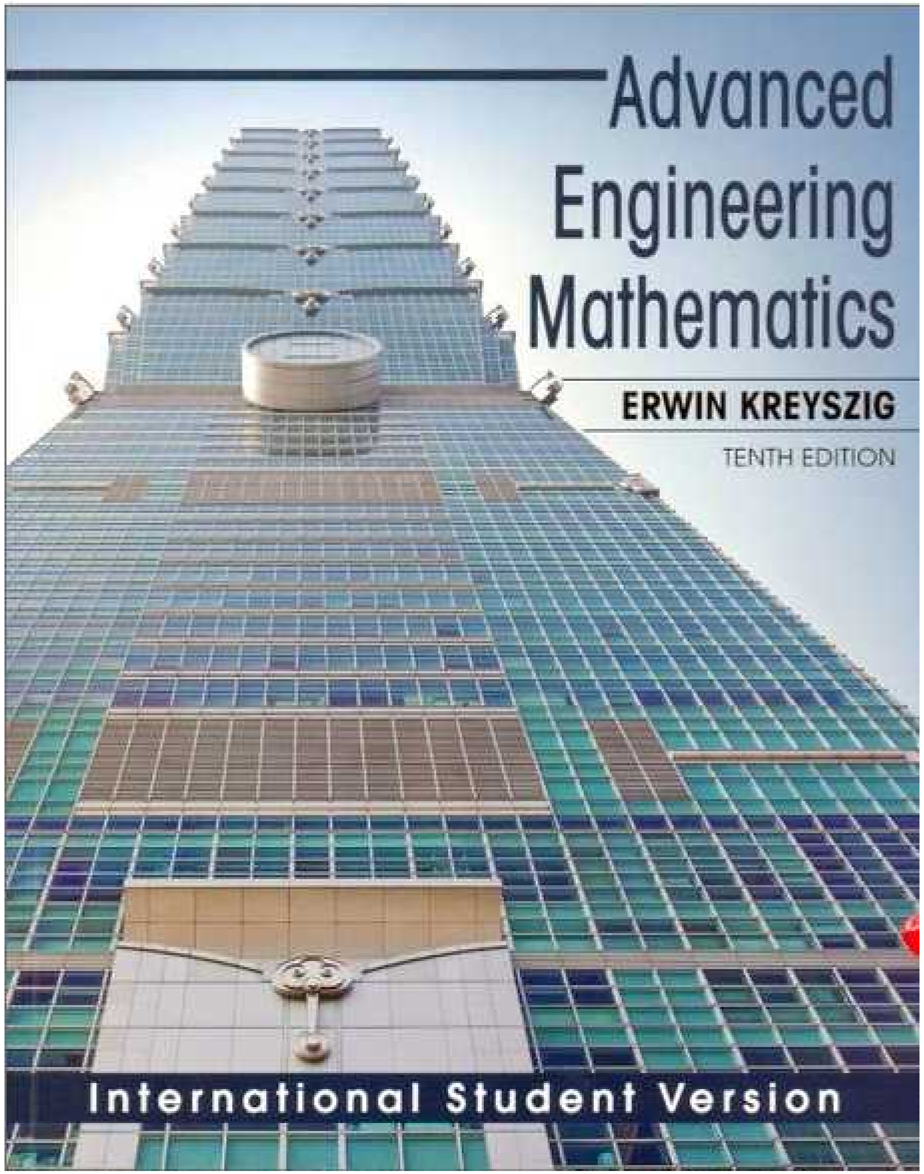


CHAPTER 15

Power Series, Taylor Series

15.1 Sequences, Series, Convergence Tests

15.2 Power Series

15.3 Functions Given by Power Series

15.4 Taylor and Maclaurin Series

15.5 Uniform Convergence, Optional

Problems

Review Questions and Problems

SUMMARY

15.0 Introduction

■ Complex power series are analogs of real power series in calculus.

■ Power series represent analytic functions.

■ Every analytic function can be represented by power series.

15.1 Sequences, Convergence

- Sequences : Obtained by assigning to each positive integer n a number z_n , called a **term** of the sequence

$$z_1, z_2, \cdots \quad or \quad \{z_1, z_2, \cdots\} \quad or \ \ briefly \ \ \{z_n\}$$

  - Term :
  - Real Sequence : Sequence whose terms are real

- Convergence
  - Convergent Sequence : Sequence that has a limit c

$$\lim_{n \to \infty} z_n = c \quad or \quad simply \ \ z_n \to c$$

  - Divergent Sequence : Sequence that does not converge

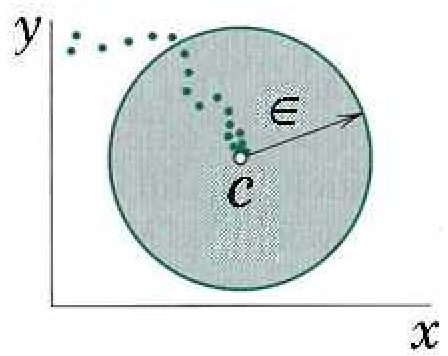
15.1 Definition of limit

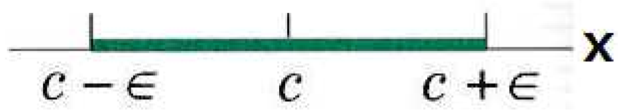
Definition of limit

$\lim_{n\to\infty} z_n = c$ means that for every ε>0 we can find an N such that

$$(1) \qquad |z_n - c| < \epsilon \quad for \;\; all \;\; n > N$$

Geometrically, all terms z_n with $n > N$ lie in the open disk of radius ϵ and center c.





