San Francisco State University

Engineering 305

Laboratory #6 – Fourier transform

Purpose

In this lab, you will investigate the Fourier transform and the frequency response.

Background reading includes:

- Lathi, Chapter 5
- Holton notes, Unit 6-7

Background

In this lab examine the Fourier transform its relation to the notion of the frequency response. We start with a discussion of computing the frequency response theoretically and end up with a method of measuring the frequency response that you could use in the laboratory.

Fourier transform

In the previous laboratory, we asserted that the Fourier series can be used to represent arbitrary time functions over an interval given time interval of length *T*. In exercise 5d of the last assignment, we argued that in the limit as $T \to \infty$, the number of Fourier series coefficients in any given range of frequencies also goes to infinity, so that we arrive at the expressions for the Fourier transform

$$\begin{aligned} x(t) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \qquad \text{Synthesis} \\ X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-jk\omega t}\,dt \qquad \text{Analysis} \end{aligned} \tag{L6.1}$$

The Fourier transform, $X(\omega)$, is a frequency transformation of a function, *x(t),* that is valid for all time. The Fourier transformation can be taken of any signal.

The Fourier transformation of the impulse response of a system, *h(t),* has a particular significance in the study of linear systems and is given a special name: the frequency response, $H(\omega)$.

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt \tag{L6.2}$$

The frequency response of a linear system characterizes the system completely in the frequency domain in exactly the same sense that the impulse response characterizes it in the time domain. Since there is a one to one relation between *h(t)* and $H(\omega)$, these characterizations are equivalent. However, it is often more convenient and more insightful to examine the behavior of systems in the frequency domain.

Computing the frequency response

Clearly, if you are given the impulse response, *h(t),* it is easy to compute the frequency response. For example, consider a simple circuit with input *x(t)* and output *y(t)* shown in Figure 1.

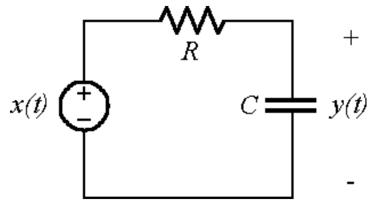


Figure 1: RC circuit

The differential equation for this system is obtained by any of a number of methods, for example by solving the circuit equations using KVL, KCL and the element constraints:

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)\,. \tag{L6.3}$$

The impulse response, obtained using methods outline in Laboratory 2, is

$$h(t) = \frac{1}{RC}e^{-t/RC}u(t)\,. \tag{L6.4}$$

The frequency response is directly calculated from Equation (L6.2) as

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \frac{1}{RC}\int_{-\infty}^{\infty} e^{-t/RC}e^{-j\omega t}u(t)dt \\ &= \frac{1}{RC}\int_{0}^{\infty} e^{-(j\omega + 1/RC)t}dt \\ &= \frac{1}{1 + j\omega RC} \end{aligned} \tag{L6.5}$$

As you can see the frequency response is clearly a complex function of ω. To visualize it, we need to plot the magnitude and phase. We can compute magnitude and phase theoretically and plot them. From Equation (L6.5), we get

$$\begin{aligned} |H(\omega)| &= \frac{1}{\sqrt{1 + (\omega RC)^2}} \\ \measuredangle H(\omega) &= -\tan^{-1}(\omega RC) \end{aligned} \tag{L6.6}$$

This frequency response can be easily plotted in Matlab. To fix ideas, let's let $R = 1$ and $C = 0.1$. The results are shown in Figure 2, in which we plot the magnitude on the upper plot and the phase on the lower plot, both on linear frequency and amplitude scales.

