Motion Planning and State Estimation in Robotics

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Overview

Course Repo

Link to Course Repo
https://github.com/arunkumar-singh/Motion_Planning_Lecture_Codes

Global Trajectory Planning

- Till now we have considered only incremental trajectory planning. That is, we were only considered about planning a trajectory for say next 0.1s or less.
- Incremental trajectory planning is myopic and greedy. The overall trajectory quality is sub-optimal
- We now move to global trajectory planning. That is, we plan the trajectory taking into account all the costs and constraints for all time steps into the future
- We will do this by modeling trajectories as polynomials.

Recall Triple Integrator System

If the input jerk j_x, j_y is constant, we get the following by integrating between time t_0 and t

$$\dddot{x} = \jmath_x, \dddot{y} = \jmath_y \tag{1}$$

$$\ddot{x}(t) = \ddot{x}_0 + \jmath_x(t - t_0), \ddot{y} = \ddot{y} + \jmath_y(t - t_0) \tag{2}$$

$$\dot{x}(t) = \dot{x}_0 + \ddot{x}_0(t - t_0) + \frac{1}{2}\jmath_x(t - t_0)^2, \dot{y}(t) = \dot{y}_0 + \ddot{y}_0(t - t_0) + \frac{1}{2}\jmath_y(t - t_0)^2 \tag{3}$$

$$x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}\ddot{x}_0(t - t_0)^2 + \frac{1}{6}\jmath_x(t - t_0)^3 \tag{4}$$

$$y(t) = y_0 + \dot{y}_0 t + \frac{1}{2}\ddot{y}_0(t - t_0)^2 + \frac{1}{6}\jmath_y(t - t_0)^3 \tag{5}$$

What if the input is not constant?

Polynomial Trajectories

Let

$$\dddot{x}(t) = 6a_1 + 24a_2 t + 60a_3 t^2, \dddot{y}(t) = 6b_1 + 24b_2 t + 60b_3 t^2 \tag{6}$$

Integrating three times, we get

$$\ddot{x}(t) = \ddot{x}_0 + 6a_1 t + 12a_2 t^2 + 20a_3 t^3 \tag{7}$$

$$\dot{x}(t) = \dot{x}_0 + \ddot{x}_0 t + 3a_1 t^2 + 4a_2 t^3 + 5a_3 t^4 \tag{8}$$

$$x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}\ddot{y}_0 t^2 + a_1 t^3 + a_2 t^4 + a_3 t^5 \tag{9}$$

$$\ddot{y}(t) = \ddot{y}_0 + 6b_1 t + 12b_2 t^2 + 20b_3 t^3 \tag{10}$$

$$\dot{y}(t) = \dot{y}_0 + \ddot{y}_0 t + 3b_1 t^2 + 4b_2 t^3 + 5b_3 t^4 \tag{11}$$

$$y(t) = y_0 + \dot{y}_0 t + \frac{1}{2}\ddot{y}_0 t^2 + b_1 t^3 + b_2 t^4 + b_3 t^5 \tag{12}$$

Polynomial Trajectories

Let

$$\dddot{x}(t) = 6a_1 + 24a_2 t + 60a_3 t^2 + 120a_4 t^3 + 210a_5 t^4 \tag{13}$$

Integrating three times, we get the following, where $x_0, \dot{x}_0, \ddot{x}_0$ are the initial position, velocity and acceleration of the robot.

$$\ddot{x}(t) = \ddot{x}_0 + 6a_1 t + 12a_2 t^2 + 20a_3 t^3 + 30a_4 t^4 + 42a_5 t^5 \tag{14}$$

$$\dot{x}(t) = \dot{x}_0 + \ddot{x}_0 t + 3a_1 t^2 + 4a_2 t^3 + 5a_3 t^4 + 6a_4 t^5 + 7a_5 t^6 \tag{15}$$

$$x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}\ddot{x}_0 t^2 + a_1 t^3 + a_2 t^4 + a_3 t^5 + a_4 t^6 + a_5 t^7 \tag{16}$$

