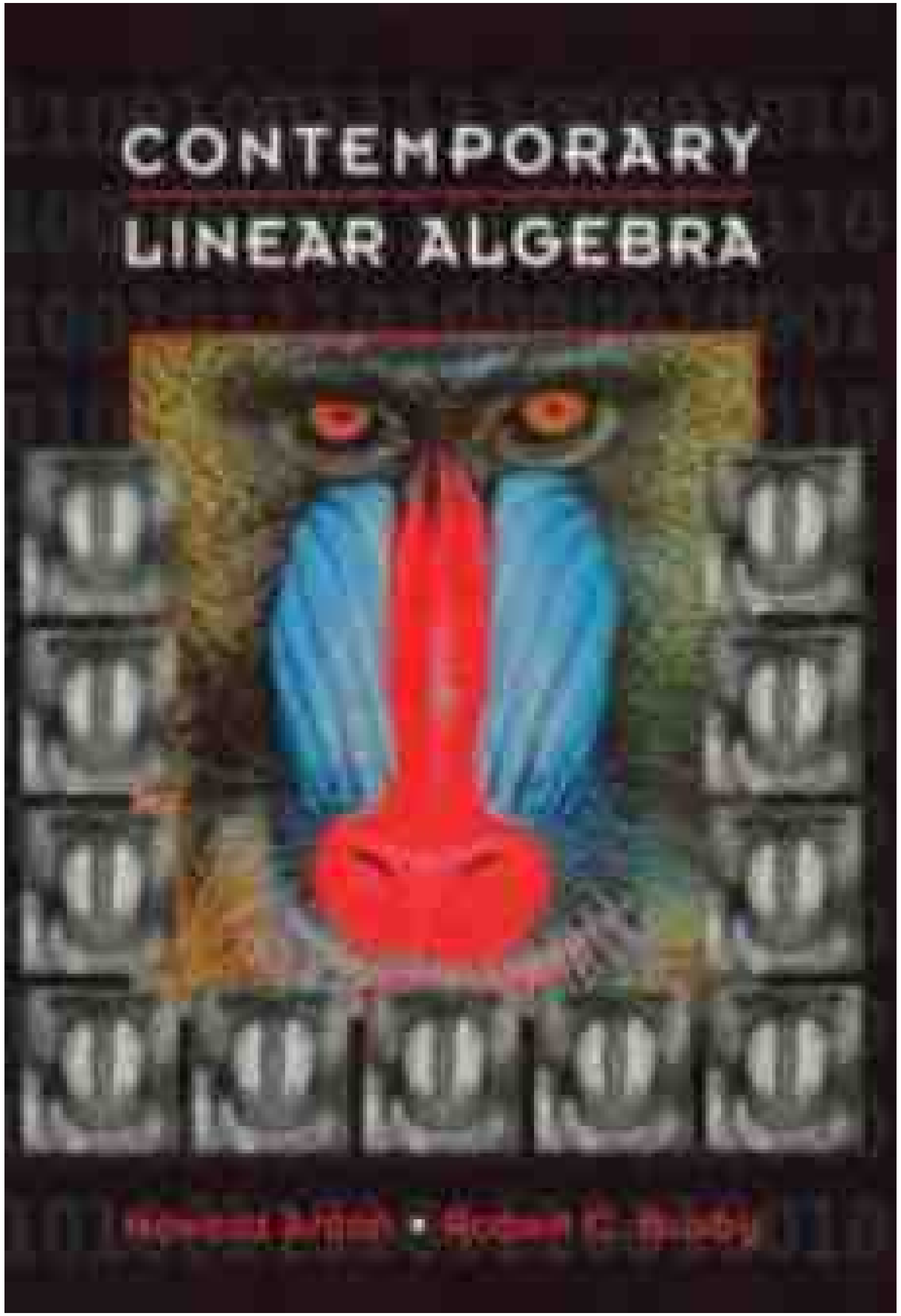


CHAPTER 2

Systems of Linear Equations

2.1 Introduction to Systems of Linear Systems

LINEAR SYSTEMS

A line in R^2 : $a_1 x + a_2 y = b$ (a_1, a_2 not both 0)

A plane in R^3 : $a_1 x + a_2 y + a_3 z = b$ (a_1, a_2, a_3 not all 0)

A linear equation in R^n :

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

$a_1, a_2, \cdots, a_n, b$: constatnts

$a_1, a_2, \cdots, a_n$: not all zero

A Homogeneous Linear Equation :

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

Example 1 Linear Equations

Linear equations:

involve : only the first power variables

not involve : trigonometric, logarithmic, exponential functions, etc.

Examples of linear equations:

$$x + 3y = 7 \qquad x_1 - 2x_2 - 3x_3 + x_4 = 0 \qquad x_1 + x_2 + \cdots + x_n = 1$$

Examples of nonlinear equations:

$$x + 3y^2 = 7 \qquad 3x + 2y - xy = 5$$

$$\sin x + y = 0 \qquad \sqrt{x_1} + 2x_2 + x_3 = 1$$

System of Linear Equations or *a Linear System*

A system of linear equations or a ***linear system*** :

Example :

$$4x_1 - x_2 + 3x_3 = -1$$
$$3x_1 + x_2 + 9x_3 = 0 \tag{5}$$

A general linear system of m equations in the n unknowns $x_1, x_2, \cdots, x_n$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned} \tag{6}$$

Linear Systems with Two and Three Unknowns

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

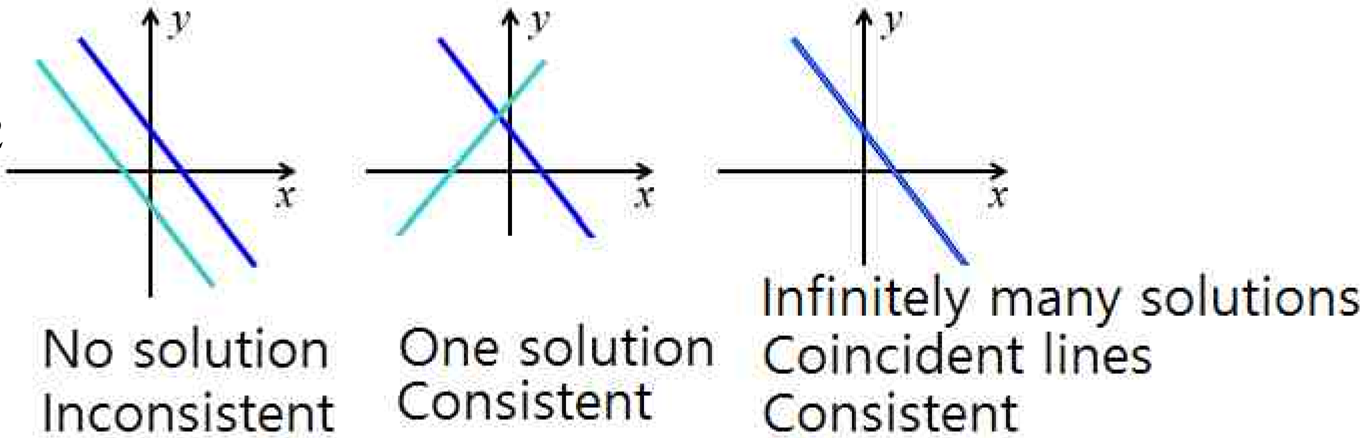


Fig. 2.1.1

1. Parallel and distinct: no intersection and consequently no solution
2. Intersect at only one point: exactly one solution
3. Coincide : infinitely many solutions

