San Francisco State University

Engineering 305

Laboratory #7 – Modulation

Purpose

In this laboratory, you will investigate some important properties of the Fourier transform.

Background reading includes:

- Lathi, Chapter 5
- Holton notes, Unit 6-7

Background

In this lab we examine some important properties of the Fourier transform, specifically filtering and modulation.

Fourier transform and properties

We start by recalling the expressions for the Fourier transform

$$\begin{aligned} x(t) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \qquad \text{Synthesis} \\ X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-jk\omega t}\,dt \qquad \text{Analysis} \end{aligned} \tag{L7.1}$$

Among the important properties of the Fourier transform are the convolution and modulation properties

<u>Convolution</u>

Consider a linear time-invariant system with input *x(t)*, impulse response *h(t)* and output *y(t)*.

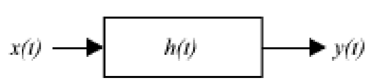

Figure 1: Linear time-invariant system

The convolution property states that

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau\,. \tag{L7.2}$$

In the frequency domain, the relation between the Fourier transforms of *x(t)*, *h(t)* and *y(t)* is

$$Y(\omega) = X(\omega)H(\omega) \tag{L7.3}$$

That is, convolution in the time domain corresponds to multiplication in the frequency domain. This property underlies the essence of filtering.

<u>Modulation</u>

The modulation property states that if we multiply two time functions, *x(t)* and *c(t)*,

$$y(t) = x(t)c(t)\,, \tag{L7.4}$$

the Fourier transform of the product is the convolution of the Fourier transforms of *x(t)* and *c(t)* in the frequency domain:

$$Y(\omega) = \tfrac{1}{2\pi}\left\{X(\omega) * C(\omega)\right\}. \tag{L7.5}$$

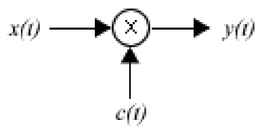

Figure 2: Modulation

This property underlies most of modern communication theory. Specifically, let’s consider the complex time function

$$c(t) = e^{j\omega_0 t}, \tag{L7.6}$$

where ω_0 is a fixed frequency. The Fourier transform of *c(t)* is

$$C(\omega) = 2\pi\delta(\omega - \omega_0), \tag{L7.7}$$

Hence, if we multiply an arbitrary signal, *x(t)*, by *c(t)*, the transform of the output is

$$\begin{aligned} Y(\omega) &= \tfrac{1}{2\pi} X(\omega) * C(\omega) \\ &= \tfrac{1}{\cancel{2\pi}} X(\omega) * \cancel{2\pi}\,\delta(\omega - \omega_0). \\ &= X(\omega - \omega_0) \end{aligned} \tag{L7.8}$$

That is, the spectrum of the output signal is just the spectrum of the input signal translated in frequency by an amount ω_0. By itself this result is of little practical value, because *c(t)* in Equation (L7.6) is complex. However, consider

$$c(t) = \cos(\omega_0 t) = \tfrac{1}{2} e^{j\omega_0 t} + \tfrac{1}{2} e^{-j\omega_0 t}. \tag{L7.9}$$

The Fourier transform is

$$C(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0), \tag{L7.10}$$

so the result of modulating *x(t)* by a cosine is

$$y(t) = x(t)\cos(\omega_0 t), \tag{L7.11}$$

whose Fourier transform is easily shown to be

$$Y(\omega) = \tfrac{1}{2} X(\omega - \omega_0) + \tfrac{1}{2} X(\omega + \omega_0). \tag{L7.12}$$

Thus, the result of modulation with a cosine is the translation of the spectrum of the input both up and down in frequency by amounts $\pm\omega_0$, as shown in Figure 3. In general, the input signal to be modulated, *x(t)*, is called the *baseband* signal, since it is centered around a frequency of $\Omega = 0$. The modulating signal, *c(t)*, is called the *carrier*. When the baseband signal is modulated by the carrier, the resulting modulated signal has a frequency spectrum that can be completely disjoint from the original spectrum; that is, all the energy in *y(t)* is found at frequencies outside of the baseband.

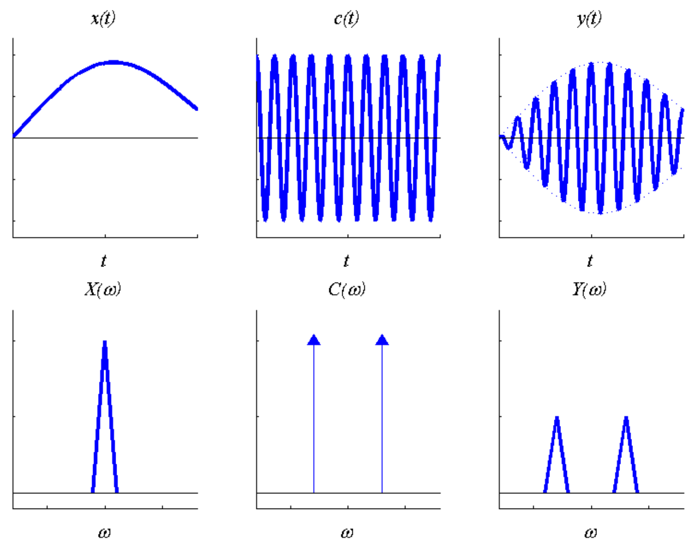

Figure 3: Modulation by a cosine

In a similar fashion, we can show that modulation of *x(t)* by $\sin(\omega_0 t)$ yields the spectrum

$$Y_s(\omega) = \tfrac{1}{2j} X(\omega - \omega_0) - \tfrac{1}{2j} X(\omega + \omega_0) \quad \text{(L7.13)}$$

