Introduction to Discrete Math

Felipe P. Vista IV

Course Outline

- Mathematical Thinking
  - Convincing Arguments, Find Example, **Optimality**, Recursion & Induction, Logic, Invariants
- Probability & Combinatronics
  - Basic Counting, Binomial Coeff, Advanced Counting, Probability, **Random Variables**

Mathematical Thinking – How to Find an Example

OPTIMALITY

- **Warm-up: Producing Chocolate**
- Subset without x and $100 - x$
- Rooks on a Chessboard
- Knights on a Chessboard
- Bishops on a Chessboard
- Subset without x and $2x$

Maximizing Profit

Situation: A factory produces milk chocolate ($10 per box) and dark chocolate ($30 per box). The daily demands are 500 and 200 boxes for milk and dark chocolate, respectively. The factory produces 600 boxes of chocolate per day.

- What is the optimum production plan?

Consulting

- A consulting company claims that the maximum profit per day is $10 000
- How can they convince the factory that this profit is indeed optimal (i.e., maximum)?
- They need to show two things:
  1. *Existential statement:*
     There exists a plan achieving profit of $10 000
  2. *Universal statement:*
     All plans give profit at most $10 000

Existential Production Plan

- **Plan**: produce 400 boxes of milk chocolate and 200 boxes of dark chocolate per day
  - It is indeed a valid plan:
    - no more than 500 boxes of milk chocolate
    - no more than 200 boxes of dark chocolate
    - no more than 600 boxes
- Profit: $400 \times 10 + 200 \times 30 =$ **10 000**

Universal Part: All Plans are Not Better

- Let M and D be the number of boxes of milk and dark chocolate, respectively, produced per day
  - $M \le 500$, $D \le 200$, $M + D \le 600$ – volume / qty
- Want to show that $10M + 30D \le 10\,000$ / [illegible]
- Sum up the inequalities
  - $10M + 10D \le 6\,000$
  - $20D \le 4\,000$

Summary

- A proof of the fact that some value **α** is optimal usually consists of two parts:
  1. *Existential statement:* Plan x = α
     There exists a solution achieving the value **α**
  2. *Universal statement:* All plan ≤ α
     All solutions achieve the value not greater than **α**
- In this lesson, we'll see several proofs of optimality, following the same pattern

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Problem

- What is the maximum number of two-digit integers (10, 11, … , 99) that one can select, if it is not allowed to select simultaneously x and y such that $x + y = 100$?

$41 + 59 = 100$

Solving the Problem

- If one takes 10, then one cannot take 90, and vice versa
  - Similarly, 11 and 89 cannot be taken simultaneously
  - 40 pairs: (10, 90), (11, 89), . . . , (49, 51)
  - 10 numbers without pairs: 50, 91, 92, … , 99
- Definitely no more than 40 + 10 = 50 numbers 45
- Optimal solution: 50, ~~51~~, … , 99 ES

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Once Again, Formally

Theorem: The maximum number of two-digit numbers such that no two of them sum up to 100 is 50.

Proof:

- Solution of size 50:
  50, 51, … , 99 (the sum of any two is greater than 100)
- Any solution should exclude at least one number from each of the 40 pairs. Hence, at most $90 - 40 = 50$ numbers

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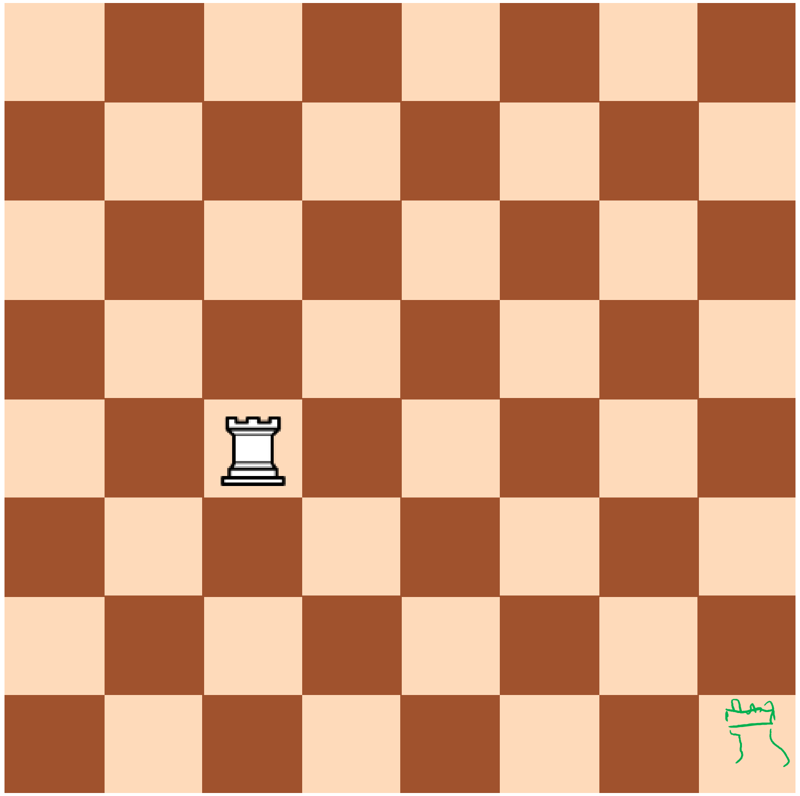
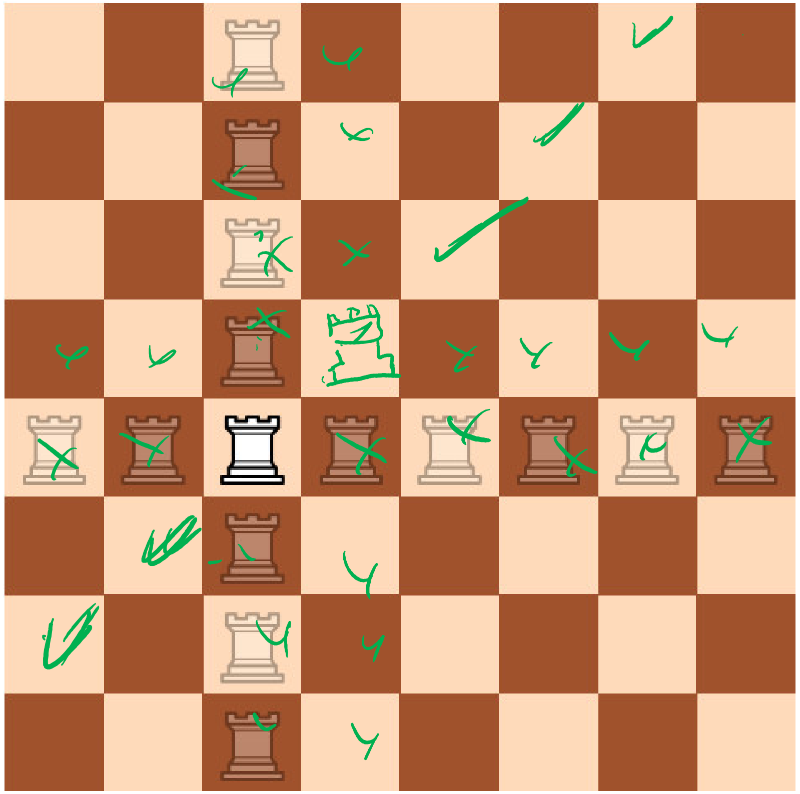
Chess Rook



A chess rook moves:

- horizontally
- diagonally

Chess Rook



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A chess rook moves:

- horizontally
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Maximum Number of Rooks

Problem: What is the maximum number of rooks on a chessboard such that no two attack each other?

