UNIT - III

4 FRAUNHOFER DIFFRACTION

Introduction

According to geometrical optics light-travels in straight lines which could be easily explained on the basis of Newton's corpuscular theory. But the wave theory of light in its original form as proposed by Huygen was not successful in explaining the observed phenomenon that light appears to travel in straight lines. A careful investigation of the fact however reveals that light suffers some deviation from its straight path in passing close to the edges of opaque obstacles and narrow slits. Hence some of the light does bend into the geometrical shadow and its intensity falls of rapidly. This deviation is extremely small when the wavelength of light is small in comparison to the dimensions of the obstacles or slit. But the deviation becomes noticeable when the dimensions of the obstacles or the slit are comparable to the wavelength of light. Thus mathematically one can express the condition for the failure of light to travel in straight lines as $\lambda \leq a$, where λ is the wavelength of light and a is the dimension of the obstacle.

For example, Let us suppose that wave of light diverging from a narrow slit S, pass an obstacle AB with a straight edge A parallel to the slit. If light travels in a straight line we must get perfect darkness below the point O and uniform illumination above the point O on the screen CD kept at a distance.

But on careful observation one can see that light enters into the geometrical shadow (below the point O) and intensity of light varies above the point O. This shows that, like sound, light bends at the corner of the obstacle. The term diffraction is given to all phenomena like this and this can be easily explained by wave nature of light.

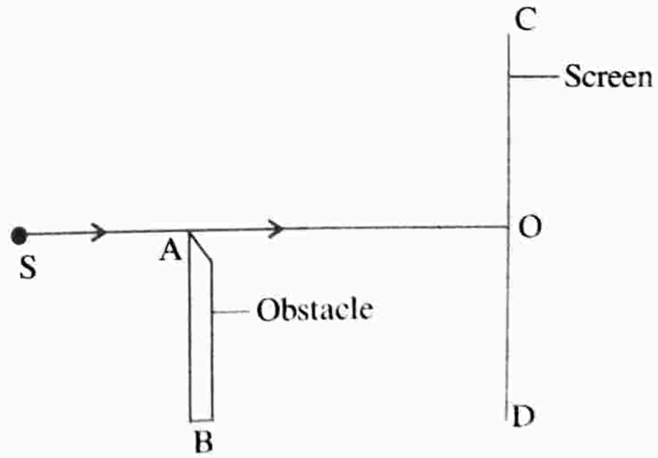


Figure 4.1

Thus, *diffraction is defined as the totality of the phenomena which are due to the wave nature of light and are observed in its propagation in media with sharply defined inhomogeneities.*

In its narrower sense *diffraction refers to the bending of light around small opaque obstacle and their encroachment within the geometrical shadow.*

The phenomenon of diffraction of light was first observed by Italian Scientist Grimaldi (1618-1668) in 1665 and was later on studied by Newton with the help of corpuscular theory. Newton attempted to interpret diffraction effect as due to attractive or repulsive forces exerted by edges of obstacles on flying corpuscles so as to deflect them from their rectilinear path, but he failed. In terms of wave theory the first attempt to explain the diffraction phenomenon was made by Thomas Young (1773-1829), who attributed to it to the interference between the direct light waves which pass near the edge of the obstacle and the wave of light reflected at grazing incidence from the edge. This could not explain the penetration of light within the geometrical shadow. Moreover according to Young's explanation the intensity distribution of light on the screen should depend upon (i) the sharpness of the edge (ii) its degree of polish and (iii) its material. But Augustin Jean Fresnel (1788 - 1827) in 1815 after a series of experiments proved that the diffraction pattern was independent of above three factors as long as the material of the obstacle is opaque.

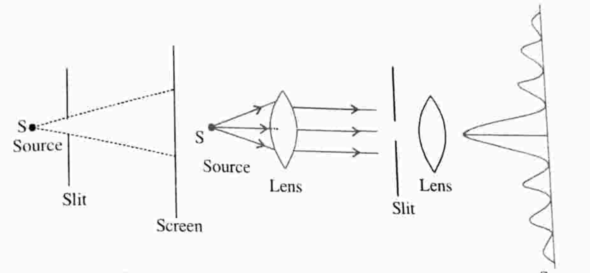
The phenomenon of diffraction was satisfactorily explained, by giving a rigorous mathematical treatment, by Fresnel. According to him *the phenomenon of diffraction is due to the mutual interference of secondary wavelets originating from various points of wave front which are not blocked off by an obstacle or allowed to pass through a slit.* That is instead of finding the new wave front by constructing the envelope of these secondary wavelets, we must find, by the principle of superposition, their resultant at every point of the screen taking account of their relative amplitudes and phases. Fresnel thus applied Huygens principle of secondary wavelets in conjunction with the principle of interference and calculated the position of fringes formed on the screen in general agreement with the observed diffraction pattern.

The diffraction phenomena are usually divided into two categories. (i) Fresnel diffraction and (ii) Fraunhofer diffraction.

In the Fresnel class of diffraction the source of light and screen are in general at a finite distance from the diffracting element (obstacle or slit). In other words the incident wave front and diffracted wave front are spherical or cylindrical. (see figure below) In the Fraunhofer class of diffraction the source and the screen are at infinite distances from the obstacle. In other words the incident wave front and diffracted wave front are plane. This can be easily achieved by placing the source on the focal plane of a convex lens and placing the screen on the focal plane of another convex

lens. The two lenses effectively move the source and the screen to infinity because the first lens makes the light beam parallel and the second lens makes the screen receive a parallel beam of light.

Owing to this it is easier to calculate the intensity distribution of a Fraunhofer diffraction pattern. Moreover it is easy to observe Fraunhofer diffraction pattern by using a spectrometer. The collimator renders a parallel beam of light and the telescope receives parallel beam of light in its focal plane. The diffracting aperture is placed on the prism table.



Fresnel diffraction **Fraunhofer diffraction**

Figure 4.2

Fraunhofer single slit diffraction

Consider a rectangular slit AB of width a. S is a narrow slit illuminated by a monochromatic light of wavelength λ. The slit S is kept at the focal plane of a converging lens L_1. The lens L_1 makes the wave front into a parallel wave front which falls on the slit AB. i.e. For the slit AB the source S is at infinity. The light after passing through AB is incident on the converging lens L_2 and is focussed on a screen XY kept on the focal plane of the lens L_2.

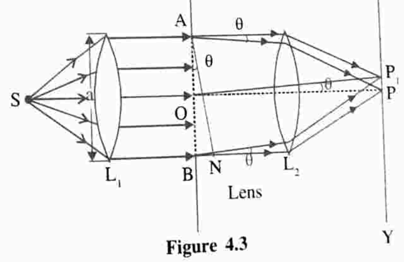


Figure 4.3

According to rectilinear propagation of light a bright image of the slit is expected at the centre P of the screen. But in practice we get a diffraction pattern i.e. a central maximum at P flanked by a number of dark and bright fringes called secondary minima and maxima on either side of O. It is seen that the width of the central maxima is twice as that of a secondary maximum and the intensity of secondary maxima goes on decreasing with the order of maxima.

According to Huygens wave theory when light falls on the slit it becomes a source of secondary wavelets. This wavelets are initially in phase and spread out in all directions. Now consider a point P on the screen such that S, O and P are in straight line. The secondary wavelets travelling in the direction parallel to OP are focussed at P. Since point P is optically equidistant from all points on the slit AB, all the secondary wavelets from AB reach the point P in the same phase and the point will have maximum intensity.

Now consider a point P_1 on the screen at which wavelets travelling in a direction making an angle θ with OP are brought to focus by the lens. The wavelets from different parts of the slit will not reach point P in phase, although they are initially in phase. This is because they cover unequal distances in reaching point P_1.

The wavelets from points A and B will have a path difference

$$BN = a\sin\theta$$

[since from the ΔABN, $\frac{BN}{AB} = \sin\theta$ i.e. $BN = AB\sin\theta = a\sin\theta$]

If this path difference is equal to λ, then P_1 is a point of minimum intensity. This is because the whole aperture AB can be imagined to be divided into two equal parts. i.e. OA and OB, then the path difference between the secondary waves from A and O will be $\frac{\lambda}{2}$. Similarly for every point in the upper half OA there is a corresponding point in the lower half OB and the path difference between the secondary waves from these points is $\frac{\lambda}{2}$. Thus they produce destructive interference and the point P_1 is of minimum intensity.

$\therefore$ For minimum $\quad a\sin\theta = \lambda$ ...... (1)

Similarly if $BN = 2\lambda$, then the aperture can be imagined to be divided into four equal parts. Then the waves from the corresponding points of two halves between A to O, or O to B will differ in path by $\frac{\lambda}{2}$ and this again gives the position of minimum intensity.

Thus for second minimum

$$a\sin\theta = 2\lambda \qquad \text{...... (2)}$$

In, general, the various minima will occur when the path difference between extreme waves is an integral multiple of λ.

i.e. $\quad a\sin\theta_n = n\lambda$ ...... (3)

or $\quad \sin\theta_n = \frac{n\lambda}{a}$

Since θ_n is small $\quad \sin\theta_n \simeq \theta_n$

$\therefore \quad \theta_n = \frac{n\lambda}{a}$ ...... (4)

θ_n gives the angular width of n^{th} order minima.

If the path difference BN is an odd multiple of $\frac{\lambda}{2}$ we will have secondary maxima in those directions.