15.1 Examples: Convergent, Divergent, Complex

EXAMPLE 1 Convergent and Divergent Sequences

The sequence $\{i^n/n\} = \{i, -1/2, -i/3, 1/4, \cdots\}$s convergent with limit 0.

The sequence $\{i^n\} = \{i, -1, -i, 1, \cdots\}$s divergent and so is $\{z_n\}$ with $z_n = (1+i)^n$

EXAMPLE 2 Sequences of the Real and Imaginary Parts

The sequence $\{z_n\}$ with $z_n = x_n + iy_n = 1 - 1/n^2 + i(2 + 4/n)$:onverges.

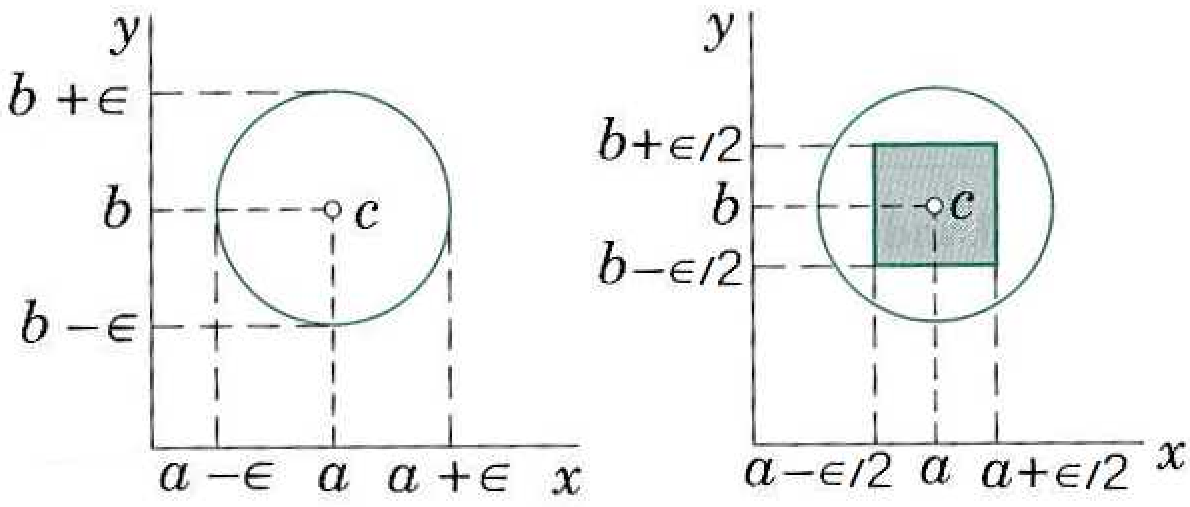
$$\lim_{n\to\infty} z_n = 1 + i2 = c$$

15.1 THEOREM 1 Complex Sequences

THEOREM 1 Sequences of the Real and Imaginary Parts

A complex sequence $\{z_n\} = \{x_n + iy_n\}$ converges to $a + ib$ if and only if $x_n \to a,\ y_n \to b$

PROOF



15.1 Terminology: **Sequence, Partial Sum, Series**

Series

Sequence: $z_1, z_2, \cdots z_m, \cdots$

Partial sum: $s_1 = z_1,\ s_2 = z_1 + z_2,\ s_3 = z_1 + z_2 + z_3,\ \cdots$

$$(2) \qquad s_n = z_1 + z_2 + \cdots + z_n \qquad (n = 1, 2, \cdots)$$

s_n :**nth partial sum** of the *infinite series* or **series**

$$(3) \qquad \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$$

A **series** is the sum of the terms of a sequence.

15.1 Convergent/Divergent Series

A **convergent series** is one whose sequence of partial sums converges.

$$\lim_{n\to\infty} s_n = s$$

Then, we write $s = \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$

A **divergent series:** a series that is not convergent

Remainder of the series s after the term z_n:

$$(4) \quad R_n = z_{n+1} + z_{n+2} + z_{n+3} + \cdots$$

15.1 Error

$$s = \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$$

If the series converges, then

$$s = s_n + R_n, \quad R_n = s - s_n \quad s_n \to s \quad R_n \to 0$$

Error: $|R_n|$

The error can be as small as possible by choosing n large enough.

15.1 THEOREM 2 Real and Imaginary Parts

THEOREM 2 Real and Imaginary Parts

$$(3) \qquad \sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots \qquad\qquad z_m = x_m + iy_m$$

The series (3) converges to the sum $s = u + iv$
Iff(if and only if)

$$x_m \to u \quad and \quad y_m \to v$$

PROOF

15.1 THEOREM 3 Divergence

Tests for Convergence and Divergence of Series

THEOREM 3 Divergence

If $z_1 + z_2 + \cdots$ converges, then $\lim_{m \to \infty} z_m = 0$

Hence, if $\lim_{m \to \infty} z_m \neq 0$, then the series diverges.

PROOF

$$\lim_{m \to \infty} z_m = \lim_{m \to \infty} (s_m - s_{m-1})$$

$$= \lim_{m \to \infty} s_m - \lim_{m \to \infty} s_{m-1} = s - s = 0$$

15.1 CAUTION! In Using Theorem 3

CAUTION!

$\lim_{m \to \infty} z_m = 0$ is *necessary* for convergence

but *not sufficient*.

