = 1.24\times10^{4}\,\text{eV}.$$

Example 5

Obtain the de Broglie wavelength of a neutron of kinetic energy 150eV. Would a neutron beam of the 150eV energy be suitable for crystal diffraction.

Solution

$$m_n = 1.675\times10^{-27}\,\text{kg}.$$

$$K.E = 150\,\text{eV} = 150\times1.6\times10^{-19}\,\text{J}$$

i.e.

$$\frac{p^2}{2m_n} = 150\times1.6\times10^{-19}$$

$$p = \sqrt{2\times1.675\times10^{-27}\times150\times1.6\times10^{-19}}$$

$$p = 28.35\times10^{-23}$$

$$\therefore\quad \lambda = \frac{h}{p} = \frac{6.62\times10^{-34}}{28.35\times10^{-23}} = 2.33\times10^{-12}\,\text{m}$$

i.e.

$$\lambda = 0.0233\text{Å}$$

For crystal diffraction to take place the wavelength of incident probe should be of the order of interatomic spacing. Since the interatomic spacing is about hundred times greater than the wavelength obtained, it is not suitable for diffraction experiments.

Note : $K.E = \frac{1}{2}mv^2 = \frac{m^2v^2}{2m} = \frac{p^2}{2m}$ for nonrelativistic particle

$K.E = mc^2 - m_0c^2 = E - m_0c^2$, with $E = \sqrt{p^2c^2 + m_0^2c^4}$ for a nonrelativistic particle.

It may be noted that in example 1 the wavelength calculated are too small to be observed in the laboratory. But in examples 2 and 3 the wavelengths are of the same order as atomic or nuclear sizes. This shows that for particles of atomic or nuclear size, the wave behaviour will be observable.

Experimental evidence for de Broglie waves

De Broglies hypothesis attributed a wave like character to matter. According to this a beam of particles each travelling with a momentum p show properties of a wave, the wavelength λ is given by the de Broglie's relation. In 1927 two American physicists Clinton Joseph Davisson and Lester Germer independently confirmed de Broglies hypothesis by studying the diffraction pattern of electrons reflected from a nickel surface. This was followed by experiments of G.P. Thomson of England in 1928 which showed the formation of diffraction rings upon the passage of electrons through thin metals. [Davisson and G.P. Thomson received Nobel prize in 1938 for their experimental discovery of the diffraction of electrons by crystal. J.J. Thomson, father of G.P. Thomson, was also awarded Nobel prize in 1906 for his theoretical and experimental investigations on the conduction of electricity by gas. People wittingly say father discovered electron as a particle son discovered that it is a wave].

Davisson and Germer experiment

The experimental arrangement consists of an electron gun from which a collimated beam of electrons emerge. These mono energetic electrons fall on the target (a single crystal of nickel) get scattered in all directions. The intensity of the scattered electrons was measured as a function of angle ϕ, the angle between the incident and the scattered beams.

A graph was plotted between the angle ϕ and the intensity of the scattered beam for different energies of the incident beam. It is observed that a bumb begins to appear in the curve of 44V electron incident beam. This bumb moves upward as the voltage increases and attains maximum for 54V at angle $\phi = 50^\circ$. Above 54V the bumb again diminishes. The bumb at 54V offers the evidence for the existence of waves associated with electrons as predicted by de Broglie.

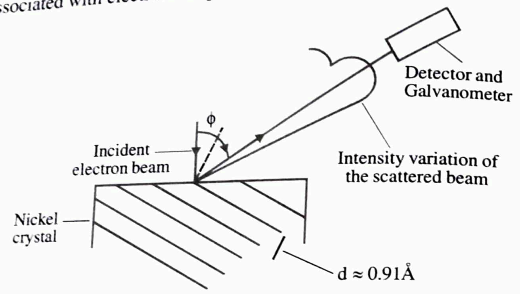


Figure 3.2

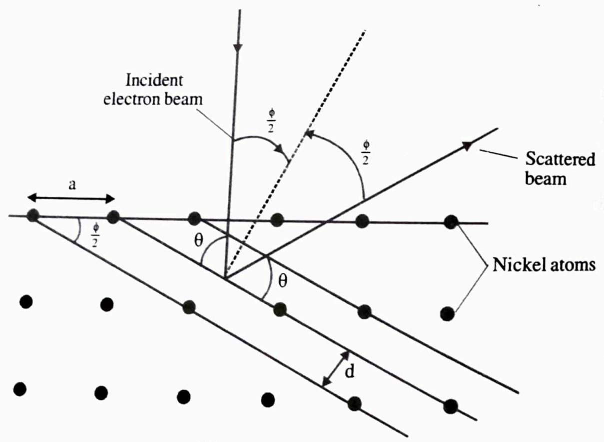


Figure 3.3

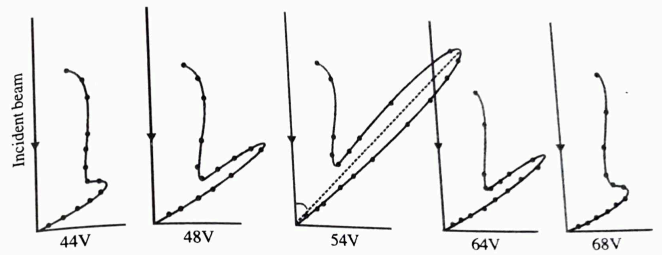


Figure 3.4

It is known from the study of X-rays that a nickel crystal acts as a grating element. According to Bragg's equation

$$n\lambda = 2d\sin\theta \qquad \text{.........(7)}$$

For first order diffraction $n = 1$

$$\lambda = 2d\sin\theta$$

For nickel crystal the distance (d) between the atomic planes is 0.91Å.

i.e. $\quad d = 0.91\text{Å}$.

The angle of incidence $\theta = \frac{(180-\phi)}{2}$

For $\phi = 50^\circ$, $\theta = \frac{(180-50)}{2} = 65^\circ$

$$\therefore \quad \lambda = 2d\sin\theta$$

$$\lambda = 2\times 0.91\times \sin 65 = 1.65\text{Å}$$

According to de-Broglies formula

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m\cdot KE}}$$

Substituting for $h = 6.62\times10^{-34}$ Js, $m = 9.1\times10^{-31}$ kg and $K\cdot E = 54$V, we get

$$\lambda = \frac{6.62\times10^{-34}}{\sqrt{2\times9.1\times10^{-31}\times54}} = 1.66\text{Å}$$

This is in remarkable agreement with the observed wavelength 1.65Å. This experiment convincingly established the de-Broglie's concept of wave nature of matter.

Note : For nickel the distance (a) between two atoms is 2.15Å.

i.e. $a = 2.15\text{Å}$

From the figure we have

$$\frac{d}{a} = \sin\frac{\phi}{2} \quad \text{where d is the interplanar distance}$$

$$\therefore \quad d = a\sin\frac{\phi}{2} = 2.15 \times \sin\left(\frac{50}{2}\right) \approx .909\text{Å}$$

$$\simeq 0.91\text{Å}$$

Example 6

In the Davisson-Germer experiment using a Ni crystal a second order beam is observed at angle of 55°. For what accelerating voltage does this occur.

