Griffith University

3130CIT Theory of Computation

(Based on slides by Harald Søndergaard of
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Variants of Turing machines

Variants of Turing Machines

To try to make the Turing machine more powerful we could add to its features:

- Let its tape extend indefinitely in both directions.
- Let its tape have multiple tracks.
- Let there be several tapes, each with its independent tape head.
- Add nondeterminism.

It turns out that none of these increase a Turing machine's capabilities as a recogniser.

Multitape Machines

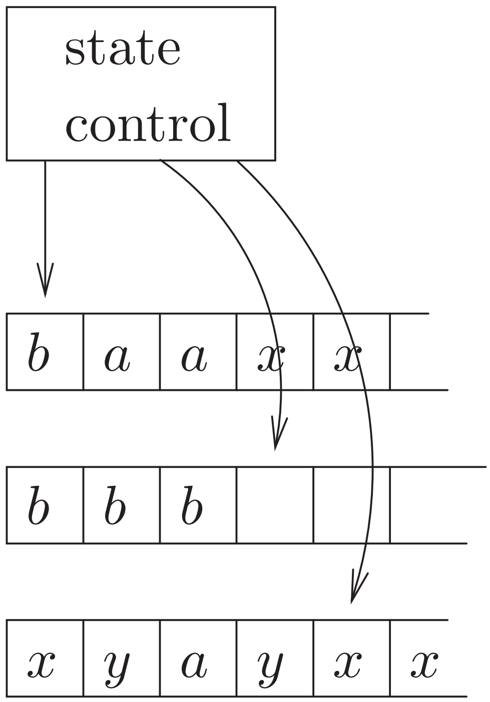
A multitape Turing machine has k tapes. It takes its input on tape 1, other tapes are initially blank.

The transition function now has type

$$\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$$

It specifies how the k tape heads behave when the machine is in state q_i, reading $a_1, \ldots a_k$:

$$\delta(q_i, a_1, \ldots, a_k) = (q_j, (b_1, \ldots, b_k), (d_1, \ldots, d_k))$$

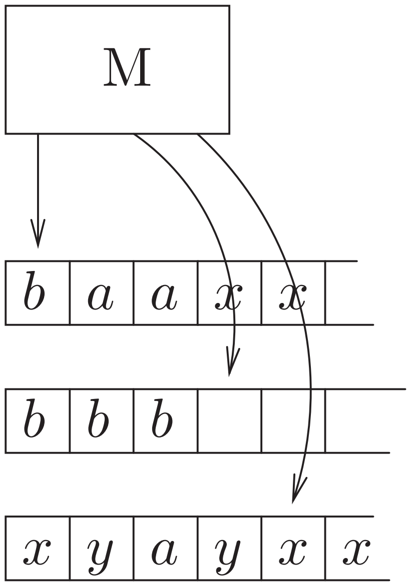


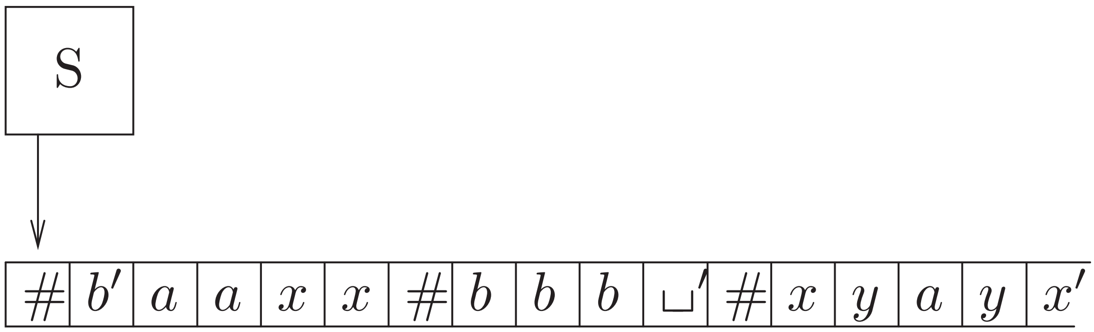
Multitape Machines (cont.)

Theorem: A language is Turing recognisable iff some multitape Turing machine recognises it.

Proof sketch: We show how to simulate a multitape machine M by a standard Turing machine S.

The standard machine has tape alphabet $\{\#\} \cup \Gamma \cup \Gamma'$ where $\#$ is a separator, not in $\Gamma \cup \Gamma'$, where there is a one-to-one correspondence between symbols in Γ and (marked) symbols in Γ'.





S reorganises its input $x_1 x_2 \cdots x_n$ into

$$\#x_1' x_2 \cdots x_n \underbrace{\#\sqcup'\# \cdots \#\sqcup'\#}_{k-1 \text{ times}}$$

Note how symbols of Γ' represent marked symbols from Γ, which denote the positions of the tape heads in the multitape machine.

