11.9 Fourier Transform. Discrete and Fast Fourier Transforms (푸리에 변환. 이산 및 고속 푸리에 변환)

$$f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} [f(v)\cos\omega v\cos\omega x + f(v)\sin\omega v\sin\omega x]\,dv\,d\omega$$

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}[f(v)\cos\omega v\cos\omega x+f(v)\sin\omega v\sin\omega x]\,dv\,d\omega$$

$$=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}f(v)\cos(\omega v-\omega x)\,dv\,d\omega$$

(1*) $$=\frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty}f(v)\cos(\omega x-\omega v)\,dv\right]d\omega$$

(1) $$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(v)\cos(\omega x-\omega v)\,dv\right]d\omega$$ ∵Even wrt ω

(2) $$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(v)\sin(\omega x-\omega v)\,dv\right]d\omega=0$$

[Because $\int_{-\infty}^{\infty}f(v)\sin(\omega x-\omega v)\,d\omega=0$]

$$(1)\ f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(v)\cos(\omega x-\omega v)dv\right]d\omega$$

$$(2)\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(v)\sin(\omega x-\omega v)dv\right]d\omega=0$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(v)\cos(\omega x-\omega v)dv+i\int_{-\infty}^{\infty}f(v)\sin(\omega x-\omega v)dv\right]d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(v)e^{i(\omega x-\omega v)}dv\right]d\omega$$

$$(4)\qquad =\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(v)e^{i\omega(x-v)}dv\right]d\omega$$

Complex Fourier Integral:

$$(4) \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) e^{i\omega(x-v)}\, dv \right] d\omega$$

Fourier Transform and Inverse Fast Fourier Transform (푸리에 변환과 역변환)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)e^{i\omega(x-v)}dv\,d\omega \quad (4)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-i\omega v}dv\right]e^{i\omega x}d\omega \quad (5)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)\,e^{i\omega x}d\omega$$

$$where\ \ \hat{f}(\omega) = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-i\omega v}dv\right] \quad (6)$$

: Fourier Transform of f

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(v)e^{-i\omega v}dv\right]e^{i\omega x}d\omega \quad \textbf{(5)}$$

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(v)e^{-i\omega v}dv \quad \textbf{(6)}$$

Fourier Transform: $\mathcal{F}(f)=\hat{f}(\omega)=\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(v)e^{-i\omega v}dv\right] \quad (6)$

Inverse Fourier T : $\mathcal{F}^{-1}(\hat{f})=f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)\,e^{i\omega x}d\omega \quad (7)$

Theorem1 Existence of the Fourier Transform

Sufficient condition for the existence of the Fourier transform (6) are

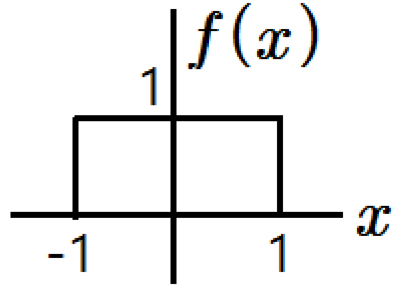
1. f(x) is piecewise continuous on every finite interval
2. f(x) is absolutely integrable on the x-axis

Fourier Transform:

$$\mathcal{F}(f) = \hat{f}(\omega) = \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} dv \right] \qquad (6)$$

Ex. 1 Fourier Transform

Find the transform of f(x)=1 if |x|<1 and f(x)=0 otherwise.



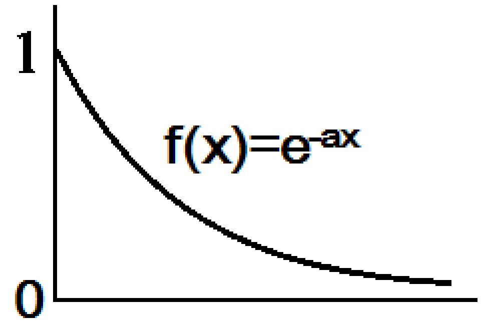
Sol.

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \cdot \left. \frac{e^{-iwx}}{-iw} \right|_{-1}^{1}$$

$$= \frac{1}{-iw\sqrt{2\pi}} \left(e^{-iw} - e^{iw} \right) = \frac{-2i \sin w}{-iw\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}$$

Example 2 Fourier Transform

Find the transform of $f(x)=e^{-ax}$ if $x>0$ and $f(x)=0$ if $x<0$; here $a>0$.



