Introduction to Discrete Math

Felipe P. Vista IV

Reminder

- Everybody, make sure that your name in ZOOM is in the following format:
  - Ex: 202054321 Juan Dela Cruz

Not changing your name to this format

* you **will** be marked Absent * → absent?

Course Outline

- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion, Logic, Invariants
- Probability & Combinatronics
  - Counting, Probability, Random Variables
- Graph Theory
  - Graphs (cycles, classes, parameters)
- Number Theory & Cryptography
  - Arithmetic in modular form
  - Intro to Cryptography

Mathematical Thinking – Combinatronics & Probability

Advanced Counting

PROBLEMS IN COMBINATRONICS

- Distributing Assignments to People
- Distribution of Candies to Kids
- Numbers with Fixed Sum of Digits
- Numbers with Non-Increasing Digits
- Splitting into Working Groups

Distributing Assignments to People

Problem

Suppose there are *4* people and *9* different assignments. Each person should receive one assignment. Assignments for different people should be different. How many ways are there to do it?

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- We have count selections of assignments for 4 people
- In this problem, people are different, so selection is ordered
- No assignment given to two persons simultaneously, so no repetitions
- Hence, we are dealing with k-permutations

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Persons	1	2	3	4
Number of options				

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- Answer is $9x8x7x6 = 3,024$

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- Answer is $9x8x7x6 = 3,024$
- Need to count permutations

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There are *4* people & *9* different assignments. Need to distribute all assignments among them. No assignment should be assigned to two people. Every person can be given arbitrary number of assignments from *0* to *9*. How many ways are there to do it?

* arbitrary – random choice, no reason or system

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- Can try to look at persons one by one

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- The first person assigned arbitrary subset

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  - *we know how to count number of subsets*

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  - *we know how to count number of subsets*
- For second person, the number of options left

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- Can try to look at persons one by one
- The first person assigned arbitrary subset
  - *we know how to count number of subsets*
- For second person, the number of options left
  - *depends on what we chose for the first person*

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- Idea: Look from the other point of view

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- Idea: Look from the other point of view
- Do not give assignments to people

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- Seems tricky
- Idea: Look from the other point of view
- Do not give assignments to people
  - *Assign people to assignments instead*

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Number of options									

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- So the answer is $4^9=262\ 144$

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- So the answer is $4^9 = 262\ 144$
- Just needed to count tuples
- But need also to look from the other side

- Distributing Assignments to People
- Distribution of Candies to Kids
- Numbers with Fixed Sum of Digits
- Numbers with Non-Increasing Digits
- Splitting into Working Groups

Distribution of Candies to Kids

Problem

There are *15* identical candies. How many ways are there to distribute them among *7* kids?

- Given each candy, we choose one of 7 kids

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  - *Order do not matter*

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- Given each candy, we choose one of 7 kids
- Repetitions are allowed
  - *Same kid can receive a candy several times*
- Candies are identical
  - *Order do not matter*
- We are dealing in combinations with repetitions!

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- Number of kids is the number of options
- Hence, answer is:

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There are *15* identical candies. How many ways are there to distribute them among *7* kids?

- Number of candies is the size of a combination $= k$
- Number of kids is the number of options $= n$
- Hence, answer is:

$$\binom{k+n-1}{n-1} \qquad 15+7-1 \Rightarrow \binom{21}{6}$$

Distribution of Candies to Kids

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There are *15* identical candies. How many ways are there to distribute them among *7* kids?

- Number of candies is the size of a combination
- Number of kids is the number of options
- Hence, answer is:

$$\binom{15+(7-1)}{(7-1)} = \binom{21}{6} = 54,264$$

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- But we can reduce this problem to the previous one

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- But we can reduce this problem to the previous one
- We have to give each kid at least one candy
- So, let us just do it!

Distribution of Candies to Kids

Problem

There are *15* identical candies. How many ways are there to distribute them among *7* kids in such a way that each kid receives at least *1* candy?

- Now, we are left with *15* - *7* = *8* candies
- We can distribute them same as previous problem!
- Combination with repetitions,
  - *Combination size: 8; Options size: 7*
- So, answer is

$$\binom{8+(7-1)}{(7-1)} = \binom{14}{6} = 3{,}003$$

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- Compare answer to previous problem: $54{,}264$

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- So, answer is: $\binom{8+(7-1)}{(7-1)} = \binom{14}{6} = 3{,}003$
- Compare answer to previous problem: $54{,}264$
- Lots of ways to share candies that will leave some of the kids without a candy

Problems in Combinatronics

- Distributing Assignments to People
- Distribution of Candies to Kids
- Numbers with Fixed Sum of Digits
- Numbers with Non-Increasing Digits
- Splitting into Working Groups

Numbers with Fixed Sum of Digits

Problem

How many non-negative integer numbers are there below $10,000$ such that their sum of digits is equal to 9?

