A FULLY POLYNOMIAL APPROXIMATION SCHEME FOR THE TOTAL TARDINESS PROBLEM *

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A fully polynomial approximation scheme is presented for the problem of sequencing jobs for processing by a single machine so as to minimize total tardiness. This result is obtained by modifying the author's pseudopolynomial algorithm for the same problem.

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We describe a fully polynomial approximation scheme for the problem of sequencing jobs for processing by a single machine so as to minimize total tardiness.

The sequencing problem, $1\|\Sigma T_j$ in the notation of [2], is as follows. There are n jobs, where job j has a specified integer *processing time* p_j and a *due date* d_j, for $j = 1, 2, \ldots, n$. Any given sequence of the jobs determines a *completion time*, C_j for each job j, when processing of the first job is started at time $t = 0$. Our objective is to find a sequence which minimizes total tardiness, where the *tardiness* of job j is, by definition,

$$T_j = \max\{0, C_j - d_j\}.$$

This problem is not known to be NP-complete, nor has a polynomial-bounded algorithm been found for it, despite the efforts of many investigators. However a pseudopolynomial algorithm, with worst case running time $O(n^4P)$, where $P = \Sigma p_j$, is known [3]. The results in this paper are obtained by modifying the pseudopolynomial algorithm, knowledge of which is assumed.

The *fully polynomial approximation scheme* we shall describe has the following properties. For any given instance of the total tardiness problem and any given $\varepsilon > 0$, an approximate solution is obtained with total tardiness T_A such that

$$T_A - T^* \leqslant \varepsilon T^*,$$

where T^* is the minimum possible total tardiness. Moreover, the worst-case running time for the computation is bounded by a polynomial (of fixed degree) in n and in $1/\varepsilon$.

It is well-known that a given set of n jobs can be sequenced so as to obtain zero total tardiness if and only if this is possible when the jobs are sequenced in EDD (Earliest Due Date) order, i.e. in nondecreasing order of due dates. Let T_{EDD} denote the total tardiness and let $T_{\max} = \max\{T_j\}$ for a given EDD order. It is known that

$$T_{\max} \leqslant T^* \leqslant T_{\text{EDD}} \leqslant nT_{\max}.$$

Now, using the notation of [3], let $T(S, t)$ denote the maximum total tardiness for a subset S of jobs, with starting time t. For any given subset S, there is an easily computed time t^* such that

$$T(S, t) = 0 \quad \text{for } t \leqslant t^*,$$

$$T(S, t) > 0 \quad \text{for } t > t^*.$$

Moreover, it is easily seen that

$$T(S, t^* + \Delta) \geqslant \Delta \quad \text{for } \Delta \geqslant 0.$$

It follows that in executing the pseudopolynomial algorithm one need actually compute $T(S, t)$ only for $t^* \leqslant t \leqslant nT_{\max}$. Accordingly, $nT_{\max}$ can be sub-

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stituted for P in the running time bound of $O(n^4P)$, yielding $O(n^5T_{max})$.

Now let us replace the given processing times p_j by rescaled processing times,

$$q_j = \left\lfloor \frac{p_j}{K} \right\rfloor, \tag{1}$$

where K is a suitably chosen scale factor. Due dates d_j are also replaced by new due dates d_j/K (with no rounding). Suppose we compute an optimal sequence with respect to these rescaled processing times and due dates. Let T_A' denote the total tardiness for this sequence with respect to processing times $p_j' = Kq_j$ (and original due dates d_j) and T_A denote the total tardiness with respect to the original processing times p_j. Making use of the inequality

$$Kq_j \leqslant p_j < K(q_j + 1),$$

we see that

$$T_A' \leqslant T^* \leqslant T_A < T_A + K\frac{n(n+1)}{2},$$

from which it follows that

$$T^* - T_A < K\frac{n(n+1)}{2}.$$

We want it to be the case that $T^* - T_A \leqslant \varepsilon T^*$. If we choose K to be such that

$$K = \frac{2\varepsilon}{n(n+1)} T_{max}, \tag{2}$$

we shall achieve the stronger result that $T^* - T_A < \varepsilon T_{max}$. Moreover, for this choice of K, the time bound of $O(n^5T_{max}/K)$ becomes $O(n^7/\varepsilon)$, as required to make the approximation scheme fully polynomial.

To recapitulate, the fully polynomial approximation scheme is as follows. Determine T_{max} for an EDD order. If $T_{max} = 0$, stop; $T^* = 0$ and an EDD order achieves T^*. Otherwise, use the value of T_{max} to determine K by (2). Rescale processing times by (1) and divide all due dates by K. Apply the pseudopolynomial algorithm of [2] to the new processing times and due dates. The sequence computed by the algorithm is a suitable approximation, i.e. one for which $T_A - T^* < \varepsilon T_{max} \leqslant \varepsilon T^*$.

It should be noted that Korte and Schrader [1] have given a characterization of a class of problems involving independence systems for which fully polynomial approximation schemes exist. There is no apparent relation between the problems considered in [1] and the total tardiness problem dealt with in this paper.

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