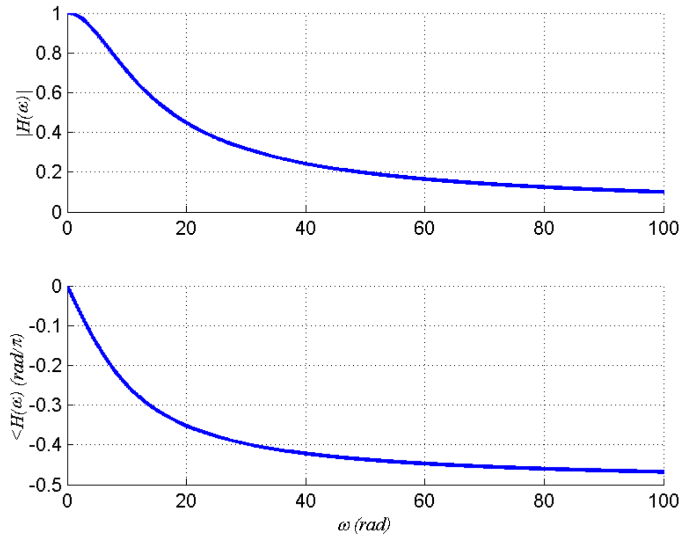


Figure 2: Frequency response of RC circuit on a linear scale

Clearly the magnitude of the frequency response is large at low frequencies and falls of at high frequencies, and the phase starts at 0 and goes to $-\pi/2$ as frequency increases. It turns out to be much more valuable to plot the frequency response on a log-log scale. Specifically, we will plot $20\log_{10}(|H(\omega)|)$ vs. $\log_{10}(\omega)$ on the upper plot and $\measuredangle H(\omega)$ vs. $\log_{10}(\omega)$ on the lower plot, yielding Figure 3.

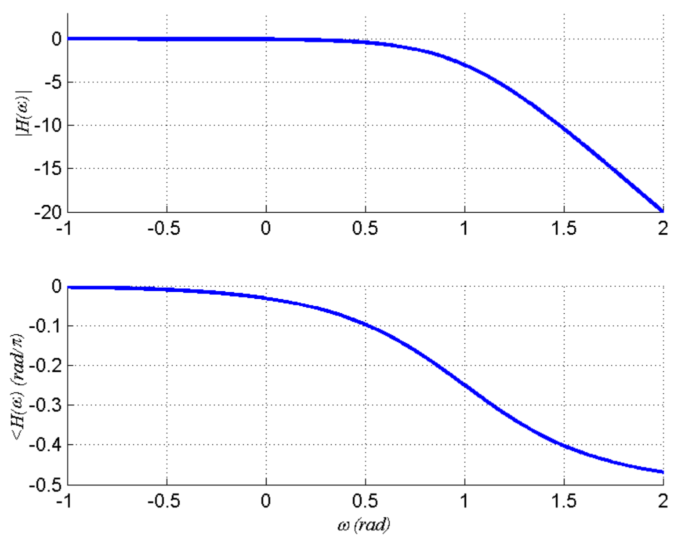


Figure 3: Frequency response of RC circuit on a log-log scale

This is a bit more informative. You can see that the magnitude of the response has basically two different behaviors. For frequencies below the cutoff frequency, $\omega = {}^{1}\!/_{RC} = 10$, the magnitude of the frequency response if 0 on a log scale which means 1unity on a linear scale, and the phase is zero. For frequencies above the cutoff frequency, the response drops linearly on a log-log scale with a slope of 1.

In the preceding treatment, we calculated the frequency response directly from the impulse response using Equation (L6.2). It is also possible to obtain the frequency response directly from the differential equation. First, we recall a couple of properties of Fourier transforms:

1. Convolution property. For any system with input *x(t)*, output *y(t)* and impulse response *h(t)*,

$$y(t) = x(t) * h(t)\,, \tag{L6.7}$$

The convolution property of Fourier transforms tells us that

$$Y(\omega) = X(\omega)H(\omega) \tag{L6.8}$$

hence,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}\,. \tag{L6.9}$$

2. Derivative property. Using the notation that $\Im\{y(t)\}$ means the Fourier transform of *y(t)* , we can easily show that the Fourier transform of the derivative of *y(t)* is

$$\Im\left\{\frac{dy(t)}{dt}\right\} = j\omega\,\Im\{y(t)\} = j\omega Y(\omega)\,. \tag{L6.10}$$

