CHAPTER 7
VIRTUAL WORK

BuildPix/(c)Construction Photography/Photoshoot

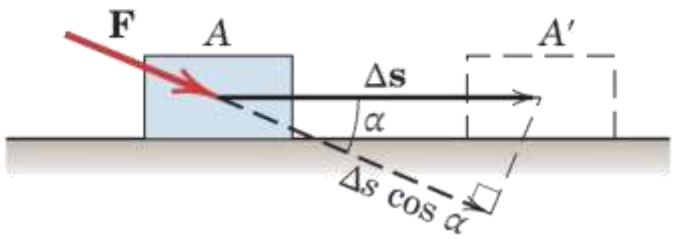
CHAPTER OUTLINE

Article 7/1 Introduction

- Introduction

Article 7/2 Work

- Work of a Force
  - Illustration

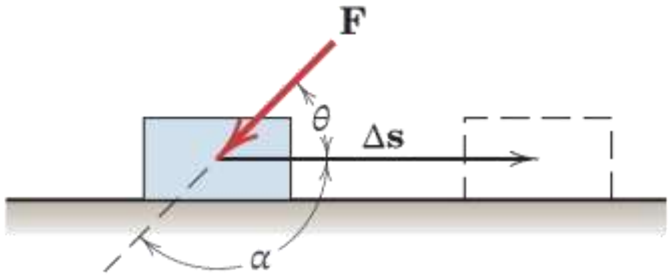


  - Calculation
    - $U = (F \cos \alpha)\Delta s = F(\Delta s \cos \alpha)$
  - Scalar Quantity

Article 7/2 – Sign Convention

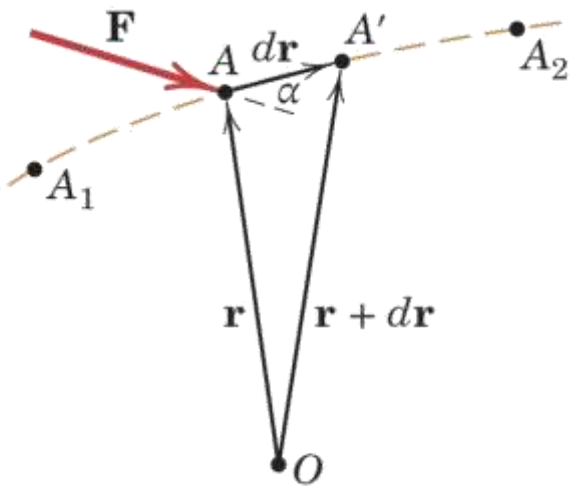
- Positive Work
  - Working component of *F* is in the same direction as the displacement.

- Negative Work
  - Working component of *F* is in the opposite direction as the displacement.

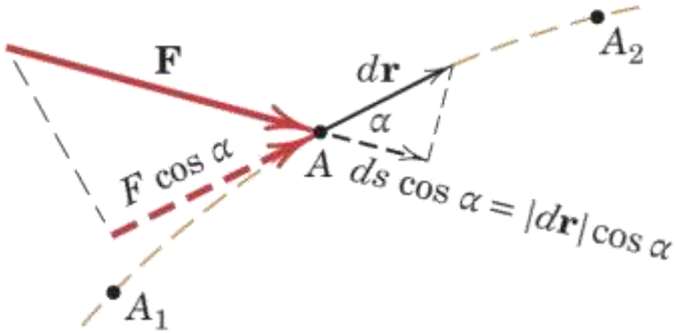


Article 7/2 – General Situation

- General Illustration of Work by a Force

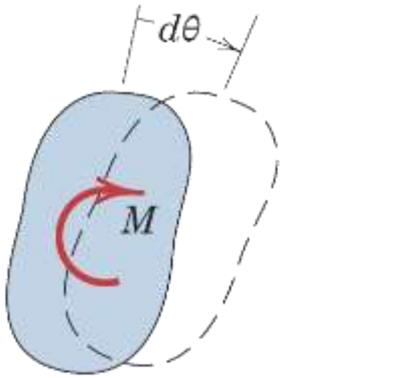


- General Expression for Work by a Force
  - $dU = \mathbf{F} \cdot d\mathbf{r}$

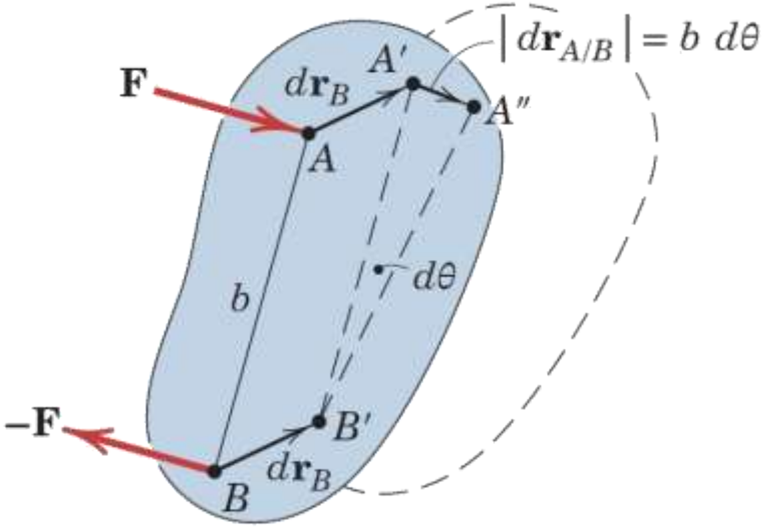


Article 7/2 – Work of a Couple

- General Illustration of Work by a Couple



- General Expression for Work by a Couple
  - $dU = M\,d\theta$



- Sign Convention
  - Positive if couple acts in the same sense as rotation.
  - Negative if couple acts in the opposite sense as rotation.

Article 7/2 – Dimensions and Units of Work

- Dimensions
  - (Force) × (Distance)

- SI Units
  - Joule (J)
  - J = N·m

- U.S. Customary Units
  - foot-pound (ft-lb)
  - Not to be confused with moment which is pound-foot (lb-ft) and a vector quantity, whereas work is a scalar quantity.

Article 7/2 – Virtual Work

- Virtual Displacement
  - Does not actually exist, but is assumed to exist.
  - Infinitesimal in size.
  - Consistent with the constrains of the system.
  - Instantaneous in time.

- Virtual Work done by a Force and Couple
  - $\delta U = \mathbf{F} \cdot d\mathbf{r}$ or $\delta U = F\, \delta s \cos \alpha$
  - $\delta U = M\, \delta\theta$

Article 7/3 Equilibrium

- Equilibrium of a Particle

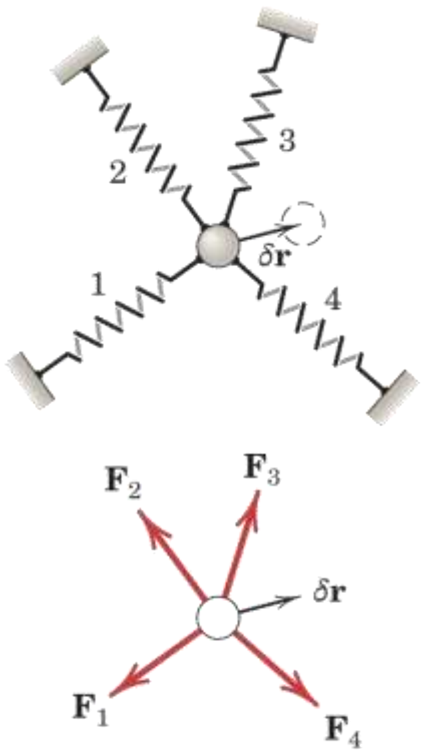
- Virtual Work done on the Particle

$$\delta U = \mathbf{F}_1 \cdot \delta\mathbf{r} + \mathbf{F}_2 \cdot \delta\mathbf{r} + \mathbf{F}_3 \cdot \delta\mathbf{r} + \cdots = \Sigma\mathbf{F} \cdot \delta\mathbf{r}$$

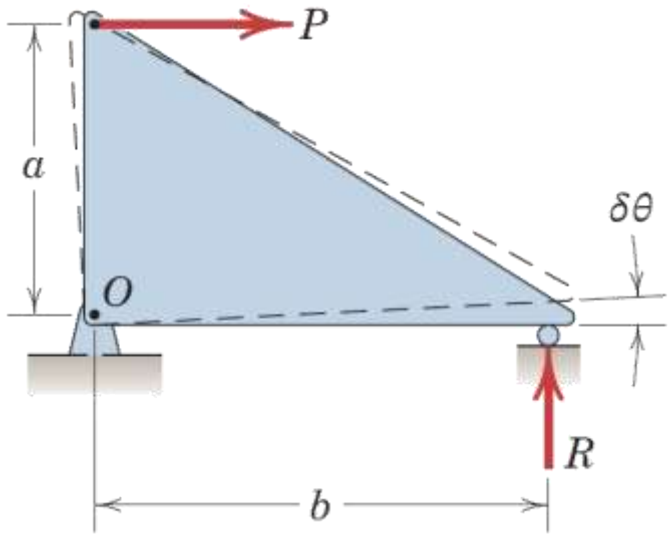
- Scalar Expression of Virtual Work

$$\delta U = \Sigma\mathbf{F} \cdot \delta\mathbf{r} = (\mathbf{i}\,\Sigma F_x + \mathbf{j}\,\Sigma F_y + \mathbf{k}\,\Sigma F_z) \cdot (\mathbf{i}\,\delta x + \mathbf{j}\,\delta y + \mathbf{k}\,\delta z)$$
$$= \Sigma F_x\,\delta x + \Sigma F_y\,\delta y + \Sigma F_z\,\delta z = 0$$