Sol.

$$\mathcal{F}(f)=\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-ax}e^{-i\omega x}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(a+i\omega)x}dx=\frac{1}{\sqrt{2\pi}}\left.\frac{e^{-(a+i\omega)x}}{-(a+i\omega)}\right|_{0}^{\infty}$$

$$=\frac{1}{\sqrt{2\pi}\,(a+i\omega)}$$

Physical Interpretation: Spectrum(물리적 해석)

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \qquad (7)$$

1. Spectral representation: f(x) is a superposition sinusoidal oscillations f all possible frequencies. In optics, light is a superposition of colors(frequencies).

2. $\hat{f}(\omega)$ is the intensity of f(x), spectral density, in the frequency interval ω~ ω+Δω.

   $|\hat{f}(\omega)|^2$ is the energy density in the frequency interval ω~ ω+Δω.

3. Total energy of a physical system= $\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega$

An Example

Harmonic Oscillator: $my'' + ky = 0$

$$y' \times (my'' + ky = 0) \qquad my'y'' + kyy' = 0$$

$$\int : \tfrac{1}{2} m (y')^2 + \tfrac{1}{2} ky^2 = \tfrac{1}{2} mv^2 + \tfrac{1}{2} ky^2 = E_0 : Total \quad Energy$$

$$my'' + ky = 0$$

$$y = a_1 \cos\omega_0 x + b_1 \sin\omega_0 x = c_1 e^{i\omega_0 x} + c_{-1} e^{-i\omega_0 x}, \quad \omega_0^2 = k/m$$

$$= A + B \quad where \quad A = c_1 e^{-i\omega_0 x}, \quad B = c_{-1} e^{i\omega_0 x}$$

$$y' = A' + B' = i\omega_0 (A - B)$$

$$E_0 = \tfrac{1}{2} mv^2 + \tfrac{1}{2} ky^2 = \tfrac{1}{2} m [i\omega_0 (A - B)]^2 + \tfrac{1}{2} k (A + B)^2$$

$$= - \tfrac{1}{2} (m\omega_0)^2 (A - B)^2 + \tfrac{1}{2} k (A + B)^2$$

$$= 2kAB = 2k (c_1 e^{-i\omega_0 x})(c_{-1} e^{i\omega_0 x}) = 2kc_1 c_{-1} = 2k|c_1|^2$$

Theorem 2 Linearity of the Fourier Transform

The Fourier transform is a linear operation; that is, for any functions f(x) and g(x) whose Fourier transforms exist and constants a and b,

$$\mathfrak{F}(af+bg)=a\mathfrak{F}(f)+b\mathfrak{F}(g) \qquad (8)$$

Proof.

$$\begin{aligned}
\mathfrak{F}(af+bg) &= aF(f)+bF(g)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af(x)+bg(x)]e^{-i\omega x}dx\\
&= a\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx+b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x)e^{-i\omega x}dx\\
&= a\mathfrak{F}(f)+b\mathfrak{F}(g)
\end{aligned}$$

Theorem 3 Fourier Transform of the derivative of f(x)

Let f(x) be continuous on the x-axis and f(x)→0 as |x|→∞. Furthermore, let f'(x) be absolutely integrable on the x-axis. Then

$$\mathcal{F}\{f'(x)\} = i\omega\,\mathcal{F}\{f(x)\} \qquad (9)$$

$$\mathcal{F}\{f''(x)\} = -\omega^2\,\mathcal{F}\{f(x)\} \qquad (10)$$

Proof.

$$\begin{aligned}
\mathcal{F}\{f'(x)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{-i\omega x}dx \\
&= \frac{1}{\sqrt{2\pi}}\left[f(x)e^{-i\omega x}\Big|_{-\infty}^{\infty} - (-i\omega)\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx\right] \\
&= 0 + i\omega\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \\
&= i\omega\mathcal{F}\{f(x)\}
\end{aligned}$$

Example 3 Application of the Theorem 3

Find the Fourier transform of $x\exp(-x^2)$ from Table III, Sec. 11.10.

Table III, Sec. 11.10

$$\mathcal{F}\{\exp(-ax^2)\} = \frac{1}{\sqrt{2a}}\exp(-\omega^2/4a)$$

Sol.

$$\begin{aligned}\mathcal{F}\{x\exp(-x^2)\} &= \mathcal{F}\{(-1/2)[\exp(-x^2)]'\}\\ &= (-1/2)\,i\omega\,\mathcal{F}\{\exp(-x^2)\}\\ &= -\frac{1}{2}\,i\omega\,\frac{1}{\sqrt{2}}\exp(-\omega^2/4)\\ &= -\frac{i\omega}{2\sqrt{2}}\exp(-\omega^2/4)\end{aligned}$$