Multitape Machines (cont.)

To simulate a move of M, S scans its tape to determine the marked symbols. S then scans the tape again, updating it according to M's transition function.

If a "virtual head" of M moves to a $\#$, S shifts that symbol, and every symbol after it, one cell to the right. In the vacant cell it writes $\sqcup$. It then continues to apply M's transition function.

Nondeterministic Turing Machines

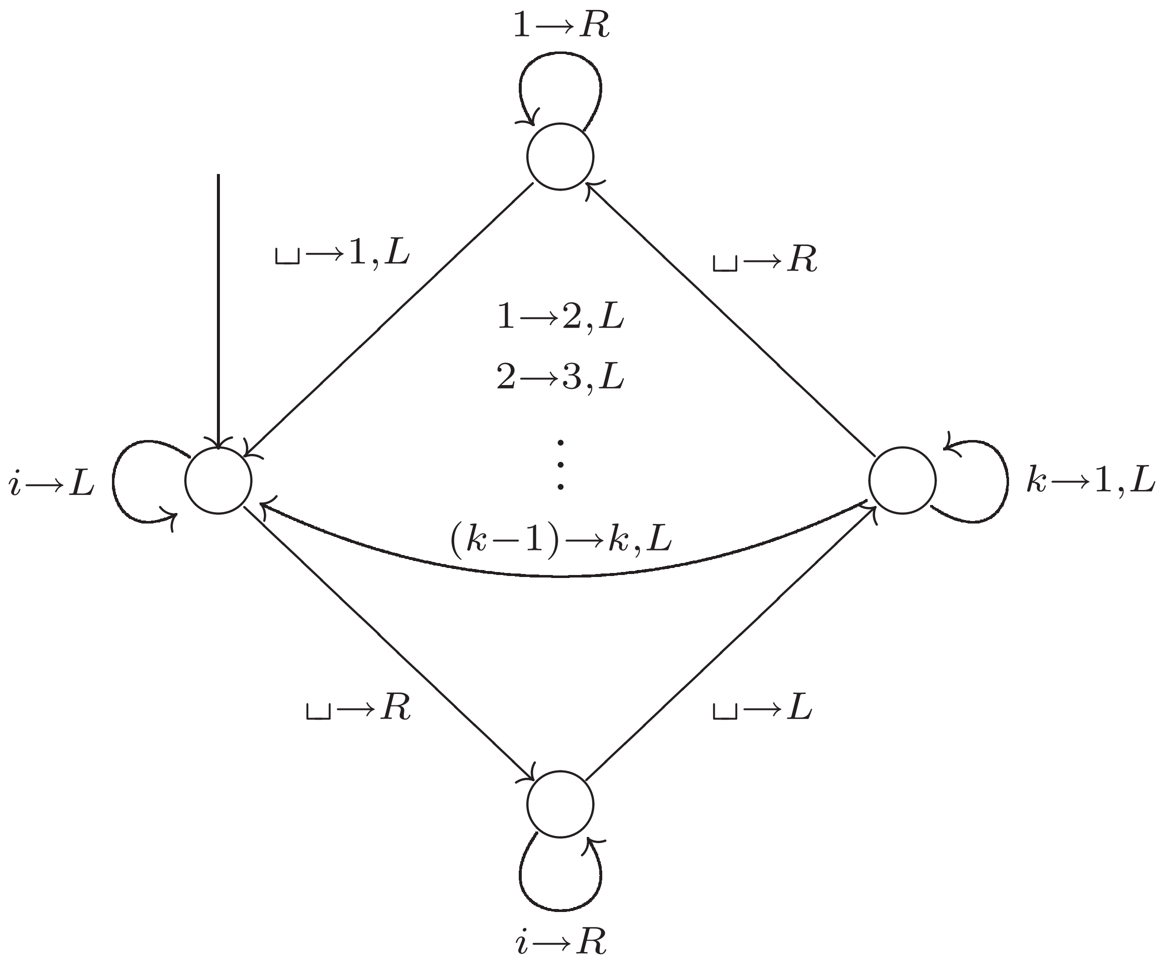
A nondeterministic Turing machine has a transition function of type

$$\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

If some computation branch lead to 'accept' then the machine accepts its input. This is the same type of nondeterminism as NFAs possess.

Nondet Turing Machines (cont.)

First, here is a deterministic machine to generate $\{1, \ldots, k\}^*$, in order of increasing length.



Try running this for $k = 3$.

Simulating Nondeterminism

Theorem: A language is Turing recognisable iff some nondeterministic Turing machine recognises it.