Solution

$\phi = 55^\circ,\ a = 2.15\text{Å}$

$$\therefore \quad d = a\sin\frac{\phi}{2} = 2.15 \times \sin\frac{55}{2} = 0.993\text{Å}$$

$$\theta = \frac{180 - \phi}{2} = 62.5$$

Using $2d\sin\theta = n\lambda \quad n = 2$

$$\lambda = d\sin\theta$$

$$\lambda = 0.993 \times \sin 62.5$$

$$\lambda = 0.881\text{Å}$$

From $$\lambda = \frac{12.27\text{Å}}{\sqrt{V}}$$

$$V = \frac{12.27^2}{\lambda^2} = \frac{12.27^2}{0.881^2}$$

$$V = 193.97 \text{ volts.}$$

Example 7

A certain crystal is cut so that the rows of atoms on its surface are separated by a distance of 0.352 nm. A beam of electrons is accelerated through a potential difference of 175V and is incidently normal on the surface. At what angle diffraction beam would be found for n = 2.

Solution

$a = 0.352\text{ nm},\ V = 175\text{v}$

Using $$\lambda = \frac{12.27\text{Å}}{\sqrt{V}} = \frac{12.27}{\sqrt{175}} = 0.927\text{Å}$$

Using $a\sin\phi = n\lambda$, here $n = 2$

$$\sin\phi = \frac{2 \times 0.927\text{Å}}{3.52\text{Å}}$$

$$\sin\phi = 0.527$$

$$\phi = 31.78^\circ$$

Note $$n\lambda = 2d\sin\theta \qquad \text{..... (1)}$$

$$\theta = \left(\frac{180 - \phi}{2}\right)$$

$$\therefore \quad n\lambda = 2d\sin\left(\frac{180 - \phi}{2}\right)$$

$$n\lambda = 2d\cos\frac{\phi}{2}$$

But $$d = a\sin\frac{\phi}{2}$$

$$\therefore \quad n\lambda = 2a\sin\frac{\phi}{2}\cos\frac{\phi}{2}$$

$$n\lambda = a\sin\phi \qquad \text{..... (2)}$$

This shows that eqs (1) and (2) are the same but in terms of different parameters.

Uncertainty relationships for classical waves

In classical mechanics the dynamical variables like position x and momentum p of a particle can be measured very accurately. But when we come to microscopic (quantum) particles we use de Broglie waves to describe particles. The amplitude of the wave tells us about the position of the particle. Suppose we consider a harmonic wave (sine wave) to describe the wave associated with a particle. This leaves us in a very uncomfortable situation (figure 3.5). Because a pure harmonic wave necessarily extends from $-\infty$ to ∞. Instead of a harmonic wave suppose we use a wave pulse (wave packet) (figure 3.5) to represent a wave associated with a particle, it does a good job of locating the particle within a small region of space but not precisely.

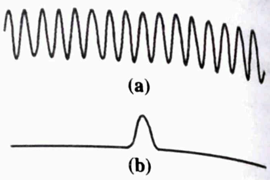


Figure 3.5
(a) A pure sine wave which extends from $-\infty$ to $+\infty$
(b) A narrow wave packet

This brings another problem. The wave pulse does not have a well defined wavelength so also momentum $\left(p = \frac{h}{\lambda}\right)$. This shows that in quantum mechanics we cannot precisely determine the position and momentum accurately. If we make wavelength more and more precise the position will become less and less precise. We can improve our knowledge of locating the position of the particle only at the expense of our knowledge of its momentum. Thus definitely there is a relation between position and momentum. We go for that.

Consider a small wave packet localised to a small region of space Δx. (see figure below). Let us try to measure the wavelength of this wave packet by placing a metre scale along the wave. Let the uncertainty (error) in the measurement be $\Delta\lambda$. This uncertainty will be a small fraction of λ. So we can write

$$\Delta\lambda = \varepsilon\lambda$$

ε is a very small quantity very much less than 1.

The size of the wave packet is roughly one wavelength. So $\Delta x \approx \lambda$

$$\therefore \quad \Delta x \Delta\lambda \approx \varepsilon\lambda^2 \qquad \text{..... (8)}$$

For a given wavelength, we can write

$$\Delta x \propto \frac{1}{\Delta\lambda}$$

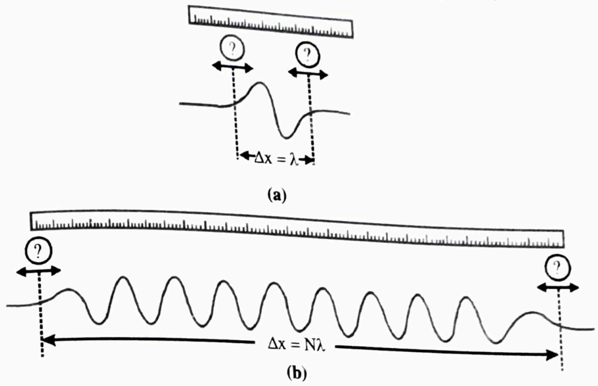


Figure 3.6
(a) Measuring the wavelength of a wave represented by a small wave packet of length roughly one wavelength.
(b) Measuring the wavelength of a wave represented by a large wave packet consisting of N waves.

It shows that when the size of the wave packet (Δx) is small, $\Delta\lambda$ the uncertainty in the measurement of wavelength is large.

Suppose our wave packet consists of N waves, then

$$\Delta(N\lambda) = \varepsilon\lambda$$

or

$$\Delta\lambda = \frac{\varepsilon\lambda}{N}$$

But

$$\Delta x = N\lambda$$

Then

$$\Delta\lambda\, \Delta x = \varepsilon\lambda^2$$

This is exactly the same as in the case of the smaller wave packet. This shows that equation 8 is the fundamental property of classical waves. It is the uncertainty relationships for classical waves.

Example 8

In a measurement of water waves 10 wave cycles are counted in a distance of 196 cm estimate the minimum uncertainty in the wavelength that might be obtained from this experiment. Take $\varepsilon = 0.1$

Solution

Wavelength of the wave $= \frac{196}{10} = 19.6\,\text{cm}$

$$\Delta x = 196\,\text{cm}, \quad \varepsilon = 0.1$$

Using $$\Delta x \Delta\lambda = \varepsilon\lambda^2$$

$$\Delta\lambda = \frac{\varepsilon\lambda^2}{\Delta x} = 0.1 \times \frac{(19.6)^2}{196} = 0.2\,\text{cm}$$

So the wavelength ranges from 19.5 to 19.7 cm

$\therefore$ $$\lambda = 19.6 \pm 0.1\,\text{cm}$$

The frequency-time uncertainty relationship

The classical uncertainty relation can also be put in another form. This time we measure the wavelength of the wave packet in terms of its time period. Instead of a metre scale earlier we use two clocks to measure the time period of wavelength. Now the size of the wave packet (see figure below) is its time duration, so that $\Delta t \approx T$. The uncertainty in the measurement time period T is

$$\Delta T = \varepsilon T$$

Then $$\Delta t\,\Delta T = \varepsilon T^2 \qquad \text{..... (9)}$$

This is the second uncertainty relation for classical waves.