1. Consistent: at least one solution
2. Inconsistent: no solution

Linear Systems with Two and Three Unknowns

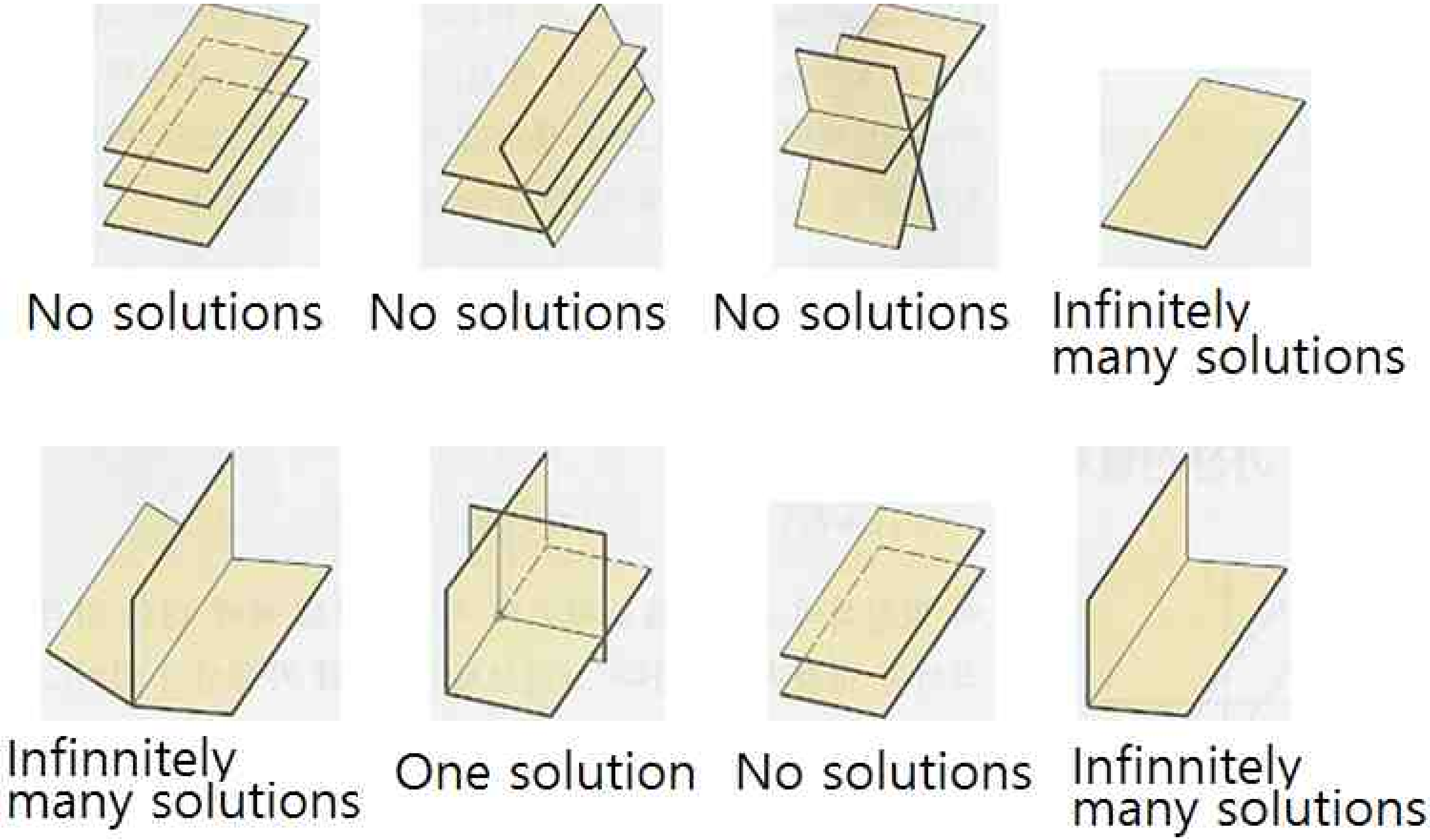


Fig. 2.1.2

Theorem 2.1.1

Theorem 2.1.1 *Every system of linear equations has zero, one, or infinitely many solutions: there are no other possibilities.*

Example 2 a linear system.

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$$

➡ One solution: (7/3, 4/3)

Example 3 a linear system.

$$\begin{aligned} x + y &= 4 \\ 3x + 3y &= 6 \end{aligned}$$

➡ No solution

Example 4 a linear system.

$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$

➡ Infinitely many solutions

Example 5 a linear system.

$$\begin{aligned} x - y + 2z &= 5 \\ 2x - 2y + 4z &= 10 \\ 3x - 3y + 6z &= 15 \end{aligned}$$

➡ Infinitely many solutions

Augmented Matrices and Elementary Row Operations

Linear system → Augmented Matrix 붙인 행렬

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
\qquad
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{bmatrix}
$$

Example :

Linear system → Augmented Matrix

$$
\begin{aligned}
x + y + 2z &= 9 \\
2x + 4y - 3z &= 1 \\
3x + 6y - 5z &= 0
\end{aligned}
\qquad
\begin{bmatrix}
1 & 1 & 2 & 9 \\
2 & 4 & -3 & 1 \\
3 & 6 & -5 & 0
\end{bmatrix}
$$

Augmented Matrices and Elementary Row Operations-cont

The basic method for solving a linear system is to use three types of operations.

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a multiple of one equation to another.

For linear systems represented by an augmented matrix,

Elementary row operations on a matrix(기본 행연산)

1. Multiply a row through by a nonzero constant.
2. Interchange rows.
3. Add a multiple of one row to another.

Example 6 Using Elementary Row Operations and Aug. Matrix

Solve the linear system.

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Sol.

$$R_1 \times (-2) + R_2 : \quad \begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$$R_1 \times (-3) + R_3 : \quad \begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Example 6 Using Elementary Row Operations -cont

$$R_1\times(-3)+R_3: \quad \begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned} \Leftrightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$R_2\times(1/2): \quad \begin{aligned} x + y + 2z &= 9 \\ y - 7/2z &= -17/2 \\ 3y - 11z &= -27 \end{aligned} \Leftrightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$R_2\times(-3)+R_3: \quad \begin{aligned} x + y + 2z &= 9 \\ y - 7/2z &= -17/2 \\ -1/2z &= -3/2 \end{aligned} \Leftrightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -1/2 & -3/2 \end{bmatrix}$$

$$R_3\times(-2): \quad \begin{aligned} x + y + 2z &= 9 \\ y - 7/2z &= -17/2 \\ z &= 3 \end{aligned} \Leftrightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Example 6 Using Elementary Row Operations -cont

$$\begin{aligned} x + y + 2z &= 9 \\ y - 7/2z &= -17/2 \\ z &= 3 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$R_2 \times (-1) + R_1: \quad \begin{aligned} x \quad + 11/2\, z &= 35/2 \\ y - 7/2\, z &= -17/2 \\ z &= 3 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 0 & 11/2 & 35/2 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} R_3 \times (-11/2) + R_1: \quad x \quad &= 1 \\ R_3 \times (7/2) + R_2: \quad y \quad &= 2 \\ z &= 3 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Example 7 Linear Cobminatins

Express w=(9, 1, 0) as a linear combination of v_1=(1,2,3), v_2=(1,4,6) and v_3=(2,-3, -5).