$\therefore$ In general for secondary maxima

$$a\sin\theta_n = (2n+1)\frac{\lambda}{2}$$

or $\quad \sin\theta_n = \frac{(2n+1)\lambda}{2a}$

or $\quad \theta_n = (2n+1)\frac{\lambda}{2a}$ ...... (5)

When $n = 1, 2, 3, \ldots\ldots$ we get first, second, third ...... secondary maxima respectively. Thus the diffraction pattern due to a single slit consists of a central bright maxima at P followed by secondary minima and maxima alternately on both sides.

Width of central maximum

If D is the distance between slit and the screen and y is the distance of the first secondary minimum from the point P, then

$$\theta = \frac{y}{D} \qquad \left(\because \text{Angle} = \frac{\text{Arc}}{\text{Radius}}\right)$$

Using equation (4), we get

$$\theta = \frac{\lambda}{a}$$

$$\therefore \quad \frac{y}{D} = \frac{\lambda}{a} \quad \text{or} \quad y = \frac{\lambda D}{a} \qquad \ldots\ldots (6)$$

$\therefore$ The distance between the first secondary minimum on either side of the central maximum $= 2y = \frac{2\lambda D}{a}$. This is nothing but the width of the central maximum.

Thus, the width of the central maximum = 2y

$$= \frac{2\lambda D}{a} \qquad \ldots\ldots (7)$$

Equation (7) shows that the width of the central maximum (i) is proportional to the wavelength of the light used. i.e., width will be large for red colour (longer wavelength) and small for violet colour (shorter waver length) and (ii) is inversely proportional to width of the slit i.e. if the slit is narrow width is large and vice versa.

When the lens L_2 is very near to the slit $D \approx f$ then $2y = \frac{2\lambda f}{a}$. (see problem 5)

Note : The diffraction pattern consists of alternate bright and dark bands when monochromatic light is used. When white light is used, the central maximum is white and the rest of the diffraction bands are coloured.

Example 1

A monochromatic light of wavelength 5000Å from a distant source falls on a slit 0.5 mm wide. What is the distance between the two dark bands on each side of the central bright band of the diffraction pattern observed on a screen placed 2 m from the slit.

Solution

$$a = 0.5 \text{ mm} = 5 \times 10^{-4} \text{m}$$
$$\lambda = 5000\text{Å} = 5000 \times 10^{-10} \text{m}$$
$$D = 2 \text{ m}$$

The distance between the two dark bands on either side of the central band $= \frac{2\lambda D}{a}$

$$= \frac{2 \times 5000 \times 10^{-10} \times 2}{5 \times 10^{-4}} = 0.4 \times 10^{-3} \text{m}$$

Example 2

Find the half angular width of the central bright maximum in the Fraunhofer diffraction pattern of a slit of width 12×10^{-5} cm when the slit is illuminated by monochromatic wavellength 6000Å.

Solution

$$a = 12 \times 10^{-5} \text{ cm} = 12 \times 10^{-7} \text{ m}$$
$$\lambda = 6000\text{Å} = 6000 \times 10^{-10} \text{m}$$

using $\quad \sin\theta = \frac{\lambda}{a}$

$$\sin\theta = \frac{6000 \times 10^{-10}}{12 \times 10^{-7}} = \frac{1}{2}$$

$$\therefore \quad \theta = 30^\circ$$

Example 3

Determine the angular separation between the central maximum and the first order maximum of the diffraction pattern due to a single slit of width 0.25 mm. when light of wavelength 5890Å is incident on it normally.

Solution

$$a = 0.25 \text{ mm} = 0.25 \times 10^{-3} \text{ m}$$
$$\lambda = 5890\text{Å} = 5890 \times 10^{-10} \text{ m}$$

For the n^{th} maximum $\quad a\sin\theta = (2n+1)\frac{\lambda}{2}$

For the first order maximum $n = 1$

$$\therefore \quad a \sin\theta = \frac{3\lambda}{2}$$

or $\quad a \sin\theta = \frac{3\lambda}{2a} = \frac{3 \times 5890 \times 10^{-10}}{2 \times 0.25 \times 10^{-3}}$

$$\sin\theta = 3.53 \times 10^{-3}$$
$$\theta = \sin^{-1}(0.00353)$$
$$\theta = 0.2^\circ.$$

Example 4

A single slit illuminated by red light of 6500Å wavelength gives first order Fraunhofer diffraction minima that subtends angle of 4.2° with the axis. How wide is the slit.

Solution

$$\lambda = 6500\text{Å} = 6500\times10^{-10}\,\text{m}$$

$$\theta = 2.1^\circ = 2.1\times\frac{\pi}{180} = 0.0366 \text{ radian}$$

For first order minimum, we have

$$\sin\theta = \frac{\lambda}{a}$$

Since θ is small $\sin\theta \approx \theta$

$$\therefore \quad \theta = \frac{\lambda}{a}$$

or $$a = \frac{\lambda}{\theta} = \frac{6500\times10^{-10}}{0.0366}$$

$$a = 1.78\times10^{-5}\,\text{m}\cdot$$

Intensity distribution in the diffraction pattern of a single slit

To find the resultant intensity on the screen, suppose that the plane wave front on the slit AB is divided into a large number of strips. Then the resultant amplitude due to all the strips can be determined by the vector polygon method.

Let $a_1, a_2, a_3 \ldots\ldots, a_n$ be the amplitudes and differing in phase by $\phi_1, \phi_2, \phi_3 \ldots\ldots, \phi_n$ of strips into which the slit AB is divided. Then the resultant amplitude due to all the waves can be determined (see figure below)

If the number of amplitudes is very large (say n) differing progressively in their initial phase, the polygon is reduced to a continuous curve. Also if the amplitudes $a_1, a_2, \ldots\ldots, a_n$ are all equal $(= a)$ and the differences are also equal the curve becomes an arc EG of a circle of radius r as shown in figure (below). ET and HG represent the directions of initial and final vectors. Then the chord EG represents resultant amplitude due to all the wavelets. Let 2ϕ be the angle between first and n^{th} amplitude we get

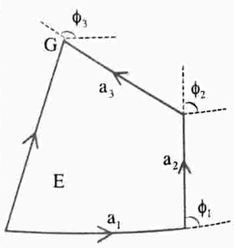


Figure 4.4

$$\phi = \angle GEH = \frac{1}{2}\angle EOG = \frac{1}{2}\frac{\text{Arc EG}}{r} = \frac{1}{2}\cdot\frac{na}{r}$$

or $$2r = \frac{na}{\phi} \qquad \ldots\ldots(1)$$

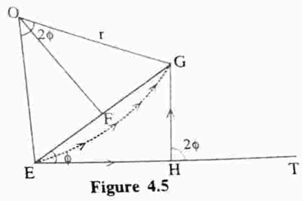


Figure 4.5

Also the resultant amplitude A = chord EG

From the ΔOFG, $\frac{FG}{OG} = \sin\phi$

i.e. $FG = OG\sin\phi = r\sin\phi$

$\therefore$ A = chord EG = EF + FG = 2FG = 2r sin ϕ

Putting the value of 2r from eq (1), we get

$$A = \frac{na}{\phi}\sin\phi$$

If all the vibrations are in phase or initial phases are zero, the resultant would be na which is the maximum value of the amplitude and is represented by A_o.

Thus we have $$A = A_o\frac{\sin\phi}{\phi} \qquad \ldots\ldots(2)$$

This gives the amplitude at any point on the screen.

$\therefore$ The intensity at any point on the screen

$$I = A_o^2\frac{\sin^2\phi}{\phi^2}$$

or $$I = I_o\frac{\sin^2\phi}{\phi^2} \qquad \ldots\ldots(3)$$

where $I_o = A_o^2$.

It shows that the intensity at any point on the screen is proportional to $\frac{\sin^2\phi}{\phi^2}$, where ϕ depends on the angle of diffraction. The path difference between the waves emanating from the point A and B of the slit is $a \sin\theta$ and the phase difference between the waves from A and B is 2ϕ. Thus we have

$$2\phi = \frac{2\pi}{\lambda} a \sin\theta \qquad \ldots\ldots (4)$$

[$\because$ phase difference $= \frac{2\pi}{\lambda}$ path difference]

Intensity distribution curve

It gives the variation of intensity with angle of diffraction. For this find the value of $\frac{\sin^2\phi}{\phi^2}$ for different values of θ by using eqn. 4. Then a graph is plotted between I and θ. See figure below.

1. When $\theta = 0$ (at the centre of the screen) we have $\phi = 0$

We know that when $\phi \to 0$, $\frac{\sin\phi}{\phi} = 1$

$$\therefore \quad \frac{\sin^2\phi}{\phi^2} = 1$$

Therefore the intensity at the central maximum $I = I_o$

2. When $\theta = \frac{\lambda}{a}$

From eqn. 4, we have

$$\phi = \pi$$

$$\therefore \quad I = 0$$

This gives the first minimum. Similarly for all values of $\theta = \pm n\frac{\lambda}{a}$.

3. When $\theta = \frac{3\lambda}{2a}$

$$\phi = \frac{3\pi}{2} \text{ (using eqn. 4)}$$

$$\therefore \quad I = I_o \frac{\sin^2 3\pi/2}{(3\pi/2)^2} = \frac{I_o}{22} \quad \left(\because \sin\frac{3\pi}{2} = -1\right)$$

i.e. This intensity is $\frac{1}{22}$ th of the intensity at the centre.

