UNIT TWO

2 GENERAL RELATIVITY AND COSMOLOGY

Introduction

To study the behaviour of matter there are two fundamental theories in physics. They are

(i) Newtonian theory of gravitation - which describes the behaviour of one mass point (gravitational field) and

(ii) the electrodynamics - which describes the behaviour of charged matter in the presence of electromagnetic field.

The special theory of relativity had its origin in the development of electrodynamics, while the general theory of relativity is the relativistic theory of gravitation.

The special theory of relativity only accounts for inertial systems, in the regions of free space where gravitational effects are neglected. In those systems the law of inertia holds good and the physical laws retain the same form. The special theory of relativity does not account for non-inertial (accelerated) systems. For example the clock paradox and universal phenomenon of gravitation could not be accounted by special theory of relativity. Thus naturally we wish to extend the principle of relativity in such a way that it may hold even for non-inertial systems and consequently the extended theory may explain the non-inertial phenomenon like clock paradox and particularly the phenomenon of gravitation. This is because **when S.T.R is extended to accelerated systems we can easily bring the effect of gravitation. The extended theory is known as the general theory of relativity**. In developing the general theory of relativity it is helpful to analyse the predictions of special theory of relativity with respect to the phenomenon of gravitation.

Principle of equivalence is the keystone of general theory of relativity enunciation by Albert Einstein in 1915.

The principle of equivalence

This is actually the principle of equivalence of gravitation and inertia. **It states that there is no way to distinguish locally the motion produced by inertial forces (acceleration, recoil, centrifugal forces) from motion produced by gravitational force.**

Einstein arrived at this conclusion with the following gedanken (thought) experiment.

In accordance with his usual mode of creative thought, Einstein set the stage with an imaginary situation. He pictured an immensely high building and inside it an elevator that had slipped from its cables and is falling freely. Within the elevator a group of physicists, undisturbed by any suspicion that their ride might end in disaster, are performing experiments. They take objects from their pockets, a fountain pen, a coin, a bunch of keys, and release them from their grasp. Nothing happens. The pen, the coin, the keys appear to the men in the elevator to remain posed in mid-air-because all of them are falling, along with the elevator and the men, at precisely the same rate in accordance with Newton's law of gravitation. Since the men in the elevator are unaware of their predicament, however, they may explain these peculiar happenings by a different assumption. They may believe they have been magically transported outside the gravitational field of the earth and are in fact poised somewhere in empty space. And they have good grounds for such a belief. If one of them jumps from the floor he floats smoothly towards the ceiling with a velocity just proportional to the vigour of his jump. If he pushes his pen or his keys in any direction, they continue to move uniformly in that direction until they hit the wall of the car. Everything apparently obeys Newton's law of inertia, and continues in its state of rest or of uniform motion in a straight line. The elevator has somehow become an inertial system, and there is no way for the men inside it to tell whether they are falling in a gravitational field or are simply floating in empty space, free from all external forces.

Einstein now shifts the scene, The physicists are still in the elevator, but this time they really *are* in empty space, far away from the attractive power of any celestial body. A cable is attached to the roof of the elevator; some supernatural force begins reeling in the cable; and the elevator travels "upward" with constant acceleration, i.e., progressively faster and faster. Again the men in the car have no idea where they are, and again they perform experiments to evaluate their situation. This time they notice that their feet press solidly against the floor. If they jump they do not float to the ceiling, for the floor comes up beneath them. If they release objects from their hands, the objects appear to "fall". If they toss objects in a horizontal direction they do not move uniformly in a straight line, but describe a parabolic curve with respect to the floor. And so the scientists, who have no idea that their windowless car actually is climbing through interstellar space, conclude that they are situated in quite ordinary circumstances in a stationary room rigidly attached to the earth and affected in normal measure by the force of gravity. There is really no way for them to tell whether they are at rest in a gravitational field or ascending with constant acceleration through outer space where there is no gravity at all.

The same dilemma would confront them if their room were attached to the rim of a huge rotating merry-go-round set in outer space. They would feel a strange force trying to pull them away from the centre of the merry-go-round, and a sophisticated outside observer would quickly identify this force as inertia (or, as it is termed in the case of rotating objects, centrifugal force). But the men inside the room, who as usual are unaware of their odd predicament, would once again attribute the force to gravity. For if the interior of their room is empty and unadorned, there will be nothing to tell them which is the floor and which is the ceiling except the force that pulls them towards one of its interior surfaces. So what a detached observer would call the "outside wall" of the rotating room becomes the "floor" of the room for the men inside. A moment's reflection shows that there is no "up" or "down" in empty space. What we on earth call "down" is simply the direction of gravity. To a man on the sun it would appear that the Australians, Africans, and Argentines are hanging by their heels from the southern hemisphere. By, the same token, Admiral Byrd's flight over the South Pole was a geometrical fiction. actually he flew *under* it - upside down. And so the men inside the room on the merry-go-round will find that all their experiments produce exactly the same results as the ones they performed when their room was being swept "upward" through space. Their feet stay firmly on the "floor." Solid objects "fall". And once again they attribute these phenomena to the force of gravity and believe themselves at rest in a gravitational field.

From these fanciful occurances Einstein draw a conclusion of great theortical importance. To physicists it is known as the principle of equivalence of gravitation and inertia.

The above discussion shows that an accelerated frame can bring the effect of graitational force. For example an elevator moving in outer space moving upward with an acceleration $a = g$; bring gravitional effect.

In the statement of the principle of equivalence, we used the word "there is no way to distinguish locally.............., locally we mean that there is no way to distinguish from within a sufficiently confined system. Our elevator is such a system. In other words the equivalence principle applies only to local systems i.e., only for small systems inertial and gravitational forces are indistinguishable. For non-local systems they are distinguishable.

Einstein realised that the principle of equivalence applied not only to mechanical systems but to all experiments, even ones based on electromagnetic radiations. Principle of equivalence predicts a change in frequency of a light wave falling in the earth's gravity.

To illustrate this consider an elevator moving up with acceleration a. At the top of the elevator there is a light source that emits a wave of frequency υ. At the bottom of the elevator and a distance H away there is a detector that observes the wave and measures its frequency. When the light wave is emitted in the accelerating elevator, the source has speed v, which is assumed to small compared with the speed of light c. When the wave is detected by the detector after a time $t \simeq \frac{H}{c}$, the floor is moving with a speed v + at. In effect there is a relative speed $\Delta v = at$ between the source and the detector, so there is Doppler shift in frequency given by

$$\upsilon' = \upsilon\sqrt{\frac{1+\frac{\Delta v}{c}}{1-\frac{\Delta v}{c}}} = \upsilon\left(1+\frac{\Delta v}{c}\right)^{1/2}\left(1-\frac{\Delta v}{c}\right)^{1/2}$$

$$\upsilon' \simeq \upsilon\left(1+\frac{\Delta v}{2c}\right)\left(1+\frac{\Delta v}{2c}\right)$$

$$\upsilon' \simeq \upsilon\left[1+\frac{\Delta v}{2c}+\frac{\Delta v}{2c}+\left(\frac{\Delta v}{2c}\right)^2\right]$$

Neglecting $\left(\frac{\Delta v}{2c}\right)^2$ we get

$$\upsilon' \simeq \upsilon\left(1+\frac{\Delta v}{c}\right)$$

$\therefore$

$$\frac{\upsilon'-\upsilon}{\upsilon} = \frac{\Delta v}{c}$$

$$\frac{\upsilon'-\upsilon}{\upsilon} = \frac{at}{c} \qquad (\because \quad \Delta v = at)$$

If the acceleration of the elevator a = g

$$\frac{\Delta\upsilon}{\upsilon} = \frac{gt}{c}$$

$$\frac{\Delta \upsilon}{\upsilon} = \frac{gH}{c^2} \qquad \left(\because \quad t = \frac{H}{c}\right)$$

Thus we can say that the principle of equivalence predicts a change in frequency of a light wave falling in the earth's gravity. This has been verified experimentally by Pound and Rebka in 1959. Further the frequency of radiation emitted by satellites and received by ground stations have confirmed this prediction to a precision of about 1 part in 10^4. This is made use of in the global positioning system (GPS). GPS relies on frequency measurements on the surface of the earth from transmitters in orbiting satellites, its accuracy depends on applying correction due to the gravitational frequency shift predicted by general relativity. Without this correction, errors in the GPS locating system of roughly 10 km per day would accumulate.

The equation for frequency shift can be re-written as

$$\frac{\Delta \upsilon}{\upsilon} = \frac{mgH}{mc^2}$$

$\frac{mgH}{m}$ is the difference in gravitational potential energy per unit mass (ΔV) between the source and the detector.

Thus $$\frac{\Delta \upsilon}{\upsilon} = \frac{\Delta V}{c^2}$$

suppose light leaving the surface of a star of mass M and radius R. The gravitational potential at the surface is $V = -\frac{GM}{R}$. If the light is observed on the earth, where the gravitational potential is negligible compared with that of the star, the frequency shift is

$$\frac{\Delta \upsilon}{\upsilon} = \frac{\Delta V}{c^2} = -\frac{GM}{Rc^2}$$

photons climbing out of a star's gravitational field lose energy and therefore shifted to smaller frequencies or longer wavelengths.

i.e. the photon has a lower frequency at the earth, corresponding to its less in energy as it leaves the field of the star. A photon in the visible region of the spectrum is thus shifted towards the red end and this phenomenon is known as gravitational red shift.

Example 1

The Lyman α line in the hydrogen spectrum has a wavelength of 121.5mm. Find the change in wavelength of this line in the solar spectrum due to the gravitational field. $G = 6.67\times10^{-11}Nm^2kg^{-2}$

$$M_\odot = 1.99\times10^{30}\,kg \text{ and } R_\odot = 6.96\times10^8\,m$$

Solution

We have $\dfrac{\Delta\upsilon}{\upsilon} = -\dfrac{GM}{Rc^2}$

Using $\upsilon = \dfrac{c}{\lambda}$

$$\Delta\upsilon = -\frac{c}{\lambda^2}\Delta\lambda$$

$$\therefore \quad \frac{\Delta\upsilon}{\upsilon} = -\frac{\Delta\lambda}{\lambda}$$

or $\dfrac{\Delta\lambda}{\lambda} = -\dfrac{\Delta\upsilon}{\upsilon} = \dfrac{GM}{Rc^2}$

substituting the values, we get

$$\frac{\Delta\lambda}{121.5\times10^{-9}} = -\frac{6.67\times10^{-11}\times1.99\times10^{30}}{6.96\times10^8\times(3\times10^8)^2}$$

$$\frac{\Delta\lambda}{121.5\times10^{-9}} = 2.119\times10^{-6}$$

or $\Delta\lambda = 2.119\times10^{-6}\times121.5\times10^{-9}$

$$\Delta\lambda = 0.257\text{pm}$$

Example 2

The increase in energy of a fallen photon was first observed in 1960 by Pound and Rebka at Harvard. Find the change in frequency of a photon of red light whose original frequency is $7.3\times10^{14}\,Hz$ when it falls through 22.5 m

solution

We have $\frac{\Delta\upsilon}{\upsilon} = \frac{gH}{c^2}$

Substituting the values, we get

$$\frac{\Delta\upsilon}{7.3\times10^{14}} = \frac{9.8\times22.5}{(3\times10^8)^2}$$

or
$$\Delta\upsilon = \frac{9.8\times22.5}{9\times10^{16}}\times7.3\times10^{14}$$

$$\Delta\upsilon = 1.79\text{Hz}$$

General theory of relativity

In Newtonian mechanics space and time were kept apart whereas in relativity space and time are intimately coupled via Lorentz transformation and called them as space time. General relativity is a theory of geometry. The motion of a particle is determined by the properties of space and time coordinates through which it moves. The equivalence between accelerated motion and gravity suggests a relationship between space time coordinates and gravity. In classical description, we would say that the presence of matter sets up a gravitational field, which then determines how objects move in response to that field. **According to general relativity the presence of matter causes space time to warp or curve, the motion of particles is determined by the shape of the curvature of space time**. General relativity gives us a procedure for calculating the curvature of space time. Roughly this is given by

$$\textbf{Curvature of space} = \frac{8\pi G}{c^4}\ \textbf{energy momentum}$$

where G is the universal gravitational constant. This equation is called Einstein's field equation in general relativity. When there is no mass, energy momentum is zero so the curvature is zero and space is flat. In the limiting case $c \to \infty$ (classical mechanics) and in the limit of weak gravitational field the Einstein's field equation becomes Newton's law of gravitation.

In non-technical terms what is actually Einstein's theory of gravitation. The gravitation of Einstein is entirely different from the gravitation of Newton. It is not a force. Einstein's law of gravitation contains nothing about force. It describes the behaviour of objects in a gravitational field - the planets, for example - not in terms of attraction but simply in terms of the paths they follow.