Sol.

Let $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$(9,1,0) = c_1(1,2,3) + c_2(1,4,6) + c_3(2,-3,-5)$$

$$\begin{aligned} c_1 + c_2 + 2c_3 &= 9 \\ 2c_1 + 4c_2 - 3c_3 &= 1 \\ 3c_1 + 6c_2 - 5c_3 &= 0 \end{aligned}$$

$$c_1 = 1,\ c_2 = 2,\ c_3 = 3$$

$$\mathbf{w} = \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3$$

Considerations in Solving Linear Systems

Echelon Forms(행 사다리꼴)

In example 6, $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$: reduced echelon form
기약 행 사다리꼴

$$\Rightarrow \quad \begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

Reduced Row Echelon Form(기약 행 사다리꼴)

A Matrix in Reduced Row Echelon Form

1. If a row does not consist entire of zeros, then the first nonzero member in the row is 1. We call this a *leading* 1.
2. Any rows that consist entirely of zeros are placed at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else.

A Matrix in Row Echelon Form 행 사다리꼴

A matrix that satisfies the first three properties.

Example 1 Row Echelon and Reduced Row Echelon Form

Determine whether the matrices are row echelon form or reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Sol.

Matrices in the first row: RREF
Matrices in the second row: REF

Example 2 More on Row Echelon and RREF

Determine whether the matrices are row echelon form or reduced row echelon form.

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad \begin{bmatrix}1&0&0&*\\0&1&0&*\\0&0&1&*\\0&0&0&0\end{bmatrix},\quad \begin{bmatrix}1&0&*&*\\0&1&*&*\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \begin{bmatrix}0&1&*&0&0&0&*&*&0&*\\0&0&0&1&0&0&*&*&0&*\\0&0&0&0&1&0&*&*&0&*\\0&0&0&0&0&1&*&*&0&*\\0&0&0&0&0&0&0&0&1&*\end{bmatrix}$$

$$\begin{bmatrix}1&*&*&*\\0&1&*&*\\0&0&1&*\\0&0&0&1\end{bmatrix},\quad \begin{bmatrix}1&*&*&*\\0&1&*&*\\0&0&1&*\\0&0&0&0\end{bmatrix},\quad \begin{bmatrix}1&*&*&*\\0&1&*&*\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \begin{bmatrix}0&1&*&*&*&*&*&*&*&*\\0&0&0&1&*&*&*&*&*&*\\0&0&0&0&1&*&*&*&*&*\\0&0&0&0&0&1&*&*&*&*\\0&0&0&0&0&0&0&0&1&*\end{bmatrix}$$

Sol.

Matrices in the first row: RREF
Matrices in the second row: REF

Example 3 Unique Solution

Show that a linear system, expressed by the augmented matrix at the right by elementary row operations, has a unique solution.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

Sol.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= 3 \\ x_2 &= -1 \\ x_3 &= 0 \\ x_4 &= 5 \end{aligned}$$

Example 4 Linear Systems in Three Unknowns

Solve the following systems.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Sol.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ \boxed{0 \quad 0 \quad 0 \quad 1} \end{bmatrix} \longrightarrow 0x + 0y + 0z = 1$ ➔ **No solution !**

Example 4 Linear Systems in Three Unknowns

(b)

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x \quad + 3z &= -1 \\ y - 4z &= 2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= -1 - 3z \\ y &= 2 + 4z \end{aligned}$$

z : free variable
(자유 변수)

x, y : leading variable
(선행 변수)

Let the free variable $z = t$

Then, $x = -1 - 3t$

$y = 2 + 4t$

$z = t$

In vector expression,

$$\begin{aligned} (x, y, z) &= (-1 - 3t,\ 2 + 4t,\ t) \\ &= (-1, 2, 0) + t(-3, 4, 1) \end{aligned}$$

The line passing (-1, 2, 0) and parallel to (-3, 4, 1)

Example 4 Linear Systems in Three Unknowns

(c) $\begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

y, z : free variables

x : leading variable

$x - 5y + z = 4 \rightarrow x = 4 + 5y - z$

Let the free variables

$$y = s$$
$$z = t$$

Then, $x = 4 + 5s - t$

$$y = s$$
$$z = t$$

In vector expression, $(x, y, z) = (4 + 5s - t,\ \ s,\ \ t)$

$$= (4,0,0) + s(5,1,0) + t(-1,0,1)$$

The plane passing (4, 0, 0) and parallel to (5, 1, 0) and (-1, 0, 1)

General Solutions As Linear Combinations of Column Vectors

A set of parametric equations(or their vector equivalent) for the solution set of a linear system is commonly called a ***general solution***. See the solutions in Example 4 (b) and (c).

Usually, it is desirable to express a general as a linear combination of column vectors.

A general solution=constant vector + variable vector 1
+ variable vector 2 + …

For example, in the previous example 4,

(b): $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

(c): $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Gauss-Jordan and Gaussian Elimination

Step 1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in step 1.

$$R_1 \leftrightarrow R_2 \qquad \begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 3. Leading 1

$$(1/2)\times R_1 \rightarrow R_1: \qquad \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Gauss-Jordan and Gaussian Elimination-cont

Step 4. Make all entries below leading 1 become zeros.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \Rightarrow (-2)\times R_1 + R_3 \to R_3 : \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Step 5. Repeat step1-step 4 until the *entire* matrix is in row echelon form.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Leftmost nonzero column in the submatrix

$$(-1/2)\times R_2 \to R_2 : \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Gauss-Jordan and Gaussian Elimination-cont

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \Rightarrow (-5)\times R_2 + R_3 \to R_3 : \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}$$

$$(2)\times R_3 \to R_3 : \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Step 6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$(7/2)\times R_3 + R_2 \to R_2 : \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Gauss-Jordan and Gaussian Elimination-cont

Gaussian Elimination : Steps from 1 to 5,
Forward phase only

Gauss-Jordan Elimination : Steps from 1 to 6,
Forward phase and Backward phase

Some Facts about Echelon Forms

1. Every matrix has a unique reduced row echelon form.

2. Row echelon forms are not unique.

Pivot positions: the positions with leading 1's

Pivot columns: the columns with the leading 1's

Example 5 Solving a Linear System by Gauss-Jordan Elimination

Solve the following linear system by Gauss-Jordan elimination.

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad\qquad - 2x_5 \qquad\quad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 \qquad + 15x_6 &= 5 \\ 2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

Sol.