Proof sketch: We need to show that every nondeterministic Turing machine N can be simulated by a deterministic Turing machine D.

We show how it can be simulated by a 3-tape machine.

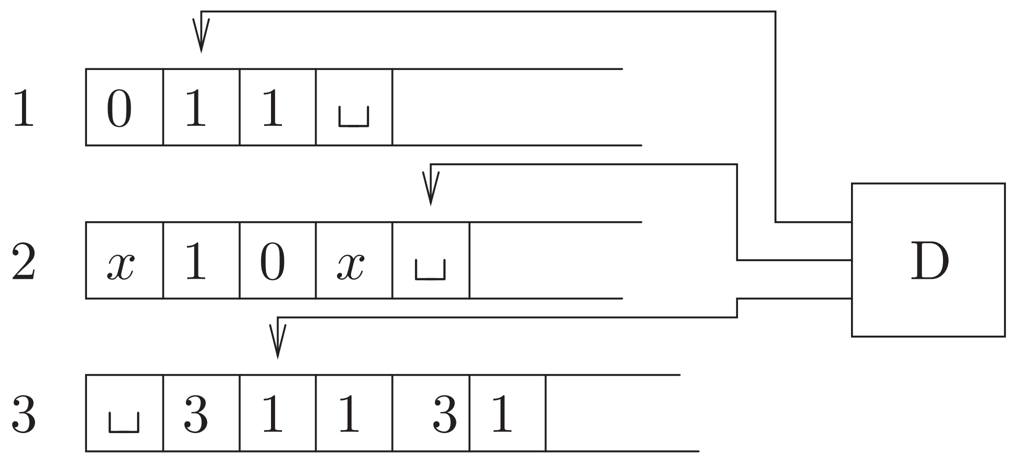
Let k be the largest number of choices, according to N's transition function, for any state/symbol combination.

Tape 1 contains the input.

Tape 3 holds progressively longer and longer sequences from $\{1, \ldots, k\}^*$.

Tape 2 is used to simulate N's behaviour for each fixed sequence of choices given by tape 3.

Simulating Nondeterminism (cont.)



1. Initially tape 1 contains input w.
   The other two tapes are empty.

2. Overwrite tape 2 by w.

3. Use tape 2 to simulate N. Tape 3 dictates how N should make its choices. If tape 3 gets exhausted, go to step 4. If N says *accept*, accept.

4. Generate the next "choice" string on tape 3.
   Go to step 2.

Enumerators

The Turing machine we built to generate all strings in $\{1, \ldots, k\}^*$ is an example of an *enumerator*.

We could imagine it being attached to a printer, and it would print all the strings in $\{1, \ldots, k\}^*$, one after the other, never terminating.

For an enumerator to enumerate a language L, for each $w \in L$, it must eventually print w. It is allowed to print w as often as it wants, and the strings can come in any order.

The reason why we also call Turing recognisable languages *recursively enumerable* is the following theorem.

Enumerators (cont.)

Theorem: L is Turing recognisable iff some enumerator enumerates L.

Proof: Let E enumerate L. Then we can build a Turing machine recognising L as follows:

1. Let w be the input.
2. Simulate E. For each string s output by E, if $s = w$, accept.

Conversely, let M recognise L. Then we can build an enumerator E by elaborating the enumerator from a few slides back: We can enumerate Σ^*, producing $s_1, s_2, \ldots$ Here is what E does:

1. Let $i = 1$.
2. Simulate M for i steps on each of $s_1, \ldots s_i$.
3. For each accepting computation, print that s.
4. Increment i and go to step 2.

Back to Algorithms

Hilbert's tenth problem (1900): Find an algorithm that determines whether a polynomial has an integral root.

As it turned out (Matijasevič 1970) no such algorithm exists.

This fact, however, can only be shown once we have a formal definition of what an algorithm *is*.

We need to argue that

$$\{p \mid p \text{ is a polynomial with integral root}\}$$

is not *decidable*.

The Church-Turing Thesis

$$\boxed{\begin{array}{c}\text{Computable}\\=\\\text{what a Turing machine can compute}\end{array}}$$

Note that we cannot hope to *prove* the Church-Turing thesis.

On the other hand, advances in physics could conceivably make the thesis false, in that some weird physical device might decide Turing machine halting, say.

The Church-Turing Thesis (cont.)

Note that

$$\{p \mid p \text{ is a polynomial with integral root}\}$$

is Turing *recognisable*.

To see this, here is how we can build a Turing machine M to recognise it.

Say the variables in p are x, y, and z.

M can enumerate all integer triples (i, j, k).

So M can evaluate p on each value triple (i, j, k) in turn.

If any of these evaluations give 0, M says *accept*.