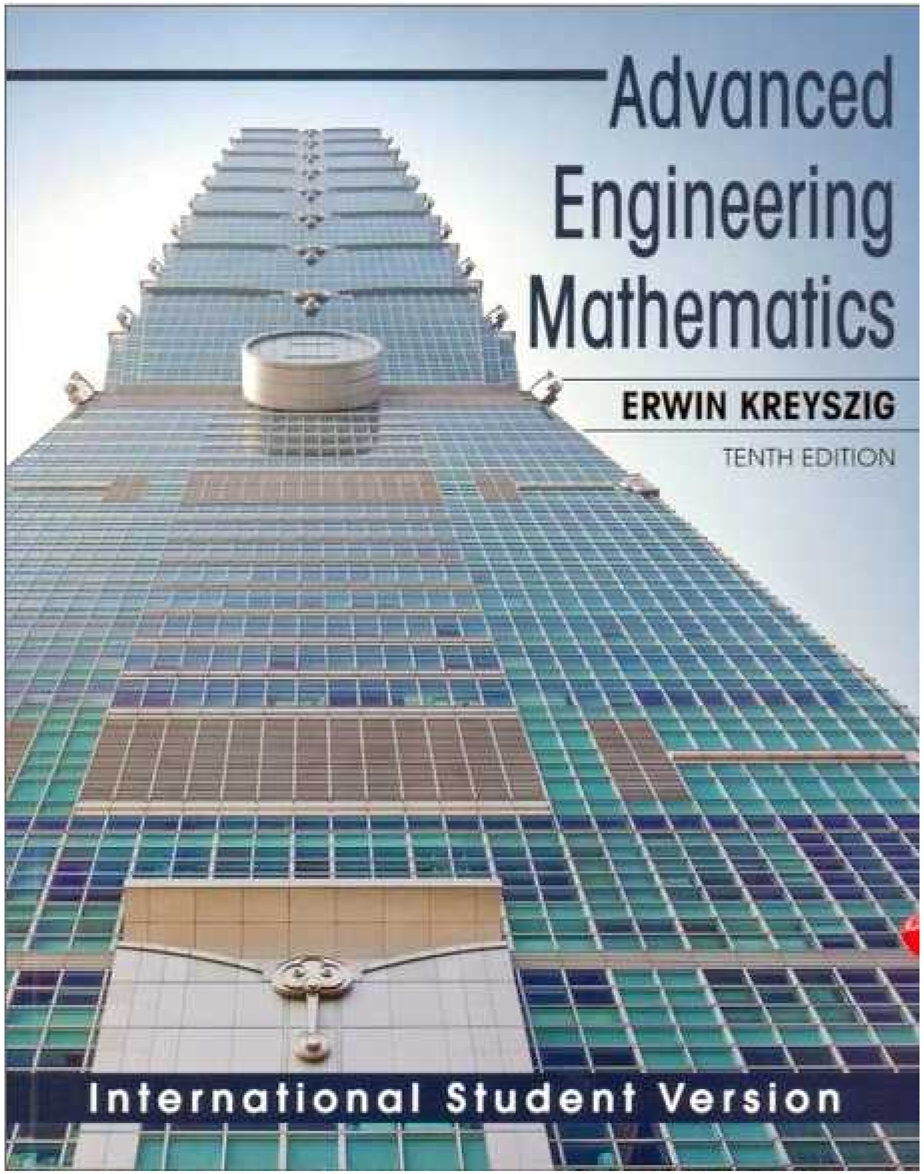
Advanced
Engineering
Mathematics
ERWIN KREYSZIG
TENTH EDITION
International Student Version


ERWIN KREYSZIG
Advanced Engineering Mathematics, Tenth Edition
KREYSZIG
공업수학
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Fourier 해석 · 편미분방정식 · 복소해석
수치해석 · 최적화, 그래프
WILEY

Engineering Mathematics 2

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Classes: 16:00-17:50 Mon, 16:00-16:50 Wed
Classroom: Inmoon2-505

On-line class will be provided until notified otherwise because of COVID-19 virus.