Aug. Matrix

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix} \begin{matrix} \\ :(-2)\times R_1 + R_2 \to R_2 \\ \\ :(-2)\times R_1 + R_4 \to R_4 \end{matrix}$$

$$\Rightarrow \begin{matrix} \\ (-1)\times R_2 \to R_2 : \\ (-5)\times R_2 + R_3 \to R_3 : \\ (-4)\times R_2 + R_4 \to R_4 : \end{matrix} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

Example 5 Solving a Linear System - conti

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ :(1/6)\times R_3 \to R_3 \\ : R_3 \leftrightarrow R_4 \end{matrix}$$

Row Echelon Form

$$\begin{matrix} (2)\times R_2 + R_1 \to R_1 : \\ (-3)\times R_3 + R_2 \to R_2 : \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form

He corresponding system of equation is

$$\begin{aligned} x_1 + 3x_2 \quad + 4x_4 + 2x_5 &= 0 \\ x_3 + 2x_4 &= 0 \\ x_6 &= 1/3 \end{aligned} \Rightarrow \begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= 1/3 \end{aligned}$$

Example 5 Solving a Linear System - conti

$$x_1 = -3x_2 - 4x_4 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = 1/3$$

$$x_2 \to r \quad x_4 \to s \quad x_5 \to t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r-4s-2t \\ r \\ -2s \\ s \\ t \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix} + \begin{bmatrix} -3r-4s-2t \\ r \\ -2s \\ s \\ t \\ 0 \end{bmatrix}$$

Expressed as a linear combination of column vectors

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix} + r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Back Substitution(역 대입)

Some examples of vectors in higher-dimensional spaces

- Experimental data: (measured value1, mv_2, …, mv_n)
- Storage and warehousing:
  (# trucks in storage1, # in storage2, …, # in storage n)
- Electrical Circuits: $v=(v_1, v_2, v_3, v_4)$
- Graphical Images:
  (x-coordinate, y-coordinate, hue, saturation, brightness)
- Economics: $(s_1, s_2, …, s_n)$, s_n: the value for the sector n
- Mechanical systems: Assume six particles move at time t.
  $(x_1, x_2, …, x_6, v_1, v_2, …, v_6, t)$

Example 6 Gauss Elimination and Back Substitution

Solve the linear system in Example 5 using the row echelon form produced by Gaussian elimination.

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Sol.

$$\begin{aligned} x_1 + 3x_2 \quad + 4x_4 + 2x_5 \quad &= 0 \\ x_3 + 2x_4 \quad &= 0 \\ x_6 &= 1/3 \end{aligned}$$

Step 1. Solve the equations for the leading variables.

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \\ x_3 &= 1 - 2x_4 - 3x_6 \\ x_6 &= 1/3 \end{aligned}$$

Example 6 Gauss Elimination and Back Substitution - cont

Step 2. Beginning with the bottom equation and working upward, successively substitute each equation into all the equation above it.

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \\ x_3 &= 1 - 2x_4 - 3x_6 \\ x_6 &= 1/3 \end{aligned} \xrightarrow{x_6 = 1/3} \begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= 1/3 \end{aligned}$$

$$\xrightarrow{x_3 = -2x_4} \begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= 1/3 \end{aligned}$$

Step 3. Assign arbitrary values for the free variables, if any.

$$\xrightarrow{x_2 = r,\ x_4 = s,\ x_5 = t} \begin{aligned} x_1 &= -3r - 4s - 2t, \\ x_2 &= r, \\ x_3 &= -2s, \end{aligned} \qquad \begin{aligned} x_4 &= s, \\ x_5 &= t, \\ x_6 &= 1/3 \end{aligned}$$

Homogeneous Linear Systems

A Homogeneous Linear System(동차선형계): *m* eqs. in *n* unknowns

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0
\end{array}
\qquad (8)
$$

Every homogeneous linear system is consistent, since (8) has the trivial solution.

$x_1 = 0, \quad x_2 = 0, \quad \cdots, \quad x_n = 0$: trivial solution(자명한 해)

All other solutions, if any, are called nontrivial solutions.(자명하지 않은 해)

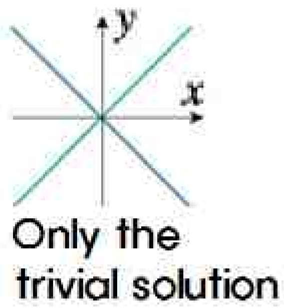
$$x_1 = s_1, \quad x_2 = s_2, \quad \ldots, \quad x_n = s_n$$

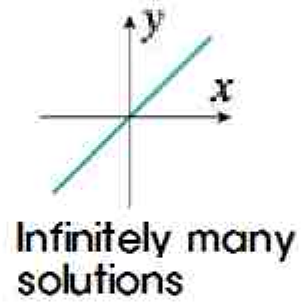
Homogeneity: scalar multiples of a solution is also a solution

$x_1 = ts_1, \quad x_2 = ts_2, \quad \ldots, \quad x_n = ts_n$ (t : scalar)

Theorem 2.2.1Solutions of a Homogeneous Linear System

Theorem 2.2.1 *A homogeneous linear system has only the trivial solution or it has infinitely many solutions; there are no other possibilities.*





Example 7 Homogeneous System with Nontrivial Solutions

Use Gauss-Jordan elimination to solve the homogeneous linear system

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\
5x_3 + 10x_4 \qquad\quad + 15x_6 &= 0 \\
2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}$$

Sol.

Augmented Matrix
$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix} \qquad (9)$$

Matrix in RREF
$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (10) \Rightarrow \quad \begin{aligned} x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad &= 0 \\ x_3 + 2x_4 \qquad\qquad &= 0 \\ x_6 &= 0 \end{aligned} \qquad (11)$$

Example 7 Homogeneous System - conti

$$\begin{aligned} x_1 + 3x_2 \quad + 4x_4 + 2x_5 \quad &= 0 \\ x_3 + 2x_4 \quad\quad &= 0 \\ x_6 &= 0 \end{aligned} \tag{11}$$

Solving for the leading variables:

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= 0 \end{aligned} \tag{12}$$

Assign arbitrary values for the free variables: $x_2 = r, \quad x_4 = s, \quad x_5 = t$

Find a general solution:

$$\begin{aligned} &x_1 = -3r - 4s - 2t, \; x_2 = r, \; x_3 = -2s, \\ &x_4 = s, \qquad\qquad\quad x_5 = t, \; x_6 = 0 \end{aligned} \tag{13}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{15}$$

Example 7 Homogeneous System - conti

$$\begin{aligned} x_1 + 3x_2 \quad + 4x_4 + 2x_5 \quad &= 0 \\ x_3 + 2x_4 \quad &= 0 \\ x_6 &= 0 \end{aligned} \qquad (11)$$

Solving for the leading variables:

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= 0 \end{aligned} \qquad (12)$$

Assign arbitrary values for the free variables:

$$x_2 = r, \quad x_4 = s, \quad x_5 = t$$

Example 7 Homogeneous System - conti

Find a general solution:

Parametric expression:

$$x_1 = -3r - 4s - 2t,\ x_2 = r,\ x_3 = -2s,\ x_4 = s,\ x_5 = t,\ x_6 = 0 \quad (13)$$

Vector expression:

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5, x_6) &= (-3r - 4s - 2t,\ r,\ -2s,\ s,\ t,\ 0) \\ &= r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) \\ &\quad + t(-2, 0, 0, 0, 1, 0)\end{aligned} \quad (14)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad (15)$$

The Dimension Theorem for Homogeneous Linear Systems

Example 7 shows two important points about solving linear homogeneous systems:

1. Elementary row operations do not alter columns of zeros. Thus, the final column of the augmented matrix and the RREF is a zero vector.