To Einstein, gravitation is simply a part of inertia; the movement of stars and the planets stem from their inherent inertia, and the courses they follow are determined by the metric properties of space. Although this sounds very abstract and even paradoxical, it becomes quite clear as soon as one dismisses the notion that bodies of matter can exert physical force $\left(F = \frac{GMm}{r^2}\right)$ on each other across millions of kilometres of empty space. This concept of "action at a distance" has troubled scientists since Newton's day. It led to particular difficulty in understanding electric and magnetic phenomena. Today scientists no longer say that a magnet "attracts" a piece of iron by some kind of mysterious but instantaneous action-at-a distance. They say rather that the magnet creates a certain physical condition in the space around it, which they term a magnetic field; and that this magnetic field then acts upon the iron and makes it behave in a certain predictable fashion. Students in any elementary science course know what a magnetic field looks like, because it can be rendered visible by the simple process of shaking iron filings on to a piece of stiff paper held above a magnet. A magnetic field and an electrical field are physical realities. They have a definite structure, and their structure is described by the field equations of James Clerk Maxwell which pointed the way towards all the discoveries in electrical and radio engineering of the past century. A gravitational field is as much of a physical reality as an electromagnetic field, and its structure is defined by the field equations of Albert Einstein.

Just as Maxwell and Faraday assumed that a magnet creates certain properties in surrounding space, so Einstein concluded that stats, moons, and other celestial objects individually determine the properties of the space around them. And just as the movement of a piece of iron in a magnetic field is guided by the structure of the field, so the path of any body in a gravitational field is determined by the geometry of that field. The distinction between Newton's and Einstein's ideas about gravitation has sometimes been illustrated by picturing a little boy playing marbles in a city lot. The ground is very uneven, ridged with bumps and hollows. An observer in an office ten stories above the street would not be able to see these irregularities in the ground. Noticing that the marbles appear to avoid some sections of the ground and move towards other sections, he might assume that a "force" is operating which repels the marbles from certain spots and attracts them towards others. But another observer on the ground would instantly perceive that the path of the marbles is simply governed by the curvature of the field. In this little fable Newton is the upstairs observer who imagines that a "force" is at work, and Einstein is the observer on the ground, who has no reason to make such an assumption. Einstein's gravitational laws, therefore, merely describe the field properties of the space-time con-tinuum. Specifically, one group of these laws sets forth the relation between the mass of a gravitating body and the structure of the field around it; they are called structure laws. A second group analyses the paths described by moving bodies in gravitational fields; they are the laws of motion.

Tests of general relativity

It should not be thought that Einstein's theory of gravitation is only a formal mathematical scheme. For it rests on assumptions of deep cosmic significance. And the most remarkable of these assumptions is that the universe is not a rigid and immutable edifice where independent matter is housed in independent space and time; it is on the contrary an amorphous continuum, without any fixed architecture, plastic and variable, constantly subject to change and distortion. Wherever there is matter and motion, the continuum is disturbed. Just as a fish swimming, in the sea agitates the water around it, so a star, a comet, or a galaxy distorts the geometry of the space-time through which it moves.

When applied to astronomical problems, Einstein's gravitational laws yield results that are close to those given by Newton. If the results paralleled each other in every case, scientists might tend to retain the familiar concepts of Newtonian law and write off Einstein's theory as a weird if original fancy. But a number of strange new phenomena have been discovered, and at least one old puzzle solved, solely on the basis of general relativity. The old puzzle stemmed from the eccentric behaviour of the planet mercury. Instead of revolving in its elliptical orbit with the regularity of the other planets, mercury deviates from its course each year by a slight but exasperating degree. Astronomers explored every possible factor that might cause this perturbation, but found no solution within the framework of Newtonian theory. It was not until Einstein evolved his laws of gravitation that the problem was solved. Of all the planets mercury lies closest to the sun. It is small and travels with great speed. Under Newtonian law these factors

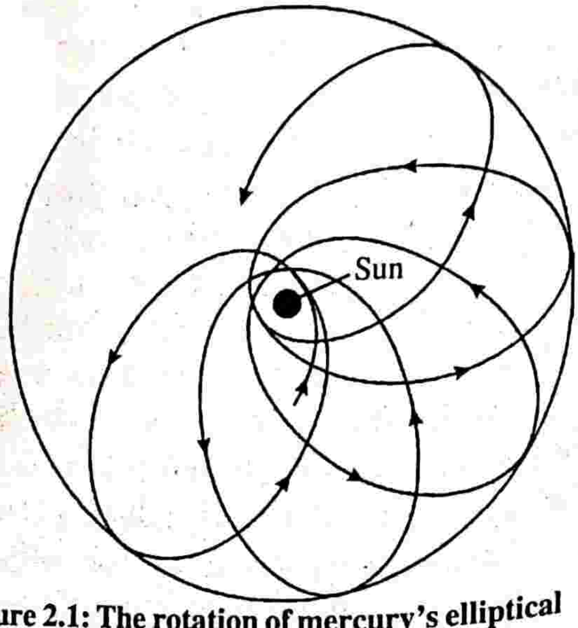


Figure 2.1: The rotation of mercury's elliptical orbit, greatly exaggerated. Actually the ellipse advances only 43 seconds of an arc per century.

should not in themselves account for the deviation; the dynamics of Mercury's movement should be basically the same as those of any other planet. But under Einstein's laws, the intensity of the sun's gravitational field and mercury's enormous speed make a difference, causing the whole ellipse of mercury's orbit to execute a slow but inexorable swing around the sun at the rate of one revolution in 3,000,000 years. This calculation is in perfect agreement with actual measurements of the planet's course. Einstein's mathematics are thus more accurate than Newton's in dealing with high velocities and strong gravitational fields.

Another prediction made by Einstein was the effect of gravitation on light.

The Sequence of thought which led Einstein to prophesy this effect began with another imaginary situation. As before, the scene opens in an elevator ascending with constant acceleration through empty space, far from any gravitational field. This time some roving interstellar gunman impulsively fires a bullet at the elevator. The bullet hits the side of the car, passes clean through and emerges from the far wall at a point a little below the point at which it penetrated the first wall. The reason for this is evident to the marksman on the outside. He knows that the bullet flew in a straight line, obeying Newton's law of Inertia; but while it traversed the distance between the two walls of the car, the whole elevator travelled "upward" a certain distance, causing the second bullet hole to appear not opposite the first one but slightly nearer the floor. However, the observers inside the elevator, having no idea where in the universe they are, interpret the situation differently. Aware that on earth any missile describes a parabolic curve towards the ground, they simply conclude that they are at rest in a gravitational field and that the bullet which passed through their car was describing a perfectly normal curve with respect to the floor.

A moment later as the car continues upwards through space a beam of light is suddenly flashed through an aperture in the side of the car. Since the velocity of light is great, the beam traverses the distance between its point of entrance and the opposite wall in a very small fraction of a second. Neverthe-less, the car travels upwards a certain distance in that interval, so the beam strikes the far wall a tiny fraction of an inch below the point at which it entered. If the observers within the car are equipped with sufficiently delicate instruments of measurement, they will be able to compute the curvature of the beam. But the question is, how will they explain it? They are still unaware of the motion of their car and believe themselves at rest in a gravitational field. If they cling to Newtonian principles, they will be completely baffled because they will insist that light rays always travel in a straight line. But if they are familiar with the special theory of relativity they will remember that energy has mass in accordance with the equation $m = E/c^2$. Since light is a form of energy they will deduce that light has mass and will therefore be affected by a gravitational field. Hence the curvature of the beam.

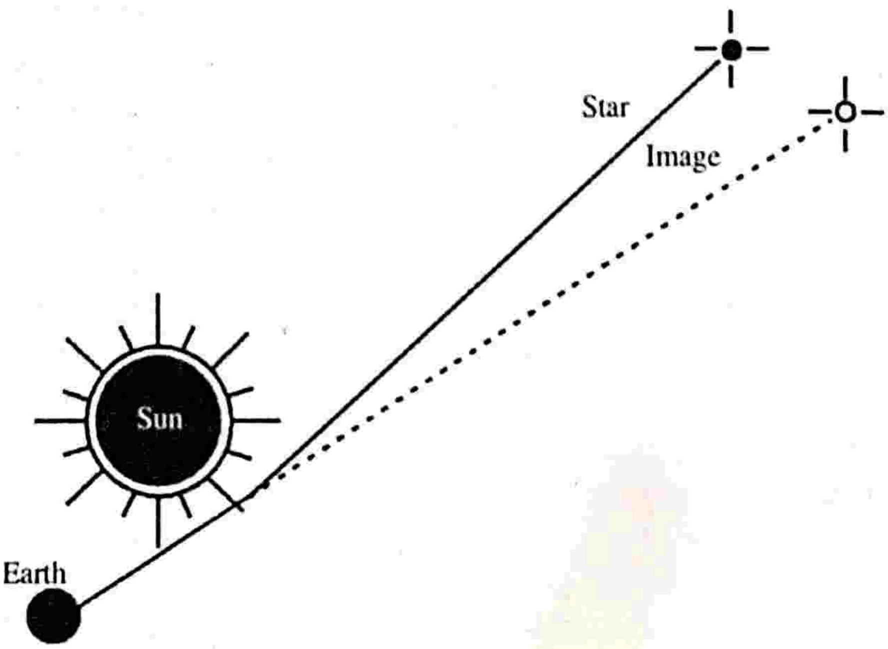


Figure 2.2: The deflection of starlight in the gravitational field of the sun. Since the light from a star in the neighbourhood of the sun's disk is bent in wards, towards the sun, as it passes through the sun's gravitational field, the image of the star appears to observers on earth to be shifted outwards and away from the sun.

From these purely theoretical considerations Einstein concluded that light, like any material object, travels in a curve when passing through the gravitational field of a massive body. He suggested that his theory could be put to test by observing the path of starlight in the gravitational field of the sun. Since the stars are invisible by day, there is only one occasion when sun and stars can be seen together in the sky, and that is during an eclipse. Einstein proposed, therefore, that photographs be taken of the stars immediately bordering the darkened face of the sun during an eclipse and compared with photographs of those same stars made at another time.

According to his theory, the light from the stars surrounding the sun should be bent inwards, towards the sun, in traversing the sun's gravitational field; hence the *images* of those stars should appear to observers on earth to be shifted outwards from their usual positions in the sky. Einstein calculated the degree of deflection that should be observed and predicted that for the stars closest to the sun the deviation would be about 1.75 seconds of an arc. Since he staked his whole general theory of relativity on this test, men of science throughout the world anxiously awaited the findings of expeditions which journeyed to equatorial regions to photograph the eclipse of 29 May 1919. When their pictures were developed and examined, the deflection of the starlight in the gravitational field of the sun was found to average

1.64 seconds — a figure as close to perfect agreement with Einstein's prediction as the accuracy of instruments allowed.

Another prediction made by Einstein on the basis of general relativity pertained to time. Having shown how the properties of space are affected by a gravitational field, Einstein reached the conclusion by analogous but somewhat more involved reasoning that time intervals also vary with the gravitational field. A clock transported to the sun should run at a slightly slower rhythm than on earth. And a radiating solar atom should emit light of slightly lower frequency than an atom of the same element on earth. The difference in wavelength would in this case be immeasurably small. But there are in the universe gravitational fields stronger than the sun's. One of these surrounds the freak star known as the "companion of Sirius" — a white dwarf composed of matter in a state of such fantastic density that 1 cubic inch of it would weigh a ton on earth. Because of its great mass, this extraordinary dwarf, which is only three times larger than the earth, has a gravitational field potent enough to perturb the movements of Sirius, seventy times its size. Its field is also powerful enough to slow down the frequency of its own radiation by a measurable degree, and spectroscopic observations have indeed proved that the frequency of light emitted by Sirius' companion is reduced by the exact amount predicted by Einstein. The shift of wavelength in the spectrum of this star is known to astronomers as "the Einstein Effect" and constitutes an additional verification of general relativity.

Stellar evolution

The predictions of general relativity were experimentally verified by the presence of large gravitational field produced by the sun. Another remarkable experimental verification of general relativity is due to much more gravitational field produced by the collapse of stars. On the basis of general relativity we predicted the stellar evolution and their collapses. A star collapsed into more compact objects like white dwarfs, neutron stars and blackholes etc. in their final stage. We predicted these and experimentally detected them provided addition verification for the presence of gravitational field and the general theory of relativity. Stellar evolution will be discussed extensively in the forth coming chapter.

The expansion of the universe

The picture of the origin of the universe began with the formulation of general theory of relativity in 1915 by Albert Einstein. Solving the field equations Einstein got an expanding universe. But he believed that the universe is static. So he modified his theory to make this possible by introducing a so called cosmological constant into his equations (one of the greatest blunders that Einstein ever committed, he himself admitted this). That is Einstein introduced a new antigravity force unlike other forces did not come from any particular source but was built into the very fabric of space time. He claimed that space -time had an inbuilt tendency to expand and this could be made to balance exactly the attraction of all the matter in the universe, so that a static universe would result. Einstein and other physicists were looking for ways of avoiding general relativities prediction of a non-static universe but one man the Russian physicist and mathematician Alexander Friedmann instead set about explaining it.