Demodulation

Modulation translates all the frequencies in the modulated signal back into the baseband. The scheme of demodulation is shown in Figure 4. The modulated signal is multiplied by a carrier of the same frequency and phase with which the signal was originally created to create an intermediate signal, *r(t)*, which is then lowpass filtered to remove the frequency components outside of the baseband, yielding a signal, *z(t)*, which we hope will be related to *x(t)*.

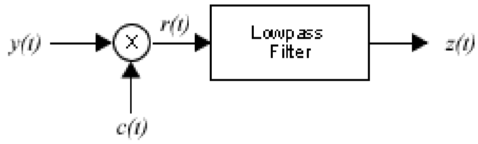

Figure 4: Demodulation scheme

The derivation is simple. Clearly,

$$r(t) = y(t)c(t)\,, \quad \text{(L7.14)}$$

and by the modulation property

$$R(\omega) = \tfrac{1}{2} Y(\omega - \omega_0) + \tfrac{1}{2} Y(\omega + \omega_0)\,. \quad \text{(L7.15)}$$

Substituting Equation (L7.12) yields

$$R(\omega) = \tfrac{1}{4} X(\omega - 2\omega_0) + \tfrac{1}{2} X(\omega) + \tfrac{1}{4} X(\omega + 2\omega_0) \quad \text{(L7.16)}$$

Which is then lowpass filtered to remove the sideband components centered at $\pm 2\omega_0$ to yield

$$Z(\omega) = \tfrac{1}{2} X(\omega) \quad \text{(L7.17)}$$

Thus,

$$z(t) = \tfrac{1}{2} x(t)\,, \quad \text{(L7.18)}$$

so the demodulated signal is just the same as the original signal, except for a gain factor, which we can easily fix by giving the filter a gain of two. Figure 5 shows the pictures in the time and frequency domains:

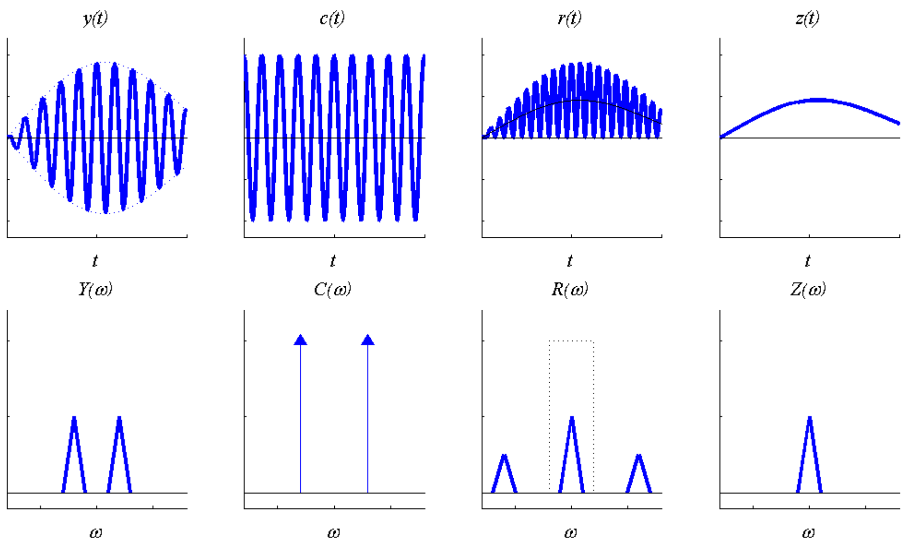

Figure 5: Demodulation

Examples of modulation

In this laboratory, we consider basic amplitude modulation/demodulation, as described above, as well as a couple of other techniques: frequency domain multiplexing (FDM) and quadrature amplitude modulation (QAM).

Frequency-domain multiplexing

What happens if we want to transmit a number of baseband signals at the same time, all of which have the same range of baseband frequencies? Let's say we have three signals, $x_1(t)$, $x_2(t)$ and $x_3(t)$ which have frequency spectra, $X_1(\omega)$, $X_2(\omega)$ and $X_3(\omega)$ respectively. If we simply add these signals together and transmit them, they will overlap completely in both the time and frequency domains, so we will be unable to demodulate them. Instead, we will move each of these signals to a different place in the frequency spectrum by modulating each one by a cosine carrier of a different frequency. The resulting sum of these modulated signals still overlaps in the time domain, but the components of $x_1(t)$, $x_2(t)$ and $x_3(t)$ are disjoint in the frequency domain, so we can, in principal demodulate them and recover each of the three signals separately.