4. When $\theta = \frac{5\lambda}{2a}$

$$\phi = \frac{5\pi}{2}$$

$$\therefore \quad I = I_o \frac{\sin^2 5\pi/2}{(5\pi/2)^2} = \frac{I_o}{61.6} \quad \left(\because \sin\frac{5\pi}{2} = 1\right)$$

In general when $\theta = \pm(2n+1)\frac{\lambda}{2a}$. We get positions of secondary maxima. When n increases intensity of secondary maxima is in rapidly decreasing order. The intensity distribution curve is shown below

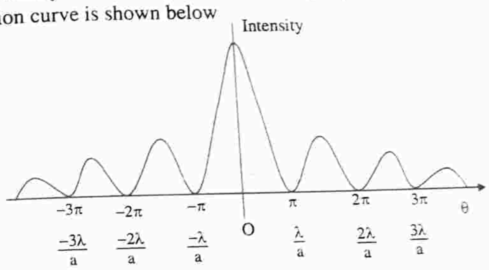


Figure 4.6

Diffraction by a circular aperture

In the Fraunhofer single slit diffraction we could see that when a plane wave is incident on a narrow slit of with a, then the emergent wave spreads out along the width of the slit with angular divergence $\frac{\lambda}{a}$ (see eqn 4 page 5). Now we can discuss the diffraction pattern due to a circular aperture. Suppose a plane wave is incident normally on the circular aperture and a lens whose diameter is much larger than that of the aperture is placed close to the aperture. Fraunhofer diffraction pattern is observed on the focal plane of the lens. It is due to the cylindrical symmetry of the system the diffraction pattern consists of a central bright disc called Airy disc surrounded by alternate bright and dark rings of rapidly decreasing intensity termed as Airy rings. Theory shows that for a circular opening the polar zones are some what less effective, hence the radius of the dark ring is slightly greater than half the width of central fringe for a slit. For a circular opening of diameter d

$$\sin\theta = 1.22\frac{\lambda}{d}$$

or $$\theta = 1.22\frac{\lambda}{d}$$

It gives the direction of first minimum where θ is the angle subtended at the centre of the aperture by the radius of the first dark ring. If f is the focal length of the lens used or the distance of the screen from the aperture, the radius of the central maximum or the radius of the first dark ring is

$$r = \theta f = \frac{1.22\lambda f}{d}$$

$$\left(\because \frac{\text{Arc}}{\text{Radius}} = \text{angle i.e., } \frac{r}{f} = \theta\right)$$

This is also called the radius of the central maximum.

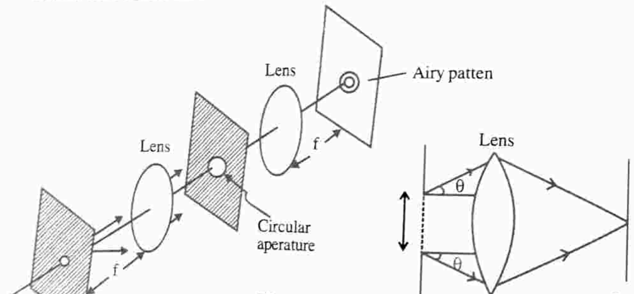


Figure 4.7

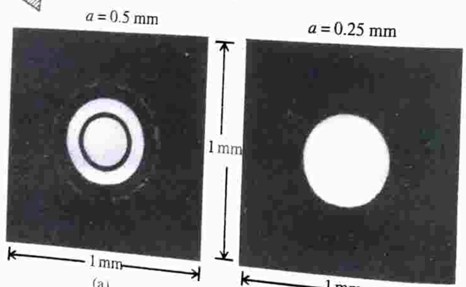


Figure 4.8

Note : It may be noted that with increase in diameter of the aperture, the radius of the central bright decreases. see example 6

Example 5

Calculate the radius of the first dark ring of the diffraction pattern produced by a circular aperture of radius 0.01 cm at the focal plane of a convex lens of focal length 10cm. Wavelength of light used 5×10^{-7} m.

Solution

The first dark ring occurs at

$$\theta = 1.22\frac{\lambda}{d} = \frac{1.22\times5\times10^{-7}}{0.02\times10^{-2}} = 3.05\times10^{-3} \text{ radian.}$$

$\therefore$ The radius of the first dark ring is

$$= \theta f = 3.05\times10^{-3}\times10\times10^{-2} = 3.05\times10^{-4}\,\text{m}.$$

Example 6

Compare the radii of two Fraunhofer diffraction patterns due to two circular apertures 0.25mm and 0.5mm with the same source of wave length and lens.

Solution

We have $r \;\alpha\; \frac{1}{d}$

$$\therefore \qquad \frac{r_1}{r_2} = \frac{d_2}{d_1} = \frac{2\times0.5}{2\times0.25} = 2$$

or $$r_1 = 2r_2$$

Intensity distribution in the Fraunbofer diffraction pattern of a circular aperture

The intensity distribution due to a circular aperture is given by

$$I = I_o\left[2\frac{J_1(\upsilon)}{\upsilon}\right]^2$$

where $$\upsilon = \frac{2\pi}{\lambda} a \sin\alpha$$

Here a is the radius of the circular aperture, λ the wavelength light used and α is the angle of diffraction i.e., α is the angle defined by a point on a screen below the circular aperture relative to the normal through the centre point. v is related to the phase of the wave on the distant screen at angle θ relative to the phase of the wave incident on the aperture at the point (r, θ) (Actually $i\upsilon r\cos\theta$ is the phase of the wave).

I_0 is the intensity at $\alpha = 0$ (which represents central maximum) and $J_1(\upsilon)$ is known as the Bessel function of the first order. The variation of Bessel function of order one is like a damped sine wave. (see figure below).

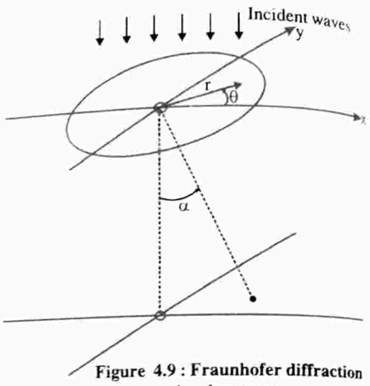
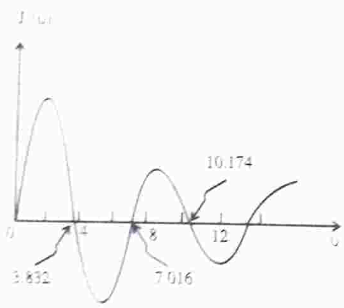
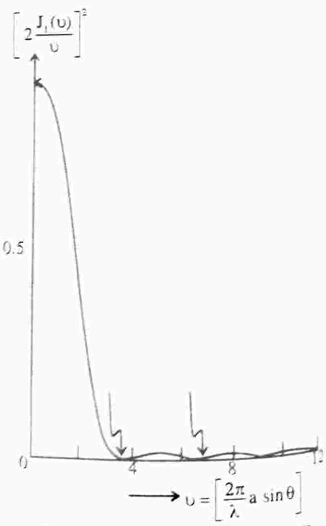


Figure 4.9 : Fraunhofer diffraction - circular aperture



The variation of $J_1(\upsilon)$ with υ

Figure 4.10



Intensity variation a with airy pattern (property of Bessel function)

Figure 4.11

At central maximum $\alpha = 0$

$$\therefore \quad J_1(0) = 0$$

But $$\lim_{\upsilon \to 0} \frac{2I_1(\upsilon)}{\upsilon} = 1$$

Thus we get $I = I_0$ at the centre. Other zeros of $J_1(\upsilon)$ occur at

$$\upsilon = 3.8317,\ 7.0156,\ 10.1735,\ 13.3237,\ 16.4706 \ldots\ldots$$

Below we have platted $\left[\frac{J_1(\upsilon)}{\upsilon}\right]^2$ versus υ which gives the intensity distribution corresponding to the Airy pattern. Thus the successive dark rings in the Airy pattern will correspond to zeros of $J_1(\upsilon)$.

i.e., $$\upsilon = \frac{2\pi}{\lambda} a \sin\alpha$$

$$= 3.8317,\ 7.0156,\ 10.1735,\ \ldots\ldots \text{etc.}$$

or $$\frac{2\pi}{\lambda} a \sin\alpha = 3.8317 \text{ for the first dark}$$

or $$\sin\alpha = \frac{3.8317\lambda}{2\pi a}$$

or $$\alpha = \frac{3.8317\lambda}{2\pi a} = \frac{1.22\lambda}{2a} = \frac{1.22\lambda}{d}$$

(since $2a = d$, diameter of the circular aperture)

For the second dark

$$\frac{2\pi}{\lambda} a \sin\alpha = 7.0156$$

or $$\sin\alpha = \frac{7.0156\lambda}{2\pi a} = \frac{2.234\lambda}{2a}$$

$$\alpha \simeq \frac{2.234\lambda}{d}$$

For example if $a = 0.5$cm and $\lambda = 5.5 \times 10^{-7}$ m (green light), then the first dark ring will occur at an angle

$$\alpha = \frac{1.22\lambda}{d} = \frac{1.22 \times 5.5 \times 10^{-7}}{1 \times 10^{-2}}$$

$$\alpha = 6.71 \times 10^{-5} \text{ radian}$$

$$\alpha = 6.71 \times 10^{-5} \times \frac{180}{\pi} \text{ degree}$$

$$= 38.465 \times 10^{-4} \text{ degree}$$

$$= 13.84 \text{ seconds.}$$

This shows that the bending or spreading of the light ray is extremely small. If this analysis had been known in the seventieth century, the arguments against the wave theory of light would have collapsed.