Evaluation: Midterm(35%), Final(45%),
Homework(10%), Attendance(10%)

Attendance will be checked only by the download of class material at ieilms_old.jbnu.ac.kr.
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If not, your class attendance may be regarded as absence.

Tests: Midterm(10.25), Final(12.13), Open material

Advanced Engineering Mathematics 2

Covered Topics

Textbook: Advanced Engineering Mathematics, 10th Ed.,
Erwin Kreyszig, Wiley(범한서적)

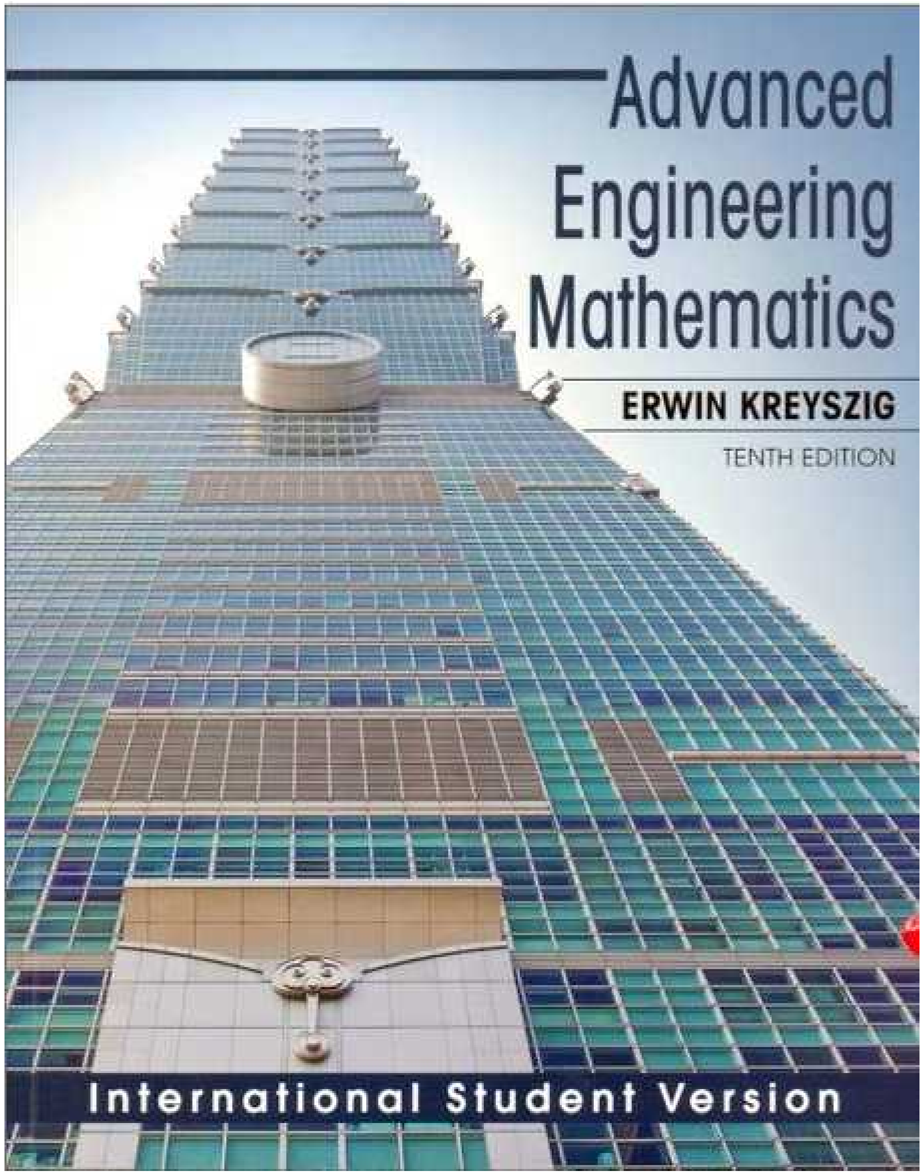
Ch. 11 Fourier Analysis

Ch. 13 Complex Numbers and Functions.
Complex Differentiation

Ch. 14 Complex Integration

Ch. 15 Power Series. Taylor Series

Ch. 16 Laurent Series. Residue Integration



CHAPTER 11

Fourier Analysis

11.1 Fourier Series

11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions

11.3 Forced Oscillations

11.4 Approximation by Trigonometric Functions

11.5 Sturm-Liouville Problems. Orthogonal Functions

11.6 Orthogonal Series. Generalized Fourier Series

11.7 Fourier Integral

11.8 Fourier Cosine and Sine Transforms

11.9 Fourier Transform. Discrete and Fast Fourier Transform

11.10 Tables of Transforms

Ch. 11 Fourier Analysis

This chapter on Fourier analysis covers three broad areas:

- Fourier series
- More general series called Sturm-Liouville expansion
- Fourier integrals and transforms

Fourier series

- Infinite series designed to represent general periodic functions in terms of simple functions, namely, cosines and sines
- Very important to the engineer and physicist because they allow the solution of ODEs in connection with forced oscillations and approximation of periodic functions
- More universal than Taylor series expansions because many discontinuous periodic functions can be developed in Fourier series but do not have Taylor series expansions

11.1 Fourier Series(푸리에 급수)

Periodic function f(x)

- $f(x)$ is defined for all real x (perhaps except at some points)
- There is some positive number p, such that $f(x+p) = f(x)$ for all x.
- p : A **period** of $f(x)$

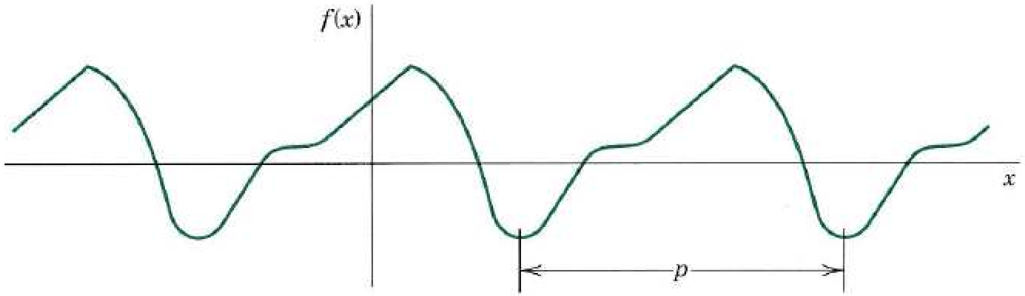


Fig 258 Periodic function of period p

- If $f(x)$ has period p, for any integer n, then $f(x+np) = f(x)$ for all x.

- If $f(x)$ and $g(x)$ have period p, then $af(x)+bg(x)$ with any constants and b also has the period p.

Examples of functions that are not periodic: $x, x^2, x^3,$ cosh x, ln x
Familiar periodic functions : cosine and sine

Trigonometric system:

(3) 1, cos$x$, sin$x$, cos$2x$, sin$2x$, …., cosnx, sinnx, …

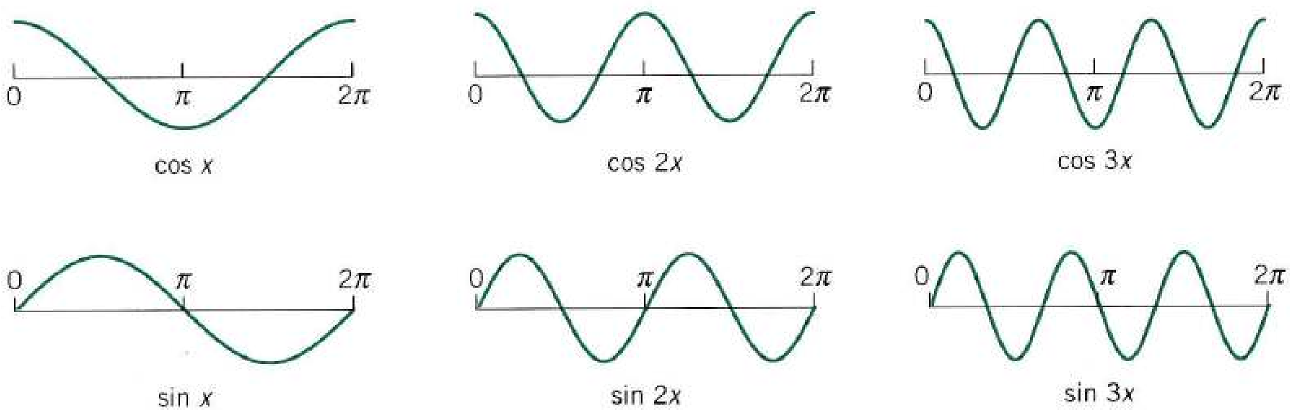


Fig. 259

Trigonometric series:

$$(4) \qquad a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + a_3 \cos 3x + b_3 \sin 3x + \cdots$$
$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Coefficients of the series: $a_0, \ a_1, \ b_1, \ a_2, \ b_2, \cdots$

If the series (4) converges, the series has the period of 2π.

If f(x) is a function of period 2π, then the function can be represented by the series (4).

$$(5) \qquad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$(5) \qquad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

The coefficients, called Fourier coefficients of f(x), are given by the Euler formula.

$$(0) \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$(6) \qquad (a) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx \quad n = 1, 2, \cdots$$

$$(b) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx \quad n = 1, 2, \cdots$$

Ex. 1 Periodic Rectangular Wave(Fig. 260)

Find the Fourier coefficients of the following periodic function.

$$(7) \qquad f(x) = \begin{cases} -k & if \quad -\pi < x < 0 \\ k & if \quad 0 < x < \pi \end{cases} \quad and \quad f(x+2\pi) = f(x)$$

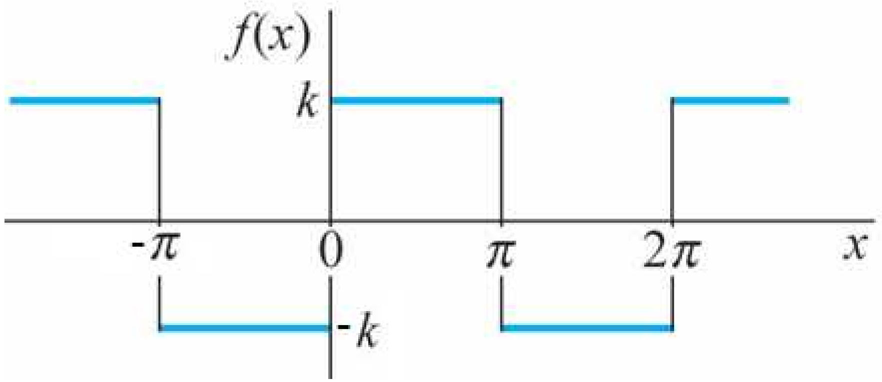
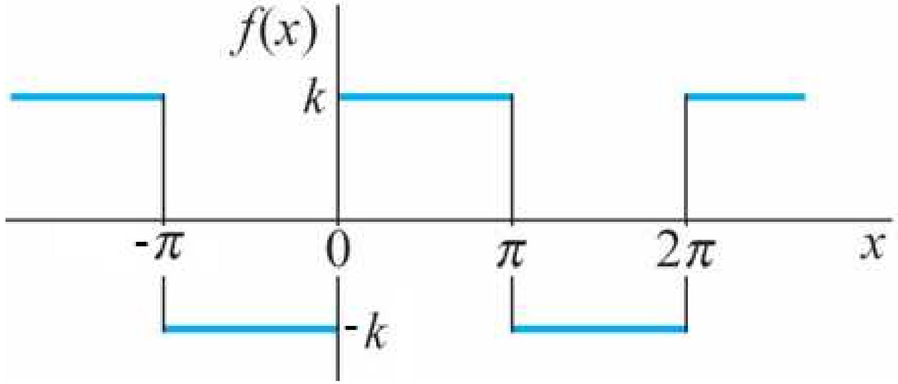