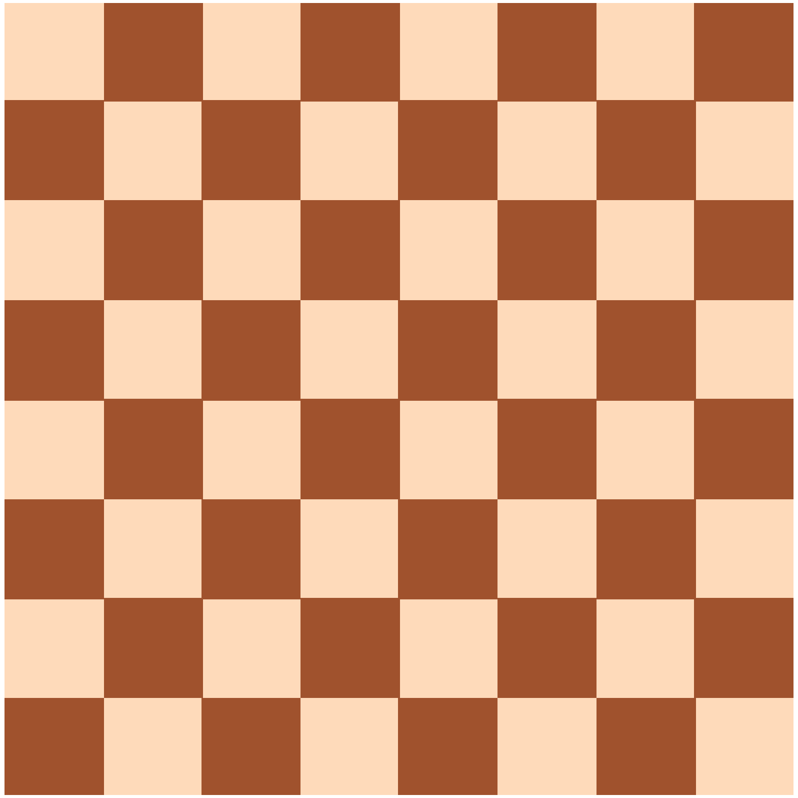
Recall: The Pigeonhole Principle

If n pigeons are placed into m boxes and $m < n$, then there is a box containing more than one pigeon

Solving the Problem

- There should be at most one rook in each row
- Hence, the total number of rooks is at most 8
- In other words, if the number of rooks is greater than the number of rows, then, by the pigeon hole principle, there is a row containing at least two rooks (and these two rooks attack each other)

Solution



- Placing 8 rooks is not difficult

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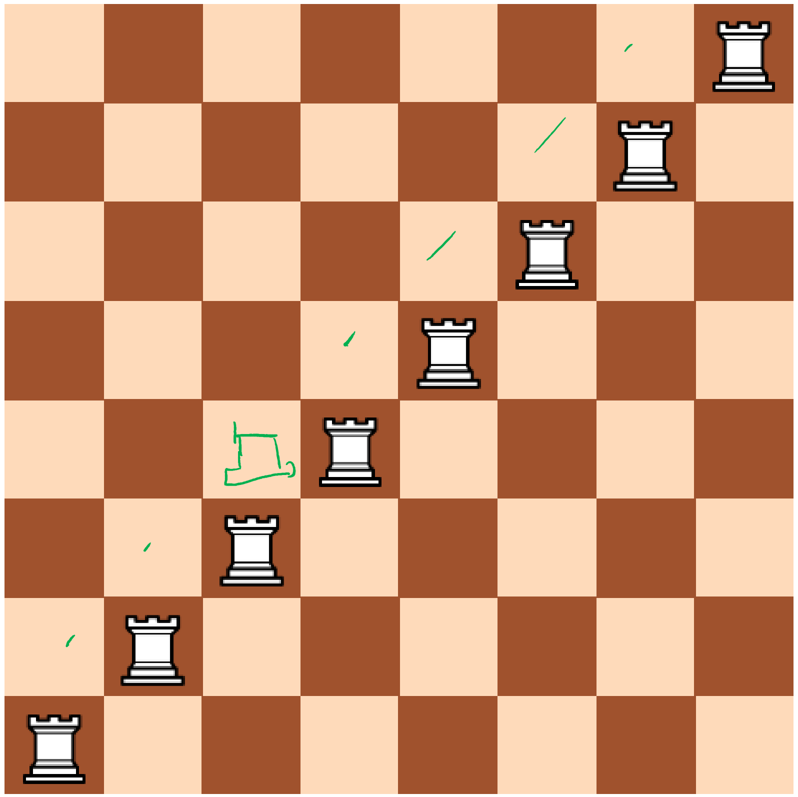

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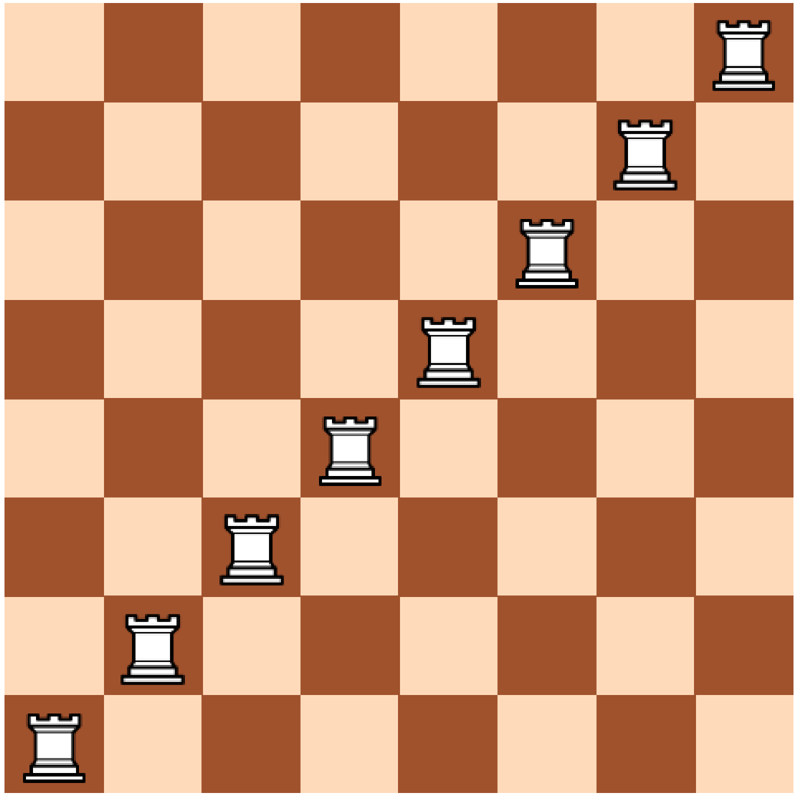