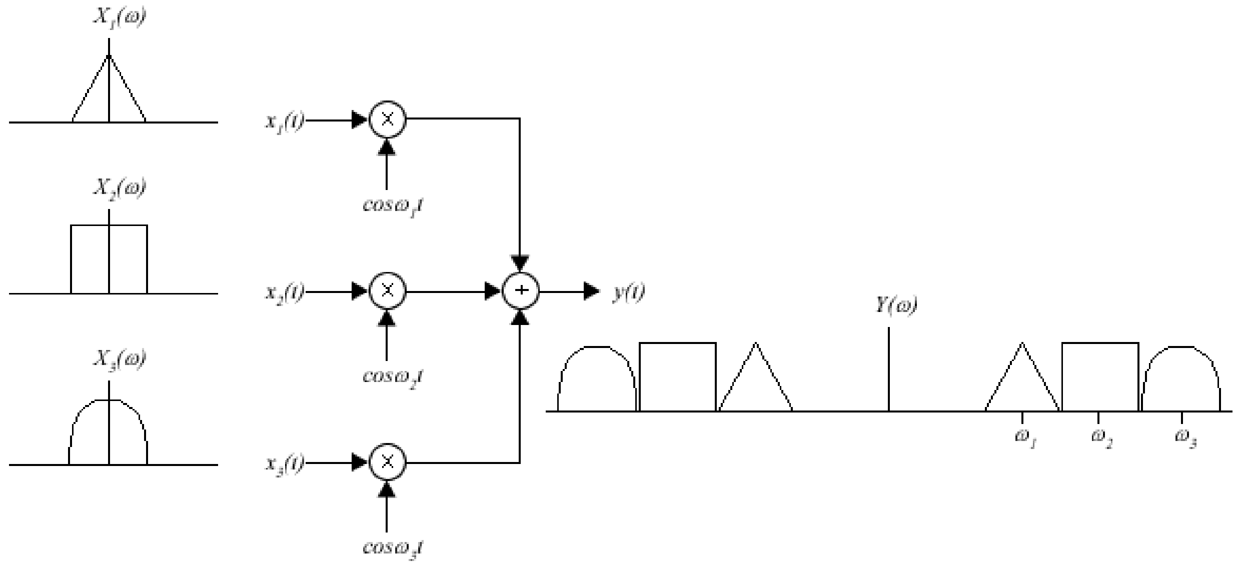

Figure 6: Frequency domain multiplexing - modulation

To recover any particular component from a frequency-domain multiplexed signal, we follow exactly the demodulation scheme shown in Figure 4; multiply the modulated signal by a cosine and lowpass filter.

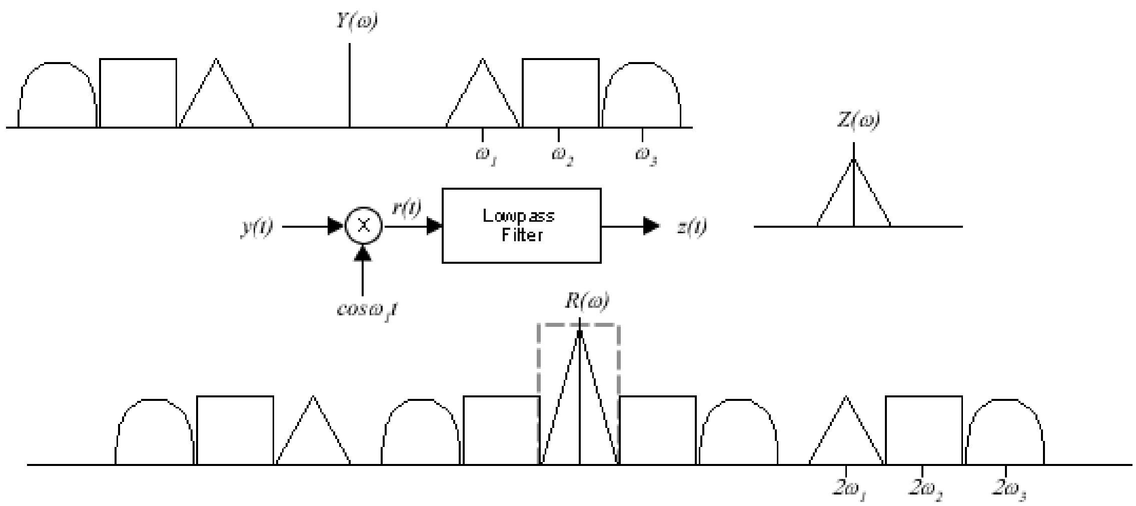

Figure 7: Demodulation of frequency-domain multiplexed signals

Quadature-amplitude modulation (QAM)

Quadature-amplitude modulation is a modulation scheme whereby two signals that have the *same* basebandwidth are modulated with carriers at the *same* frequency and can still be correctly demodulated. The scheme is shown in Figure 8.

