

# Spread and Basket Option Pricing

## Applied Stochastic Processes (FIN 514)

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- Options written on multiple underlying assets
- Extension of Black-Scholes model from uni-variate to multi-variate
- Write a MC pricing routine with control variate
- Analytic approximations (to be used as control variates)
  - Normal model price
  - Kirk's approximation (Margrabe's formula)
  - Geometric basket option

# Background: popular derivatives in non-vanilla class

Spread options:  $(S_1 - S_2 - K)^+$

- Crack spread option:  $(P \text{ of oil products} - P \text{ of oil} - K)^+$
- Spark spread option:  $(P \text{ of electricity} - P \text{ of gas} - K)^+$
- Non-inversion note: digital call on rates term-structure,  $(30y - 2y)^+$

Basket options:  $(\sum w_k S_k - K)^+$  with  $w_k > 0$

- Popular as OTC derivatives in FX and commodities market
- Index options

Asian options (path-dependent):  $(\sum_1^N S(t_k)/N - K)^+$

- Efficient hedge over average cost, safe from market manipulation.
- Fed fund swaps: daily compounded trade-averaged FF rate
- China interest swaps: 3-month average of 7-day repo rate as a standard floating rate (cf. 3-month Libor)

# Problem setup

- $N$  asset prices,  $S_k(t)$ , following the correlated geometric Brownian motions (GBM)

$$\frac{dS_k(t)}{S_k(t)} = (r - q_k) dt + \sigma_k dW_k(t) \quad \text{for } 1 \leq k \leq N$$

for volatilities  $\sigma_k$ , dividend rates  $q_k$ , risk-free rate  $r$  and BMs  $W_k(t)$ , the correlation  $\mathbb{E}\{dW_k(t)dW_j(t)\} = \rho_{kj}dt$  ( $\rho_{kk} = 1$ ).

- $N$  observation times:  $t_k$  ( $1 \leq k \leq N$ ), with expiry at  $T$
- The payout of the option at the maturity  $T$

$$C(T) = \left( \sum_{k=1}^N w_k S_k(t_k) - K \right)^+$$

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- **European basket option:**

$w_k > 0$  and  $t_k = T$  for all  $k$ ;  $N < 10$

- **European spread option:**

$w_k < 0$  for some  $k$ ,  $t_k = T$  for all  $k$ ;  $N = 2$

- **Discretely monitored Asian option** (covered later):

$w_k = 1/N$ ,  $0 \leq t_1 < \dots < t_N = T$  and  $S_k(t)$ 's are identical;  $N \gg 10$

# Challenges

## Mathematical problem

Multi-dimensional integration over the domain of positive payoff

## Difficulties

- The lognormal RV sum is neither lognormal nor has analytic distribution.
- Numerical valuation is cursed by dimensionality:  $O(M^N)$   
e.g.  $h = 0.1$  grid between  $\pm 7$  std. dev. for 4 assets:  $140^4 \approx 400e6$
- Monte-Carlo simulation is used for pricing in industry and academics.

## Quote from Broadie and Detemple (2004, MS Survey)

*"Many problems are effectively exponential  $\dots$ . Efficient and convergent methods for pricing high-dimensional and path-dependent American securities depend on the development of new algorithms, not faster computers."*

# Normal model approximation

- Spread Option:  $\sigma_{n1} = \sigma_1 S_1(0)$ ,  $\sigma_{n2} = \sigma_2 S_2(0)$

$$\text{Var}(S_1(T) - S_2(T)) = (\sigma_{n1}^2 + \sigma_{n2}^2 - 2\rho\sigma_{n1}\sigma_{n2}) \cdot T$$

$$\sigma_n = \sqrt{\sigma_{n1}^2 + \sigma_{n2}^2 - 2\rho\sigma_{n1}\sigma_{n2}}$$

and use the normal model formula.

- Basket Option:  $\sigma_{nj} = \sigma_j S_j(0)$  or  $= \sigma_j F_j(0)$

$$\text{Var}\left(\sum w_k S_k(T)\right) = \left(\sum_j w_j^2 \sigma_{nj}^2 + 2 \sum_{jk} \rho_{jk} w_j w_k \sigma_{nj} \sigma_{nk}\right) \cdot T$$

$$\Sigma_{jk} = \rho_{jk} \sigma_{nj} \sigma_{nk}, \quad \sigma_n = \sqrt{\mathbf{w}^T \Sigma \mathbf{w}}$$

and use the normal model formula.

# Normal model approximation for Control Variate

- Use the result as a control variate of Spread and Asian option:

$$C^{CV}(T, K) = C^{MC}(T, K) + \left( C_N^{Exact}(T, K) - C_N^{MC}(T, K) \right)$$

Use the same sequence of RNs for  $C^{MC}$  and  $C_N^{MC}$ . [R Demo]

- **Homework Set 3:**

Implement Monte-Carlo pricer for Spread and Basket options with control variate with normal model price.

- Final project: implement other analytic approximation methods or CV methods (e.g., Kirk's approximation for spread options)



# Exchange option: Margrabe's formula

- Option to exchange one asset  $S_1$  for another  $S_2$ :  $(S_1(T) - S_2(T))^+$
- Spread option with zero strike  $K = 0$ :  $(S_1(T) - S_2(T) - 0)^+$
- Max (best-of) option:  $\max(S_1, S_2) = S_2 + \max(S_1 - S_2, 0)$ .