We can write similar expression for y as well

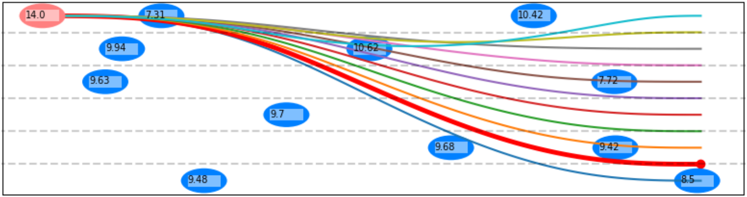
$$\ddot{y}(t) = \ddot{y}_0 + 6b_1 t + 12b_2 t^2 + 20b_3 t^3 + 30b_4 t^4 + 42b_5 t^5 \tag{17}$$

$$\dot{y}(t) = \dot{y}_0 + \ddot{y}_0 t + 3b_1 t^2 + 4b_2 t^3 + 5b_3 t^4 + 6b_4 t^5 + 7b_5 t^6 \tag{18}$$

$$y(t) = y_0 + \dot{y}_0 t + \frac{1}{2}\ddot{y}_0 t^2 + b_1 t^3 + b_2 t^4 + b_3 t^5 + b_4 t^6 + b_5 t^7 \tag{19}$$

Computing the right a's and b's

- Different a's and b's lead to different trajectories.
- Global trajectory planning can be thought as the problem of computing the right a's and b's for your problem setting.



(a)

Figure: Each trajectory corresponds to a different set of a's and b's

Polynomial Trajectories in Matrix Form

Lets start with the 5^{th} order polynomial.

During numerical computations, we are mostly interested in knowing trajectory values at specific time instants like $t_0, t_1, t_2 \dots t_f$.

$$x(t_0) = x_0 \tag{20}$$

$$x(t_1) = x_0 + \dot{x}_0 t_1 + \frac{1}{2}\ddot{x}_0 t_1^2 + a_1 t_1^3 + a_2 t_1^4 + a_3 t_1^5 \tag{21}$$

$$x(t_2) = x_0 + \dot{x}_0 t_2 + \frac{1}{2}\ddot{x}_0 t_2^2 + a_1 t_2^3 + a_2 t_2^4 + a_3 t_2^5 \tag{22}$$

$$\dots\dots\dots\dots\dots \tag{23}$$

$$x(t_f) = x_0 + \dot{x}_0 t_f + \frac{1}{2}\ddot{x}_0 t_f^2 + a_1 t_f^3 + a_2 t_f^4 + a_3 t_f^5 \tag{24}$$

Polynomial Trajectories in Matrix Form

In matrix form, we have

$$x(t) = \begin{bmatrix} x(t_1) \\ x(t_2) \\ x(t_3) \\ \dots \\ x(t_f) \end{bmatrix} = \mathbf{A}_x \widetilde{\mathbf{c}_x} + \mathbf{P}_x \mathbf{c}_x, \tag{25}$$

where,

$$\mathbf{A}_x = \begin{bmatrix} 1.0 & t_1 & 0.5t_1^2 \\ 1.0 & t_2 & 0.5t_2^2 \\ 1.0 & t_3 & 0.5t_3^2 \\ \dots\dots & & \\ 1.0 & t_f & 0.5t_f^2 \end{bmatrix}, \widetilde{\mathbf{c}}_x = \begin{bmatrix} x_0 \\ \dot{x}_0 \\ \ddot{x}_0 \end{bmatrix} \tag{26}$$

$$\mathbf{P}_x = \begin{bmatrix} t_1^3 & t_1^4 & t_1^5 \\ t_2^3 & t_2^4 & t_2^5 \\ t_3^3 & t_3^4 & t_3^5 \\ \dots\dots & & \\ t_f^3 & t_f^4 & t_f^5 \end{bmatrix}, \mathbf{c}_x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{27}$$

Polynomial Trajectories in Matrix Form

Similarly, we get

$$\dot{x}(t) = \begin{bmatrix} \dot{x}(t_1) \\ \dot{x}(t_2) \\ \dot{x}(t_3) \\ \dots \\ \dot{x}(t_f) \end{bmatrix} = \dot{\mathbf{A}}_x \widetilde{\mathbf{c}_x} + \dot{\mathbf{P}}_x \mathbf{c}_x, \tag{28}$$

where,

$$\dot{\mathbf{A}}_x = \begin{bmatrix} 0.0 & 1.0 & t_1 \\ 0.0 & 1.0 & t_2 \\ 0.0 & 1.0 & t_3 \\ \dots\dots & & \\ 0.0 & 1.0 & t_4 \end{bmatrix}, \widetilde{\mathbf{c}_x} = \begin{bmatrix} x_0 \\ \dot{x}_0 \\ \ddot{x}_0 \end{bmatrix} \tag{29}$$

$$\dot{\mathbf{P}}_x = \begin{bmatrix} 3t_1^2 & 4t_1^3 & 5t_1^4 \\ 3t_2^2 & 4t_2^3 & 5t_2^4 \\ 3t_2^2 & 4t_2^3 & 5t_2^4 \\ \dots\dots & & \\ 3t_f^2 & 4t_f^3 & 5t_f^4 \end{bmatrix}, \mathbf{c}_x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{30}$$