- Result



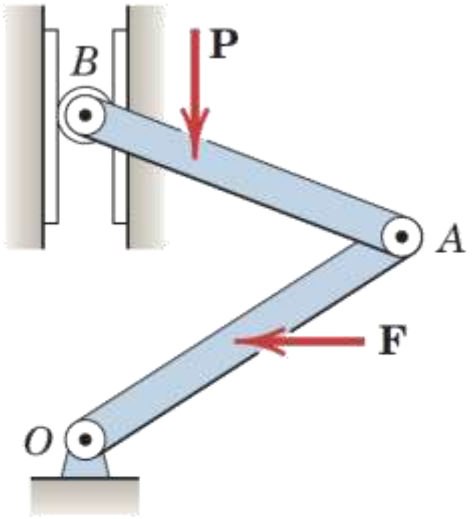
Article 7/3 – Equilibrium of a Rigid Body



- Equilibrium of a Rigid Body
- Determine the Roller Force, *R*
- Virtual Work Calculation
  - $\delta U = 0 = Rb\ \delta\theta - Pa\ \delta\theta$
- Result
  - $Pa - Rb = 0$
  - Equivalent to Moment Summation about Hinge *O*

Article 7/3 – Equilibrium of Ideal Systems of Rigid Bodies

- Ideal System
  - Friction is Neglected
  - Connections do not Absorb Energy



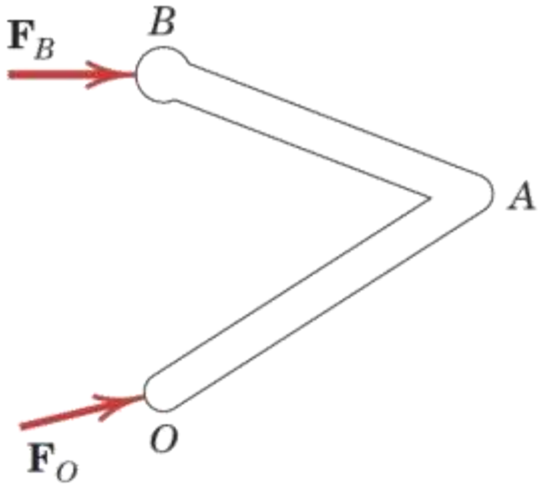
(a) Active forces

- Example

- Active Forces
  - Perform Virtual Work

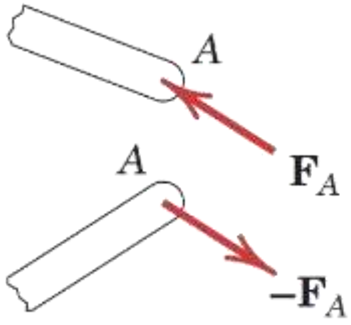
Article 7/3 – Other Types of Forces to Consider

- Reactive Forces
  - Act at Fixed Locations
  - Do not Perform Virtual Work



(b) Reactive forces

- Internal Forces
  - Occur in Equal and Opposite Pairs
  - Net Virtual Work is Zero



(c) Internal forces

Article 7/3 – Principle of Virtual Work (1 of 2)

- The Principle Stated
  - The virtual work done by external active forces on an ideal mechanical system in equilibrium is zero for any and all virtual displacements consistent with the constraints.
- Mathematical Expression
  - $\delta U = 0$

Article 7/3 – Principle of Virtual Work (2 of 2)

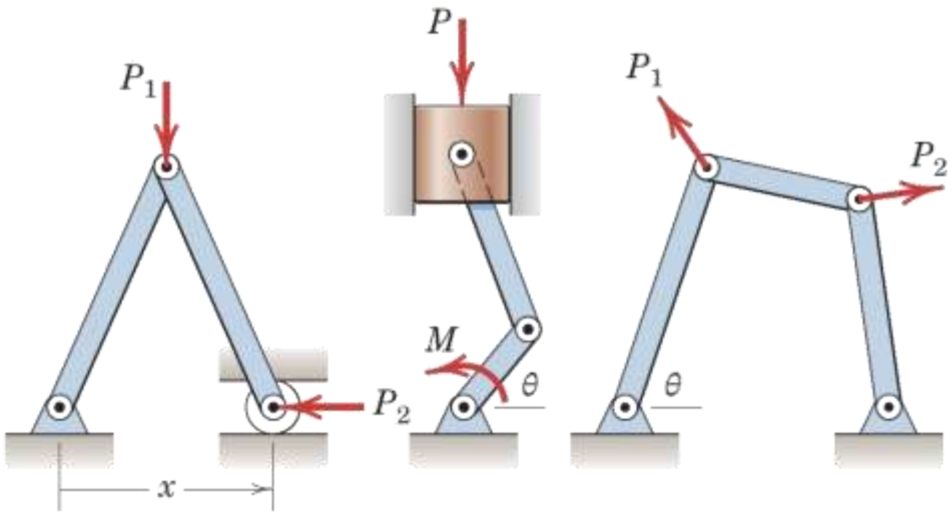
- Advantages of Virtual Work
  - It is not necessary to dismember ideal systems to establish the relations between active forces.
  - The relations between active forces may be determined directly without reference to the reactive forces.

- General Comments
  - Method of virtual work requires internal friction forces do negligible work, or else the work by friction must be considered.
  - Construct an active-force diagram which shows only the *active forces* which act on the body. The reactive forces and internal forces do not perform any virtual work and do not need to be included.

Article 7/3 – Degrees of Freedom (1 of 2)

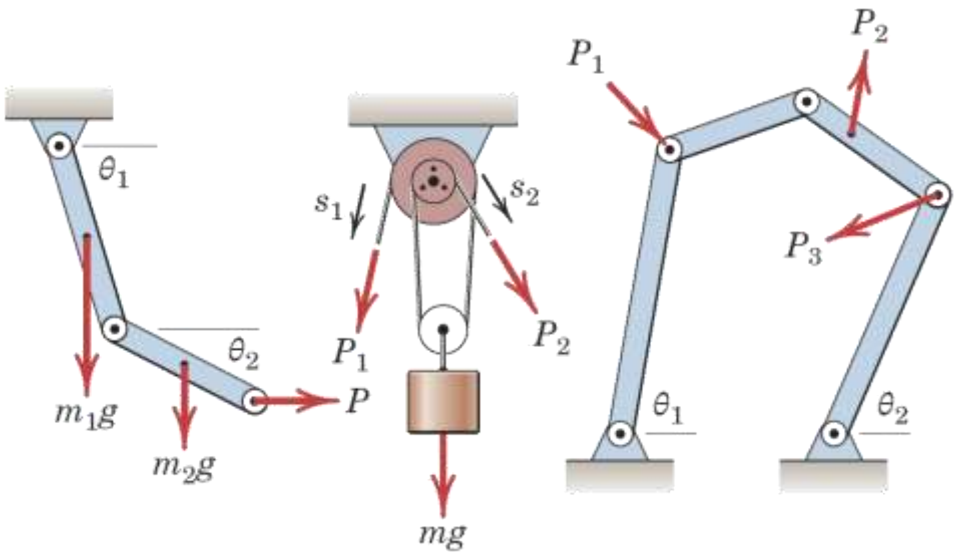
- Definition
  - The number of degrees of freedom of a mechanical system is the number of independent coordinates needed to specify completely the configuration of the system.
  - The principle of virtual work may be applied as many times as there are degrees of freedom.
  - With each application, allow only one independent coordinate to change at a time while holding the others constant.

- One-Degree-of-Freedom Systems



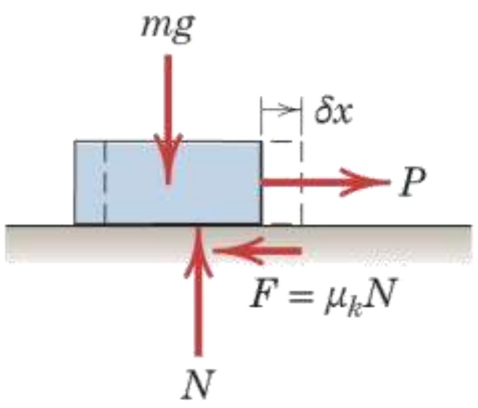
Article 7/3 – Degrees of Freedom (2 of 2)

- Two-Degree-of-Freedom Systems



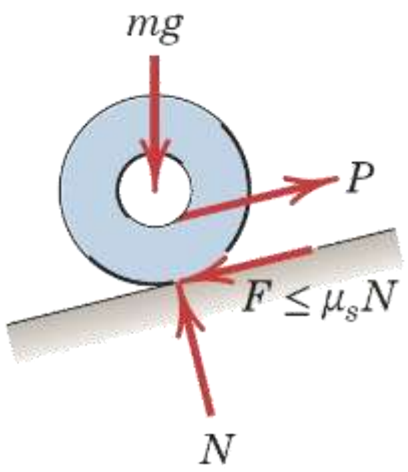
Article 7/3 – Systems with Friction (1 of 2)

- Effect of Friction
  - Dissipates Positive Work by External Forces
  - Negative Work which cannot be Regained