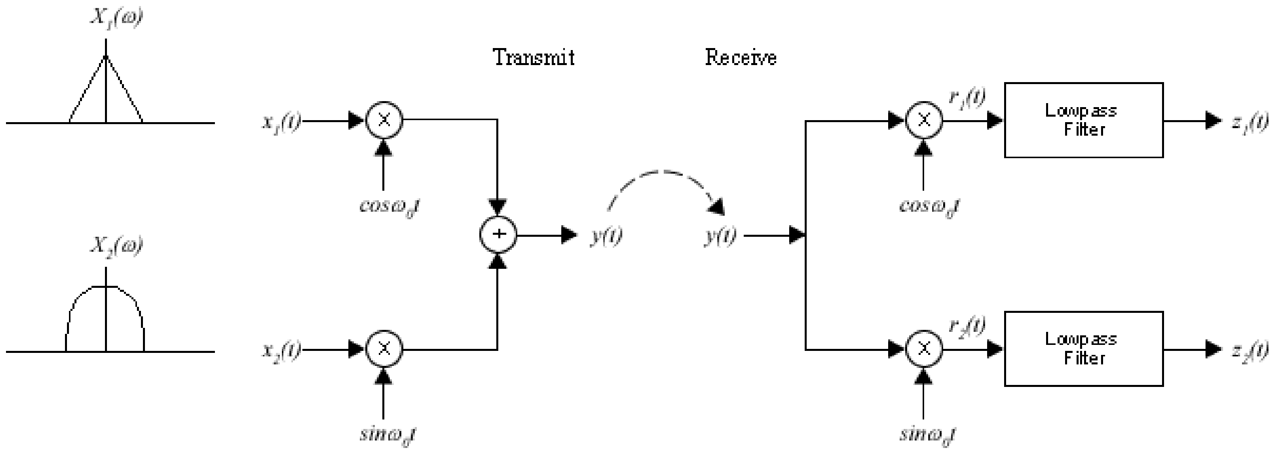

Figure 8: Quadrature amplitude modulation and demodulation

The two baseband signals, $x_1(t)$ and $x_2(t)$ have frequency spectra, $X_1(\omega)$ and $X_2(\omega)$ that completely overlap. These two signals are modulated by two carriers of the same frequency, ω_0. Although the frequency of the carriers is the same, the phase is different: one carrier is a sine and the other is a cosine. The phase of the sine is $\pi/2$ -- one quadrant -- out of phase with respect to the cosine. Hence this is called quadature amplitude modulation. The result of this modulation method is easily analyzed. Clearly,

$$y(t) = x_1(t)\cos\omega_0 t + x_2(t)\sin\omega_0 t\,. \tag{L7.19}$$

Applying Equations (L7.12) and (L7.13), we get

$$Y(\omega) = \tfrac{1}{2}X_1(\omega-\omega_0) + \tfrac{1}{2}X_1(\omega+\omega_0) + \tfrac{1}{2j}X_1(\omega-\omega_0) - \tfrac{1}{2j}X_1(\omega+\omega_0)\,. \tag{L7.20}$$

We demodulate this signal with the same two carriers. Just concentrating on the top limb of Figure 8,

$$\begin{aligned} r_1(t) &= y(t)\cos\omega_0 t \\ &= \left(x_1(t)\cos\omega_0 t + x_2(t)\sin\omega_0 t\right)\cos\omega_0 t \\ &= x_1(t)\cos^2\omega_0 t + x_2(t)\sin\omega_0 t\cos\omega_0 t \\ &= x_1(t)\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\omega_0 t\right) + x_2(t)\left(\tfrac{1}{2}\sin 2\omega_0 t\right) \end{aligned}\,. \tag{L7.21}$$

In the frequency domain:

$$R_1(\omega) = \tfrac{1}{2}Y(\omega-\omega_0) + \tfrac{1}{2}Y(\omega+\omega_0)\,. \tag{L7.22}$$

Substituting Equation (L7.20), we get

$$\begin{aligned} R_1(\omega) &= \tfrac{1}{2}Y(\omega-\omega_0) + \tfrac{1}{2}Y(\omega+\omega_0) \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}X_1(\omega-2\omega_0) + \tfrac{1}{2}X_1(\omega) + \tfrac{1}{2j}X_1(\omega-2\omega_0) - \cancel{\tfrac{1}{2j}X_1(\omega)}\right) \\ &\quad + \tfrac{1}{2}\left(\tfrac{1}{2}X_1(\omega) + \tfrac{1}{2}X_1(\omega+2\omega_0) + \cancel{\tfrac{1}{2j}X_1(\omega)} - \tfrac{1}{2j}X_1(\omega+2\omega_0)\right), \\ &= \tfrac{1}{2}X_1(\omega) + \tfrac{1}{2}\left(\tfrac{1}{2}X_1(\omega-2\omega_0) + \tfrac{1}{2}X_1(\omega+2\omega_0)\right) \\ &\quad + \tfrac{1}{2}\left(\tfrac{1}{2j}X_1(\omega+2\omega_0) - \tfrac{1}{2j}X_1(\omega-2\omega_0)\right) \end{aligned} \tag{L7.23}$$