Friedmann made two very simple assumptions about the universe, one is that the universe looks identical in which ever direction we look, the second one is that this would also be true if we were observing the universe from anywhere else. From these two assumptions in 1922 Friedmann showed that our universe is non-static. Few years later (1929) the American astronomer Edwin Hubble experimentally proved that we are living in an expanding universe. The discovery that the universe is expanding was one of the greatest intellectual revolutions of the twentieth century.

The evidence for the expansion of the universe comes from the observed change in the wavelength of the light emitted by distant galaxies. According to relativistic Doppler effect in terms of wavelength, we can write

$$\lambda' = \lambda\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

where λ is the wavelength emitted by the galaxy in its own frame, λ' is the wavelength we measure on earth and v is the relative velocity between the source of light and the observer.

The light emitted by a star such as the sun has a continuous spectrum. As the light passes through the stars atmosphere some of it is absorbed by the gases in the atmosphere, so the continuous emission spectrum has a few dark absorption lines super imposed. Comparison between the known wavelengths of these lines (measured on earth for sources at rest relative to the observer) and the Doppler shifted wavelengths enables us to calculate the speed of the star from the above equation.

If stars are moving away from us wavelength observed is found to be increased. i.e, wavelengths shifted towards the longer wavelength (red) and stars moving towards us wavelength observed is found to be shifted towards shorter wavelength (blue). The average speed of these stars in our galaxy relative to earth is about $3\times10^3 ms^{-1}$. The change in wavelength for these stars is very small. Light from nearby galaxies also show small change in red shift or blue shift.

However when we observe light from distant galaxies all are found to be red shifted. From these we concluded that galaxies are receding from us. According to Friedmann's assumption we can conclude that any other observer in the universe would draw the same conclusion. Combining the experimental observation and the assumption of Friedmann, in general we can say that galaxies would be observed to recede from every point in the universe. In other words our universe is expanding.

Hubble's law

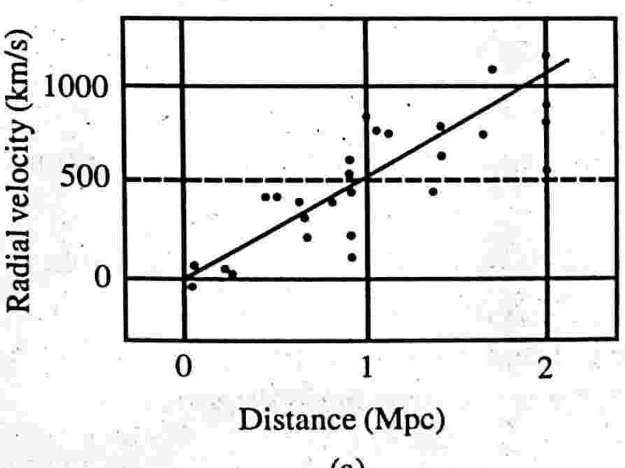
In 1920 the American astronomer Hubble Edwin Powel (1889-1953) started using the 100 inch (1.5m) telescope on Mount Wilson in California. From this observation he concluded that spiral nebulae are not nebulae but distant galaxies. By measuring the red shift of these distant galaxies he calculated the speeds of these galaxies. At the same time he also devised a method to calculate the distances of these galaxies. Hubble observed 46 galaxies and their distances and speeds. From this he made two remarkable conclusions. The galaxies are moving away from us and the further away a galaxy is, the faster it is moving away. **This proportionality between the recessional speed (v) of galaxies and its distance is known as Hubble law.**

i.e., $v \propto d$

$$v = H_0 d$$

The proportionality constant H_0 is known as the Hubble parameter.

The Hubbles plot of v versus d is shown in figure below. It is due to the uncertainties in the measurement of the distance points obtained are found to be scattered. However it gives a strong indication that v and d are linear.



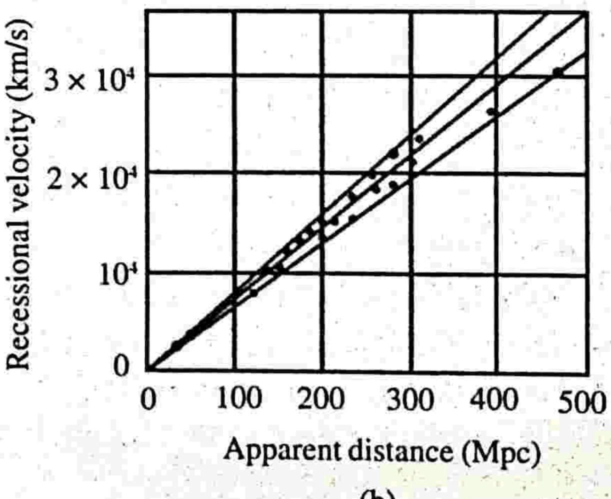


Figure 2.3

More modern data based on observing supernovas in distant galaxies are shown in figure above. From this the value of H_0 was calculated. The slope v-d graph gives H_0. The value of Hubble parameter calculated to be

$$H_0 = 72 \text{kms}^{-1} (\text{Mpc})^{-1}$$

$$1\text{Mpc} = \text{Million parsec}$$

$$1 \text{ Parsec} = 3.08 \times 10^{16} \text{m}$$

$$\therefore \qquad H_0 = \frac{72 \times 10^3 \text{ms}^{-1}}{3.08 \times 10^{22} \text{m}} = 23.38 \times 10^{-19} \text{s}^{-1}.$$

The Hubble parameter has the dimension of inverse time.

The inverse of H_0 gives

$$\frac{1}{H_0} = \frac{1}{23.38 \times 10^{-19}} \text{s}$$

$$\frac{1}{H_0} = \frac{100 \times 10^{17}}{23.38} \text{s} = 4.28 \times 10^{17} \text{s}$$

or $$\frac{1}{H_0} = 4.28 \times 10^{17} \text{s}$$

or $$\frac{1}{H_0} = \frac{4.28 \times 10^{17}}{3.15 \times 10^{7}} \text{ years}$$

$$\frac{1}{H_0} = 13.59 \times 10^{9} \text{ years}$$

Very surprisingly you can see that the inverse of Hubble parameter is almost equal to age of the universe.

Hubble's law was enunciated in the year 1929. After about 20 years Hubble installed a 200 inch (3m) Palomer telescope. Through this telescope he observed distant galaxies in all directions and concluded that what Friedmann said was perfectly correct, it follows that distribution matter is uniform in all directions.

Note: $1\ AU = 1.496\times10^{11}\ m$

$1\ ly = 9.46\times10^{15}\ m$

$1\ pc = 3.26 ly = 3.08\times10^{16}\ m$

Relation between v and d

Consider the universe represented by the three dimensional coordinate system shown in figure below, where each point is a galaxy, with the earth at the origin we can determine the distance of each galaxy. Consider two galaxies I and II. Let d_1 be the distance of galaxy I and d_2 that of galaxy II from the origin. If this universe were to expand with all points becoming farther apart. Then d_1 goes to d_1' and d_2 goes to d_2' (see fig. below). So we can write

$$d_1' = k\, d_1$$

$$d_2' = k\, d_2$$

or in general we can write

$$d' = k\, d$$

$\therefore$ The recessional velocity of any galaxy is

$$v = \frac{d' - d}{t}$$

where t is the time in which the distance d goes to d'.

or $$v = \frac{kd - d}{t} = \frac{d(k-1)}{t}$$

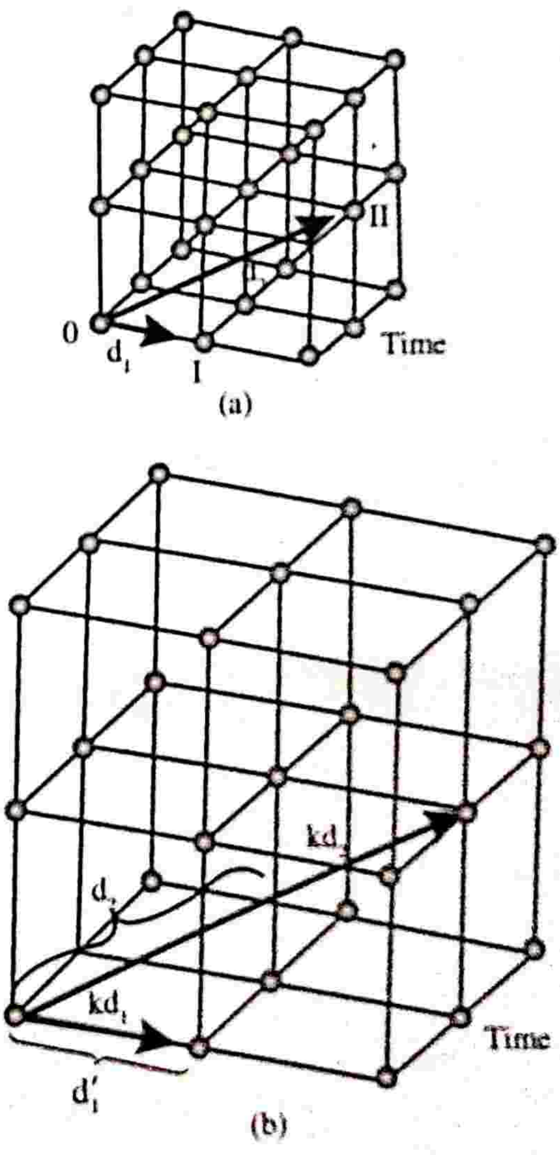


Figure 2.4: **The expansion of a coordinate space, showing that the apparent speed of recession depends on the distance, d_2 is greater than d_1, and d_2 increases faster than d_1**

If we compare two galaxies, we have

$$\frac{v_1}{v_2} = \frac{d_1}{d_2}$$

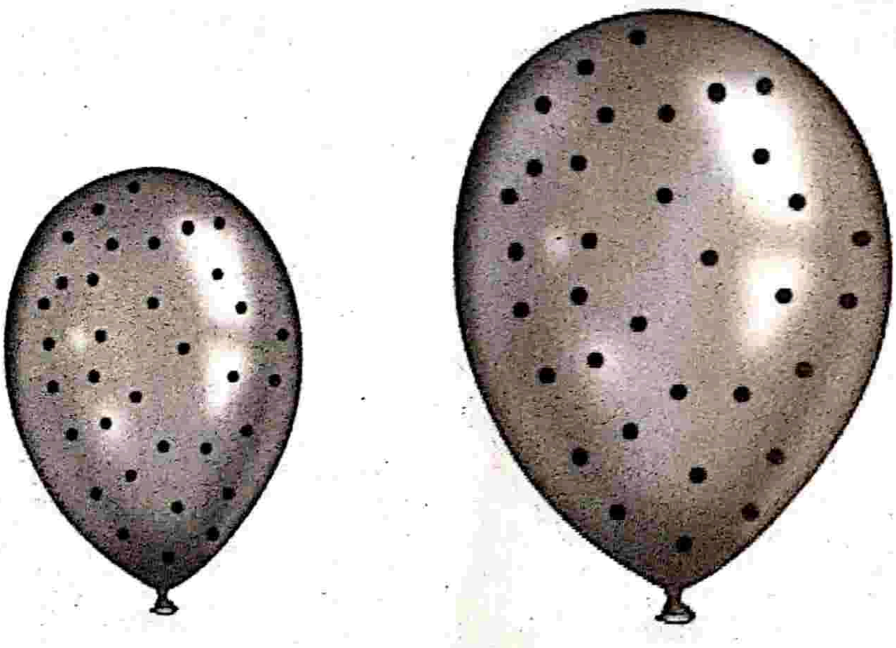
This is identical with Hubble's law

$$v = H_0 d$$

i.e. $v \alpha d$

This shows that if distance of galaxy is large recessional velocity of the galaxy is also large.

We can demonstrate the expanding universe with an analogy. Take a spotted balloon. Each spot represents a galaxy. When we inflate the balloon, one can see that each spot is moving away from the others. (see figure below)



(a) (b)

Figure 2.5: As a balloon is inflated, every observer on the surface experiences a velocity distance relationship of the form of the Hubble law

Consider another analogy. Take a loaf of raisin bread and place it in an oven. As the bread expands every raisin observes all the others to be moving away from it and the speed of recession increases with the separation. See figure below.

So far we arrived at the expansion of the universe on the basis of relativistic

Doppler effect. Actually correct interpretation of the cosmological red shift should come from general theory of relativity. According to general relativity the shift in wavelength is caused by a stretching of the entire fabric of space-time. Imagine small photos of galaxies glued to a rubber sheet. As the sheet is stretched, the distance between the galaxies increases. This stretching of the space between the galaxies causes the wavelength of light signal from one galaxy to increase. At low speeds Doppler red shift gives correct results. However for very large cosmological red shifts, a more correct analysis must be based on the stretching model. According to stretching model

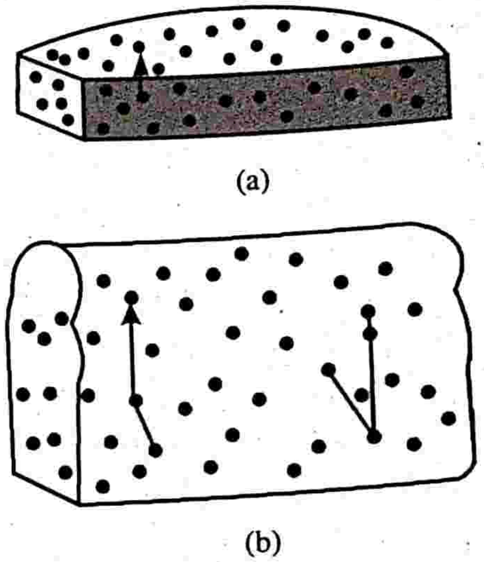


Figure 2.6: Another system in which the Habble law is valid

$$\frac{\lambda'}{\lambda} = \frac{R_0}{R}$$

where R_0 represents size of the universe at present and R represents the size of the universe at the time light was emitted. λ' is the wavelength of light received now and λ is the wavelength light when it was emitted.