Convolution(합성곱)

The convolution f*g of functions f and g is defined by

$$h(x) = (f*g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)\,dp \tag{11}$$

$$= \int_{-\infty}^{\infty} f(x-p)g(p)\,dp$$

Theorem 4 Convolution Theorem(합성곱 정리)

Suppose that f(x) and g(x) are piecewise continuous, bounded, and absolutely integrable on the x-axis. Then

(12) $$\mathcal{F}\{f*g\} = \sqrt{2\pi}\ \mathcal{F}\{f\}\ \mathcal{F}\{g\}$$

Proof.

$$(f*g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)\,dp = \int_{-\infty}^{\infty} f(x-p)g(p)\,dp$$

$$\mathcal{F}\{f*g\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(p)g(x-p)\,dp\right] e^{-i\omega x}\ dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p)g(x-p)e^{-i\omega x}dx\ dp$$

$$\mathcal{F}\{f*g\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(p)g(x-p)e^{-i\omega x}dx\,dp$$

$$\Downarrow \quad Let\ \ x-p=q,\ \ then\ \ x=p+q$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(p)g(q)e^{-i\omega(p+q)}dq\,dp$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(p)e^{-i\omega p}dp\int_{-\infty}^{\infty}g(q)e^{-i\omega q}dq$$

$$=\sqrt{2\pi}\ \mathcal{F}(f)\ \mathcal{F}(g)$$

(12) $$\mathcal{F}\{f*g\} = \sqrt{2\pi}\ \mathcal{F}\{f\}\ \mathcal{F}\{g\}$$

Inverse Fourier Transform

(7) $$\mathcal{F}^{-1}(\hat{f}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega$$

(13) $$(f*g)(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)\, \hat{g}(\omega)\, e^{i\omega x}\, d\omega$$

Discrete Fourier Transform(DFT), Fast FT(FFT)

Let f(x) be periodic, for simplicity of period 2π.

Assume that N measurements are taken over the interval $0 \leqq x \leqq 2\pi$ at the following regularly spaced points

(14) $$x_k = \frac{2\pi k}{N}, \quad k = 0, 1, \cdots, N-1$$

Let's find a complex trigonometric polynomial q(x)

(15) $$q(x) = \sum_{n=0}^{N-1} c_n e^{inx_k}$$

that interpolates f(x)at the nodes (14), that is, $q(x_k)=f(x_k)$.

(16) $$f(x_k) = f_k = q(x_k) = \sum_{n=0}^{N-1} c_n e^{inx_k}, \quad k = 0, 1, \cdots, N-1$$

Hence we must determine the coefficients $c_0, \ldots, c_{N-1}$ such that (16) holds.

(16) $$f_k = \sum_{n=0}^{N-1} c_n e^{inx_k}, \quad k = 0, 1, \cdots, N-1$$

$$f_k e^{-imx_k} = e^{-imx_k} \sum_{n=0}^{N-1} c_n e^{inx_k} = \sum_{n=0}^{N-1} c_n e^{i(n-m)x_k}$$

(17)
$$\begin{aligned}\sum_{k=0}^{N-1} f_k e^{-imx_k} &= \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} c_n e^{i(n-m)x_k} \\ &= \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} e^{i(n-m)2\pi k/N} \\ &= \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} [e^{i(n-m)2\pi/N}]^k = \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} r^k\end{aligned}$$

$$where \;\; r = e^{i(n-m)2\pi/N}$$

$$\sum_{k=0}^{N-1} r^k = \begin{cases} N, & r = 1 \\ \dfrac{1-r^n}{1-r}, & r \neq 1 \end{cases} \qquad r = e^{i(n-m)2\pi/N}$$

$$\begin{cases} n = m: & r = 1 \\ n \neq m: & r \neq 1 \end{cases}$$

$$\therefore \begin{cases} n = m: & \displaystyle\sum_{k=0}^{N-1} r^k = \sum_{k=0}^{N-1} 1 = N \\ n \neq m: & \displaystyle\sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r} = \frac{1-[e^{i(n-m)2\pi/N}]^N}{1-e^{i(n-m)2\pi/N}} = 0 \end{cases}$$

(17) $$\sum_{k=0}^{N-1} f_k e^{-imx_k} = \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} [e^{i(n-m)2\pi/N}]^k = \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} r^k$$

$$\sum_{k=0}^{N-1} r^k = \begin{cases} N, & n = m \\ 0, & n \neq m \end{cases}$$

$$\therefore \sum_{k=0}^{N-1} f_k e^{-imx_k} = \sum_{n=0}^{N-1} c_n \sum_{k=0}^{N-1} r^k = Nc_m$$

(18*) $$c_n = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-inx_k}$$

$$\text{where } f_k = f(x_k),\ \ k = 0, 1, \cdots, N-1$$

(18*) $$c_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-inx_k}$$

FFT를 이용하여 c_n을 구할 때 N이 연속적으로 반으로 감소하므로 다음 식이 일반적임

Since computation of the c_n (by the FFT) involve successive halfing of the problem size N, it is practical to drop the 1/N, and define the DFT of the given signal by the following equation.