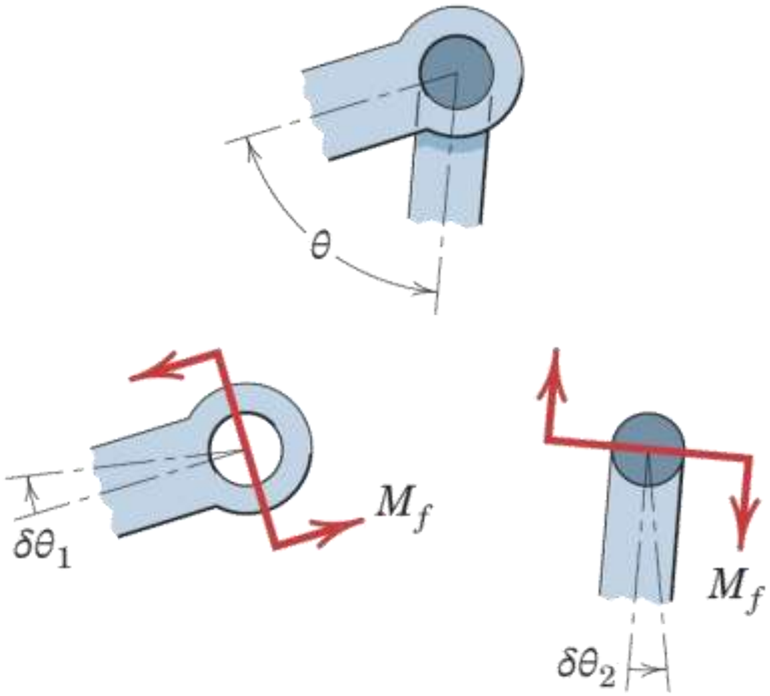
- Illustration with Sliding Block

- Situation of a Rolling Wheel



Article 7/3 – Systems with Friction (2 of 2)

- Friction at a Pinned Connection



Article 7/3 – Mechanical Efficiency (1 of 3)

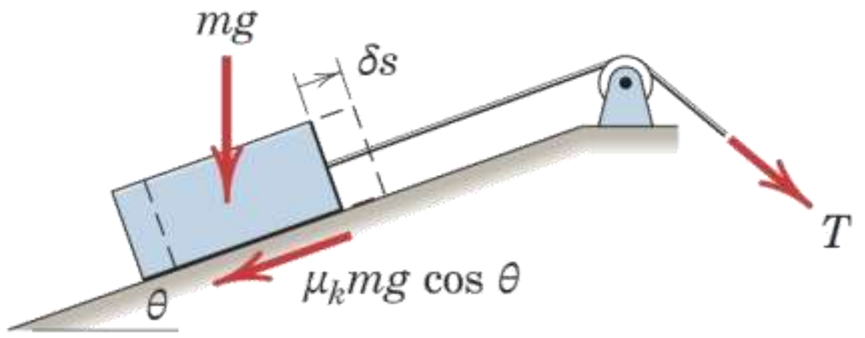
- Efficiency Defined
  - Efficiency is the ratio of the output work from a machine and the input work to the machine.

- Mathematical Expression

$$e = \frac{\text{output work}}{\text{input work}}$$

Article 7/3 – Mechanical Efficiency (2 of 3)

- Example 1: Block Moving up an Incline

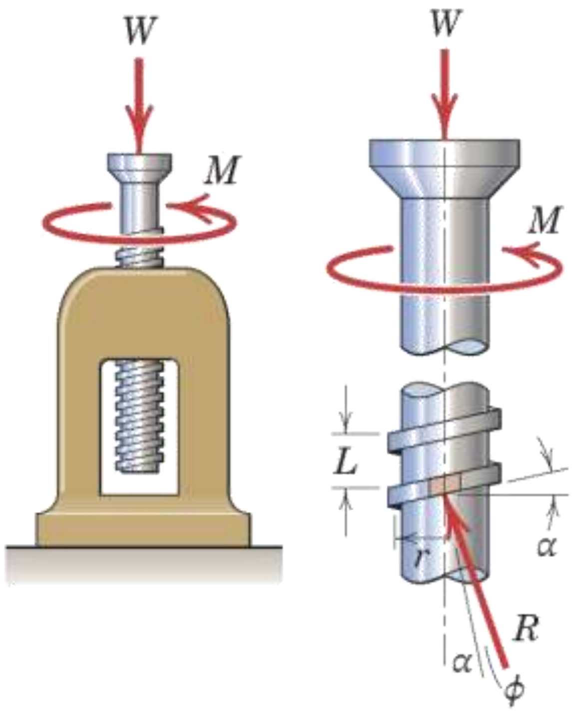


- Efficiency

$$e = \frac{mg\,\delta s \sin\theta}{mg(\sin\theta + \mu_k \cos\theta)\,\delta s} = \frac{1}{1 + \mu_k \cot\theta}$$

Article 7/3 – Mechanical Efficiency (3 of 3)

- Example 2: Screw Jack



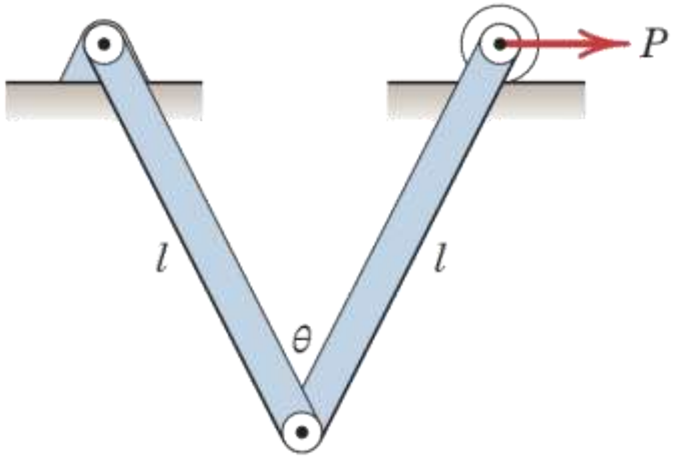
- Efficiency

$$e = \frac{Wr\,\delta\theta \tan\alpha}{Wr\,\delta\theta \tan(\alpha+\phi)} = \frac{\tan\alpha}{\tan(\alpha+\phi)}$$

Article 7/3 – Sample Problem 7/1 (1 of 3)

- Problem Statement

  Each of the two uniform hinged bars has a mass *m* and a length *l*, and is supported and loaded as shown. For a given force *P* determine the angle θ for equilibrium.



Article 7/3 – Sample Problem 7/1 (2 of 3)

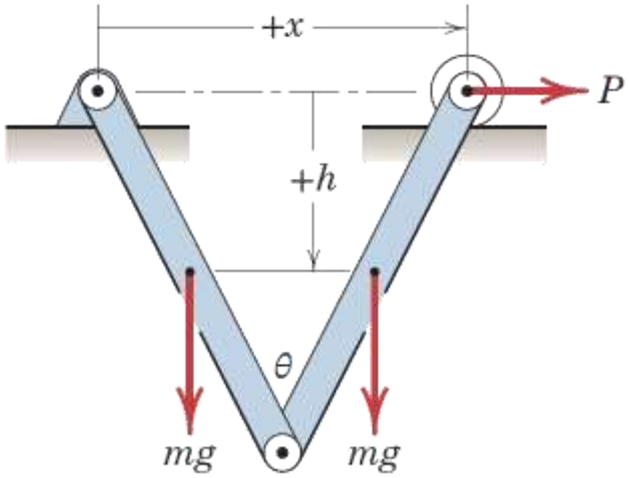
- Solution

The principle of virtual work requires that the total work of all external active forces be zero for any virtual displacement consistent with the constraints. Thus, for a movement δx the virtual work becomes

$$[\delta U = 0] \qquad P\,\delta x + 2mg\,\delta h = 0 \quad ①$$

We now express each of these virtual displacements in terms of the variable θ, the required quantity. Hence,

$$x = 2l \sin\frac{\theta}{2} \qquad \text{and} \qquad \delta x = l\cos\frac{\theta}{2}\,\delta\theta$$



① Note carefully that with x positive to the right δx is also positive to the right in the direction of P, so that the virtual work is $P(+\delta x)$. With h positive down δh is also mathematically positive down in the direction of mg, so that the correct mathematical expression for the work is $mg(+\delta h)$. When we express δh in terms of $\delta\theta$ in the next step, δh will have a negative sign, thus bringing our mathematical expression into agreement with the physical observation that the weight mg does negative work as each center of mass moves upward with an increase in x and θ.

Article 7/3 – Sample Problem 7/1 (3 of 3)

- Solution (cont.)