Convergence $\;\overset{\Longrightarrow}{\not\Longleftarrow}\;$ $\lim_{m \to \infty} z_m = 0$

Example: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

$$z_m = \frac{1}{m} \qquad \lim_{m \to \infty} z_m = \lim_{m \to \infty} \frac{1}{m} = 0$$

However, the series diverges.

15.1 THEOREM 4 Cauchy's Convergence Principle

THEOREM 4 Cauchy's Convergence Principle for Series

A series $z_1 + z_2 + \cdots$ converges if and only if for any ε>0 there exists an N such that

$$(5) \qquad |z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \epsilon$$

for every $n > N$ *and* $p = 1, 2, \cdots$

PROOF is omitted.

15.1 Absolute/Conditional Convergence

Absolute Convergence

A series is absolute convergent if the following series is convergent.

$$\sum_{m=1}^{\infty} |z_m| = |z_1| + |z_2| + \cdots$$

Conditional Convergence

If $z_1 + z_2 + \cdots$ converges but $|z_1| + |z_2| + \cdots$ diverges, then the series $z_1 + z_2 + \cdots$ is called, more precisely, *conditionally convergent*.

15.1 EXAMPLE 3 A Conditionally Convergent Series

EXAMPLE 3 A Conditionally Convergent Series

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges conditionally.

If a series is absolutely convergent, then the series converges.

$$|z_{n+1} + \cdots + z_{n+p}| \leq |z_{n+1}| + \cdots + |z_{n+p}|$$

Since the series is absolutely convergent, by Cauchy's convergence principle, for any ε>0 there exists N such that $|z_{n+1}| + \cdots + |z_{n+p}| < \epsilon$

15.1 Absolute Convergence ⇨ Conditional Convergence

If a series is absolutely convergent, then the series converges.

$$|z_{n+1} + \cdots + z_{n+p}| \le |z_{n+1}| + \cdots + |z_{n+p}|$$

Since the series is absolutely convergent, by Cauchy's convergence principle, for any ε>0 there exists N such that

$$|z_{n+1}| + \cdots + |z_{n+p}| < \epsilon$$

$$\therefore |z_{n+1} + \cdots + z_{n+p}| \le \epsilon$$

Thus, the series is convergent by Cauchy's convergence principle.

15.1 THEOREM 5 Comparison Test

THEOREM 5 Comparison Test

If $|z_1| < b_1, \ |z_2| < b_2, \cdots$ and $b_1 + b_2 + \cdots$ converges,

then $z_1 + z_2 + \cdots$ converges absolutely.

PROOF

15.1 THEOREM 6

THEOREM 6 Geometric Series

The geometric series

$$(6^*) \qquad \sum_{m=0}^{\infty} q^m = 1 + q + q^2 + \cdots$$

$$= \begin{cases} 1/(1-q) & if \quad |q| < 1 \\ diverges & if \quad |q| \geq 1 \end{cases}$$

PROOF

$$s_n = 1 + q + q^2 + \cdots + q^n$$

$$-)\ qs_n = \qquad q + q^2 + \cdots + q^n + q^{n+1}$$

$$s_n - qs_n = 1 - q^{n+1}$$

15.1 THEOREM 6-cont

$$
\begin{array}{rl}
s_n = & 1 + q + q^2 + \cdots + q^n \\
-)\; qs_n = & \quad\;\; q + q^2 + \cdots + q^n + q^{n+1} \\
\hline
\end{array}
$$

$$s_n - qs_n = 1 - q^{n+1}$$

$$s_n(1-q) = 1 - q^{n+1}$$

$$s_n = \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q} - \frac{q^{n+1}}{1-q}$$

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-q^{n+1}}{1-q}$$

$$= \begin{cases} 1/(1-q) & if \;\; |q| < 1 \\ diverges & if \;\; |q| \geq 1 \end{cases}$$

15.1 THEOREM 7 Ratio Test

THEOREM 7 Ratio Test

$$(7) \qquad \left| \frac{z_{n+1}}{z_n} \right| \leq q < 1 \quad for \ all \ n > N$$

⇒ The series $z_1 + z_2 + \cdots$ converges absolutely.

$$(8) \qquad \left| \frac{z_{n+1}}{z_n} \right| \geq 1 \quad for \ all \ n > N$$

⇒ The series $z_1 + z_2 + \cdots$ diverges.

15.1 THEOREM 7 Ratio Test-proof

PROOF

15.1 THEOREM 7 Ratio Test-Caution

CAUTION!

$$(7) \qquad \left|\frac{z_{n+1}}{z_n}\right| \leq q < 1 \quad for \;\; all \;\; n > N$$

The inequality (7) implies $\left|z_{n+1}/z_n\right| < 1$,
but this does *not* imply convergence.

$$z_n = 1/n, \qquad |z_{n+1}/z_n| = n/(n+1) < 1$$

However, the series $z_1 + z_2 + \cdots$ diverges.

15.1 THEOREM 8 Ratio Test

THEOREM 8 Ratio Test

Consider a series $z_1 + z_2 + \cdots$ with

$$z_n \neq 0 \quad \text{and} \quad \lim_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right| = L .$$

Then, the series $z_1 + z_2 + \cdots$:

(a) If $L < 1$, the series converges absolutely.

(b) If $L > 1$, the series diverges.

(c) If $L = 1$, the series may converge or diverge.