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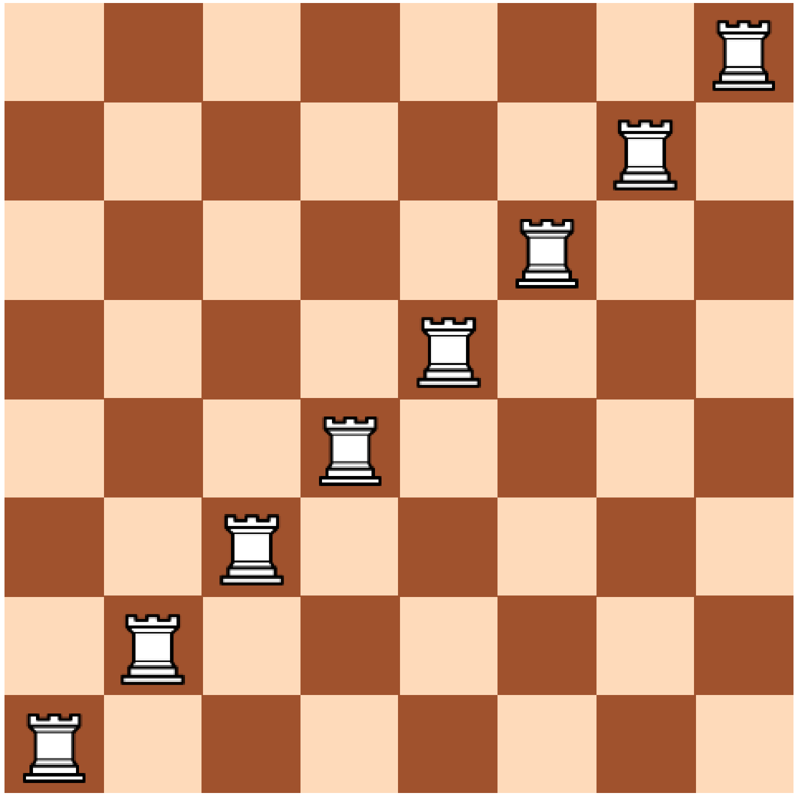

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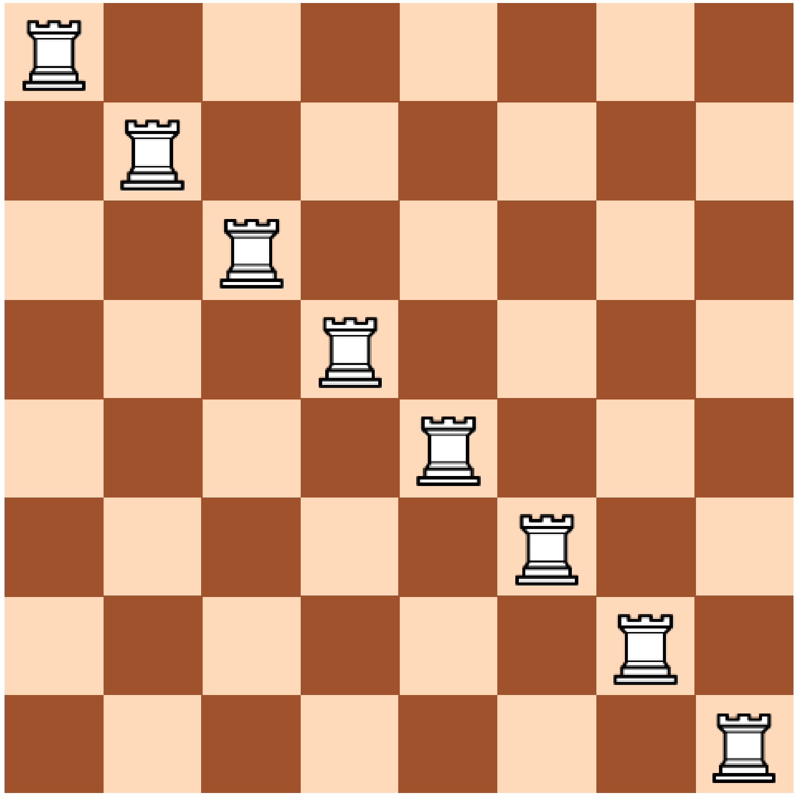

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- **Knights on a Chessboard**
- Bishops on a Chessboard
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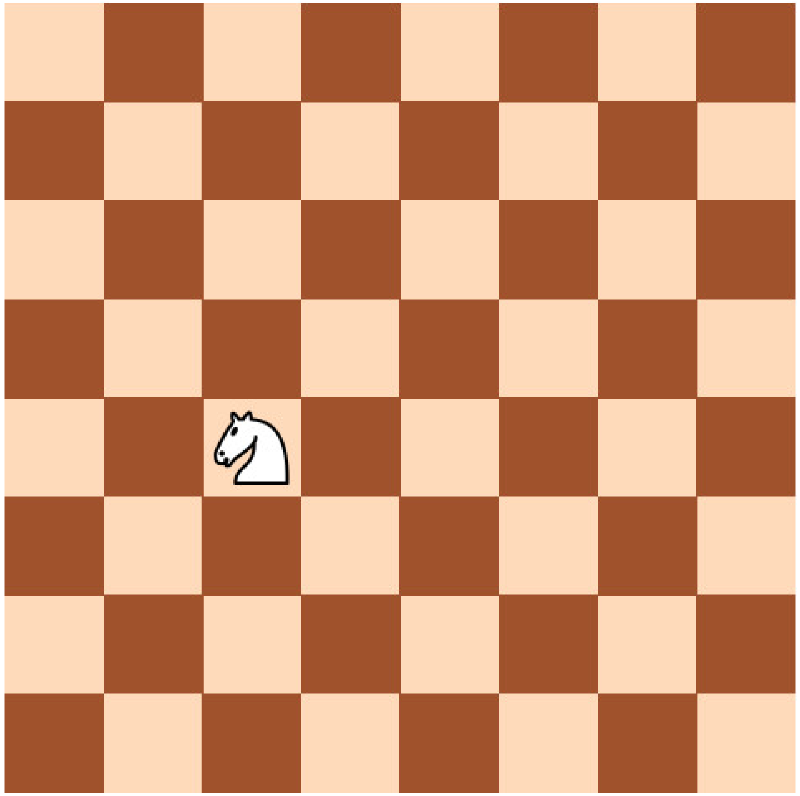
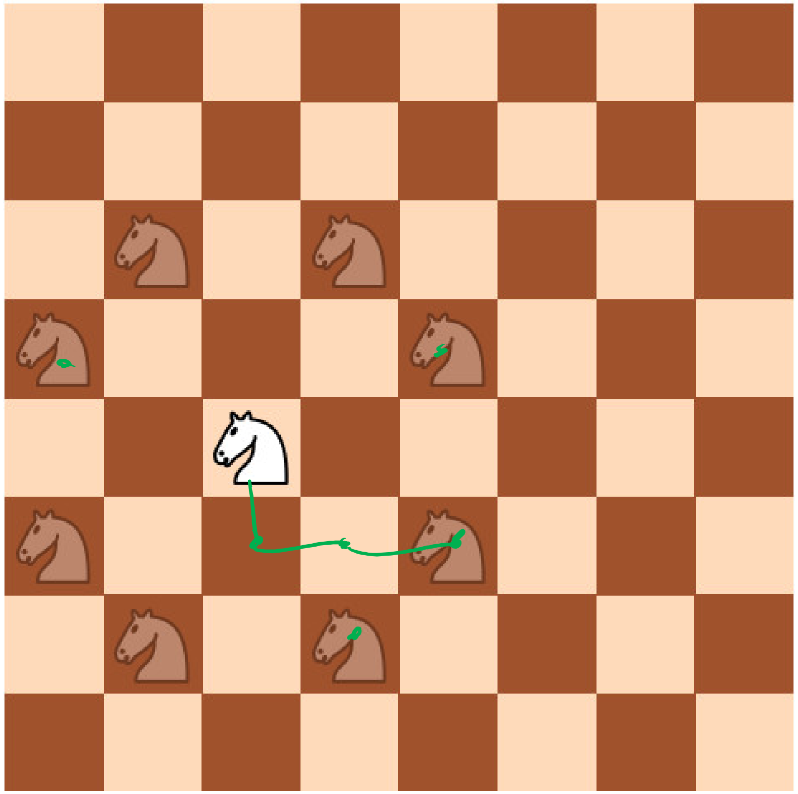
Chess Knight



A chess knight moves:

- In an L-shape

Chess Knight



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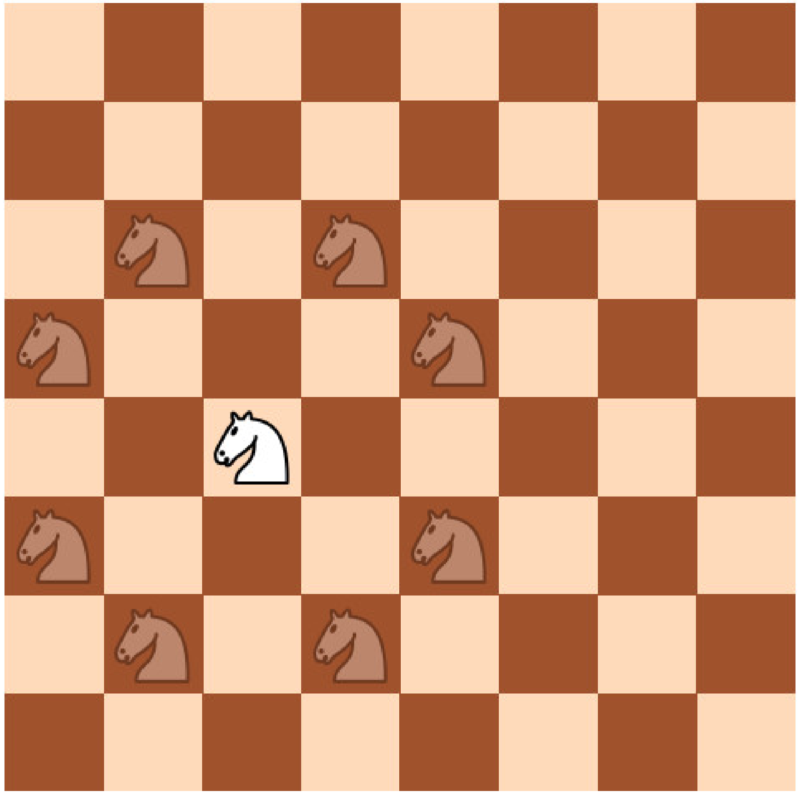
Maximum Number of ~~Rooks~~

Problem: What is the maximum number of knights on a chessboard such that no two attack each other?

Speculating



- A knight attacks only cells of opposite colors
- There is a solution with 32 knights:
  - place knights on all white cells

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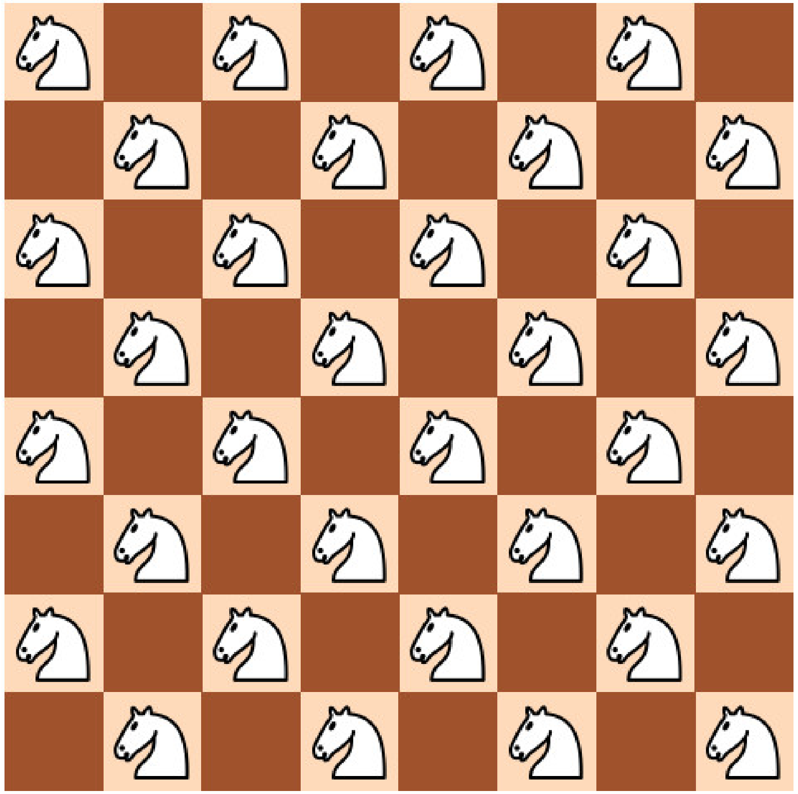

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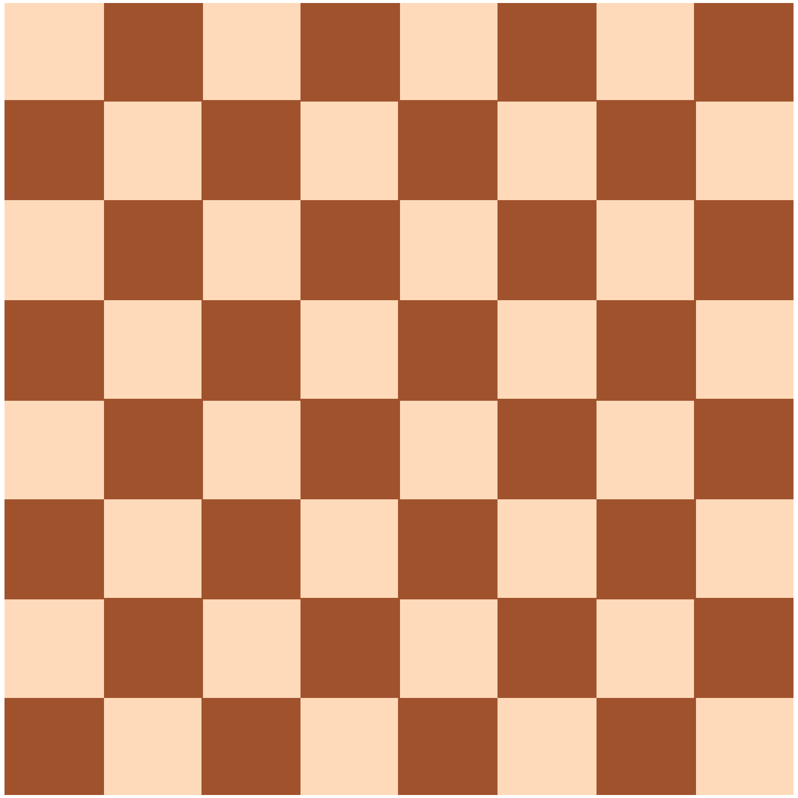
Speculating Further



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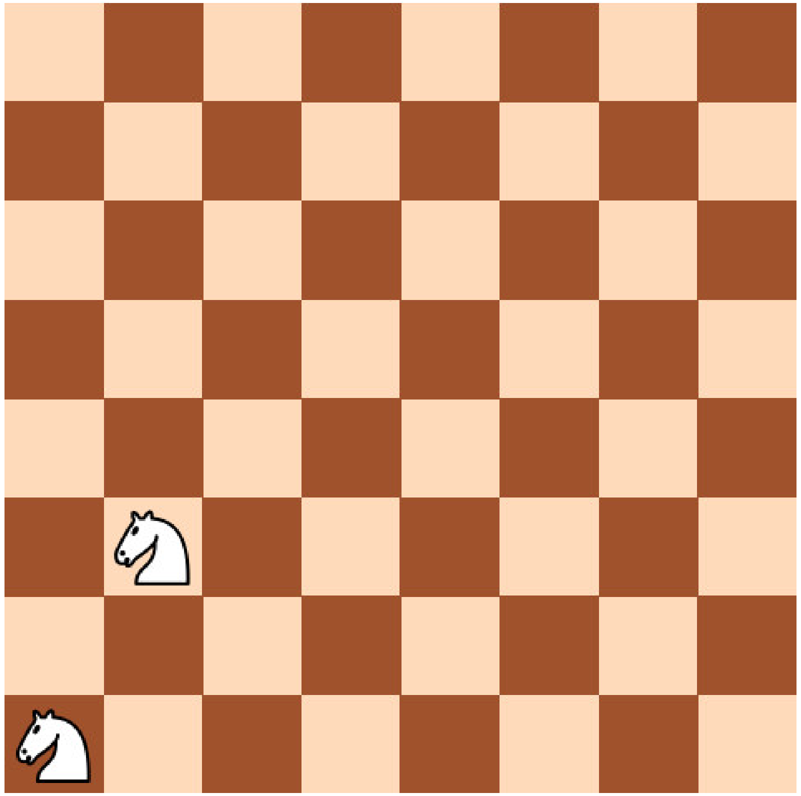