Example 2

A distant galaxy in the constellation Hydra is receding from the earth at $6.12 \times 10^7 \, ms^{-1}$. By how much is a green spectral line of wavelength 500nm emitted by the galaxy shifted towards the red end of the spectrum. Also calculate the galaxy distance. Take $\frac{1}{H_0} = 14 \times 10^9$ years.

Solution

$$\lambda = 500 \times 10^{-9} m$$

$$v = 6.12 \times 10^7 ms^{-1}$$

$$\therefore \quad \frac{v}{c} = \frac{6.12 \times 10^7}{3 \times 10^8} = 0.204$$

$$\text{Using} \quad \lambda' = \lambda\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = 500\times10^{-9}\sqrt{\frac{1+0.204}{1-0.204}}$$

$$\lambda' = 500\times10^{-9}\sqrt{\frac{1.204}{0.796}} = 500\times10^{-9}\times1.23$$

$$\lambda' = 615\text{nm}$$

$\therefore$ The shift in wavelength $= 615 - 500\text{nm} = 115\text{nm}$

Using Hubble's law

$$v = H_0 d$$

$$d = \frac{v}{H_0} = 0.204c\times14\times10^9 \text{ years}$$

$$d = 0.204\times14\times10^9 \text{ light years}$$

$$d = 2.856\times10^9 \text{ light years}$$

Example 3

Use Hubble law to estimate the wavelength of the 590nm sodium line as observed emitted from galaxies whose distance from us is (a) 1.0×10^6 light years (b) 1.0×10^9 light years. Take $H_0 = 72\text{kms}^{-1}(\text{Mpc})^{-1}$

Solution

$$\lambda = 590\times10^{-9}\text{m}$$

a) $\quad d = 1.0\times10^6$ light years

Using $\quad v = H_0 d \quad$ ..... (1)

$$H_0 = 72\text{kms}^{-1}(\text{Mpc})^{-1}$$

1 parsec = 3.26 ly

$$\therefore \quad d = \frac{1.0\times10^6}{3.26}\text{ parsec} = 0.307\times10^6 \text{ parsec}$$

$$d = 0.307\text{Mpc}$$

Put this in equation 1, we get

$$v = 72\frac{\text{km}}{\text{s}}\cdot\frac{1}{\text{Mpc}}\cdot 0.307\text{Mpc}$$

$$v = 22.1\frac{\text{km}}{\text{s}} = 22.1\times10^{3}\,\text{ms}^{-1}$$

$$\therefore \quad \frac{v}{c} = \frac{22.1\times10^{3}}{3\times10^{8}} = 7.37\times10^{-5}$$

Using $\quad \lambda' = \lambda\sqrt{\dfrac{1+\dfrac{v}{c}}{1-\dfrac{v}{c}}} = 590\times10^{-9}\sqrt{\dfrac{1+7.37\times10^{-5}}{1-7.37\times10^{-5}}}$

$$\therefore \quad \lambda' \simeq 590\times10^{-9} = 590\text{nm}$$

b) $\quad d = 1.0\times10^{9}\,\text{ly} = \dfrac{1.0\times10^{9}}{3.26}\,\text{pc}$

$$d = 3.07\times10^{8}\,\text{pc} = 307\text{Mpc}$$

$$v = H_0 d$$

$$v = 72\frac{\text{km}}{\text{s Mpc}}\cdot 307\text{Mpc}$$

$$v = 22104\frac{\text{km}}{\text{s}} = 2.21\times10^{7}\,\text{ms}^{-1}$$

$$\frac{v}{c} = \frac{2.21\times10^{7}}{3\times10^{8}} = 7.37\times10^{-2}$$

$$\therefore \quad \lambda' = \lambda\sqrt{\frac{1+\dfrac{v}{c}}{1-\dfrac{v}{c}}} = 590\times10^{-9}\sqrt{\frac{1+7.37\times10^{-2}}{1-7.37\times10^{-2}}}$$

$$\lambda' = 590\times10^{-9}\sqrt{\frac{1.0737}{0.9263}}$$

$$\lambda' = 590 \times 10^{-9} \times 1.08$$

$$\lambda' = 637.2 \times 10^{-9}\,\text{m}$$

$$\lambda' = 637.2\,\text{nm}$$

The cosmic microwave background radiation

In 1923 Friedmann showed that our universe is non-static. Few years later (1929) Hubble experimentally confirmed the expansion of the universe. In 1940 George Gamow, a student of Freidmann, suggested that if we are living in an expanding universe and when we go back in time we get the early universe which must be hot and dense. Far enough back in time, the universe would have been too hot for stable matter to form. Its composition was then a gas of particles and photons.

The unstable particle eventually decayed to stable ones and the stable particles eventually clumped together to form matter. The photons that filled the universe remained but their wavelengths were stretched by the continuing expansion of the universe. Today those photons have a much lower temperature but they still uniformly fill the universe. This was first suggested by two American scientists. Bob Dickse and Jim Peebles. They argued that if Gamow said is correct we should still be able to see the glow of the early universe, because light form very distant parts of it would only be reaching us now. This **glow of the early universe is known as the cosmic back ground radiation.**

This picture of a hot early stage of the universe was first put forward by the scientist George Gamow in a famous paper written in 1948 with his student Ralph Alpher. Gamow had quite a sense of humour- he persuaded the nuclear scientist Hans Bethe to add his name to the paper to make the list of authors "Alpher, Bethe, Gamow'' like the first three letters of the Greek alphabet. Alpha, beta, gamma. Particularly appropriate for a paper on the beginning of the universe! In this paper they made the remarkable prediction that radiation (in the form of photons) from the very hot early stages of the universe should still be around today, but with its temperature reduced to only a few degrees above absolute zero(-273°C). At the time of publishing this paper, not much was known about the nuclear reactions of protons and neutrons. So predictions made for the proportions of various elements in the early universe were rather inaccurate. Gamow and others predicted that the background radiation would be at a temperature of the order of 5K to 10K and energy of the order of 10^{-3} eV or a wavelength of order 1mm. As this wavelength lies in the microwave region. The background radiation is called microwave back ground radiation. So they could not predict the temperature or energy of the back ground radiation correctly. But these calculations have been repeated in the light of better knowledge and now agree very well what we observe.

Calculation of the energy of the microwave background radiation

Consider the whole universe as a black body emitting radiation at the temperature T. From this we can calculate the average energy per photon at temperature T.

According to Planck's radiation law, the number of photons in the frequency range υ and $\upsilon+d\upsilon$ is

$$n(\upsilon)d\upsilon = \frac{8\pi\upsilon^2 V d\upsilon}{c^3}\frac{1}{e^{\frac{h\upsilon}{kT}}-1} \qquad \text{..... (1)}$$

Since each photon has energy $h\upsilon$, the energy density

$$u(\upsilon) = \frac{\text{Total energy}}{\text{Volume}} = \frac{n(\upsilon)h\upsilon}{V}$$

$$\therefore \quad u(\upsilon)d\upsilon = \frac{8\pi h}{c^3}\frac{\upsilon^3 d\upsilon}{e^{\frac{h\upsilon}{kT}}-1} \qquad \text{..... (2)}$$

To get the total energy density we have to integrate the above equation over all possible frequencies i.e, from $\upsilon=0$ to $\upsilon=\infty$

Thus $$u = \int_0^\infty u(\upsilon)d\upsilon = \frac{8\pi h}{c^3}\int_0^\infty \frac{\upsilon^3 d\upsilon}{e^{\frac{h\upsilon}{kT}}-1}$$

Put $\frac{h\upsilon}{kT} = x$, then $\frac{hd\upsilon}{kT} = dx$

$$\therefore \quad u = \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x-1}dx$$

The value of $\int_0^\infty \frac{x^3}{e^x-1}dx = \frac{\pi^4}{15}$

$$u = \frac{8\pi^5 k^4 T^4}{15c^3h^3} \qquad \text{..... (3)}$$

Integrating equations (1) from $\upsilon=0$ to $\upsilon=\infty$ we get the total number of photons.

$$N=\int_0^\infty n(\upsilon)d\upsilon=\frac{8\pi V}{c^3}\int_0^\infty \frac{\upsilon^2 d\upsilon}{e^{\frac{h\upsilon}{kT}}-1}$$

Put $\frac{h\upsilon}{kT}=x$, then $\frac{hd\upsilon}{kT}=dx$, we get

$$N=\frac{8\pi}{c^3}\left(\frac{kT}{h}\right)^3 V\int_0^\infty \frac{x^2dx}{e^x-1}$$

Then value of $\int_0^\infty \frac{x^2dx}{e^x-1}=2.404$ (See example 4)

$$\therefore \quad \frac{N}{V}=\frac{8\pi k^3T^3}{c^3h^3}\cdot 2.404$$

$$\frac{N}{V}=\frac{19.23\pi k^3T^3}{c^3h^3} \qquad .....(4)$$

$\therefore$ The average energy of the photon

$$\bar{E}=\frac{\text{Total energy}}{\text{Total number}}=\frac{U}{N}$$

or

$$\bar{E}=\frac{U}{V\frac{N}{V}}=\frac{u}{\frac{N}{V}}$$

$$\bar{E}=\frac{8\pi^5k^4T^4}{15c^3h^3}\cdot\frac{c^3h^3}{19.23\pi k^3T^3}$$

$$\bar{E}=\frac{8}{15\times 19.23}\pi^4 kT$$

Substituting the value of $k=0.8625\times 10^{-4}\,eVK^{-1}$

we get $$\bar{E} = \frac{8\times\pi^4\times 0.8625\times 10^{-4}T}{15\times 19.23}\text{eV}.$$

$$\bar{E} = 2.325\times 10^{-4}T\ \text{eV}.$$

This is the expression for average energy per photon in terms of temperature T.

Now we look at the experimental evidence for the existence of microwave background radiation and the determination of its temperature.

Put $\upsilon = \frac{c}{\lambda}$, then $d\upsilon = -\frac{c}{\lambda^2}d\lambda$ in equation (2), we get

$$u(\lambda)d\lambda = \frac{8\pi hc\ \lambda^{-5}d\lambda}{e^{\frac{hc}{\lambda kT}}-1}$$

This shows that energy density u at any wavelength λ is enough for the determination of temperature T. But the radiation actually will be a spectrum, so to measure temperature we require measurement over a range of wavelengths.

The first experimental evidence for the existence of this microwave radiation was obtained in the year 1965. In this year two American physicists at the Bell telephone laboratories in New Jersey Arno Penzias and Rober Wilson, used a microwave antenna tuned to a wavelength of 7.35 cm. At this wavelength they recorded an annoying hiss from their antenna that could not be eliminated, no matter how much care they took in refining the measurements. After pains taking efforts to eleminate to the noise they concluded that it was coming from no identifiable source and was striking their antenna from all directions, day and night, summer and winter. From the radiant energy at that wavelength they deduced a temperature of $3.1\pm 1.0K$. It was later concluded that the radiation was nothing but the microwave background radiation, the remnants of Bing-Bang. For this discovery Penzias and Wilson shared the Nobel Prize in Physics.

The temperature of the back ground radiation deduced by Penzias and Wilson was not that much accurate. This is because no precise data were available below a wavelength of 1 cm due to atmospheric absorption. After that several sophisticated and refined experiments have conducted. The most recent measurements were made with the Cosmic Background Explorer (COBE) satellite, which was launched into earth orbit in the year 1989 and the Wilkinson Microwave Anisotropy (WMAP) satellite which was launched into solar orbit in the year 2001. The COBE and WMAP satellites were able to obtain very precise data on the intensity of the background radiation in the wavelength range between 1 cm and 0.05 cm. The results from the COBE satellite was plotted shown in figure 2.6. The data points fall precisely on the solid line which is calculated from equation (2) for a temperature of T = 2.725 K. For this remarkable result achieved by John Mather and George smoot were awarded the 2006 Nobel Prize in Physics.

