3.3 Realizable finite hypothesis classes

We make some assumptions to make PAC more concrete.

Assumptions:

(i) Finite hypothesis space: $\mathcal{H}$ is finite.

(ii) Realizable: $\exists\, h^* \in \mathcal{H}$ s.t. $L(h^*) = 0$

[Theorem 4] realizable finite hypothesis class.

Assume $\mathcal{H} = \{h : \mathcal{X} \to \mathcal{Y}\}$, $\ell((x,y),h) = \mathbb{1}[y \neq h(x)]$. Let p^* be any distribution over $\mathcal{X} \times \mathcal{Y}$. Assume Assumptions (i) & (ii) hold. Let $\hat{h}$ be ERM, then the following two equivalent statements hold:

① with prob. $\geq 1-\delta$, $L(\hat{h}) \leq \frac{1}{n}(\log|\mathcal{H}| + \log(1/\delta))$

② with prob. $\geq 1-\delta$: $n \geq (\log|\mathcal{H}| + \log(1/\delta))/\varepsilon \Rightarrow L(\hat{h}) \leq \varepsilon$

Pf: Idea: bound the probability of $\{L(\hat{h}) > \varepsilon\}$.

Let $B = \{h \in \mathcal{H} : L(h) > \varepsilon\}$, then we have

$$P\{L(\hat{h}) > \varepsilon\} = P\{\hat{h} \in B\}$$

Since $\hat{L}(h^*) = L(h^*) = 0$, then $\hat{L}(\hat{h}) = 0$, we have

$$P\{\hat{h} \in B\} \leq P\{h \in B : \hat{L}(h) = 0\}$$

Step 1: bound $P\{\hat{L}(h) = 0\}$ for a fixed $h \in B$.

Since data is i.i.d., for $h \in B$:

$$P\{\hat{L}(h) = 0\} = (1 - L(h))^n \leq (1-\varepsilon)^n \leq e^{-\varepsilon n},$$

Step 2: Union bound for any $h \in B$

$$P\{h \in B : \hat{L}(h) = 0\} \leq \sum_{h \in B} P\{\hat{L}(h) = 0\}$$

$$\leq |\mathcal{H}| e^{-\varepsilon n} =: \delta.$$

Taking logs of the last equality and rearranging:

$$\epsilon = \frac{1}{n}\left(\log|H| + \log(1/\delta)\right) \qquad \square$$

Note: (i) $P\{\hat{L}(h) = 0\} = (1 - L(h))^n$ is because h does not err with prob. $= 1 - L(h)$ on each sample.

Since
$$\begin{aligned} L(h) &= E_{(x,y)\sim p^*}[\ell((x,y),h)] \\ &= \int \mathbb{1}\{y \neq h(x)\}\, dp^* \\ &= p^*\{y \neq h(x)\} \end{aligned}$$