San Francisco State University

Engineering 315

Laboratory #3 – Linearity and Time Invariance

Purpose

In this lab, you will investigate the response of linear and time-invariant systems to signals.

Background reading includes:

- Lathi, Chapter2
- Holton notes, Unit 3

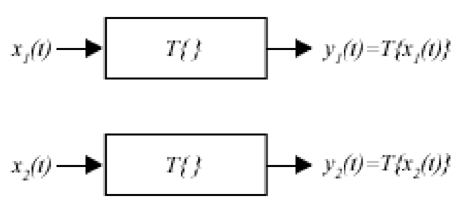
Background

Linearity

Any system, whether linear or not, can be viewed as a transformation, $T\{\ \}$, that maps an input, $x(t)$, to an output, $y(t)$:

$$y(t) = T\{x(t)\}. \tag{L3.1}$$

Linear systems satisfy both the *additivity* and *scaling* properties. In order to test for linearity, we first measure the output of the system to two inputs, $x_1(t)$ and $x_2(t)$, applied separately. The outputs are $y_1(t) = T\{x_1(t)\}$ and $y_2(t) = T\{x_2(t)\}$, respectively:



The additivity property

The additivity property states that if $T\{x_1(t)\} = y_1(t)$ and $T\{x_2(t)\} = y_2(t)$, then

$$T\{x_1(t) + x_2(t)\} = T\{x_1(t)\} + T\{x_2(t)\} = y_1(t) + y_2(t) \tag{L3.2}$$

We can think about the additivity property in terms of two experiments. In Experiment #1, we add the two inputs, $x_1(t)$ and $x_2(t)$, and use the sum as the input to the system, yielding an output that we'll call $e_1(t)$. In the second experiment, we put each input into the system separately, yielding $y_1(t) = T\{x_1(t)\}$ and $y_2(t) = T\{x_2(t)\}$, as above, and then add the two outputs, yielding $e_2(t)$. Graphically, we have this:

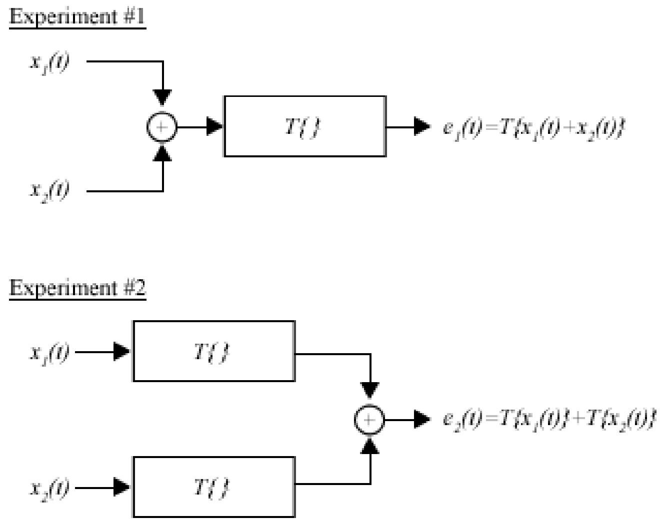


Figure 1: Test for the additivity property of linear systems

If the results of these two experiments are the same, that is, if $T\{x_1(t)+x_2(t)\}=T\{x_1(t)\}+T\{x_2(t)\}$, then the system satisfies the additivity property. In words, the additivity property is satisfied if that the transformation of the sum of inputs (Experiment #1) is the same as the sum of the transformation of inputs (Experiment #2).

The scaling property

The scaling property states that if $T\{x(t)\}=y(t)$, then

$$T\{k\,x_1(t)\}=kT\{x(t)\}=k\,y(t)\,. \tag{L3.3}$$

Again, we can view this as the result of two experiments. In Experiment #1, we scale the input, $x(t)$, by a constant, *k*, before putting this into the system, yielding an output that we'll call $e_1(t)$. In the second experiment, we input $x(t)$ into the system, yielding $y(t)=T\{x(t)\}$, and then scale this output by constant, *k*, yielding $e_2(t)$. Graphically, we have this:

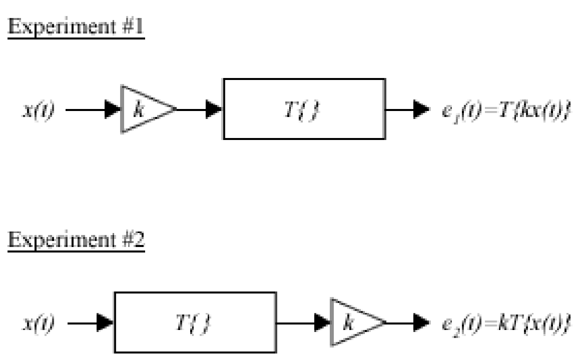


Figure 2: Test for the scaling property of linear systems

If the results of these two experiments are the same, that is, if $T\{k\,x(t)\}=kT\{x(t)\}$, then the system satisfies the scaling property. In words, the scaling property is satisfied if the transformation of the scaled input (Experiment #1) is the same as the scaled transformation of input (Experiment #2).

The additivity and scaling tests can be combined into a single test. Given $y_1(t)=T\{x_1(t)\}$ and $y_2(t)=T\{x_2(t)\}$, and two constants, a_1 and a_2, then linearity is satisfied if

$$T\left\{a_1x_1(t)+a_2x_2(t)\right\}=a_1T\left\{x_1(t)\right\}+a_2T\left\{x_2(t)\right\}=a_1y_1(t)+a_2y_2(t) \qquad \text{(L3.4)}$$

Time invariance

A system is time invariant if the output of the system doesn't depend on the absolute time at which the input is presented to the system. We can test for time invariance by performing two experiments. In Experiment #1, we delay the input, $x(t)$, by a constant, τ, before putting it into the system, yielding an output that we'll call $e_1(t)=T\left\{x(t-\tau)\right\}$. In the second experiment, we input $x(t)$ into the system, yielding $y(t)=T\left\{x(t)\right\}$, and then scale this output by constant, τ, yielding $e_2(t)=y(t-\tau)$. Graphically, we have this:

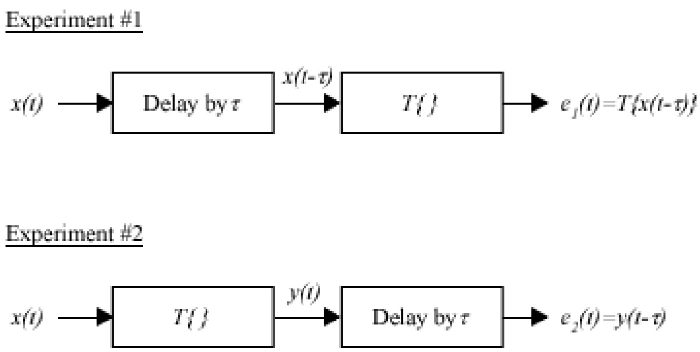


Figure 3: Test for time invariance of linear systems

If the results of these two experiments are the same, then the system is time invariant. In words, the time invariant property is satisfied if the transformation of the delayed input (Experiment #1) is the same as the delayed transformation of input (Experiment #2).

Assignment

In this lab we will test the linearity and time-invariance properties of various systems using Matlab. For each of the following systems, determine if the system could satisfy the additivity property, the scaling property and/or the time-invariance property:

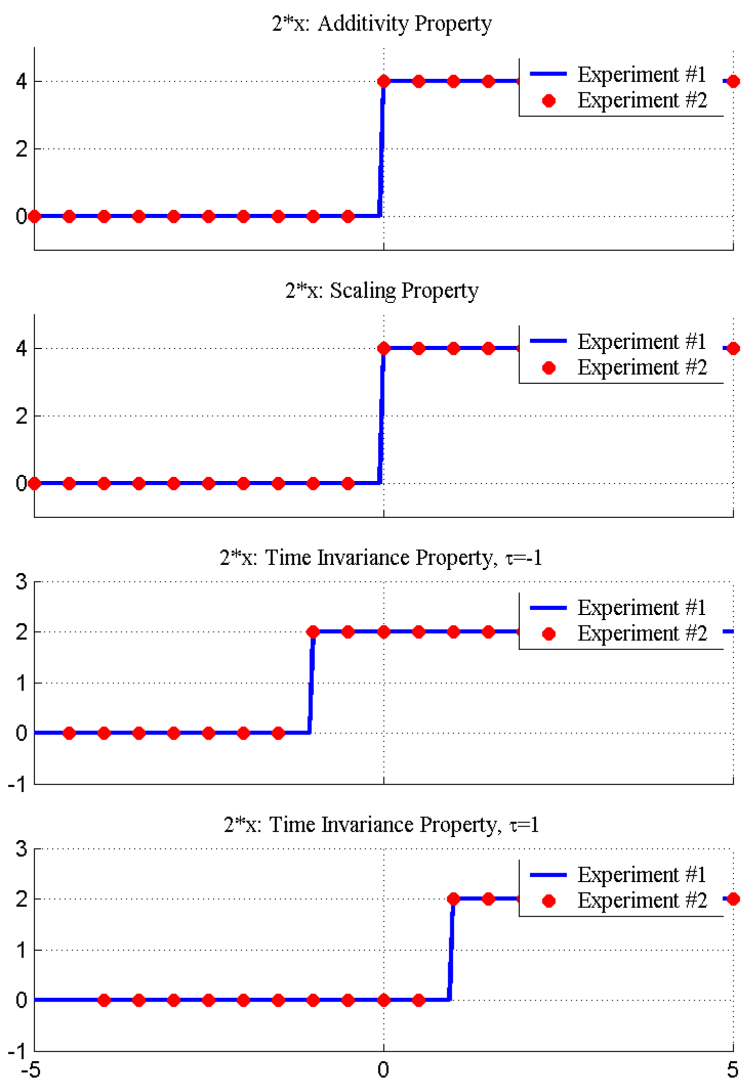
a) $y(t)=2x(t)$
b) $y(t)=x(t)+1$
c) $y(t)=\ln x(t)$
d) $y(t)=x^2(t)$
e) $y'(t)+2y(t)=2x(t)$
f) $y(t)=x(t)u(t)$
g) $y(t)=tu(t)$

We will test linearity and time-invariance using the step function you developed in Lab#1. Of course testing with a single function can't definitively prove that the system is linear or time-invariant, however, if the test for linearity fails with any single input, this definitively disproves that the system could be linear. So, what we will be able to say is either a) the system *could be* linear or b) the system *is not* linear. The task is to make one page of plots for each of the systems listed above. Each plot will have three panels.

- The first panel will show the test for the additivity property of linear systems. In blue and red, respectively, show the result of Experiment#1 and Experiment#2 as illustrated in Figure 1. I've used a step of height one for both $x_1(t)$ and $x_2(t)$ and have selected the range of t to be $-5\le t\le 5$.
- The second panel will show the test for the scaling property of linear systems. In blue and red, respectively, show the result of Experiment#1 and Experiment#2 as illustrated in Figure 2.

- The third and fourth panels will show the test for the time invariance of linear systems. In blue and red, respectively, show the result of Experiment#1 and Experiment#2 as illustrated in Figure 3. Here, you will want to try two shifts (i.e. values of τ) for each experiment. The third panel should show a positive τ, and the fourth panel a negative τ. For Experiment #1, you have to time-shift the input, that is, $u(t-\tau)$. For Experiment #2, you have to time-shift the output. You can do this just by computing *y(t)* and shifting the time abscissa of the plot in the `plot` command.

Here's what I get for the first example:



One suggestion for how to program at least some of these is the following. Here's a portion of the function I used to produce the above plots:

```
function plotlti(T)
% LTI Check linearity and time invariance of systems
%     T is a string which gives the transformation of the system
%     i.e. T = '2*x' would indicate y(t)=2*x(t)
```

```
%
%     Your Name Date

clf
t1 = -5:0.05:5; % fine time scale
t2 = -5:0.5:5;  % coarse time scale

% Additivity Experiment #1
x1 = u(t1);     % create two steps
x2 = u(t1);
x = x1 + x2;    % add them
e1 = eval(T);   % apply transformation

% Additivity Experiment #2
x = u(t2);      % create a step
y1 = eval(T);   % transform it to form y1
y2 = y1;        % use the same output as y2
e2 = y1 + y2;   % add them up
% ... plotting code here ...
```

To produce the plots, I just called `plotlti('2*x')`. Note the use of the `eval` function. This handy Matlab function evaluates any *string* in the current Matlab workspace as if you had typed that string into the command line. I can use the same program to get plots for parts a)-d). For example, to evaluate part b), I'd just call `plotlti('x+1')`. I have to do a bit more work for the other parts. Note also the use of the `legend` function, which helps label the different parts of each plot.