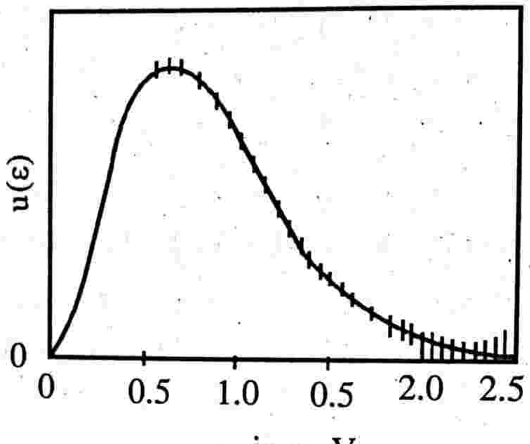


Figure 2.6

$$u(\upsilon) = \frac{8\pi h\,\upsilon^3}{c^3(e^{\frac{h\upsilon}{kT}} - 1)}$$

Put $h\upsilon = \varepsilon$

$$u(\varepsilon) \frac{8\pi\varepsilon^3}{h^2c^3(e^{\frac{\varepsilon}{kT}} - 1)}$$

Using the deduced value of T = 2.7K in equation (4), we can see that there are about 4.0×10^8 photon /m³. The average energy per photon is

$$\bar{E} = 2.325\times10^{-4}\times2.7\text{eV}$$

$$\bar{E} = 6.3\times10^{-4}\text{eV}$$

Calculating the number of photons of the back ground radiation plays a vital role in Bing Bang cosmology. This is because after the Bing Bang the ratio of nucleaons (protons and neutrons) to photons is found to be almost constant till late. (after nearly 14×10^9 years).

Example 4

Evaluate the integral $\int_0^\infty \frac{x^2dx}{e^x - 1}$ given $\sum_{n=1}^{\infty}\frac{1}{n^3} = \frac{\pi^3}{25.76}$

Solution

$$I = \int_0^\infty \frac{x^2dx}{e^x - 1} = \int_0^\infty \frac{x^2dx}{e^x(1 - e^{-x})}$$

$$I = \int_0^\infty \frac{e^{-x}x^2dx}{1 - e^{-x}} = \int_0^\infty e^{-x}x^2(1 - e^{-x})^{-1}dx$$

Using

$$(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n, \quad x < 1$$

$$I = \int_0^{\infty} e^{-x} x^2 \sum_{n=0}^{\infty} e^{-nx}$$

$$I = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-(n+1)x} x^2 dx$$

Using the standard integral $\int_0^{\infty} e^{-ax} x^n dx = n!(a)^{-n-1}$

$$\therefore \qquad I = \sum_{n=0}^{\infty} 2!(n+1)^{-2-1}$$

$$I = 2\sum_{n=0}^{\infty} \frac{1}{(n+1)^3} = 2\sum_{n=1}^{\infty} \frac{1}{n^3}$$

$$I = 2.\frac{\pi^3}{25.76} = 2.404$$

Dark Matter

A galaxy is a huge collection of nebulae (gases), stellar remnants and billions of stars and their solar system held together by gravity. There are six types of galaxies. They are spiral, elliptical, active, dwarf and irregular galaxies. Many galaxies have spiral structure with a bright central regions containing most of the galaxies mass and several arms in a flat disc. The entire structure rotate about an axis perpendicular to the plane of the disc. We belong to milky way (spiral) galaxy, where sun is in one of the spiral arms at a distance of 8.5 kpc from the centre and has a tangential velocity of 220 kms^{-1}. At this speed it takes about 240 million years for a complete rotation. During the life time of the solar system of about 4.5 billion years, the sun has made about 20 revolutions.

To analyse the motion of stars, in the galaxy we can make use of Kepler's harmonic law of motion $(T^2 \alpha\, r^3)$ because stars in the galaxy are bound due to gravitational force. We assume that the gravitational force on the sun is due to masses at the centre of the galaxy. The other stars in the spiral arm, whose mass contribution to the force on the sun is negligibly small. According to Kepler's third law, we have

$$T^2 = \frac{4\pi^2}{GM} r^3$$

where M is the mass contained within region of radius r.

Using $T = \frac{2\pi r}{v}$ we get

$$v = \sqrt{\frac{GM}{r}}$$

Put $v = 220\text{kms}^{-1} = 220 \times 10^3 \text{ms}^{-1}$

$r = 8.5\text{kpc} = 8.5 \times 10^3 \times 3.08 \times 10^{16} = 26.18 \times 10^{19}\text{m}$

$$\therefore \quad M = \frac{v^2 r}{G} = \frac{(220 \times 10^3)^2 \times 26.18 \times 10^{19}\text{m}}{6.67 \times 10^{-11}}$$

$$M = \frac{22^2 \times 10^8 \times 26.18 \times 10^{19}}{6.67 \times 10^{-11}}$$

$$M = \frac{22^2 \times 26.18}{6.67} \times 10^{38}\text{kg}$$

$$M = 1.9 \times 10^{41}\text{kg}$$

$$M_{\odot} \simeq 1.9 \times 10^{30}\text{kg}$$

$$M = \frac{1.9 \times 10^{41}}{1.9 \times 10^{30}} \approx 10^{11} \text{ solar mass.}$$

This shows that a mass equivalent of 10^{11} solar masses lies within the Sun's orbit.

According to this model, we expect stars beyond the Sun to have tangential velocity that decreases with r. Within the solar system all planets exactly and precisely follow this model. However we observe that v is almost constant or perhaps increases slightly for stars beyond the Sun. See figure below. Other galaxies also show the same effect. This shows that either the model is not correct or there is something missing.

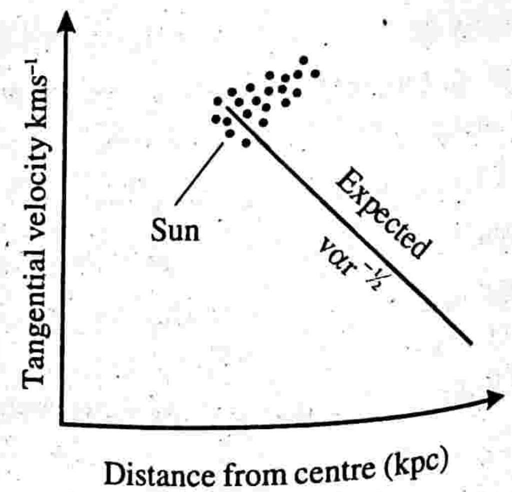


Figure 2.7

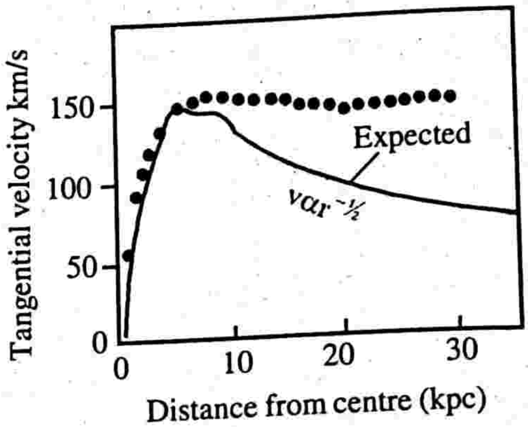


Figure 2.8

The tangential speeds of stars in distant galaxies can be measured by the Doppler shift of their light. If we are viewing a galaxy along the plane of the disc, then one side will always be moving towards us and the other will always be moving away from us. From the difference between the Doppler shifts of the light from the two sides of the galaxy we can calculate the rotational speed from this measurement, we can determine how the tangential velocity of the galaxy depends of the distance from its centre it is observed that the tangential velocity found to be constant throughout the visible part of the galaxy. See figure 2.8.

In other words the results obtained are not consistent with Kepler's law. In order to maintain the status quo of Kepler's law as well as the observational evidence that v is independent of r, there is a simple way. if we assume that M is a linear function of r, both are satisfied. But this assumption is not correct because most of the masses are concentrated in the central region of galaxy.

To resolve this it has been concluded that there is a large quantity of invisible matter in galaxy which contributes to gravitational force. This invisible matter is called dark matter. To supply the required gravitational force this dark matter must have atleast 10 times the mass of the visible matter in the galaxy. That is more than 90% of the matter in the galaxy is in the invisible form.

Dark Matter

Dark matter is a term used to describe the invisible form of matter that can be inferred to exist from its gravitational effects, but does not emit or absorb detectable amount of light.

The adjective dark is used since it (almost) neither emits nor absorbs electromagnetic radiations.

According to the present cosmological evidence, **dark matter is that something which provides the gravitational attractive force that keeps this together and explains how the cohesion of stars, galaxies and even the galactic clusters is possible.**

The evidence for the existence of dark matter comes from the observation of light from distant galaxies that passes by a cluster of galaxies on its way to earth. This light is deflected by the gravitational field of the cluster in a process known as gravitational lensing. From the amount of deflection of the light, it is possible to deduce the quantity of matter in the cluster. The results of these observations show that there is much more matter in the cluster than we would deduce from the luminous matter alone provide evidence for the existence of dark matter.

Finally we discuss what kind of objects make up this dark matter. There two speculation in this regard. One is that dark matter is a Massive Compact Halo Objects (MACHO) which includes black holes, neutron stars, white dwarfs, brown dwarfs etc. The other is that dark matter is a Weakly Interacting Massive Particles(WIMPS) which include neutrinos, magnetic monopoles etc. The major difference between the two types of objects is that MACHOs are made from baryons while WIMPS are non-baryonic matter current theories suggest that the most of the dark matter is of the non-baryonic form. But no examples of this types of matter have yet been produced in any laboratory on earth except neutrions.

It is not nice to go without mentioning dark energy though it is not in the syllubs.

Dark energy

It is a hypothetical form of energy that permeats all space and enters a negative pressure, so as the universe expands, the pressure increase and causes the universe to expand at an ever increasing rate.

Dark energy is a kind of repulsive gravity, actually pushing the universe apart. The effect of dark energy is small for objects of the size of galaxies and stars but is critical for understanding the large scale structure of the universe.

Are dark matter and dark energy related

It is natural to conjecture the dark matter and dark energy as two different manifestations of the same physical quantity according to Einstein's mass energy equivalence. But the two do not seem to be related to each other.

Dark matter is the force that keeps the universe together and explains how the cohesion of stars, galaxies and even the galactic clusters is possible. Dark energy, on the other hand, is the force responsible for the acceleration of the expansion of the universe at an ever increasing rate. The influence of dark matter is attractive where as that of dark energy is repulsive. Thus dark matter and dark energy are two competing forces in our universe.

Cosmology and general relativity

So far we have been dealing with general theory of relativity and its predictions. Now we discuss another area where general relativity has proved to be useful almost from its inception. This is **the branch of astronomy dealing with the large scale structure of the universe as a whole: briefly called cosmology.**

Cosmology deals with the study of the universe on the large scale, including its origin, evolution and future. For this study we require relativity (both), quantum theory, the fundamental results from atomic and molecular physics, statistical physics, thermodynamics, nuclear physics and particle physics.

In general relativity the governing equation is the Einstein's field equation, which reads

$$\text{Curvature of space} = \frac{8\pi G}{c^4}\text{ mass energy density.}$$

This equation describes the entire universe. Since our aim is to draw cosmological predictions, we are not interested in the local variations in energy density but rather in the large scale variations of the energy density. The average density of the entire universe over a distance that is large compared with the spacing between galaxies is called large scale variations in the density.

It is due to the expansion of the universe, the density of the universe is not a constant i.e, density ρ is a function of time. Solving Einstein's field equation for the large scale structure of the universe we get

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi}{3}G\rho R^2 - Kc^2$$

This is known as Friedmann equation. Here R(t) represents the size of the universe at time t, ρ represents the mass-energy density of the universe at time t. K is a constant which gives geometrical structure of the universe.

For $K = 0$, the universe is flat

$K = 1$, the universe is curved and closed

$k = -1$, the universe is curved and open.

To solve Friedmann equation we have to specify the value of K. On the large scale; our universe seems to be flat, so we take $K = 0$.

$$\therefore \quad \left(\frac{dR}{dt}\right)^2 = \frac{8\pi}{3}G\rho R^2 \qquad \text{.....(1)}$$

The mass energy density ρ includes both the matter and radiation present in the universe. But the present universe is dominated by matter and the contribution of radiation to ρ is negligible.

$$\rho_m = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$$

As M is constant $\rho_m \; \alpha \; R^{-3}$

Putting this value in the above equation (1), we get

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi}{3} G \; \frac{M}{\frac{4}{3}\pi R^3} R^2$$

$$\left(\frac{dR}{dt}\right)^2 = 2\frac{GM}{R}$$

or
$$\frac{dR}{dt} = \frac{\sqrt{2GM}}{R^{\frac{1}{2}}}$$

$$R^{\frac{1}{2}} dR = \sqrt{2GM}\; dt$$

Integrating, we get

$$\frac{R^{\frac{3}{2}}}{\frac{3}{2}} = \sqrt{2GM}\; t$$

or
$$R^{\frac{3}{2}} = \frac{3}{2}\sqrt{2GM}\; t$$

$$R(t) = \left(\left(\frac{3}{2}\right)\sqrt{2GM}\right)^{\frac{2}{3}} t^{\frac{2}{3}} = At^{\frac{2}{3}}$$

$\therefore$
$$\frac{dR}{dt} = \frac{2}{3} At^{\frac{-1}{3}}$$

Putting the value of R(t) and $\frac{dR}{dt}$ in equation (1), we get

$$\left(\frac{2}{3}A\,t^{\frac{-1}{3}}\right)^2 = \frac{8\pi}{3}G\rho_m\,A^2\,t^{4/3}$$

$$\frac{4}{9}A^2\,t^{\frac{-2}{3}} = \frac{8\pi}{3}G\rho_m\,A^2\,t^{4/3}$$

$$\therefore \qquad t = \frac{1}{\sqrt{6\pi G\rho_m}}.$$ This is the expression for the matter dominated universe.