15.1 THEOREM 8 Ratio Test-proof

PROOF

15.1 THEOREM 8 Ratio Test-case (c)

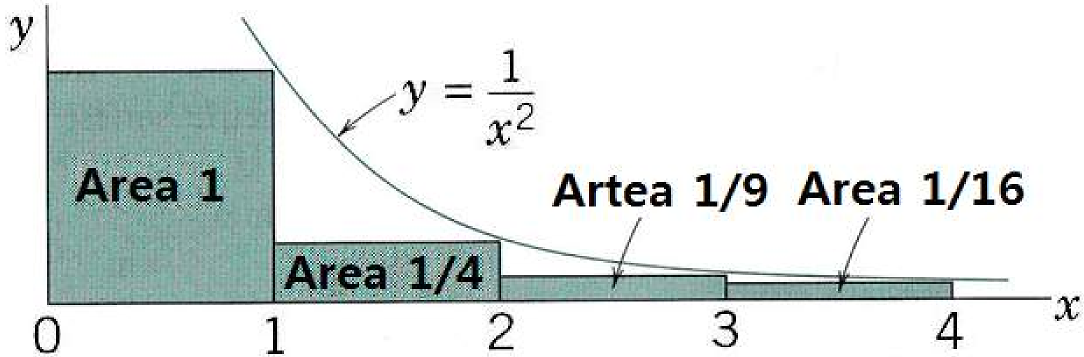
(c) If $L = 1$, the series may converge or diverge.

$$z_n = 1/n: \quad L = \lim_{n\to\infty} |z_{n+1}/z_n| = \lim_{n\to\infty} |n/(n+1)| = 1$$

The series $\sum z_n$ diverges.

$$z_n = 1/n^2: \quad L = \lim_{n\to\infty} |z_{n+1}/z_n| = \lim_{n\to\infty} |n^2/(n+1)^2| = 1$$

The series $\sum z_n$ converges.



EXAMPLE 4 Ratio Test

Determine whether the following series converges or diverges.

$$\sum_{m=0}^{\infty} \frac{(100+75i)^n}{n!} = 1 + (100+75i) + \frac{1}{2!}(100+75i)^2 + \cdots$$

Sol.

$$\left|\frac{z_{n+1}}{z_n}\right| = \frac{|(100+75i)^{n+1}/(n+1)!|}{|(100+75i)^n/n!|}$$

$$= \left|\frac{(100+75i)^{n+1}}{(100+75i)^n}\right| \frac{n!}{(n+1)!}$$

$$= \frac{|100+75i|}{n+1} \rightarrow L = 0$$

Converges since L<1

EXAMPLE 5 Theorem 7 More General Than Theorem 8

Let $a_n = i/2^{3n}$ and $b_n = 1/2^{3n+1}$.

Is the following series convergent or divergent?

$$a_0 + b_0 + a_1 + b_1 + a_2 + b_2 + \cdots = i + \frac{1}{2} + \frac{i}{8} + \frac{1}{16} + \frac{i}{64} + \frac{1}{128} + \cdots$$

Sol.

$$L_1 = \lim_{n\to\infty}\left|\frac{z_{2n+1}}{z_{2n}}\right| = \frac{1}{4} \qquad L_2 = \lim_{n\to\infty}\left|\frac{z_{2n+2}}{z_{2n+1}}\right| = \frac{1}{2}$$

$$L = \max\{L_1,\ L_2\} = \max\{1/2,\ 1/4\} = 1/2$$

The series converges by Theorem 7 but not by Theorem 8.

15.1 THEOREM 9 Root Test

THEOREM 9 Root Test

$$(9) \quad \sqrt[n]{|z_n|} \le q < 1$$ for all n>N, then $\sum_{1}^{\infty} z_n$ converges.

$$(10) \quad \sqrt[n]{|z_n|} \ge 1$$ for infinitely many n, then $\sum_{1}^{\infty} z_n$ diverges.

PROOF

$$(9) \quad \sqrt[n]{|z_n|} \le q < 1 \;\; then \;\; |z_n| \le q^n \;\; for \;\; n > N$$

$$\sum_{1}^{\infty} |z_n| = \sum_{1}^{N-1} |z_n| + \sum_{N}^{\infty} |z_n| \le \sum_{1}^{N-1} |z_n| + \sum_{N}^{\infty} q^n$$

15.1 THEOREM 10 Root Test

THEOREM 10 Root Test

If a series $z_1 + z_2 + \cdots$ is such that $\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$ then

(a) The series converges absolutely if L<1.
(b) The series diverges if L>1.
(c) If L=1, the test fails; that is, no conclusion is possible.
If L=1 and the limit approaches strictly from above then the series diverges.

PROOF

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PROBLEM SET 15.1

1. Is the given sequence bounded? Convergent? Find its limit points.
$z_n = (1 + i)^{2n}/2^n$

$$z_n = (1+i)^{2n}/2^n = \left[\frac{(1+i)^2}{2}\right]^n = i^n$$

$$|z_n| = |i^n| = 1$$

bounded

Diverges

PROBLEM SET 15.1

2. Is the given sequence bounded? Convergent? Find its limit points.

$z_n = (3 + 4i)^n / n!$

$$|z_n| = |(3 + 4i)^n / n!| = 5^n / n!$$

$$= \frac{5}{1}\frac{5}{2}\frac{5}{3}\frac{5}{4}\frac{5}{5}\frac{5}{6}\cdots\cdots\frac{5}{n}$$

$$< \frac{5}{1}\frac{5}{2}\frac{5}{3}\frac{5}{4}\frac{5}{5}\frac{5}{n} \longrightarrow 0$$