Polynomial Trajectories in Matrix Form

Similarly, we get

$$\ddot{x}(t) = \begin{bmatrix} \ddot{x}(t_1) \\ \ddot{x}(t_2) \\ \ddot{x}(t_3) \\ \dots \\ \ddot{x}(t_f) \end{bmatrix} = \ddot{\mathbf{A}}_x \widetilde{\mathbf{c}_x} + \ddot{\mathbf{P}}_x \mathbf{c}_x, \tag{31}$$

where,

$$\ddot{\mathbf{A}}_x = \begin{bmatrix} 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 1.0 \\ 0.0 & 0.0 & 1.0 \\ \dots\dots & & \\ 0.0 & 0.0 & 1.0 \end{bmatrix}, \widetilde{\mathbf{c}}_x = \begin{bmatrix} x_0 \\ \dot{x}_0 \\ \ddot{x}_0 \end{bmatrix} \tag{32}$$

$$\ddot{\mathbf{P}}_x = \begin{bmatrix} 6t_1 & 12t_1^2 & 20t_1^4 \\ 6t_2 & 12t_2^2 & 20t_2^4 \\ 6t_3 & 12t_3^2 & 20t_3^4 \\ \dots\dots & & \\ 6t_f & 12t_f^2 & 20t_f^4 \end{bmatrix}, \mathbf{c}_x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{33}$$

Point to Point Trajectory in Matrix Form

$$\overbrace{\begin{bmatrix} x(t_f) \\ \dot{x}(t_f) \\ \ddot{x}(t_f) \end{bmatrix}}^{\mathbf{s}_{end}} = \overbrace{\begin{bmatrix} {}^n\mathbf{A}_x \\ {}^n\dot{\mathbf{A}}_x \\ {}^n\ddot{\mathbf{A}}_x \end{bmatrix}}^{\mathbf{G}_{end}} \widetilde{\mathbf{c}}_x + \overbrace{\begin{bmatrix} {}^n\mathbf{P}_x \\ {}^n\dot{\mathbf{P}}_x \\ {}^n\ddot{\mathbf{P}}_x \end{bmatrix}}^{\mathbf{H}_{end}} \mathbf{c}_x \tag{34}$$

Solution is given by

$$\mathbf{c}_x = \mathbf{H}_{end}^{-1}(\mathbf{s}_{end} - \mathbf{G}_{end}\widetilde{\mathbf{c}}_x) \tag{35}$$

Polynomial Form for a 7^{th} degree polynomial

$$x(t) = \begin{bmatrix} x(t_1) \\ x(t_2) \\ x(t_3) \\ \dots \\ x(t_f) \end{bmatrix} = \mathbf{A}_x \widetilde{\mathbf{c}_x} + \mathbf{P}_x \mathbf{c}_x, \tag{36}$$

where,

$$\mathbf{A}_x = \begin{bmatrix} 1.0 & t_1 & 0.5t_1^2 \\ 1.0 & t_2 & 0.5t_2^2 \\ 1.0 & t_3 & 0.5t_3^2 \\ \dots\dots & & \\ 1.0 & t_f & 0.5t_f^2 \end{bmatrix}, \widetilde{\mathbf{c}_x} = \begin{bmatrix} x_0 \\ \dot{x}_0 \\ \ddot{x}_0 \end{bmatrix} \tag{37}$$

$$\mathbf{P}_x = \begin{bmatrix} t_1^3 & t_1^4 & t_1^5 & t_1^6 & t_1^7 \\ t_2^3 & t_2^4 & t_2^5 & t_2^6 & t_2^7 \\ t_3^3 & t_3^4 & t_3^5 & t_3^6 & t_3^7 \\ \dots\dots & & & & \\ t_f^3 & t_f^4 & t_f^5 & t_f^6 & t_f^7 \end{bmatrix}, \mathbf{c}_x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \tag{38}$$

