Stochastic Finance (2016-17 M3)
Mid-term Exam, Mar 23, 2017

BM stands for Brownian motion. **RN** and **RV** stand for random number and random variable respectively.

1. **[4 points] Standard BM**
   If B_t is a standard BM, determine whether each of the followings is a standard BM or not. Provide a brief reason for your answer.
   (a) $4B_{t/2}$ **Answer: No**. $\text{Var}(4B_{t/2}) = 16 \times t/2 = 8t \neq t$.
   (b) $tB_{1/t}$ with $B_0 = 1$ **Answer: No**. B_0 should be 0.
   (c) $2(B_{1+t/4} - B_1)$ **Answer: Yes**. $B_{1+t/4} - B_1$ is equivalent to $B_{t/4}$ and $2B_{t/4}$ is equivalent to B_t.
   (d) $\sqrt{t}Z$ for a standard normal RV Z **Answer: No**. For any value of Z, $\sqrt{t}Z$ is not a stochastic process. For example, $\sqrt{s}Z$ and $(\sqrt{t} - \sqrt{s})Z$ for $s < t$ are correlated.

2. **[2 points] Martingale related to BM**
   If B_t is a standard BM, find the value of the coefficient λ in order for each of the following expressions to be a martingale.
   (a) $B_{at}^2 - \lambda t$ **Answer:** $\lambda t = E(B_{at}^2) = at$. Therefore, $\lambda = a$.
   (b) $\exp(-B_{at} + \lambda t)$ **Answer:** $\sqrt{a}B_t$ is a BM equivalent to $-B_{at}$. Therefore, $\lambda = -a/2$.

3. **[3 pts] Average of BM path**
   If B_t for $0 \leq t \leq 1$ is a standard BM, what is the distribution of the average of the BM values observed at three different times, $T = 1/3, 2/3$ and 1,

$$A = \frac{1}{3}\Big(B_{\frac{1}{3}} + B_{\frac{2}{3}} + B_1\Big)?$$

   Please make sure to provide the mean and the standard deviation of the distribution.

   Answer

$$\begin{aligned}
\text{Var}\Big(\frac{1}{3}(B_{1/3} + B_{2/3} + B_1)\Big) &= \frac{1}{9}E\Big((B_{1/3} + B_{2/3} + B_1)^2\Big) \\
&= \frac{1}{9}E\Big(B_{1/3}^2 + B_{2/3}^2 + B_1^2 + 2B_{1/3}(B_{2/3} + B_1) + 2B_{2/3}B_1\Big) \\
&= \frac{1}{9}\Big(\frac{1}{3} + \frac{2}{3} + 1 + 2 \cdot \frac{1}{3} \cdot 2 + 2 \cdot \frac{2}{3}\Big) = \frac{1}{9}\frac{14}{3} = \frac{14}{27}.
\end{aligned}$$

4. **[8 points] Generating RNs for correlated BMs**
   Throughout this problem, assume that X_t and Y_t are two independent standard BMs.

(a) Other than the examples we covered in the class, there are many ways to create standard BMs. A linear combination of the two BMs with the coefficients a and b,

$$W_t = aX_t + bY_t$$

is also a BM. (No need to prove it.) What is the condition for a and b under which W_t is a **standard** BM.

(b) What is the correlation between X_t and W_t? We have not defined the correlation of two BMs yet, so simply compute the correlation of the two distributions of the BMs at $t = 1$, i.e, X_1 and W_1. (In fact, the correlation is same for any time t.) You do not have to use the answer of (a).

(c) Assume that $\{z_k\}$ for $k = 1, 2, \cdots$ is a sequence of standard normal RVs, i.e., $N(0, 1)$, which are generated from computer (e.g., using Box-Muller algorithm). Use $\{z_k\}$ to generate RNs for X_t for a fixed time t.

(d) Assume that we have two standard BMs, X_t and W_t, which have correlation ρ. How can you generate the pairs of RNs for X_t and W_t for a fixed time t?

Answer

(a) $\mathrm{Var}(W_t) = a^2\mathrm{Var}(X_t) + b^2\mathrm{Var}(Y_t) = (a^2 + b^2)t$ should be t. Therefore, $a^2 + b^2 = 1$.