which you should recognize as just the transform of Equation (L7.22). Draw the spectrum of this, and note that the spectrum contains one term at the baseband, $\tfrac{1}{2}X_1(\omega)$, which corresponds to $\tfrac{1}{2}x_1(t)$, plus a number of terms centered around $\pm 2\omega_0$, which correspond to $\tfrac{1}{2}x_1(t)\cos 2\omega_0 t$ and $\tfrac{1}{2}x_1(t)\sin 2\omega_0 t$. Clearly, when $R_1(\omega)$ is lowpass filtered with a filter that only passes the baseband, the terms centered around $\pm 2\omega_0$ will be filtered out, leaving only $\frac{1}{2}x_1(t)$. Hence, we conclude that $z_1(t) = \frac{1}{2}x_1(t)$ and you can also easily show that $z_2(t) = \frac{1}{2}x_2(t)$.

Assignment

In this assignment, we will investigate basic principles of modulation/demodulation, frequency domain multiplexing and quadrature amplitude modulation.

1) Basic modulation and demodulation
   a. Simple sine and cosine. Here we will investigate the modulation and demodulation of a simple baseband waveform with a carrier as shown in Figures 3 and 5. We will specifically examine what happens when we change the phase of the demodulating signal with respect to the phase of the original modulating signal.
      i. You will create a function called `modcos` that demonstrates modulation and demodulation. Here is a specification of the function:

```
function modcos(theta, f0)
%   MODCOS Plot modulation and demodulation of a cosine
%          with varying angles for demodulation. This function
%          plots four panels
%          1) The first panel plots x(t) = cos(2*pi*1*t) where
%             t is the interval between -0.25 and 0.25 seconds
%             sampled at 8000 samples/sec.
%          2) The second panel plots y(t)=x(t)c(t),
%             where c(t) = cos(2*pi*100*t).
%             This is the modulated signal.
%          3) The third panel plots r(t)=y(t)cos(2*pi*100*t+theta*pi)
%             where theta is an angle (default 0)
%             This is the demodulated signal before lowpass filtering.
%          4) The fourth panel plots the lowpass filtered version of
%             r(t) using lsim. The filter selected is a second order
%             Butterworth lowpass filter with cutoff frequency
%             f0 (default 20).
```

You can create the Butterworth filter with this little Matlab program, `makefilt`. You can examine at the frequency response of this filter using Matlab's `bode` or `ltiview` function. To do the filtering, create the filter with `makefilt(f0, 2)`, then filter using `z=lsim(filt, r, t)`. Here is the result of `modcos(0, 20)`:

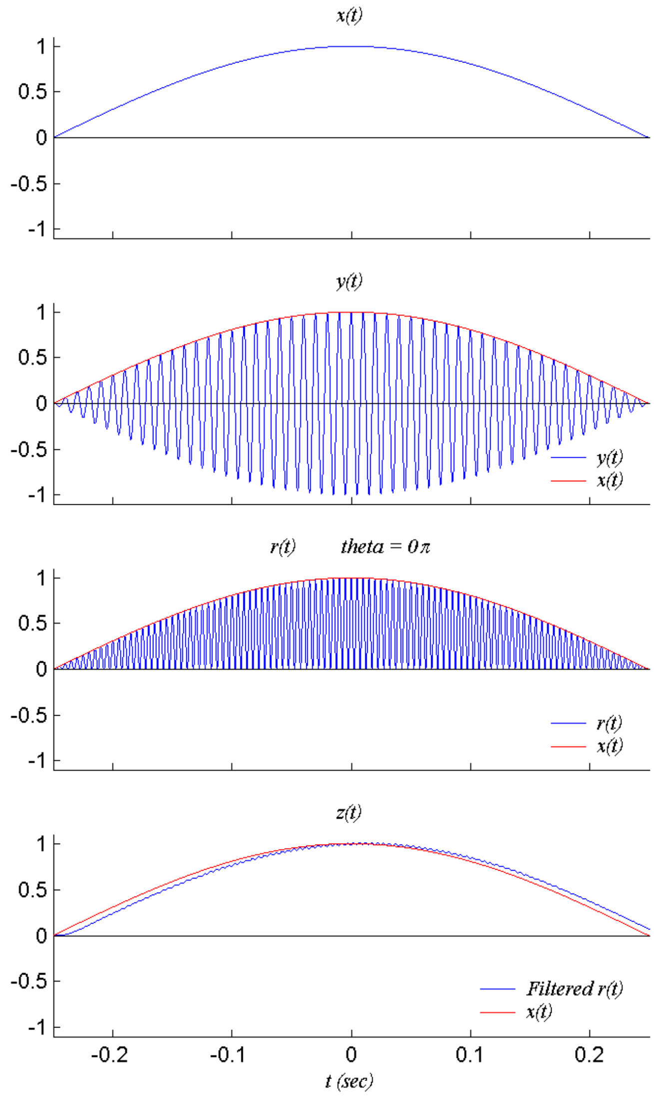

Figure 9: Modulation and demodulation with a cosine

ii. Make plots of `modcos(0, 20)`, `modcos(0.25, 20)`, `modcos(0.5, 20)`, `modcos(0.75, 20)` and `modcos(1, 20)`. Explain what is happening in panel #4.