The Fourier transform of higher order derivatives of *y(t)* are equally easily obtained:

$$\Im\left\{\frac{d^n y(t)}{dt^n}\right\} = (j\omega)^n Y(\omega)\,. \tag{L6.11}$$

To obtain the frequency response directly from the differential equation of Equation (L6.3), we take the Fourier transform of both sides of the equation and use the linearity and derivative properties to obtain:

$$\Im\left\{\frac{dy(t)}{dt} + \frac{1}{RC}y(t)\right\} = \Im\left\{\frac{1}{RC}x(t)\right\},$$

so

$$j\omega Y(\omega) + \frac{1}{RC}Y(\omega) = \frac{1}{RC}X(\omega)\,. \tag{L6.12}$$

Collecting terms we get

$$Y(\omega)\left(j\omega + \frac{1}{RC}\right) = \frac{1}{RC}X(\omega)\,.$$

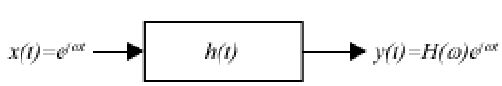
Finally, forming the ratio of $Y(\omega)$ and $X(\omega)$ and applying Equation (L6.9) gives

$$H(\omega) = \frac{1}{1 + j\omega RC}\,, \tag{L6.13}$$

which is exactly the same as we obtained in Equation (L6.5).

Eigenfunctions

In the previous lab, we discussed the orthogonality property of the complex exponential basis function, $e^{j\omega t}$, where ω is the angular frequency, and showed how summations of these orthogonal functions could be used to represent arbitrary functions in a Fourier series. In this section, we will look at the eigenfunction property of complex exponential basis function and see how it leads to the notion of the frequency response. Consider a linear system with input *x(t)*, output *y(t)*, and impulse response *h(t)*.



If the input to the system is a complex exponential at a particular frequency, ω, that is, $x(t) = e^{j\omega t}$, then the output of the system, *y(t)*, is also a complex exponential of exactly the same frequency, but with a different magnitude and phase. The proof is simple:

$$\begin{aligned} y(t) &= x(t) * h(t) = \int_{-\infty}^{-\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{-\infty} h(\tau)e^{j\omega(t-\tau)}d\tau = e^{j\omega t} \underbrace{\int_{-\infty}^{-\infty} h(\tau)e^{-j\omega\tau}d\tau}_{H(\omega)} . \\ &= H(\omega)e^{j\omega t} = \left|H(\omega)\right| e^{j\left(\omega t + \measuredangle H(\omega)\right)} \end{aligned} \qquad \text{(L6.14)}$$

$H(\omega)$ is a complex function of ω that depends on the characteristics of the system. As you can see from Equation (L6.14), $H(\omega)$ is the Fourier transform of the impulse response, *h(t)*, that is, the frequency response of the system.

At any fixed value of ω, $H(\omega)$ is a complex constant that is independent of time. So, if the input to the system is a complex exponential of frequency ω, i.e. $x(t) = e^{j\omega t}$, then output $y(t) = H(\omega)e^{j\omega t}$ is also complex exponential of exactly the same frequency, ω, just multiplied by the constant $H(\omega)$. All the time-dependent behavior of the output is determined by $e^{j\omega t}$. The magnitude and phase of the output is determined by $H(\omega)$. $x(t) = e^{j\omega t}$ is called an *eigenfunction* of the system because the input and output are the same except for a constant, $H(\omega)$, which is called the *eigenvalue*.