Equation 9 can also be written in terms of frequency

We have $\nu = \frac{1}{T}$

$$\Delta\nu = -\frac{1}{T^2}\Delta T \text{ or } \Delta T = T^2\Delta\nu$$

putting this in equation a by ignoring the negative sign, we get

$$\Delta t T^2 \Delta\nu = \varepsilon T^2$$

or $$\Delta t\,\Delta\nu = \varepsilon \qquad \text{..... (10)}$$

It shows that longer the duration (Δt) of wave packet, the more precisely we can measure its frequency.

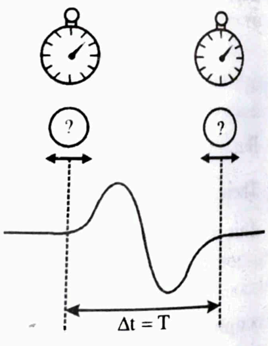


Figure 3.7
Measuring the period of a wave represented by a small wave packet of duration roughly one period.

Example 9

Sound waves travel through air at a speed of 330 ms^{-1}. A whistle blast at a frequency of about 1.0 kHz lasts for 0.2 s.

a) Over what distance in space does the wave train representing the sound extend.
b) What is the wavelength of the sound.
c) Estimate the precision with which an observer could measure the wavelength.
d) Estimate the precision with which an observer could measure the frequency.

Solution

$$v = 330\,\text{ms}^{-1}, \quad \nu = 1.0\,\text{kHz}, \quad \Delta t = 2\text{s}$$

a) $$\Delta x = v\Delta t$$
$$\Delta x = 330 \times 2 = 660\,\text{m}$$

b) $$\lambda = \frac{v}{\nu} = \frac{330}{10^3} = 0.33\,\text{m}$$

c) Using $$\Delta x\,\Delta\lambda = \varepsilon\lambda^2$$
$$\Delta\lambda = \frac{\varepsilon\lambda^2}{\Delta x} = \frac{0.1 \times (0.33)^2}{660} = 0.017\,\text{mm}$$

d) Using $$\Delta\nu\,\Delta t \simeq \varepsilon$$
$$\Delta\nu \simeq \frac{\varepsilon}{\Delta t} = \frac{0.1}{2} = 0.05\,\text{Hz}$$

Example 10

A stone is dropped into water creates a disturbance that lasts for 4s at the point of impact. The wave speed is 25 cms^{-1}. (a) Over what distance on the surface of water does the group of waves extend. (b) An observer counts 12 wave crests in the group. Estimate the precision with which the wavelength can be determined.

Solution

$$v = 25\,\text{cms}^{-1}, \; \Delta t = 4.0\,\text{s}$$

a) $$\Delta x = v\Delta t = 25 \times 10^{-2} \times 4 = 1\text{m}$$

b) $$\lambda = \frac{1\text{m}}{12} = 0.083\,\text{m}$$

Using $$\Delta x\,\Delta\lambda = \varepsilon\lambda^2$$

$$\Delta\lambda = \frac{\varepsilon\lambda^2}{\Delta x}$$

$$\Delta\lambda = 0.1 \times \frac{(0.083)^2}{1} = 0.00069\,\text{m}$$

$$\Delta\lambda = 0.069\,\text{cm}$$

Heisenberg's Uncertainty relationships

The uncertainty relationships of classical waves is a general one. It can be applied to all waves. Now we apply them to de Broglie waves.

We have $\Delta x\,\Delta\lambda = \varepsilon\lambda^2$

According to de Broglie $p = \frac{h}{\lambda}$

Taking differentials on both sides, we get

$$dp = -\frac{h}{\lambda^2}d\lambda$$

$$\therefore \quad \Delta\lambda = \frac{\lambda^2 \Delta p}{h}$$

Put this in the uncertainty relation, we get

$$\Delta x \frac{\lambda^2 \Delta p}{h} = \varepsilon\lambda^2$$

or

$$\Delta x\,\Delta p = \varepsilon h \qquad \text{..... (11)}$$

Quantum mechanics provides a formal procedure for calculating Δx and Δp for wave packets corresponding to different physical situations. The smallest value of the product turns out to be $\frac{h}{4\pi}$ for a particular wave packet. For all other wave packets this value is larger. Thus we can write

$$\Delta x\,\Delta p = \frac{h}{4\pi} \qquad \text{..... (12)}$$

$\frac{h}{2\pi}$ is denoted by $\hbar$ (h-bar or h cross) called Dirac constant. Now the uncertainty relationship turns out to be

$$\Delta x\,\Delta p_x \geq \frac{\hbar}{2} \qquad \text{..... (13)}$$

Adding a subscript to p indicating that uncertainties measured along x-direction similarly we can write

$$\Delta y\,\Delta p_y \geq \frac{\hbar}{2} \qquad \text{..... (14)}$$

and

$$\Delta z\,\Delta p_z \geq \frac{\hbar}{2} \qquad \text{..... (15)}$$

Equations 13, 14 and 15 are called Heisenbergs uncertainty relationships.

Note

(i) Comparing equations 11 and 12 we get $\varepsilon = \frac{1}{4\pi}$

(ii) For rough estimates we can use $\Delta x\,\Delta p_x \sim \hbar$ similarly others.

Proof of Heisenberg's uncertainty based on hypothetical experiment

Consider a stream of monoenergetic microparticles is moving along the x-axis as shown in figure below. The momentum of all the particles in the stream are well defined. Each particle carries a momentum p_x in x-direction and zero in y-direction. Therefore the uncertainty in y-component of the momentum $\Delta p_y = 0$. However, the location of the particle in y-direction is absolutely uncertain. That is $\Delta y = \infty$. To find the location of the particle, let a slit of width a be placed at x = 0, in its path. As the particle emerges from the slit at x = 0 it must have been localised in the y-direction within an uncertainty $\Delta y = a$ when the slit is made very narrower then a

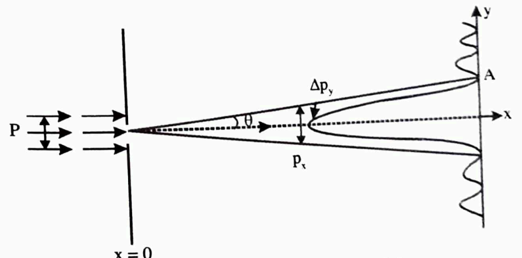


Figure 3.8: Uncertainty principle

also becomes very smaller and the uncertainty in the y-position of the particle may be reduced. However a reduction in the width of the slit, causes diffraction of particle-waves. The width of the diffraction pattern obtained on the screen is defined by the angular separation 2θ between the two minima on either side of $x = 0$ on the screen. It is given by

$$\sin\theta = \frac{\lambda}{a} = \frac{\lambda}{\Delta y}$$

or

$$\theta = \frac{\lambda}{\Delta y} \qquad \text{..... (17)}$$

Because of the diffraction effect at the slit, the particle acquires a small component in the momentum Δp_y in the y-direction. The original momentum in the x-direction p_x decreases so that the resultant momentum p_x remains constant. The original momentum of the particle in the y-direction was accurately to be zero. Therefore Δp_y is the uncertainty in the y-component of the momentum.

From the figure

$$\tan\theta = \frac{\Delta p_y}{p_x}$$

$$\theta \simeq \frac{\Delta p_y}{p_x} \qquad \text{..... (18)}$$

comparing the equations, 17 and 18 we get

$$\frac{\Delta p_y}{p_x} = \frac{\lambda}{\Delta y}$$

or

$$\Delta y\,\Delta p_y = \lambda p_x$$

But $\lambda = \dfrac{h}{p_x}$ we get

$$\Delta y\,\Delta p_y = h$$

This is roughly the uncertainty principle.