Solving for the coefficients of a 7^{th} order polynomial

We can follow a similar procedure to derive the $\dot{\mathbf{P}}_x, \ddot{\mathbf{P}}_x$ for the 7^{th} order polynomial.
Lets now re-write the linear equation for solving the 7^{th} order polynomial.

$$\overbrace{\begin{bmatrix} x(t_f) \\ \dot{x}(t_f) \\ \ddot{x}(t_f) \end{bmatrix}}^{\mathbf{s}_{end}} = \overbrace{\begin{bmatrix} {}^{n}\mathbf{A}_x \\ {}^{n}\dot{\mathbf{A}}_x \\ {}^{n}\ddot{\mathbf{A}}_x \end{bmatrix}}^{\mathbf{G}_{end}} \widetilde{\mathbf{c}}_x + \overbrace{\begin{bmatrix} {}^{n}\mathbf{P}_x \\ {}^{n}\dot{\mathbf{P}}_x \\ {}^{n}\ddot{\mathbf{P}}_x \end{bmatrix}}^{\mathbf{H}_{end}} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \tag{39}$$

We cannot solve the above equation since the matrix $\mathbf{H}_{end}$ is not square.

Incorporating intermediate way-point trajectory

We need to append the rows corresponding to intermediate points. Suppose, the intermediate points happen at time t_i, t_j. So, we need to extract row i and j from matrix $\mathbf{A}_x$ and $\mathbf{P}_x$.

$$\overbrace{\begin{bmatrix} \mathbf{s}_{mid} \\ \mathbf{s}_{end} \end{bmatrix}}^{\mathbf{s}} = \overbrace{\begin{bmatrix} \mathbf{G}_{mid} \\ \mathbf{G}_{end} \end{bmatrix}}^{\mathbf{G}} \widetilde{\mathbf{c}}_x + \overbrace{\begin{bmatrix} \mathbf{H}_{mid} \\ \mathbf{H}_{end} \end{bmatrix}}^{\mathbf{H}} \mathbf{c}_x \tag{40}$$

$$\mathbf{G}_{mid} = \begin{bmatrix} {}^i\mathbf{A}_x \\ {}^j\mathbf{A}_x \end{bmatrix}, \mathbf{H}_{mid} = \begin{bmatrix} {}^i\mathbf{P}_x \\ {}^j\mathbf{P}_x \end{bmatrix}, \mathbf{s}_{mid} = \begin{bmatrix} x(t_1) \\ x(t_2) \end{bmatrix} \tag{41}$$

Solution is given by

$$\mathbf{c}_x = \mathbf{H}^{-1}(\mathbf{s} - \mathbf{G}\widetilde{\mathbf{c}}_x) \tag{42}$$

Linear Equation Solving is not Enough

- Computing trajectory coefficients through solving linear equations cannot be used when the matrix $\mathbf{H}$ is not invertible. This in turn can happen when the trajectory polynomial has more coefficients than specified equations.
- Using higher degree polynomial with more trajectory coefficients gives us a great deal of flexibility but then we have to find out of way of computing those coefficients.
- One way of computing trajectory coefficients is through the optimization route.

Introducing Optimality

- We want the robots to behave in a particular way and not just go from point A to point B
- The most intuitive way of describing robot behavior is in terms of cost functions. For example, move with a particular reference velocity as much as possible.
- Most trajectory planning problems are framed as optimization problems

Computing minimum of functions

$$\min f(s) = s^2 - 6s + 4 \tag{43}$$

To solve this minimization, we take the derivative with respect to x and equate it to zero

$$2s - 6 = 0 \tag{44}$$

Multi variable function

$$\min f(s_1, s_2) = 2s_1^2 + 2s_1s_2 + 2s_2^2 - 6s_1 \tag{45}$$

We compute partial derivatives and equate them to zero

$$\bigtriangledown f_{s_1} = 4s_1 + 2s_2 - 6 = 0 \tag{46}$$
$$\bigtriangledown f_{s_2} = 2s_1 + 4s_2 = 0 \tag{47}$$
$$\tag{48}$$

