

Stochastic-alpha-beta-rho (SABR) model

Applied Stochastic Processes (FIN 514)

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The project overview

SABR Model

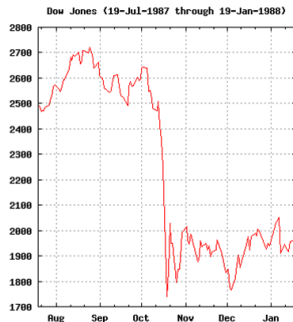
- One of the most popular **stochastic volatility (SV)** model
- Heavily used in trading options for interest rate and FX
- Explains volatility skew/smile with minimal and intuitive parameters

Project Goal

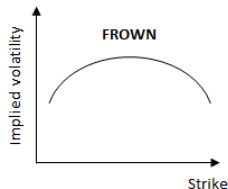
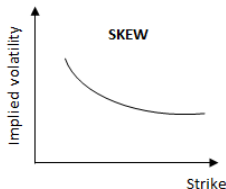
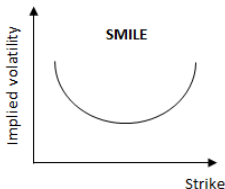
- Implement the approximation formula by Hagan (provided)
- Implement option pricing with Euler method and MC
- Implement the probabilistic method by Kennedy et al (2012)
- Implement a smile calibration routine based on the method of Kennedy et al (2012)

Background: volatility skew/smile

- Black Monday crash in 1987:
DJIA -22.6% in one day!
- Overall 'short gamma' due to the portfolio insurance (put on equity index)
- Market values (down-side) tail event higher than before.
- Market sees volatility skew/smile



(From Wikipedia)



Why need model for smile? challenges in risk management

- Option trading desk (market-making/sell-side) usually accumulates option positions with different strikes.
- Under BSM model,
 - Vol σ fixed under spot change $S_0 \rightarrow S_0 + \Delta$.
 - Risk-management is easy: delta and vega clearly defined
 - One can hedge delta (with underlying stock) and vega (with ATM option)
 - However, **the OTM option prices/risks are not correct!**
- BSM model with different σ to each option K ?
 - How do we fix the volatilities?
 - Sticky strike rule $\sigma = \sigma(K)$ vs sticky delta rule $\sigma = \sigma(S_0 - K)$.
 - Need to characterize the smile with a few minimal parameters.
- For better risk management, we need models which can capture the volatility smile.

How to model smile? Local volatility (LV)

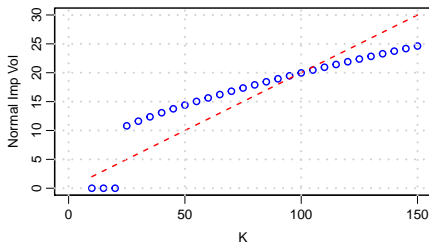
- Volatility depending on the 'current location' of S_t :

$$\text{BSM: } \frac{dS_t}{S_t} = \sigma f(S_t) dW_t \quad \text{Normal: } dS_t = \sigma_n f_n(S_t) dW_t$$

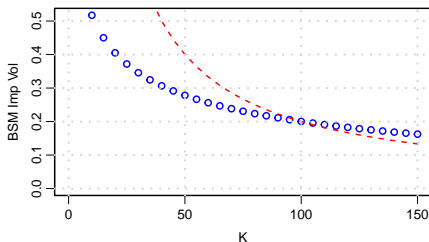
- BSM model:** a trivial case with $f(x) = 1$. However, it is a local vol model under normal volatility ($f_n(x) = x$).
- Normal model:** a trivial case with $f_n(x) = 1$. However, it is a local vol model under BSM volatility ($f(x) = 1/x$).
- What is the implied normal volatility of the Black-Scholes price on varying K ? What is the relation between the implied volatility and the local vol?
- The implied volatility is the volatility average of the in-the-money paths
- Exercise 1:** Chart the normal implied vol of the prices under BSM model for typical parameter sets. Measure the slope, $\partial\sigma(K)/\partial K$, at the money.