iii. Make plots of `modcos(0, 20)`, `modcos(0, 50), modcos(0, 100)` and `modcos(0, 1000)`. Explain what is happening in panel #4.

iv. Now change the program so that $x(t) = \cos(2\pi t + \theta) - \frac{1}{2}$ and repeat part ii. Notice that the cosine is being modulated by an envelope that goes both positive and negative, so that *y(t)* is symmetrical about to time axis. When *y(t)* is demodulated, the demodulated waveform, *r(t),* is not symmetrical and the lowpassed version of *r(t)* recovers the original waveform (at least when $\theta = 0$).

b. Complex waveform. Repeat this same process with a complex waveform: a sample of speech.

i. Download this simple waveform: hello.wav. This waveform is a speech sample, $x(t)$, which has been bandlimited to less than 5 kHz in frequency. It has been sampled at 22.05 kHz. Use `[x, fs] = wavread('hello')` to load it into memory. `y` is the data and `fs` is the sample rate.

ii. Modulate this waveform with a cosine at 8 kHz. To form $y(t) = x(t)\cos(2\pi \cdot 8000t)$. This means you have to produce a cosine of the same length as the speech, sampled at the same sample rate. Play the sound using the Matlab command `soundsc(x, fs)`. What do you hear? This is the modulated signal and it should sound very crummy.

iii. Demodulate the waveform with the same carrier to form $r(t) = y(t)\cos(2\pi \cdot 8000t)$ and play that. You should be able to hear the base band plus the sidebands which constitute the baseband modulated at twice the carrier frequency, which is what you expect from Equation (L7.16).

iv. Now filter the modulated waveform at 4000 Hz using a sharp Butterworth filter, which you create with `filt=makefilt(4000, 10)`, Then filter using `z=lsim(filt, r, t)` to form $z(t)$. Now listen to the result and compare it with the original signal. The sidebands should be gone.

You should create the following picture to show that you have successfully accomplished this task:

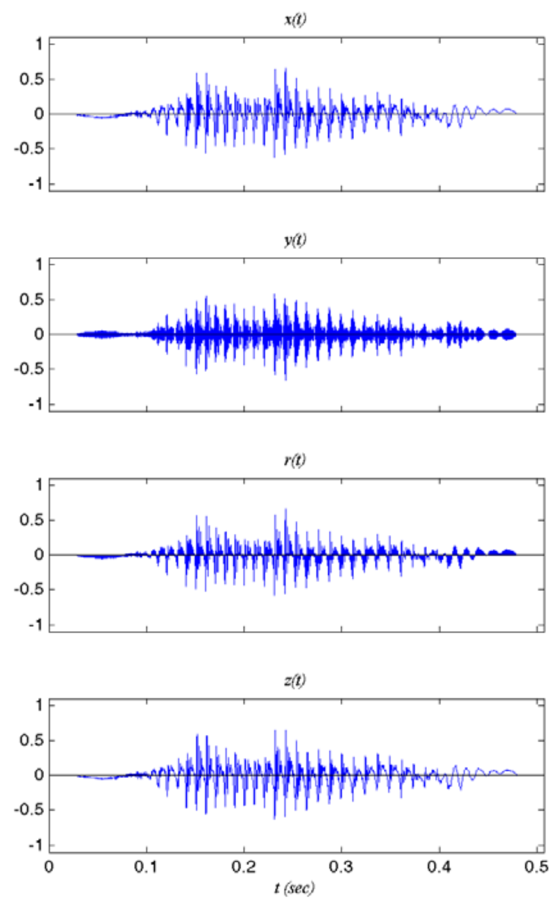

Figure 10: Modulation of speech

2) Frequency domain multiplexing

a. Simple sine and cosine. Create a couple of cosines about 1 second long at frequencies $f_1 = 200$ and $f_2 = 400$ Hz. Sample these at cosines at 8000 Hz to form $x_1(t)$ and $x_2(t)$. Modulate $x_1(t)$ with a carrier at 1000 Hz and modulate $x_2(t)$ with a carrier at 2000 Hz and add the results to form a single signal, *y(t)*. This is the frequency-domain multiplexed signal created using the plan of Figure 6. Look at the result and listen to it. Now demodulate this signal using the plan of Figure 7. You will want to create a Butterworth filter of appropriate bandwidth, and you ought to be able to use the same filter for both demodulated signals. What choices did you make for the bandwidth of this filter? You should see something similar to the plot in Figure 10.