The second uncertainty relation

We have

$$\Delta E\,\Delta t = \varepsilon h$$

we found that the minimum uncertainty wave packet gives $\varepsilon = \dfrac{1}{4\pi}$

$$\Delta E\,\Delta t = \frac{h}{4\pi}$$

$\therefore$

$$\Delta E\,\Delta t = \frac{\hbar}{2} \qquad \text{..... (19)}$$

or

This is the second Heisenberg's uncertainty principle.

This tells us that the more precisely we try to determine the time coordinate of the particle, the less precisely we know its energy. For example if a particle has a very short life time between its creation and decay $(\Delta t \to 0)$ its rest energy will be quite uncertain $(\Delta E \to \infty)$. Conversely the rest energy of a stable particle $(\Delta t = \infty)$ can in principle be measured with good precision $(\Delta E = 0)$.

Note: For rough estimates we can use $\Delta t\,\Delta E \sim \hbar$

Heisenberg's uncertainty principle - I

Statement

It is impossible to determine, precisely and simultaneously, the position and momentum of a microscopic particle.

Mathematical statement

In any simultaneous determination of the position and momentum of a microscopic particle, the product of the corresponding uncertainties is equal to or greater than $\frac{\hbar}{2}$.

Heisenberg's uncertainty principle - II

Statement

It is impossible to determine, precisely and simultaneously, the energy and the time coordinate of a microscopic particle.

Mathematical statement

In any simultaneous determination of the energy and time coordinate of a microscopic particle, the product of the corresponding uncertainties is equal to or greater than $\frac{\hbar}{2}$.

Example 11

The speed of an electron is measured to within an uncertainty of $2.0 \times 10^4\ \text{ms}^{-1}$ what is the size of smallest region of space in which electron can be confined.

Solution

$$\Delta v = 2.0\times10^{4}\,\text{ms}^{-1}$$

Using
$$\Delta x\,\Delta p = \hbar$$

$$\Delta x = \frac{\hbar}{\Delta p} = \frac{\hbar}{m\Delta v} = \frac{1.05\times10^{-34}}{9.11\times10^{-31}\times2\times10^{4}}$$

$$\Delta x \approx 5.8\text{nm}$$

Example 12

An electron is confined to a region of space of the size of an atom (0.1 nm).

a) what is the uncertainty in the momentum of the electron.

b) what is the kinetic energy of an electron with a momentum equal to Δp.

Solution

$$\Delta x = 0.1\text{nm}$$

a) Using
$$\Delta x\,\Delta p \approx \hbar$$

$$\Delta p \approx \frac{\hbar}{\Delta x} = \frac{1.05\times10^{-34}}{0.1\times10^{-9}}$$

$$\Delta p \approx 1.05\times10^{-24}\,\text{kg ms}^{-1}$$

b) $$K = \frac{(\Delta p)^2}{2m} = \frac{(1.05\times10^{-24})^2}{2\times9.11\times10^{-31}} = \frac{1.05^2}{18.22}\times10^{-17}\,\text{J}$$

$$K = 6.05\times10^{-19}\,\text{J}$$

or
$$K = \frac{6.05\times10^{-19}}{1.6\times10^{-19}}\,\text{eV}$$

$$K = 3.78\,\text{eV}$$

Example 13

Calculate the minimum uncertainty in the position of an electron weighing 9.1×10^{-31} kg and moving with an uncertainty in speed of $3\times10^{7}\,\text{ms}^{-1}$.

Solution

$$m_0 = 9.1\times10^{-31}\,\text{kg}$$

$$v = 3\times10^{7}\,\text{ms}^{-1}$$

Momentum of the electron $p = mv$

$$p = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$p = \frac{9.1\times10^{-31}\times3\times10^{7}}{\sqrt{1-\frac{1}{100}}} \qquad \left[\because\ \frac{v^2}{c^2} = \frac{(3\times10^{7})^2}{(3\times10^{8})^2}\right]$$

$$p = 27.46\times10^{-24}\,\text{kgms}^{-1}$$

The uncertainty in momentum can at the most be equal to this value of the momentum itself

i.e. $(\Delta p)_{max} = 27.46\times10^{-24}\,\text{kgms}^{-1}$

We have
$$\Delta x\,\Delta p \geq \frac{\hbar}{2}$$

$$\therefore \quad (\Delta x)_{min}(\Delta p)_{max} \geq \frac{\hbar}{2}$$

$$(\Delta x)_{min} \geq \frac{\hbar}{2(\Delta p)_{max}}$$

$$\geq \frac{1.05\times10^{-34}}{2\times27.46\times10^{-24}}$$

$$(\Delta x)_{min} \geq 0.0192\text{Å}.$$

Example 14

Estimate the size of the hydrogen atom and the ground state energy from the uncertainty principle.

Solution

Assume that the spread (uncertainty) in position of the electron in a hydrogen atom is $\Delta x \equiv a$

From the uncertainty principle, we have

$$\Delta x\,\Delta p \geq \hbar$$

$$\Delta p \geq \frac{\hbar}{\Delta x} = \frac{\hbar}{a}$$

If T denotes the K.E of the electron and ΔT, the uncertainty in the K.E, we get

$$T = \frac{p^2}{2m}$$

$$\Delta T = \frac{\Delta p^2}{2m} = \frac{\hbar^2}{2ma^2}$$

Similarly if V denote the P.E. of the electron, we may write the uncertainty in the P.E as

$$\Delta V = -\frac{e^2}{4\pi\varepsilon_o a}$$

$$\therefore \quad \Delta E = \Delta T + \Delta V$$

where ΔE is the uncertainty in total energy.

$$\therefore \quad \Delta E = \frac{\hbar^2}{2ma^2} - \frac{e^2}{4\pi\varepsilon_o a}$$

The ground state of the atom will correspond to the minimum value of E to which $\frac{d}{da}\Delta E = 0$

$$\therefore \quad \frac{d}{da}\Delta E = -\frac{\hbar^2}{ma^3} + \frac{e^2}{4\pi\varepsilon_o a^2}$$

when $\frac{d}{da}\Delta E = 0,\ a = a_0$.

$$0 = -\frac{\hbar^2}{ma_0^3} + \frac{e^2}{4\pi\varepsilon_o a_0^2}$$

or

$$\frac{\hbar^2}{ma_0^3} = \frac{e^2}{4\pi\varepsilon_o a_0^2}$$

$$\frac{\hbar^2}{ma_0^3} = \frac{e^2}{4\pi\varepsilon_o}$$

$$\therefore \quad a_0 = \frac{4\pi\varepsilon_o\hbar^2}{e^2 m} = \frac{(1.054\times10^{-3/4})^2}{9\times10^9(1.6\times10^{-19})^2\times9.1\times10^{-31}}$$

a_0 corresponds to the minimum value of a_0 which gives the Bohr radius.