Case: $S_0 = 100, \sigma = 20\% (\sigma_n = 20), r = q = 0$:

- Implied normal vol for constant BSM vol ($\sigma = 20\%$):



- Implied normal vol for constant normal vol ($\sigma_n = 20$):



Displaced GBM (shifted BSM) model

- A quick local vol model
- 'Displaced asset price' $S_t + L$ follows GBM:

$$dS_t = \sigma_L(S_t + L) dW_t$$

- Somewhere between normal ($L \rightarrow \infty$) and log-normal model ($L = 0$).
- Can reuse BS formula with $S_0 + L \rightarrow S_0$ and $K + L \rightarrow K$.
- Calibration of σ_L (ATM option price on target):

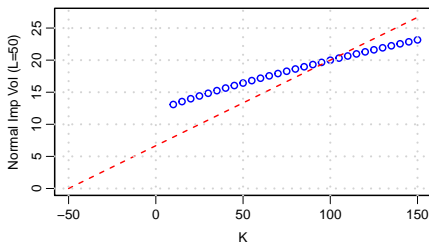
$$\sigma_n \approx \sigma_L(S_0 + L) \approx \sigma S_0$$

But, needs an exact calibration of σ_L for a given σ_{BS} .

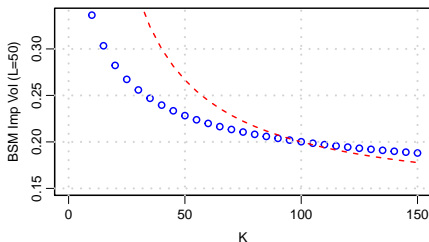
- **Exercise 2:** Chart the BSM implied vol of the prices under displaced GBM model. Using the implemented implied vol function, exactly calibrate σ_L to the ATM price.

Case: $S_0 = 100, L = 50, \sigma = 20\%, r = q = 0$:

- $\sigma_L = \sigma S_0 / (S_0 + L) = 13.33\%$
- Implied normal vol: (red line: $\sigma_L(K + L)$)



- Implied BSM vol: (red line: $\sigma_L(K + L)/K$)



How to model smile? Stochastic volatility (SV)

- Volatility changing over time:

$$\text{BSM: } \frac{dS_t}{S_t} = \sigma_t dW_t \quad \text{Normal: } dS_t = \sigma_t dW_t$$

- Many models proposed (mostly for BSM). For $dW_t dZ_t = \rho dt$,
 - Hull-White (SABR):

$$\frac{d\sigma_t}{\sigma_t} = \alpha dZ_t$$

- Heston: $V_t = \sigma_t^2$ follows Cox-Ingersoll-Ross (CIR) process,

$$dV_t = \kappa(V_\infty - V_t)dt + \alpha\sqrt{V_t}dZ_t$$

- SV model correctly captures the smile, α for curvature and ρ for skewness.

Stochastic- α, β, ρ model SDE:

$$dS_t = \sigma_t S_t^\beta dW_t$$

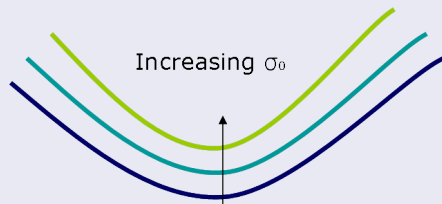
$$d\sigma_t = \alpha \sigma_t dZ_t$$

$$dW_t dZ_t = \rho dt$$

- Parameters: $\sigma_0, \alpha, \beta, \rho$.
- σ_0 : overall volatility, calibrated to ATM implied vol
- β : elasticity or 'backbone'. (Normal: $\beta = 0$, BSM: $\beta = 1$)
- α : volatility of volatility, σ following a GBM
- ρ : correlation between asset price and volatility