The angular spread of the beam is approximately given by

$$\Delta\alpha = \frac{1}{2}\,\frac{1.22\lambda}{D} = \frac{0.61\lambda}{D} \approx \frac{\lambda}{D}$$

i.e.,
$$\Delta\alpha \approx \frac{\lambda}{\text{Linear dimension of the aperture}}$$

For calculating the angular spread we took the spread of first dark ring since 84% of the energy is contained within the first dark ring. The above formula gives us a very interesting practical application. Inorder to receive large amount of energy from a loud speaker by an observer the diffraction spread must be small, so the diameter of the aperture of the speaker must be large. In other words to obtain more directionality of sound waves one should use a loud speaker of large aperture.

Resolving power

When two objects are very close together they may appear as one and it may be difficult for the eye to see them as separate. The eye can see two objects as separate, if the angle subtended by them at the eye is greater than one minute which is the resolving limit of the normal eye. When the two nearby objects subtend an angle less than one minute at the eye, they may be seen as separate by using optical instruments. *The method of separating two nearby objects is called resolution and the ability of an optical instrument to produce distinctly separate images of two objects very close together is called its resolving power.*

Resolving power is normally measured as the reciprocal of the smallest angle subtended at the objective of optical instrument by two point objects, which can just be distinguished as separate.

The optical instruments such as telescopes and microscopes are used to resolve

two nearby object points where as prisms and gratings are used to resolve two nearby spectral lines. The former one is called geometrical resolution and the latter is called spectral resolution.

Rayleigh's criterion for resolution

When a beam of light from a point object passes through the objective of a telescope, the lens acts like a circular aperture and produces a diffraction pattern instead of a point object. This diffraction pattern is a bright disc surrounded by alternate dark and bright rings it is known as Airy's disc. If there are two point objects lying close to each other, then two diffraction patterns are produced, which may overlap on each other and it may be difficult to distinguish them as separate.

To obtain a measure of the resolving power of a instrument Lord Rayleigh after a detailed study of the resultant intensity distribution in the diffraction patterns of closely situated equally bright point sources suggested a criterion for resolution. It suggested that *two diffraction patterns of closely situated equally bright point sources may be regarded as separate if the central maximum of one coincides with the first minimum of the other.* This is equivalent to the condition that distance between the centres at the patterns should be equal to the radius of the central diffraction disc.

This criterion of resolutions is equally applicable to the resolution of spectral lines of equal intensity. Thus two close lines in a spectrum are said to be just resolved when the central maximum of the diffraction pattern of one falls upon the first minimum of the diffraction pattern of the other (see figure below)

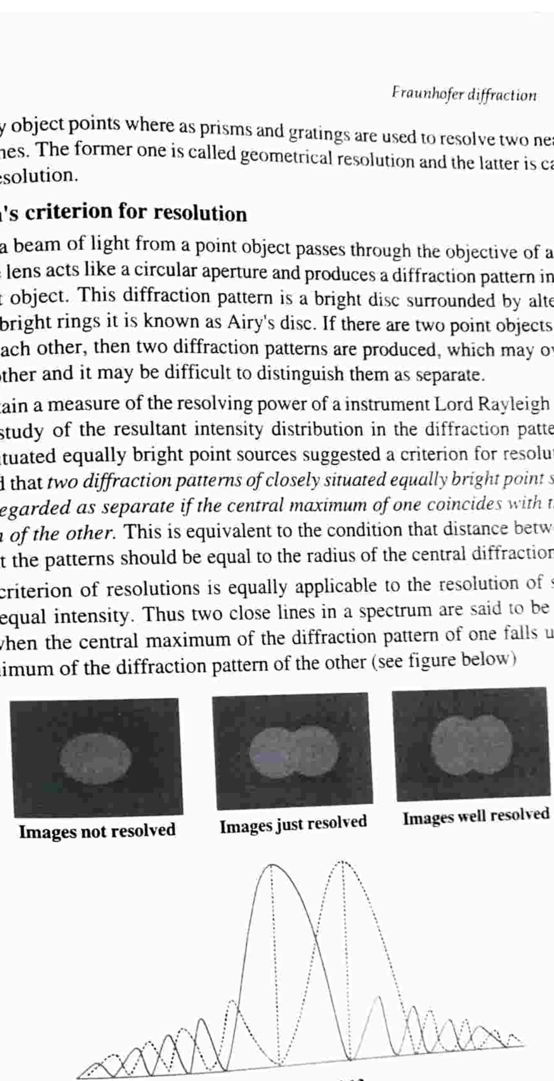


Figure 4.12

It is to be noted that the diffraction effects set a theoretical limit to the resolving power of any optical instrument.

Resolving power of a telescope

The ability of an optical instrument expressed in numerical measure to resolve the images of two near by points is termed as its resolving power. For every telescope there is a resolving limit $(d\theta)$. The limit of resolution of a telescope is given by

$$d\theta = \frac{1.22\lambda}{d}$$

where λ is the wavelength of light used and d is the aperture of the telescope objective.

The reciprocal of the limit of resolution measures the resolving power (R.P)

i.e., $$R.P = \frac{1}{d\theta} = \frac{d}{1.22\lambda}$$

If f is the focal length of the telescope objective, then

$$d\theta = \frac{r}{f} = \frac{1.22\lambda}{d}$$

or $$r = \frac{1.22\lambda f}{d}$$

where r is the radius of the central bright image. The diameter of the first dark ring is equal to the diameter of the central image. The central bright disc is the Airy's disc. For example, let two distant stars subtend angle of 1 second of an arc at the objective of the telescope. If $\lambda = 5500Å$ (our eye is most sensitive to this green yellow colour in the visible region). Then the diameter of the objective required for just resolution can be calculated from the equation

$$d\theta = \frac{1.22\lambda}{d}$$

$$\therefore \quad d = \frac{1.22\lambda}{d\theta} = \frac{1.22 \times 5500 \times 10^{-10}}{4.85 \times 10^{-6}}$$

$$d = 13.9\text{cm}$$

$$d\theta = 1'' = 4.85 \times 10^{-6} \text{ radian}$$

Note : The resolving power of a telescope increases with increase in diameter of the objective. With the increase in diameter of the objective, the effect of spherical aberration becomes appreciable, so to minimise this, central portion of the telescope is covered with a stop. This doesnot effect the resolving power.

Example 7

The diameter of the largest telescope objective is $80''$. Calculate the angular separation of the object that it can resolve. Assume $\lambda = 5500Å$.

Solution

$$d = 80'' = 80 \times 2.5\text{cm} = 200\text{ cm} = 2\text{m}$$

$$\lambda = 5500Å = 5500 \times 10^{-10}\text{m}$$

$$d\theta = \frac{1.22\lambda}{d} = \frac{1.22 \times 5500 \times 10^{-10}}{2}$$

$$d\theta = 0.335 \times 10^{-6} \text{ radian}$$

$$d\theta = \frac{0.335 \times 10^{-6}}{4.85 \times 10^{-6}} = 0.069 \text{ seconds of arc.}$$

Example 8

Calculate the diameter of a telescope lens if a resolution of 0.1 seconds arc is required at $\lambda = 6 \times 10^{-7}\text{m}$.

Solution

$$d\theta = 0.1 \text{ seconds of arc}$$

$$d\theta = 0.1 \times 4.85 \times 10^{-6} \text{ radian}$$

$$\lambda = 6 \times 10^{-7}\text{m}$$

We have $$d\theta = \frac{1.22\lambda}{d}$$

or $$d = \frac{1.22\lambda}{d\theta} = \frac{1.22 \times 6 \times 10^{-7}}{4.85 \times 10^{-7}}$$

$$= 1.509\text{m}.$$

Note : If we assume that the angular resolution of the human eye is primarily due to diffraction effects then

$$d\theta = \frac{\lambda}{d}$$

If $\lambda = 5.5 \times 10^{-7}\text{m}$, and $d = 2\text{mm}$ (pupil diameter).

$$\therefore \quad d\theta = \frac{5.5\times10^{-7}}{2\times10^{-3}} = 2.75\times10^{-4} \text{ radian}$$

If the objects are at a distance of 20m. To resolve the two objects, they must be separated by

$$2.75\times10^{-4}\times20 = 5.5\text{mm}. \text{ (see also problem 11)}$$

Two slit Fraunhofer diffraction

Consider two parallel slits AB and CD of equal width a and separated by a distance b. Let a plane wave front be incident normally on the slits. The rays emerging out the slits are collected on a screen by using a convex lens. All the wavelets reacting at P travel equal optical paths, therefore they are in same phase hence get a maximum at P.

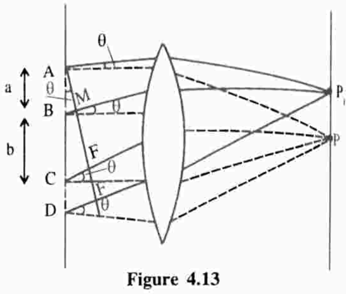


Figure 4.13

In this case diffraction pattern is due to two aspects. (i) The diffraction due to secondary waves from the two slits individually and (ii) the interference phenomenon due to the secondary waves emanating from the corresponding parts at the two slits.

Diffraction maxima and minima

Consider the secondary waves travelling in a direction inclined at an angle θ with the direction of the incident wave. If the path difference BM is equal to λ, the wave length of light, then θ gives the direction of diffraction minimum. Considering the wavefront on AB to be made up of two halves, the path difference between the corresponding points of the upper and lower half is equal to $\frac{\lambda}{2}$. Therefore the effect at P_1 due to the waves from CD is also minimum.