Fig. 260

Sol.

Find a_0, a_n, and b_n by the Euler formula given by (6).

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-k)\,dx + \int_{0}^{\pi} k\,dx\right]$$
$$= \frac{1}{2\pi}\left[-kx\big|_{-\pi}^{0} + kx\big|_{0}^{\pi}\right] = 0$$



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0\left[\because f(x)\cos nx \;\; is\;\; an \;\; odd\;\; function\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\cdot 2\int_{0}^{\pi} f(x)\sin nx\,dx$$
$$= \frac{2}{\pi}k\frac{\cos nx}{-n}\bigg|_{0}^{\pi} = \frac{2k}{n\pi}(1-\cos n\pi) = \begin{cases} 4k/(n\pi), & for\;\; odd\;\; n \\ 0, & for\;\; even\;\; n \end{cases}$$

$$(5)\quad f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= 0 + \sum_{n=1}^{\infty}\left[0 \cdot \cos nx + \frac{2k}{n\pi}(1 - \cos n\pi)\sin nx\right]$$

$$= \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

Partial sum(Fourier Polynomial):

$$S_1 = \frac{4k}{\pi}\sin x$$

$$S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right)$$

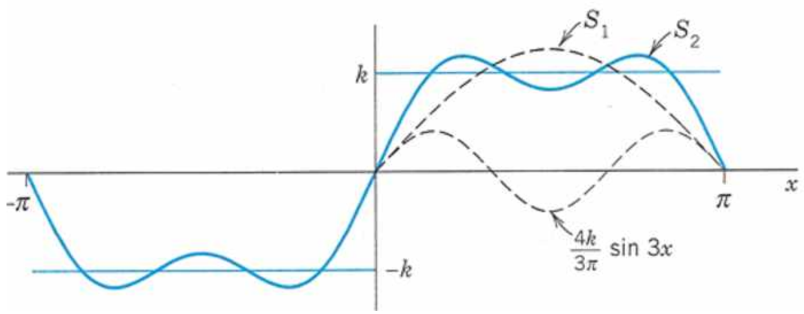
$$S_3 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right)$$

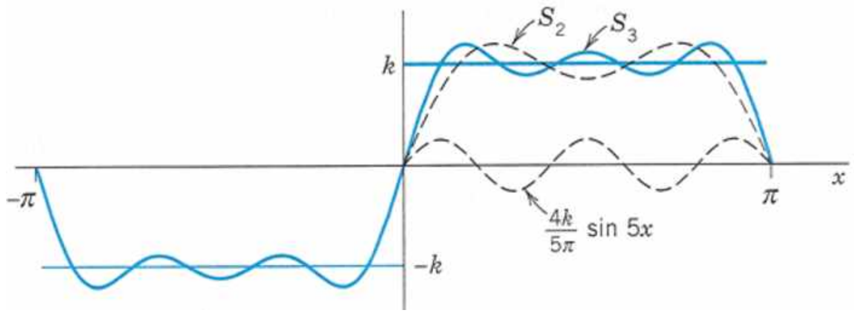
Partial sum:

$$S_1 = \frac{4k}{\pi}\sin x$$

$$S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right)$$

$$S_3 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right)$$





The value of π by Leibniz in 1673.

$$(5)\quad f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right)$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - + \cdots$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right)$$