Multi-variable quadratic function in Matrix form

$$f(s_1, s_2) = 2s_1^2 + 2s_1 s_2 + 2s_2^2 - 6s_1$$

$$\Rightarrow f(s_1, s_2) = \frac{1}{2} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}^T \overbrace{\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}}^{\mathbf{Q}} \overbrace{\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}}^{\mathbf{s}} + \underbrace{\begin{bmatrix} -6 \\ 0 \end{bmatrix}^T}_{\mathbf{q}} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

$$f(s_1, s_2) = \frac{1}{2}\mathbf{s}^T \mathbf{Q} \mathbf{s} + \mathbf{q}^T \mathbf{s} \tag{49}$$

The derivative (or gradient) in matrix form is given by

$$\bigtriangledown f = \mathbf{Q}\mathbf{s} + \mathbf{q} = 0 \Rightarrow \mathbf{s} = -\mathbf{Q}^{-1}\mathbf{q} \tag{50}$$

Multiple Quadratic Costs

$$
\begin{aligned}
\frac{1}{2}\mathbf{s}^T\mathbf{Q}_1\mathbf{s} + \mathbf{q}_1^T\mathbf{s} + \frac{1}{2}\mathbf{s}^T\mathbf{Q}_2\mathbf{s} + \mathbf{q}_2^T\mathbf{s} \\
= \frac{1}{2}\mathbf{s}^T\mathbf{Q}\mathbf{s} + \mathbf{q}^T\mathbf{s} \\
\mathbf{Q} = \mathbf{Q}_1 + \mathbf{Q}_2, \mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2
\end{aligned}
\tag{51}
$$

The derivative (or gradient) in matrix form is given by

$$
\mathbf{Q}\mathbf{s} + \mathbf{q} = 0 \Rightarrow \mathbf{s} = -\mathbf{Q}^{-1}\mathbf{q} \tag{52}
$$

Least Squares Cost

Consider the following cost function where a_{ij}, d_i are given constants.

$$\min f(s_1, s_2) = \frac{1}{2}((a_{11}s_1 + a_{12}s_2 - d_1)^2 + (a_{21}s_1 + a_{22}s_2 - d_2)^2 + (a_{31}s_1 + a_{32}s_2 - d_3)^2) \tag{53}$$

$$\begin{aligned} f(s_1, s_2) &= \frac{1}{2} \| \overbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}}^{\mathbf{A}} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} - \overbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}^{\mathbf{b}} \|_2^2 \\ f(s_1, s_2) &= \mathbf{s}^T\mathbf{Q}\mathbf{s} + \mathbf{q}^T\mathbf{s} \\ \mathbf{Q} &= \mathbf{A}^T\mathbf{A}, \mathbf{q} = -\mathbf{A}^T\mathbf{b} \end{aligned} \tag{54}$$

Again, solution is given by

$$\mathbf{Q}\mathbf{s} + \mathbf{q} = 0 \Rightarrow \mathbf{s} = -\mathbf{Q}^{-1}\mathbf{q} \tag{55}$$

Constructing Cost Functions

Suppose, you want to choose s_1, s_2 such that their sum is as close as possible to 1 and s_2 is as close as possible to 0.5

Constructing Cost Functions

Suppose, you want to choose s_1, s_2 such that their sum is as close as possible to 1 and s_2 is as close as possible to 0.5

$$f(s_1, s_2) = (s_1 + s_2 - 1)^2 + (s_2 - 0.5)^2 \tag{56}$$

Constructing Cost Functions

Suppose, you want to choose s_1, s_2 such that their sum is as close as possible to 1 and magnitude of s_2 is as small as possible

Constructing Cost Functions

Suppose, you want to choose s_1, s_2 such that their sum is as close as possible to 1 and magnitude of s_2 is as small as possible

$$f(s_1, s_2) = (s_1 + s_2 - 1)^2 + (s_2)^2 \tag{57}$$

Constructing Cost function

Construct a cost function such that the end position, velocity, acceleration is as close as possible to $x_f, \dot{x}_f, \ddot{x}_f$

$$f_{end} = \frac{1}{2}((x(t_f) - x_f)^2 + (\dot{x}(t_f) - \dot{x}_f)^2 + (\ddot{x}(t_f) - \ddot{x}_f)^2) \tag{58}$$

