# Applied Stochastic Processes (M1, 2016–17)
# Mid-term Exam (Oct 14, 2016)

You can use the following functions in your answers without further evaluation,

$$\text{Standard normal PDF: } \phi(x) = e^{-x^2/2}/\sqrt{2\pi}$$
$$\text{Standard normal CDF: } \Phi(x) = \textstyle\int_{-\infty}^{x} \phi(s)ds.$$

1. **[3 pts] Itô's calculus**
   Which of the following quantity is same as $(dB_t)^2$ ? **Answer** (b) $(dB_t)^2 = dt$
   (a) $\sqrt{dt}$ (b) $dt$ (c) $dt/2$ (d) $(dt)^2/2$

2. **[3 points] Martingale, Polya's urn**
   A box has 1 red ball and 9 blue balls. Pick up one ball randomly. If it is red, put it back and add one more red ball into the box. If it is blue, put it back and add one more blue ball into the box. If $Y_n$ is the proportion of the red balls in the box after the process is repeated $n$ times ($Y_0 = 0.1$), show that $\{Y_n\}$ is a martingale. So what is the expected number of the red balls after you repeat the process 100 times?

   **Answer**

   $$Y_{n+1} = \begin{cases} \frac{(n+10)Y_n+1}{n+11} & \text{if a red ball is picked with probability } Y_n \\ \frac{(n+10)Y_n}{n+11} & \text{if a blue ball is picked with probability } 1-Y_n \end{cases}$$

   Therefore,

   $$E(Y_{n+1}|Y_n) = \frac{(n+10)Y_n+1}{n+11}Y_n + \frac{(n+10)Y_n}{n+11}(1-Y_n) = Y_n,$$

   so $Y_n$ is a martigale. The expected number of the red balls at $n = 100$ is

   $$E\big((10+100)Y_{100}\big) = 110\,Y_0 = 110 \times 0.1 = 11.$$

3. **[3 points] Wald's equation**
   When $\{X_k\}$ are independent identically distributed random variable and $N$ is a random variable taking positive integer values, Wald's equation says

   $$E(X_1 + X_2 + \cdots X_N) = E(N)\,E(X_1)$$

   if either (i) $N$ is independent from $\{X_k\}$ or (ii) $N$ is a stopping time with respect to $\{X_k\}$.

   Consider an example where $X_k = 0$ or $1$ with 50% and 50% probability and $N$ is given as

   $$N = X_1 + X_2 + 1.$$

   Obviously, $E(X_k) = 1/2$ and $E(N) = 2/2 + 1 = 2$. Find $E(X_1 + X_2 + \cdots + X_N)$ and explain why Wald's equation does not hold in this example.

   **Answer**
   We can branch on the first scenarios (with probability 1/4) depending on the outcome of $X_1$ and $X_2$: 0–0, 0–1, 1–0 and 1–1.

   $$\begin{aligned} E(X_1 + X_2 + \cdots + X_N) &= \frac{1}{4}E(0) + \frac{1}{4}E(0+1) + \frac{1}{4}E(1+0) + \frac{1}{4}E(1+1+X_3) \\ &= \frac{1}{4}(0+1+1+2+\frac{1}{2}) = \frac{9}{8} \neq E(X_k)E(N) = 1. \end{aligned}$$

   Wald's equation does not hold because $N = X_1 + X_2 + 1$ is neither
   (i) independent from $\{X_k\}$: $N$ is defined via $X_1$ and $X_2$
   nor (ii) a stopping time w.r.t. $\{X_k\}$: it looks into the future, i.e., $X_2$ at $k = 1$.

4. **[3 points] Volatility on holidays**
In the class, we covered the relation between the daily volatility $\sigma_d$ and the annual volatility $\sigma_y$ as

$$\sigma_y = \sqrt{256}\,\sigma_d = 16\,\sigma_d,$$

where we assume the stock price is moving only on the 256 trading days in one year.

In this problem, we want to make it slightly more complicated. The stock price is usually more volatile on Mondays because new information, e.g., news related to the stock, economy, etc, is accumulated over the weekend and make the price move on Monday when the stock market is open. What would be the relation between the daily volatility $\sigma_d$ and the annual volatility $\sigma_y$ if we assume Monday's price is 50% more volatile than the other trading days, i.e., $1.5\,\sigma_d$ on Mondays? Assume there are 52 Mondays in one year (so the rest of the trading days are $256 - 52 = 204$).