The impact of parameters

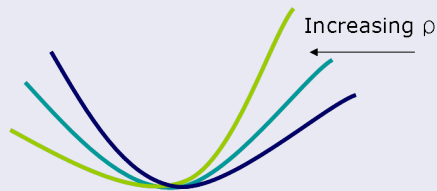
Starting vol σ_0



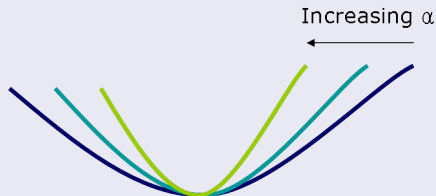
Backbone β

- Fixed or infrequently changed
- BSM model: $\beta = 1$ (Equity, FX)
- Normal model: $\beta = 0$ (Interest Rate)

Correlation ρ



'Vol of vol' α



Implied vol formula (Hagan et al, 2002)

The first few terms of the 'Taylor expansion' near $\alpha\sqrt{T} \approx 0$.

$$\begin{aligned} \sigma_B(K, f) &= \frac{\alpha}{(fK)^{(1-\beta)/2} \left\{ 1 + \frac{(1-\beta)^2}{24} \log^2 f/K + \frac{(1-\beta)^4}{1920} \log^4 f/K + \dots \right\}} \cdot \left(\frac{z}{\kappa(z)} \right) \\ &\cdot \left\{ 1 + \left[\frac{(1-\beta)^2}{24} \frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4} \frac{\rho\beta v\alpha}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24} v^2 \right] t_{ex} + \dots \right\} \end{aligned} \quad (2.17a)$$

Here

$$z = \frac{v}{\alpha} (fK)^{(1-\beta)/2} \log f/K, \quad (2.17b)$$

and $\kappa(z)$ is defined by

$$\kappa(z) = \log \left\{ \frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1 - \rho} \right\}. \quad (2.17c)$$

Success of SABR model

- Volatility smile information encoded into three parameters σ_0, α, ρ !!
- Vega (volatility) risk managed by the three parameters rather than each individual vol.
- Three implied vols (or option prices) on the smile can calibrate the parameters. → An effective interpolation method for implied volatility (or option price)

Limitation of Hagan's formula

- Arbitrage is equivalent to some event happening with negative probability. The price of a derivative paying \$1 on that event is negative (should be free at most)!
- Digital (Call/Put) option from call spread

$$P(S_T > K) = D(K) = \frac{C(K) - C(K + \Delta K)}{\Delta K} = -\frac{\partial C(K)}{\partial K}$$

- When $\alpha\sqrt{T} \gg 1$, Hagan's formula sometimes implies $P(S_T > K) < 0$ from

$$C(K, \sigma(K)) < C(K + \Delta K, \sigma(K + \Delta K)).$$

The volatility effect $\sigma(K + \Delta K)$ overcomes (should NOT!) the moneyness effect $K + \Delta K$.

MC Simulation (Euler method)

- Unlike normal or BSM model (as in spread/basket option project), we can not jump directly from $t = 0$ to T .
- Divide the interval $[0, T]$ into N small steps, $t_k = (k/N)T$ and $\Delta t_k = T/N$ and simulate each time step,

$$S_t : \begin{cases} \beta = 0 : S_{t_{k+1}} = S_{t_k} + \sigma_{t_k} Z_1 \sqrt{\Delta t_k} \\ \beta = 1 : \log S_{t_{k+1}} = \log S_{t_k} + \sigma_{t_k} \sqrt{\Delta t_k} Z_1 - \frac{1}{2} \sigma_{t_k}^2 \Delta t_k, \end{cases}$$
$$\sigma_t : \sigma_{t_{k+1}} = \sigma_{t_k} \exp \left(\alpha \sqrt{\Delta t_k} Z_2 - \frac{1}{2} \alpha^2 \Delta t_k \right),$$

where $Z_1, Z_2 \sim N(0, 1)$ with correlation ρ .