We have $\Delta E = \frac{\hbar^2}{2ma^2} - \frac{e^2}{4\pi\varepsilon_0 a}$

When ΔE is minimum $a = a_0$

$$(\Delta E)_{min} = \frac{\hbar^2}{2ma_0^2} - \frac{e^2}{4\pi\varepsilon_0 a_0}$$

$$(\Delta E)_{min} = \frac{\hbar^2 m^2 e^4}{2m(4\pi\varepsilon_0)^2\hbar^4} - \frac{e^2 me^2}{(4\pi\varepsilon_0)^2\hbar^2}$$

$$(\Delta E)_{min} = \frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2} - \frac{me^4}{(4\pi\varepsilon_0)^2\hbar^2}$$

$$(\Delta E)_{min} = -\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}$$

$$(\Delta E)_{min} = -\frac{9.1\times10^{-31}\times(1.6\times10^{-19})^4}{2\times(1.054\times10^{-34})^2}\times(9\times10^9)^2$$

$$= -2.174\times10^{-18}\,J$$

or

$$(\Delta E)_{min} = \frac{2.174\times10^{-18}}{1.6\times10^{-19}} = -13.6eV$$

Which is indeed the ground state energy.

Example 15

Using the uncertainty principle estimate the ground state energy of the harmonic oscillator.

Solution

For a linear harmonic oscillator

$$E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$$

$$E \approx \Delta E$$

Assume. $p \approx \Delta p,\ x \approx \Delta x = a$

$$\therefore \quad \Delta E = \frac{\Delta p^2}{2m} + \frac{1}{2}m\omega^2\Delta x^2$$

$$\Delta E = \frac{\Delta p^2}{2m} + \frac{1}{2}m\omega^2 a^2$$

Using $\Delta x\Delta p \geq \hbar$, $\Delta p \geq \frac{\hbar}{\Delta x} = \frac{\hbar}{a}$

$$\therefore \quad \Delta E \simeq \frac{\hbar^2}{2ma^2} + \frac{1}{2}m\omega^2 a^2 \qquad \text{..... (1)}$$

Minimising the energy

$$\frac{d}{da}\Delta E = -\frac{\hbar^2}{ma^3} + m\omega^2 a$$

when $\frac{d}{da}(\Delta E) = 0 \quad a = a_0$

$$0 = -\frac{\hbar^2}{ma_0^3} + m\omega^2 a_0$$

$$\frac{\hbar^2}{ma_0^3} = m\omega^2 a_0$$

$$a_0^4 = \frac{\hbar^2}{m^2\omega^2}$$

or
$$a_0^2 = \frac{\hbar}{m\omega}$$

Substituting this in equation (1), we get

$$(\Delta E)_{min} \simeq +\frac{\hbar^2}{2m\left(\frac{\hbar}{m\omega}\right)} + \frac{m\omega^2\hbar}{2m\omega}$$

$$(\Delta E)_{min} = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{2} = \hbar\omega$$

$$(\Delta E)_{min} \simeq \hbar\omega$$

Example 16

Using uncertainty principle show that electrons cannot exist inside a nucleus.

Solution

We know that the size of a nucleus is of the order of 10^{-14}m. If electron is present inside a nucleus, the uncertainty in its position is also of the order of $10^{-14}\,m(\Delta x)$

From the uncertainty principle

$$\Delta x\Delta p \geq \hbar$$

where Δp is the uncertainty in its momentum.

$$\therefore \quad \Delta p \geq \frac{\hbar}{2\Delta x} = \frac{1.054\times10^{-34}}{10^{-14}}$$

$$= 1.054\times10^{-20}$$

i.e.
$$\Delta p \geq 1.054\times10^{-20}\,\text{kgms}^{-1}$$

If this is the uncertainty in a nuclear electrons momentum, the momentum p itself must be at least comparable in magnitude

i.e.
$$p \simeq 1.054\times10^{-20}\,\text{kgms}^{-1}$$

K.E of the electron, $K.E = mc^2 - m_0c^2$

or
$$K.E = E - m_0c^2$$

$$K.E = \sqrt{p^2c^2 + m_0^2c^4} - m_0c^2$$

$$pc = 1.054\times10^{-20}\times3\times10^8 = 3.162\times10^{-12}\,J$$

$$m_0c^2 = 9.1\times10^{-31}\times9\times10^{16} = 8.19\times10^{-15}\,J$$

Since $pc \gg m_0c^2$, K.E can be approximated to

$$K.E \approx pc = 3.162\times10^{-12}\,J$$

or $$K.E = \frac{3.162\times10^{-12}}{1.6\times10^{-19}}\,eV = 19.76\times10^6\,eV$$

or $$K.E \approx 20\,MeV$$

Thus if the electron is to be present inside a nucleus, it must have a kinetic energy of the order of 20 MeV. However experiments indicate that an electron associated even with unstable atoms never have more than a fraction of this energy. Thus we can conclude that electrons cannot exist in a nucleus.

Example 17

Using uncertainty principle show that electrons can exist in an atom.

Solution

Taking hydrogen as an example we can prove this. The size of hydrogen atom is $0.5\times10^{-10}\,m$. Hence the uncertainty in locating of the position of an electron in the considered atom is of the order of $0.5\times10^{-10}\,m$.

i.e. $$\Delta x = 0.5\times10^{-10}\,m$$

Using $$\Delta x\Delta p \geq \frac{\hbar}{2}$$

$$\therefore \quad \Delta p \geq \frac{\hbar}{2\Delta x} = \frac{1.054\times10^{-34}}{2\times0.5\times10^{-10}} = 1.054\times10^{-24}\,kg\,ms^{-1}.$$

Thus electrons momentum (p) is of the order of $1.054\times10^{-34}\,kg\,ms^{-1}$.

$$\therefore \quad \text{K.E of the electrum} = \frac{p^2}{2m} = \frac{(1.054\times10^{-24})^2}{2\times9.1\times10^{-31}}$$

$$K.E = 6.1\times10^{-19}\,J$$

or $$K.E = \frac{6.1\times10^{-19}}{1.6\times10^{-19}}\,eV = 3.81\,eV$$

in this kinetic energy an electron can certainly be confined within an atom. Remember that for hydrogen atom $E_n = \frac{13.6}{n^2}\,eV$

Example 18

The mean life time of the eta meson is known to be 7×10^{-19} s. If the rest mass of this particle is 549 MeV, what is the uncertainty in its rest mass?

Solution

Let m_0 be the rest mass of the eta meson. We are given that

$$m_0c^2 = 549\,MeV = 549\times1.6\times10^{-13}\,J$$

$$\therefore \quad m_0 = \frac{549\times1.6\times10^{-13}}{(3\times10^8)^2}\,kg = 97.6\times10^{-29}\,kg = 9.76\times10^{-28}\,kg$$

By uncertainty principle, we have

$$\Delta E\Delta t \geq \frac{\hbar}{2}$$

$$\therefore \quad \Delta E \geq \frac{\hbar}{2(7\times10^{-19})}\,\text{joule} = \frac{1.054\times10^{-34}}{14\times10^{-19}}\,J = 7.53\times10^{-18}\,\text{joule}$$

We can take $\Delta E \sim c^2(\Delta m_0)$

$$\therefore \quad (\Delta m_0) \sim \frac{\Delta E}{c^2} = \frac{7.53\times10^{-18}}{9\times10^{16}}\,kg = 8.37\times10^{-35}\,kg$$

Expressed in eV, this is equivalent to

$$\frac{7.53\times10^{-18}}{1.6\times10^{-19}} = 47\,eV$$

Thus the uncertainty in the rest mass of the particle is of the order of 47 eV.