(b)

$$\mathrm{Corr}(W_t, X_t) = \frac{\mathrm{Cov}(X_t, W_t)}{\sqrt{\mathrm{Var}(X_t)\mathrm{Var}(W_t)}} = \frac{at}{\sqrt{t \cdot (a^2 + b^2)t}} = \frac{a}{\sqrt{a^2 + b^2}}$$

(c) $\{\sqrt{t}\, z_k\}$ is the RNs for X_t.

(d) We can rewrite W_t as $W_t = \rho X_t + \sqrt{1 - \rho^2}Y_t$. Therefore, the random numbers for X_t and W_t can be generated as

$$(\sqrt{t}\, z_1,\ \rho\sqrt{t}\, z_1 + \sqrt{1 - \rho^2}\sqrt{t}\, z_2)$$
$$(\sqrt{t}\, z_3,\ \rho\sqrt{t}\, z_3 + \sqrt{1 - \rho^2}\sqrt{t}\, z_4)$$
$$\cdots$$
$$(\sqrt{t}\, z_{2k-1},\ \rho\sqrt{t}\, z_{2k-1} + \sqrt{1 - \rho^2}\sqrt{t}\, z_{2k})$$

5. **[3 points] Wald's equation**

When $\{X_k\}$ are independent identically distributed random variable and N is a random variable taking positive integer values, Wald's equation says

$$E(X_1 + X_2 + \cdots X_N) = E(N)\, E(X_1)$$

if either (i) N is independent from $\{X_k\}$ or (ii) N is a stopping time with respect to $\{X_k\}$.

Consider an example where $X_k = 0$ or 1 with 50% and 50% probability and N is given as

$$N = X_2 + 1.$$

Obviously, $E(X_k) = 1/2$ and $E(N) = 1/2 + 1 = 3/2$. Find $E(X_1 + X_2 + \cdots + X_N)$ and explain why Wald's equation does not hold in this example. If N is given instead as

$$N = X_1 + 1,$$

does Wald's equation hold? Is N a stopping time?

Answer
Branching on the value of X_2:

$$N = \begin{cases} 1 & (X_2 = 0, \quad \text{Prob} = 1/2) \\ 2 & (X_2 = 1, \quad \text{Prob} = 1/2), \end{cases}.$$

we compute

$$\begin{aligned} E(X_1 + X_2 + \cdots + X_N) &= \frac{1}{2}E(X_1) + \frac{1}{2}E(X_1 + 1) \\ &= \frac{1}{2}(\frac{1}{2} + \frac{1}{2} + 1) = 1 \neq E(X_k)E(N) = \frac{3}{4}. \end{aligned}$$

Wald's equation does not hold because $N = X_1 + X_2 + 1$ is
(i) not independent from $\{X_k\}$ as N is defined via X_2 and (ii) not a stopping time w.r.t. $\{X_k\}$ because we need future information X_2 in order to determine N (X_1 is not enough to tell $N = 1$ or not.)

If $N = X_1 + 1$, N is a stopping time. Based on X_1 we can tell whether $N = 1$ or not: $N = 1$ if $X_1 = 0$ and $N \neq 1$ (in fact, $N = 2$) if $X_1 = 1$. Therefore Wald's inequality should hold. We can directly verify that

$$E(X_1 + X_2 + \cdots + X_N) = \frac{1}{2}E(X_1) + \frac{1}{2}E(X_1 + X_2) = \frac{1}{2}(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}) = \frac{3}{4}.$$

6. [**5 points**] Please circle the most appropriate item to you in each of the following questions.

   (a) The difficulty of this course is (i) easy (ii) appropriate (iii) difficult.

   (b) Compared to the other required courses in the 1st year, the load of this course is (i) lower (ii) similar (iii) higher.

   (c) If this course was **not** a required course, you would (i) still register (ii) not register.

   (d) Professor's preparation for the course and communication with students are (i) satisfactory (ii) acceptable (iii) unsatisfactory.

   (e) To your opinion (not professor's), this course is going to be (i) useful (ii) not useful for your future career path.