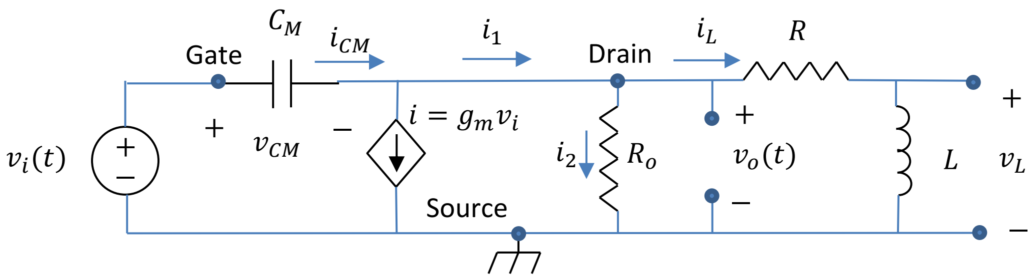
A. A small signal model of an N type MOSFET driving an inductive load is shown below. Use circuit theory to find the system differential equations in the state space matrix form as in eq. (5 – 1) and (5 – 2) in Handout 5. Remember: there is going to be one first order differential equation per storage element (i.e. an inductor or a capacitor).
   - What is the order of the system?
   - What are the state variables?



List of constants and variables:
- $v_i(t) =$ System input (volts).
- $v_o(t) =$ System output (volts).
- $g_m =$ Transconductance (constant) (S) = (Ω^{-1})
- $C_M = c_{gd} =$ Miller capacitance (constant) (F). It is between input and output and creates an undesirable feedback.
- $R_o =$ Output resistance (constant) (Ω).
- $R =$ Load resistance (constant) (Ω).
- $L =$ Load inductance (constant) (H).

ANSWER A:
- First identify the energy storage elements and write their equations:

$$C_M\colon\ i_{CM} = C_M \frac{dv_{CM}}{dt}\ ; \qquad L\colon\ v_L = L\frac{di_L}{dt}$$

- State variables are those whose derivatives exist; hence the state vector is:

$$\mathbf{x} = \begin{bmatrix} v_{CM} \\ i_L \end{bmatrix}$$

- Rewrite energy storage elements equations with derivatives alone on the left hand side:

$$C_M\colon \frac{dv_{CM}}{dt} = \frac{1}{C_M}\, i_{CM}\ ; \qquad L\colon\ \frac{di_L}{dt} = \frac{1}{L} v_L \Longrightarrow \frac{d}{dt}\begin{bmatrix} v_{CM} \\ i_L \end{bmatrix} = \begin{bmatrix} \frac{1}{C_M}\, i_{CM} \\ \frac{1}{L} v_L \end{bmatrix} \qquad \boxed{(1)}$$

- Now find the right hand side, i.e. i_{CM} and v_L, in terms of state variables v_{CM}, i_L and input v_i from above circuit.

- To do so, first write the general state space equations (input is v_i, output is v_o):

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}\begin{bmatrix} v_{CM} \\ i_L \end{bmatrix} = \mathbf{A}\begin{bmatrix} v_{CM} \\ i_L \end{bmatrix} + \mathbf{B}v_i$$

$$\mathbf{y} = v_o = \mathbf{C}\begin{bmatrix} v_{CM} \\ i_L \end{bmatrix} + \mathbf{D}v_i$$

- Note the dimensions of variables: input size = 1, output size = 1, state vector size = 2; hence: $\mathbf{A}$ is 2 X 2, $\mathbf{B}$ is 2 X 1, $\mathbf{C}$ is 1 X 2 and $\mathbf{D}$ is 1 X 1; substitute equation 1 for the derivatives:

$$\frac{d}{dt}\begin{bmatrix} v_{CM} \\ i_L \end{bmatrix} = \begin{bmatrix} \frac{1}{C_M} i_{CM} \\ \frac{1}{L} v_L \end{bmatrix} = \overbrace{\begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}}^{\mathbf{A}} \begin{bmatrix} v_{CM} \\ i_L \end{bmatrix} + \overbrace{\begin{bmatrix} \cdots \\ \cdots \end{bmatrix}}^{\mathbf{B}} v_i \tag{2}$$

$$v_o = \overbrace{\begin{bmatrix} \cdots & \cdots \end{bmatrix}}^{\mathbf{C}} \begin{bmatrix} v_{CM} \\ i_L \end{bmatrix} + \overbrace{\begin{bmatrix} \cdots \end{bmatrix}}^{\mathbf{D}} v_i \tag{3}$$

- It is easier to begin with an equation for v_o as v_o is used in future; apply Kirchhoff's voltage law (KVL) around the outer loop:

$$v_i = v_{CM} + v_o \Longrightarrow v_o = -v_{CM} + v_i \tag{4}$$

- From above, the matrix expression for output v_o may be written:

$$v_o = \overbrace{\begin{bmatrix} -1 & 0 \end{bmatrix}}^{\mathbf{C}} \begin{bmatrix} v_{CM} \\ i_L \end{bmatrix} + \overbrace{\begin{bmatrix} 1 \end{bmatrix}}^{\mathbf{D}} v_i \tag{5}$$