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- Consider numbers as sequences of 4 digits

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- Consider numbers as sequences of *4* digits
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  - *But already unclear for the second one*

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  - *But already unclear for the second one*
- Idea: Look from the other side

Numbers with Fixed Sum of Digits

* * * *

- There are four positions

Numbers with Fixed Sum of Digits

* * * *

- There are four positions
- We split the weight *9* among them

Numbers with Fixed Sum of Digits

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- Assume they are "*0*" in the beginning, then add "*1*" nine times

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- Each time we pick one of the digits to increase
- Order does not matter, there are repetitions
- We have combinations of size *9* among *4* options
- The answer is $\binom{9+(4-1)}{(4-1)} = \binom{12}{3} = 220$

Numbers with Fixed Sum of Digits

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- Combinations of size *10* from *4* options
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- Is everything ok? Let's check!

Numbers with Fixed Sum of Digits

The following code searches through all combinations:

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import itertools as it
count = 0
for d in it.product(range(10), repeat = 4):
    if sum(d) == 10:
        count += 1
print(count)
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What went wrong?

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- But digits should be at most *9*
- What should we know

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- With our approach, we can assign all ten ones to the same position
- But digits should be at most *9*
- What should we know
- Just subtract the number of things that we should not have counted

Numbers with Fixed Sum of Digits

* * * *

- What we should not have counted

Numbers with Fixed Sum of Digits

* * * *

- What we should not have counted
- Assignments of all ten ones to the same digit

Numbers with Fixed Sum of Digits

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- What we should not have counted
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- But there are just *4* of them!

Numbers with Fixed Sum of Digits

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- What we should not have counted
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- So our previous answer was off by just *4*
- The correct answer is $286 - 4 = 282$

- Distributing Assignments to People
- Distribution of Candies to Kids
- Numbers with Fixed Sum of Digits
- Numbers with Non-Increasing Digits
- Splitting into Working Groups

Numbers with Non-Increasing Digits

Problem

How many four-digit numbers are there such that their digits are not increasing? Three-digit numbers are also four-digit, they just start with 0.

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  - *but number of options for second number depends on first number*
- Idea: Look from the other side

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- Once we pick four digits, our number is uniquely determined

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- The answer is

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- The answer is $\binom{4+(10-1)}{(10-1)} = \binom{13}{9} = 715$

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There are 12 students in the class. How many ways are there to split them into working groups of size 2 to work on the same assignment?

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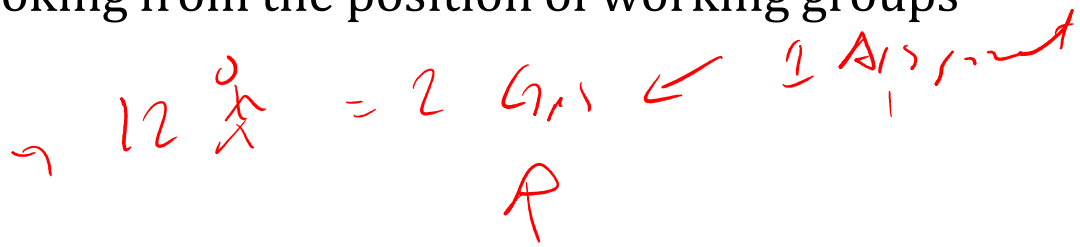
- There are several ways to solve it
- But we need to combine several ideas!

Splitting into Work Groups

- One solution goes by looking from the position of working groups



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- Are we done? No!

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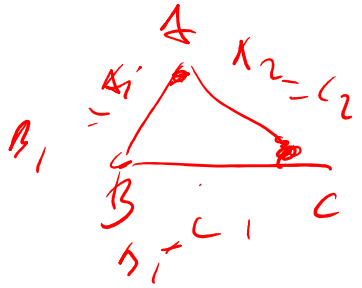
- Enumerate people as numbers 1 to 12
- For example, we count this splitting into groups
- Order between groups also do not matter
- What to do?
- Apply old idea!
  - We have counted each splitting 6! times, for each permutation of 6 groups



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Summary

- We have discussed several standard settings in Combinatronics
- Vast majority of counting problems we meet in practice fall into these setting
- There are more complicated situations that we have not discussed
- There are situations that are so complicated, they stay unresolved
- Next topic, we proceed to Probability

Thank you.