In contrast, the early universe was dominated by radiation, the contribution to ρ from matter is negligible. Here radiation density $\rho_r \;\alpha\; R^{-4}$. This is because from Plank's radiation law

energy density $u \;\alpha\; \dfrac{d\lambda}{\lambda^5}$

when all the wavelengths scale with R $d\lambda \;\alpha\; R$ and $\lambda^5 \;\alpha\; R^5$

so $\rho_r \;\alpha\; R^{-4}$

proceeding as before we get

$$R(t) = A'\,t^{\frac{1}{2}}$$

and $$t = \sqrt{\frac{32}{\pi G\rho_r}}.$$ This is the expression for the radiation dominated universe.

Now we introduce the Hubble parameter which can be defined as time variation of the scale factor R.

i.e., $$H = \frac{1}{R}\frac{dR}{dt}$$

If the universe has been expanding at a constant rate, i.e., $R \;\alpha\; t$

or $$R = Bt$$

$$\frac{dR}{dt} = B$$

$$\therefore \qquad H = \frac{1}{R}\frac{dR}{dt} = \frac{B}{Bt} = \frac{1}{t}$$

or $$t = H^{-1}$$

In this case the age of the universe is the reciprocal of the Hubbles parameter. For the matter dominated universe, we have

$$R(t) = A\,t^{2/3}$$

$$\frac{dR}{dt} = A\cdot\frac{2}{3}\,t^{-1/3}$$

$$\therefore \qquad H = \frac{1}{R}\frac{dR}{dt} = \frac{A\cdot\frac{2}{3}\,t^{-1/3}}{A\,t^{2/3}}$$

$$H = \frac{2}{3t}$$

or $$t = \frac{2}{3}H^{-1}$$

For the radiation dominated universe, we have

$$R(t) = A^{1/2}\,t^{1/2}$$

we get $$t = \frac{1}{2}H^{-1}$$

This shows that in either case we can take H^{-1} as rough estimate measure of the age of the universe at any time.

The above discussion shows that we can characterise the universe by several parameter such as K, ρ, R(t) and H. The challenge to the observational astronomer is to obtain data on the distribution and motion of the stars and galaxies that can be analysed to obtain values for the above parameters.

The big bang cosmology

Our present universe is characterised by a relatively low temperature (2.7K) and a low density of particles. In other words present universe is matter dominated. This structure and evolution of the universe are dominated by gravitational force. As our universe is expanding and cooling, in the distant past universe must have been characterised by a higher temperature and greater density of particles. Let us imagine that we go back in time and examine the universe at earlier times, even before the formation of stars and galaxies. At some time in the history, the temperature of universe must have been high to ionise atoms. At that time universe consisted of a plasma of electrons and positive ions and electromagnetic force was important in determining the structure of the universe. As we go back again in time the temperature of the universe must have been enough to ionise nuclie. This time universe consisted of electrons, protons and neutrons along with radiation. In this time the strong nuclear force was important in determining the evolution of the universe. At still earlier times the weak interaction played a significant role.

If we try to go back in time still further universe consists of leptons and quarks. In this time quarks, leptons and radiations were important in determining the evolution of the universe. Since we do not know much about quarks and their interaction. So we can't describe this very early state of the universe. Quarks and their interaction are now not subjects of particle physicists and they are in the hot pursuit of discovering it. The signatures of quarks have been observed at Geneva-Switzerland some year back. We now strongly believe that one day we will be able to understand the interactions and their properties completely, then we can go back still in earlier times. **Eventually we reach a fundamental barrier when the universe had an age of 10^{-43}s, which is known as Planck time. None of the theories in physics gives us any clue about the structure of the universe before Planck time.**

Later than Planck time but before the condensation of bulk matter the universe consisted of particles, antiparticles and radiation was almost in thermal equilibrium temperature T. The universe at this time is radiation dominated. To evaluate the temperature T at this time we make use of the relation.

$$t = \sqrt{\frac{3}{32\pi G\rho_r}}$$

Using $\rho_r = \frac{u}{c^2}$ and $u = \frac{8\pi^5 k^4}{15\, c^3 h^3} T^4$

or $$\rho_r = \frac{8\pi^5 k^4 T^4}{15\, c^5 h^3}$$

$$\therefore \quad t = \sqrt{\frac{3\times 15 c^5 h^3}{32\pi G\; 8\pi^5 k^4 T^4}}$$

$$t = \sqrt{\frac{45}{256}\,\frac{c^5 h^3}{G\pi^6 k^4}\cdot\frac{1}{T^2}}$$

$$\therefore \qquad \left(\frac{45}{256}\frac{c^5h^3}{G\pi^6k^4}\right)^{1/4}\cdot\frac{1}{t^{1/2}}=\frac{A}{t^{1/2}}\text{ kelvin}$$

substituting the values of c, h, G and k

we get $\quad A \simeq 1.5\times10^{10}$

$$\therefore \qquad T=\frac{1.5\times10^{10}}{t^{1/2}}K$$

The equation relates the age of the early universe to its temperature.

The radiation of the early universe consisted of high energy photons, whose average energy is estimated as kT. The interactions between the radiation and matter can be represented by our familiar processes.

Photons $\rightarrow$ particle + antiparticle

Particle + Antiparticle $\rightarrow$ Photons

For example $\qquad 2\gamma \rightarrow e^+ + e^-$

$$e^+ + e^- \rightarrow 2\gamma$$

From conservation of energy and momentum

$$2m_ec^2 = 2h\upsilon$$

or $\qquad m_ec^2 = h\upsilon = 0.511\text{ MeV},$ minimum value of photon

Example 5

At what age did the universe cool below the threshold temperature for (a) nucleon production (b) pi-meson production. $m_p = 1.67\times10^{-27}$ kg and $m_\pi = 140$MeV

Solution

a) To calculate the threshold frequency, we can make use of

$$mc^2 = kT$$

$$T=\frac{mc^2}{k}=\frac{1.67\times10^{-27}\times(3\times10^8)^2}{1.38\times10^{-23}}$$

$$T=\frac{1.67\times9\times10^{-11}}{1.38\times10^{-23}}=1.09\times10^{13}K$$

Using $T = \frac{1.5\times10^{10}}{t^{1/2}}$

$$t = \frac{(1.5\times10^{10})^2}{T^2} = \frac{2.25\times10^{20}}{1.09^2\times10^{26}}$$

$$t = 1.89\times10^{-6}s$$

b) The mass of pi-meson $m_\pi = 140$ MeV

$$\therefore \quad T = \frac{mc^2}{k} = \frac{140\times10^6 eV}{8.617\times10^{-5} eV/K}$$

$$T = 1.62\times10^{12} K$$

Age of the universe $t = \frac{(1.5\times10^{10})^2}{(1.62\times10^{12})^2}$

$$t = \frac{2.25\times10^{20}}{1.62^2\times10^{24}} = \frac{2.25}{1.62^2}\times10^{-4}s$$

$$t = 8.57\times10^{-5}s$$

Example 6

At what temperature was the universe hot enough to permit the photons to produce K mesons (500 MeV) ? At what age did the universe have this temperature.

Solution

$$T = \frac{mc^2}{k} = \frac{500\times10^6 eV}{8.617\times10^{-5} eVK^{-1}} = 5.8\times10^{12} K.$$

$$t = \left(\frac{1.5\times10^{10}}{5.8\times10^{12}}\right)^2 = 6.68\times10^{-6}s$$

We begin to analyse the evolution of the universe from $1\mu s$ (age of the universe)

1) At $t = 1\mu s$

The temperature of the universe can be calculated to be $T = \frac{1.5\times10^{10}}{t^{1/2}} K$

i.e., $$T = \frac{1.5 \times 10^{10}}{(10^{-6})^{1/2}} = 1.5 \times 10^{13} K$$

The energy of the particles is = kT

$$= 8.617 \times 10^{-5} \times 1.5 \times 10^{13} eV$$

$$\simeq 1300 MeV$$

At present the temperature of the background radiation is 2.7K and the observable radius of the universe is 10^{26} m. At $1\mu s$, the temperature is raised to 1.5×10^{13}K.

At 2.7 K, the size is 10^{26} m

At 1K, the size is $10^{26} \times 2.7$

At 1.5×10^{13}K, the size is $\dfrac{10^{26} \times 2.7}{1.5 \times 10^{13}}$

$$= 1.8 \times 10^{13} m$$

So the size of the universe at $1\mu s$ is about the present size of the solar system (10^{13}m). At energy about 1300MeV, the particle present are $p, \bar{p}, n, \bar{n}, e^-, e^+, \mu^+, \bar{\mu}, \pi^0, \pi^-, \pi^+$, photons, neutrinos and antineutrinos etc. So particle creation and annihilation may occur, so the number of particles is almost equal to the number of antiparticles. Further, the number of photons is roughly equal to the number of nucleons, which in turn roughly equal to the number of electrons. But the relative number of protons and neutrons is determined by three factors.

(i) Boltzmann factor $e^{-\frac{\Delta E}{kT}}$ decides which one is larger in number.

$$\Delta E = (m_n - m_p)c^2 = (939.56 - 938.27) MeV \simeq 1.3 MeV$$

$$kT = 8.617 \times 10^{-15} \frac{eV}{K} \cdot 1.5 \times 10^{13} K = 13 \times 10^8 eV$$

$$kT = 1300 MeV. \text{ So } e^{-\frac{\Delta E}{kT}} = e^{-\frac{1}{1000}} = 0.999 = 1$$

Thus number of the protons is almost equal to the number of neutrons.

(ii) The nuclear reactions such as $n + \upsilon_e \rightleftharpoons p + e^-$, $n + e^+ \rightleftharpoons p + \bar{\upsilon}_e$ neutrons and protons are produced in both directions as long as there are plenty of e^-, e^+, υ_e and $\bar{\upsilon}_e$

(iii) The neutron decay time is about 10 minutes. So when $t < 1s$, there is not enough time for the neutron to decay. All the above three factors keep the neutron to proton ratio close to 1.

2) At $t = 10^{-2}s$ we can calculate $T = 1.5 \times 10^{11} K$ and $kT = 13MeV$. This shows that photons have too little energy to produce pions and muons and because of pion and muon life times are much shorter than $10^{-2}s$, they might have decayed into electrons, positrons and neutrinos. At this energy pair production of nucleons and antinucleons no longer occurs, but nucleon-antinucleon annihilation continues. But pair production of electrons and positrons can still occur. So at $t = 10^{-2}s$, universe consists of p, n, e^-, e^+, photons and neutrinos. The neutron to proton ratio remains almost about 1.

3) At $t = 1s$

The temperature of the universe becomes $T = 1.5 \times 10^{10} K$ and $kT = 1.3MeV$.

The neutron - proton ratio

$$\frac{N_n}{N_p} = e^{-\frac{\Delta E}{kT}} = e^{-\frac{1.3MeV}{1.3MeV}} = e^{-1} = 0.37$$

$\therefore$ The relatative number of protons $\dfrac{N_p}{N_p + N_n} = \dfrac{1}{1 + \dfrac{N_n}{N_p}} = \dfrac{1}{1 + 0.37} = 0.73$

This shows that there are 73% protons and 27% neutrons. During this period the influence of the neutrinos has been decreasing. To convert a proton to a neutron by capturing an antineutrino $(\bar{\upsilon}_e + p \rightarrow n + e^+)$, antineutrino having energy atleast 1.8 MeV above the mean neutrino energy (1.3MeV) at this temperature. This begins the time of neutrino decoupling. Since the capturing of neutrinos no longer occur, it begins to fill the entire universe and neutrinos cool along with the expansion of the universe. These primordial neutrinos presently have almost the same density of microwave photons but a slightly lower temperature t = 2K.

4) At $t = 6s$

Now the temperature of the universe becomes $T = \frac{1.5\times10^{10}}{6^{1/2}} K = 6.12\times10^{9} K$ and the average energy

$$kT = 8.617\times10^{-5}\times6.12\times10^{9} = 4.99\times10^{-5}$$
$$= 0.499 \text{ MeV}$$

This energy is not sufficient to produce electron-positron pair. But electron-positron annihilation continues and almost all positrons and 99.999% electrons ares are annihilated. Since the electrons have too little energy (0.499 MeV), conversion of proton to neutron cannot occur. However the weak interaction process that influences the relative number of protons and neutrons is the radioactive decay of the neutron. Since the decay time of neutron is 10 minutes this will not occur. Now there are about 84% protons and 16% neutrons. i.e., proton number is about 5 times the neutrons at this stage there are no remaining positrons or antinucleons because particle-anti particle annihilation has reduced the number of nucleons while the photon number remains the same and there are about $10^{9}N$ photons and the same number of neutrinos.

i.e., at $t = 6s$

The number of protons $= N$

The number of electrons $= N$

The number of neutrons $= \frac{N}{5}$

The number of photons $= 10^{9}N$

The number of neutrions $= 10^{9}N$

The formation of nuclei and atoms

In this section we discuss at what temperature and age of the universe, the formation of nuclei and atoms begin to occur.