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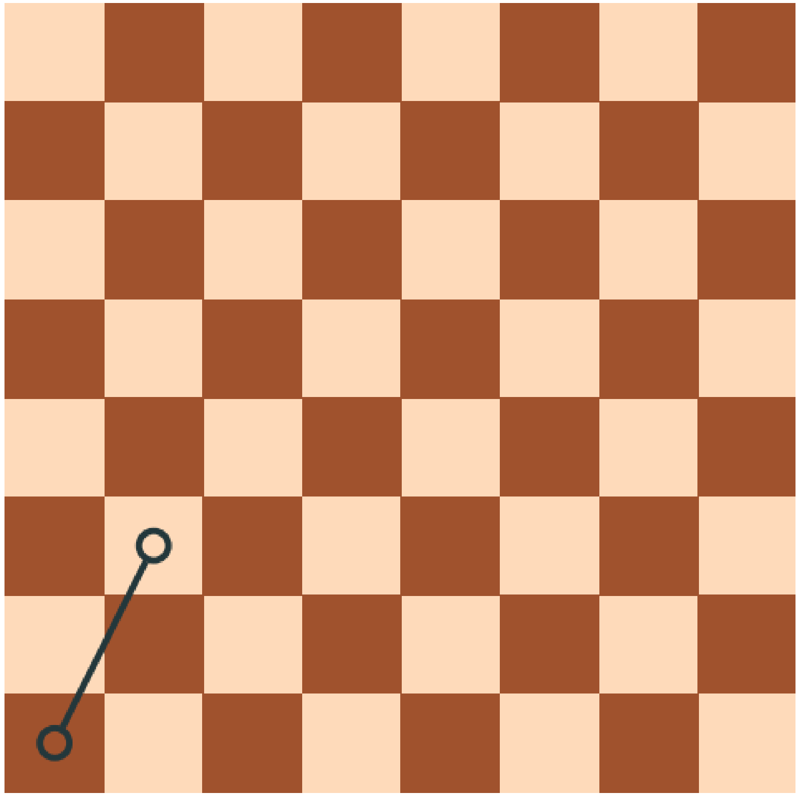

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- let's try...

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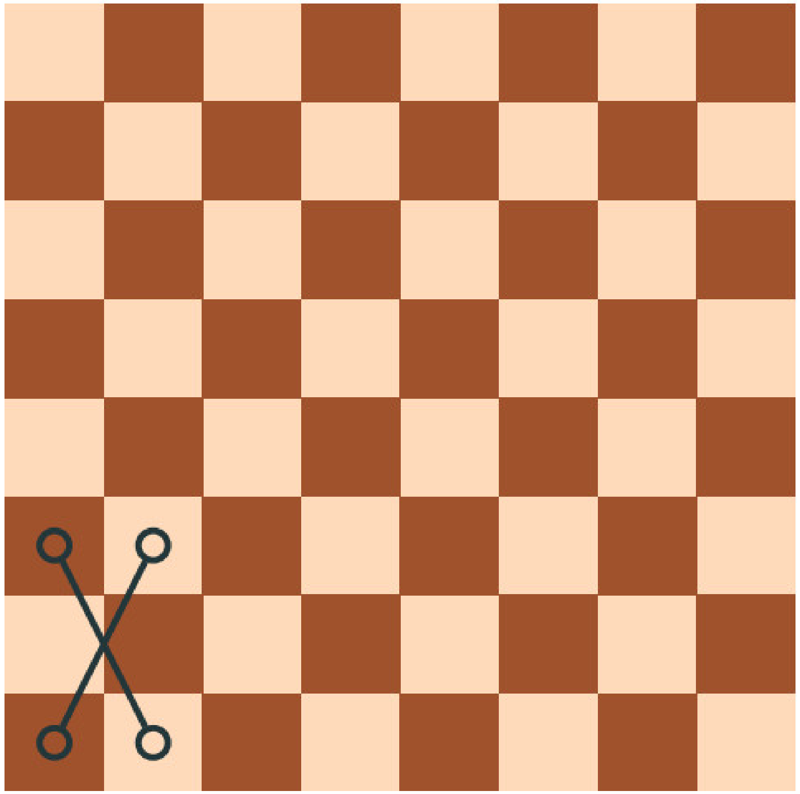

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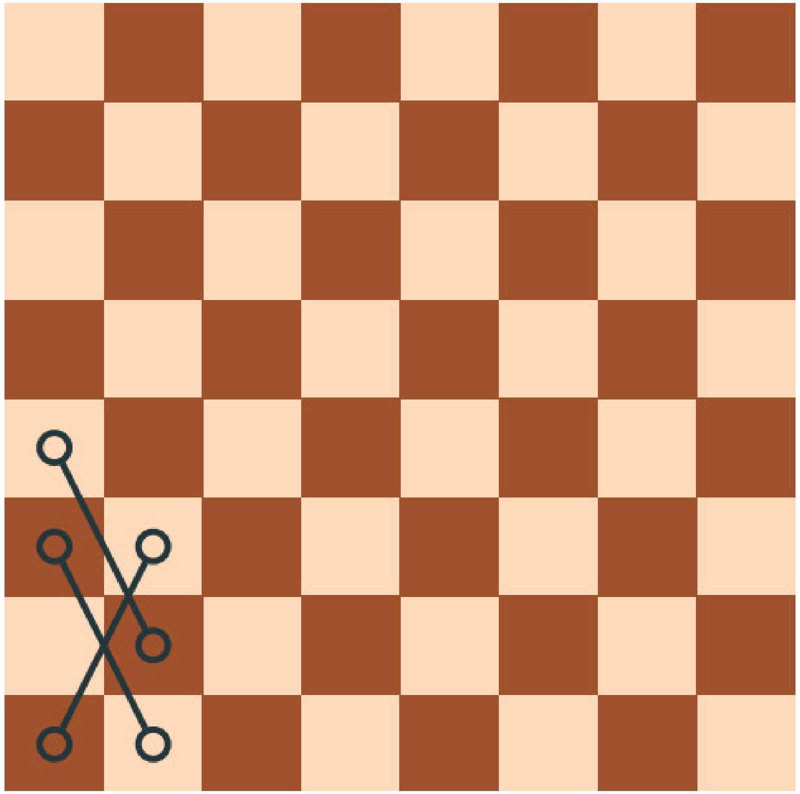