bounded

$$\lim_{n \to \infty} z_n = 0.$$

PROBLEM SET 15.1

3. Is the given sequence bounded? Convergent? Find its limit points.

$z_n = n\pi/(4 + 2ni)$

$$|z_n| = \frac{n\pi}{\sqrt{16+(2n)^2}} = \frac{\pi}{\sqrt{16/n^2+4}} \rightarrow \frac{\pi}{2}$$

bounded

$$z_n = \frac{n\pi}{4 + 2ni} = \frac{\pi}{4/n + 2i} \rightarrow \frac{\pi}{2i}$$

$$\lim_{n\to\infty} z_n = \frac{\pi}{2i}$$

PROBLEM SET 15.1

4. Is the given sequence bounded? Convergent? Find its limit points.

$z_n = (1 + 2i)^n$

$$|z_n| = |(1 + 2i)^n| = |(\sqrt{5})^n| \to \infty$$

unbounded

divergent

PROBLEM SET 15.1

5. Is the given sequence bounded? Convergent? Find its limit points.

$z_n = (-1)^n + 10i$

$$|z_n| = |(-1)^n + 10i| = \sqrt{101}$$

bounded

divergent

PROBLEM SET 15.1

6. Is the given sequence bounded? Convergent? Find its limit points.

$z_n = (\cos n\pi i)/n$

$$\left|z_n\right| = \left|\frac{\cos n\pi i}{n}\right| = \frac{1}{n}\ \frac{e^{n\pi} + e^{-n\pi}}{2} \ \rightarrow \infty$$

Unbounded

divergent

7. Is the given sequence bounded? Convergent? Find its limit points

$z_n = n^2 + i/n^2$

Unbounded

divergent

PROBLEM SET 15.1

8. Is the given sequence bounded? Convergent? Find its limit points.

$z_n = [(1 + 3i)/\sqrt{10}]^n$

$$|z_n| = \left|\left[\frac{1+3i}{\sqrt{10}}\right]^n\right| = \left[\frac{\sqrt{10}}{\sqrt{10}}\right]^n = 1$$

Bounded

Divergent

PROBLEM SET 15.1

9. Is the given sequence bounded? Convergent? Find its limit points.

$z_n = (3 + 3i)^{-n}$

$$|z_n| = \left|\frac{1}{(3 + 3i)^{-n}}\right| = \frac{1}{(3\sqrt{2})^n} < \frac{1}{3}$$

Bounded

Convergent

PROBLEM SET 15.1

10. Is the given sequence bounded? Convergent? Find its limit points.

$$z_n = \sin\left(\tfrac{1}{4}n\pi\right) + i^n$$

$$|z_n| \leq 2$$

Bounded

Divergent

PROBLEM SET 15.1

16. Is the given series convergent or divergent? Give a reason.

$$\sum_{n=0}^{\infty} \frac{(20 + 30i)^n}{n!}$$

Use Ratio test.

$$L = \lim_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right| = \lim_{n \to \infty} \frac{10\sqrt{13}}{n+1} = 0 < 1$$

Thus, converges absolutely by Theorem 8.

PROBLEM SET 15.1

17. Is the given series convergent or divergent? Give a reason.

$$\sum_{n=2}^{\infty} \frac{(-i)^n}{\ln n}$$

Use comparison test.

$$\left|\frac{(-i)^n}{\ln n}\right| > \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges,}$$

Thus, the series diverges by Theorem 5.

PROBLEM SET 15.1

18. Is the given series convergent or divergent? Give a reason.

$$\sum_{n=1}^{\infty} n^2 \left(\frac{i}{4}\right)^n$$

Use Ratio test.

$$L = \lim_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^2}{n^2} \frac{(i/4)^{n+1}}{(i/4)^n} \right| = \frac{1}{4} < 1$$

Thus, converges absolutely by Theorem 8.

PROBLEM SET 15.1

19. Is the given series convergent or divergent? Give a reason.

$$\sum_{n=0}^{\infty} \frac{i^n}{n^2 - i}$$

Use comparison test.

$$\left|\frac{i^n}{n^2 - i}\right| < \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges.}$$

Thus, the series converges by Theorem 5.

PROBLEM SET 15.1

20. Is the given series convergent or divergent? Give a reason.

$$\sum_{n=0}^{\infty} \frac{n+i}{3n^2+2i}$$

Use comparison test.

$$\left|\frac{n+i}{3n^2+2i}\right| = \frac{\sqrt{n^2+1}}{\sqrt{9n^4+4}} > \frac{\sqrt{n^2+1}}{3n^2+2} > \frac{n}{3n^2+2}$$

and $\displaystyle\sum_{n=0}^{\infty} \frac{n}{3n^2+2}$ diverges.

Thus, the series diverges by Theorem 5.

PROBLEM SET 15.1

21. Is the given series convergent or divergent? Give a reason.

$$\sum_{n=0}^{\infty} \frac{(\pi + \pi i)^{2n+1}}{(2n+1)!}$$

Use Ratio test.

$$L = \lim_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right| = \lim_{n \to \infty} \left| \frac{(\pi + \pi i)^{2n+3}}{(2n+3)!} \frac{(2n+1)!}{(\pi + \pi i)^{2n+1}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(\pi + \pi i)^2}{(2n+3)(2n+2)} \right| \to 0$$

Thus, converges absolutely by Theorem 8.

PROBLEM SET 15.1

23. Is the given series convergent or divergent?

$$\sum_{n=0}^{\infty} \frac{(-1)^n(1+i)^{2n}}{(2n)!}$$

Use Ratio test.

$$L = \lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(1+i)^{2n+2}}{(2n+2)!}\,\frac{(2n)!}{(-1)^n(1+i)^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(1+i)^2}{(2n+1)(2n+2)}\right| \to 0$$

Thus, converges absolutely by Theorem 8.