We found that the age of the universe goes from 10^{-6}s to 6s all the time protons, neutrons and photons were present. suppose neutrons and protons collide it is possible to form deuterons (^{2}H).

i.e., $n + p \rightarrow {}^{2}H + \gamma$

and also $\gamma + {}^{2}H \rightarrow n + p$.

we know that the binding energy of deuteron is 2.22 MeV. Thus in order to form deuterons the energy of photons must be less than 2.22 MeV. If the photons have energy greater than 2.22 MeV it will break up the deuterons formed. The temperature corresponding to 2.22 MeV is,

$$T = \frac{2.22\text{MeV}}{8.617\times10^{-5}}\text{eVK}^{-1}$$

$$T = \frac{2.22\times10^{6}}{8.617\times10^{-5}} = 2.57\times10^{10}\text{K}$$

$$T \simeq 2.5\times10^{10}\text{K}$$

This shows that when the temperature falls below 2.5×10^{15}K stable deuterons will be formed. However this does not occur. This is because photons do not have single energy.

The photon energy distribution is a spectrum. A small fraction of photons has energies above 2.22 MeV and this will continue to break the deuterons. See figure 2.9.

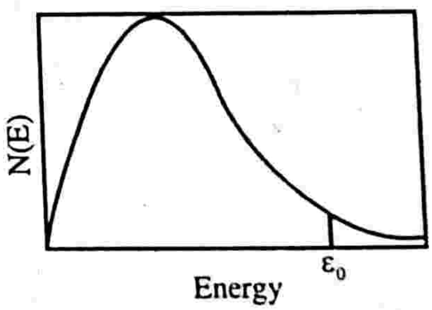


Figure 2.9: The thermal radiation spectrum. The photons above $\varepsilon_0 = 2.22$ MeV are sufficiently energetic to break apart deuterium unclei

Before matter and antimatter annihilation occured, there were about as many photons as nucleons and antinucleons, but after $t = 10^{-2}$s the ratio of nucleons to photons is about 10^{-9}. Out of these about $\frac{1}{6}\times10^{-6}$ of the nucleons are neutrons. If the fraction of photons above 2.22 MeV is greater than $\frac{1}{6}\times10^{-6}$, there will be atleast one energetic photon per neutron, which effectively prevents deuteron formation. Our next aim is to calculate to what temperature the photons must cool before fewer than $\frac{1}{6}\times10^{-9}$ of them are above 2.22 MeV.

We have the relation

$$n(\upsilon)d\upsilon = \frac{8\pi\upsilon^2 V d\upsilon}{c^3}\frac{1}{e^{h\upsilon/KT}}-1$$

$$\frac{n(\upsilon)d\upsilon}{V} = \frac{8\pi\upsilon^2 d\upsilon}{c^3 e^{h\upsilon/Kt} - 1}$$

Put $h\upsilon = \varepsilon$, $hd\upsilon = d\varepsilon$

$$\frac{n(\upsilon)d\upsilon}{V} = \frac{8\pi\ \varepsilon^2\ d\varepsilon}{h^3c^3} \quad \frac{1}{e^{\frac{\varepsilon}{kT}} - 1}$$

since $\quad \varepsilon \gg kT, \ \frac{1}{e^{\frac{\varepsilon}{kT}} - 1} \simeq e^{-\frac{\varepsilon}{kT}}$

$$\frac{n(\upsilon)d\upsilon}{V} = \frac{8\pi\ \varepsilon^2}{h^3c^3}\ e^{-\frac{\varepsilon}{kT}} d\varepsilon$$

Integrating this we get the total number density

$$\frac{N}{V} = \int_0^\infty \frac{n(\upsilon)d\upsilon}{V} = \int_0^\infty \frac{8\pi\ \varepsilon^2}{h^3c^3}\ e^{-\frac{\varepsilon}{kT}} d\varepsilon$$

$$\frac{N}{V} = \int_0^{\varepsilon_0} \frac{8\pi\ \varepsilon^2\ e^{-\frac{\varepsilon}{kT}} d\varepsilon}{h^3c^3} + \int_{\varepsilon_0}^\infty \frac{8\pi}{h^3c^3} \varepsilon^2 e^{-\frac{\varepsilon}{kT}} d\varepsilon$$

Where $\varepsilon_0 = 2.22$ MeV

Thus

$$\left(\frac{N}{V}\right)_{\varepsilon > \varepsilon_0} = \frac{8\pi}{h^3c^3} \int_{\varepsilon_0}^\infty \varepsilon^2 e^{-\frac{\varepsilon}{kT}} d\varepsilon$$

Integrating by parts we get

$$\left(\frac{N}{V}\right)_{\varepsilon > \varepsilon_0} = \frac{8\pi}{h^3c^3} k^3T^3\ e^{-\frac{\varepsilon_0}{kT}} \left[\left(\frac{\varepsilon_0}{kT}\right)^2 + \frac{2\varepsilon_0}{kT} + 2\right]$$

Recall our expression for $\frac{N}{V} = \frac{19.23\ \pi\ k^3\ T^3}{c^3\ h^3}$

$$\therefore \qquad \frac{\left(\frac{N}{V}\right)_{\varepsilon>\varepsilon_0}}{\left(\frac{N}{V}\right)} = \frac{8}{19.23} e^{-\frac{\varepsilon_0}{kT}} \left[\left(\frac{\varepsilon_0}{kT}\right)^2 + 2\frac{\varepsilon_0}{kT} + 2 \right]$$

$$= 0.42\, e^{-\frac{\varepsilon_0}{kT}} \left[\left(\frac{\varepsilon_0}{kT}\right)^2 + 2\frac{\varepsilon_0}{kT} + 2 \right]$$

To prevent deuteron formation, the fraction must be $\frac{1}{6} \times 10^{-9}$

$$\therefore \qquad \frac{1}{6} \times 10^{-9} = 0.42 e^{-\frac{\varepsilon_0}{kT}} \left[\left(\frac{\varepsilon_0}{kT}\right)^2 + 2\frac{\varepsilon_0}{kT} + 2 \right]$$

Put $\quad \frac{\varepsilon_0}{kT} = x$

$$\frac{1}{6} \times 10^{-9} = 0.42 e^{-x} (x^2 + 2x + 2)$$

or

$$e^x = 6 \times 0.42 \times 10^9 (x^2 + 2x + 2)$$

$$e^x = 2.52 \times 10^9 (x^2 + 2x + 2)$$

Solving this we get $x = 28$

Thus, $\quad \frac{\varepsilon_0}{kT} = 28$

$$T = \frac{\varepsilon_0}{k \times 28} = \frac{2.22 \text{MeV}}{8.617 \times 10^{-5} \frac{\text{eV}}{\text{K}} \times 28}$$

$$T = \frac{2.22 \times 10^6}{8.617 \times 10^{-5} \times 28} = \frac{222 \times 10^9}{8.617 \times 28} \text{K}$$

$$T = 9.2 \times 10^8 \text{K} = 9 \times 10^8 \text{K}$$

This shows that when $T > 9\times10^{8}K$, the number of photons with $\varepsilon > 2.22MeV$ the deuteron formation is prevented. When $T < 9\times10^{8}K$ deuterons are produced.

The age of the universe corresponding to this temperature

$$t = \left(\frac{1.5\times10^{10}}{T}\right)^2$$

$$t = \left(\frac{1.5\times10^{10}}{9\times10^{8}}\right)^2 = 277s \simeq 250s$$

This shows that after about 250s deuterons begin to form. Then deuterons react with protons and neutrons available to give helium and tritium.

$$^{2}H + p \rightarrow {}^{3}He + \gamma$$

and $$^{2}H + n \rightarrow {}^{3}H + \gamma$$

The energies of the formation of these nuclei are 5.49 MeV and 6.26 MeV respectively. The reaction continues to give ^{4}He

$$^{3}He + n \rightarrow {}^{4}He + \gamma$$

and $$^{3}H + p \rightarrow {}^{4}He + \gamma$$

There are no stable nuclei with A = 5, so no further reactions of this sort are possible. It is also not possible to combine two helium ^{4}He to form beryllium ^{8}Be. Since Be is highly unstable. It would be possible to form stable ^{6}Li and ^{7}Li, but these are very small in quantities relative to H and He. From Li further reactions are possible such as $^{7}Li + {}^{4}He \rightarrow {}^{11}B$ and so forth, but these occur in still small quantities. The end products ^{2}H and He along with the left over original protons make up about 99.9999% of the nuclei after the era of nuclear reactions.

When the age of the universe is t = 250s, the original 16% neutrons present at t = 6s had beta-decayed to about 12%, leaving 8% protons. Most of the ^{2}H, ^{3}H and ^{3}He were transformed into heavier nuclei. So we can assume the universe to be composed mostly of ^{1}H and ^{4}He nuclei. Of the N nucleons present at t = 250s, 12% (0.12N) were neutrons and 0.88N were protons. The 0.12N neutrons combined with 0.12N protons forming 0.06N ^{4}He and leaving $0.88N - 0.12N = 0.76N$ protons.

The universe then consisted of 0.82N nuclei, of which 0.06N (7.3%) were ^4He and 0.76N (92.7%) were protons. Helium is about four times as massive as hydrogen. So by mass the universe is about 24% helium.

At this point the universe began long and uneventful period of cooling. during which the strong interactions ceased to be of importance.

The final step in the evolution of the primitive universe is the formation of neutral hydrogen and helium atoms from the 1 H, 2 H, 3He and 4He and the free electrons. In the case of hydrogen, this takes place when the photon energy drops below 13.6eV. Otherwise atoms will be ionised by the radiation. There are still about 10^9 photons for every proton and so we must wait for the radiation to cool until the fraction of photons above 13.6eV is less than about 10^{-9}. As before we can solve for T, for the fraction $F = 10^{-9}$. We get $\frac{\varepsilon_0}{kT} = 6$ with $\varepsilon_0 = 13.6\text{eV}$. $\therefore T = 6070\text{K}$ which occur at time $6.1 \times 10^{12}\text{s} = 190,0000$ years. This estimate is only a rough one, because for those calculation we considered only energy density of the radiation present in the universe. But as the universe cools, the contribution of matter to the energy density becomes more significant, so the temperature drops more slowly than we would estimate. This contribution may increase this time by about a factor o 2 to about 3,80,000 years, and the radiation temperature is decreased by about a factor of $\sqrt{2}$, to $T = 4300$ K.

After the formation of neutral atoms, there are virtually no charged particles left in the universe and the radiation left is not energetic enough to ionise the atoms. This is the time of decoupling of the radiation field from the matter and now the electro magnetism, the third of our basic forces is no longer important in shaping the evolution of the universe. This point onwards gravity plays its role to shape the universe.

The time after 380,000 years had been comparatively uneventful, from the point of view of cosmology. Density fluctuations of the hydrogen and helium triggered the condensation of galaxies and then first generation stars were born. Supernova explosions of the material from these stars permitted the formation of second generation systems, among which planets formed from the rocky debris.

Meanwhile, the decoupled radiation field, unaffected by the gravitational coming and going matter, began the long journey that eventually took it cooled again by a factor of 1600. Then everything could be observed by the radio telescope of 20th century.

The remarkable story of the evolution of the universe are summarised in figure below.