Thus the first minimum due to diffraction at a single slit is given by $a\sin\theta_{d1} = \lambda$

or $\quad \sin\theta_{d1} = \frac{\lambda}{a}$

In general n^{th} minimum is given by

$$a\sin\theta_{dn} = n\lambda$$

or $\quad \sin\theta_{dn} = \frac{n\lambda}{a}$

Similarly n^{th} maximum due to diffraction is given by

$$\sin\theta'_{dn} = (2n+1)\frac{\lambda}{a}$$

Interference maxima and minima

If we consider a point P_1 on the screen the resultant amplitude will be the sum of the amplitudes of the wavelets coming from the two slits. Since the wavelets reaching at P_1 from the two extreme ends of the two slits differ in path and therefore there is a phase difference between them. Depending upon the path difference, P_1 may be a maximum point or minimum point.

For P_1 to be minimum, the path difference between the wavelets from A and that from the corresponding point C should be $\frac{\lambda}{2}$ or $(2n+1)\frac{\lambda}{2}$

i.e., $\quad$ when $CF = AC\sin\theta = (2n+1)\frac{\lambda}{2}$

or $\quad (a+b)\sin\theta = (2n+1)\frac{\lambda}{2}.$

This shows that in addition to the diffraction phenomenon occurring at the two slits individually, phenomenon of interference also occurs due to the wavelets coming out of the corresponding points of the two slits. The directions of the minima due to interference phenomena are given by putting.

$$n = 0, 1, 2, 3 \ldots..$$

$$\therefore \quad (a+b)\sin\theta_{in} = (2n+1)\frac{\lambda}{2}$$

$$\text{or} \quad \sin\theta_{in} = \frac{(2n+1)\lambda}{(a+b)\,2} \qquad \ldots\ldots (1)$$

Similarly the directions of maxima due to interference phenomena a are given by

$$(a+b)\sin\theta_{in} = n\lambda$$

$$\text{or} \quad \sin\theta'_{in} = \frac{n\lambda}{a+b} \qquad \ldots\ldots (2)$$

(i stands for interference.)

From eqn (1) we get

$$\sin\theta_2 - \sin\theta_1 = \frac{\lambda}{a+b} \qquad \ldots\ldots (3)$$

From eqn (2) we get

$$\sin\theta'_2 - \sin\theta'_1 = \frac{\lambda}{a+b} \qquad \ldots\ldots (4)$$

Equations 3 and 4 show that the angular separation between any two consecutive minima or maxima is equal to $\frac{\lambda}{a+b}$, i.e., the angular separation is inversely proportional to $(a+b)$, the distance between two slits.

Note : Comparing equations for $\sin\theta_1$ and $\sin\theta_d$ we see that θ_i is very much smaller than θ_d which shows that a number of interference minima lie in between corresponding diffraction minima.

Intensity distribution on the screen due to diffraction from two slits

According to Huygen's principle every point in the slits AB and CD sends out secondary wavelets in all directions. From the theory of diffraction at a slit, the resultant amplitude due to wavelets diffracted from each slit, in θ direction is given by $\frac{A_o \sin\phi}{\phi}$, where $\phi = \frac{\pi}{\lambda} a \sin\theta$ (see eqn 4 page 127).

Obviously one can consider the two slits AB and CD as equivalent to two coherent sources placed at their middle points. Each of these is sending a wavelet of amplitude $\frac{A_o \sin\phi}{\phi}$. Let the two phase difference between the two wavelets reacting at P_1 is δ. The path difference between two waves resulting in interference pattern is $(a+b)\sin\theta$.

$\therefore$ The phase difference $\delta = \frac{2\pi}{\lambda}(a+b)\sin\theta$. The resultant amplitude (R) can be found out by the method of vector amplitude diagram.

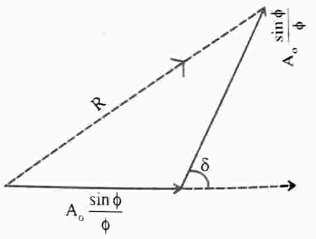


Figure 4.14

$$R^2 = \left(\frac{A_o \sin\phi}{\phi}\right)^2 + \left(\frac{A_o \sin\phi}{\phi}\right)^2 + 2\left(\frac{A_o \sin\phi}{\phi}\right)\left(\frac{A_o \sin\phi}{\phi}\right)\cos\delta$$

$$R^2 = 2A_0^2 \frac{\sin^2\phi}{\phi}(1+\cos\delta)$$

$$R^2 = 4A_0^2 \frac{\sin^2\phi}{\phi^2}\cos^2\frac{\delta}{2} \quad \text{Put } \beta = \frac{\delta}{2}$$

or $$R^2 = 4A_0^2 \frac{\sin^2\phi}{\phi^2}\cos^2\beta.$$

$\therefore$ The resultant intensity

$$I \,\alpha\, R^2 = 4A_0^2 \frac{\sin^2\phi}{\phi^2}\cos^2\beta$$

or $$I = 4I_o \frac{\sin^2\phi}{\phi^2}\cos^2\beta, \text{ with } kA_0^2 = I_o$$

i.e., The resultant intensity I is proportional to $4A_0^2 \frac{\sin^2\phi}{\phi^2}\cos^2\beta$. Thus the resultant intensity in the diffraction pattern depends on two factors.

(i) $A_0^2 \frac{\sin^2\phi}{\phi^2}$ which gives diffraction pattern due to each single slit and

(ii) $\cos^2\beta$ which gives the interference pattern due to light waves of the same amplitude from the two slits.

$\frac{\sin^2\phi}{\phi^2}$ gives a central maxima in the direction $\theta = 0$ $(\delta = 0)$. On either side of it alternately minima and subsidiary maxima of decreasing intensity are observed.

Positions of maxima and minima

Positions of minima are given by

$$\sin\phi = 0$$

or $$\phi = \pm n\pi$$

or $$\frac{\pi}{\lambda} a \sin\phi = \pm n\pi \text{ where } n = 1, 2, 3\ldots\ldots$$

Thus minima are obtained for

$$\phi = \pm\pi, \pm 2\pi, \pm 3\pi \ldots\ldots$$

Positions of subsidiary maxima are given by

$$\sin\phi = \pm(2n+1)\frac{\pi}{2}$$

Thus maxima are obtained for

$$\phi = \pm 3\frac{\pi}{2},\ \pm 5\frac{\pi}{2},\ \pm 7\frac{\pi}{2},\ ....$$

$\cos^2\beta$, the interference term gives a set of equidistant dark and bright fringes. The maxima i.e., the bright fringes are obtained in the directions given by

$$\cos^2\beta = 1$$

or $$\beta = \pm n\pi$$

or $$\frac{\pi}{\lambda}(a+b)\sin\theta = \pm n\pi$$

Corresponding to n = 0, 1, 2, 3 .... the maxima are called as first order, second order, third order .... maxima.

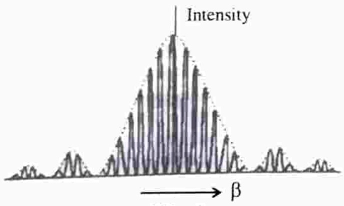
Positions of minima are given by

$$\cos^2\beta = 0$$

or $$\beta = \pm(2n+1)\frac{\pi}{2}$$

or $$\beta = \pm\frac{\pi}{2},\ \pm 3\frac{\pi}{2},\ \pm 5\frac{\pi}{2},\ ....$$

The resultant diffraction pattern is shown in figure.



Double slit diffraction pattern

Figure 4.15

Missing orders of interference maxima in double slit diffraction pattern

If the width of the slits be 'a' and the opaque portion between the slits be 'b'. Depending upon the relative values of a and b some order of interference maxima are absent from the resultant pattern.

For interference maxima, we have

$$(a+b)\sin\theta = \pm m\lambda$$

For diffraction minima, we have

$$a\sin\theta = \pm n\lambda$$

If for certain values of θ the above equations are satisfied simultaneously then

the interference maxima will coincide with the diffraction minima thus, corresponding orders of interference maxima become absent.

(i) If a = b, then

$$2a\sin\theta = \pm m\lambda$$

and $$a\sin\theta = \pm n\lambda$$

or $$m = 2n$$

If n = 1, 2, 3 .... then m = 2, 4, 6....... Hence the 2nd, 4th, 6th ...... order interference maxima coincide with 1st, 2nd, 3rd order diffraction minima and hence they will be absent

(ii) If b = 2a

$$3a\sin\theta = m\lambda$$

and $$a\sin\theta = n\lambda$$

or $$m = 3n$$

If n = 1, 2, 3 ..... then m = 3, 6, 9 ...... Hence the 3rd, 6th, 9th interference maxima coincide with 1st, 2nd, 3rd diffraction minima and they will be absent.

(iii) If a + b = 0 or b = 0, the two slits join and all the orders of the interference maxima will be missing.

N slit diffraction pattern

If there are N slits each of with a. The distance between two consecutive slits is assumed to be b. As before the resultant intensity can be calculated to be

$$I = I_o\left(\frac{\sin^2\phi}{\phi^2}\right)\left(\frac{\sin^2 N\beta}{\sin^2\beta}\right)$$

The term $\dfrac{\sin^2\phi}{\phi^2}$ represents this diffraction pattern due to a single slit. The term $\dfrac{\sin^2 N\beta}{\sin^2\beta}$ represents the interferance effects due to the secondary waves from the N slits.