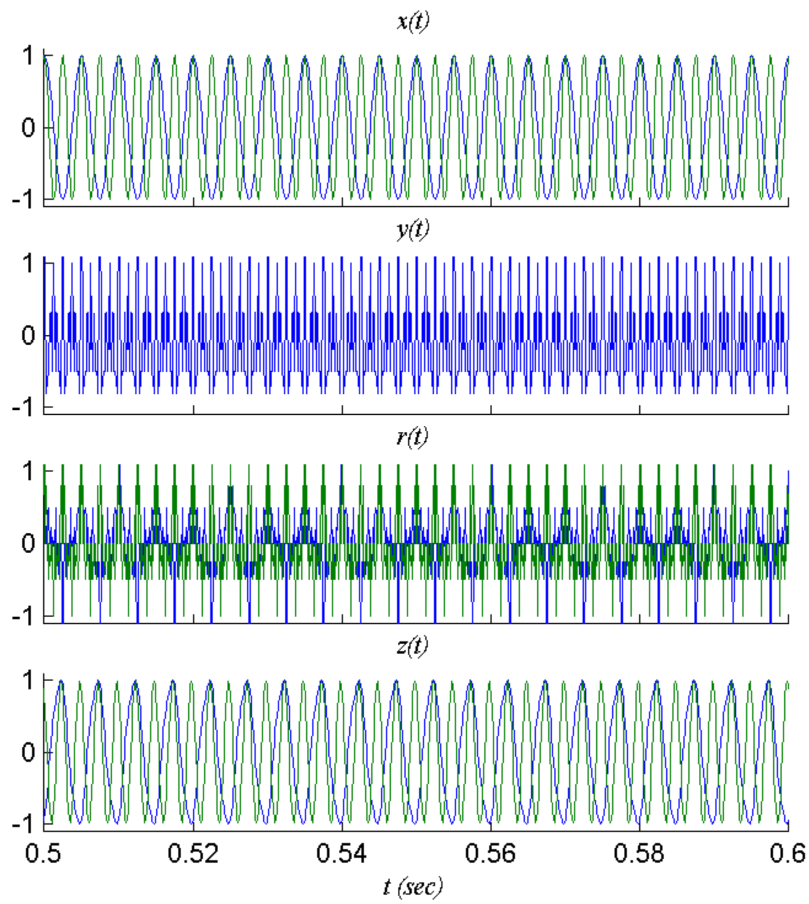

Figure 11: Frequency-domain modulation of cosines

b. Complex waveform. Repeat part a, except with two speech waveforms. You have already downloaded the first waveform, [hello.wav](). The second one, [goodbye.wav](), is of the same length. Both are sampled at 22.05 kHz. Modulate one of the waveforms with a 3kHz cosine and the other one with a 9 kHz cosine. Demodulate and filter with a 20^{th} order Butterworth filter and a cutoff of about 2500 Hz. Look and listen. You should get something that looks like the plot shown

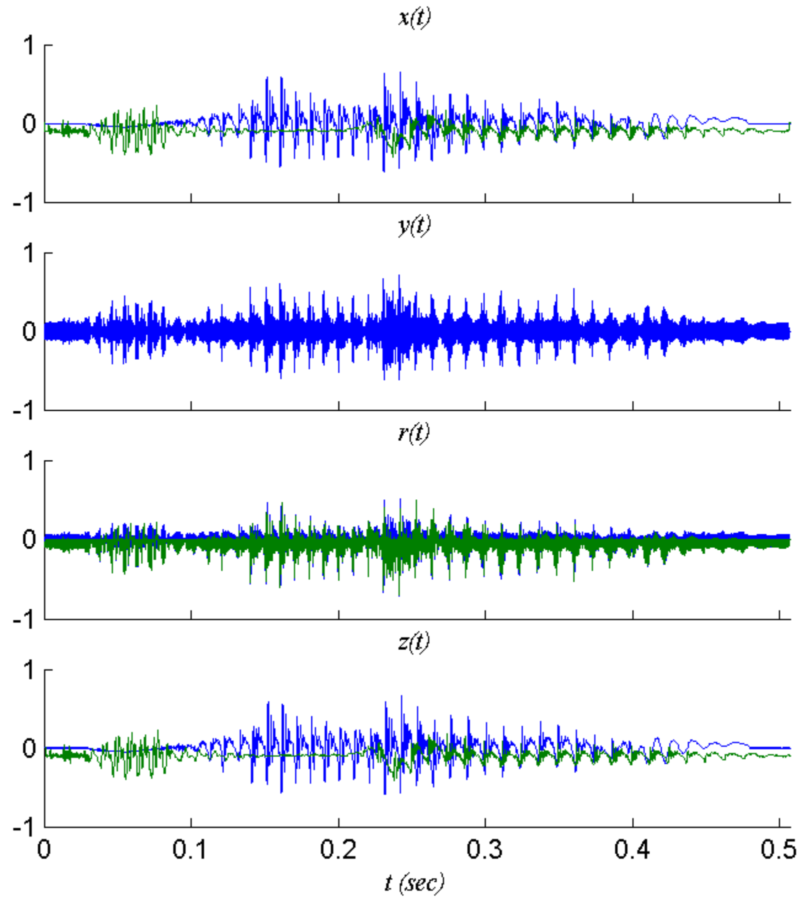

Figure 12: Frequency-domain modulation of speech waveforms

3) Quadrature amplitude modulation
   a. Simple sine and cosine. Repeat part 2a, but do quadrature amplitude modulation. You should only have to modify your program a little. Choose a single carrier frequency, $f_c = 2000$ Hz. The picture should look like that shown in Figure 13.