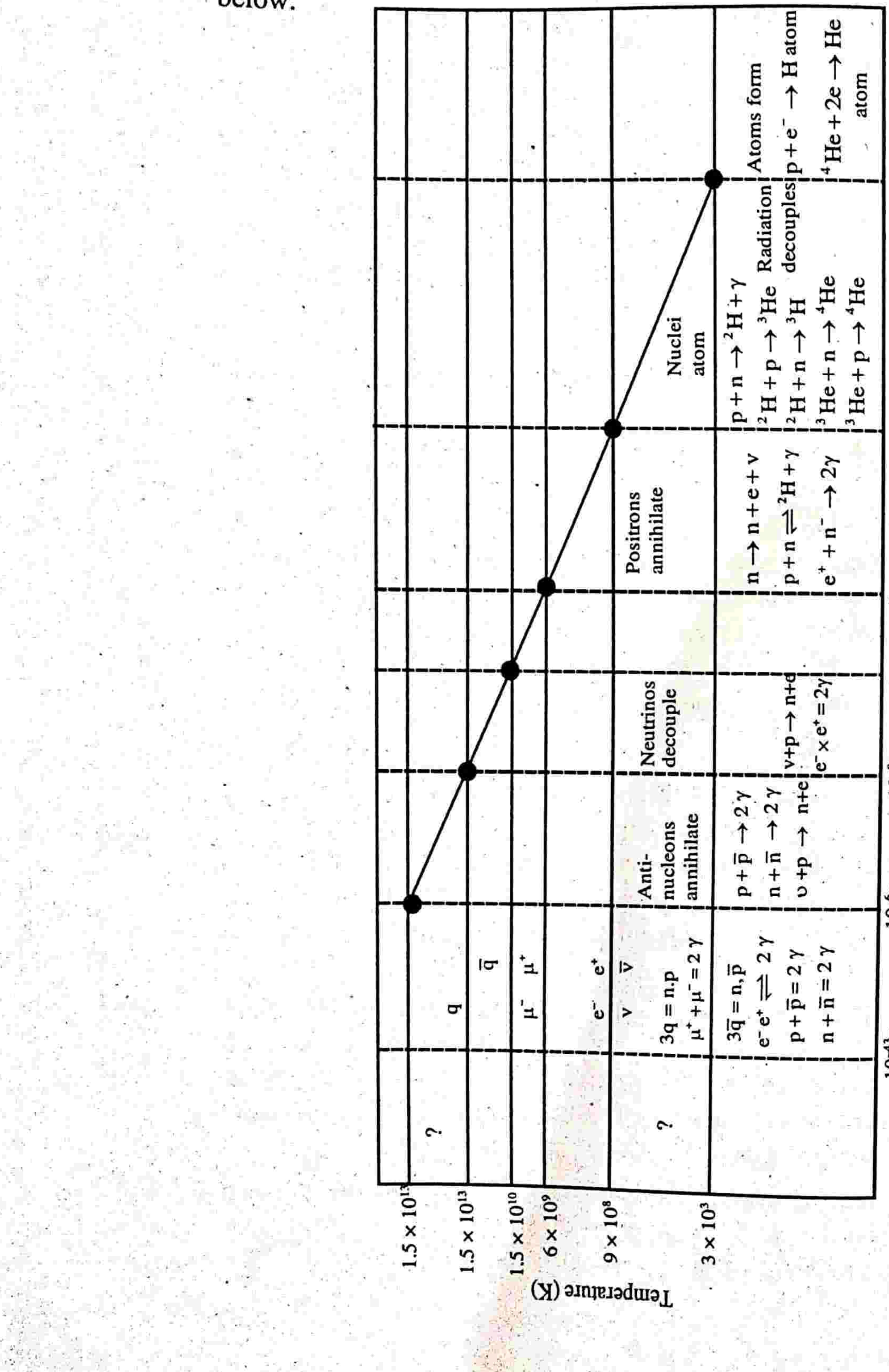


Figure 2.10: Evolution of the universe according to the Big Bang cosmology

IMPORTANT FORMULAE

1. Doppler shift in frequency of light

$$\frac{\upsilon' - \upsilon}{\upsilon} = \frac{\Delta v}{c}$$

2. Expression for frequency shift when a light wave falling in the earths gravity.

(i) $\frac{\Delta\upsilon}{\upsilon} = \frac{mgH}{mc^2}$

(ii) $\frac{\Delta\upsilon}{\upsilon} = -\frac{GM}{Rc^2}$

(iii) $\frac{\Delta\lambda}{\lambda} = \frac{GM}{Rc^2}$

3. Relation between space – time curvature and energy momentum.

Curvature of space $= \frac{8\pi G}{c^2}$ energy momentum.

4. Relativistic Doppler effect of light

$$\lambda' = \lambda\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

5. Hubble's law:

$V = H_0 d, \quad V = H_o = 72\text{kms}^{-1}(\text{Mpc})^{-1}$

6. Planck's radiation law:

$$n(\upsilon)d\upsilon = \frac{8\pi V}{c^3}\upsilon^2 \frac{d\upsilon}{e^{\frac{h\upsilon}{KT}} - 1}$$

$$u(\upsilon)d\upsilon = \frac{8\pi h}{c^3}\frac{\upsilon^3 d\upsilon}{e^{\frac{h\upsilon}{KT}} - 1}$$

$$u(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{\frac{h\upsilon}{\lambda KT}} - 1}$$

7. Average energy of the photon:

$$\bar{E} = \frac{8}{15 \times 9.23}\pi^4 kT$$

10. Friedmann equation:

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi}{3}G\rho R^2 - Kc^2$$

$K = 0,$ the universe is flate

$K = 1,$ the universe is curved and closed

$K = -1,$ the universe is curved and open

11. For a flat universe – Matter dominated.

$$R(t) = At^{2/3}$$

12. Relation between Hubble parameter and size of the universe.

$$H = \frac{1}{R}\frac{dR}{dt}$$

13. For the radiation dominated universe.

$$R(t) = A^{1/2}t^{1/2}.$$

14. Rought estimate of the age of the universe.

$$t = \frac{1}{H}$$

15. Relation between age and temperature of the universe.

$$T = \frac{1.5\times10^{10}}{t^{1/2}}$$

16. Relation between mass of the particle produced and temperature of the universe.

$$mc^2 = kT$$

17. Boltzsmann factor $= e^{-\frac{\Delta E}{KT}} = \frac{N_n}{N_p}$

where $\qquad \Delta E = (m_n - m_p)c^2$

UNIVERSITY MODEL QUESTIONS

Section A

(*Answer questions in about* **two** *or* **three** *sentences*)

Short answer type questions

1. What is meant by principle of covariance?
2. What is meant by principle of equivalence?
3. What is general theory of relativity?
4. What were the predictions of general theory of relativity?
5. Briefly explain how accelerated frames can take into account the effect of gravity.

6. Write down the equation for frequency shift when a light wave falling in earth's gravity and explain the symbols used.
7. What is gravitational red shift?
8. How does gravity affect the space time in general relativity.
9. How does the motion of a particle occurs in space-time according to general theory of relativity.
10. Write down the relation between curvature of space and energy-momentum in general relativity.
11. Briefly explain how does Einstein's field equation become Newtons law of gravitation.
12. Distinguish between Newton's law of gravitation and Einstein's law of gravitation.
13. How does light bend in gravitational field?
14. What is cosmological constant?
15. Solving field equations Einstin got an expanding universe. What was the mistake he committed?
16. What were the two assumptions made by Friedmann to arrive at the conclusion that the whole universe is expanding?
17. How did we arrive at the conclusion that the universe is expanding?
18. What is red shift?
19. What is blue shift?
20. What is Hubble's law?
21. What is the relation between recessional speed and distance.
22. Demonstrate the expanding universe with an analogy.
23. What is cosmic microwave background radiation?
24. How did Gamow arrive at the prediction of the cosmic microwave background radiation.
25. What was the contribution of Penzias and Wilson with regard to microwave background radiation.
26. Name two satellites sent into the earths orbit to study microwave background radiation.
27. Is microwave background radiation a reality? Justify?
28. What is Bing-Bang.
29. What is the evidence of Bing-Bang.
30. What is dark matter?
31. What is the function of dark matter.
32 Distinguish between dark matter and dark energy.
33. What is cosmology.
34. What is Planck time.
35. Write down Friedmann equation and explain the symbols used.
36. Distinguish between matter dominated and radiation dominated universe.

37. What is bing-bang cosmology?
38. Write down the relation between age and temperature of the universe.
39. What is the connection between the formation of nuclei and atoms with age and temperature of universe.
40. What is the age of the universe when protons, neutrons were present?

Section B

(Answer questions in a paragraph of about half a page to one page)

Paragraph / Problem type questions

1. Explain how did Einstein arrive at the principle of equivalence.
2. Distinguish between S.T.R and G.T.R.
3. Derive an expression for gravitational frequency shift.
4. Distinguish between Newton's law of gravitation and Einstein's law of gravitation.
5. Explain three major prediations of general theory of relativity.
6. Explain the phenomenon of bending of light rays in the presence of gravitational field.
7. Explain the gravitational red shift and blue shift.
8. Show that inverse of Hubble's parameter is almost equal to the age of the universe.
9. Write a brief note on Friedmann universe and Einstein universe.
10. State and explain Hubble's law.
11. Explain how did Hubble arrived at the conclusion that the whole universe is expanding.
12. Write a brief note on COBE and WMAP.
13. The sun's mass is 2.0×10^{30} kg and its radius is 7.0×10^{-8} m. Find the approximate gravitational red shift in light of wavelength 500 nm emitted by the Sun. [1.06 pm]
14. Find the approximate gravitational red shift in 500 nm light emitted by a white dwarf star whose mass is that of the Sun but whose radius is that earth, 6.4×10^{6} m.
15. A satellite is in orbit at an altitude of 150 km, we wish to communicate with it using a radio signal of frequency 10^9 Hz. What is the gravitational change in frequency between a ground station and the satellite? Assume g is a constant. [1.63×10^{-2} Hz]
16. Light from a certain galaxy is red shifted so that the wavelength of one of its characteristics spectral lines is doubled. Assume the validity of Hubble's law, calculate the distance to this galaxy. [8.15×10^{3} ly]
17. From the expression for energy density of the thermal radiation.
    a) find the energy at which the maximum of the radiation energy spectrum occurs.
    b) Evaluate the peak photon energy of the 2.7K microwave background radiation.

$$\begin{bmatrix} E = 2.821\,kT \\ E = 6.64\times10^{-4}\,eV \end{bmatrix}$$

18. Evaluate $\int_0^\infty \frac{x^3}{e^x - 1} dx$ $\left[\frac{\pi^4}{15}\right]$

19. Calculate the age of the universe when deutrons were produced. Take $T < 9\times10^8 K$ [250 s]

20. What was the age of the universe when the nucleons consisted of 60% protons and 40% neutrons. Given that $\Delta E = (m_n - m_p)c^2 = 1.3 MeV$ [0.17 s]

Section C

(*Answer questions in about* **one** *or* **two** *pages*)

Long answer type questions

1. Derive an expression for the energy of the microwave background radiation.
2. Establish that the inverse of Hubbles parameter is a rough estimate of measure of the age of the universe.

Hints to solutions

13. $\frac{\lambda' - \lambda}{\lambda} = \frac{GM}{Rc^2} = \frac{6.67\times10^{-11}\times2\times10^{30}}{8} = 0.2117\times10^{-5}$

$\lambda' - \lambda = 0.2117\times10^{-5}\times500\times10^{-9} m$

$= 1.058\times10^{-12} m \simeq 1.06 m$ ·

14. $\lambda' - \lambda = \frac{GM}{Rc^2}\lambda = \frac{6.67\times10^{-17}\times2\times10^{30}\times500\times10^{-9}}{6.4\times10^6\times(3\times10^8)^2}$

$= 0.1157 nm$

15. $\frac{\Delta\upsilon}{\upsilon} = \frac{gH}{c^2} = \frac{9.8\times150\times10^3}{(3\times10^8)^2} = 16.33\times10^{-12}$

$\Delta\upsilon = 16.33\times10^{-12}\times10^9 = 1.633\times10^{-2} Hz.$

16. $\lambda' - \lambda = \sqrt{\frac{1+\frac{V}{c}}{1-\frac{V}{c}}}, \lambda' = 2\lambda$ (given)

$$2 = \sqrt{\frac{1+\frac{V}{c}}{1-\frac{V}{c}}} \quad \because \; V = \frac{3}{5}c$$

Using $v = H_o d$

$$d = \frac{v}{H_o} = \frac{3}{5}\frac{c}{H_o} = 8.15 \times 10^3 \text{ly}$$

Take $H_o = 72 \text{kms}^{-1}(\text{Mpc})^{-1}$

1 parsec = 3.08×10^{16}m

1 ly = 9.46×10^{15}m.

17. a) $u(\upsilon)d\upsilon = \dfrac{8\pi h}{c^3}\dfrac{\upsilon^3}{e^{\frac{h\upsilon}{kT}}-1}d\upsilon,\ \varepsilon = h\upsilon$

$$u(\varepsilon) = \frac{8\pi\varepsilon^3}{(hc)^3}\frac{1}{e^{\frac{\varepsilon}{kT}}-1}$$

Find $\dfrac{du}{d\varepsilon} = 0$, we get $e^{-x} = 1 - \dfrac{x}{3}$ where $x = \dfrac{\varepsilon}{kT}$ gives $x = 2.841$

$\varepsilon = 2.821kT$

b) Use $\varepsilon = 2.821kT$

18. See example 4

19. Use $t = \left(\dfrac{1.5 \times 10^{10}}{T}\right)^2$

20. $\dfrac{N_p}{N_n} = \dfrac{0.6}{0.4} = 1.5$

Use $\dfrac{N_n}{N_p} = e^{-\frac{\Delta E}{kT}}$

$$\frac{\Delta E}{kT} = \ln(1.5) = 0.41$$

$$T = \frac{\Delta E}{0.41k} = \frac{1.3\text{MeV}}{0.41 \times 8.62 \times 10^{-5}\text{eVK}^{-1}}$$

$T = 3.7 \times 10^{10}$K

Then use $t = \left(\dfrac{1.5 \times 10^{10}}{T}\right)^2 = 0.17$s