2. Rows of zeros, corresponding to the following equation, may be neglected.

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 = 0$$

The Dimension Theorem for Homogeneous Linear Systems

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0
\end{array}
\quad \Longrightarrow \quad
\begin{array}{llll}
x_{k1} & & & + \sum(\,) = 0 \\
& x_{k2} & & + \sum(\,) = 0 \\
& & \ddots & \quad \vdots \\
& & & x_{kr} + \sum(\,) = 0
\end{array}
\qquad (16)
$$

Sum of free variables

Theorem 2.2.2 *(Dimension Theorem for Homogeneous Systems)*
If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has n-r free variables.

Theorem 2.2.3 *A homogeneous linear system with more unknowns than equations has infinitely many solutions.*

Stability, Roundoff Error, Partial Pivoting

Large scale linear systems are solved on computers.

Roundoff errors, introduced by approximations, may degrade the answer. Algorithms in which this tends to happen are said to be *unstable*.

Thus, in the practical implementation of Gauss-Jordan and Gaussian elimination, it is standard to perform a row interchange in at each step to put the entry with the largest absolute value into the pivot position before dividing to introduce a leading 1. This is called *partial pivoting*.

$$
\begin{aligned}
0.0001x + 1.000y &= 1.000 \\
1.000x - 1.000y &= 0.000
\end{aligned}
\quad \xrightarrow{\text{Partial Pivoting}} \quad
\begin{aligned}
1.000x - 1.000y &= 0.000 \\
0.0001x + 1.000y &= 1.000
\end{aligned}
$$

2.3 Applications Of Linear Systems

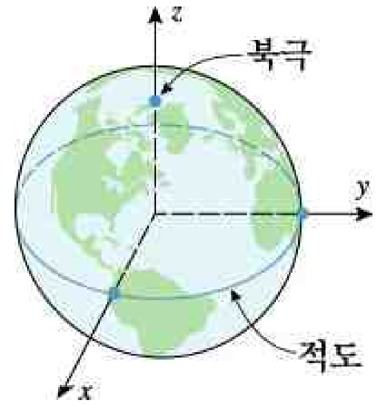
Global Positioning

GPS(Global Positioning System) is the system

- used by the military, ships, airplane pilots, surveyors, utility companies, automobiles, and hikers to locate current positions by communicating with a system of satellites
- operated by the U.S. Department of Defense
- nominally uses 24 satellites that orbit the Earth every 12 hours at a height of about 11,000miles.
- These satellites move in six orbital planes that have been chosen to make between five and eight satellites visible from any point on Earth.

2.3 Applications Of Linear Systems

The unit of the distance is Earth radius.



$$x^2 + y^2 + z^2 = 1$$

Speed of light: 0.469 Earth radii/10ms

$$d = 0.469(t - t_0)$$

Satellites position: (x_0, y_0, z_0), t_0 : time sending signal

Ship's position: (x, y, z), t : time receiving signal

$$d = 0.469(t - t_0) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 0.22(t - t_0)^2$$

Theoretically, three satellites are sufficient to locate the position of a ship. The ships, however, do not have clocks to compute t for global positioning. Thus the variable t must be regarded as an unknown.

Example 1 Global Positioning

Suppose that a ship received the following data from four satellites in which coordinates are measured in Earth radii and time in hundredth of a second after midnight.

Determine the position of the ship.

Satellite	Satellite Position	Time
1	(1.12, 2.10, 1.40)	1.06
2	(0.00, 1.53, 2.30)	0.56
3	(1.40, 1.12, 2.10)	1.16
4	(2.30, 0.00, 1.53)	0.75

Sol.

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=0.22(t-t_0)^2$$

From sat. 1: $(x-1.12)^2+(y-2.10)^2+(z-1.40)^2=0.22(t-1.06)^2$

$$2.24x+4.20y+2.80z-0.466t=x^2+y^2+z^2-0.22t^2+7.377$$

Similarly for other satellites,

$$3.06y+4.60z-0.246t=x^2+y^2+z^2-0.22t^2+7.562$$

$$2.80x+2.24y+4.20z-0.510t=x^2+y^2+z^2-0.22t^2+7.328$$

$$4.60x+3.06z-0.330t=x^2+y^2+z^2-0.22t^2+7.507$$

Example 1 Global Positioning-conti

$$
\begin{aligned}
2.24x + 4.20y + 2.80z - 0.466t &= x^2 + y^2 + z^2 - 0.22t^2 + 7.377 \\
3.06y + 4.60z - 0.246t &= x^2 + y^2 + z^2 - 0.22t^2 + 7.562 \\
2.80x + 2.24y + 4.20z - 0.510t &= x^2 + y^2 + z^2 - 0.22t^2 + 7.328 \\
4.60x \qquad\qquad + 3.06z - 0.330t &= x^2 + y^2 + z^2 - 0.22t^2 + 7.507
\end{aligned}
$$

$$
\begin{aligned}
R_1\text{-}R_2:&\quad 2.24x + 1.14y - 1.80z - 0.220t = -0.185 \\
R_1\text{-}R_3:&\quad -0.56x + 1.96y - 1.40z + 0.044t = 0.049 \\
R_1\text{-}R_4:&\quad -2.36x + 4.20y - 0.26z - 0.136t = -0.130
\end{aligned}
$$

$$
\begin{bmatrix}
1 & 0 & 0 & -0.153 & -0.139 \\
0 & 1 & 0 & -0.128 & -0.118 \\
0 & 0 & 1 & -0.149 & -0.144
\end{bmatrix}
\Rightarrow
\begin{aligned}
x &= -0.139 + 0.153t \\
y &= -0.118 + 0.128t \\
z &= -0.144 + 0.149t
\end{aligned}
$$

Example 1 Global Positioning-conti

$$2.24x + 4.20y + 2.80z - 0.466t = x^2 + y^2 + z^2 - 0.22t^2 + 7.377$$

$$x = -0.139 + 0.153t, \quad y = -0.118 + 0.128t, \quad z = -0.144 + 0.149t$$

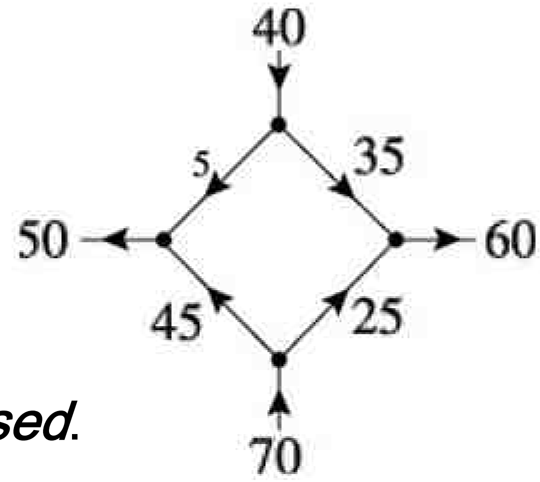
$$8.639 - 0.945t - 0.158t^2 = 0$$

$$t = 4.985$$

The position of the ship:

$$(x, y, z) = (-0.139 + 0.153t,\ -0.118 + 0.128t,\ -0.144 + 0.149t)$$
$$= (0.624,\ 0.519,\ 0.598)$$

Network Analysis



A *network* is a set of *branches* through which something flows.