THEOREM 1 Orthogonality of the Trigonometric System (3)

The trigonometric system (3) is orthogonal on the interval $-\pi \leq x \leq \pi$ (hence on any other interval of length 2π); that is

$$
\begin{aligned}
&(a)\quad \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \qquad n \neq m \\
(9)\quad &(b)\quad \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \qquad n \neq m \\
&(c)\quad \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \qquad n \neq m \;\; or \;\; n = m
\end{aligned}
$$

PROOF

Functions f_1 and f_2 are orthogonal to each other on [a, b] iff

$$\int_a^b f_1(x) f_2(x) \, dx = 0$$

Formula in the Appendix App.3 (11):

$$\sin x \sin y = (-1/2)[\cos(x+y) - \cos(x-y)]$$
$$\cos x \cos y = (1/2)[\cos(x+y) + \cos(x-y)]$$
$$\sin x \cos y = (1/2)[\sin(x+y) + \sin(x-y)]$$

$$(a)\ \int_{-\pi}^{\pi} \cos nx \cos mx \, dx \qquad n \neq m$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(n+m)x + \cos(n-m)x]\, dx$$

$$= \frac{1}{2}\left[\frac{\sin(n+m)x}{n+m} + \frac{\sin(n-m)x}{n-m}\right]_{-\pi}^{\pi} = 0$$

$$(b)\ \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \qquad n \neq m$$

$$= -\frac{1}{2}\int_{-\pi}^{\pi} [\cos(n+m)x - \cos(n-m)x]\, dx$$

$$= -\frac{1}{2}\left[\frac{\sin(n+m)x}{n+m} - \frac{\sin(n-m)x}{n-m}\right]_{-\pi}^{\pi} = 0$$

$$(c)\ \int_{-\pi}^{\pi} \sin nx \cos mx \, dx \qquad n \neq m$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\sin(n+m)x + \sin(n-m)x]\, dx$$

$$= -\frac{1}{2}\left[\frac{\cos(n+m)x}{n+m} + \frac{\cos(n-m)x}{n-m}\right]_{-\pi}^{\pi} = 0$$

Application of Theorem 1 to the Fourier Series (5)

The Fourier coefficient a_0:

$$(5)\quad f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]dx$$

Assume termwise integration

$$= a_0\int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi} \cos nx dx + b_n\int_{-\pi}^{\pi} \sin nx dx\right)$$

$$= 2\pi a_0$$

$$\therefore a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \qquad (6.0)$$

The Fourier coefficient a_m:

$$(5)\quad f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$(10)\quad \int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cos mx dx$$

$$= a_0 \cdot \int_{-\pi}^{\pi} \cos mx dx + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_n \cos nx \cos mx dx$$

$$+ \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} b_n \sin nx \cos mx dx$$

$$= a_0 \cdot 0 + a_m \int_{-\pi}^{\pi} \cos^2 mx dx + 0 = a_m \cdot 2\int_{0}^{\pi}\frac{1+\cos 2x}{2}dx = a_m \pi$$

$$\therefore a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx \qquad (6.a)$$

The Fourier coefficient b_m:

$$(5)\quad f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$(11)\quad \int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\sin mx dx$$

$$= \int_{-\pi}^{\pi} a_0 \sin mx dx + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} a_n \cos nx \sin mx dx$$

$$+ \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} b_n \sin nx \sin mx dx$$

$$= a_0 \cdot 0 + 0 + b_m \int_{-\pi}^{\pi} \sin^2 mx dx$$

$$= b_m \cdot 2\int_{0}^{\pi}\frac{1-\cos 2x}{2}dx = b_m \pi$$

$$\therefore b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx \qquad (6.b)$$