- Typically, $\Delta t_k \approx 0.25$. For $T = 30$, $N = 120$, quite time-consuming.
- Any good control variate?

$$C(K) = \frac{1}{N} \sum_{i=1}^N (S_T^{(i)} - K)^+$$

Stochastic integral of σ_t

From Itô's lemma,

$$d\sigma_t = \alpha\sigma_t dZ_t \quad \rightarrow \quad d\log \sigma_t = -\frac{1}{2}\alpha^2 dt + \alpha dZ_t$$

and we know the final distribution,

$$\sigma_T = \sigma_0 \exp\left(-\frac{1}{2}\alpha^2 T + \alpha Z_T\right).$$

We also know

$$\alpha \int_0^T \sigma_t dZ_t = \sigma_T - \sigma_0 = \sigma_0 \exp\left(-\frac{1}{2}\alpha^2 T + \alpha Z_T\right) - \sigma_0,$$

which will be useful for the integration of S_t .

Stochastic integral of S_t (normal: $\beta = 0$)

Writing the SDE in a de-correlated form,

$$dS_t = \sigma_t \left(\rho dZ_t + \sqrt{1 - \rho^2} dW_t \right) \quad \text{for} \quad dW_t dZ_t = 0.$$

Integrating S_t , we get so far as

$$\begin{aligned} S_T - S_0 &= \rho \int_0^T \sigma_t dZ_t + \sqrt{1 - \rho^2} \int_0^T \sigma_t dW_t \\ &= \frac{\rho}{\alpha} (\sigma_T - \sigma_0) + \sqrt{1 - \rho^2} \boxed{\int_0^T \sigma_t dW_t} \end{aligned}$$

From Itô's Isometry, the **integrated variance** of the box is $V_T := \int_0^T \sigma_t^2 dt$.
With some more work, the box can be expressed as

$$\int_0^T \sigma_t dW_t = W \sqrt{V_T}, \text{ where } W \sim N(0, 1) \text{ independent from } V_T \text{ and } \sigma_T$$

Conditional MC method (normal $\beta = 0$)

Given (σ_T, V_T) , the distribution of S_T is

$$S_T = S_0 + \frac{\rho}{\alpha}(\sigma_T - \sigma_0) + \sqrt{(1 - \rho^2)V_T} W$$

and the option price is from the normal model:

$$C_N(\sigma_T, V_T) = C_N \left(S_0 := S_0 + \frac{\rho}{\alpha}(\sigma_T - \sigma_0), \sigma_N := \sqrt{(1 - \rho^2)V_T/T} \right)$$

Then the option is obtained as an expectation over MC simulation of V_T :

$$C_{\beta=0} = E(C_N(\sigma_T, V_T)), \quad \text{where} \quad V_T = \sum_k \sigma_{t_k}^2 \Delta t_k$$

In this way, no need for simulating S_t . Given (σ_T, V_T) the price is exact, therefore MC variance is much smaller than that of brute-force MC.

Conditional MC method (BSM $\beta = 1$)

Given (σ_T, V_T) , the distribution of S_T is

$$\log(S_T/S_0) = \frac{\rho}{\alpha}(\sigma_T - \sigma_0) - \frac{1}{2}V_T + \sqrt{1 - \rho^2} W \sqrt{V_T}$$

and the option price is from the BSM model:

$$C_{BSM}(\sigma_T, V_T) = C_{BSM} \left(S_0 e^{\frac{\rho}{\alpha}(\sigma_T - \sigma_0) - \frac{\rho^2}{2}V_T}, \sqrt{(1 - \rho^2)V_T/T} \right)$$

Then the option is obtained as an expectation over MC simulation of V_T :

$$C_{\beta=1} = E(C_{BSM}(\sigma_T, V_T)), \quad \text{where} \quad V_T = \sum_k \sigma_{t_k}^2 \Delta t_k$$

In this way, no need for simulating S_t . Given (σ_T, V_T) the price is exact, therefore MC variance is much smaller than that of brute-force MC.

