Quantum Chaotic Environments, the Butterfly Effect, and Decoherence

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We investigate the sensitivity of quantum systems that are chaotic in a classical limit to small perturbations of their equations of motion. This sensitivity, originally studied in the context of defining quantum chaos, is relevant to decoherence when the environment has a chaotic classical counterpart.

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The exponential divergence of two trajectories evolving under identical equations of motion from slightly different initial conditions—the famous butterfly effect—is a fingerprint of chaos in classical mechanics. However, an analogous definition of "quantum chaos" based on evolution in Hilbert space is problematic. By unitarity, the overlap between two evolving wave functions—a natural indicator of distance between them—is preserved with time; hence there is no divergence. Peres [1] has suggested an alternative approach, in which one considers two trajectories evolving from identical initial conditions but under slightly different equations of motion, rather than the other way around. Classically, even for small perturbations [2], one generically expects rapid divergence when the systems are chaotic, as the perturbation (i.e., the difference between equations of motion) soon introduces a small displacement between the trajectories. Quantally, the overlap between the wave functions begins at unity, then decays with time, and the rate of this decay—a measure of the sensitivity of quantum evolution to perturbations in the equations of motion—has been suggested as a signature of quantum chaos [3].

The sensitivity of quantum evolution also plays an important role in *environment-induced decoherence* [4,5]. As an illustrative example, consider a composite system consisting of a two-state *spin* (S) and a generic *environment* ($\mathcal{E}$), governed by a Hamiltonian of the form

$$\hat{H}_{\mathcal{E}S} = \hat{H}_{\mathcal{E}}\hat{I}_S + \hat{V}_{\mathcal{E}}^{+}|+\rangle\langle+|_S + \hat{V}_{\mathcal{E}}^{-}|-\rangle\langle-|_S. \quad (1)$$

Here the identity $\hat{I}_S$ and the two projection operators act on the Hilbert space of the spin, whereas $\hat{V}_{\mathcal{E}}^{\pm}$ and $\hat{H}_{\mathcal{E}}$ act on that of the environment. We view $\hat{H}_{\mathcal{E}}$ as the "bare" Hamiltonian for the environment and $\hat{V}_{\mathcal{E}}^{\pm}$ as a perturbative coupling to the state of the spin. An initial state $\Psi(0) = \psi_{\mathcal{E}}(0)[\alpha|+\rangle_S + \beta|-\rangle_S]$ evolves into

$$\Psi(t) = \alpha\psi_{\mathcal{E}}^{+}(t)|+\rangle_S + \beta\psi_{\mathcal{E}}^{-}(t)|-\rangle_S, \quad (2)$$

where the evolution of $\psi_{\mathcal{E}}^{\pm}(t)$ is generated by $\hat{H}_{\mathcal{E}} + \hat{V}_{\mathcal{E}}^{\pm}$. The initially pure state of the spin, $\alpha|+\rangle_S + \beta|-\rangle_S$, evolves into a mixture of the pointer states $\{|+\rangle_S, |-\rangle_S\}$ as a result of monitoring by the environment: once the overlap $|\langle\psi_{\mathcal{E}}^{-}|\psi_{\mathcal{E}}^{+}\rangle|^2$ becomes negligible, the state of the spin can be described in terms of classical probabilities rather than quantum amplitudes. We are thus motivated to ask, *what limits are placed on the sensitivity of a quantum system to perturbations in its equations of motion?* The aim of this Letter is to provide an answer to this question, with emphasis on chaotic systems.

The object of our study will be a pair of wave functions, $\psi_1(t)$ and $\psi_2(t)$, identical at $t = 0$, that evolve under slightly different Hamiltonians, $\hat{H}_1$ and $\hat{H}_2 \equiv \hat{H}_1 + \hat{V}$, respectively. In the context of Eq. (1), $\hat{H}_1$ and $\hat{H}_2$ correspond to $\hat{H}_{\mathcal{E}} + \hat{V}_{\mathcal{E}}^{-}$ and $\hat{H}_{\mathcal{E}} + \hat{V}_{\mathcal{E}}^{+}$, so that $\hat{V} = \hat{V}_{\mathcal{E}}^{+} - \hat{V}_{\mathcal{E}}^{-}$. As in Refs. [3,6], our measure of "sensitivity" will be the rate of decay of the quantum overlap

$$O_q(t) = |\langle\psi_1(t)|\psi_2(t)\rangle|^2. \quad (3)$$

We first derive, from the uncertainty principle, a general bound on this rate. We will then clarify the difference in robustness between classical chaotic systems and their quantum counterparts, by comparing the smallest structures found in classical probability distributions to those found in quantum Wigner functions. Finally, we will illustrate central issues with a numerical example, placing bounds on the time needed for a (classically chaotic) quantum environment to decohere a quantum system of interest.

