Applied Stochastic Processes (2017–18 Module 1 Fall) Mid-term Exam Solutions

BM stands for Brownian motion. **RN** and **RV** stand for random number and random variable respectively. You can use the following functions in your answers without further evaluation,

$$\text{Standard normal PDF: } n(x) = e^{-x^2/2}/\sqrt{2\pi}$$
$$\text{Standard normal CDF: } N(x) = \int_{-\infty}^{x} n(s)ds.$$

1. **[7 pts] Poisson process**
   In Poisson process, the CDF for the arrival time t is given as $F(t) = 1 - e^{-\lambda t}$ for the arrival rate λ.

   (a) From a uniform RV U, generate RN for the **conditional** arrival time t conditional on that the next arrival is after some time t_0, (i.e., $t > t_0$)
   Answer The RV for unconditional arrival time t can be simulated as $t = -(1/\lambda)\log U$, where U is a uniform RV. From the memoryless property, t conditional on $t \ge t_0$ can be simulated as
   $$t = t_0 - (1/\lambda)\log U.$$

   (b) Assume that the default of a company follows the Poisson process with the arrival rate λ. In the credit default swap (CDS) on the company, party A pays (to B) premium continuously at the rate p (i.e., pays $p\,dt$ during a time period dt) until the maturity T or the company's default whichever comes first, and party B pays (to A) \$1 when the company defaults. What is the fair premium rate p (which makes the NPVs of both parties equal)? Assume that the risk-free rate is zero, i.e., $r = 0$ (although the problem becomes more interesting if $r > 0$).
   Answer
   $$\begin{aligned}
   \text{NPV of party A} &= \text{NPV of party B}\\
   \int_0^T 1\cdot \lambda e^{-\lambda t}dt &= \int_0^T pt\cdot\lambda e^{-\lambda t}dt + pT\cdot e^{-\lambda T}\\
   1 - e^{-\lambda T} &= p\left[-t\,e^{-\lambda t} - \frac{1}{\lambda}e^{-\lambda t}\right]_{t=0}^{T} + pT\cdot e^{-\lambda T}\\
   1 - e^{-\lambda T} &= \frac{p}{\lambda}(1 - e^{-\lambda T})
   \end{aligned}$$
   Therefore the fair premium value is $p = \lambda$.

2. **[6 points] Asian option**
   Asian option is an option where the payoff at maturity T is derived from the average of the underlying prices at a given set of times before and at the maturity,
   $$\left(\frac{1}{N}\sum_{k=1}^{N} S(t_k) - K\right)^+ \quad \text{for} \quad 0 < t_1 < \cdots < t_N = T.$$
   When $N = 3$ and $t_k = (k+3)/3$ ($T = 2$), find the price of the at-the-money Asian option. Assume the underlying stock price follows an arithmetic BM process (normal model), $dS_t = \sigma\, dW_t$, and the option price is given as $C = 0.4\,\sigma\sqrt{T}$ (at-the-money strike, zero interest rate

and zero divided rate). How much the price of this Asian option is cheaper (or more expensive) than that of the European option with the same maturity ($T = 2$) and the same volatility?

Answer

$$\begin{aligned}
&\mathrm{Var}\Big(\frac{1}{3}(B_{4/3} + B_{5/3} + B_2)\Big) = \frac{1}{9}E\Big((B_{4/3} + B_{5/3} + B_2)^2\Big) \\
&= \frac{1}{9}E\Big(B_{4/3}^2 + B_{5/3}^2 + B_2^2 + 2B_{4/3}(B_{5/3} + B_2) + 2B_{5/3}B_2\Big) \\
&= \frac{1}{9}\Big(\frac{4}{3} + \frac{5}{3} + 2 + 2\cdot\frac{4}{3}\cdot 2 + 2\cdot\frac{5}{3}\Big) = \frac{1}{9}\cdot\frac{41}{3} = \frac{41}{27}.
\end{aligned}$$

The option price is given as

$$\text{Price of Asian option} = 0.4\sqrt{\frac{41}{27}}\,\sigma.$$

Asian option is about 12.8% ($= 1 - \sqrt{41/54}$) cheaper than European option with the same expiry, $0.4\sqrt{2}\,\sigma$.

3. **[7 pts] Bessel process**
The distribution of the following RV,

$$Q = \|(Z_1 + \mu_1, \cdots, Z_n + \mu_n)\|^2 = (Z_1 + \mu_1)^2 + \cdots + (Z_n + \mu_n)^2,$$
$$\text{with} \quad \mu = \mu_1^2 + \cdots \mu_n^2$$

where $Z_1, \cdots, Z_n$ are independent standard normal RVs, is defined as *non-central chi square* (χ^2, pronounced as *kai*) distribution with degree n and non-centrality parameter $\mu \geq 0$, denoted by $Q \sim \chi^2(n, \mu)$. Thanks to radial symmetry, the distribution is completely determined by $\mu = \mu_1^2 + \cdots + \mu_n^2$. The χ^2 distribution is an important subject of statistics, so the PDF and CDF is well-known although the computation is still challenging from some cases. The degree n can be generalized to any positive real number (i.e., not only integers).

On the other hand, the **squared** Bessel process with dimension n is defined as

$$X_t = \|(B_{1t}, \cdots, B_{nt})\|^2 = B_{1t}^2 + \cdots + B_{nt}^2$$

where $B_{1t}, \cdots, B_{nt}$ are n independent standard BMs. Therefore the distribution of X_t given X_s ($s < t$) follows a scaled non-central χ^2 distribution,

$$X_t = (t - s)\,Q \quad \text{where} \quad Q \sim \chi^2\left(n,\ \frac{X_s}{t - s}\right)$$

(No need to prove this for the remaining questions. Just use it.)