It is very interesting to note that when N = 1, the above equation reduces to the intensity of single slit diffraction pattern.

i.e., $I = I_o \dfrac{\sin^2\phi}{\phi^2}$

when $N = 2$ (two slits)

$$I = I_o \frac{\sin^2\phi}{\phi^2}\frac{\sin^2 2\beta}{\sin^2\beta}$$

or $$I = I_o \frac{\sin^2\phi}{\phi^2}\frac{(2\sin\beta\cos\beta)^2}{\sin^2\beta}$$

$$I = 4I_o \frac{\sin^2\phi}{\phi^2}\cos^2\beta.$$

This is nothing but the intensity distribution of two slit diffraction pattern.

Principal maxima

The interference term $\dfrac{\sin^2 N\beta}{\sin^2\beta}$ possesses maximum values for $\beta = 0, \pi, 2\pi, \ldots\ldots = m\pi$ which is equal to N^2.

When $$\lim_{\beta\to m\pi}\frac{\sin N\beta}{\sin\beta} = \lim_{\beta\to m\pi}\frac{\frac{d}{d\beta}(\sin N\beta)}{\frac{d}{d\beta}(\sin\beta)}$$

$$= \lim_{\beta\to m\pi}\frac{N(\cos N\beta)}{\cos\beta} = \pm N.$$

$\therefore$ $$\frac{\sin^2 N\beta}{\sin^2\beta} = N^2 \text{ when } \beta \to m\pi$$

These maxima correspond in position to those of the double slit, since for the above values of β

$$(a+b)\sin\phi = m\lambda \ (m = 0, 1, 2, \ldots\ldots)$$

For $\beta = \pm m\pi$, the intensity is maximum and is given by

$$I = N^2 I_o \frac{\sin^2\phi}{\phi^2}$$

where $\phi = \dfrac{\pi a\sin\theta}{\lambda} = \dfrac{\pi a}{\lambda}\dfrac{m\lambda}{(a+b)} = \dfrac{\pi am}{(a+b)}$

Such maxima are called principal maxima.

Minima and secondary maxima

Now we will see that for what values of β, the function $\dfrac{\sin^2 N\beta}{\sin^2\beta}$ will become minima. Actually the numerator of the function becomes zero more often than the denominator. This occurs at the values $N\beta = 0, \pi, 2\pi, \ldots$ or in general $\beta = p\pi$. In the special cases when $p = 0, N, 2N, \ldots$, β will be $0, \pi, 2\pi, \ldots$ so for those values the denominator will also vanish, as a result $\dfrac{\sin^2 N\beta}{\sin^2\beta} = N^2$ and we have the principal maxima described above. The other values of p (other than 0, N, 2N ....) give zero intensity since for these the denominator does not vanish at the same time. Hence the condition for minimum is $N\beta = p\pi$, excluding those values for which $p = mN$, m being the order.

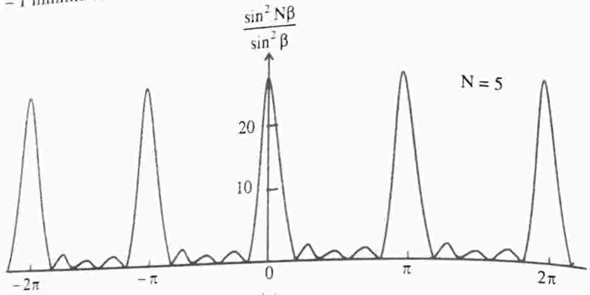
or $$\beta = \frac{p\pi}{N} \ (p \neq 0, N, 2N, \ldots\ldots)$$

These values of β correspond to path differences

$$(a+b)\sin\theta = \frac{\lambda}{N}, \frac{2\lambda}{N}, \frac{3\lambda}{N}, \ldots\ldots \frac{(N-1)\lambda}{N}, \frac{(N+1)\lambda}{N} \ldots\ldots$$

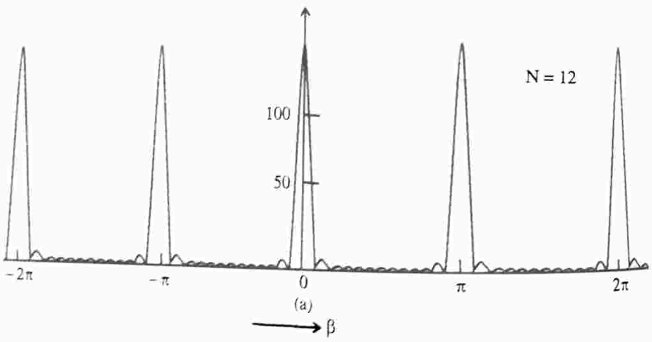
This is the condition of minima. Here we omitted the values $0, \dfrac{N\lambda}{N}, \dfrac{2N\lambda}{N}, \ldots\ldots$ for which $(a+b)\sin\theta = m\lambda$ corresponds to principal maxima.

The above discussion shows that between two adjacent principal maxima there are $N - 1$ minima. For example say 0 and $\dfrac{N\lambda}{N}$ are two adjacent principal maxima. In between these two we can see $\dfrac{\lambda}{N}, \dfrac{2\lambda}{N}, \ldots\ldots \dfrac{(N-1)\lambda}{N}$, which corresponds to minimum and the number is $N - 1$. Between the other minima the intensity rises again then falls. Thus secondary maxima is produced. The intensity of secondary maxima is much less than the principal maxima. Between two adjacent minima we can see one secondary maxima, i.e., in between $\frac{\lambda}{N}$ and $\frac{2\lambda}{N}$ we can $\frac{3\lambda}{2N}$ which corresponds to secondary minimum. In general there are N – 2 secondary maxima in between N – 1 minima or two principal maxima. See figure below.



(a)

Figure 4.16



(a)

Figure 4.17

If N = 5, the minima is 4 and secondary maxima is 3.

Angular width of principal maxima

The diffraction pattern produced by N slits for m^{th} order principal maximum occurs at

$$(a+b)\sin\theta_m = m\lambda,\ m = 0, 1, 2...... \qquad ......(1)$$

The first minima on either side occurs at

$$(a+b)\sin\theta = \pm\frac{\lambda}{N} \qquad ......(2)$$

If $\theta_m + \Delta\theta_{1m}$ and $\theta_m - \Delta\theta_{1m}$ represent the angles of diffraction on either side of principal maxima. Then

$$(a+b)\sin(\theta_m \pm \Delta\theta_{1m}) = m\lambda \pm \frac{\lambda}{N} \qquad ......(3)$$

$$\sin(\theta_m \pm \Delta\theta_{1m}) = \sin\theta_m \cos\Delta\theta_{1m} \pm \cos\theta_m \sin\Delta\theta_{1m}$$

$$\cos\Delta\theta_{1m} \approx 1 \text{ and } \sin\Delta\theta_{1m} \approx \Delta\theta_{1m}$$

or $$\sin(\theta_m \pm \Delta\theta_{1m}) = \sin\theta_m \pm \Delta\theta_{1m}\cos\theta_m$$

Now eqn 3 becomes

$$(a+b)(\sin\theta_m \pm \Delta\theta_{1m}\cos\theta_m)$$

$$= m\lambda \pm \frac{\lambda}{N}$$

or $$(a+b)\,\Delta\theta_{1n}\cos\theta_m = \frac{\lambda}{N}$$

or $$\Delta\theta_{1m} = \frac{\lambda}{N(a+b)\cos\theta_m} \qquad ......(4)$$

Which shows that when N increases the angular width decreases. Moreover when N increases the intensity of principal maximum $(I \propto N^2)$ also increases. This means that for large N, we get a sharply peaked diffraction pattern (see figure above).

Example 9

Light of wavelength 5500Å falls normally on a double slit. If the width of the slit is 22×10^{-7}m and the opaque gap between the slits of width 44×10^{-7}m. Finding the missing order for the interference maxima.

Solution

$$a = 22\times10^{-7}\,m,\ b = 44\times10^{-7}\,m$$

For interference maxima ......(1)

$$(a+b)\sin\theta = n\lambda$$

For diffraction minima

$$a \sin\theta = m\lambda \qquad \text{......(2)}$$

$$\frac{(a+b)}{a} = \frac{n}{m}$$

or

$$n = m\frac{(a+b)}{a} = m\,\frac{22\times10^{-7}+44\times10^{-7}}{22\times10^{-7}}$$

or

$$n = 3m$$

If we put m = 1, 2, 3 ......

Then n = 3, 6, 9 ......

Therefore 3[rd], 6[th] and 9[th] order interference maxima are missing in 1[st], 2[nd], and 3[rd] order diffraction minima. When m = 1 n = 3. Thus in the central diffraction maxima we will have five interference maxima, for $n = 0, \pm 1, \pm 2$.

Example 10

Consider a set of two slits each of width 5×10^{-4} m and separated by a distance 0.1cm illuminated by a monochromatic light of wavelength 6328×10^{-10} m. If a convex lens of focal length 10 cm is placed beyond the double slit arrangement. Calculate the positions of the minima inside the first diffraction minimum.

Solution

$$a = 5\times10^{-4}\,\text{m}, \quad a+b = 0.1\times10^{-2}\,\text{m}$$

$$f = 10\times10^{-2}\,\text{m}$$

The first diffraction minimum occurs at $a\sin\theta = \lambda$. ......(1)

For the interference minima, we have

$$(a+b)\sin\theta = \left(n+\frac{1}{2}\right)\lambda \qquad \text{......(2)}$$

There will be two minima inside first diffraction minimum. If x_n be the position of n[th] order minima on the screen, then

$$(a+b)\frac{x_n}{f} = \left(n+\frac{1}{2}\right)\lambda$$

or

$$x_n = \frac{2f}{a+b}\lambda\left(n+\frac{1}{2}\right)$$

For $n = 0$ $\quad x_0 = \dfrac{2\times10^{-1}\times6328\times10^{-10}}{2\times0.1\times10^{-2}} = 0.03164\,\text{mm}$

For $n = 1$ $\quad x_1 = 0.0939\,\text{mm}$.