- To find the first row of $\mathbf{A}$ and $\mathbf{B}$ matrices in equation (2), apply Kirchhoff's current law (KCL) at the node right after the capacitor (see figure on page 1):

$$i_{CM} = i + i_1; \quad i = g_m v_i \Longrightarrow i_{CM} = g_m v_i + i_1 \tag{6}$$

- Apply KCL at "Drain" node in the figure:

$$i_1 = i_2 + i_L; \quad i_2 = \frac{v_o}{R_o} \Longrightarrow \quad i_1 = \frac{v_o}{R_o} + i_L \tag{7}$$

- Substitute for: v_o from (4):

$$i_1 = \frac{-v_{CM} + v_i}{R_o} + i_L \tag{8}$$

- Substitute (8) in (6):

$$i_{CM} = g_m v_i + i_1 \Longrightarrow i_{CM} = g_m v_i + \frac{-v_{CM} + v_i}{R_o} + i_L \tag{9}$$

- Factor v_i and rewrite (9) in proper variables order:

$$i_{CM} = -\frac{1}{R_o} v_{CM} + i_L + \left(g_m + \frac{1}{R_o}\right) v_i \tag{10}$$

- Divide (10) by $1/C_M$:

$$\frac{1}{C_M} i_{CM} = -\frac{1}{R_o C_M} v_{CM} + \frac{1}{C_M} i_L + \left(\frac{g_m}{C_M} + \frac{1}{R_o C_M}\right) v_i \tag{11}$$

- The first row of $\mathbf{A}$ and $\mathbf{B}$ matrices in equation (2) may now be filled:

$$\frac{d}{dt}\begin{bmatrix} v_{CM} \\ i_L \end{bmatrix} = \begin{bmatrix} \frac{1}{C_M} i_{CM} \\ \frac{1}{L} v_L \end{bmatrix} = \overbrace{\begin{bmatrix} -\frac{1}{R_o C_M} & \frac{1}{C_M} \\ \cdots & \cdots \end{bmatrix}}^{\mathbf{A}} \begin{bmatrix} v_{CM} \\ i_L \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{g_m}{C_M} + \frac{1}{R_o C_M} \\ \cdots \end{bmatrix}}^{\mathbf{B}} v_i \tag{12}$$

- It remains to find v_L in terms of v_{CM}, i_L, v_i. Note that:

$$v_o = i_L R + v_L \Longrightarrow \qquad v_L = v_o - i_L R \tag{13}$$

- Substitute for v_o from (4):

$$v_L = v_o - i_L R;\ v_o = -v_{CM} + v_i \Longrightarrow \quad v_L = -v_{CM} + v_i - i_L R \tag{14}$$

- Divide above by L and rewrite in proper variables order:

$$\frac{1}{L} v_L = -\frac{1}{L} v_{CM} - \frac{R}{L} i_L + \frac{1}{L} v_i \tag{15}$$

- The 2nd row of (12) may now be filled:

$$\frac{d}{dt}\begin{bmatrix} v_{CM} \\ i_L \end{bmatrix} = \begin{bmatrix} \frac{1}{C_M} i_{CM} \\ \frac{1}{L} v_L \end{bmatrix} = \overbrace{\begin{bmatrix} -\frac{1}{R_o C_M} & \frac{1}{C_M} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}}^{\mathbf{A}} \begin{bmatrix} v_{CM} \\ i_L \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{g_m}{C_M} + \frac{1}{R_o C_M} \\ \frac{1}{L} \end{bmatrix}}^{\mathbf{B}} v_i \tag{16}$$

- Final answer:

$$\underbrace{\frac{d}{dt}\begin{bmatrix} v_{CM} \\ i_L \end{bmatrix}}_{d\mathbf{x}/dt} = \overbrace{\begin{bmatrix} -\frac{1}{R_o C_M} & \frac{1}{C_M} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}}^{\mathbf{A}} \underbrace{\begin{bmatrix} v_{CM} \\ i_L \end{bmatrix}}_{\mathbf{x}} + \overbrace{\begin{bmatrix} \frac{g_m}{C_M} + \frac{1}{R_o C_M} \\ \frac{1}{L} \end{bmatrix}}^{\mathbf{B}} \underbrace{v_i}_{\mathbf{u}}\ ; \qquad \underbrace{v_o}_{\mathbf{y}} = \overbrace{\begin{bmatrix} -1 & 0 \end{bmatrix}}^{\mathbf{C}} \underbrace{\begin{bmatrix} v_{CM} \\ i_L \end{bmatrix}}_{\mathbf{x}} + \overbrace{[1]}^{\mathbf{D}} \underbrace{v_i}_{\mathbf{u}}$$