DFT: $$\hat{f}_n = Nc_n = \sum_{k=0}^{N-1} f_k e^{-inx_k} = \sum_{k=0}^{N-1} f_k e^{-in2\pi k/N} \quad (18)$$

DFT of the given signal $f = [f_0 \quad \cdots \quad f_{N-1}]^T$ is defined by the vector $\hat{f} = [\hat{f}_0 \quad \cdots \quad \hat{f}_{N-1}]^T$

DFT: $$\hat{f}_n = Nc_n = \sum_{k=0}^{N-1} f_k e^{-inx_k} = \sum_{k=0}^{N-1} f_k e^{-in2\pi k/N} \qquad (18)$$

$$\begin{aligned}
n &= 0 &: \hat{f}_0 &= f_0 + f_1 + \cdots + f_{N-1} \\
n &= 1 &: \hat{f}_1 &= f_0 + f_1 e^{-i2\pi/N} + \cdots + f_{N-1}[e^{-i2\pi/N}]^{N-1} \\
n &= 2 &: \hat{f}_2 &= f_0 + f_1 e^{-i4\pi/N} + \cdots + f_{N-1}[e^{-i4\pi/N}]^{N-1} \\
&\vdots \\
n &= N-1 &: \hat{f}_{N-1} &= f_0 + f_1 e^{-i(N-1)2\pi/N} + \cdots + f_{N-1}[e^{-i(N-1)2\pi/N}]^{N-1}
\end{aligned}$$

In matrix form,

$$\hat{f} = \begin{bmatrix} \hat{f}_0 \\ \vdots \\ \hat{f}_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & \end{bmatrix} \begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix} = F_N f$$

$$\hat{f} = \begin{bmatrix} \hat{f}_0 \\ \vdots \\ \hat{f}_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & \end{bmatrix} \begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix} = F_N f$$

F_N: Fourier matrix

$$\hat{f} = F_N f$$

where

$$\hat{f} = [\hat{f}_0\ \hat{f}_1\ \hat{f}_2 \cdots \hat{f}_{N-1}]' \quad f = [f_0\ f_1\ f_2 \cdots f_{N-1}]'$$

$$F_N = [e_{nk}]$$

$$e_{nk} = e^{-inx_k} = e^{-i2\pi nk/N} = \mathrm{w}^{nk}, \quad \mathrm{w} = \mathrm{w}_N = e^{-2\pi i/N} \tag{19}$$

$$F_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \mathrm{w}_N^1 & \mathrm{w}_N^2 & \cdots & \mathrm{w}_N^{\mathrm{N}-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \mathrm{w}_N^{\mathrm{N}-1} & [\mathrm{w}_N^{\mathrm{N}-1}]^2 & \cdots & [\mathrm{w}_N^{\mathrm{N}-1}]^{\mathrm{N}-1} \end{bmatrix}$$

Example 4 Discrete Fourier Transform(DFT), N=4

Find the DFT of sampled values f=[0 1 4 9]^T.

Sol.

$$\mathrm{w} = \mathrm{w}_4 = e^{-2\pi i/4} = -i$$

$$F_4 = \begin{bmatrix} \mathrm{w}_4^0 & \mathrm{w}_4^0 & \mathrm{w}_4^0 & \mathrm{w}_4^0 \\ \mathrm{w}_4^0 & \mathrm{w}_4^1 & \mathrm{w}_4^2 & \mathrm{w}_4^3 \\ \mathrm{w}_4^0 & \mathrm{w}_4^2 & \mathrm{w}_4^4 & \mathrm{w}_4^6 \\ \mathrm{w}_4^0 & \mathrm{w}_4^3 & \mathrm{w}_4^6 & \mathrm{w}_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & (-i)^2 & (-i)^3 \\ 1 & (-i)^2 & [(-i)^2]^2 & [(-i)^2]^3 \\ 1 & (-i)^3 & [(-i)^3]^2 & [(-i)^3]^3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

$$
\begin{aligned}
\hat{f} &= F_N f \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} \\
&= \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix}
\end{aligned}
$$