THEOREM 2 Representation by a Fourier Series

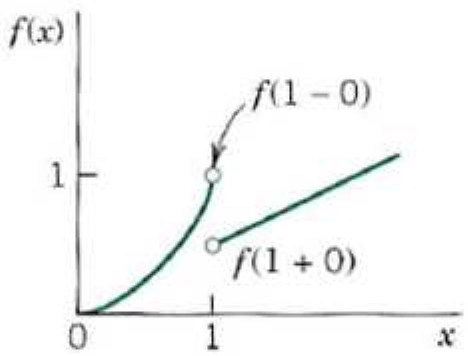
Let $f(x)$ be periodic with period 2π and piecewise continuous in the interval $-\pi \le x \le \pi$. Furthermore, let $f(x)$ have a left- and right-hand derivatives at each point of that interval. Then the Fourier series (5) of $f(x)$ with the coefficients in (6) converges. Its sum is $f(x)$, except at point x_0 where $f(x)$ is discontinuous. There the sum of the series is the average of left- and right-hand limits of $f(x)$ at x_0.

PROOF

Omitted!

Left- and Right-hand limits

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x/2 & \text{if } x \geqq 1 \end{cases}$$



$$f(1-0) = 1,$$

$$f(1+0) = \tfrac{1}{2}$$

EX 2 Convergence at a Jump as Indicated in Theorem 2

Show that values of the Fourier series for the following function agree with Theorem 2.

$$(7)\qquad f(x)=\begin{cases}-k & if \quad -\pi<x<0\\ k & if \quad 0<x<\pi\end{cases} \quad and \quad f(x+2\pi)=f(x)$$

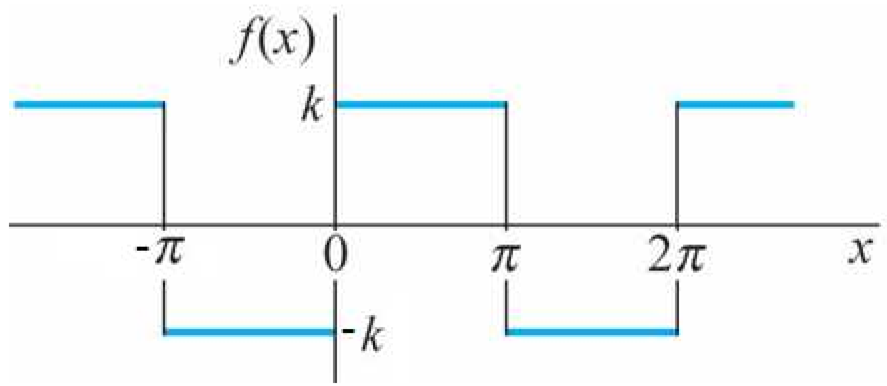


Fig. 260

Sol.

$$(5)\quad f(x)=\frac{4k}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\cdots\right)$$

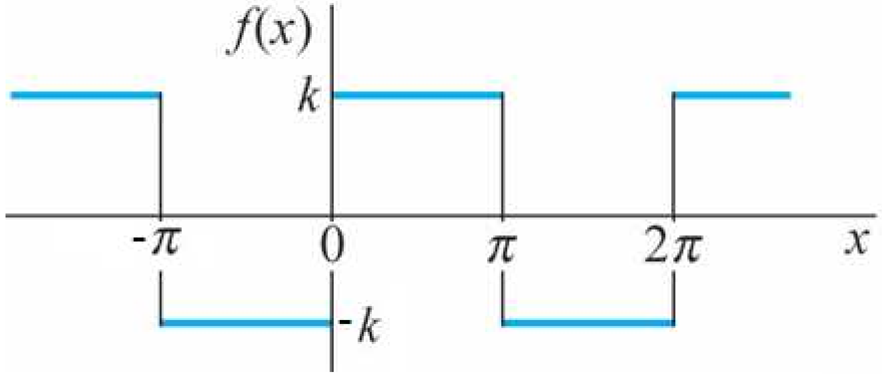


Fig. 260

$$(5)\quad f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

$$f(0) = \frac{4k}{\pi}(0+0+0+\cdots) = 0$$

$$\textit{The average of } f(0^-) \textit{ and } f(0^+) = \frac{k-k}{2} = 0$$

Similarly, the relation holds at ∓nπ.