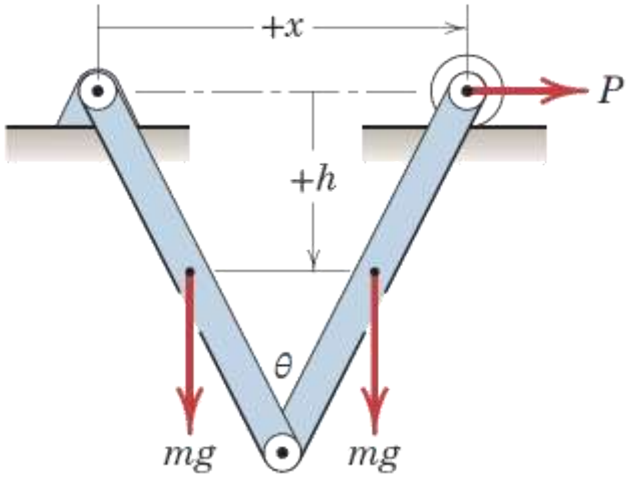
$$h = \frac{l}{2}\cos\frac{\theta}{2} \qquad \text{and} \qquad \delta h = -\frac{l}{4}\sin\frac{\theta}{2}\,\delta\theta \quad ②$$

Substitution into the equation of virtual work gives us

$$Pl\cos\frac{\theta}{2}\,\delta\theta - 2mg\frac{l}{4}\sin\frac{\theta}{2}\,\delta\theta = 0$$

from which we get

$$\tan\frac{\theta}{2} = \frac{2P}{mg} \qquad \text{or} \qquad \theta = 2\tan^{-1}\frac{2P}{mg} \qquad \textit{Ans.}$$

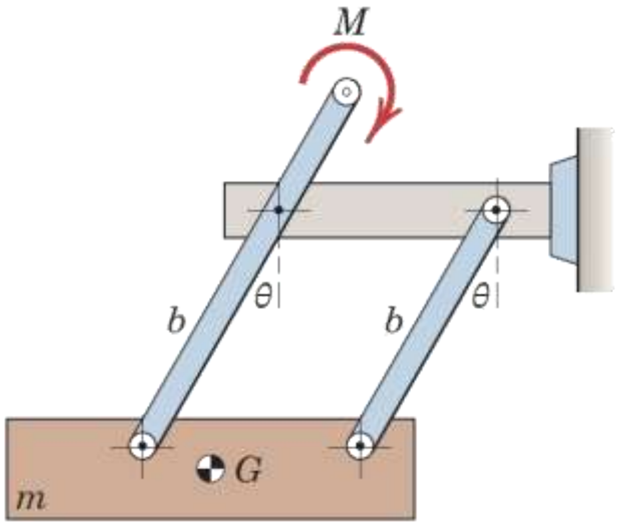


② We obtain δh and δx with the same mathematical rules of differentiation with which we may obtain dh and dx.

Article 7/3 – Sample Problem 7/2 (1 of 3)

- Problem Statement

  The mass m is brought to an equilibrium position by the application of the couple M to the end of one of the two parallel links which are hinged as shown. The links have negligible mass, and all friction is assumed to be absent. Determine the expression for the equilibrium angle θ assumed by the links with the vertical for a given value of M. Consider the alternative of a solution by force and moment equilibrium.



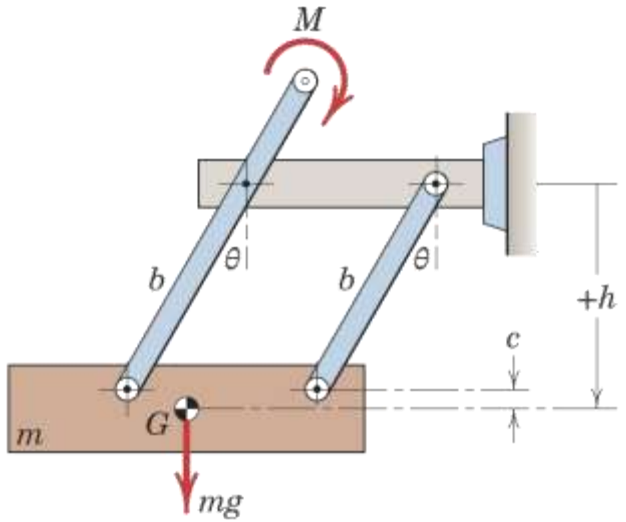
Article 7/3 – Sample Problem 7/2 (2 of 3)

- Solution

The vertical position of the center of mass G is designated by the distance h below the fixed horizontal reference line and is $h = b\cos\theta + c$. The work done by mg during a movement δh in the direction of mg is

$$\begin{aligned} +mg\,\delta h &= mg\,\delta(b\cos\theta + c) \\ &= mg(-b\sin\theta\,\delta\theta + 0) \\ &= -mgb\sin\theta\,\delta\theta \end{aligned}$$

The minus sign shows that the work is negative for a positive value of $\delta\theta$. ① The constant c drops out since its variation is zero.



① Again, as in Sample Problem 7/1, we are consistent mathematically with our definition of work, and we see that the algebraic sign of the resulting expression agrees with the physical change.

Article 7/3 – Sample Problem 7/2 (3 of 3)

- Solution (cont.)

With θ measured positive in the clockwise sense, $\delta\theta$ is also positive clockwise. Thus, the work done by the clockwise couple M is $+M\,\delta\theta$. Substitution into the virtual-work equation gives us

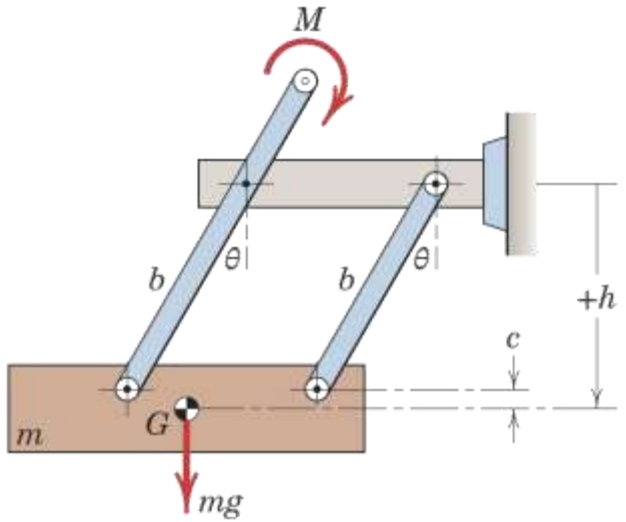
$[\delta U = 0]$ $$M\,\delta\theta + mg\,\delta h = 0$$

which yields

$$M\,\delta\theta = mgb \sin\theta\,\delta\theta$$

$$\theta = \sin^{-1}\frac{M}{mgb}$$ *Ans.*

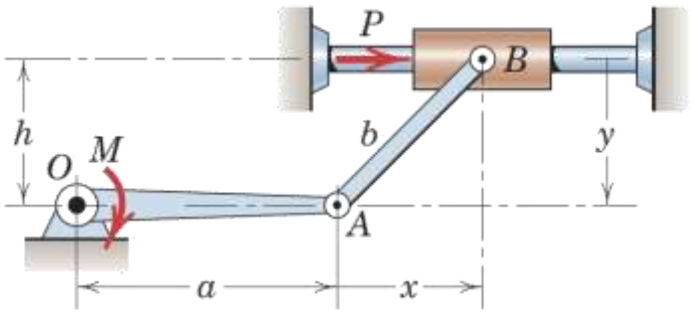
Inasmuch as $\sin\theta$ cannot exceed unity, we see that for equilibrium, M is limited to values that do not exceed mgb.



Article 7/3 – Sample Problem 7/3 (1 of 3)

- Problem Statement

  For link *OA* in the horizontal position shown, determine the force *P* on the sliding collar which will prevent *OA* from rotating under the action of the couple *M*. Neglect the mass of the moving parts.



Article 7/3 – Sample Problem 7/3 (2 of 3)

- Solution

We will give the crank OA a small clockwise angular movement $\delta\theta$ as our virtual displacement and determine the resulting virtual work done by M and P. From the horizontal position of the crank, the angular movement gives a downward displacement of A equal to

$$\delta y = a\ \delta\theta \quad ①$$

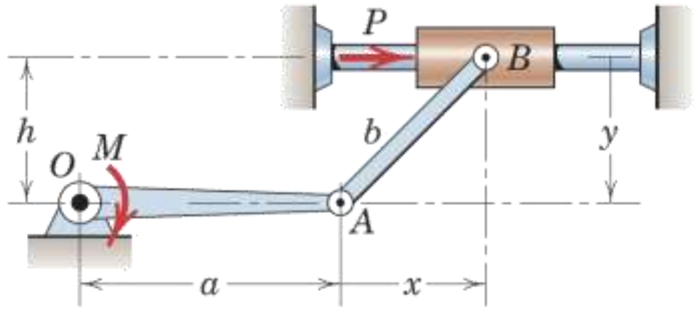
where $\delta\theta$ is, of course, expressed in radians.

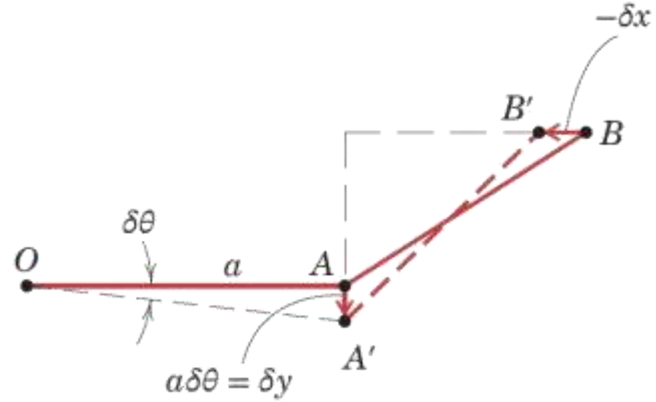
From the right triangle for which link AB is the constant hypotenuse we may write

$$b^2 = x^2 + y^2$$

We now take the differential of the equation and get

$$0 = 2x\ \delta x + 2y\ \delta y \qquad \text{or} \qquad \delta x = -\frac{y}{x}\ \delta y \quad ②$$





① Note that the displacement $a\ \delta\theta$ of point A would no longer equal δy if the crank OA were not in a horizontal position.