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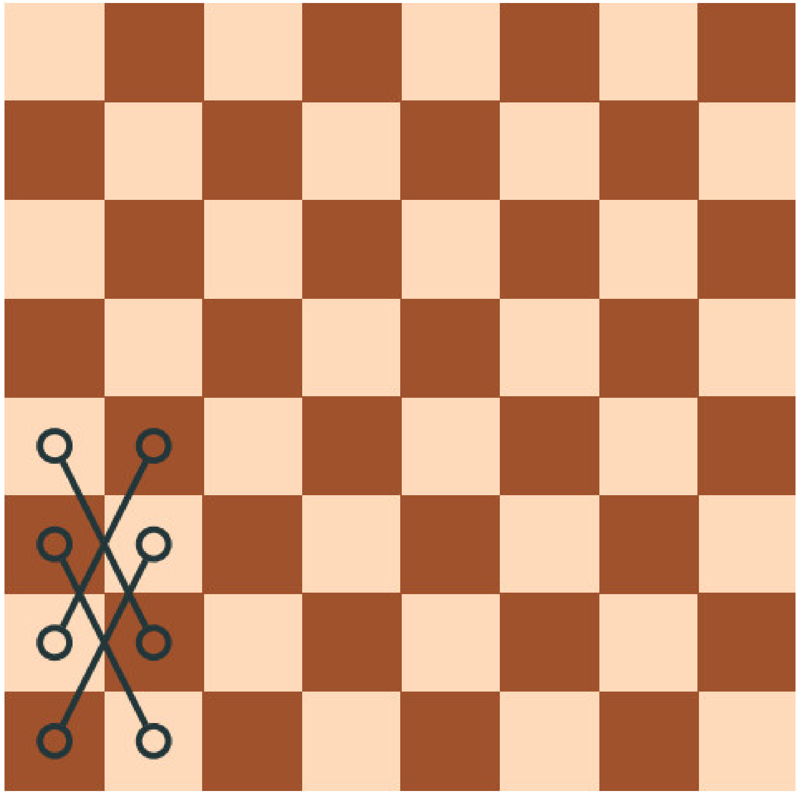

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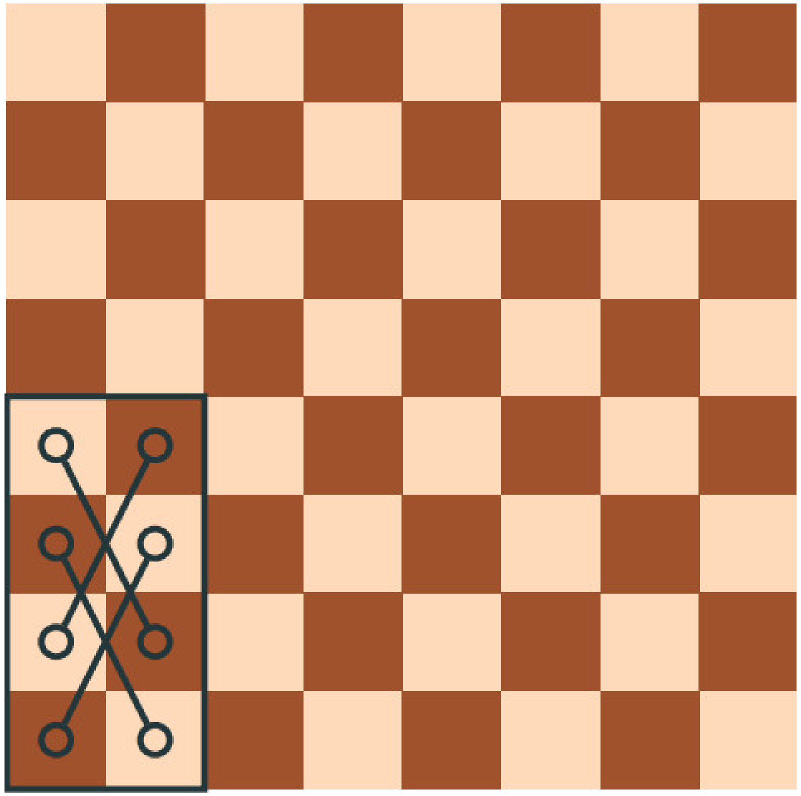

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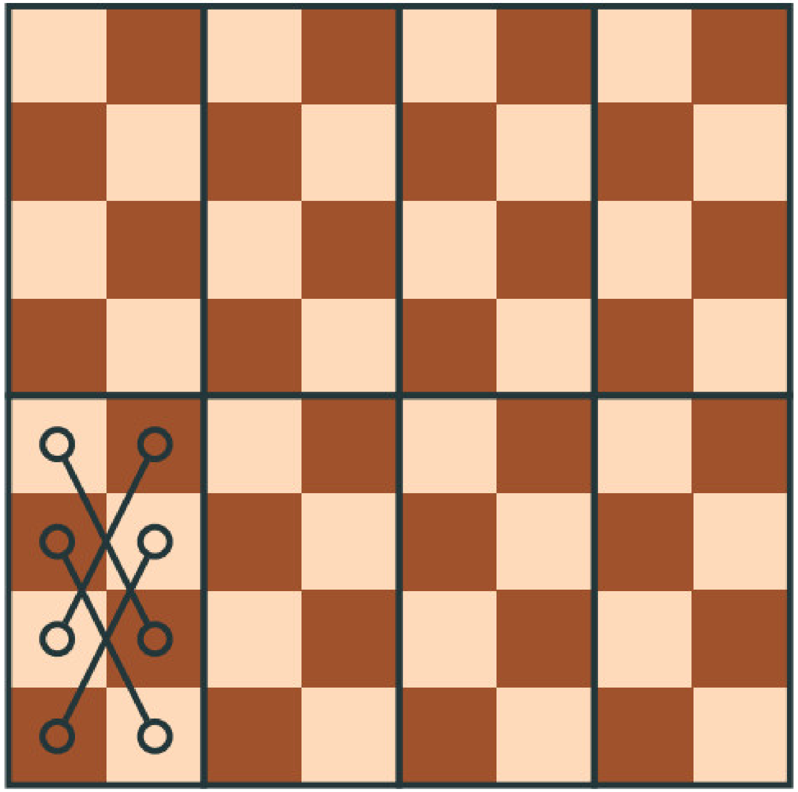

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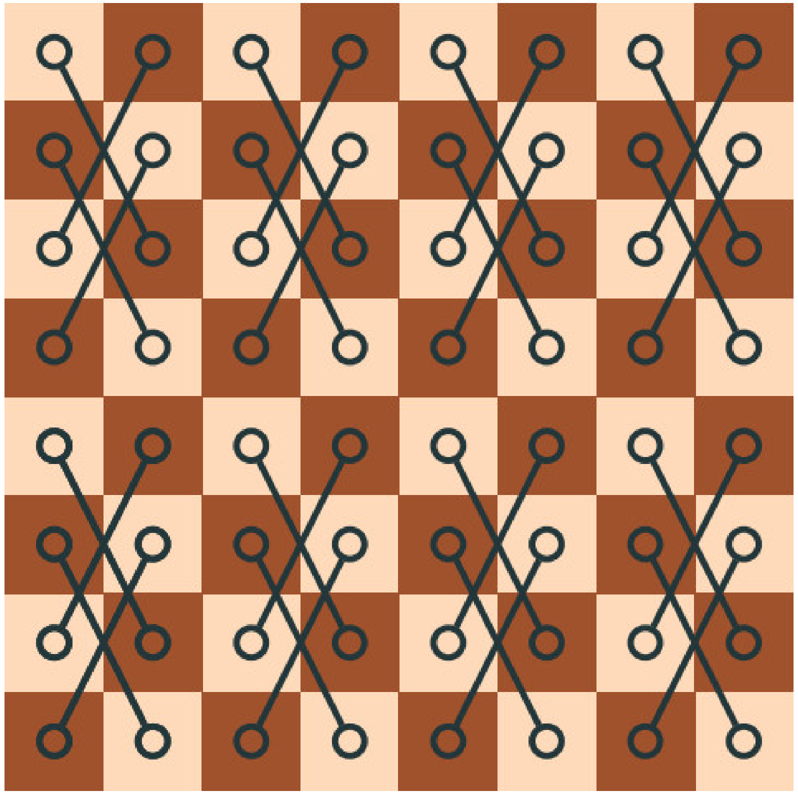