Quantum lower bound for overlap decay and decoherence time.—Using the projection operator $\hat{P}(t) = |\psi_2\rangle\langle\psi_2|$ to rewrite the above-defined quantum overlap as $O_q(t) = \langle\psi_1|\hat{P}|\psi_1\rangle$, and applying the Schrödinger equation in the Heisenberg picture, we obtain

$$\frac{dO_q}{dt} = -\frac{i}{\hbar}\langle[\hat{V}, \hat{P}]\rangle, \quad (4)$$

where $\langle\cdots\rangle \equiv \langle\psi_1|\cdots|\psi_1\rangle$. Now, the uncertainty relation for $\hat{V}$ and $\hat{P}$ is $\Delta V\Delta P \geq |\langle[\hat{V}, \hat{P}]\rangle|/2$, where $(\Delta V)^2 = \langle\hat{V}^2\rangle - \langle\hat{V}\rangle^2$ is the variance of the operator $\hat{V}$ in the state ψ_1, and similarly $(\Delta P)^2 = \langle\hat{P}^2\rangle - \langle\hat{P}\rangle^2 = O_q - O_q^2$. Combining this with (4) gives

$$-\frac{dO_q}{dt} \leq \left|\frac{dO_q}{dt}\right| \leq \frac{2}{\hbar}\Delta V(O_q - O_q^2)^{1/2}, \quad (5)$$

leading, after some algebra, to the inequality

$$O_q(t) \geq \cos^2\left(\frac{1}{\hbar}\int_0^t \Delta V dt'\right) \equiv \cos^2\phi(t), \quad (6)$$

valid until $\phi(t)$ (which never decreases) reaches $\pi/2$. Note that by reversing the roles of ψ_1 and ψ_2 in this argument, we typically obtain a quantitatively different, though equally valid, result. We can therefore view ΔV appearing in (6) as the spread of $\hat{V}$ in either state ψ_1 or ψ_2, whichever gives the tighter bound.

When ψ_1 and ψ_2 represent states of a quantum environment (as discussed above), then (6) gives a lower bound on the decoherence times, defined by $O_q(\tau_D) \approx 0$:

$$\tau_D \gtrsim \pi\hbar/2\overline{\Delta V}. \tag{7}$$

Here $\overline{\Delta V}$ is a typical value of ΔV, which characterizes the perturbation caused in the evolving quantum environment by the coupling to the system which it decoheres.

Quantum and classical overlap in terms of phase space distributions.— Apart from studying the sensitivity of quantum evolution in its own right, we would like to compare it with classical sensitivity, particularly in the case of chaotic evolution. We will work with functions in phase space as these transparently suggest a classical counterpart of the quantum overlap $O_q(t)$.

Equation (3) can be rewritten as

$$O_q(t) = 2\pi\hbar \int W_1(x, p, t)W_2(x, p, t)dxdp, \tag{8}$$

where the W_i's are Wigner functions [7] corresponding to ψ_1 and ψ_2, evolving under $\hat{H}_1$ and $\hat{H}_2$ [8]. Let us now consider two classical phase space distributions, $L_1(x, p, t)$ and $L_2(x, p, t)$, obeying the Liouville equation under the respective classical Hamiltonians $H_1(x, p)$ and $H_2(x, p)$. We set the initial conditions for the L's to be the same as those for the W's: $L_1 = L_2 = W_1 = W_2$ at $t = 0$. In view of (8) it is natural to define a classical overlap,

$$O_c(t) = 2\pi\hbar \int L_1(x, p, t)L_2(x, p, t)dxdp, \tag{9}$$

where the (arbitrary) normalization factor $2\pi\hbar$ was chosen so that $O_c(0) = O_q(0) = 1$. By comparing the decay times of $O_q(t)$ and $O_c(t)$, we now have a setup for comparing quantum and classical sensitivity to perturbations in the equations of motion. Because of the relevance to decoherence, we will refer to these as the quantum and classical *decoherence times,* though in the classical case this is just convenient nomenclature.