(a) Show that the **squared** Bessel process satisfies

$$dX_t = 2\sqrt{X_t}\,dW_t + n\,dt$$

Answer:

$$dX_t = \sum_{k=1}^{n}\left(2B_{kt}\,dB_{kt} + \frac{1}{2}\cdot 2dt\right) = 2\sqrt{X_t}\,dW_t + n\,dt,$$

where we use $\sum_k B_{kt}\,dB_{kt} = \sqrt{\sum_k B_{kt}^2}\,dW_t = \sqrt{X_t}\,dW_t$ for an independent BM W_t.

(b) Show that the Bessel process defined as $R_t = \sqrt{X_t}$ satisfies

$$dR_t = dW_t + \frac{n-1}{2}\frac{dt}{R_t}.$$

Answer: Applying Itô's lemma,

$$dR_t = \frac{d\,X_t}{2\sqrt{X_t}} - \frac{(dX_t)^2}{8X_t\sqrt{X_t}} = \frac{2R_t\,dW_t + n\,dt}{2R_t} - \frac{(2R_t\,dW_t)^2}{8R_t^3} = dW_t + \frac{n-1}{2}\frac{dt}{R_t}.$$

(c) The SDE for the CEV process for $0 < \beta \le 1$ is given as

$$dS_t = \sigma\, S_t^\beta\, dW_t.$$

Show that the CEV process can be reduced to the Bessel process defined in (b). Express the distribution of S_t in terms of S_0 and $Q \sim \chi^2(n, \mu)$. Clearly state the corresponding values for n and μ? (If σ makes the problem difficult for you, you may assume $\sigma = 1$ to solve the problem. But you will get a partial credit.)
Answer: We apply Itô's lemma to $Y_t = S_t^{1-\beta}/(1-\beta)$:

$$dY_t = S_t^{-\beta} dS_t + \frac{1}{2}(-\beta S_t^{-1-\beta})(dS_t)^2 = \sigma dW_t + \frac{\beta\,\sigma^2}{2(1-\beta)}\frac{dt}{Y_t}.$$

The σ can be absorbed to t by introducing the variance $\tau = \sigma^2 t$,

$$dY_{\tau/\sigma^2} = dW_\tau + \frac{\beta}{2(1-\beta)}\frac{d\tau}{Y_{\tau/\sigma^2}}$$

Therefore $Y_{\tau/\sigma^2}/\tau$ follows χ^2 distribution with $n = 1/(1-\beta)$ from $(n-1)/2 = \beta/2(1-\beta)$ and $\mu = Y_0/\tau$.

$$\frac{Y_{\tau/\sigma^2}}{\tau} = Q \quad \text{where} \quad Q \sim \chi^2\left(\frac{1}{1-\beta},\ \frac{S_0^{1-\beta}}{(1-\beta)\,\sigma^2 t}\right).$$

Finally, replacing $\tau = \sigma^2 t$ and $Y_t = S_t^{1-\beta}/(1-\beta)$,

$$\frac{S_t^{1-\beta}}{(1-\beta)\,\sigma^2 t} = Q \quad \text{or} \quad S_t = \left((1-\beta)\sigma^2 t\, Q\right)^{1/(1-\beta)}$$

4. **[5 pts] CIR process**
The CIR process given as

$$dX_t = a(X_\infty - X_t)dt + \sigma\sqrt{X_t}\,dB_t$$

was originally proposed to model the dynamics of interest rate by Cox, Ingersoll and Ross. The process was also used to model the variance V_t in the Heston stochastic volatility model:

$$dV_t = \kappa(V_\infty - V_t)dt + \alpha\sqrt{V_t}dZ_t.$$

Applying the similar change of variable used in Ornstein-Uhlenbeck process, show that the CIR process (either in X_t or V_t) can be represented in terms of the **squared** Bessel process in 3.(a). Clearly state the corresponding dimension n of the squared Bessel process.

Answer: We apply the change of variable, $Y_t = e^{at} X_t$, from O-U process. Then, Y_t satisfy

$$dY_\tau = aX_\infty e^{at}\, dt + \sqrt{X_t}\, \sigma e^{at}\, dB_t = aX_\infty e^{at}\, dt + 2\sqrt{Y_t}\, \frac{\sigma e^{at/2}}{2}\, dB_t.$$

Now we also introduce a new time variable from the variance of the BM,

$$\tau = \int_0^t \left(\frac{\sigma e^{at/2}}{2} \right)^2 ds = \frac{\sigma^2}{4a}(e^{at} - 1), \quad d\tau = \frac{\sigma^2 e^{at}}{4} dt$$

Define $\bar{Y}_\tau = Y_t$, then the process $\bar{Y}_\tau$ follows

$$d\bar{Y}_\tau = \frac{4aX_\infty}{\sigma^2}\, d\tau + 2\sqrt{Y_t}\, dB_\tau,$$

which is the squared Bessel process with dimension $n = 4aX_\infty/\sigma^2$. Finally the original process X_t can be expressed in terms of the **squared** Bessel process $\bar{Y}_\tau$ with dimension $n = 4aX_\infty/\sigma^2$:

$$X_t = e^{-at}\, \bar{Y}_{\sigma^2(e^{at}-1)/(4a)}.$$