Stochastic Finance (2016–17 M3)
Final Exam, April 20, 2017

BM stands for Brownian motion. **RN** and **RV** stand for random number and random variable respectively.

1. [**4 points**] **Stochastic calculus** Find **all** surviving terms in stochastic calculus

   (a) $dB_t \cdot dt$
   (b) $(dB_t)^2$
   (c) $dx \cdot dt$
   (d) $dB_t^1 \cdot dB_t^2$ for the two independent BMs, B_t^1 and B_t^2

   Answer (b) $(dB_t)^2 = dt$

2. [**6 points**] **Option pricing under the BSM and normal model** Assume that a stock's daily price change is 1.5% of the current price. What is the annual volatility of the stock under the Black-Scholes-Merton model and the normal model? What is the price of the at-the-money call option maturing in 3 months under the two models? Assume that $S_0 = 100$, $r = q = 0$ and there are 256 trading days in one year. You may use the following CDF values for the standard normal distribution $N(z)$.

z	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16
$N(z)$	0.508	0.516	0.524	0.532	0.540	0.548	0.556	0.564

   Answer

$$\sigma_{\text{BS}} = 1.5\% \times \sqrt{265} = 24\%, \quad \sigma_{\text{N}} = 24\% \times S_0 = 24$$

   Under the BSM model:

$$d_1 = -\frac{\sigma_{\text{BS}}\sqrt{1/4}}{2} = 0.06, \quad d_2 = 0.06$$
$$C_0 = S_0 N(d_1) - KN(d_2) = 100N(0.06) + 100(1 - N(0.06)) = 4.8$$

   Under the normal model:

$$C_0 = 0.4 \times \sigma_{\text{N}} \times \sqrt{1/4} = 4.8$$

3. [**5 points**] **Option delta under the BSM model** By directly computing the derivative, show that the delta of a call option (i.e., sensitivity with respect to the underlying stock price S_0) is

$$D = \frac{\partial C_0}{\partial S_0} = N(d_1) \quad \text{with} \quad d_1 = \frac{\log(S_0\, e^{rT}/K)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}.$$

   Since the terms d_1 and d_2 are defined through S_0, you should also differentiate $d1$ and d_2 rather than treating them as constants.

   Answer Using the properties

$$\frac{\partial d_1}{\partial S_0} = \frac{\partial d_2}{\partial S_0} = \frac{\partial}{\partial S_0}\log S_0 = \frac{1}{S_0\sigma\sqrt{T}}$$

   and

$$d_1^2 - d_2^2 = (A+B)^2 - (A-B)^2 = 4AB = 2\log(S_0\, e^{rT}/K),$$

   we compute the delta as

$$\begin{aligned}
D &= \frac{\partial}{\partial S_0}\left(S_0 N(d_1) - e^{-rT} K N(d_2)\right) \\
&= N(d_1) + S_0 n(d_1)\frac{\partial d_1}{\partial S_0} - e^{-rT} K n(d_2)\frac{\partial d_2}{\partial S_0} \\
&= N(d_1) + \frac{n(d_1)}{\sigma\sqrt{T}}\left(1 - e^{(d_1^2 - d_2^2)/2}\frac{K}{S_0 e^{rT}}\right) \\
&= N(d_1) + \frac{n(d_1)}{\sigma\sqrt{T}}\left(1 - \frac{S_0 e^{rT}}{K}\cdot\frac{K}{S_0 e^{rT}}\right) = N(d_1).
\end{aligned}$$

4. [9 points] **Volatility of a stock denominated in difference currency** If you invest in a stock listed in a foreign country, you are exposed to the risk of both the stock price (in the foreign currency) and the foreign exchange rate. We are going to see volatility of the value of the stock in the domestic currency. Assume that you invested in Amazon's stock (NASDAQ ticker `AMZN`, which is currently about 900 USD) and the volatility is 20%. Also assume that the volatility of the foreign exchange rate (currency code `USDCNY`, X CNY = 1 USD, which is 6.89 recently) and the volatility is 8%. Also assume that the correlation between the stock price change and the FX rate change is given as ρ (not necessarily independent!). The SDEs for the processes can be written as

$$\begin{aligned} \text{Stock price:} \quad & \frac{dS_t}{S_t} = r_S dt + \sigma_S dB_t^S \quad \text{where} \quad \sigma_S = 20\% \\ \text{FX rate:} \quad & \frac{dF_t}{F_t} = r_F dt + \sigma_F dB_t^F \quad \text{where} \quad \sigma_F = 8\% \\ \text{Correlation:} \quad & dB_t^S \cdot dB_t^F = \rho\, dt. \end{aligned}$$