The smallest structures of phase space distributions.— A central hypothesis of this Letter is that the time scale for the decay of the overlaps O_c and O_q is determined primarily by the size of the smallest structures in the corresponding phase space distributions, with particular relevance when the classical evolution is chaotic. In both cases we have two initially identical phase space distributions (the L's or W's) evolving with time while slowly accumulating a relative displacement due to the perturbation; a substantial decay of overlap occurs when this displacement is large enough that the two functions no longer "sit one on top of the other." Clearly, this depends not only on the rate at which the functions move apart, but also on the local smallness of their structure, as this determines the degree of displacement needed to kill the overlap. The difference between the decay of overlap in the classical, chaotic case, and in its quantum counterpart, arises because of the qualitatively different mechanisms governing the emergence of small-scale details in the corresponding phase space distribution.

In the classical case, the size of local structure in L_1 and L_2 shrinks exponentially with time, due to the stretching and folding associated with chaotic evolution: the probability distributions become thin and elongated, with a local width decreasing as $\exp(-\lambda t)$, where λ is the largest Lyapunov exponent. As there is no lower bound to this smallness, it is clear that the decay time will be set predominantly by the Lyapunov time. By contrast, there are limits on the fineness of detail that can develop in the Wigner function, W; W of a superposition of two identical Gaussians separated by ΔX—a Schrödinger cat-like state—exhibits interference fringes in momentum on a scale $\delta p \simeq \hbar/\Delta X$ [4]. More generally, when spread over an area $A = \Delta X\Delta P$ in two-dimensional phase space, W exhibits local structure on scales $\delta p \simeq \hbar/\Delta X$, $\delta x \simeq \hbar/\Delta P$ [9,10]. The corresponding phase space scale is associated with the sub-Planck action $a \sim \hbar^2/A$, which has physical consequences [10]. Most notably for our purposes, the decay of O_q occurs when the relative displacement of W_1 and W_2 is such that the product of their smallest scale fringes gives rise to a rapidly oscillating contribution to the integrand in Eq. (8). Two examples serve to build up intuition related to these issues.

Example I.—Let us assume that identical wave functions ψ_1 and ψ_2 are superpositions of N Gaussians: $\tilde{G}_j \sim \exp[-(x - x_j)^2/\xi^2]\exp(ip_jx/\hbar)$

$$\psi_1(x) = \psi_2(x) = \sum_{j=1}^{N} \tilde{G}_j(x; x_j, p_j).$$

The corresponding Wigner functions W_1 and W_2 consist of N coherent-state Gaussians G_j, centered at points (x_j, p_j), as well as pairwise interference terms $G_{j,k}$:

$$W = \sum_{j=1}^{N} G_j + \sum_{j<k} G_{j,k}.$$

We assume (following [10]) that the coherent-state Gaussians are sparse: each pair G_j and G_k is well separated by the "distance" $d_{j,k}$ in phase space. The interference term $G_{j,k}$ is then another Gaussian located halfway between G_j and G_k, modulated by an oscillatory factor of frequency $d_{j,k}/\hbar$ and twice the amplitude of G_j [4]. The overlap (8) of W_1 and W_2 then works out to be $O_q \approx Ng + (N - 1)Ng$, where $g = \int G_j^2 dxdp$. The first term corresponds to the overlap deposited in the Gaussians G_j, the second in the interference terms $G_{j,k}$. Thus, for large N *most of the overlap resides in the interference terms.* Suppose we now displace W_2 relative to W_1 by a distance at least on the order of the size of a typical interference

fringe, but small compared to the size of the G_j's. (In a dynamical setting, this relative displacement is achieved by evolution under slightly different Hamiltonians, a situation we study numerically below.) The contribution to O_q from a given G_j will be nearly unaffected, but the product of $G_{j,k}$ with the displaced image of itself will oscillate rapidly and contribute little to the integral in Eq. (8). The net result is $O_q \sim 1/N \ll 1$. This is somewhat counterintuitive: for a fixed number of Gaussians of fixed size, we can increase the sensitivity of the system—as measured by the perturbation needed to suppress the overlap $|\langle\psi_1|\psi_2\rangle|^2$—simply by increasing the average distance between the Gaussians, or equivalently the total area occupied in phase space.