Inverse Discrete Fourier Transform(IDFT)

DFT: $f \longrightarrow \hat{f}$ $\quad \hat{f} = F_N f$

IDFT: $\hat{f} \longrightarrow f$ $\quad f = F_N^{-1} \hat{f}$

$$(21a) \quad \overline{F}_n F_N = F_N \overline{F}_n = N\mathrm{I}, \quad \mathrm{I}: N \times N \ \textit{Unit Matrix}$$

$$(21b) \quad F_N^{-1} = \frac{1}{N} \overline{F}_N$$

Proof of (21): Next slide

$$(21a)\quad \overline{F}_n F_N = F_N \overline{F}_n = N\mathrm{I}, \quad \mathrm{I}: N \times N \ \textit{Unit Matrix}$$

$$(21b)\quad F_N^{-1} = \frac{1}{N}\overline{F}_N$$

Proof of (21)

$$G_N = \overline{F}_n F_N = [g_{jk}]$$

$g_{jk} =$ j-th row of $\overline{F}_n \times$ k-th column of F_N

$$= (\overline{w}^j w^k)^0 + (\overline{w}^j w^k)^1 + (\overline{w}^j w^k)^2 + \cdots + (\overline{w}^j w^k)^{N-1}$$

$$= W^0 + W^1 + W^2 + \cdots + W^{N-1}$$

where $W = \overline{w^j} w^k = e^{i(2\pi j/N)} e^{-i(2\pi k/N)}$

$$= e^{-i2\pi(k-j)/N}$$

$$= \sum_{m=0}^{N-1} W^m$$

$$W = e^{-i2\pi(k-j)/N} \Rightarrow \begin{cases} j \neq k: & W \neq 1 \\ j = k: & W = 1 \end{cases}$$

$$g_{jk} = \sum_{k=0}^{N-1} W^k = \begin{cases} j \neq k: & g_{jk} = \dfrac{W^0(1-W^N)}{1-W} = 0 \\ j = k: & W = 1, \quad \therefore g_{jk} = N \end{cases}$$

$$\Downarrow$$

$$G_N = \overline{F}_n F_N = [g_{jk}] = N\mathrm{I}$$

Fast Fourier Transform(FFT)

DFT: $$c_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-inx_k} \qquad (18^*)$$

For DFT,
Required Number of Operations: $O(N^2)$

FFT: $O(N \log_2 N)$

$$e_{nk} = e^{-inx_k} = e^{-i2\pi nk/N} = \mathrm{w}^{nk}, \quad \mathrm{w} = \mathrm{w}_N = e^{-2\pi i/N} \qquad (19)$$

N=2M

$$\mathrm{w}_N^2 = \mathrm{w}_{2M}^2 = (e^{-2\pi i/N})^2 = e^{-4\pi i/N} = e^{-2\pi i/M} = \mathrm{w}_M$$

Split the vector $f = [f_0 \;\; f_1 \;\; f_2 \;\; \cdots \;\; f_{N-1}]'$ into two vectors f_{ev} and f_{od} with M=N/2 components each.

$$f_{ev} = [f_0 \;\; f_2 \;\; \cdots \;\; f_{N-2}]'$$
$$f_{od} = [f_1 \;\; f_3 \;\; \cdots \;\; f_{N-1}]'$$

$$f_{ev} = [f_0 \ f_2 \ \cdots \ f_{N-2}]'$$

$$\xrightarrow{\text{DFT}} \hat{f}_{ev} = [\hat{f}_{ev,0} \ \hat{f}_{ev,1} \ \cdots \ \hat{f}_{ev,M-1}]' = F_M f_{ev}$$

$$f_{od} = [f_1 \ f_3 \ \cdots \ f_{N-1}]'$$

$$\xrightarrow{\text{DFT}} \hat{f}_{od} = [\hat{f}_{od,0} \ \hat{f}_{od,1} \ \cdots \ \hat{f}_{od,M-1}]' = F_M f_{od}$$

$$\Downarrow$$

$$(22) \quad \begin{aligned} &(a) \quad \hat{f}_n = \hat{f}_{ev,n} + \omega_M^n \hat{f}_{od,n} \qquad n = 0, \cdots, M-1 \\ &(b) \quad \hat{f}_{n+M} = \hat{f}_{ev,n} - \omega_M^n \hat{f}_{od,n} \qquad n = 0, \cdots, M-1 \end{aligned}$$