Constructing Cost function

Construct a cost function to keep the acceleration as small as possible at all times

$$f_{acc} = \frac{1}{2}\sum_t \ddot{x}(t)^2 \tag{59}$$

$$f_{acc} = \frac{1}{2}\sum_k ({}^k\ddot{\mathbf{A}}_x\widetilde{\mathbf{c}}_x + {}^k\ddot{\mathbf{P}}_x\mathbf{c}_x)^2$$

$$f_{acc} = \frac{1}{2}\|\ddot{\mathbf{A}}_x\widetilde{\mathbf{c}}_x + \ddot{\mathbf{P}}_x\mathbf{c}_x\|_2^2 \tag{60}$$

$$f_{acc} = \frac{1}{2}\mathbf{c}_x^T\mathbf{Q}_{acc}\mathbf{c}_x + \mathbf{q}_{acc}^T\mathbf{c}_x, \tag{61}$$

where

$$\mathbf{Q}_{acc} = \ddot{\mathbf{P}}_x^T\ddot{\mathbf{P}}_x, \mathbf{q}_{acc} = \ddot{\mathbf{P}}_x^T(\ddot{\mathbf{A}}_x\widetilde{\mathbf{c}}_x) \tag{62}$$

Constructing Cost function

Construct a cost function to ensure trajectory passes as close as possible to the intermediate points x_1 and x_2

$$
\begin{aligned}
f_{mid} &= \frac{1}{2}((x(t_1) - x_1)^2 + (x(t_2) - x_2)^2) \\
f_{mid} &= \frac{1}{2}\|\mathbf{H}_{mid}\mathbf{c}_x + \mathbf{G}_{mid}\widetilde{\mathbf{c}}_x - \mathbf{s}_{mid}\|_2^2 \qquad (63)
\end{aligned}
$$

$$f_{mid} = \frac{1}{2}\mathbf{c}_x^T\mathbf{Q}_{mid}\mathbf{c}_x + \mathbf{q}_{mid}^T\mathbf{c}_x, \qquad (64)$$

where

$$\mathbf{Q}_{mid} = \mathbf{H}_{mid}^T\mathbf{H}_{mid}, \mathbf{q}_{mid} = -\mathbf{H}_{mid}^T(\mathbf{s}_{mid} - \mathbf{G}_{mid}\widetilde{\mathbf{c}}_x) \qquad (65)$$

Constructing Cost Functions

Adding the final goal cost, acceleration cost and mid-point cost together.

$$f = \frac{1}{2}((x(t_f) - x_f)^2 + (\dot{x}(t_f) - \dot{x}_f)^2 + (\ddot{x}(t_f) - \ddot{x}_f)^2) + \frac{1}{2}\sum_t \ddot{x}(t)^2 + \frac{1}{2}((x(t_1) - x_1)^2 + (x(t_2) - x_2)^2) \tag{66}$$

$$\begin{aligned} f = f_{end} + f_{mid} + f_{acc} &= \frac{1}{2}\mathbf{c}_x^T\mathbf{Q}_{end}\mathbf{c}_x + \mathbf{q}_{end}^T\mathbf{c}_x + \frac{1}{2}\mathbf{c}_x^T\mathbf{Q}_{mid}\mathbf{c}_x + \mathbf{q}_{mid}^T\mathbf{c}_x + \frac{1}{2}\mathbf{c}_x^T\mathbf{Q}_{acc}\mathbf{c}_x + \mathbf{q}_{acc}^T\mathbf{c}_x \\ f &= \frac{1}{2}\mathbf{c}_x^T\mathbf{Q}\mathbf{c}_x + \mathbf{q}^T\mathbf{c}_x, \mathbf{Q} = \mathbf{Q}_{end} + \mathbf{Q}_{mid} + \mathbf{Q}_{acc}, \mathbf{q} = \mathbf{q}_{end} + \mathbf{q}_{mid} + \mathbf{q}_{acc} \end{aligned} \tag{67}$$

Constructing Cost Functions Contd..

Construct a cost function to ensure that (i) $x(t)$ and $y(t)$ are as close *as possible* to the desired path $(\mathbf{x}_d(t), \mathbf{y}_d(t))$ *at all times* and (ii) acceleration is as small as possible at all times

$$f(\mathbf{c}_x) = \frac{1}{2}w_1 \sum_t (\mathbf{x}(t) - \mathbf{x}_d(t))^2 + \frac{1}{2}w_2 \sum_t \ddot{x}(t)^2 \tag{68}$$

$$= \frac{1}{2}w_1 \|\mathbf{A}_x \widetilde{\mathbf{c}}_x + \mathbf{P}_x \mathbf{c}_x - x_d(t)\|_2^2 + \frac{1}{2}w_2 \|\ddot{\mathbf{A}}_x \widetilde{\mathbf{c}}_x + \ddot{\mathbf{P}}_x \mathbf{c}_x\|_2^2 \tag{69}$$