Example II.—Classical probability distributions have no interference fringes. However, there is no bound on the smallness of structures resulting from chaotic stretching and folding. Let us examine a simplified model of the classical evolution of two Gaussians in phase space:

$$L_1(x, p, t = 0) = L_2(x, p, t = 0) = \frac{1}{2\pi\sigma_x\sigma_p} e^{-\frac{x^2}{2\sigma_x^2} - \frac{p^2}{2\sigma_p^2}},$$

with x and p dimensionless. We now assume that with time both L's are exponentially stretched in p and squeezed in x, in an area-preserving way: $\sigma_p(t) = \sigma\exp(\lambda t)$, $\sigma_x(t) = \sigma\exp(-\lambda t)$; furthermore, while the centroid of L_1 remains fixed at $x = p = 0$, the centroid of L_2 drifts with a constant velocity $\mathbf{v} = (v_x, v_p)$. These assumptions mock up the relevant features of chaotic evolution under slightly different Hamiltonians, where $|\mathbf{v}|$ is an indicator of the size of the perturbation. A simple calculation gives us the following overlap between L_1 and L_2:

$$O_c(t) = e^{-(v_x t\cdot e^{\lambda t}/2\sigma)^2} e^{-(v_p t\cdot e^{-\lambda t}/2\sigma)^2}. \qquad (10)$$

Unless $v_x e^{\lambda t} < v_p e^{-\lambda t}$, the first factor will dominate, and the overlap will decay to negligible values on a time scale set by λ^{-1}. We note that the above inequality will almost certainly be obeyed for $t \gg \lambda^{-1}$: $v_x/v_p < e^{-2\lambda t}$ will be violated only for perturbations almost exactly (to within an angle $\theta \sim e^{-2\lambda t}$) aligned with the direction of the extended Gaussian. Moreover, in a real chaotic system probability distributions are not just stretched but also folded, which further strengthens our conclusions, as a phase space-dependent alignment of $\mathbf{v}$ with the instantaneous forms of the probability distribution would be required to suppress decay on the Lyapunov time scale.

While both of these examples are highly simplified, we believe they capture the essential physics. We now present numerical results illustrating actual evolution.

Numerical simulation.—A time-dependent Hamiltonian that generates chaos in one dimension is

$$H = \frac{p^2}{2m} - \kappa\cos(x - l\sin(t)) + a\frac{x^2}{2}. \qquad (11)$$

(See Ref. [11].) For parameter values $m = 1$, $\kappa = 0.36$, $l = 3.8$, and $a = 0.01$, the stroboscopic Poincaré surface of section consists of four islands of stability surrounded by a chaotic sea. We have simulated both quantum and classical evolution, starting from a Gaussian distribution centered outside the regular region at $t = 0$, and evolving under H until time $t = T$, at which point a perturbation is turned on and the evolution forks into two branches governed by the Hamiltonians

$$H_\pm = \frac{p^2}{2m} - \kappa\cos[x - l\sin(T + \tau)] + a\frac{(x \pm \epsilon)^2}{2}, \qquad (12)$$

where $\epsilon = 0.5$ and $\tau \equiv t - T$. The perturbation is thus $V = H_+ - H_- = 2a\epsilon x$. The preparation time interval $(0 \le t \le T)$ allows the distributions to develop small structures in phase space. After the perturbation is turned on at $\tau = 0$, we monitor the decay of overlaps.

Figure 1 shows how the decoherence time τ_d (defined here as the time τ at which the overlap decreases to a value 0.9 [12]) depends on preparation time T for quantum and classical evolution. For short preparation times, both are equally sensitive to the perturbation applied at $\tau = 0$, reflecting the fact that the size of the smallest structure is basically the same in both cases. However, once the distributions have spread over much of the dynamically accessible area in the phase space, which occurs at $T \approx 20$, the size of interference fringes in the Wigner functions saturates, resulting in more or less constant decoherence times even for long T. The crosses in Fig. 1 denote lower bounds on the value of t_d obtained from (6) by setting $\cos^2\phi = 0.9$, with ΔV evaluated directly from the simulation. These lower bounds are very close to the actual decoherence times, indicating that the quantum states used in our evolution are close to minimum uncertainty states with respect to the uncertainty principle mentioned after (4). In the absence of structure saturation in the classical distributions L, the classical decoherence time continues to decrease with increasing preparation time. While the computational cost of the classical simulations became prohibitive for