PROBLEM SET 15.1

24. Is the given series convergent or divergent?

$$\sum_{n=1}^{\infty} \frac{(3i)^n n!}{n^n}$$

Use Ratio test.

$$L = \lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = \lim_{n\to\infty}\left| \frac{(3i)^{n+1}(n+1)!}{(n+1)^{n+1}} \frac{n^n}{(3i)^n n!}\right|$$

$$= \lim_{n\to\infty}\left| \frac{(3i)^{n+1}(n+1)\cdot n!}{(n+1)\cdot(n+1)^n} \frac{n^n}{(3i)^n n!}\right| = \lim_{n\to\infty} \frac{3n^n}{(n+1)^n}$$

$$= 3\lim_{n\to\infty} \frac{1}{(1+1/n)^n} \to \frac{3}{e}$$

Thus, diverges absolutely by Theorem 8.

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PROBLEM SET 15.1

25. Is the given series convergent or divergent? $\sum_{n=1}^{\infty} \frac{i^n}{n}$

$$\sum_{n=1}^{\infty} \left| \frac{i^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{: diverges}$$

$$\sum_{n=1}^{\infty} \frac{i^n}{n} = i\underbrace{(1-1/3+1/5-1/7+\cdots)}_{\text{Alternating series}} + \underbrace{(-1/2+1/4-1/6+1/8-\cdots)}_{\text{Alternating series}}$$

$$= i\sum_{k=0}^{\infty}\left(\frac{1}{4k+1} - \frac{1}{4k+3}\right) + \sum_{k=0}^{\infty}\left(\frac{-1}{4k+2} + \frac{1}{4k+4}\right)$$

$$= i\sum_{k=0}^{\infty} \frac{2}{16k^2+16k+3} - \sum_{k=0}^{\infty} \frac{2}{16k^2+24k+8}$$

Converges conditionally.

15.2 Power Series

A power series in powers of z-z_0:

$$(1) \qquad \sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$a_n : \ coefficients$$

$$z_0 : \ center\ of\ the\ series$$

If z_0=0:

$$(2) \qquad \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

15.2 EXAMPLE 1 Convergence in a Disk. Geometric Series

EXAMPLE 1 Convergence in a Disk. Geometric Series

The geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

converges absolutely if $|z|<1$ and diverges if $|z|\geq 1$
(see Theorem 6 in Sec. 15.1)

EXAMPLE 2 Convergence for Every z

The power series(Maclaurin series of e^z)

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

absolutely converges for every z.

Sol.

$$Ratio = \left| \frac{z^{n+1}/(n+1)!}{z^n/n!} \right| = \left| \frac{z}{n+1} \right| \to 0$$ for every z.

EXAMPLE 3 Convergence only at the Center(Useless Series)

The following power series converges only at z=0, but diverges for z≠0.

$$\sum_{n=0}^{\infty} n!z^n = 1 + z + 2z^2 + 3!z^3 + \cdots$$

Sol.

$$Ratio = \left| \frac{(n+1)!\ z^{n+1}}{n!\ z^n} \right| = |(n+1)z| \ \rightarrow \ \infty$$

$$as \ \ n \rightarrow \infty \ \ for \ \ all \ \ z \neq 0$$

15.2 THEOREM 1 Convergence of a Power Series

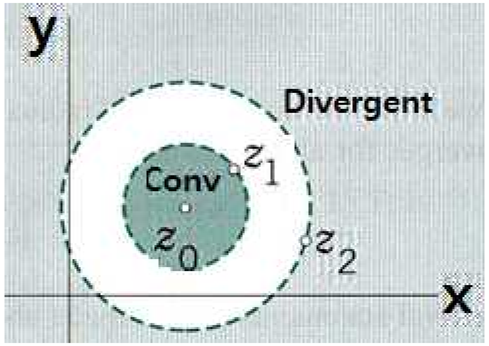
THEOREM 1 Convergence of a Power Series

(a) Every power series (1) converges at the center z_0.

(b) If (1) converges at a point $z=z_1 \neq z_0$, it converges absolutely for every z closer to z_0 than z_1, that is, $|z-z_0|<|z_1-z_0|$. See Fig. 365

(c) If (1) diverges at $z=z_2$, it diverges for every z further away from z_0 than z_3. See Fig 365

$$(1) \qquad \sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$



15.2 THEOREM 1 Proof

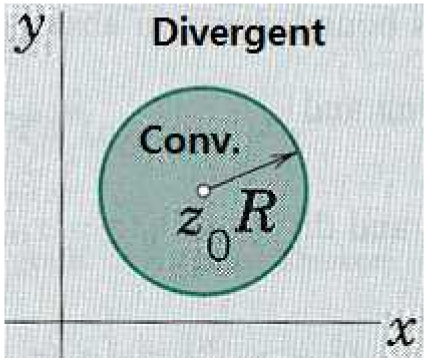
PROOF

15.2 Radius of Convergence of a Power Series

Radius of Convergence of a Power Series

Consider the smallest circle with center z_0 that includes all the points at which a given power series (1) converges. Let R denote its radius.

$$(1) \qquad \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$



Then,

$|z - z_0| = R$: Circle of convergence

R: Radius of convergence

The series (3)

converges within the circle of convergence $(4) \quad |z - z_0| < R$

diverges outside of the circle of convergence $(5) \quad |z - z_0| > R$

May converge or diverge on the circle of convergence

WILEY

EXAMPLE 4 Behavior on the Circle of Convergence

On the circle of convergence (radius=1 in all three series)

$\sum z^n/n^2$ converges everywhere since $\sum 1/n^2$ converges.

