3.11 VC dimension

[Definition 11] VC dimension

The VC dimension of a family of functions $\mathcal{H}$ with boolean outputs is the maximum number of points that can be shattered by $\mathcal{H}$: $\quad VC(\mathcal{H}) = \sup\{n : s(\mathcal{H}, n) = 2^n\}$

Note: To show a class $\mathcal{H}$ has VC dimension d,

(i) upper bound: show $d+1$ points can't be shattered;

(ii) lower bound: show d points can be shattered.

[Theorem 10] finite-dimentional function class

Let $\mathcal{F} \subseteq \{f : \mathcal{X} \to \mathbb{R}\}$. Let $\mathcal{H} = \{x \mapsto \mathbb{1}\{f(x) \geq 0\} : f \in \mathcal{F}\}$.

Then we have $\quad VC(\mathcal{H}) \leq \dim(\mathcal{F})$

Pf: for any $n > \dim(F)$, $x_1, \cdots, x_n$ are given.

Consider $M(f) := [f(x_1), \cdots, f(x_n)] \in \mathbb{R}^n$

$M := \{M(f) : f \in F\}$ is linear space. $\dim(M) \leq \dim(F)$.

Since $n > \dim(F) \geq \dim(M)$, $\exists\, 0 \neq c \subset \mathbb{R}^n$ s.t. $M(f) \cdot c = 0$ for all $f \in F$. Without loss of generality, $\{c_i > 0\} \neq \phi$. Then

$$\sum_{c_i > 0} c_i f(x_i) + \sum_{c_i \leq 0} c_i f(x_i) = 0 \qquad \text{for all } f \in F.$$

Suppose H shatters $\{x_1, \cdots, x_n\}$, we could find a $h \in H$ s.t.

$$h(x_i) = 1 \quad \text{whenever } c_i \geq 0 \quad \text{and } h(x_i) = 0 \text{ whenever } c_i < 0$$

we have $\sum_{c_i \geq 0} c_i h(x_i) + \sum_{c_i < 0} c_i h(x_i) > 0$, but $h \in F$, which is a contradiction.

Therefore, H can't shatter $\{x_1, \cdots, x_n\}$ for any choise of

[illegible] $VC(H) \leq \dim(F)$ $\square$

$x_1, \dots, x_n$, so $VC(\mathcal{H}) = \dim(\mathcal{F})$ $\blacksquare$