Example 19

It is known that an atom can remain in its excited state for a time of the order of 10^{-8} s. If the atom de-excites itself by emitting its excess energy in the form of electromagnetic radiation, find the uncertainty associated with this emitted energy. Hence estimate the minimum 'spread' in the frequency of the emitted electromagnetic radiation.

Solution

We know that, $\Delta E \Delta t \geq \frac{\hbar}{2}$

$$\therefore \quad \Delta E \geq \frac{\hbar}{2\Delta t} = \frac{1.054\times10^{-34}}{2\times10^{-8}} \text{ joule} = 0.527\times10^{-26} \text{ joule}$$

$$= \frac{0.527\times10^{-26}}{1.6\times10^{-19}} \text{ eV} \cong 3.3\times10^{-8} \text{eV}$$

Thus the minimum uncertainty in the emitted energy is 3.3×10^{-8}eV.

Now $\quad E = h\nu \quad$ so $\quad \Delta E = h\Delta\nu$

$$\therefore \quad \Delta\nu = \frac{\Delta E}{h} = \frac{0.527\times10^{-26}\,\text{J}}{6.6\times10^{-34}\,\text{Js}}$$

$$\cong 0.08\times10^{8}\,\text{s}^{-1} = 8\,\text{MHz}$$

Thus the minimum 'spread' in the frequency of the emitted radiation is (nearly) 8 MHz

A statistical interpretation of uncertainty

Whenever we have some quantity which can assume several values with various probabilities, it is customary to use the root mean square deviation. This is the uncertainty in the value of that quantity. This is equal to the measure of the width of the probability distribution.

Let momentum p be our quantity to be measured. Let $p_1, p_2, \ldots, p_n$ be the individual measurements. Then its mean value denoted by $\langle p \rangle$ is

$$\langle p \rangle = \frac{p_1 + p_2 + p_3 + \ldots, p_n}{n}$$

Each value of momentum deviates from its average value. Thus in general deviation d can be written as

$$d = p - \langle p \rangle$$

$$\therefore \quad d^2 = (p - \langle p \rangle)^2$$

$$\langle d^2 \rangle = \langle (p - \langle p \rangle)^2 \rangle$$

$\langle d^2 \rangle$ is called mean square deviation.

$$\langle d^2 \rangle = \left\langle (p^2 - 2p\langle p \rangle + \langle p \rangle^2) \right\rangle$$

$$\langle d^2 \rangle = \langle p^2 \rangle - 2\langle p \rangle \langle p \rangle + \langle p \rangle^2$$

$$\langle d^2 \rangle = \langle p^2 \rangle - 2\langle p \rangle^2 + \langle p \rangle^2$$

$$\langle d^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2$$

By definition root of the mean square deviation is called uncertainty. Here the uncertainty in momentum denoted by Δp.

$$\therefore \quad \Delta p = \sqrt{\langle d^2 \rangle}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \qquad \ldots.. (20)$$

Similarly we can write

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2}$$

Now we want to show that uncertainty is the width of the distribution.

Let us imagine that we do an experiment in which a large number of particles passes through a slit (diffraction experimental set up, see fig. 3.8) and we measure the y-component momentum of each particle after it passes through the slit. We can do this experiment simply by placing a detector at different locations on the screen where we observe diffraction pattern. The detector actually accepts particles over finite region on the screen, it measures in a range of y-component momentum. The result of the experiment is shown in figure below. The vertical scale shows the number of particles with momentum in each interval corresponding to different locations of the detector on the screen. The values are symmetrically arranged about zero, which indicates that the mean value of p_y is zero. The width of the distribution is characterised by Δp_y.

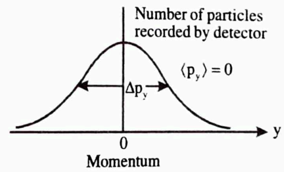


Figure 3.9 : Results that might be obtained from measuring the number of electrons in a given time interval at different locations on the screens.

Now we recall the definition of root mean square derivation (uncertainty).

We have $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$

or $\Delta p_y = \sqrt{\langle p_y^2 \rangle - \langle p_y \rangle^2}$

But $\langle p_y \rangle = 0$

$\therefore$ $\Delta p_y = \sqrt{\langle p_y^2 \rangle}$

This is usually taken to be the measure of the magnitude of p_y. From the graph it is seen that this is indeed true.

Wave packets

A wave packet is a short burst or envelope of localised wave action that travels as a unit. The amplitude of wave packet is large over a finite region of space or time and is very small outside the region.

A wave packet is formed by the superposition of two or more different waves of slightly different frequencies. As the number of waves increases the wave packet becomes more localised in space. A wave packet does not change its shape as it moves.

Phase velocity and group velocity

Phase velocity of the de Broglie waves

The velocity with which a definite phase of each individual monochromatic wave travels is called phase velocity or wave velocity.

The phase velocity cannot be measured experimentally. It is not possible to distinguish between the successive waves for a monochromatic wave since they are all exactly similar.

Once we associate a de Broglie wave with any matter we can calculate the velocity (v_p) of the wave using the relation

$$v_p = \nu\lambda$$

$$v_p = h\nu\frac{\lambda}{h} = \frac{E}{h/\lambda}$$

Using $E = mc^2$ and $\frac{h}{\lambda} = p = mv$ we get

$$v_p = \frac{mc^2}{mv} = \frac{c^2}{v} \quad \text{..... (21)}$$

where v is the velocity of the particle. According to relativity $v < c$.

$\therefore$ $v_p > c$

This shows that the velocity of the de Broglie wave is greater than the velocity of light.

i.e. de Broglie waves travel faster than light. This seems to violate special theory of relativity. To understand this we must distinguish between phase velocity (wave velocity) and group velocity.

Phase velocity is the velocity of propagation of a harmonic wave $e^{i(kx-\omega t)}$ or it is the rate at which surfaces of constant phase $(kx - \omega t = \text{constant})$ move. It is therefore called phase velocity.

We have $kx - \omega t = \text{constant}$ ..... (22)

Differentiating on both sides with respect to time t, we get

$$k\frac{dx}{dt} - \omega = 0$$

or

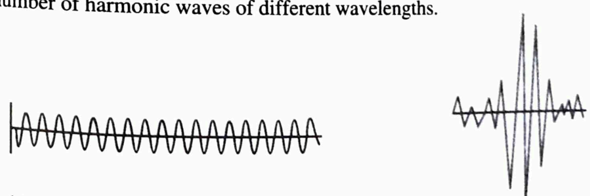
$$\frac{dx}{dt} = \frac{\omega}{k}$$

$\frac{dx}{dt}$ is the velocity of the wave called phase velocity v_p.

$$\therefore \quad v_p = \frac{\omega}{k} \quad \text{..... (23)}$$

Group velocity

We have seen that a monochromatic wave is not suitable to represent a matter wave. But a wave packet will do this. A wave packet is the superposition of large number of harmonic waves of different wavelengths.