$\sum z^n/n$ converges at -1(by Leibniz's test) but diverges at 1.

$\sum z^n$ diverges everywhere.

15.2 Leibniz's Test

Leibniz's test

Alternating series $\sum (-1)^n a_n$ converges if

(a) Alternating series

(b) a_n decreases monotonically

(c) $\lim_{n \to \infty} a_n = 0$

15.2 Notation R=∞ and R=0

Notation R=∞ and R=0

$R=\infty$: The series converges everywhere.

$R=0$: The series converges only at the center.

15.2 THEOREM 2 Radius of Convergence R

THEOREM 2 Radius of Convergence R

$$(1) \qquad \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

The radius of convergence of the series (1) is

$$(6) \qquad R = \frac{1}{L^*} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \quad \text{(Cauchy-Hadamard formula)}$$

15.2 THEOREM 2 Radius of Convergence R-Proof

PROOF

$$Ratio = \left| \frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n} \right| = \left| \frac{a_{n+1}}{a_n} \right| |z-z_0|$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |z-z_0| = L^* \, |z-z_0| = 1$$

$$R = |z-z_0| = \frac{1}{L^*} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

15.2 EXAMPLE 5 Radius of Convergence

EXAMPLE 5 Radius of Convergence

Find the radius of convergence of the following series

$$\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(z-3i)^n$$

Sol.

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty}\left[\frac{(2n)!/(n!)^2}{[2(n+1)]!/[(n+1)!]^2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(2n)!}{(2n+2)(2n+1)\cdot(2n)!}\cdot\frac{[(n+1)!]^2}{(n!)^2}\right]$$

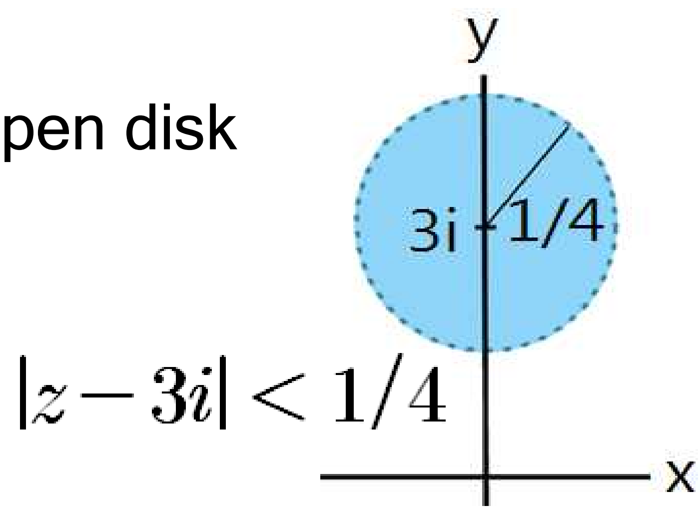
15.2 EXAMPLE 5 Radius of Convergence-conti

$$R = \lim_{n\to\infty}\left[\frac{(2n)!}{(2n+2)(2n+1)\cdot(2n)!}\cdot\frac{[(n+1)!]^2}{(n!)^2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)^2}{(2n+2)(2n+1)}\right]$$

$$= \lim_{n\to\infty}\left[\frac{n^2(1+1/n)^2}{n^2(2+2/n)(2+1/n)}\right] = \frac{1}{4}$$

Thus, the series converges in the open disk $|z-3i| < 1/4$ with the center 3i.



15.2 EXAMPLE 6 Extension of Theorem 2

EXAMPLE 6 Extension of Theorem 2

Find the radius of convergence of the following series

$$\sum_{n=0}^{\infty}\left[1+(-1)^n+\frac{1}{2^n}\right]z^n$$

Sol.

Apply Theorem 2:

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$=\lim_{n\to\infty}\left[\frac{1+(-1)^n+\frac{1}{2^n}}{1+(-1)^{n+1}+\frac{1}{2^{n+1}}}\right]$$

15.2 EXAMPLE 6 Extension of Theorem 2-conti

$$R = \lim_{n\to\infty}\left[\frac{1+(-1)^n+\dfrac{1}{2^n}}{1+(-1)^{n+1}+\dfrac{1}{2^{n+1}}}\right]$$

$$= \begin{cases} even\ \ n: \lim_{n\to\infty}\left[\dfrac{1+1+1/2^n}{1-1+1/2^{n+1}}\right] = \infty \\ odd\ \ n: \lim_{n\to\infty}\left[\dfrac{1-1+1/2^n}{1+1+1/2^{n+1}}\right] = 0 \end{cases}$$

$$= no\ \ limit$$

Theorem 2 is of no help.

15.2 EXAMPLE 6 Extension of Theorem 2-conti

Extension of Theorem 2 :

(6) $$R = \frac{1}{L^*} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$ (Cauchy-Hadamard formula)

(6*) $$R = \frac{1}{\tilde{L}} \quad where \quad \tilde{L} = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

$$\tilde{L} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|1 + (-1)^n + 2^{-n}|}$$

$$= \begin{cases} even\ \ n : \lim_{n\to\infty} \sqrt[n]{2 + 2^{-n}} = 1 \\ odd\ \ n : \lim_{n\to\infty} \sqrt[n]{1 - 1 + 2^{-n}} = 1/2 \end{cases}$$

(6**) $$R = \frac{1}{\mathrm{Max}\,(1, 1/2)} = 1$$ Thus, the series converges for $|z|<1$.

