5.5 Follow the regularized leader

Idea: add some regularization to stablize the learner.

[Algorithm 3] follow the regularized leader (FTRL)
Let $\psi: S \to \mathbb{R}$ be a function called regularizer.
Let $f_1, \cdots, f_T$: sequence of loss.
On iteration t, learner chooses w_t by

$$w_t \in \arg\min_{w \in S} \psi(w) + \sum_{i=1}^{t-1} f_i(w)$$

Note: FTL is FTRL with $\psi = 0$.

☆ Quadratic ψ linear f_t

$$\psi(w) = \frac{1}{2\eta} \|w\|_2^2 \quad , \quad f_t(w) = w \cdot z_t$$

Then FTRL optimization problem is

$$w_t = \arg\min_{w \in S} \left\{ \frac{1}{2\eta} \|w\|_2^2 - w \cdot \theta_t \right\} \quad \cdots (496)$$

where $\theta_t = -\sum_{i=1}^{t-1} z_i$ [Regrad θ_t as the direction we want to move in]

If $S = \mathbb{R}^d$, FTRL has a closed form solution:

$$w_t = \eta \theta_t$$

Rerange it:

$$w_{t+1} = w_t - \eta z_t$$

If $S \neq \mathbb{R}^d$, then FTRL requires a projection onto S,
by (496):

$$w_t \in \arg\min_{w \in S} \frac{1}{2\eta} \|w - \eta\theta_t\|_2^2 = \Pi_S(\eta\theta)$$

This is Lazy Projection since θ_t still accumulates unprojected gradients. Also known as Nesterov's dual averaging algorithm.

[Theorem 30] regret of FTRL
Let $S \subseteq \mathbb{R}^d$ be a convex set.
Let $f_1, \cdots, f_T$ be a linear loss $f_t(w) = w \cdot z_t$ for any z_t and

Let $f_1, \cdots, f_T$ be any linear loss : $f_t(w) = w \cdot z_t$ for some $z_t \in \mathbb{R}^k$.

Let $\psi(w) = \frac{1}{2\eta}\|w\|_2^2$ be quadratic regularizer for any $\eta > 0$.

Then Regret w.r.t. any $u \in S$ satisfies:

$$\mathrm{Regret}(u) \le \frac{1}{2\eta}\|u\|_2^2 + \frac{\eta}{2}\sum_{i=1}^{T}\|z_t\|_2^2 \qquad (501)$$

Tradeoff between two terms of RHS:

(i) $\frac{1}{2\eta}\|u\|_2^2$: bigger η means w_t is easier to get u.

(ii) $\frac{\eta}{2}\sum_{i=1}^{T}\|z_t\|_2^2$: smaller η means w_t's are closer and stabler

$(w_{t+1} - w_t = -\eta z_t)$.

Corollary:

Let $\|z_t\|_2 \le L$, $\|u\|_2 \le B$ for $\forall u \in S$. By (501) we have

$$\mathrm{Regret}(u) \le \frac{B^2}{2\eta} + \frac{\eta T L^2}{2} \qquad \text{for } \forall \eta > 0$$

$$\Rightarrow \quad \mathrm{Regret} \le BL\sqrt{T}$$

Now we prove a weaker result:

$$\mathrm{Regret}(u) \le \frac{1}{2\eta}\|u\|_2^2 + \eta\sum_{t=1}^{T}\|z_t\|_2^2$$

Pf: idea: reduce FTRL $\to$ FTL.

Regrad ψ as the first function. We regard the FTL of the sequence of loss : $\psi, f_1, \cdots, f_T$. By Lemma 7:

$$[\psi(w_0) - \psi(u)] + \sum_{t=1}^{T}[f_t(w_t) - f_t(u)] \le [\psi(w_0) - \psi(w_1)] + \sum_{t=1}^{T}[f_t(w_t) - f_t(w_{t+1})]$$

Canceling $\psi(w_0)$, noting $\psi(w_1) \ge 0$, we have:

$$\mathrm{Regret}(u) = \sum_{t=1}^{T}[f_t(w_t) - f_t(u)] \le \frac{1}{2\eta}\|u\|_2^2 + \sum_{t=1}^{T}[f_t(w_t) - f_t(w_{t+1})]$$

Observe that:

$$\begin{aligned} f_t(w_t) - f_t(w_{t+1}) &= z_t \cdot (w_t - w_{t+1}) && f_t = w \cdot z_t \\ &\le \|z_t\|_2 \|w_t - w_{t+1}\|_2 \\ &= \|z_t\|_2 \|\Pi_S(\eta\theta_t) - \Pi_S(\eta\theta_{t-1})\|_2 && w_t = \Pi_S(\eta\theta_t) \end{aligned}$$

$$\le \|z_t\|_2 \|\eta\theta_t - \eta\theta_{t+1}\|_2$$ projection decreases distance

$$= \eta\|z_t\|_2^2$$ $\theta_{t+1} = \theta_t - z_t$

$$\Rightarrow \text{Regret} \le \frac{1}{2\eta}\|u\|_2^2 + \eta \sum_{t=1}^{T} \|z_t\|_2^2 \qquad \square$$