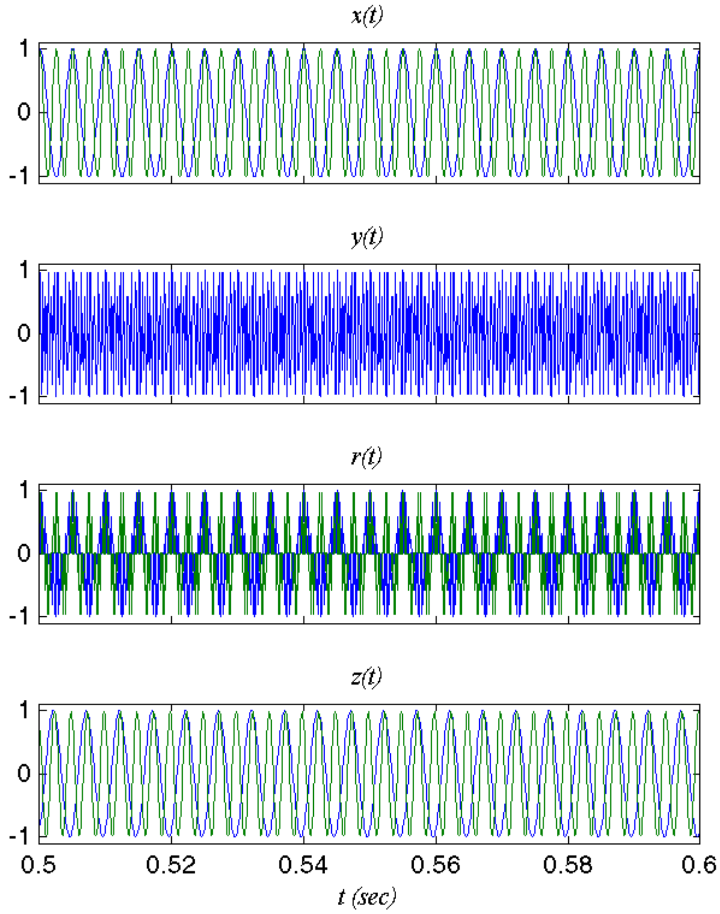

Figure 13: Quadrature-amplitude modulation of cosines

b. Complex waveform. Repeat part 2b using quadrature amplitude modulation. Choose a single carrier frequency, $f_c = 8000$ Hz. Your picture should look like that in Figure 14.

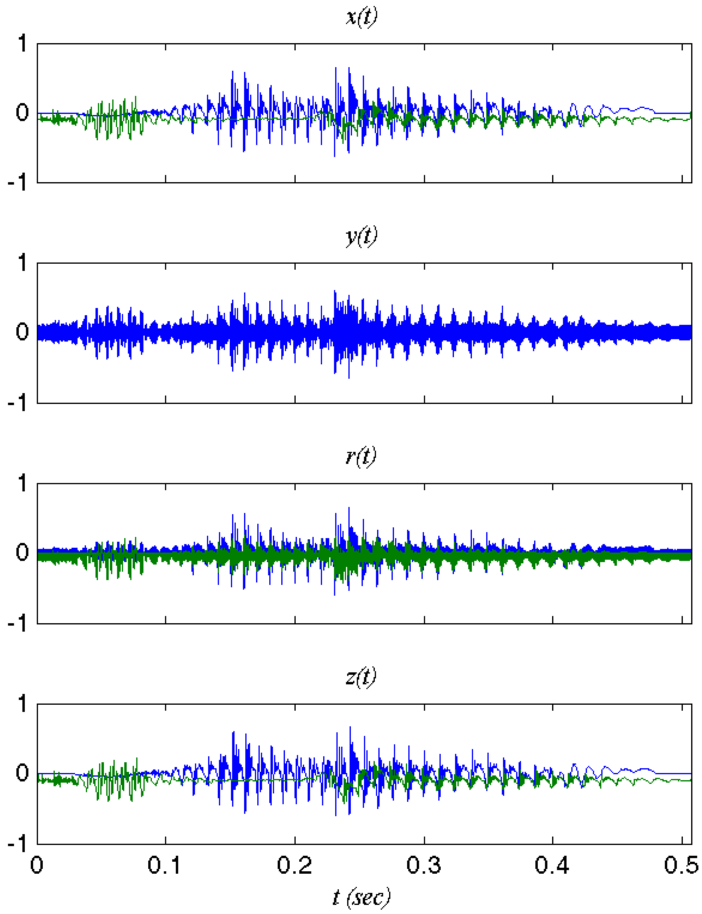

Figure 14: Quadrature-amplitude modulation of speech waveforms