(a) Single monochromatic wave train

(b) A localized wave packet formed through superposition of different waves

Figure 3.10

The velocity of a wave packet formed by a group of harmonic waves is called group velocty (v_g). Its value is

$$v_g = \frac{d\omega}{dk} \quad \text{..... (24)}$$

To see how to arrive at eqn 24, let us consider the simple case of superposition of two waves. Let the angular frequency (ω) and propagation constants (k) of the two waves differ by small amounts $\Delta\omega$ and Δk respectively. We may write the wave equations of the two waves as

$$y_1 = A\sin(\omega t - kx)$$

and

$$y_2 = A\sin[(\omega + \Delta\omega)t - (k + \Delta k)x]$$

According to the superposition principle the resultant wave

$$y = y_1 + y_2$$

$$y = A\{\sin(\omega t - kx) + \sin[(\omega + \Delta\omega)t - (k + \Delta k)x]\}$$

$$y = 2A\sin\left[\frac{(2\omega + \Delta\omega)t}{2} - \frac{(2k + \Delta k)}{2}x\right] \times \cos\frac{(\Delta\omega t - \Delta k \cdot x)}{2}$$

$$\left[\because \ \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$$y \simeq 2A\sin(\omega t - kx)\cos\left(\frac{\Delta\omega\, t - \Delta k\, x}{2}\right) \quad \text{..... (25)}$$

we used $2\omega + \Delta\omega \simeq 2\omega$

and $2k + \Delta k \approx 2k$

since $\Delta\omega$ and Δk are small compared with ω and k respectively.

The resultant wave (eqn 14) is thus a wave, whose angular frequency ω and propagation constant k, modulated by modulation frequency $\frac{\Delta\omega}{2}$ and propagation constant $\frac{\Delta k}{2}$. Thus the phase velocity of the wave is $\frac{\omega}{k}$ while the group velocity is

$$v_g = \lim_{\Delta k \to 0}\frac{(\Delta\omega/2)}{(\Delta k/2)} = \lim_{\Delta k \to 0}\frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

When ω and k have continuous spreads instead of the two values in the preceeding discussion, the group velocity is given by

$$v_g = \frac{d\omega}{dk} \quad \text{..... (26)}$$

Relation between Group velocity and Phase velocity

Consider a non-relativistic particle of mass m and having v_p and v_g as the phase velocity and group velocity respectively.

If λ is the de Broglie wavelength and υ the frequency of the wave, then

Phase velocity, $$v_p = \frac{\omega}{k} \quad \text{..... (1)}$$

Group velocity, $$v_g = \frac{d\omega}{dk} \quad \text{..... (2)}$$

Where ω is the angular velocity of the wave and k is the wave vector.

From eq(1), we have $\omega = v_p k$

$$\therefore \quad v_g = \frac{d}{dk}(v_p k) = v_p + k\frac{dv_p}{dk}$$

$$v_g = v_p + k\frac{dv_p}{d\lambda}\cdot\frac{d\lambda}{dk} \quad \text{..... (3)}$$

we have $\lambda = \frac{2\pi}{k}$

$$\therefore \quad \frac{d\lambda}{dk} = -\frac{2\pi}{k^2}$$

Put this in eq (3), we get

$$v_g = v_p + k\frac{dv_p}{d\lambda}\left(-\frac{2\pi}{k^2}\right)$$

$$v_g = v_p - \frac{2\pi}{k}\frac{dv_p}{d\lambda}$$

or $$v_g = v_p - \lambda\frac{dv_p}{d\lambda}$$

This equation shows that v_g is less than v_p when the medium is dispersive, i.e., when v_p is a function of λ. For light wave in vacuum, there is no dispersion, i.e., v is independent of λ.

Hence $\frac{dv_p}{d\lambda} = 0$

So $v_p = v_g = c$

This is true for elastic waves in a homogeneous medium.

Group velocity for de Broglie waves

Consider a de Broglie wave (wave packet) associated with a particle of rest mass m_o travelling with a speed v.

We have $\omega = 2\pi\upsilon = \frac{2\pi h\upsilon}{h} = \frac{2\pi mc^2}{h}$

or $$\omega = \frac{2\pi c^2}{h}\frac{m_o}{\sqrt{1-v^2/c^2}} \qquad \text{..... (27)}$$

and $$k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h} = \frac{2\pi}{h}mv$$

or $$k = \frac{2\pi v}{h} \times \frac{m_o}{\sqrt{1-v^2/c^2}} \qquad \text{..... (28)}$$

By definition $v_g = \frac{d\omega}{dk}$

or $$v_g = \frac{d\omega/dv}{dk/dv} \qquad \text{..... (29)}$$

From eqn 27, we get

$$\frac{d\omega}{dv} = \frac{d}{dv}\left[\frac{2\pi m_o c^2}{h}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right] = \frac{2\pi m_o c^2}{h}\left[-\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(-\frac{2v}{c^2}\right)\right]$$

$$= \frac{2\pi m_o c^2}{h}\frac{v}{c^2}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} = \frac{2\pi m_o v}{h}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

From eqn 28, we get

$$\frac{dk}{dv} = \frac{d}{dv}\left[\frac{2\pi m_o v}{h}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right]$$

$$= \frac{2\pi m_o}{h}\times\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} + \frac{2\pi m_o v}{h}\left\{-\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(-\frac{2v}{c^2}\right)\right\}$$

$$= \frac{2\pi m_o}{h}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} + \frac{2\pi m_o}{h}\frac{v^2}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}$$

$$= \frac{2\pi m_o}{h}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left[\left(1-\frac{v^2}{c^2}\right)+\frac{v^2}{c^2}\right] = \frac{2\pi m_o}{h}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

Hence $v_g = \frac{d\omega/dv}{dk/dv}$

$$= \frac{2\pi m_o v}{h}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} \times \frac{h}{2\pi}\frac{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}{m_o} = v$$

Thus we find that de Broglie wave packet associated with a moving body travels with the same velocity as the moving particle.

Example 20

Show that the group velocity and the particle velocity are the same in the case of nonrelativistic as well as relativistic particles.

Solution

We have $v_g = \frac{d\omega}{dk} = \frac{d\hbar\omega}{d\hbar k} = \frac{dE}{dp}$

For a nonrelativistic particle

$$E = \frac{p^2}{2m}$$

$$\frac{dE}{dp} = \frac{2p}{2m} = \frac{p}{m} = \frac{mv}{m} = v$$

Hence $v_g = v$.

For a relativistic particle

$$E^2 = p^2c^2 + m_o^2c^4$$

$$2E\frac{dE}{dp} = 2pc^2$$

$$\therefore \quad \frac{dE}{dp} = \frac{pc^2}{E}$$

$$p = mv = \frac{m_o v}{\sqrt{1-\frac{v^2}{c^2}}}, \quad E = mc^2 = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\therefore \quad \frac{dE}{dp} = v$$

Hence $\quad v_g = v.$

Example 21

The phase velocity of the gravity waves of wavelength λ is determined from $v_p = \sqrt{\frac{g\lambda}{2\pi}}$. Find the group velocity of the wave.

Solution

$$v_p = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{g}{k}}$$

or $$\frac{\omega}{k} = \sqrt{\frac{g}{k}}$$

or $$\omega = \sqrt{gk}$$

$$\frac{d\omega}{dk} = \sqrt{g}\cdot\frac{1}{2\sqrt{k}} = \frac{1}{2}\sqrt{\frac{g}{k}}$$

$$v_g = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{v_p}{2}$$

Example 22

Show that the group velocity of a wave is given by $v_g = \frac{d\upsilon}{d\left(\frac{1}{\lambda}\right)}$.