The *fundamental period* is the smallest positive period. Find it for

1. $\cos x$, $\sin x$,
$\cos 2x$, $\sin 2x$,
$\cos \pi x$, $\sin \pi x$,
$\cos 2\pi x$, $\sin 2\pi x$

2. $\cos nx$, $\sin nx$,

$\cos \dfrac{2\pi x}{k}$, $\sin \dfrac{2\pi x}{k}$,

$\cos \dfrac{2\pi nx}{k}$, $\sin \dfrac{2\pi nx}{k}$

3. Linear combinations of periodic functions. Vector space. If $f(x)$ and $g(x)$ have period p, show that $h(x) = af(x) + bg(x)$ has the period p (a, b, constant). Thus all functions of period p form a **vector space**.

Proof. Assume $f(x)$ and $g(x)$ have period p. Then

(A) $\qquad f(x+p) = f(x), \qquad g(x+p) = g(x).$

Now

$$\begin{aligned} h(x+p) &= af(x+p) + bg(x+p) && \text{(by definition of } h\text{)} \\ &= af(x) + bg(x) && \text{[by (A)]} \\ &= h(x) && \text{(by definition of } h\text{).} \end{aligned}$$

(B) $\qquad h(x+p) = h(x).$

Sketch or graph $f(x)$ which for $-\pi < x < \pi$ is given as follows.

6. $f(x) = |x|$

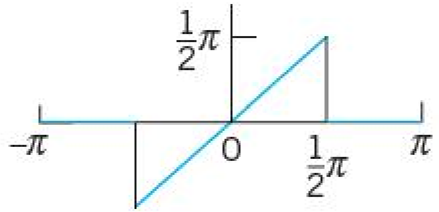
7. $f(x) = |\sin x|, \quad f(x) = \sin |x|$

8. $f(x) = e^{-|x|}, \quad f(x) = |e^{-x}|$

9. $$f(x) = \begin{cases} x & \text{if} \quad -\pi < x < 0 \\ \pi - x & \text{if} \quad 0 < x < \pi \end{cases}$$

10. $$f(x) = \begin{cases} -\cos^2 x & \text{if} \quad -\pi < x < 0 \\ \cos^2 x & \text{if} \quad 0 < x < \pi \end{cases}$$

16 Find the Fourier series of the following function, which is assumed to have the period 2π.



Sol.

(5) $$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

(0) $$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

(6) (a) $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad n = 1, 2, \cdots$$

(b) $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad n = 1, 2, \cdots$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \; dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad n = 1,2,\cdots$$

$$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,\cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad n = 1,2,\cdots$$

$$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} x\,\sin nx\,dx$$

formula for integration by parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

formula for integration by parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx \qquad \int u\,dv = uv - \int v\,du$$

$$\int x\sin nx\,dx = x\frac{-\cos nx}{n} - \int 1\cdot\frac{-\cos nx}{n}dx$$

$$= -\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\,\sin nx dx$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\frac{\pi}{2}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]_0^{\frac{\pi}{2}} = \begin{cases} \dfrac{2\sin\frac{n\pi}{2}}{n^2\pi} & (n:\text{ odd }) \\[2ex] \dfrac{-\cos\frac{n\pi}{2}}{n} & (n:\text{ even }) \end{cases}$$

(5) $$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = 0 \qquad a_n = 0$$

$$b_n = \begin{cases} \dfrac{2\sin\frac{n\pi}{2}}{n^2\pi} & (n: \text{odd}) \\ \dfrac{-\cos\frac{n\pi}{2}}{n} & (n: \text{even}) \end{cases}$$

$$= \frac{2}{\pi}\left[\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - \cdots\right]$$

$$+ \frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x + \frac{1}{6}\sin 6x - \cdots$$