For the example of Figure 1, which is described by a differential equation, the frequency response can also be obtained without calculating the Fourier transform. Consider the solution of the differential equation of the our example system, Equation(L6.3), for input $x(t) = e^{j\omega t}$. Because $e^{j\omega t}$ is an eigenfunction of the system, we guess a solution of the form $y(t) = He^{j\omega t}$, where *H* is a constant. Plugging *y(t)* into the left side of Equation (L6.3) and *x(t)* into the right side gives

$$j\omega H \cancel{e^{j\omega t}} + \frac{1}{RC} H \cancel{e^{j\omega t}} = \frac{1}{RC} \cancel{e^{j\omega t}} . \qquad \text{(L6.15)}$$

All terms of the differential equation have the same time dependence, so $e^{j\omega t}$ can be factored out, leaving

$$H = \frac{1}{1 + j\omega RC} , \qquad \text{(L6.16)}$$

which is again the same as Equation (L6.5).

Measuring the frequency response

There are a number of ways to measure the frequency response in the laboratory. At least theoretically, we could use $e^{j\omega t}$ to measure the frequency response of the system. We would just input $x(t) = e^{j\omega t}$ at various values of ω, and measure $y(t) = H(\omega)e^{j\omega t}$. Then we could measure $H(\omega)$ by dividing *y(t)* by *x(t)*. In practice, this could not be done, because we can’t produce a complex exponential function of time.

However, we *can* produce sines and cosines, and these functions are closely related to the complex exponential. For example, we have just shown that if the input to a system is $x_1(t) = e^{j\omega t}$, the output is $y_1(t) = H(\omega)e^{j\omega t}$. We can also show that if the input to a system is $x_2(t) = e^{-j\omega t}$, then the output is $y_2(t) = H(-\omega)e^{-j\omega t}$. By the linearity property of linear systems, if the input to the system is the sum of the two inputs,

$$x(t) = x_1(t) + x_2(t) = \tfrac{1}{2}e^{j\omega t} + \tfrac{1}{2}e^{-j\omega t} = \cos(\omega t)\ ,$$

then the output will be the sum of the outputs,

$$y(t) = y_1(t) + y_2(t) = \tfrac{1}{2}H(\omega)e^{j\omega t} + \tfrac{1}{2}H(-\omega)e^{-j\omega t} \tag{L6.17}$$

The input is clearly a simple function that we can produce in the lab. But what about the output? To make sense of this output, we note that for a real system (that is, for a system for which *h(t)* is real), there is a relation between between $H(\omega)$ and $H(-\omega)$. Specifically, if *h(t)* is real, then $h(t) = h^*(t)$ for all *t*. Thus, the Fourier transform of *h(t)* and $h^*(t)$ have to be equal.

$$\int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} h^*(t)e^{-j\omega t}dt \tag{L6.18}$$

However,

$$\int_{-\infty}^{\infty} h^*(t)e^{-j\omega t}dt = \left(\int_{-\infty}^{\infty} h(t)e^{j\omega t}dt\right)^* = H^*(-\omega)\ ,$$

so we conclude that $H(\omega) = H^*(-\omega)$ or equivalently, $H^*(\omega) = H(-\omega)$. Plugging this result into Equation (L6.17) gives

$$\begin{aligned} y(t) &= \tfrac{1}{2}H(\omega)e^{j\omega t} + \tfrac{1}{2}H(-\omega)e^{-j\omega t} \\ &= \tfrac{1}{2}H(\omega)e^{j\omega t} + \tfrac{1}{2}H^*(\omega)e^{-j\omega t} \\ &= \tfrac{1}{2}\left(H(\omega)e^{j\omega t} + \left(H(\omega)e^{j\omega t}\right)^*\right) \\ &= \mathrm{Re}\left\{H(\omega)e^{j\omega t}\right\} \end{aligned} \tag{L6.19}$$

Expressing $H(\omega)$ in polar form, we get

$$\begin{aligned} y(t) &= \mathrm{Re}\left\{H(\omega)e^{j\omega t}\right\} \\ &= \mathrm{Re}\left\{\left|H(\omega)\right| e^{j(\omega t + \measuredangle H(\omega))}\right\} \\ &= \left|H(\omega)\right|\cos(\omega t + \measuredangle H(\omega)) \end{aligned} \tag{L6.20}$$