We get similar expressions for y component

$$f(\mathbf{c}_y) = \frac{1}{2}w_1 \sum_t (\mathbf{y}(t) - \mathbf{y}_d(t))^2 + \frac{1}{2}w_2 \sum_t \ddot{y}(t)^2$$

$$\frac{1}{2}\|\mathbf{A}_y \widetilde{\mathbf{c}}_y + \mathbf{P}_y \mathbf{c}_y - y_d(t)\|_2^2 + \frac{1}{2}\|\ddot{\mathbf{A}}_y \widetilde{\mathbf{c}}_y + \ddot{\mathbf{P}}_y \mathbf{c}_y\|_2^2 \tag{70}$$

The trajectories are computed by solving the following two optimization problem

$$\arg\min f(\mathbf{c}_x), \arg\min f(\mathbf{c}_y) \tag{71}$$

Modeling Hard/Strict constraints in Trajectory Planning

- Modeling robot behaviors through cost functions has a downside: the weights needs to be tuned to achieve the desired behavior.
- Strict requirements like stopping at the final position or bounds in velocity, acceleration can be handled alternately through hard constraints leading to constrained optimization problem.

Constrained Optimization with Equality Constraints

$$\arg\min \frac{1}{2}\mathbf{c}_x^T\mathbf{Q}\mathbf{c}_x + \mathbf{q}^T\mathbf{c}_x \tag{72}$$

$$\mathbf{M}\mathbf{c}_x = \mathbf{n} \tag{73}$$

The solution to the above optimization problem is given in two parts

$$\boldsymbol{\lambda} = -(\mathbf{M}\mathbf{Q}^{-1}\mathbf{M}^T)^{-1}(\mathbf{n} + \mathbf{M}\mathbf{Q}^{-1}\mathbf{q}) \tag{74}$$

$$\mathbf{c}_x = -\mathbf{Q}^{-1}(\mathbf{M}^T\boldsymbol{\lambda} + \mathbf{q}) \tag{75}$$

Constrained Optimization with Equality Constraints: Using Solver

$$\arg\min \frac{1}{2}\mathbf{c}_x^T\mathbf{Q}\mathbf{c}_x + \mathbf{q}^T\mathbf{c}_x \tag{76}$$

$$\mathbf{M}\mathbf{c}_x = \mathbf{n} \tag{77}$$

We can just give the matrices and vectors to the solver to get a solution

$$\mathbf{c}_x = \text{CVXOPT}(\mathbf{Q}, \mathbf{q}, \mathbf{M}, \mathbf{n}) \tag{78}$$

Hard Constraints

Example: End position, velocity and acceleration constraints

$$x(t_f) = x_f \tag{79}$$

$$^n\mathbf{A}_x\widetilde{\mathbf{c}}_x + {^n\mathbf{P}_x}\mathbf{c}_x = x_f \tag{80}$$

$$\dot{x}(t_f) = \dot{x}_f \tag{81}$$

$$^n\dot{\mathbf{A}}_x\widetilde{\mathbf{c}}_x + {^n\dot{\mathbf{P}}_x}\mathbf{c}_x = \dot{x}_f \tag{82}$$

$$\ddot{x}(t_f) = \ddot{x}_f \tag{83}$$

$$^n\ddot{\mathbf{A}}_x\widetilde{\mathbf{c}}_x + {^n\ddot{\mathbf{P}}_x}\mathbf{c}_x = \ddot{x}_f \tag{84}$$

$$\begin{bmatrix} ^n\mathbf{A}_x\widetilde{\mathbf{c}}_x + {^n\mathbf{P}_x}\mathbf{c}_x \\ ^n\dot{\mathbf{A}}_x\widetilde{\mathbf{c}}_x + {^n\dot{\mathbf{P}}_x}\mathbf{c}_x \\ ^n\ddot{\mathbf{A}}_x\widetilde{\mathbf{c}}_x + {^n\ddot{\mathbf{P}}_x}\mathbf{c}_x \end{bmatrix} = \begin{bmatrix} x_f \\ \dot{x}_f \\ \ddot{x}_f \end{bmatrix} \tag{85}$$