② The length b is constant so that $\delta b = 0$. Notice the negative sign, which merely tells us that if one change is positive, the other must be negative.

Article 7/3 – Sample Problem 7/3 (3 of 3)

- Solution (cont.)

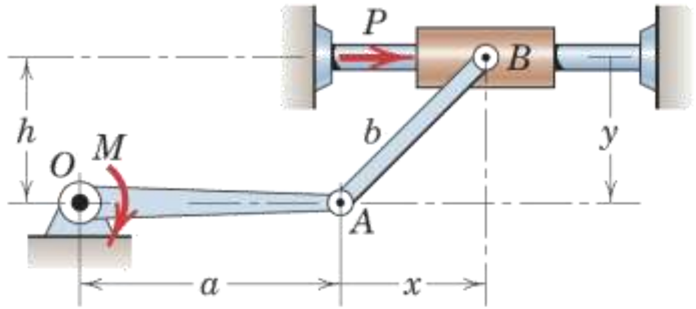
Thus,

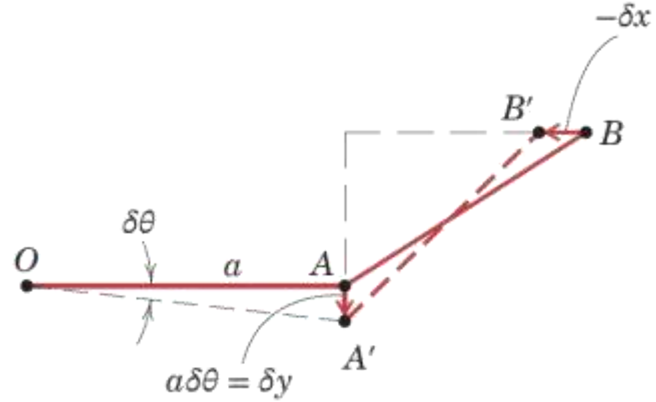
$$\delta x = -\frac{y}{x}\, a\, \delta\theta$$

and the virtual-work equation becomes

$$[\delta U = 0] \qquad M\,\delta\theta + P\,\delta x = 0 \qquad M\,\delta\theta + P\left(-\frac{y}{x}\, a\, \delta\theta\right) = 0 \quad ③$$

$$P = \frac{Mx}{ya} = \frac{Mx}{ha} \qquad\qquad \textit{Ans.}$$



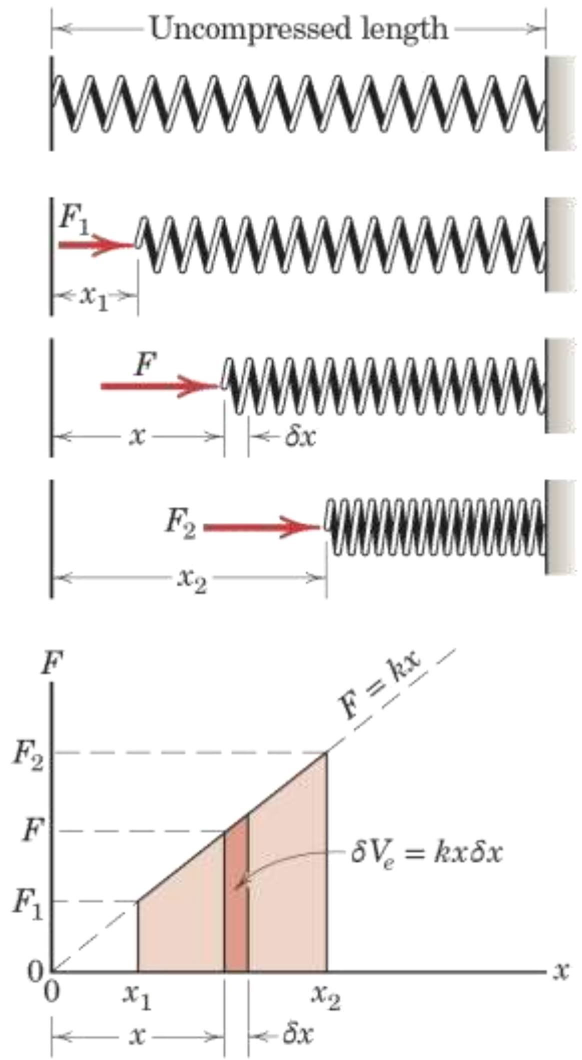


③ We could just as well use a counterclockwise virtual displacement for the crank, which would merely reverse the signs of all terms.

Article 7/4 Potential Energy and Stability

- Elastic Potential Energy

- Illustration
  - Linear Spring Stiffness, k
  - Spring Force, $F = kx$
  - Work done on the Spring, $dU = F\,dx$
  - Potential Energy, $V_e = \frac{1}{2} kx^2$
  - Change in Potential Energy, $\Delta V_e = \frac{1}{2} k(x_2^2 - x_1^2)$

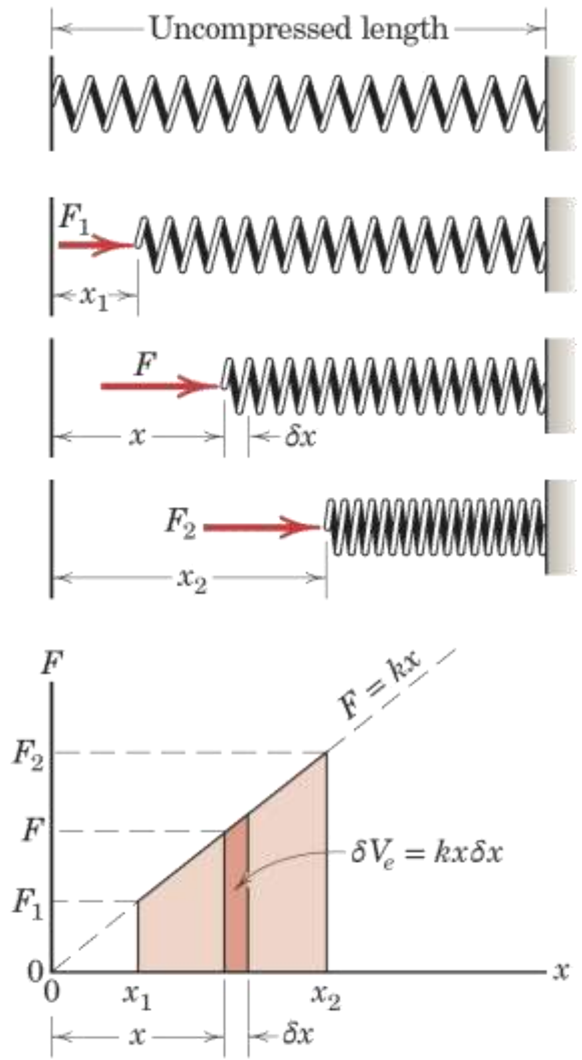


Article 7/4 – Elastic Potential Energy (cont.)

- Virtual Displacements and Potential Energy
  - Virtual Displacement, δx
  - Virtual Change in Potential Energy, $\delta V_e = F\,\delta x$

- Sign Convention
  - If the spring is **compressed**, the virtual change in elastic potential energy is **negative**.
  - If the spring is **stretched**, the virtual change in elastic potential energy is **positive**.

- The work done on the body is the negative of the potential energy change of the spring.



Article 7/4 – Torsional Springs

- A torsional spring resists rotation of a shaft or another element.

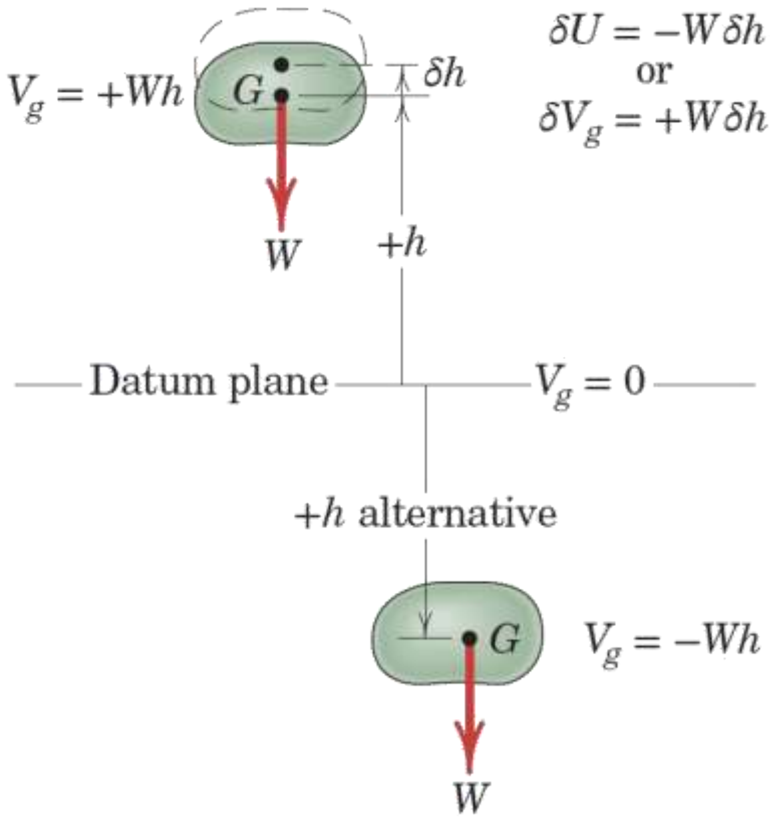
- Torsional Stiffness, k_T
  - Units of torque per radian twist (N·m/rad or lb-ft/rad)

- Elastic Potential Energy, $V_e = \frac{1}{2} k_T \theta^2$

Article 7/4 – Gravitational Potential Energy (1 of 3)

- Illustration & Definition

  The gravitational potential energy V_g is defined as the work done on the body by a force equal and opposite to the weight in bringing the body to the position under consideration from some arbitrary datum plane where the potential energy is defined to be zero.