Plane diffraction grating

Introduction

The arrangement of large number of slits of equal width separated by opaque portions arranged at equal distances is known as grating. When a wave front is incident on a grating surface light is transmitted through the slits and obstructed by opaque portions. Such a grating is called transmission grating. Joseph Fraunhofer was the first to produce a grating in 1821, which consists of 310 slits by stretching fine wires over screws. After that he succeeded in making a ruled grating over glass with a fine diamond point. With the help of this he determined the wavelengths of all principal Fraunhofer lines in solar spectrum. After Fraunhofer, in 1880 Rutherford rules a grating on metal with the help of an automatic dividing engine. Rowland developed a technique of ruling them on a concave mirror. After long experimentation R.W. Wood found that a luminised pyrax glass is suitable for the construction of large gratings.

The diffraction grating in its normal form consists of an optically plane glass plate upon which a large number of fine, equidistant and parallel lines are ruled with the help of a dividing engine. The lines are drawn very close to each other about 1000 lines per cm. The ruled width is opaque to light and are called as opacities, while the space between any two lines allows light to pass through and are called transparencies. Such a plate is called as plane transmission grating.

Gratings are used for the determination of wavelength of light. Because of the difficulties of construction of gratings, replica greetings are used in laboratories. These are made from original gratings. For this a thin layer of collidion solution is poured over the surface of a ruled grating and allowed to harden. The collidion film when stripped from the grating retains an impression of the rulings of the original grating which is fixed between two glass plates. This serves as the plane transmission grating.

Theory of plane transmission grating

Consider a plane transmission grating AB, having N parallel slits. Let a be the width of each slit and b be the width of each opaque portion. The distance between two successive slits i.e. (a + b) is called the grating element. L is a converging lens and XY is a screen kept at the focal plane of the lens L.

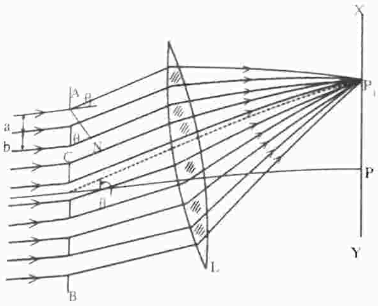


Figure 4.18

Let a plane wave front be incident normally on N slits. Then according to Huygen principle all points in each slit become a source of secondary wavelets. Most of the secondary wavelets proceeding from the slits will continue to travel in the direction of the incident light when focussed by a convex lens L they will give a line of maximum intensity on a screen placed at the focal plane of the lens. This is called central maximum P.

Let us now consider the beam which is diffracted at an angle θ. Let this beam be collected by the lens L. It is obvious that there is a path difference between the wavelets coming out from each slit and bending at an angle θ. To find the path difference between the diffracted wavelets, let us consider the wavelets originating from two consecutive points A and C. Draw N normal to the direction of diffracted beam. Therefore the path difference between the wavelets originating from A and C is

$$CN = (a+b)\sin\theta \left(\because \frac{CN}{AC} = \sin\theta \text{ and } AC = a+b\right)$$

If $\quad (a+b)\sin\theta = n\lambda \qquad \text{......(1)}$

where $n = 1, 2, 3, \ldots\ldots$ these waves reinforce i.e. all the wavelets of wavelength λ originating from the various corresponding points reinforce giving rise to a principal maxima P_1 at an angle θ. Putting $n = 1, 2, 3, \ldots\ldots$ the angles of diffraction corresponding to the principal maxima of different orders can be obtained. Exactly similar maxima are obtained above P due to the light diffracted upwards. Thus, on either side of the central maximum (P) diffraction maxima of different orders are obtained symmetrically. Thus equation (1) can be written as

$$(a+b)\sin\theta_n = \pm n\lambda \qquad \text{......(2)}$$

This shows that θ_n is different for different wavelengths.

Therefore if composite light is used it gets dispersed and in each order the principal maxima of different colours are formed at different angles. In other words on either side of the central maxima, spectra of the source are obtained due to diffraction. If n = 1, the diffraction spectrum is called first order spectrum.

If n = 2, it is called the second order spectrum and so on.

If there are N rulings in one metre of the grating, then

$$N(a+b) = 1$$

or

$$(a+b) = \frac{1}{N}$$

Hence equation (2) becomes

$$\sin\theta_n = \pm Nn\lambda \qquad \text{.....(3)}$$

This is called as grating law.

Example 11

A parallel beam of monochromatic light is allowed to be incident normally on a plane transmission grating having 5000 lines/cm and the third order spectral line is found to be diffracted through $45°$. Calculate the wavelength of light.

Solution

$$N = 5000\text{cm}^{-1} = 5000\times10^{2}\text{m}^{-1} \left(\because\ a+b=\frac{1}{N}\right)$$

$$n = 3,\ \theta = 45^\circ$$

Using $\sin\theta = Nn\lambda$

$$\lambda = \frac{N\sin\theta}{n} = \frac{\sin 45}{5000\times10^{-2}\times3}$$

$$\lambda = 4.713\times10^{-7}\text{m} = 4713\ \text{Å}$$

Example 12

Calculate the angle between the central image of clamp filament and its first diffracted image produced by a fabric with 160 threads per cm. $\lambda = 6\times10^{-5}$cm.

Solution

$N = 160\ cm^{-1} = 160 \times 10^2 m^{-1}$

$\lambda = 6 \times 10^{-5} cm = 6 \times 10^{-7} m$

$n = 1$

We have $\sin\theta = Nn\lambda$

$\sin\theta = 160 \times 10^2 \times 1 \times 6 \times 10^{-7} = 0.0096$

$\theta = 0.55^\circ = 33$ minute.

Example 13

A plane transmission grating which has 5500 lines per cm is used to produce a spectrum of light from a mercury lamp. What will be the angular separation of the two yellow mercury lines of wavelength 5770 and 5791Å when viewd in the second order

Solution

$N = 5500 \times 10^2 m^{-1}$, $\lambda_1 = 5770 \times 10^{-10} m$, $\lambda_2 = 5791 \times 10^{-10} m$

Using $\sin\theta = Nn\lambda$

For the first line, $\sin\theta_1 = 5500 \times 10^2 \times 2 \times 5770 \times 10^{-10} = 0.6347$

$\theta_1 = 39^\circ 24'$.

For the second line, $\sin\theta_2 = 5500 \times 10^2 \times 2 \times 5791 \times 10^{-10} = 0.6370$

$\theta_2 = 39^\circ 34'$.

$\therefore$ Angular separation, $\theta_2 - \theta_1 = 10'$

Example 14

With a diffraction grating arranged for normal incidence, it is found that a spectral line of the second order spectrum coincides with a spectral line of third order. If the wavelength of former is 6000Å. What is the wavelength of later.

Solution

We have $\sin\theta = Nn\lambda$.

Two lines coincide means θ is same for both.

$\therefore \quad Nn_1\lambda_1 = Nn_2\lambda_2$

or $\quad n_1\lambda_1 = n_2\lambda_2$

$2 \times 6000 = 3 \times \lambda_2$

$\therefore \quad \lambda_2 = 4000Å$

Example 15

If the grating element is $2 \times 10^{-4} cm$, how many order of spectrum are possible for a light of wavelength 650 nm.

Solution

Grating element $a + b = \frac{1}{N} = 2 \times 10^{-4} cm = 2 \times 10^{-6} m$

$\lambda = 650nm = 650 \times 10^{-9} m$

For maximum number of orders $\sin\theta = 1$

Using $\sin\theta = Nn\lambda$, we have

$1 = Nn_{max}\lambda$

or $\quad n_{max} = \frac{1}{N\lambda} = \frac{2 \times 10^{-6}}{650 \times 10^{-9}} = 3.07$

$\therefore$ order possible is only 3.

Resolving power of a grating

The resolving power of a diffraction grating is defined as its ability to show two neighbouring lines in a spectrum as separate. According to Rayleigh's criterion the resolving power of a grating is measured as the ratio of the wavelength λ of any spectral line to the smallest difference in wavelength $d\lambda$, between this line and a neighbouring line such that the two lines appear just resolved.

Thus, the resolving power of a grating $= \frac{\lambda}{d\lambda}$

In figure, XY is the grating surface and MN is the field of view of the telescope. P_1 is the n^{th} primary maximum of a spectral line of wavelength λ at an angle of diffraction θ_n. P_2 is the n^{th} primary maximum of a second spectral line of wavelength $\lambda + d\lambda$ at a diffracting angle $\theta_n + d\theta$. P_1 and P_2 are the spectral lines in the n^{th} order. These two spectral lines according to Rayleigh, will appear just resolved if the central maximum position of P_2 also corresponds to the first minimum of P_1.