To summarize, for a linear system, if the input is $x(t) = \cos(\omega t)$, the output is

$$y(t) = \left|H(\omega)\right|\cos(\omega t + \measuredangle H(\omega))\ .$$

That is, if you put a cosine of frequency ω into a linear system, the output is a cosine of the same frequency, but with a magnitude and phase determined exclusively by $H(\omega)$ at that frequency. This gives a much more intuitive explanation for why $H(\omega)$ is called the frequency response. It also means that we can measure $H(\omega)$ for any system just using cosines. In order to do this, we

1. Input $x(t) = \cos(\omega t)$
2. Measure $y(t) = \left|H(\omega)\right|\cos(\omega t + \measuredangle H(\omega))$
3. To measure $\left|H(\omega)\right|$ and $\measuredangle H(\omega)$, we compute two constants by multiplying *y(t)* by $\cos\omega t$ and $\sin\omega t$ and integrating over one period, $T = \dfrac{2\pi}{\omega}$:

$$
\begin{aligned}
z_1 &= \tfrac{1}{T}\int_0^T y(t)\cos(\omega t)dt = \tfrac{1}{T}\int_0^T |H(\omega)|\cos(\omega t + \measuredangle H(\omega))\cos(\omega t)dt \\
&= |H(\omega)|\left\{\tfrac{1}{T}\int_0^T \cos\left(\measuredangle H(\omega)\right)\cos^2 \omega t dt - \tfrac{1}{T}\int_0^T \sin\left(\measuredangle H(\omega)\right)\cos\omega t \sin\omega t dt\right\} \\
&= |H(\omega)|\left\{\cos\left(\measuredangle H(\omega)\right)\tfrac{1}{T}\int_0^T \cos^2 \omega t dt - \sin\left(\measuredangle H(\omega)\right)\tfrac{1}{T}\int_0^T \cos\omega t \sin\omega t dt\right\} \\
&= \tfrac{1}{2}|H(\omega)|\cos\left(\measuredangle H(\omega)\right) = \tfrac{1}{2}\mathrm{Re}\left\{H(\omega)\right\}
\end{aligned}
\tag{L6.21}
$$

and

$$
\begin{aligned}
z_2 &= \tfrac{1}{T}\int_0^T y(t)\sin(\omega t)dt = \tfrac{1}{T}\int_0^T |H(\omega)|\cos(\omega t + \measuredangle H(\omega))\sin(\omega t)dt \\
&= |H(\omega)|\left\{\tfrac{1}{T}\int_0^T \cos\left(\measuredangle H(\omega)\right)\cos\omega t \sin\omega t dt - \tfrac{1}{T}\int_0^T \sin\left(\measuredangle H(\omega)\right)\sin^2 \omega t dt\right\} \\
&= |H(\omega)|\left\{\cos\left(\measuredangle H(\omega)\right)\tfrac{1}{T}\int_0^T \cos\omega t \sin\omega t dt - \sin\left(\measuredangle H(\omega)\right)\tfrac{1}{T}\int_0^T \sin^2 \omega t dt\right\} \\
&= -\tfrac{1}{2}|H(\omega)|\sin\left(\measuredangle H(\omega)\right) = -\tfrac{1}{2}\mathrm{Im}\left\{H(\omega)\right\}
\end{aligned}
\tag{L6.22}
$$

4. Given z_1 and z_2 , we can compute

$$H(\omega) = 2\left(z_1 - jz_2\right) \tag{L6.23}$$

or

$$
\begin{aligned}
|H(\omega)| &= 2\sqrt{\left(z_1^2 + z_2^2\right)} \\
\measuredangle H(\omega) &= -\tan^{-1}\left(\frac{z_2}{z_1}\right)
\end{aligned}
\tag{L6.24}
$$

Assignment

In this assignment, we will calculate or measure and plot the frequency response in several ways. Consider the simple circuit of Figure 1 with input *x(t)* and output *y(t)*.

a) Computation and plotting of frequency response based on solution of differential equation.