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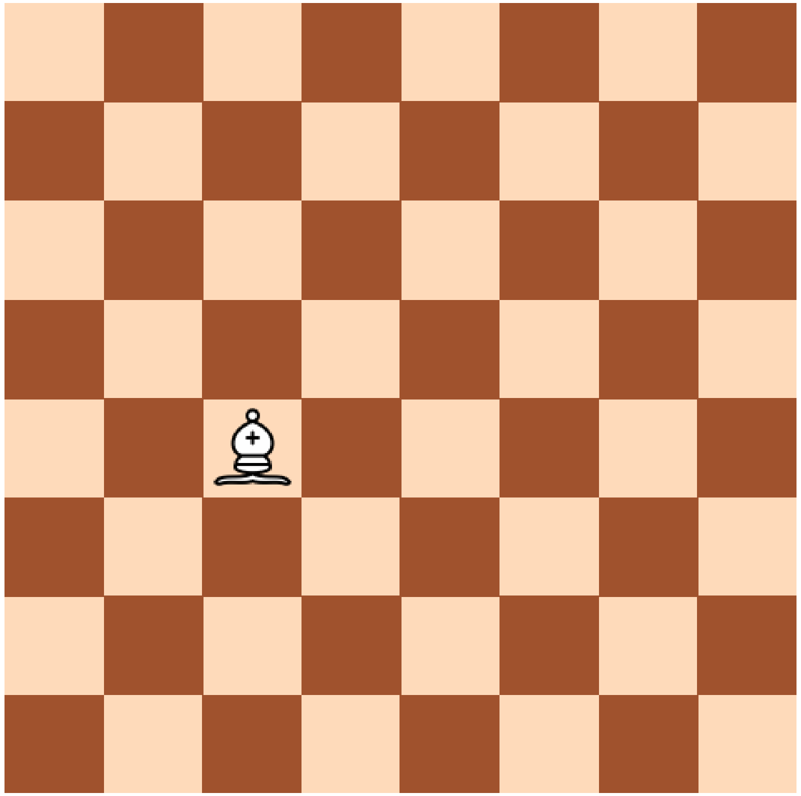
Chess Bishop



A chess bishop moves:

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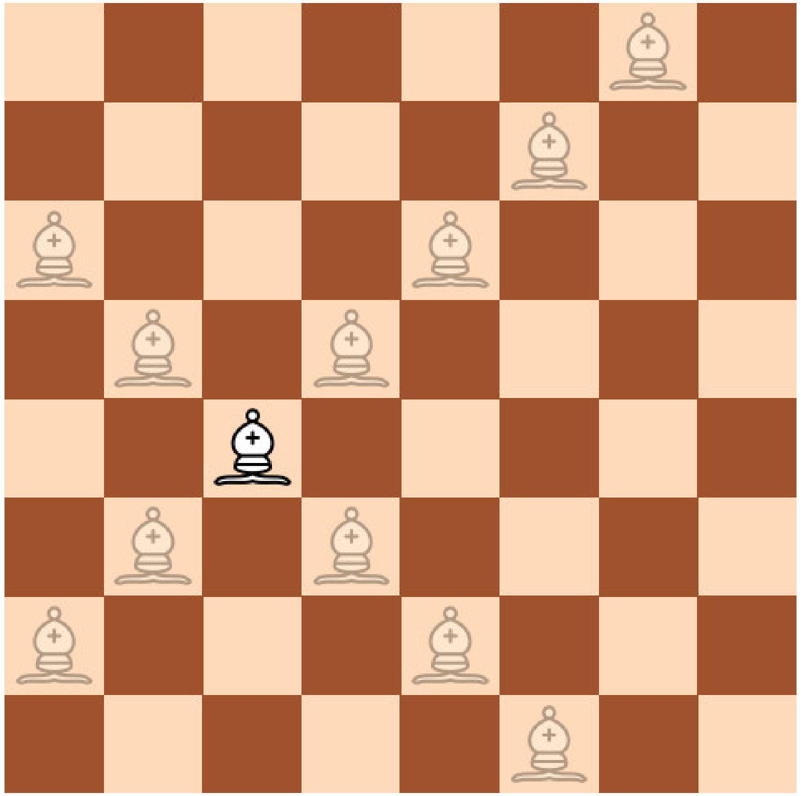
Chess Bishop



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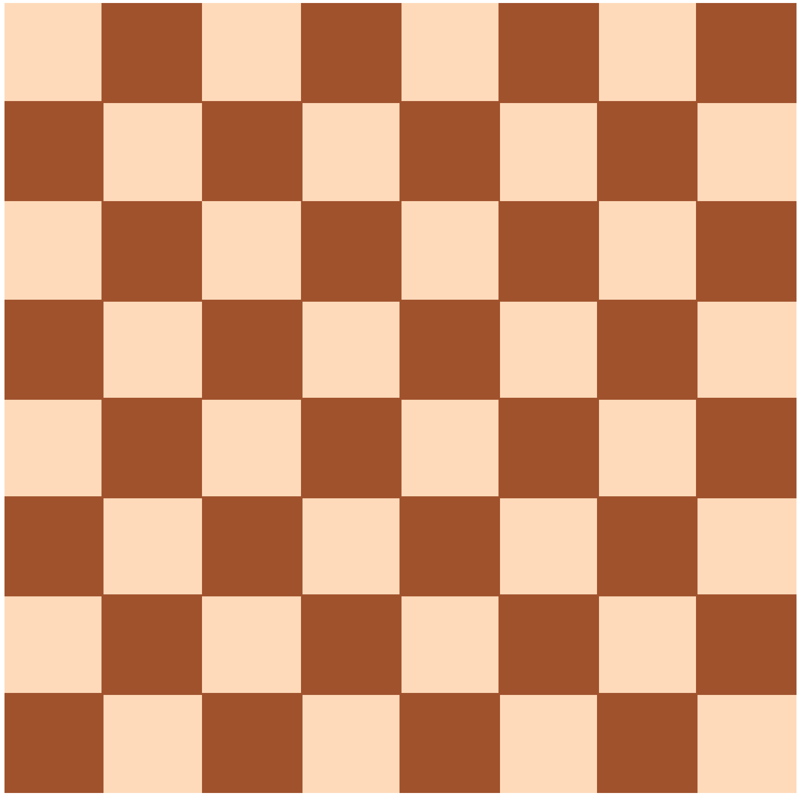
Maximum Number of Bishops

Problem: What is the maximum number of bishops on a chessboard such that no two attack each other?