Solution

We have $\quad v_g = \frac{d\omega}{dk}$

Put $\omega = 2\pi\upsilon$ and $k = \frac{2\pi}{\lambda}$

$$v_g = \frac{d(2\pi\upsilon)}{d\left(\frac{2\pi}{\lambda}\right)} = \frac{d\upsilon}{d\left(\frac{1}{\lambda}\right)}$$

Example 23

The phase velocity of ripples on a liquid surface is $\sqrt{\frac{2\pi S}{\lambda\rho}}$, where S is the surface tension and ρ the density of the liquid. Find the group velocity of the ripples.

Solution

$$v_p = \sqrt{\frac{2\pi}{\lambda}\frac{S}{\rho}} = \sqrt{\frac{kS}{\rho}}$$

$$\frac{\omega}{k} = \sqrt{\frac{kS}{\rho}} \qquad \left(\because \; v_p = \frac{\omega}{k}\right)$$

or

$$\omega = \sqrt{k^3 \frac{S}{\rho}} = \sqrt{\frac{S}{\rho}} k^{\frac{3}{2}}$$

$$\frac{d\omega}{dk} = \sqrt{\frac{S}{\rho}} \cdot \frac{3}{2} k^{\frac{1}{2}}$$

$$\frac{d\omega}{dk} = \frac{3}{2}\sqrt{\frac{Sk}{\rho}} = \frac{3}{2} v_p$$

Example 24

Show that the phase velocity of the de Broglie waves of a particle of mass m and de Broglie wavelength λ is given by

$$v_p = c\sqrt{1 + \left(\frac{mc\lambda}{h}\right)^2}$$

Solution

From relativity we have

$$E = \sqrt{p^2c^2 + m^2c^4}$$

$$E = cp\sqrt{\left(1 + \frac{m^2c^2}{p^2}\right)}$$

or

$$\frac{E}{p} = c\sqrt{1 + \frac{m^2c^2}{p^2}} \quad \text{put } p = \frac{h}{\lambda}$$

$$\frac{E}{p} = c\sqrt{1 + \frac{m^2\lambda^2c^2}{h^2}} = c\sqrt{1 + \left(\frac{m\lambda c}{h}\right)^2}$$

By definition $v_p = \frac{\omega}{k} = \frac{2\pi\upsilon}{\frac{2\pi}{\lambda}} = \frac{\upsilon}{\frac{1}{\lambda}} = \frac{h\upsilon}{\frac{h}{\lambda}} = \frac{E}{p}$

$$\therefore \quad v_p = c\sqrt{1 + \left(\frac{mc\lambda}{h}\right)^2}$$

This shows that v_p is greater than c. These waves produce dispersion in vaccum.

Example - 25

Show that phase velocity is half of group velocity.

Solution

Phase velocity is given by $v_p = \upsilon\lambda$ ..... (1)

De Broglie wavelength, $\lambda = \frac{h}{mv_g}$

Using $\frac{1}{2}mv_g^2 = h\upsilon$

$$\therefore \; \upsilon = \frac{mv_g^2}{2h}$$

Putting the values of λ and υ in equ (1), we get

$$v_p = \frac{mv_g^2}{2h} \cdot \frac{h}{mv_g} = \frac{v_g}{2}$$

Example - 26

Show that product of phase velocity and group velocity is equal to the square of velocity of light.

Solution

We have $v_p = \frac{\omega}{k} = \frac{\hbar\omega}{\hbar k} = \frac{E}{p} = \frac{mc^2}{mv}$

$$v_p = \frac{c^2}{v}$$

v is the particle velocity. Since $v = v_g$

$$v_p = \frac{c^2}{v_g}$$

$$\therefore \quad v_g v_g = c^2.$$

Probability and randomness

Any single measurement of the position or momentum can be measured with high accuracy. If so the wave like behaviour of particle cannot be observed, hence uncertainty of position or momentum plays no role at all.

In quantum mechanics instead of measuring the position or momentum once from a large number of similarly prepared systems we take large number of measurements and go for the mean of the measurements. Then automatically uncertainty of the measurement comes into play.

Even if we make similarly prepared systems for the measurement, the outcome of each measurement will be different. From these random results how can we construct a mathematical theory? The solution to this dilemma lies in the consideration of the probability of obtaining any given result from an experiment whose possible results are subject to the laws of statistics. i.e., in quantum we always take of the probability of measurements. In other words quantum theory does not predict the result of a single measurement but predicts the statistical behaviour of the system. The quantum theory developed on the basis of statistical behaviour and probability is found to agree with experimental results. It may also be noted that the random behaviour of a system governed by the laws of quantum mechanics is a fundamental aspect of nature.

Probability amplitude

In classical physics a wave is characterised by its amplitude A, wave vector k and its angular velocity ω. In quantum mechanics we deal with matter waves. According to de Broglie a particle of momentum p is to be associated with a wave of definite wave length $\lambda = \frac{h}{p}$. But we found that this leaves us in a very uncomfortable situation. To circumvent this difficulty we introduced the concept of wave packet. The wave packet is characterised by its amplitude. The amplitude is related to the likelihood of finding the particle at time. Hence called probability amplitude. That is, the amplitude is large where the particle is likely to be found and small where particle is less likely to be found. In other words the probability of finding the particle at any point depends on the amplitude of matter wave at that point.

In classical physics the intensity of any wave is proportional to the square of its amplitude. In analogy with this, in quantum mechanics, we have probability to observe particles is proportional to

$$|\text{matter wave amplitude}|^2 = |\text{wave packet amplitude}|^2$$

The mathematical tool to represent a wave packet is called wave function. It is denoted by the symbol $\psi(r, t)$ (the greek letter psi), where $r = x, y, z$.

IMPORTANT FORMULAE

1. De Broglie wave length $\lambda = \frac{h}{p}$.

   Different forms

   (i) $\lambda = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2meV}}$

   (ii) $\lambda = \frac{h}{\sqrt{3mkT}}$

2. For relativistic particle, $\lambda = \frac{hc}{\sqrt{K.E(K.E + 2m_0c^2)}}$

   or $\lambda = \frac{h}{mv} = \frac{h\sqrt{1 - v^2/c^2}}{m_0 v}$

3. Single slit diffraction

   $2d\sin\theta = n\lambda \qquad n = 1, 2, 3, \ldots$

   or $a\sin\phi = n\lambda \qquad n = 1, 2, 3, \ldots$

4. Classical position - wavelength uncertainty

   $\Delta x\, \Delta y \simeq \varepsilon\, \lambda^2$

5. Classical frequency - time uncertainty

   $\Delta \nu\, \Delta t \simeq \varepsilon$

6. Heisenberg position - momentum uncertainty

   $$\Delta x\, \Delta p_x \geq \frac{\hbar}{2},\ \Delta y\, \Delta p_y \geq \frac{\hbar}{2},\ \Delta z\, \Delta p_z \geq \frac{\hbar}{2}.$$

7. Heisenberg energy - time uncertainty

   $\Delta E\, \Delta t \geq \frac{\hbar}{2}$.

8. Uncertainty in position

   $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

9. Uncertainty in momentum

   $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$

10. Phase velocity

    $v_p = \frac{\omega}{k} = \frac{E}{p}$