Proof of (22)

$$\hat{f}_n = \sum_{k=0}^{N-1} f_k \omega_N^{nk} \quad (18)(19)$$

$$= \sum_{k=0}^{M-1} f_{2k} \omega_N^{n2k} + \sum_{k=0}^{M-1} f_{2k+1} \omega_N^{n(2k+1)}$$

$$= \sum_{k=0}^{M-1} f_{2k} \omega_M^{nk} + \omega_N^n \sum_{k=0}^{M-1} f_{2k+1} \omega_M^{nk}, \quad (\because \omega_N^2 = \omega_M)$$

$$(23) = (22a) \quad \hat{f}_n = \sum_{k=0}^{M-1} f_{2k} \omega_M^{nk} + \omega_N^n \sum_{k=0}^{M-1} f_{2k+1} \omega_M^{nk}$$

$$(22b) \quad \hat{f}_{n+M} = \sum_{k=0}^{M-1} f_{2k} \omega_M^{(n+M)k} + \omega_N^{(n+M)} \sum_{k=0}^{M-1} f_{2k+1} \omega_M^{(n+M)k}$$

$$= \sum_{k=0}^{M-1} f_{2k} \omega_M^{nk} - \omega_N^n \sum_{k=0}^{M-1} f_{2k+1} \omega_M^{nk} \quad (\because \omega_N^M = -1)$$

EXAMPLE 5 Fast Fourier Transform(FFT), N=4

Find the FFT and DFT for the signal $(f_0 \ \ f_1 \ \ f_2 \ \ f_3)$

Sol.

$$(22a) \quad \hat{f}_n = \sum_{k=0}^{M-1} f_{2k}\omega_M^{nk} + \omega_N^n \sum_{k=0}^{M-1} f_{2k+1}\omega_M^{nk}$$

$$\hat{f}_0 = \hat{f}_{ev,0} + \omega_N^0 \hat{f}_{od,0} = (f_0 + f_2) + (f_1 + f_3)$$

$$\hat{f}_1 = \hat{f}_{ev,1} + \omega_N^1 \hat{f}_{od,1} = (f_0 - f_2) - i(f_1 - f_3)$$

$$(22b) \quad \hat{f}_{n+M} = \sum_{k=0}^{M-1} f_{2k}\omega_M^{nk} - \omega_N^n \sum_{k=0}^{M-1} f_{2k+1}\omega_M^{nk}$$

$$\hat{f}_2 = \hat{f}_{ev,0} - \omega_N^0 \hat{f}_{od,0} = (f_0 + f_2) - (f_1 + f_3)$$

$$\hat{f}_3 = \hat{f}_{ev,1} - \omega_N^1 \hat{f}_{od,1} = (f_0 - f_2) - (-i)(f_1 - f_3)$$

1. By FFT

$$(f_0 \quad f_1 \quad f_2 \quad f_3)$$

$$f_{ev} = (f_0 \quad f_2) \qquad\qquad f_{od} = (f_1 \quad f_3)$$

DFT

$$\hat{f}_n = \sum_{k=0}^{N-1} f_k e^{-in2\pi k/N} \qquad (18)$$

$$\hat{f}_{ev,0} = f_0 + f_2 \qquad\qquad \hat{f}_{od,0} = f_1 + f_3$$

$$\hat{f}_{ev,1} = f_0 - f_2 \qquad\qquad \hat{f}_{od,1} = f_1 - f_3$$

$$\hat{f}_{ev,0} = f_0 + f_2 \qquad \hat{f}_{od,0} = f_1 + f_3$$
$$\hat{f}_{ev,1} = f_0 - f_2 \qquad \hat{f}_{od,1} = f_1 - f_3$$

$$\hat{f}_0 = \hat{f}_{ev,0} + \omega_N^0 \hat{f}_{od,0} = (f_0 + f_2) + (f_1 + f_3)$$
$$\hat{f}_1 = \hat{f}_{ev,1} + \omega_N^1 \hat{f}_{od,1} = (f_0 - f_2) - i(f_1 - f_3)$$
$$\hat{f}_2 = \hat{f}_{ev,0} - \omega_N^0 \hat{f}_{od,0} = (f_0 + f_2) - (f_1 + f_3)$$
$$\hat{f}_3 = \hat{f}_{ev,1} - \omega_N^1 \hat{f}_{od,1} = (f_0 - f_2) - (-i)(f_1 - f_3)$$

2. By DFT

$$w = w_4 = e^{-2\pi i/4} = -i$$

$$F_4 = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & (-i)^2 & (-i)^3 \\ 1 & (-i)^2 & [(-i)^2]^2 & [(-i)^2]^3 \\ 1 & (-i)^3 & [(-i)^3]^2 & [(-i)^3]^3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

$$\hat{f} = F_N f$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_0 + f_1 + f_2 + f_3 \\ f_0 - i f_1 - f_2 + i f_3 \\ f_0 - f_1 + f_2 - f_3 \\ f_0 + i f_1 - f_2 - i f_3 \end{bmatrix}$$

