Option Pricing Under 'Normal' Model
Stochastic Finance (FIN 519)

Jaehyuk Choi

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Stochastic Process

Under normal model, stock price follows Brownian motion (BM) with volatility σ,

$$S_t = e^{(r-q)t} S_0 + \sigma B_t \quad \left(dS_t = (r-q)e^{(r-q)t} S_0 dt + \sigma dB_t \right),$$

where r is interest rate and q is dividend rate. (The BM has normal distribution: $B_t \sim N(0,t)$.)
Under Black-Scholes-Merton (BSM) model, stock *return* follows BM

$$S_t = S_0 \exp\left((r-q-\frac{\sigma_L^2}{2})t + \sigma B_t\right) \quad \left(\frac{dS_t}{S_t} = r dt + \sigma_L dB_t \right).$$

Different names

- Normal process (vs Log-normal process)
- Arithmetic BM (vs Geometric BM)
- Bachelier model (vs Black-Scholes-Merton model)

Why study normal model?

- Some underlying assets (e.g., interest rate, spread)
  - the price can be negative,
  - The daily change is independent of the level of the price level
- More intuitive than Black-Scholes-Merton

Option Price (Call)

Underlying asset price at maturity T:

$$S_T = F + \sigma\sqrt{T}z, \quad \text{where} \quad F = e^{(r-q)T}\,S_0, \quad z \sim N(0,1)$$

Payoff:

$$\max(S_T - K, 0) = (S_T - K)^+ = (F - K + \sigma\sqrt{T}z)^+$$
$$S_T = K \quad \Rightarrow \quad z = -d = \frac{K-F}{\sigma\sqrt{T}} \quad \left(d = \frac{F-K}{\sigma\sqrt{T}}\right)$$

Option value (forward):

$$\begin{aligned} C(K) &= \int_{-d}^{\infty} (F - K + \sigma\sqrt{T}z)\, n(z)dz \\ &= (F-K)(1-N(-d)) + \sigma\sqrt{T}\, n(-d) \\ &= (F-K)N(d) + \sigma\sqrt{T}\, n(d) \end{aligned}$$

Here we used

$$\int z\, n(z)dz = \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = -n(z) + C.$$

Option value (present):

$$C_0(K) = e^{-rT} C(K)$$

Option Price (Put)

Payoff:

$$(K - S_T)^+ = (K - F - \sigma\sqrt{T}z)^+$$

$$\text{The root of } S_T = K \quad \Rightarrow \quad z = -d = \frac{K-F}{\sigma\sqrt{T}} \quad \left(d = \frac{F-K}{\sigma\sqrt{T}}\right)$$

Option value (forward)

$$\begin{aligned} P(K) &= \int_{-\infty}^{-d} (K - F - \sigma\sqrt{T}z)\, n(z) dz \\ &= (K-F)N(-d) - \sigma\sqrt{T}\, n(-d) \\ &= (K-F)N(-d) + \sigma\sqrt{T}\, n(d) \end{aligned}$$

Option value (present):

$$P_0(K) = e^{-rT} P(K)$$

Put-Call parity holds!

$$C(K) - P(K) = (F-K)N(d) - (K-F)N(-d) = (F-K)\big(N(d) + N(-d)\big) = F - K$$

If $K = F$ (at-the-money), $d = 0$ and the option prices are

$$C(K = F) = P(K = F) = \sigma\sqrt{T}n(0) = \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} \approx 0.4\,\sigma\sqrt{T}$$

$$\text{Straddle} = C + P \approx 0.8\,\sigma\sqrt{T}$$

$$C_0(K = F) = P_0(K = F) = \frac{e^{-rT}\sigma\sqrt{T}}{\sqrt{2\pi}} \approx e^{-rT}\,0.4\,\sigma\sqrt{T}$$

Therefore the option price is proportional to the *width* (or stdev) of the distribution of the future price, $\sigma\sqrt{T}$, which is consistent with the intuition. Before we derive Black-Scholes formula, let's keep this relation between the volatility and the option price in mind. Even without the Black-Scholes formula (which is somewhat complicated), this relation should give you a very good intuition.

Greeks (risks of option)

Delta: sensitivity on the underlying price

$$\frac{\partial C}{\partial F} = N(d), \quad \frac{\partial P}{\partial F} = -N(-d) \quad \left(d = \frac{F-K}{\sigma\sqrt{T}}\right)$$

$$\left(\frac{\partial C}{\partial F} - \frac{\partial P}{\partial F} = 1\right)$$

$N(d)$ measures how closely the call option price moves with the underlying stock, i.e., how much the option is in-the-money.

Gamma: convexity on the underlying price

$$\frac{\partial^2 C}{\partial F^2} = \frac{\partial^2 P}{\partial F^2} = \frac{n(d)}{\sigma\sqrt{T}}$$

Vega: sensitivity on the volatility

$$\frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = \sqrt{T}\, n(d)$$

1. Derive the (forward) price of the digital(binary) call/put option struck at K at maturity T. The digital(binary) call/put option pays \$1 if S_T is above/below the strike K, i.e. $1_{S_T \geq K} / 1_{S_T \leq K}$.
2. The payoff of the call option, $\max(S_T - K, 0)$ can be decomposed into two parts,
$$S_T \cdot 1_{S_T \geq K} \quad - \quad K \cdot 1_{S_T \geq K}.$$
The first payout is the payout of the **asset-or-nothing** call option and the second payout if the binary call option multiplied with $-K$. What is the price of the asset-or-nothing call option?
3. Using the joint distribution of B_t and B_t^*, derive the price of the call option struck at K and knock-out at K_1 $(> K)$. First, generalize the joint CDF function $P(u < B_t, v < B_t^*)$ to σB_t. Next, derive the pdf on u by taking derivative on u. Then, integrate the payoff $(S_T - K)^+$ from K to K_1. (Assume that the risk-free rate is zero, $r = 0$, so that $S_0 = F$. Otherwise the problem is too complicated.)