## Margrabe's formula

$$C_{sw} = S_1(0)N(d_+) - S_2(0)N(d_-),$$

where  $d_{\pm} = \frac{\log(S_1(0)/S_2(0))}{\sigma_R\sqrt{T}} \pm \sigma_R\sqrt{T}$  and  $\sigma_R = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$ .

## SDE on $S_1/S_2$

$$dS_k(t)/S_k(t) = r dt + \sigma_k dW_k(t) \quad (k = 1, 2), \quad dW_1 dW_2 = \rho dt$$

Applying Itô's lemma to  $S_1/S_2$ ,

$$d\left(\frac{S_1}{S_2}\right) = \frac{dS_1}{S_2} - \frac{S_1}{S_2^2} dS_2 + \frac{S_1}{S_2^3} (dS_2)^2 - \boxed{\frac{dS_1 dS_2}{S_2^2}}$$

For  $R = S_1/S_2$ ,

$$\frac{dR}{R} = (\sigma_2^2 - \rho\sigma_1\sigma_2) dt + (\sigma_1 dW_1 - \sigma_2 dW_2)$$

Alternatively,

$$d \log R = d \log S_2 - d \log S_1 = -\frac{1}{2}(\sigma_1^2 - \sigma_2^2) dt + \sigma_1 dW_1 - \sigma_2 dW_2$$

$$\frac{dR}{R} = (\cdots) + (\sigma_1 dW_1 - \sigma_2 dW_2) = \text{same result}$$

# Equivalent Martingale Measure with Numeraire $S_2$

Decorrelating the SDE on  $R$ , for  $dW'_1 dW_2 = 0$ ,

$$\frac{dR}{R} = (\sigma_2^2 - \rho\sigma_1\sigma_2) dt + \sigma_1(\rho dW_2 + \sqrt{1-\rho^2}\sigma_1 dW'_1) - \sigma_2 dW_2.$$

Now we change the measure from  $P$  (riskless saving numeraire) to  $Q$  ( $S_2$ );

$$\begin{aligned} C_{sw} &= 1 \cdot E^P \left( \frac{(S_1(T) - S_2(T))^+}{e^{rT}} \right) \\ &= S_2(0) E^Q \left( \frac{(S_1(T) - S_2(T))^+}{S_2(T)} \right) = S_2(0) E^Q \left( (R(T) - 1)^+ \right) \end{aligned}$$

Under the new measure, the standard BM is defined as

$$dW_2^P = dW_2^Q + \sigma_2 dt \quad \text{and} \quad dW_1^P = dW_1^Q.$$

Then the SDE on  $R$  becomes driftless and can be written with a single BM

$$\begin{aligned} \frac{dR}{R} &= (\sigma_2^2 - \rho\sigma_1\sigma_2) dt + \sqrt{1-\rho^2}\sigma_1 dW_1^P - (\sigma_2 - \rho\sigma_1)(dW_2^Q + \sigma_2 dt) \\ &= \sqrt{1-\rho^2}\sigma_1 dW_1^Q + (\sigma_2 - \rho\sigma_1) dW_2^Q = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} dZ^Q \end{aligned}$$

# Margrabe's exchange option formula

The switch option price is obtained from just another Black-Scholes formula on  $R(T)$  with

- $K = 1$
- $\sigma_R = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$
- prefactor (unit of options)  $S_2(0)$ .

Finally we obtain

$$C_{sw} = S_2(0) E^Q \left( (R(T) - 1)^+ \right) = S_2(0) \left( \frac{S_1(0)}{S_2(0)} N(d_+) - 1 \cdot N(d_-) \right)$$

where  $d_{\pm} = \frac{\log(S_1(0)/S_2(0))}{\sigma_R \sqrt{T}} \pm \sigma_R \sqrt{T}$  and  $\sigma_R = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$ .

## Kirk's approximation

If we assume  $S_2$  follows displaced GBM with  $L = K$ ,  $S_2^D = S_2 + K$ , then we can apply Margrabe's formula!

$$(S_1 - S_2 - K)^+ = (S_1 - \boxed{(S_2 + K)})^+ = (S_1 - S_2^D)^+$$

The volatility of  $S^D$  should be 'calibrated'. We match the local vol at ATM

$$\sigma_2^D(S_2(0) + K) = \sigma_2 S_2(0)$$

Plug in  $S_2(0) + K \rightarrow S_2(0)$  and  $\sigma_2 S_2(0)/(S_2(0) + K) \rightarrow \sigma_2$  to Margrabe:

## Kirk's approximation

$$C_{Kirk} = S_1(0)N(d_+) - (S_2(0) + K)N(d_-),$$

$$d_{\pm} = \log\left(\frac{S_1(0)}{S_2(0) + K}\right) / \sigma_R \sqrt{T} \pm \sigma_R \sqrt{T}$$

$$\sigma_R = \sqrt{\sigma_1^2 + \sigma_2'^2 - 2\rho\sigma_1\sigma_2'}, \quad \sigma_2' = \sigma_2 S_2(0)/(S_2(0) + K)$$

## Some comments on Kirk's approximation

- The method is fairly accurate for reasonable parameter range: error of  $10^{-3}$ .
- Kirk's formula can be a better control variate than the normal model price.
- What if  $K < 0$  ? Displaced GBM on  $S_1$  instead of  $S_2$ ?

$$(S_1 - S_2 - K)^+ = (\boxed{S_1 - K} - S_2)^+ = (S_1^D - S_2)^+$$