Solving the Problem



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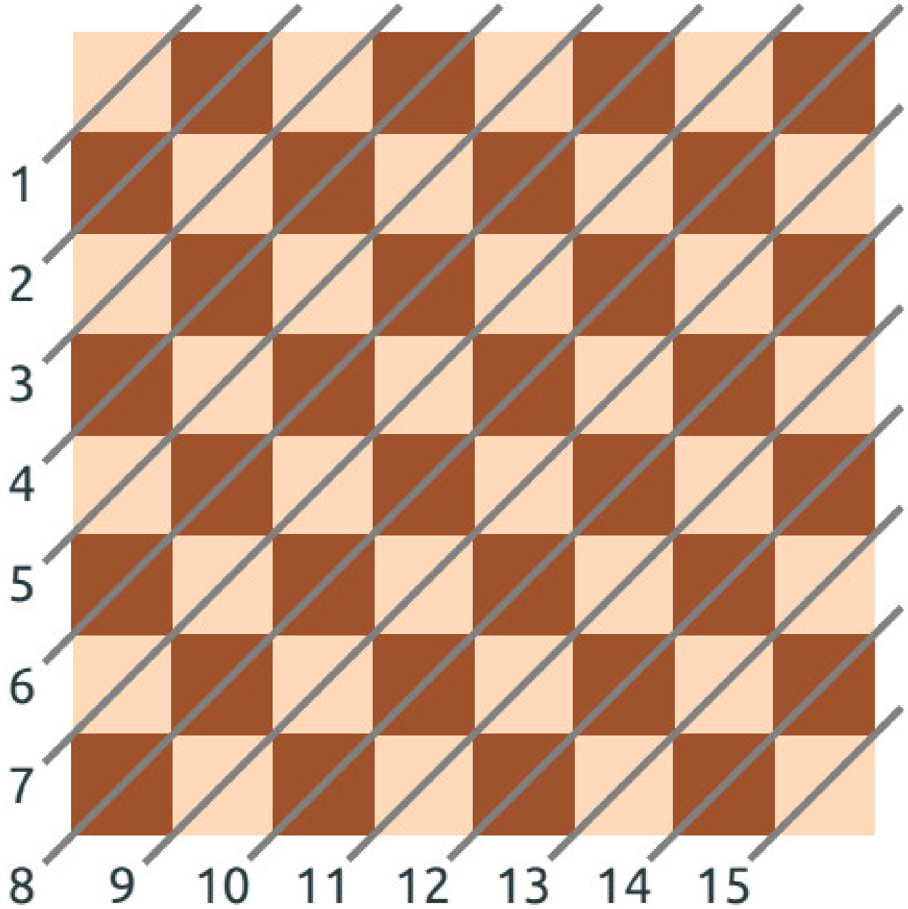

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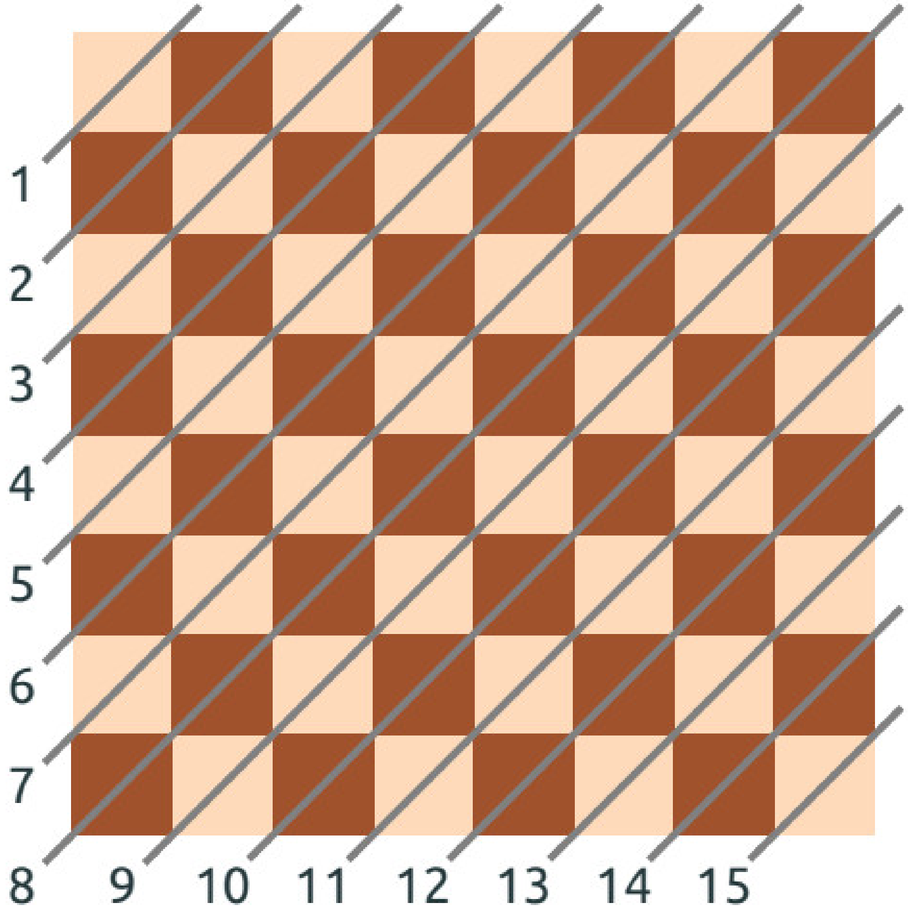
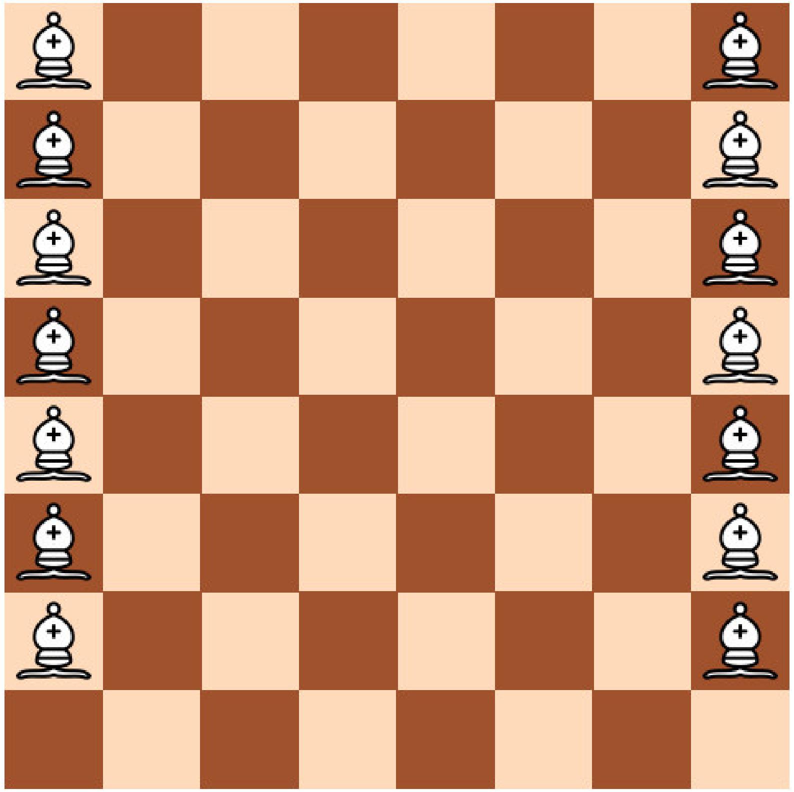


- Diagonally partition the board into 15 diagonals
- each diagonal contains at most one bishop, so at most 15 bishops
- but! The diagonals 1 and 15 cannot both contain a bishop, so at most 14

Solving the Problem



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Subset without x and $2x$

Problem: What is the *maximum* number of integers among 1, 2, … , 50 that one can select, if it is not allowed to select simultaneously x and $2x$?

If $x = 1$, $2x = 2$

$= 2$, $2x = 4$

$= 3$, $2x = 6$

Speculating

- 1 and 2 cannot be taken simultaneously
  2 and 4 cannot be taken simultaneously
- More generally, no two neighbors from the following chain can be taken simultaneously:
  - 1 − 2 − 4 − 8 − 16 − 32
  - Another chain: 3 − 6 − 12 − 24 − 48
- The integers 1, 2, … , 50 can be partitioned into such chains, each chain starting with an odd number

Partitioning into Chains

1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	···	49
2	6	10	14	18	22	26	30	34	38	42	46	50				
4	12	20	28	36	44											
8	24	40														
16	48															
32																

- Chains are independent
- To maximize number of integers taken in each chain, take every second number in the chain, starting with the first one

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- Optimum size: 33

Thank you.