The branches meet at points, called *nodes* or *junctions*.

Networks are of basic two types: *open* and *closed*.

In an open network the flow medium can enter and leave the network.

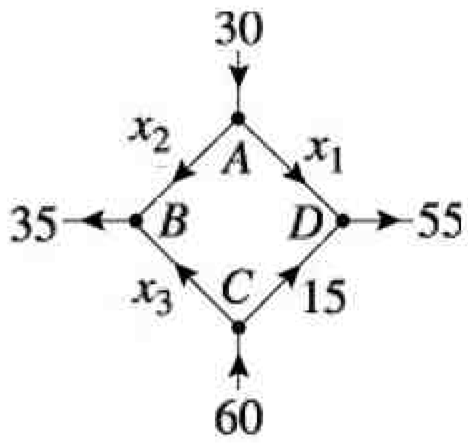
In a closed network the flow medium circulates continuously through the network.

Three basic properties of networks:

1. At any instant **One-Directional Flow** in a branch
2. **Flow Conservation** at a Node
3. **Flow Conservation** in the Network

Example 2 Network Analysis Using Linear Systems

Find the flow of rates and directions of flow.



Sol.

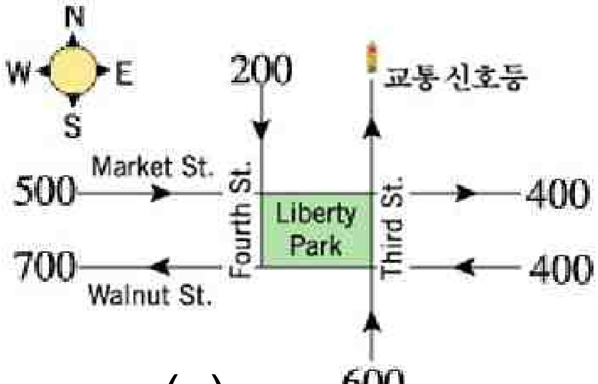
At node A: $x_1 + x_2 = 30$
At node B: $x_2 + x_3 = 35$
At node C: $x_3 + 15 = 60$
At node D: $x_1 + 15 = 55$

$$
\begin{aligned}
x_1 + x_2 \qquad &= 30 \\
x_2 + x_3 &= 35 \\
x_3 &= 45 \\
x_1 \qquad\qquad &= 40
\end{aligned}
$$

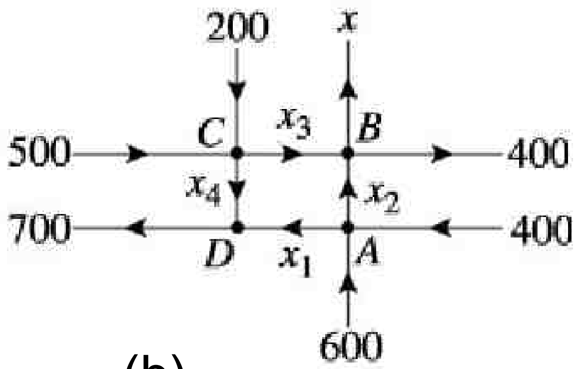
$$x_1 = 40, \quad x_2 = -10, \quad x_3 = 45$$

Example 3 Design of Traffic Patterns

Find the flow of rates and directions of flow.



(a)



(b)

Sol.

(a) Find the flow of rates and directions of flow.

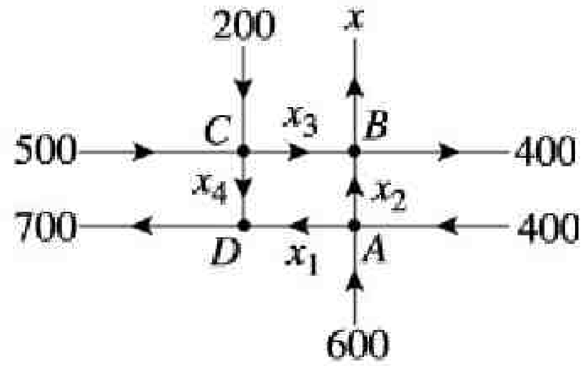
Flowing in: $500 + 400 + 600 + 200 = 1700$

Flowing out: $x + 700 + 400$

Example 3 Design of Traffic Patterns

(b) Find the flow of rates and directions of flow.

Intersection	Flow In		Flow Out
A	$400 + 600$	$=$	$x_1 + x_2$
B	$x_2 + x_3$	$=$	$400 + x$
C	$500 + 200$	$=$	$x_3 + x_4$
D	$x_1 + x_4$	$=$	700



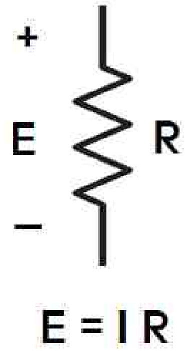
$$x_1 = 700 - t,\ x_2 = 300 + t,\ x_3 = 700 - t,\ x_4 = t \qquad (17)$$

$$0 \leq x_1 \leq 700, \quad 300 \leq x_2 \leq 1000,$$
$$0 \leq x_3 \leq 700, \quad 0 \leq x_4 \leq 700$$

Electrical Circuits

Ohm's Law:

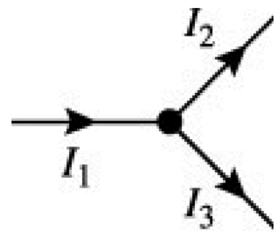
If a current I amperes passes through a resistor with a resistance of R ohms, then the resulting electrical potential drop is given by



$$E = IR$$

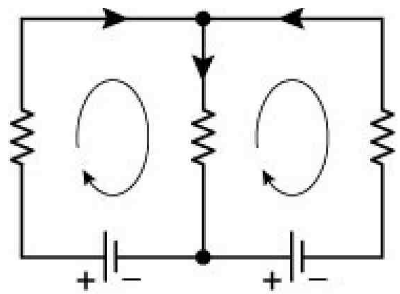
Kirchhoff's Current Law(KCL):

The algebraic sum into a node at instant is zero.



