3.12 Norm-constrained hypothesis classes

[Theorem 11] Rademacher complexity of L_2 ball.

Let $\mathcal{F} = \{ z \mapsto w \cdot z : \|w\|_2 \le B_2 \}$ bounds on weight vectors.

Assume $\mathbb{E}_{z \sim p^*}[\|Z\|_2^2] \le C_2^2$, Then

$$R_n(\mathcal{F}) \le \frac{B_2 C_2}{\sqrt{n}}$$

Pf:

$$
\begin{aligned}
R_n(\mathcal{F}) &= \frac{1}{n} \mathbb{E}\Big[\sup_{\|w\|_2 \le B_2} \sum_{i=1}^n \sigma_i (w \cdot Z_i) \Big] \\
&\le \frac{1}{n} \mathbb{E}\Big[\sup_{\|w\|_2 \le B_2} \|w\|_2 \Big\| \sum_{i=1}^n \sigma_i Z_i \Big\|_2 \Big] \quad \text{(Hölder ineq.)} \\
&\le \frac{B_2}{n} \mathbb{E}\Big[\Big\| \sum_{i=1}^n \sigma_i Z_i \Big\|_2 \Big] \\
&\le \frac{B_2}{n} \sqrt{\mathbb{E}\Big[\Big\| \sum_{i=1}^n \sigma_i Z_i \Big\|_2^2 \Big]} \quad \text{(concavity of sqrt)} \\
&= \frac{B_2}{n} \sqrt{\mathbb{E}\Big[\sum_{i=1}^n \| \sigma_i Z_i \|_2^2 \Big]} \quad \text{expectation of cross term} = 0. \\
&= \frac{B_2}{n} \sqrt{\mathbb{E}\Big[\sum_{i=1}^n \| Z_i \|_2^2} \\
&\le \frac{B_2 C_2}{\sqrt{n}}.
\end{aligned}
$$

$\square$

[Theorem 12] Rademacher complexity of L_1 ball

Assume $\|Z_i\|_\infty \le C_\infty$ with prob. 1 for all data points $i = 1, \cdots, n$.

Then $$R_n(\mathcal{F}) \le \frac{B_1 C_\infty \sqrt{2 \log(2d)}}{\sqrt{n}}$$

Pf: Key : L_1 ball is the convex hull of the following

$$W = \bigcup_{j=1}^d \{ B_1 e_j, -B_1 e_j \}$$

$$\Rightarrow R_n(\mathcal{F}) = \mathbb{E}\Big[\sup_{w \in W} \frac{1}{n} \sum_{i=1}^n \sigma_i (w \cdot Z_i) \Big]$$

We have $w \cdot Z_i \le \|w\|_1 \|Z_i\|_\infty \le B_1 C_\infty$. By Massart's finite lemma

$$R_n(\mathcal{F}) \le \sqrt{\frac{2M^2 \log|\mathcal{F}|}{n}} = \sqrt{\frac{2 B_1^2 C_\infty^2 \log(2d)}{n}}$$

$\square$