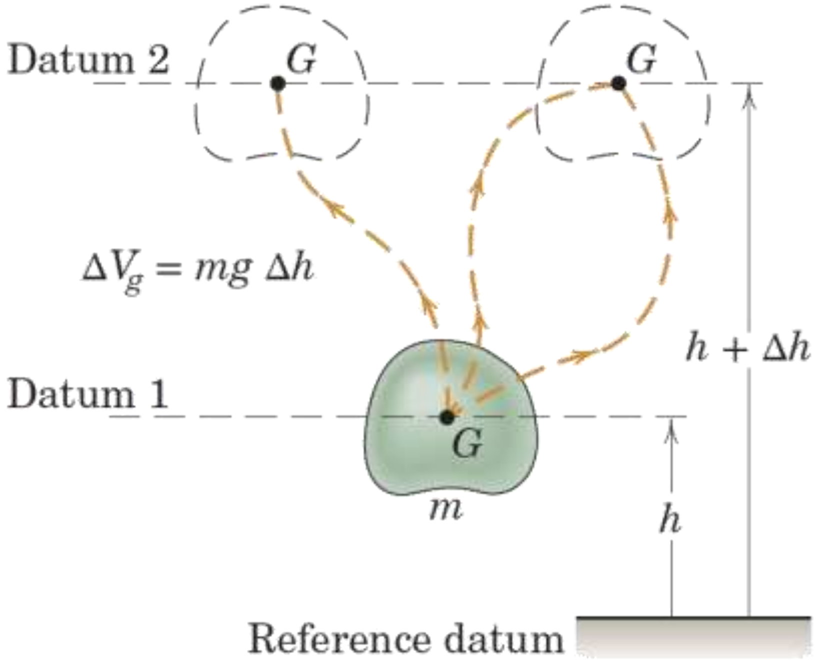


- Potential Energy, $V_g = mgh$
  - Height h is measured from the datum.
  - Above the datum is considered positive.
  - Below the datum is considered negative.

- Change in Potential Energy, $\Delta V_g = mg\,\Delta h$

Article 7/4 – Gravitational Potential Energy (2 of 2)

- Path Irrelevance

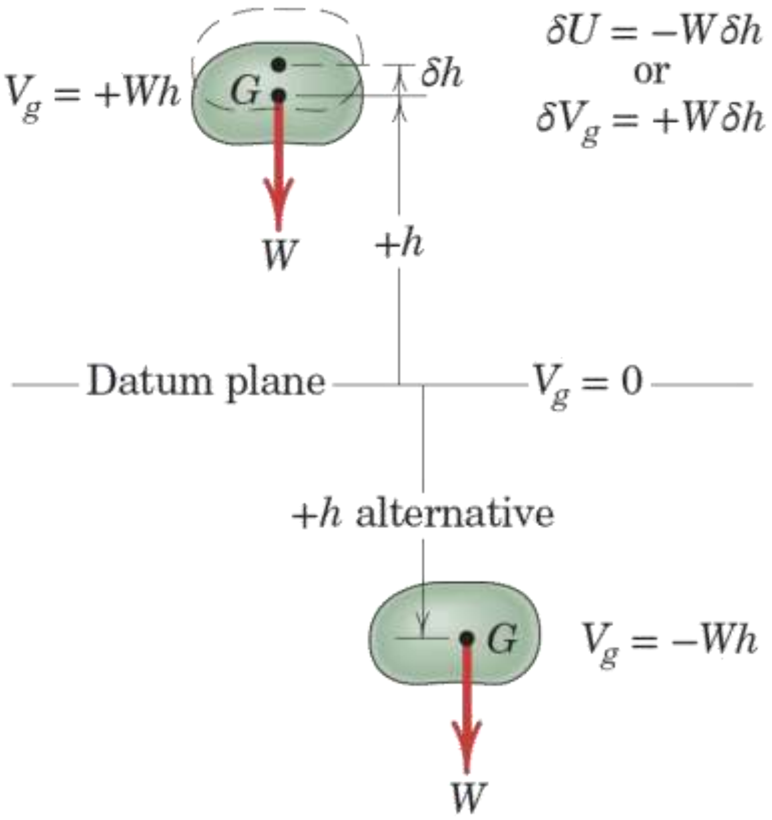


Article 7/4 – Gravitational Potential Energy (3 of 3)

- Virtual Displacements and Potential Energy
  - Virtual Displacement, δh
  - Virtual Change in Potential Energy, $\delta V_g = mg\,\delta h$

- Sign Convention
  - If the virtual displacement is **upward**, the sign of the virtual change in gravitational potential energy is **positive**.
  - If the virtual displacement is **downward**, the sign of the virtual change in gravitational potential energy is **negative**.

- The work done on the body is the negative of the potential energy change of the body.



Article 7/4 – The Principle of Virtual Work Restated

- The Equation Restated

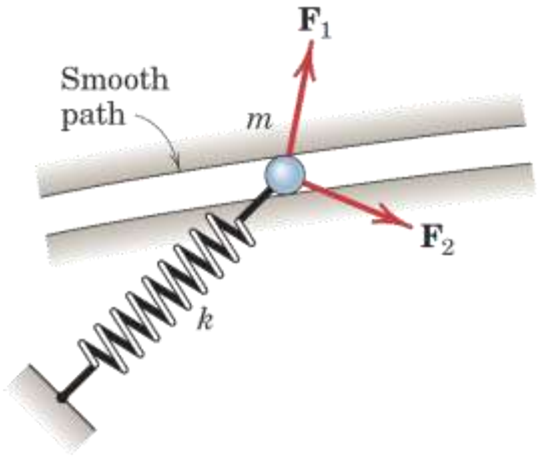
  The virtual work done by all external active forces (other than the gravitational and spring forces accounted for in the potential energy terms) on a mechanical system in equilibrium equals the corresponding change in the total elastic and gravitational potential energy of the system for any and all virtual displacements consistent with the constraints.

- Mathematical Expression: $\delta U' - (\delta V_e + \delta V_g) = 0$ or $\delta U' = \delta V$

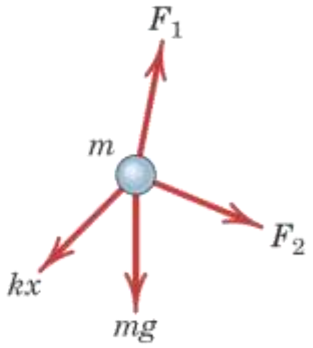
  - $\delta U'$ = sum of the work done by all active forces other than springs or weight.
  - $-\delta V_e$ = work done by the spring forces.
  - $-\delta V_g$ = work done by the weight forces.
  - δV = change in total elastic and gravitational potential energy of the system.
  - The equation above is also called the *energy equation*.

Article 7/4 – Active Force Diagrams

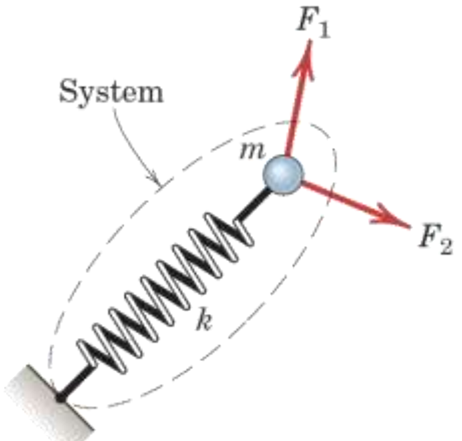
- Illustration
  - (*a*) Body of Interest
  - (*b*) Use of Equation 7/3
  - (*c*) Use of Equation 7/6



(*a*)



Eq. 7/3: $\delta U = 0$
(*b*)



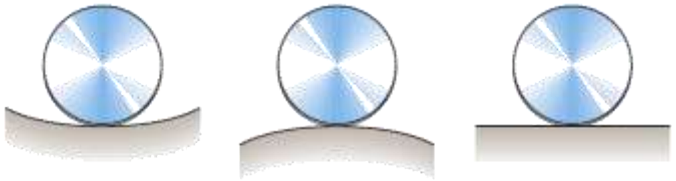
Eq. 7/6: $\delta U' = \delta V_e + \delta V_g = \delta V$
(*c*)

Article 7/4 – Stability of Equilibrium (1 of 2)

- Introduction
  - Consider a mechanical system where movement is accompanied by changes in gravitational and elastic potential energies and where no work is done on the system by nonpotential forces ($\delta U' = 0$).
  - $\delta U' - (\delta V_e + \delta V_g) = 0$ or $(\delta V_e + \delta V_g) = 0$ or $\delta (V_e + V_g) = 0$ or $\delta V = 0$
  - The total potential energy V of the system is constant or has a stationary value.
- One-Degree-of-Freedom System Implication
  - Potential energy is a function of some variable, x, and its derivatives.
  - Then…$dV/dx = 0$ for equilibrium.