Table I. Fourier Cosine Transforms

$f(x)$	$\hat{f}_c(w) = \mathcal{F}_c(f)$
$\begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\,\frac{\sin aw}{w}$
$x^{a-1} \quad (0 < a < 1)$	$\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(a)}{w^a}\cos\frac{a\pi}{2}$
$e^{-ax} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2 + w^2}\right)$
$e^{-x^2/2}$	$e^{-w^2/2}$
$e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}}\,e^{-w^2/(4a)}$
$x^n e^{-ax} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}}\,\frac{n!}{(a^2 + w^2)^{n+1}}\,\mathrm{Re}\,(a + iw)^{n+1}$
$\begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{\sqrt{2\pi}}\left[\frac{\sin a(1 - w)}{1 - w} + \frac{\sin a(1 + w)}{1 + w}\right]$
$\cos(ax^2) \quad (a > 0)$	$\frac{1}{\sqrt{2a}}\cos\left(\frac{w^2}{4a} - \frac{\pi}{4}\right)$
$\sin(ax^2) \quad (a > 0)$	$\frac{1}{\sqrt{2a}}\cos\left(\frac{w^2}{4a} + \frac{\pi}{4}\right)$
$\frac{\sin ax}{x} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}}\,(1 - u(w - a))$
$\frac{e^{-x}\sin x}{x}$	$\frac{1}{\sqrt{2\pi}}\arctan\frac{2}{w^2}$
$J_0(ax) \quad (a > 0)$	$\sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{a^2 - w^2}}\,(1 - u(w - a))$

Table II. Fourier Sine Transforms

$f(x)$	$\hat{f}_s(w) = \mathcal{F}_s(f)$	$f(x)$	$\hat{f}_s(w) = \mathcal{F}_s(f)$
$\begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos aw}{w}\right]$	$x^n e^{-ax} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}}\frac{n!}{(a^2+w^2)^{n+1}}\operatorname{Im}(a+iw)^{n+1}$
$1/\sqrt{x}$	$1/\sqrt{w}$	$xe^{-x^2/2}$	$we^{-w^2/2}$
$1/x^{3/2}$	$2\sqrt{w}$	$xe^{-ax^2} \quad (a > 0)$	$\frac{w}{(2a)^{3/2}}e^{-w^2/4a}$
$x^{a-1} \quad (0 < a < 1)$	$\sqrt{\frac{2}{\pi}}\frac{\Gamma(a)}{w^a}\sin\frac{a\pi}{2}$	$\begin{cases} \sin x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{\sqrt{2\pi}}\left[\frac{\sin a(1-w)}{1-w} - \frac{\sin a(1+w)}{1+w}\right]$
$e^{-ax} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}}\left(\frac{w}{a^2+w^2}\right)$	$\frac{\cos ax}{x} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}}\,u(w-a)$
$\frac{e^{-ax}}{x} \quad (a > 0)$	$\sqrt{\frac{2}{\pi}}\arctan\frac{w}{a}$	$\arctan\frac{2a}{x} \quad (a > 0)$	$\sqrt{2\pi}\,\frac{\sin aw}{w}e^{-aw}$

Table III. Fourier Transforms

$f(x)$	$\hat{f}(w) = \mathcal{F}(f)$	$f(x)$	$\hat{f}(w) = \mathcal{F}(f)$
$\begin{cases} 1 & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin bw}{w}$	$\begin{cases} e^{ax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{(a-iw)c} - e^{(a-iw)b}}{\sqrt{2\pi}(a - iw)}$
$\begin{cases} 1 & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{-ibw} - e^{-icw}}{iw\sqrt{2\pi}}$	$\begin{cases} e^{iax} & \text{if } -b < x < b \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin b(w - a)}{w - a}$
$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert w\rvert}}{a}$	$\begin{cases} e^{iax} & \text{if } b < x < c \\ 0 & \text{otherwise} \end{cases}$	$\frac{i}{\sqrt{2\pi}} \frac{e^{ib(a-w)} - e^{ic(a-w)}}{a - w}$
$\begin{cases} x & \text{if } 0 < x < b \\ 2x - b & \text{if } b < x < 2b \\ 0 & \text{otherwise} \end{cases}$	$\frac{-1 + 2e^{ibw} - e^{-2ibw}}{\sqrt{2\pi}w^2}$	$e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}} e^{-w^2/4a}$
$\begin{cases} e^{-ax} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} \quad (a > 0)$	$\frac{1}{\sqrt{2\pi}(a + iw)}$	$\frac{\sin ax}{x} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}}$ if $\lvert w\rvert < a$; 0 if $\lvert w\rvert > a$