**Answer**

$$\sigma_y^2 = \left(204 \times 1 + 52 \times 1.5^2\right)\sigma_d^2 = 321\sigma_d^2$$

Therefore,

$$\sigma_y = \sqrt{321}\sigma_d \approx 17.92\,\sigma_d$$

5. **[3 points] Asian option**
Asian option is an option where the payoff at maturity $T$ is derived from the average of the underlying prices at a given set of times before and at the maturity,

$$\left(\frac{1}{N}\sum_{k=1}^{N} S(t_k) - K\right)^+ \quad \text{for} \quad 0 < t_1 < \cdots < t_N = T.$$

When $N = 4$ and $t_k = k/4$ ($T = 1$) (a quarterly averaged Asian option), find the price of the Asian option. Assume the underlying stock price follows BM process, $dS(t) = \sigma\, dB(t)$, and the option price is given as $C = 0.4\,\sigma\sqrt{T}$ (at-the-money strike, zero interest rate and zero divided rate). How much the price of this Asian option is cheaper (or more expensive) to that of the European option with the same maturity and the same volatility?

**Answer**

$$\begin{aligned}
\mathrm{Var}\Big(\frac{\sigma}{4}(B_{1/4} + B_{2/4} + B_{3/4} + B_{4/4})\Big) &= \frac{\sigma^2}{16}\mathrm{Var}(B_{1/4} + B_{2/4} + B_{3/4} + B_{4/4}) \\
&= \frac{\sigma^2}{16}E\Big((B_{1/4} + B_{2/4} + B_{3/4} + B_{4/4})^2\Big) \\
&= \frac{\sigma^2}{16}E\Big(B_{1/4}^2 + B_{2/4}^2 + B_{3/4}^2 + B_{4/4}^2 + 2B_{1/4}(B_{2/4} + B_{3/4} + B_{4/4}) + 2B_{2/4}(B_{3/4} + B_{4/4}) + 2B_{3/4}B_{4/4}\Big) \\
&= \frac{\sigma^2}{16}\Big(\frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} + 2\frac{1}{4}\cdot 3 + 2\frac{2}{4}\cdot 2 + 2\frac{3}{4}\cdot 1\Big) = \frac{\sigma^2}{16}\frac{30}{4} = \frac{15\sigma^2}{32}.
\end{aligned}$$

The option price is given as

$$\text{Price of Asian option} = 0.4\sqrt{\frac{15}{32}}\,\sigma.$$

Asian option is about 31.5% ($= 1 - \sqrt{15/32}$) cheaper than European option with the same expiry, $0.4\sigma$.

6. **[4 points] Brownian motion with drift**

**Proposition** The maximum of BM with drift $\mu < 0$, $M_\mu = \max_{t\geq 0}(B_t + \mu t)$, has exponential distribution

$$P(M_\mu > x > 0) = e^{-2|\mu|x}.$$

(a) Using the above proposition, prove that,

$$\text{Prob}(B_t \leq at + b \text{ for all } t > 0) = 1 - e^{-2ab} \quad \text{for } a, b > 0$$

(b) Using a proper change of variable, scaling of BM and etc, extend the result of (a) for BM with volatility, $\sigma B_t$.

**Answer**

(a)

$$\text{Prob}(B_t \leq at + b) = \text{Prob}(B_t - at \leq b) = 1 - \text{Prob}(B_t - at > b) = 1 - e^{-2ab}$$

(b) Method 1:

$$\text{Prob}(\sigma B_t \leq at + b) = \text{Prob}(B_t \leq \frac{a}{\sigma}t + \frac{b}{\sigma}) = 1 - e^{-\frac{2ab}{\sigma^2}}$$

Method 2: $(\sigma B_t \sim B_{\sigma^2 t})$

$$\text{Prob}(\sigma B_t \leq at + b) = \text{Prob}(B_{\sigma^2 t} \leq at + b) = \text{Prob}(B_t \leq \frac{a}{\sigma^2}t + b) = 1 - e^{-\frac{2ab}{\sigma^2}}$$

7. **[4 points] SDE (modified from Exercise 9.2)**

Solve the SDE for the starting value $X_0$

$$dX_t = -t\, X_t\, dt + \sigma e^{-t^2/2}\, dB_t.$$

What is the mean and the variance of the process when $t \to \infty$ ?

**Answer**

From $e^{t^2/2}(dx + tdx) = d(e^{t^2/2}x)$, we apply Itô's lemma to our initial guess of $e^{t/2}X_t$;

$$d\Big(e^{t/2}X_t\Big) = te^{t^2/2}\, X_t\, dt + e^{t^2/2}dX_t = e^{t^2/2}\Big(tX_t dt + dX_t\Big) = \sigma dB_t$$

$$e^{t^2/2}X_t - X_0 = \sigma B_t$$

$$X_t = e^{-t^2/2}(X_0 + \sigma B_t),$$

$$E(X_t) = e^{-t^2/2}X_0 \to 0 \text{ as } t \to \infty$$

$$\text{Var}(X_t) = e^{-t^2}\sigma^2\, t \to 0 \text{ as } t \to \infty$$

8. **[3 points] SDE (Between normal and lognormal)**

Solve the SDE for the starting value $X_0$ and $0 < \beta < 1$,

$$\frac{dX_t}{X_t^\beta} = \sigma\, dB_t + \frac{\sigma^2\beta}{2}X_t^{\beta-1}\, dt$$