PROBLEM SET 15.2

6. Find the center and the radius of convergence.

$$\sum_{n=0}^{\infty} 4^n (z+1)^n$$

Center: $z = -1$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{4^n}{4^{n+1}}\right| = \frac{1}{4}$$

PROBLEM SET 15.2

7. Find the center and the radius of convergence.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(z - \frac{1}{2}\pi\right)^{2n}$$

Center: $z = \frac{1}{2}\pi$

$$R^2 = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n}{(2n)!}\cdot\frac{[2(n+1)!]}{(-1)^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^n}{(2n)!}\cdot\frac{(2n+2)(2n+1)\cdot(2n)!}{(-1)^{n+1}}\right| = \lim_{n\to\infty}(2n+2)(2n+1) = \infty$$

$R = \infty$

PROBLEM SET 15.2

8. Find the center and the radius of convergence.

$$\sum_{n=0}^{\infty} \frac{n^n}{n!}(z - \pi i)^n$$

$n=0$

Center: $z = \pi i$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n^n}{n!}\cdot\frac{(n+1)!}{(n+1)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n^n}{n!}\cdot\frac{(n+1)\,n!}{(n+1)(n+1)^n}\right|$$

$$= \lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{(1+1/n)^n} = \frac{1}{e}$$

PROBLEM SET 15.2

9. Find the center and the radius of convergence.

$$\sum_{n=0}^{\infty} \frac{n(n-1)}{3^n}(z-i)^{2n}$$

Center: $z = i$

$$R^2 = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n(n-1)}{3^n} \cdot \frac{3^{n+1}}{(n+1)n}\right| = \lim_{n\to\infty}\frac{3(n-1)}{n+1} = 3$$

$$R = \sqrt{3}$$

PROBLEM SET 15.2

10. Find the center and the radius of convergence.

$$\sum_{n=0}^{\infty} \frac{(z-2i)^n}{n^n}$$

Center: $z = 2i$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{n^n}\cdot\frac{(n+1)^{n+1}}{1}\right|$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n (n+1) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n (n+1) = \lim_{n\to\infty} e\,(n+1) = \infty$$

PROBLEM SET 15.2

11. Find the center and the radius of convergence.

$$\sum_{n=0}^{\infty} \left(\frac{2-i}{1+5i}\right) z^n$$

Center: $z = 0$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{2-i}{1+5i} \frac{1+5i}{2-i} \right| = 1$$

PROBLEM SET 15.2

12. Find the center and the radius of convergence.

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{8^n} z^n$$

Center: $z = 0$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n n}{8^n} \cdot \frac{8^{n+1}}{(-1)^{n+1}(n+1)} \right| = \lim_{n\to\infty} \frac{8n}{n+1} = 8$$

PROBLEM SET 15.2

13. Find the center and the radius of convergence.

$$\sum_{n=0}^{\infty} 16^n (z+i)^{4n}$$

Center: $z = -i$

$$R^4 = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{16^n}{16^{n+1}} \right| = \frac{1}{16}$$

$$R = \sqrt[4]{1/16} = 1/2$$

PROBLEM SET 15.2

14. Find the center and the radius of convergence.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} z^{2n}$$

Center: $z = 0$

$$R^2 = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n}{2^{2n}(n!)^2} \cdot \frac{2^{2n+2}[(n+1)!]^2}{(-1)^{n+1}} \right|$$

$$= \lim_{n\to\infty} 4(n+1)^2 = \infty$$

PROBLEM SET 15.2

15. Find the center and the radius of convergence.

$$\sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} (z - 2i)^n$$

Center: $z = 2i$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(2n)!}{4^n (n!)^2} \cdot \frac{4^{n+1} [(n+1)!]^2}{(2n+2)!} \right|$$

$$= \lim_{n\to\infty} \frac{4(n+1)^2}{(2n+2)(2n+1)} = 1$$

PROBLEM SET 15.2

16. Find the center and the radius of convergence.

$$\sum_{n=0}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n$$

Center: $z = 0$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(3n)!}{2^n (n!)^3} \bullet \frac{2^{n+1}[(n+1)!]^3}{(3n+3)!}\right|$$

$$= \lim_{n\to\infty}\frac{2(n+1)^3}{(3n+3)(3n+2)(3n+1)} = \frac{2}{27}$$

PROBLEM SET 15.2

17. Find the center and the radius of convergence.

$$\sum_{n=1}^{\infty} \frac{2^n}{n(n+1)} z^{2n+1}$$

Center: $z = 0$

$$R^2 = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{2^n}{n(n+1)} \bullet \frac{(n+1)(n+2)}{2^{n+1}}\right| = \lim_{n\to\infty}\frac{n+2}{2n} = \frac{1}{2}$$

$$R = \frac{1}{\sqrt{2}}$$

PROBLEM SET 15.2

18. Find the center and the radius of convergence.

$$\sum_{n=0}^{\infty} \frac{2(-1)^n}{\sqrt{\pi}(2n+1)n!} z^{2n+1}$$

Center: $z = 0$

$$R^2 = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{2(-1)^n}{\sqrt{\pi}\,(2n+1)n!} \cdot \frac{\sqrt{\pi}\,(2n+3)(n+1)!}{2(-1)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \frac{(2n+3)(n+1)}{2n+1} = \infty$$

$R = \infty$