For the nth primary maximum at P_1 for the wavelength λ, we have

$$(a + b)\sin\theta = n\lambda \quad \text{...... (1)}$$

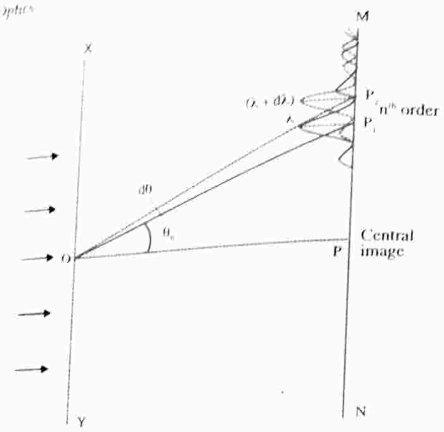


Figure 4.19

For the nth primary maximum at P_2 for the wavelength $\lambda + d\lambda$, we have

$$(a+b)\sin(\theta + d\theta) = n(\lambda + d\lambda) \qquad \text{...... (2)}$$

For the first secondary minimum at P_2 after the nth primary maximum at P_1 corresponding to λ is given by

$$(a+b)\sin(\theta + d\theta) = n\lambda + \frac{\lambda}{N'} \qquad \text{...... (3)}$$

This is obtained from equation (1). To obtain the first secondary minimum we have to introduce an additional path diffrence $\frac{\lambda}{N'}$ to primary maximum corresponding to λ, where N' is the total number of lines on the grating surface.

Comparing equations 2 and 3 we get

$$nd\lambda = \frac{\lambda}{N'}$$

or

$$\frac{\lambda}{d\lambda} = nN' \qquad \text{...... (4)}$$

Thus the resolving power is directly proportional to (i) the order of the spectrum and (ii) the total number of lines on the grating surface.

Note : It may be noted that for a given grating, the distance between the spectral lines is double in the second order spectrum than that in the first order spectrum.

Dispersive power of a grating

Dispersive power of a grating is defined as the rate of change of angle of diffraction with respect to wavelength.

If θ and $\theta + d\theta$ are the angles of diffraction of two neighbouring spectral lines of wavelength λ and $\lambda + d\lambda$, then $\frac{d\theta}{d\lambda}$ is called the dispersive power.

For normal incidence on the grating, we have

$$(a+b)\sin\theta = n\lambda$$

or

$$\sin\theta = Nn\lambda$$

Differentiating we get

$$\cos\theta \, d\theta = Nnd\lambda$$

or

$$\frac{d\theta}{d\lambda} = \frac{Nn}{\cos\theta}$$

Thus dispersive power is (i) directly proportional to the number of lines N per unit length (ii) directly proportional to the order of the spectrum and (iii) inversely proportional to the cosine of the angle of diffraction.

For small values of θ, $\cos\theta = 1$. In this case, for a given grating and for given order, the angular dispersion $d\theta$ is directly proportional to the difference of wavelength $d\lambda$, hence called normal spectrum.

Note

(i) High dispersive power refers to wide separation of spectral lines where as high resolving power refers to the ability of the instrument to show nearby spectral lines as separate ones.

(ii) It is also noted that resolving power increases with increase in the total number of lines on the grating surface where as dispersive power increases with increase in the number of lines per unit length.

Example 16

Two lines in a second order spectrum of a plane transmission grating are resolved. If the lines are due to lights of wavelengths 5890Å and 5896Å. Find the number of lines in the grating.

Solution

$$\lambda_1 = 5890Å,\ \lambda_2 = 5896Å$$

$$n = 2$$

$$\lambda_2 - \lambda_1 = d\lambda = 6Å = 6\times10^{-10}\,m$$

Using $$\frac{\lambda}{d\lambda} = nN$$

or $$N = \frac{\lambda}{nd\lambda} = \frac{5893\times10^{-10}}{2\times6\times10^{-10}}$$

$$N = 491$$

Example 17

A plane transmission grating has 14000 lines to an inch for a length of 6 inches. If the wavelength region is 5×10^{-5}cm , find the resolving power of the grating in the first order and the smallest wavelength difference that can be resolved.

Solution

$$\lambda = 5\times10^{-5}\,cm = 5\times10^{-7}\,m$$

Total number of lines $= 14000 \times 6 = 84000$

$\therefore$ Resolving power $= n\lambda = 1\times84000$

$= 84000$

Using $$\frac{\lambda}{d\lambda} = nN$$

$$\therefore \quad d\lambda = \frac{\lambda}{nN} = \frac{5\times10^{-7}}{84000}$$

$$= 0.59\times10^{-10}\,m = 0.59Å$$

Example 18

A diffraction grating which has 5000 lines/cm is used at normal incidence. Calculate the dispersive power of the grating in the second order spectrum in the wavelength region 6000Å.

Solution

$$a + b = \frac{1}{5000}\,cm = \frac{10^{-2}}{5000}\,m$$

$$n = 2,\ \lambda = 6000Å = 6000\times10^{-10}\,m$$

Using $$(a+b)\sin\theta = n\lambda$$

$$\sin\theta = \frac{n\lambda}{a+b} = \frac{2\times6\times10^{-7}}{10^{-2}}\times5000 = 0.6$$

Dispersive power $$\frac{d\theta}{d\lambda} = \frac{nN}{\cos\theta}$$

$$= \frac{2\times5000\times10^{2}}{0.8} = 1.25\times10^{6}$$

$$[\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-0.6^2} = 0.8]$$

Determination of wavelengths using grating

To determine the wavelength of a given monochromatic light a spectrometer and a grating can be made use of. For this preliminary adjustments of the spectrometer are made. The slit is made narrow. The telescope is brought in line with the collimator. The telescope is adjusted so that the point of intersection of the cross wires coincides with the fixed edge of the image of the slit. The telescope is then clamped. The vernier table is unclamped and adjusted so that the reading of the vernier I is 0° and reading of the vernier II is 180°. The vernier table is then clamped. The telescope is then unclamped and rotated exactly through 90° and then clamped. The grating is then mounted on the grating table, with its ruled surface facing the collimator. The grating table alone is rotated so that the reflected image of the slit coincides with the point of intersection of cross wires. Now the angle of incidence is 45°. The vernier scale is now unclamped and rotated exactly through 45° in such a direction that the ruled surface of the grating faces the collimator. The vernier table is then clamped. The grating is now in the normal incidence position.

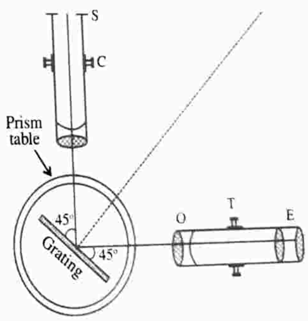


Figure 4.20

Now the telescope is brought in line with first order maximum on the left side of the central maximum. The readings on the vernier table are noted. Then the telescope is brought in line with the first order image on the right side and the vermer readings taken. The difference in the readings of each vernier gives us the value of 2θ. Thus θ is evaluated in each case and the mean θ is used for the calculations.

The angle θ gives the angle of diffraction for the first order. According to the theory of grating

$$\sin\theta = Nn\lambda$$

for $n = 1$

$$\sin\theta = N\lambda$$

or

$$\lambda = \frac{\sin\theta}{N} \quad \text{......(1)}$$

Knowing the number of lines per unit length (N) the wavelength can be found out.

The experiment can be repeated for second order maximum. If $2\theta_2$ is the difference in readings corresponding to the two images of second order on both sides of the central image, then

$$\sin\theta_2 = 2N\lambda$$

or

$$\lambda = \frac{\sin\theta_2}{2N} \quad \text{......(2)}$$

From equations (1) and (2) the mean wavelength can be found out.

UNIVERSITY MODEL QUESTIONS

Section A

(Answer questions in **two** *or* **three** *sentences)*

Short answer type questions

1. What is meant by diffraction of light?
2. What is the condition for diffraction to take place?
3. What is Fresnel diffraction?
4. What is Fraunhofer diffraction?
5. Distinguish between Fresnel diffraction and Fraunhofer diffraction.
6. Write down the condition for obtaining intensity minima and maxima in Fraunhofer single slit diffraction pattern and explain the symbols.
7. Draw the intensity distribution curve in Fraunhofer single slit diffraction pattern.
8. Write down the expression for intensity distribution in the diffraction pattern of Fraunhofer single slit experiment and explain the symbols.
9. What are Airy rings?

10. Write down the expression for intensity distribution in the Fraunhofer diffraction of a circular aperture and explain the symbols.
11. Why the diffraction of sound is more evident in daily life than light?
12. Define resolving power.
13. What is Rayleigh's criterion for resolution.
14. Write down the condition for diffraction maxima and minima in a double slit Fraunhofer diffraction and explain the symbols.
15. Write down the condition for interference maxima and minima in a double slit diffraction pattern.
16. Draw the intensity distribution pattern of double slit diffraction pattern.
17. Define principal maxima of N slit diffraction pattern.
18. What are the positions of principal maxima in an N-slit diffraction pattern.
19. For N = 5 (slits) draw the distribution function $\frac{\sin^2 N\beta}{\sin^2\beta}$ and mark principal maxima, minima and secondary maxima.
20. What is a grating?
21. What is meant by plane transmission grating?
22. State and explain grating law.
23. Define the resolving power of a grating.
24. What are the factors on which the resolving power of a grating depend?
25. Define the dispersive power of a grating and write down an expression for it.
26. Distinguish between resolving power and dispersive power.
27. Why is grating spectrum called a normal spectrum?
28. Why does a grating have closely spaced rulings?

Section B

(Answer questions in a **paragraph** *of about half a page to one page)*

Paragraph / Problem type questions

1. Derive an expression for the width of the central maximum of Fraunhofer single slit diffraction.
2. Explain the diffraction by circular aperture.
3. Derive a condition for the missing orders of interference maxima in double slit diffraction pattern.
4. Derive an expression for angular width of principal maxima in an N slit diffraction pattern.
5. Explain how gratings are prepared in laboratories.