**Answer**

We apply Itô's lemma to $X_t^{1-\beta}$,

$$\begin{aligned}
d(X_t^{1-\beta}) &= (1-\beta)X_t^{-\beta}dX_t - \frac{1}{2}(1-\beta)X_t^{-1-\beta}\beta(dX_t)^2 \\
&= (1-\beta)X_t^{-\beta}dX_t - \frac{1}{2}(1-\beta)\beta X_t^{-1-\beta}\cdot X_t^{2\beta}\sigma^2 dt \\
&= (1-\beta)\Big(\sigma\, dB_t + \frac{\sigma^2\beta}{2}X_t^{\beta-1}\, dt\Big) - \frac{1}{2}(1-\beta)\beta X_t^{-1-\beta}\big(X_t^\beta\sigma\big)^2 dt \\
&= (1-\beta)\sigma dB_t.
\end{aligned}$$

Therefore,

$$X_t^{1-\beta} = X_0^{1-\beta} + (1-\beta)\sigma B_t$$
$$X_t = \left(X_0^{1-\beta} + (1-\beta)\sigma B_t\right)^{1/(1-\beta)}.$$

9. [4 points] **Joint distribution of $B_t^m$ and $B_t$**
We are going to derive the price of the binary call option with knock-out (down-and-out) feature under BM (normal) model. Assume that the underlying stock follows the process $S_t = S_0 + \sigma B_t$. The option will pay you \$1 at the expiry $T$ if $S_T > K$ for a strike price $K$ **and** the stock price $S_t$ has never been below $K_O$ for $K_O < \min(S_0, K)$ anytime before the expiry, $0 \le t \le T$. In other words, this option knocks out (expires worthless) if $S_t$ falls below $K_O$ any time before the expiry $T$.
(a) In the class (and in the textbook), we derived the joint CDF for $B_T$ and $B_T^M = \max_{0\le t\le T} B_t$,

$$P(B_T^M < v,\ B_T < u) = \Phi(u/\sqrt{t}) - \Phi((u-2v)/\sqrt{t}), \quad v \ge \max(0, u)$$

Using this result, derive the joint CDF for BM with volatility, $\sigma B_t$,

$$P(\sigma B_T^M < v,\ \sigma B_T < u).$$

(b) Using the symmetry that $-\sigma B_t$ has the same distribution as $\sigma B_t$, drive the probability

$$P(\sigma B_T^m > v, \sigma B_T > u), \quad v \le \min(0, u)$$

where $B_T^m$ is the running **minimum**, $B_T^m = \min_{0\le t\le T} B_t$.
(c) Finally find the price of the binary call option with knock-out feature? Assume that interest rate and divided rate is zero. How much is this derivative cheaper (or more expensive) than the regular binary call option **without** knock-out feature?

**Answer**
(a)

$$P(\sigma B_T^M < v,\ \sigma B_T < u) = P(B_T^M < v/\sigma,\ B_T < u/\sigma) = \Phi\Big(\frac{u}{\sigma\sqrt{t}}\Big) - \Phi\Big(\frac{u-2v}{\sigma\sqrt{t}}\Big)$$

(b) Under the reflection, $B_t \to -B_t$, we get

$$(-B)_T^m = \min_{0\le t\le T}(-B_t) = -\max_{0\le t\le T} B_t = -B_T^M.$$

Applying the reflection to the probability,

$$\begin{aligned} P(\sigma B_T^m > v,\ \sigma B_T > u) &= P(\sigma(-B)_T^m > v,\ \sigma(-B_T) > u) \\ &= P(-B_T^M > v/\sigma,\ -B_T > u/\sigma) \\ &= P(B_T^M < -v/\sigma,\ B_T < -u/\sigma) \\ &= \Phi\Big(\frac{-u}{\sigma\sqrt{t}}\Big) - \Phi\Big(\frac{2v-u}{\sigma\sqrt{t}}\Big). \end{aligned}$$

(c) Plug in $K - S_0 \to u$ and $K_O - S_0 \to v$ to get

$$\text{Price with Knockout} = \Phi\Big(\frac{S_0 - K}{\sigma\sqrt{t}}\Big) - \Phi\Big(\frac{2K_O - K - S_0}{\sigma\sqrt{t}}\Big)$$

The price without knockout is the first term, so the knockout feature is cheapening the price by $\Phi\Big(\frac{2K_O-K-S_0}{\sigma\sqrt{t}}\Big)$.