SUMMARY OF CHAPTER 11

Fourier Series(period=2π)

(1*) $$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad n = 1, 2, \cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad n = 1, 2, \cdots$$

SUMMARY OF CHAPTER 11-cont

Fourier Series(period=2L)

(1)
$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

SUMMARY OF CHAPTER 11-cont

(1) $$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

If f(x) is even,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{0}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 0$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

SUMMARY OF CHAPTER 11-cont

(1) $$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

If f(x) is odd,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

SUMMARY OF CHAPTER 11-cont

Fourier Integral:

(3) $$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega$$

(4) $$where \quad A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv$$

Fourier Integral in Complex Form:

SUMMARY OF CHAPTER 11-cont

Complex Fourier Integral:

(5) $$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)e^{i\omega(x-v)})dv\right]d\omega$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)\,e^{i\omega x}\,d\omega$$

where

(6) $$\hat{f}(\omega) = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-i\omega v}dv\right]$$

SUMMARY OF CHAPTER 11-cont

Fourier Transform:

(6) $$\hat{f}(\omega) = \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-i\omega v} dv \right]$$

Inverse Fourier Transform:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x} d\omega$$

SUMMARY OF CHAPTER 11-cont

Fourier Cosine Transform:

(7) $$\mathcal{F}_c(f) = \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx$$

Inverse Fourier Cosine Transform:

$$f(x) = \mathcal{F}_c^{-1}[\hat{f}_c(\omega)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos \omega x \, d\omega$$

SUMMARY OF CHAPTER 11-cont

Fourier Sine Transform:

(8) $$\mathcal{F}_S(f) = \hat{f}_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx$$

Inverse Fourier Sine Transform:

$$f(x) = \mathcal{F}_S^{-1}[\hat{f}_S(\omega)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_S(\omega) \sin \omega x \, d\omega$$

SUMMARY OF CHAPTER 11-cont

Discrete Fourier Transform:

$$f = [f_0 \quad \cdots \quad f_{N-1}]^T \xrightarrow{\text{DFT}} \hat{f} = [\hat{f}_0 \quad \cdots \quad \hat{f}_{N-1}]^T$$

$$\hat{f}_n = Nc_n = \sum_{k=0}^{N-1} f_k e^{-inx_k} = \sum_{k=0}^{N-1} f_k e^{-in2\pi k/N} \quad (18)$$

In matrix form,

$$\hat{f} = \begin{bmatrix} \hat{f}_0 \\ \vdots \\ \hat{f}_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & \end{bmatrix} \begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix} = F_N f$$

$$F_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \mathrm{w}_N^1 & \mathrm{w}_N^2 & \cdots & \mathrm{w}_N^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \mathrm{w}_N^{N-1} & [\mathrm{w}_N^{N-1}]^2 & \cdots & [\mathrm{w}_N^{N-1}]^{N-1} \end{bmatrix}, \quad \mathrm{w} = \mathrm{w}_N = e^{-2\pi i/N}$$

SUMMARY OF CHAPTER 11-cont

Fast Fourier Transform:

$$f = [f_0 \quad f_1 \quad f_2 \; f_3 \; \cdots \; f_{N-1}]$$

$$f_{ev} = [f_0 \;\; f_2 \;\; \cdots \;\; f_{N-2}] \qquad f_{od} = [f_1 \;\; f_3 \;\; \cdots \;\; f_{N-1}]$$

DFT

$$\hat{f}_n = \sum_{k=0}^{N-1} f_k e^{-in2\pi k/N} \qquad (18)$$

$$(22) \quad \begin{aligned} &(a) \quad \hat{f}_n = \hat{f}_{ev,n} + \omega_M^n \hat{f}_{od,n} \quad n = 0, \cdots, M-1 \\ &(b) \quad \hat{f}_{n+M} = \hat{f}_{ev,n} - \omega_M^n \hat{f}_{od,n} \quad n = 0, \cdots, M-1 \end{aligned}$$

Homework for Chapter 11

11.1 16, 18

11.2 24, 26

11.7 7, 10

11.8 1, 9

11.9 2,7

Due: Nov. 16

Send your solution by email to twjeong@jbnu.ac.kr

File name of your solution: AEM2_your-name_Ch11