B. (Use MATLAB) Substitute the following numbers for the constants in the system state space differential equations and find the system transfer function $G(s) = \frac{V_o(s)}{V_i(s)}$ by equation (5 – 7) in Handout 5 (all numbers: 5 significant digits i.e. use "`vpa(…, 5)`" command:

- $g_m = 0.005$ S.
- $C_M = 20 \times 10^{-12}$ F.
- $R_o = 200\ \Omega$
- $R = 1000\ \Omega$
- $L = 10^{-4}$ H

C. When v_i is a step input of 1 volt, find the system response $v_o(t)$ by finding the inverse Laplace transform of $V_o(s)$ symbolically by MATLAB. Plot system response v_o vs. time using MATLAB control system toolbox: first define the transfer function by "`tf`" command and then use "`step`" command to plot the response (don't specify the final time in step command).

COMMENTS:

- As seen below, the step response is negative even though the input is a positive unit step; this is because the circuit is an inverting amplifier.
- Compare the symbolic results to that of the control system toolbox. Note that $y(t)$ obtained symbolically is an exact closed form solution which is not available in control system toolbox.

PART B:

```
syms s t        % Laplace symbols.

% Circuit Data:
gm = 0.005;
cm = 20e-12;
ro = 200;
r = 1000;
l = 1e-4;

% A,B,C,D matrices:
A = [-1/(cm*ro), 1/cm; -1/l, -r/l]
B = [gm/cm + 1/(ro*cm); 1/l]
C = [-1, 0]
D = 1

Gs = C * (s*eye(2) - A)^-1 * B + D;
Gs = simplifyFraction(Gs);
vpa(Gs, 5);
disp(' ******************  G(s):')
pretty(ans)

% Make the leading coefficient in denominator = 1:

[NN, DD] = numden(Gs)   % Separate the numerator and denominator.
DDcs = coeffs(DD)       % Find denominator coefficients of powers of s.
DD = DD/DDcs(3)         % Divide both denominator and numerator by the
```

```
                        % coeff of s^2.
NN = NN/DDcs(3)

DD = collect(DD)
NN = collect(NN)

% Reconstruct G(s):
Gs = NN/DD

% Answer part B:

disp(' ***************** Simplified G(s):')
vpa(Gs, 5)
pretty(ans)
%
```

```
A =

   1.0e+10 *

   -0.0250    5.0000
   -0.0000   -0.0010


B =

   1.0e+08 *

    5.0000
    0.0001


C =

    -1     0


D =

     1

 ******************  G(s):
             7  2            15               22
  (- 3.3554 10  s  + 8.0531 10   s + 8.3886 10  ) 1.0
- ----------------------------------------------------
              7  2            15               23
      3.3554 10  s  + 8.7242 10   s + 1.0066 10


NN =

33554432*s^2 - 8053063680000001*s - 83886080000000010000000
```

```
DD =

33554432*s^2 + 8724152320000001*s + 100663296000000010000000


DDcs =

[ 100663296000000010000000, 8724152320000001, 33554432]


DD =

s^2 + (8724152320000001*s)/33554432 + 786432000000000078125/262144


NN =

s^2 - (8053063680000001*s)/33554432 - 655360000000000078125/262144


DD =

s^2 + (8724152320000001*s)/33554432 + 786432000000000078125/262144


NN =

s^2 - (8053063680000001*s)/33554432 - 655360000000000078125/262144


Gs =

-(- s^2 + (8053063680000001*s)/33554432 + 655360000000000078125/262144)/(s^2 +
(8724152320000001*s)/33554432 + 786432000000000078125/262144)

 ***************** Simplified G(s):

ans =

-(1.0*(- 1.0*s^2 + 2.4e8*s + 2.5e15))/(s^2 + 2.6e8*s + 3.0e15)

          2         8           15
  (- 1.0 s  + 2.4 10  s + 2.5 10  ) 1.0
- -------------------------------------
         2         8           15
        s  + 2.6 10  s + 3.0 10
```

Part C:

```
Ys = Gs * 1/s
Ys = vpa(Ys,5);
yt = ilaplace(Ys);
```

```
% Answer C:
yt = vpa(yt,5)

% Plotting using control system toolbox ss, tf and step commands:

Gss = ss(A,B,C,D);      % Define state space system.
Gs = tf(Gss)            % Find its transfer function.
step(Gs)                % Plot step response.
```

```
Ys =

-(- s^2 + (8053063680000001*s)/33554432 + 655360000000000078125/262144)/(s*(s^2 +
(8724152320000001*s)/33554432 + 786432000000000078125/262144))

yt =

2.0264*exp(-2.479e8*t) - 0.19305*exp(-1.2102e7*t) - 0.83333

Gs =

  s^2 - 2.4e08 s - 2.5e15
  -----------------------
   s^2 + 2.6e08 s + 3e15

Continuous-time transfer function.
```