As described in Lab 2, we can describe the differential equation in terms of two arrays `a` and `b`. For example, in the system of Figure 1 with $R = 1$ and $C = 0.1$,

```
a = [1 10];
b = 10;
```

Our task in this part is to create a Matlab function to calculate and plot the frequency response on a log-log scale. We base this solution on solving the differential equation, as outlined in the background section of this lab. You will write a Matlab function

```
function freqde(b, a)
% FREQDE   Frequency response of linear system
```

```
%          based on solution of differential equation
```

The result of this function should be exactly as shown if Figure 3. You should choose recognize that for any linear constant-coefficient differential equation $H(\omega)$ can be expressed as the ratio of two polynomials in powers of $j\omega$. The numerator polynomial is given by the `b` array and the denominator polynomial by the `a` array. You can easily evaluate polynomials using the Matlab `polyval` function if you want.

i. Once you have written this function, use it to understand the relation between the impulse response and the frequency response for a first order system. Produce plots of the impulse response and frequency response for a system described by Equation (L6.3) for $RC = 0.1$, 1 and 10. What is the relation between the cutoff frequency of the frequency response and the rise-time of the impulse response?

ii. In Lab #2, we discussed the classic second order system described by equation

$$\frac{d^2y(t)}{dt^2} + 2\xi\omega_0\frac{dy(t)}{dt} + \omega_0^2 y(t) = \omega_0^2 x(t) \;,$$

where ω_0 is called the *natural frequency* and ξ is the *damping factor*. We showed that there are four regions of behavior: *overdamped* ($\xi > 1$), *underdamped* ($\xi < 1$), *critically damped* ($\xi = 1$) and *undamped* ($\xi = 0$).

a) Let $\omega_0 = 10$. Produce plots of the impulse response and frequency response for ξ=0.1, 0.2, 0.5, 0.8, 1, 2, 5, 8 and 10.

b) Now let $\xi = 0.1$. Produce plots of the impulse response and frequency response for ω_0 =0.1, 1 and 10.

iii. Use your `freqde` function it to visualize the impulse and frequency response of the following systems, which are exactly the same as those you studied in Lab 2.

a. $y' + 5y = 5x(t)$

b. $y'' + 3y' + 2y = 2x(t)$

c. $y'' + y' + y = x(t)$

d. $y'' + 2y' + y = x(t)$

e. $y'' + y = x(t)$

f. $y''' + 6y'' + 11y' + 6y = 6x(t)$

g. $y''' + 4y'' + 21y' + 34y = 34x(t)$

b) <u>Computation and plotting of frequency response based on response of system to a cosine</u>.

In this part, we will use to create a Matlab function to measure and plot the frequency response on a log-log scale, as outlined in the background section of this lab.

You will write a Matlab function

```
function freqcos(b, a)
% FREQCOS   Frequency response of linear system
%          based on measurement of response to cosine
```

The result of this function should be the same as that we obtained before, but the measurement strategy is different, and corresponds exactly to what you would do in the laboratory if you had a circuit and a function generator. Here’s what you want to do.

i. Create a Matlab array, `c`, of a cosine of frequency ω. Also create an array, `s`, of a sine of the same frequency. Select the sampling interval of each array such that the array has 10 periods of 256 points per period, for a total of 2560 points.

ii. Measure the response of the differential equation, *y(t)*, to the cosine with the Matlab `lsim` function.

```
y = lsim(sys, c, t);
```

where the `sys` is obtained using the Matlab `tf` function

```
sys = tf(b, a);
```

iii. Calculate z_1 and z_2 using Equations (L6.21) and (L6.22). We will compute the integral numerically as the mean of the product of `y` and sine or cos. Then use Equation (L6.24) to compute $|H(\omega)|$ and $\measuredangle H(\omega)$.