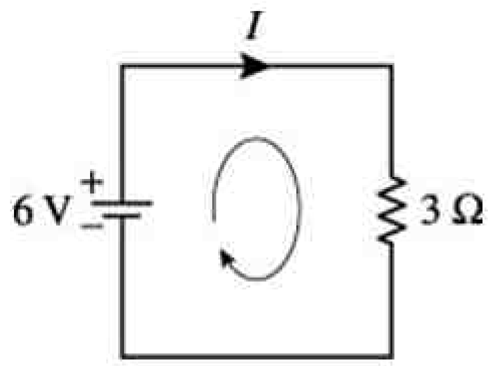
Kirchhoff's Voltage Law(KVL):

The algebraic sum of the voltages around any loop in a circuit is identically zero for all time.



Example 4 A Circuit with One Closed Loop

Determine the current.



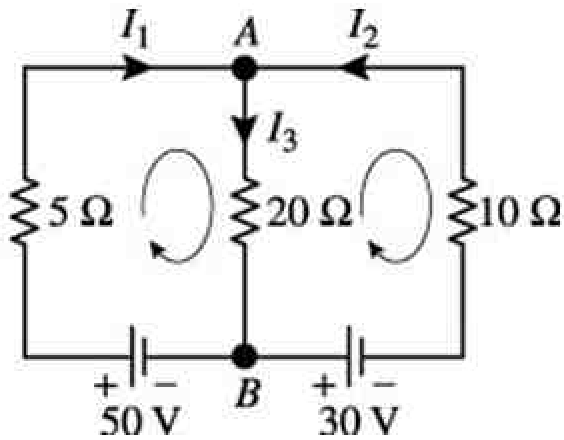
Sol.

Loop equation by KVL: $6 - 3I = 0$

$$I = 2 \text{ [A]}$$

Example 5 A Circuit with Three Closed Loops

Determine the currents.



Sol.

Current equation by KCL:

$$I_1 + I_2 - I_3 = 0$$

$$50 - 5I_1 - 20I_3 = 0$$

$$30 + 20I_3 + 10I_2 = 0$$

$$\begin{array}{rcrcrcr} I_1 & + & I_2 & - & I_3 & = & 0 \\ 5I_1 & & & + & 20I_3 & = & 50 \\ & & 10I_2 & + & 20I_3 & = & -30 \end{array}$$

$$I_1 = 6\ [\mathrm{A}],\ I_2 = -5\ [\mathrm{A}],\ I_3 = 1\ [\mathrm{A}]$$

Balancing Chemical Equations

Balance the following chemical equation

$$CH_4 + O_2 \longrightarrow CO_2 + H_2O \tag{10}$$

Sol.

$$x_1\,(CH_4) + x_2\,(O_2) \longrightarrow x_3\,(CO_2) + x_4\,(H_2O) \tag{11}$$

	Left Side		Right Side
Carbon	x_1	$=$	x_3
Hydrogen	$4x_1$	$=$	$2x_4$
Oxygen	$2x_2$	$=$	$2x_3 + x_4$

$$\begin{aligned}
x_1 \quad\quad - x_3 \quad\quad &= 0 \\
4x_1 \quad\quad\quad - 2x_4 &= 0 \\
2x_2 - 2x_3 - x_4 &= 0
\end{aligned}$$

Balancing Chemical Equations-conti

$$\begin{aligned} x_1 \quad - \; x_3 \qquad &= 0 \\ 4x_1 \qquad\quad - 2x_4 &= 0 \\ 2x_2 - 2x_3 - \; x_4 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{bmatrix}$$

$$x_1 = t/2, \quad x_2 = t, \quad x_3 = t/2, \quad x_4 = t$$

Example 6 Balancing Chemical Equations Using Linear Systems

Balance the following chemical equation

$$\underset{\text{hydrochloric acid}}{HCl} + \underset{\text{sodium phosphate}}{Na_3PO_4} \longrightarrow \underset{\text{phosphoric acid}}{H_3PO_4} + \underset{\text{sodium chloride}}{NaCl}$$

Sol.

$$x_1\,(HCl) + x_2\,(Na_3PO_4) \longrightarrow x_3\,(H_3PO_4) + x_4\,(NaCl) \tag{13}$$

Equating the number atoms:

$$\begin{aligned} 1x_1 &= 3x_3 && : \text{H} \\ 1x_1 &= 1x_4 && : \text{Cl} \\ 3x_2 &= 1x_4 && : \text{Na} \\ 1x_2 &= 1x_3 && : \text{P} \\ 4x_2 &= 4x_3 && : \text{O} \end{aligned}$$

Example 6 Balancing Chemical Equations - conti

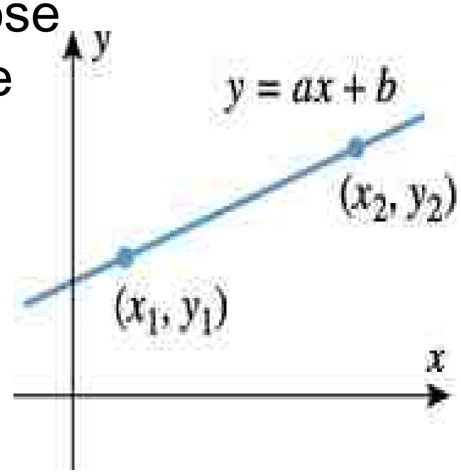
$$
\begin{aligned}
1x_1 &= 3x_3 \\
1x_1 &= 1x_4 \\
3x_2 &= 1x_4 \\
1x_2 &= 1x_3 \\
4x_2 &= 4x_3
\end{aligned}
\qquad
\begin{array}{rrrrl}
x_1 & & -3x_3 & & =0 \\
x_1 & & & -x_4 & =0 \\
 & 3x_2 & & -x_4 & =0 \\
 & x_2 & -\ x_3 & & =0 \\
 & 4x_2 & -4x_3 & & =0
\end{array}
$$

$$
\begin{bmatrix}
1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & -\frac{1}{3} & 0 \\
0 & 0 & 1 & -\frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
x_1 = t, \quad x_2 = t/3, \\
x_3 = t/3, \quad x_4 = t
$$

Polynomial Interpolation

Polynomial interpolation is to find a polynomial whose graph passes through a specified set of points in the plane.