(a) Derive the SDE for the stock value in CNY. That is, calculate $d(F_t\, S_t)$.

(b) Re-write the SDE with a single BM, let's say B_t^{FS}, and find the volatility of the stock price in CNY. This is the extension of our mid-term exam problem. You need to find c such that

$$c\, dB_t^{FS} = a\, dB_t^S + b\, dB_t^F \quad \text{with} \quad dB_t^S \cdot dB_t^F = \rho\, dt$$

(c) What is the minimum/maximum value of the combined volatility? Under which scenario?

Answer

(a)

$$\begin{aligned} d(F_t\, S_t) &= S_t dF_t + F_t dS_t = S_t \cdot F_t(r_F dt + \sigma_F dB_t^F) + F_t \cdot S_t(r_S dt + \sigma_S dB_t^S) + F_t dB_t^F \cdot S_t dB_t^S \\ \frac{d(F_t\, S_t)}{F_t\, S_t} &= (r_F + r_S + \rho\sigma_F\sigma_S)dt + \sigma_F dB_t^F + \sigma_S dB_t^S \end{aligned}$$

(b) Since,

$$(\sigma_F dB_t^F + \sigma_S dB_t^S)^2 = (\sigma_F^2 + \sigma_S^2 + 2\rho\sigma_F\sigma_S)\, dt,$$

we can re-write the SDE as

$$\frac{d(F_t\, S_t)}{F_t\, S_t} = (r_F + r_S + \rho\sigma_F\sigma_S)dt + \sqrt{\sigma_F^2 + \sigma_S^2 + 2\rho\sigma_F\sigma_S}\; dB_t^{FS}.$$

The volatility is

$$\sigma_{FS} = \sqrt{\sigma_F^2 + \sigma_S^2 + 2\rho\sigma_F\sigma_S}.$$

(c) The maximum is 20%+8%=28% when $\rho = 100\%$ and the minimum is 20%-8%=12% when $\rho = -100\%$.

5. [6 points] **Solving SDE** Solve the modified Cox-Ingersoll-Ross (CIR) model.

$$dX_t = \frac{\sigma^2}{4} dt + \sigma\sqrt{X_t}\, dB_t.$$

The original CIR model has the drift term $a(b - X_t)dt$ instead of $(\sigma^2/4)dt$ and was originally proposed for the movements of interest rates.

Answer First we remove $\sqrt{X_t}$ from the coefficient of dB_t

$$\frac{dX_t}{\sqrt{X_t}} = \frac{\sigma^2}{4\sqrt{X_t}} dt + \sigma\, dB_t$$

and we solve either $dx/\sqrt{x}$ or $dx/\sqrt{x} = \sigma^2 dt/(4\sqrt{x})$ for our guess. It turns out that the first candidate $dx/\sqrt{x}$ works. Since $\int dx/\sqrt{x} = 2\sqrt{x}$,

$$d\left(2\sqrt{X_t}\right) = \frac{dX_t}{\sqrt{X_t}} - \frac{(dX_t)^2}{4X_t\sqrt{X_t}} = \frac{\sigma^2}{4\sqrt{X_t}} dt + \sigma\, dB_t - \frac{\sigma^2 X_t}{4X_t\sqrt{X_t}} dt = \sigma dB_t$$

$$2\sqrt{X_t} - 2\sqrt{X_0} = \sigma B_t \quad \Rightarrow \quad X_t = \left(\sqrt{X_0} + \frac{\sigma B_t}{2}\right)^2$$