Article 7/4 – Stability of Equilibrium (2 of 2)

- Conditions on Equilibrium



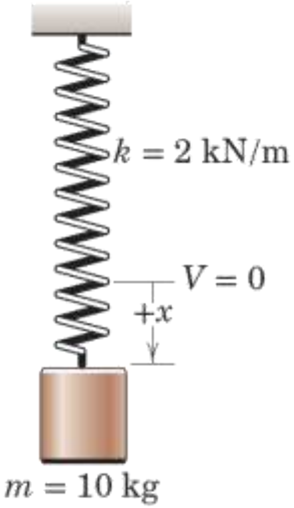
  - Stable: Total Potential Energy is a Minimum
  - Unstable: Total Potential Energy is a Maximum
  - Neutral: Total Potential Energy is a Constant
  - Mathematical Summary

Equilibrium	$\dfrac{dV}{dx} = 0$
Stable	$\dfrac{d^2V}{dx^2} > 0$
Unstable	$\dfrac{d^2V}{dx^2} < 0$

Article 7/4 – Sample Problem 7/4 (1 of 3)

- Problem Statement

  The 10-kg cylinder is suspended by the spring, which has a stiffness of 2 kN/m. Plot the potential energy *V* of the system and show that it is minimum at the equilibrium position.



Article 7/4 – Sample Problem 7/4 (2 of 3)

- Solution

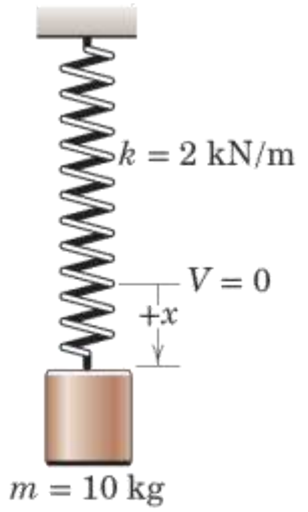
(Although the equilibrium position in this simple problem is clearly where the force in the spring equals the weight mg, we will proceed as though this fact were unknown in order to illustrate the energy relationships in the simplest way.) We choose the datum plane for zero potential energy at the position where the spring is unextended. ①

The elastic potential energy for an arbitrary position x is $V_e = \frac{1}{2}kx^2$ and the gravitational potential energy is $-mgx$, so that the total potential energy is

$$[V = V_e + V_g] \qquad V = \frac{1}{2}kx^2 - mgx$$

Equilibrium occurs where

$$\left[\frac{dV}{dx} = 0\right] \qquad \frac{dV}{dx} = kx - mg = 0 \qquad x = mg/k$$



① The choice is arbitrary but simplifies the algebra.

Article 7/4 – Sample Problem 7/4 (3 of 3)

- Solution (cont.)

Although we know in this simple case that the equilibrium is stable, we prove it by evaluating the sign of the second derivative of V at the equilibrium position. Thus, $d^2V/dx^2 = k$, which is positive, proving that the equilibrium is stable.

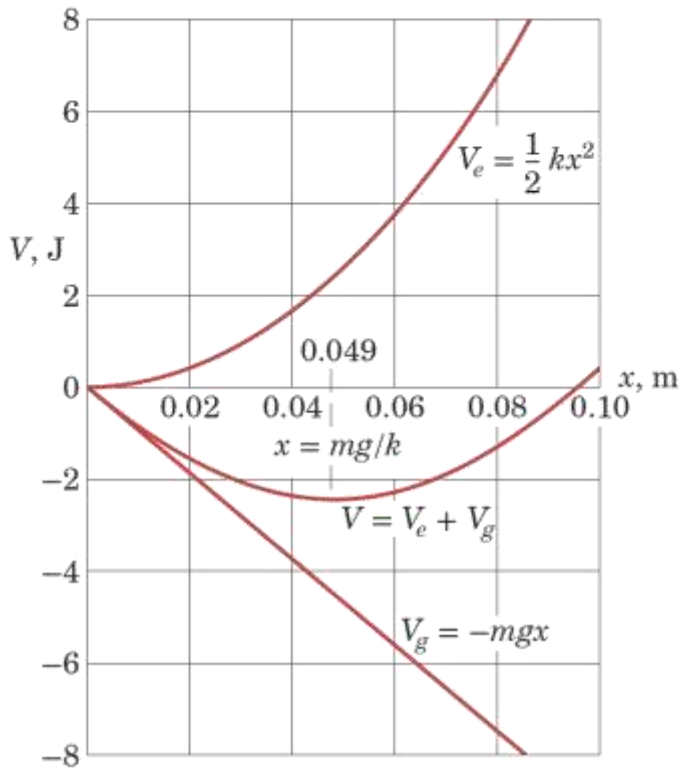
Substituting numerical values gives

$$V = \frac{1}{2}(2000)x^2 - 10(9.81)x$$

expressed in joules, and the equilibrium value of x is

$$x = 10(9.81)/2000 = 0.0490 \text{ m} \qquad \text{or} \qquad 49.0 \text{ mm} \qquad \textit{Ans.}$$

We calculate V for various values of x and plot V versus x as shown. The minimum value of V occurs at $x = 0.0490$ m where $dV/dx = 0$ and d^2V/dx^2 is positive. ②

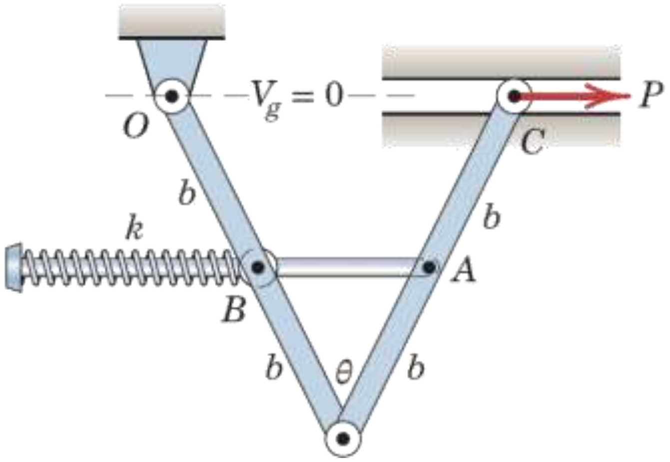


② We could have chosen different datum planes for V_e and V_g without affecting our conclusions. Such a change would merely shift the separate curves for V_e and V_g up or down but would not affect the position of the minimum value of V.

Article 7/4 – Sample Problem 7/5 (1 of 3)

- Problem Statement

  The two uniform links, each of mass m, are in the vertical plane and are connected and constrained as shown. As the angle θ between the links increases with the application of the horizontal force P, the light rod, which is connected at A and passes through a pivoted collar at B, compresses the spring of stiffness k. If the spring is uncompressed in the position where $\theta = 0$, determine the force P which will produce equilibrium at the angle θ.



Article 7/4 – Sample Problem 7/5 (2 of 3)

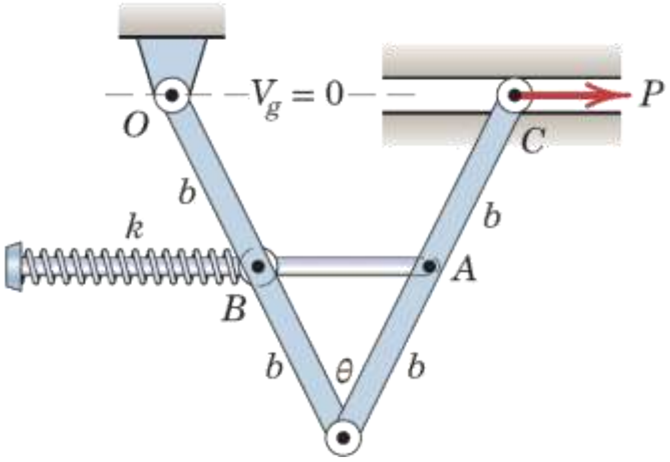
- Solution

The given sketch serves as the active-force diagram of the system. The compression x of the spring is the distance which A has moved away from B, which is $x = 2b \sin \theta/2$. Thus, the elastic potential energy of the spring is

$$[V_e = \tfrac{1}{2}kx^2] \qquad V_e = \frac{1}{2}k\left(2b \sin\frac{\theta}{2}\right)^2 = 2kb^2 \sin^2\frac{\theta}{2}$$

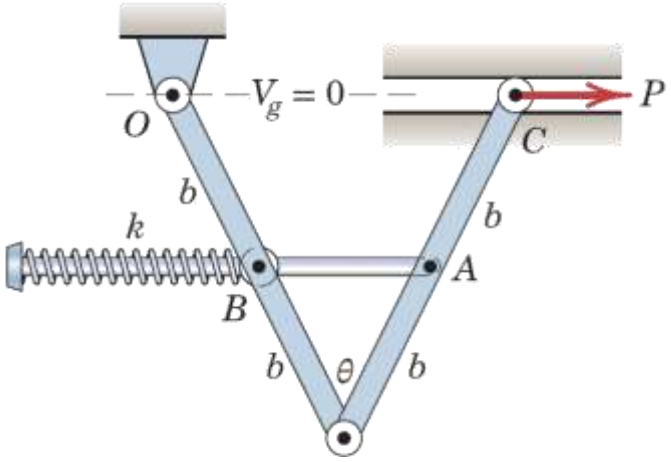
With the datum for zero gravitational potential energy taken through the support at O for convenience, the expression for V_g becomes

$$[V_g = mgh] \qquad V_g = 2mg\left(-b \cos\frac{\theta}{2}\right)$$



Article 7/4 – Sample Problem 7/5 (2 of 3)

- Solution (cont.)