Simplest example: to find a line passing through

$$(x_1, y_1),\ (x_2, y_2)$$

For n points : $(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3),\ \cdots,\ (x_n, y_n)$ (16)

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \tag{17}$$

Theorem 2.3.1 *(Polynomial Interpolation) Given any n points in the xy-plane that have distinct x-coordinates, there is a unique polynomial of degree n-1 or less whose graph passes through those n points.*

Proof

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$\begin{array}{ccccccccc}
a_0 & + & a_1 x_1 & + & a_2 x_1^2 & + \cdots + & a_{n-1} x_1^{n-1} & = & y_1 \\
a_0 & + & a_1 x_2 & + & a_2 x_2^2 & + \cdots + & a_{n-1} x_2^{n-1} & = & y_2 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
a_0 & + & a_1 x_n & + & a_2 x_n^2 & + \cdots + & a_{n-1} x_n^{n-1} & = & y_n
\end{array}$$

Theorem 2.3.1 Polynomial Interpolation-conti

$$
\begin{array}{ccccccccc}
a_0 & + & a_1x_1 & + a_2x_1^2 & + \cdots + & a_{n-1}x_1^{n-1} & = & y_1 \\
a_0 & + & a_1x_2 & + a_2x_2^2 & + \cdots + & a_{n-1}x_2^{n-1} & = & y_2 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
a_0 & + & a_1x_n & + a_2x_n^2 & + \cdots + & a_{n-1}x_n^{n-1} & = & y_n
\end{array}
$$

$$
\begin{bmatrix}
1 & x_1 & x_1^1 & \cdots & x_1^{n-1} & y_1 \\
1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & y_2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
1 & x_n & x_n^2 & \cdots & x_n^{n-1} & y_n
\end{bmatrix}
$$

Example 7 Polynomial Interpolation by Gauss-Jordan Elimination

Find a polynomial whose graph passes the following points

$$(1, 3), \quad (2, -2), \quad (3, -5), \quad (4, 0)$$

Sol.

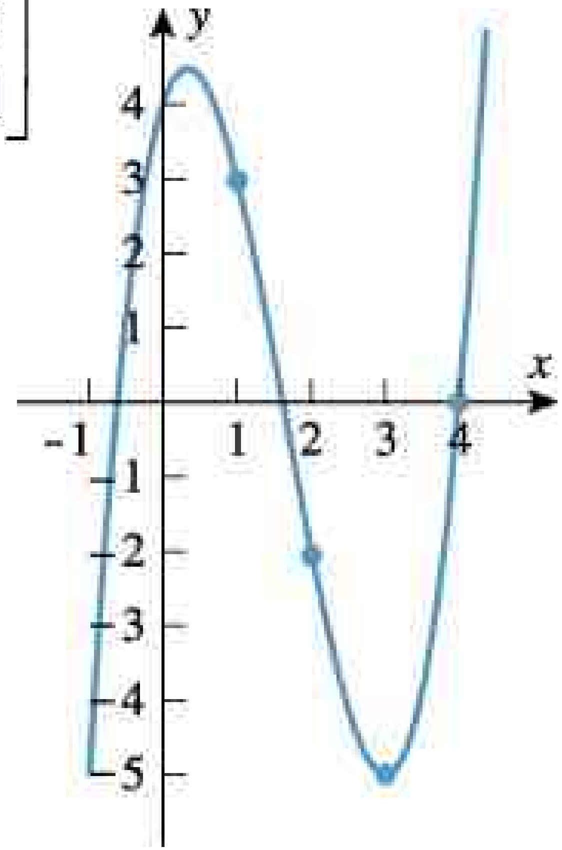
$$p(x) = y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & y_1 \\ 1 & x_2 & x_2^2 & x_2^3 & y_2 \\ 1 & x_3 & x_3^2 & x_3^3 & y_3 \\ 1 & x_4 & x_4^2 & x_4^3 & y_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{bmatrix}$$

Example 7 Polynomial Interpolation - conti

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$a_0 = 4,\ a_1 = 3,\ a_2 = -5,\ a_3 = 1$$



Example 8 Approximate Integration

Find a polynomial approximating the integrand of the following integration.

$$\int_0^1 \sin\left(\frac{\pi x^2}{2}\right) dx$$

Sol.

Approximating the integrand at the following five points.

$$x_0 = 0, \quad x_1 = 0.25, \quad x_2 = 0.5, \quad x_3 = 0.75, \quad x_4 = 1$$

Let $f(x) = \sin\left(\frac{\pi x^2}{2}\right)$

$$f(0) = 0, \quad f(0.25) = 0.098017, \quad f(0.5) = 0.382683,$$
$$f(0.75) = 0.77301, \quad f(1) = 1$$

Five points: $(0,0)$, $(0.25, 0.098017)$, $(0.5, 0.382683)$,
$(0.75, 0.77301)$, $(1, 1)$

Example 8 Approximate Integration

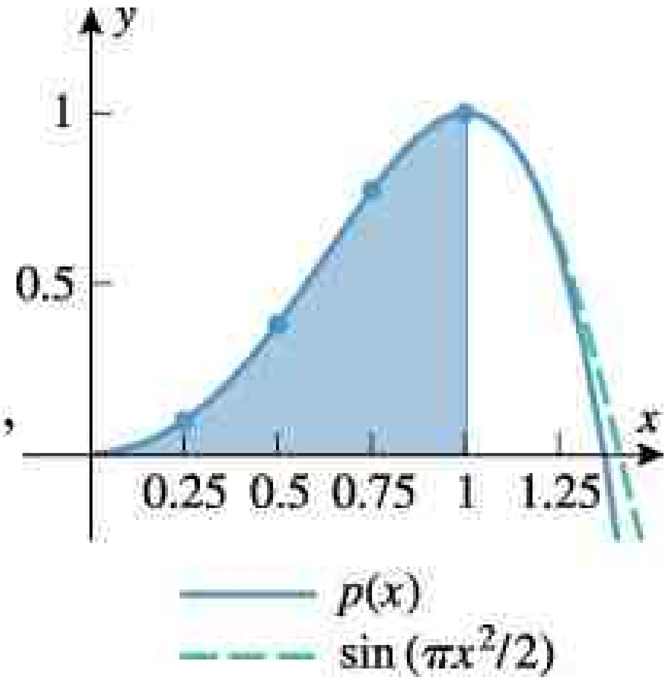
(0,0), (0.25, 0.098017), (0.5, 0.382683),
(0.75, 0.77301), (1, 1)

$$p(x) = y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$0 = a_0 + a_1 0 \quad + a_2 0^2 \quad + a_3 0^3 \quad + a_4 0^4$$
$$0.098017 = a_0 + a_1 0.25 + a_2 0.25^2 + a_3 0.25^3 + a_4 0.25^4$$
$$0.382683 = a_0 + a_1 0.5 \ + a_2 0.5^2 \ + a_3 0.5^3 \ + a_4 0.5^4$$
$$0.77301 = a_0 + a_1 0.75 + a_2 0.75^2 + a_3 0.75^3 + a_4 0.75^4$$
$$1 = a_0 + a_1 1 \quad + a_2 1^2 \quad + a_3 1^3 \quad + a_4 1^4$$

Example 8 Approximate Integration

$$\begin{bmatrix} 1 & 0 & 0^2 & 0^3 & 0^4 & 0 \\ 1 & 0.25 & 0.25^2 & 0.25^3 & 0.25^4 & 0.098017 \\ 1 & 0.5 & 0.5^2 & 0.5^3 & 0.5^4 & 0.382683 \\ 1 & 0.75 & 0.75^2 & 0.75^3 & 0.75^4 & 0.77301 \\ 1 & 1 & 1^2 & 1^3 & 1^4 & 1 \end{bmatrix}$$



$a_0 = 0$, $a_1 = 0.098796$, $a_2 = 0.762356$,
$a_3 = 2.14429$, $a_4 = -2.00544$

$$p(x) = 0.098796x + 0.762356x^2 + 2.14429x^3 - 2.00544x^4$$