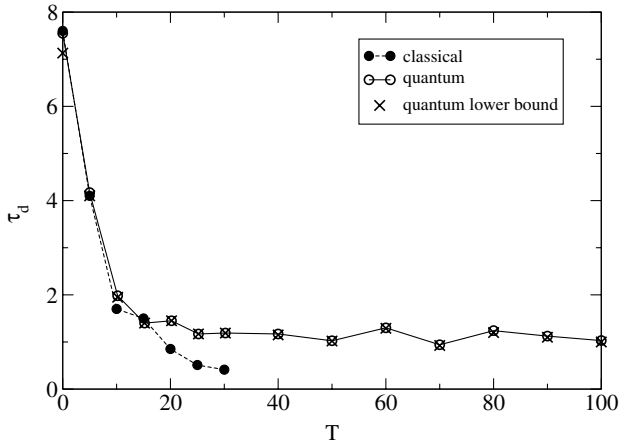

FIG. 1. Time after which the overlap decreases to 0.9 versus preparation time for classical and quantum environments.

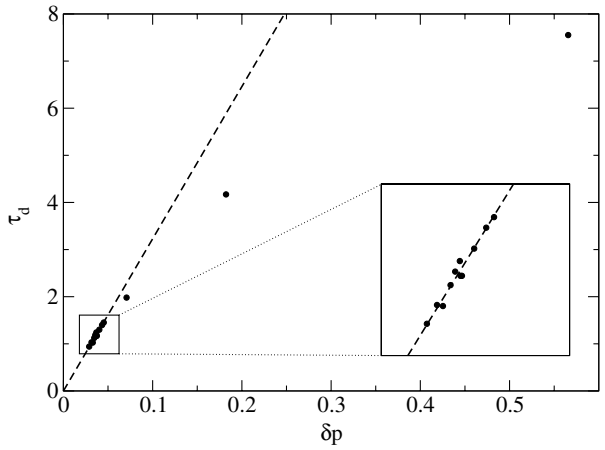

FIG. 2. Time after which overlap falls by 10% versus δp.

$T > 30$, the observed decrease in τ_d is consistent with a rapid approach toward zero.

To further investigate the relevance of sub-Planck structure in the Wigner function to the decay of O_q, let us define the spread of the system in position as $\Delta X = (\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2)^{1/2}$, and in momentum $\Delta P = (\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2)^{1/2}$, with averages $\langle \cdots \rangle$ taken at $\tau = 0$. This translates to interference fringes of size $\delta x \approx \hbar/\Delta P$ in position and $\delta p \approx \hbar/\Delta X$ in momentum. Note that in the quantum counterpart of a classically chaotic system the dominant effect of the exponential instability is the decrease of δx and δp in size. This is caused by the exponential spreading of the effective support of the state throughout the available phase space, until saturation occurs. Using our data, we have determined δx and δp for each of the 14 preparation times T shown in Fig. 1. For the parameter values and form of the perturbation we have chosen, we have found that δp is more relevant than δx for the decay of $O_q(t)$. Therefore in Fig. 2 we plot the dependence of τ_d on δp. The linear dependence observed for small values of δp (large T) is not unexpected. Recall that decoherence is achieved when the displacement of W_2 relative to W_1 becomes comparable to δp. In the regime of small δp, this occurs sufficiently rapidly that the value of δp does not change much during the process. Hence, given a constant rate of relative drift in the momentum direction (due to the form of our perturbation, $\hat{V} = 2a\epsilon\hat{x}$), we expect $\tau_d \propto \delta p$. On the other hand, when δp is initially large (small T), then the decoherence time will also be large, and δp itself will decrease during this time; hence, we expect in this regime to obtain τ_d smaller than suggested by the initial value of δp, in agreement with the three highest data points shown in Fig. 2. Note that all of these conclusions relied on choosing an initial state with support contained in a chaotic region of the phase space of our system. If we had chosen an initial state residing in one of the regular islands, chaos would not be relevant, as illustrated by somewhat analogous numerical studies of entropy production (see, e.g., [13]).

Conclusions.—We have investigated the sensitivity of classical chaotic systems and their quantum counterparts to perturbations in their equations of motion. We have derived a lower bound for the decay of $O_q(t)$. The sensitivity (in both the classical and quantum cases) is set by the size of the smallest structure of the related phase space functions. We have discussed the relevance of these results to the ability of quantum environments to rapidly decohere systems to which they are coupled.

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