The distance between O and C is $4b \sin \theta/2$, so that the virtual work done by P is

$$\delta U' = P\,\delta\left(4b \sin \frac{\theta}{2}\right) = 2Pb \cos \frac{\theta}{2}\,\delta\theta$$

The virtual-work equation now gives

$[\delta U' = \delta V_e + \delta V_g]$

$$2Pb \cos \frac{\theta}{2}\,\delta\theta = \delta\left(2kb^2 \sin^2 \frac{\theta}{2}\right) + \delta\left(-2mgb \cos \frac{\theta}{2}\right)$$

$$= 2kb^2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\,\delta\theta + mgb \sin \frac{\theta}{2}\,\delta\theta$$

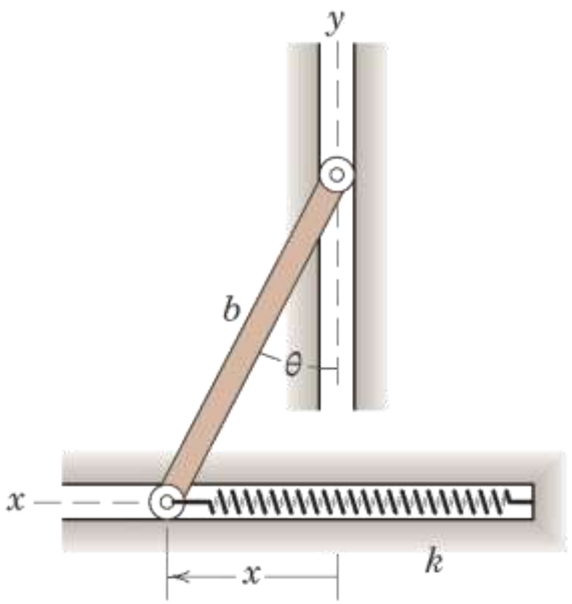
Simplifying gives finally

$$P = kb \sin \frac{\theta}{2} + \frac{1}{2} mg \tan \frac{\theta}{2} \qquad \textit{Ans.}$$

Article 7/4 – Sample Problem 7/6 (1 of 4)

- Problem Statement

  The ends of the uniform bar of mass *m* slide freely in the horizontal and vertical guides. Examine the stability conditions for the positions of equilibrium. The spring of stiffness *k* is undeformed when $x = 0$.



Article 7/4 – Sample Problem 7/6 (2 of 4)

- Solution

The system consists of the spring and the bar. Since there are no external active forces, the given sketch serves as the active-force diagram. ① We will take the x-axis as the datum for zero gravitational potential energy. In the displaced position the elastic and gravitational potential energies are

$$V_e = \frac{1}{2}kx^2 = \frac{1}{2}kb^2 \sin^2\theta \qquad \text{and} \qquad V_g = mg\,\frac{b}{2}\cos\theta$$

The total potential energy is then

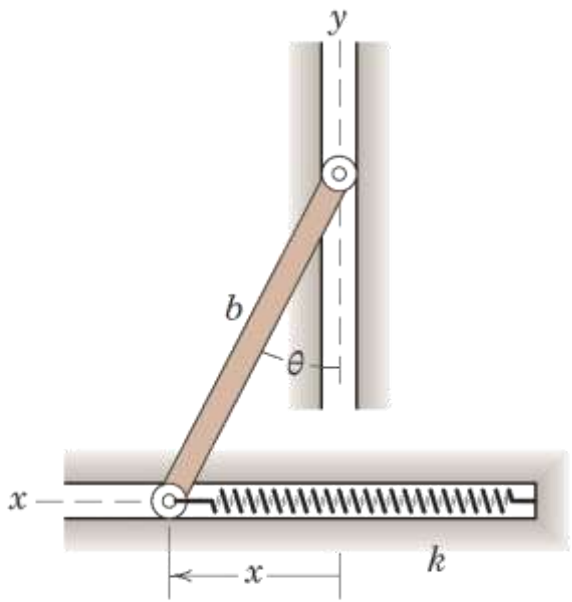
$$V = V_e + V_g = \frac{1}{2}kb^2 \sin^2\theta + \frac{1}{2}mgb\cos\theta$$

Equilibrium occurs for $dV/d\theta = 0$ so that

$$\frac{dV}{d\theta} = kb^2 \sin\theta\cos\theta - \frac{1}{2}mgb\sin\theta = (kb^2\cos\theta - \frac{1}{2}mgb)\sin\theta = 0$$

The two solutions to this equation are given by

$$\sin\theta = 0 \qquad \text{and} \qquad \cos\theta = \frac{mg}{2kb} \quad ②$$



① With no external active forces there is no $\delta U'$ term, and $\delta V = 0$ is equivalent to $dV/d\theta = 0$.

② Be careful not to overlook the solution $\theta = 0$ given by $\sin\theta = 0$.

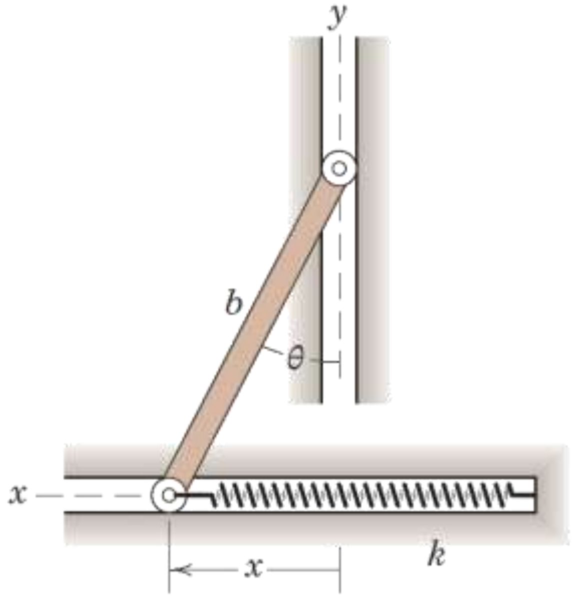
Article 7/4 – Sample Problem 7/6 (3 of 4)

- Stability Equation

We now determine the stability by examining the sign of the second derivative of V for each of the two equilibrium positions. The second derivative is

$$\frac{d^2V}{d\theta^2} = kb^2(\cos^2\theta - \sin^2\theta) - \tfrac{1}{2}mgb\cos\theta$$

$$= kb^2(2\cos^2\theta - 1) - \tfrac{1}{2}mgb\cos\theta$$



- Stability Solution I

$\sin\theta = 0, \theta = 0$

$$\frac{d^2V}{d\theta^2} = kb^2(2-1) - \tfrac{1}{2}mgb = kb^2\left(1 - \frac{mg}{2kb}\right)$$

$= \text{positive (stable)} \qquad \text{if } k > mg/2b$

$= \text{negative (unstable)} \qquad \text{if } k < mg/2b \qquad$ *Ans.*

Thus, if the spring is sufficiently stiff, the bar will return to the vertical position even though there is no force in the spring at that position. ③

③ We might not have anticipated this result without the mathematical analysis of the stability.

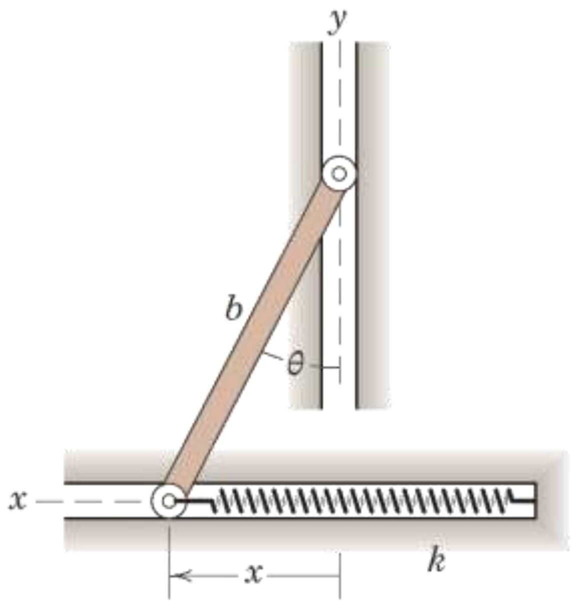
Article 7/4 – Sample Problem 7/6 (3 of 4)

- Stability Solution II

$$\cos\theta = \frac{mg}{2kb}, \theta = \cos^{-1}\frac{mg}{2kb}$$

$$\frac{d^2V}{d\theta^2} = kb^2\left[2\left(\frac{mg}{2kb}\right)^2 - 1\right] - \frac{1}{2}mgb\left(\frac{mg}{2kb}\right) = kb^2\left[\left(\frac{mg}{2kb}\right)^2 - 1\right] \quad Ans.$$

Since the cosine must be less than unity, we see that this solution is limited to the case where $k > mg/2b$, which makes the second derivative of V negative. Thus, equilibrium for Solution II is never stable. ④ If $k < mg/2b$, we no longer have Solution II since the spring will be too weak to maintain equilibrium at a value of θ between 0 and 90°.



④ Again, without the benefit of the mathematical analysis of the stability we might have supposed erroneously that the bar could come to rest in a stable equilibrium position for some value of θ between 0 and 90°.