$$\mathbf{M} = \mathbf{H}_{end} = \begin{bmatrix} ^n\mathbf{P}_x \\ \dot{\mathbf{P}}_x \\ ^n\ddot{\mathbf{P}}_x \end{bmatrix}, \mathbf{n} = \mathbf{s}_{end} - \mathbf{G}_{end}\widetilde{\mathbf{c}}_x = \begin{bmatrix} x_f \\ \dot{x}_f \\ \ddot{x}_f \end{bmatrix} - \begin{bmatrix} ^n\mathbf{A}_x\widetilde{\mathbf{c}}_x \\ ^n\dot{\mathbf{A}}_x\widetilde{\mathbf{c}}_x \\ ^n\ddot{\mathbf{A}}_x\widetilde{\mathbf{c}}_x \end{bmatrix} \tag{86}$$

Constrained Optimization with Hard Inequality Constraints

$$\arg\min \frac{1}{2}\mathbf{c}_x^T \mathbf{Q}\mathbf{c}_x + \mathbf{q}^T\mathbf{c}_x \tag{87}$$

$$\mathbf{M}_{ineq}\mathbf{c}_x \leq \mathbf{n}_{ineq} \tag{88}$$

Example: bounds on velocity, acceleration

$$v_x^{min} \leq \dot{\mathbf{A}}_x\widetilde{\mathbf{c}}_x + \dot{\mathbf{P}}_x\mathbf{c}_x \leq v_x^{max} \tag{89}$$

$$a_x^{min} \leq \dot{\mathbf{A}}_{\dot{x}}\widetilde{\mathbf{c}}_x + \ddot{\mathbf{P}}_x\mathbf{c}_x \leq a_x^{max} \tag{90}$$

$$\tag{91}$$

$$\mathbf{M}_{ineq} = \begin{bmatrix} \dot{\mathbf{P}}_x \\ -\dot{\mathbf{P}}_x \\ \ddot{\mathbf{P}}_x \\ -\ddot{\mathbf{P}}_x \end{bmatrix}, \mathbf{n}_{ineq} = \begin{bmatrix} v_x^{max} - \dot{\mathbf{A}}_x\widetilde{\mathbf{c}}_x \\ -v_x^{min} + \dot{\mathbf{A}}_x\widetilde{\mathbf{c}}_x \\ a_x^{max} - \ddot{\mathbf{A}}_x\widetilde{\mathbf{c}}_x \\ -a_x^{min} + \ddot{\mathbf{A}}_x\widetilde{\mathbf{c}}_x \end{bmatrix} \tag{92}$$

Solving x and y components together

- Till now, we have considered costs and constraints that are de-coupled along the x and y axis of motion.
- But what if we have constraints of the following form

$$x(t_f) + y(t_f) = d \tag{93}$$

The above equation implies that the final point of the trajectory can be anywhere on a line.

Solving x and y components together

Suppose, we have a general problem of the following form

$$\min \frac{1}{2}\mathbf{c}_x^T\mathbf{Q}_x\mathbf{c}_x + \mathbf{q}_x^T\mathbf{c}_x + \frac{1}{2}\mathbf{c}_y^T\mathbf{Q}_y\mathbf{c}_y + \mathbf{q}_y^T\mathbf{c}_y \tag{94}$$

$$\mathbf{M}_x\mathbf{c}_x + \mathbf{M}_y\mathbf{c}_y = d \tag{95}$$

For solving problems like the one shown above where both x, y are coupled, we need to form a different representation where we stack $\mathbf{c}_x, \mathbf{c}_y$ together.

$$\frac{1}{2}\mathbf{c}_{xy}^T\mathbf{Q}_{xy}\mathbf{c}_x + \mathbf{q}_{xy}^T\mathbf{c}_{xy} \tag{96}$$

$$\mathbf{M}_{xy}\mathbf{c}_{xy} = d \tag{97}$$

$$\mathbf{Q}_{xy} = \begin{bmatrix} \mathbf{Q}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_y \end{bmatrix}, \mathbf{q}_{xy} = \begin{bmatrix} \mathbf{q}_x \\ \mathbf{q}_y \end{bmatrix}, \mathbf{c}_{xy} = \begin{bmatrix} \mathbf{c}_x \\ \mathbf{c}_y \end{bmatrix}, \mathbf{M}_{xy} = \begin{bmatrix} \mathbf{M}_x & \mathbf{M}_y \end{bmatrix} \tag{98}$$