The conditional distribution of V_T on σ_T (Kennedy et al)

The conditional mean of V_T on σ_T is known as

$$E(V_t|\sigma_T) = \frac{\sigma_0^2\sqrt{T}}{2\alpha} \frac{N(d_\alpha + \alpha\sqrt{T}) - N(d_\alpha - \alpha\sqrt{T})}{n(d_\alpha + \alpha\sqrt{T})}$$
$$\text{for } d_\alpha = \log(\sigma_T/\sigma_0)/(\alpha\sqrt{T}).$$

The distribution of S_T is approximated

$$S_T = S_0 + \frac{\rho}{\alpha}(\sigma_T - \sigma_0) + \sqrt{1 - \rho^2} \eta(\sigma_T) W \sqrt{T}$$

for $\eta(\sigma_T) = E(V_t|\sigma_T)/\sqrt{T}$. For a given σ_T , S_T follows a normal distribution, so we now the option

$$C_N(\sigma_T) = C_N \left(S_0 := S_0 + \frac{\rho}{\alpha}(\sigma_T - \sigma_0), \sigma_N := \sqrt{1 - \rho^2} \eta(\sigma_T) \right)$$

Option price as an integration (Kennedy et al)

$$\begin{aligned}C_{\beta=0} &= E\left((S_T - K)^+\right) = E\left((S_T - K)^+ | \sigma_T\right) = E\left(C_N(\sigma_T)\right) \\&= \int_{-\infty}^{\infty} C_N\left(S_0 + \frac{\rho}{\alpha}(\sigma_T(z) - \sigma_0), \sqrt{1 - \rho^2} \eta(\sigma_T(z))\right) n(z) dz \\&\quad \text{where } \sigma_T(z) = \sigma_0 \exp\left(-\frac{1}{2}\alpha^2 T + \alpha\sqrt{T} z\right)\end{aligned}$$

Using Gauss-Hermite quadrature (GHQ), [Py Demo]

$$C_{\beta=0} = \sum_m C_N\left(S_0 + \frac{\rho}{\alpha}(\sigma_T(z_m) - \sigma_0), \sqrt{1 - \rho^2} \eta(z_m)\right) w_m$$

for some points $\{z_m\}$ and weights $\{w_m\}$, and $\eta(z_m) := \eta(\sigma_T(z_m))$.

The results are similar:

$$\log(S_T/S_0) = \frac{\rho}{\alpha}(\sigma_T - \sigma_0) - \frac{1}{2}V_T + \sqrt{1 - \rho^2} W \sqrt{V_T}$$

$$C_{\beta=1} = \sum_m C_{BS} \left(S_0 e^{\frac{\rho}{\alpha}(\sigma_T(z_m) - \sigma_0) - \frac{\rho^2}{2}\eta(z_m)}, \sqrt{1 - \rho^2} \eta(z_m) \right) w_m$$

for some points $\{z_m\}$ and weights $\{w_m\}$.

- Implement the method of Kennedy et al and compare it against the Monte Carlo result for both normal ($\beta = 0$) and BSM backbone ($\beta = 1$).

- When β is given (0 or 1), three parameters, σ_0 , ρ and α , can be calibrated to three option prices (or implied volatilities), typically at $K = S_0$ (ATM), $S_0 - \Delta$ and $S_0 + \Delta$.

$$\text{SABR}(\sigma_0, \rho, \alpha) \rightarrow \sigma(S_0), \sigma(S_0 - \Delta), \sigma(S_0 + \Delta)$$

- Write a calibration routine in R to solve σ_0 , ρ and α .

Validation

- $C_{\beta=0}$ should converge to the normal model price when α is very small.
- Test against the result from Korn & Tang (Wilmott)

Homework

- First focus on the normal backbone $\beta = 0$.
- Make sure to use antithetic method (create Z , then add $-Z$).
- Short (1 page) write-up briefly explaining the code.
- Reproduce the graphs in 'The impact of parameters'. Fix your normal implied vol at ATM.
- Your script should be self-complete and should run without error.