Introduction to Discrete Math

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Course Outline

- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion & Induction, Logic, Invariants
- Probability & Combinatronics
  - Basic Counting, Binomial Coeff, Advanced Counting, Probability, Random Variables

Probability & Combinatronics – Random Variables

RANDOM VARIABLES & EXPECTATIONS

- **Random Variables**
- Average
- Expectation

Random Variables

- We have studied probability distributions
- Studied events (subsets of outcomes) and their probabilities
- Events correspond to yes or no questions
- It is important to study numerical characteristics of random outcomes
- So we introduce random variables

Random Variables

- Random variable f is a variable whose value is determined by a random experiment
- We have probability distribution on the finite set X of k outcomes
- Outcomes have probabilities $p_1, \dots, p_k$
- To define f we assign a number a_i to each outcome
- Then f has value a_i with probability p_i

Random Variables

- Looks familiar?
  - We have already done this!
- Outcomes of the dice throw are labeled by numbers

Random Variables

Other examples:

- Tossing a coin: heads=0, tail=1
- An age of a random person in the class
- Grade of a random person in the class
- Sum of outcomes of two dice throws

- Random Variables
- **Average**
- Expectation

Average

What is an average salary in a country?

- The total salary of all population divided by the number of employees
- This is the standard notion of average
- It is called arithmetic mean in mathematics

Example

Problem: Student got scores 78, 72 and 87 on three tests. What is the average score?

- Have to add all scores and divide by their number

Example

Problem: Student got scores 78, 72 and 87 on three tests. What is the average score?

78 72 87

5

78 75 80

79 75 79

- Have to add all scores and divide by their number
- Here it is:

$$\frac{78+72+87}{3}=\frac{237}{3}=79$$

- We got lucky and the answer is integer
  - but this is not guaranteed

HR Management Strategy

Problem: Suppose HR management in some company uses the following strategy: fire everyone who performs below average. What will be a result of such strategy?

~~1 98~~
~~2 95~~
~~3 85~~
~~4 85~~
~~5 80~~
6 100
~~7 59~~
~~8 84~~
~~9 90~~ 99.4%
~~10 73~~

- Might sound reasonable, but ...
- Unless everyone works equally (extremely rare case) ...
- There is always someone who works below average!
  - If we fire them, the average performance will grow
- New people get below average!
  - Everyone will be fired except one best employee

≃ 98% | 95% | 99%

Average Outcome of Dice Throw

Problem: Suppose we throw a dice many times. What is the average outcome?

- Can we give a precise answer?
  - No, it is a random variable
- But we can give an approximation that is good with high probability

Average Outcome of Dice Throw

1 → 6

- Suppose we throw a dice n times for a very large n
- Then among outcomes there are approximately $n/6$ ones, $n/6$ twos, and so on…
- The sum of results is then approximately

$$\frac{n}{6}\times 1+\frac{n}{6}\times 2+\frac{n}{6}\times 3+\frac{n}{6}\times 4+\frac{n}{6}\times 5+\frac{n}{6}\times 6$$

$$=\frac{n(1+2+3+4+5+6)}{6}=\frac{21n}{6}=3.5n$$

- The average can be obtained by dividing with number of throws n
- Thus the average is approximately: $\frac{3.5n}{n}=3.5$
- This is an expected value or expectation of a dice throw

- Random Variables
- Average
- **Expectation**

Expectation

Let's consider the general case

- Suppose we have a random variable f on the distribution with 4 outcomes
- Probabilities of outcomes are p_1, p_2, p_3, p_4
- Values of f are a_1, a_2, a_3, a_4
- Let's repeat the random experiment many times

Expectation

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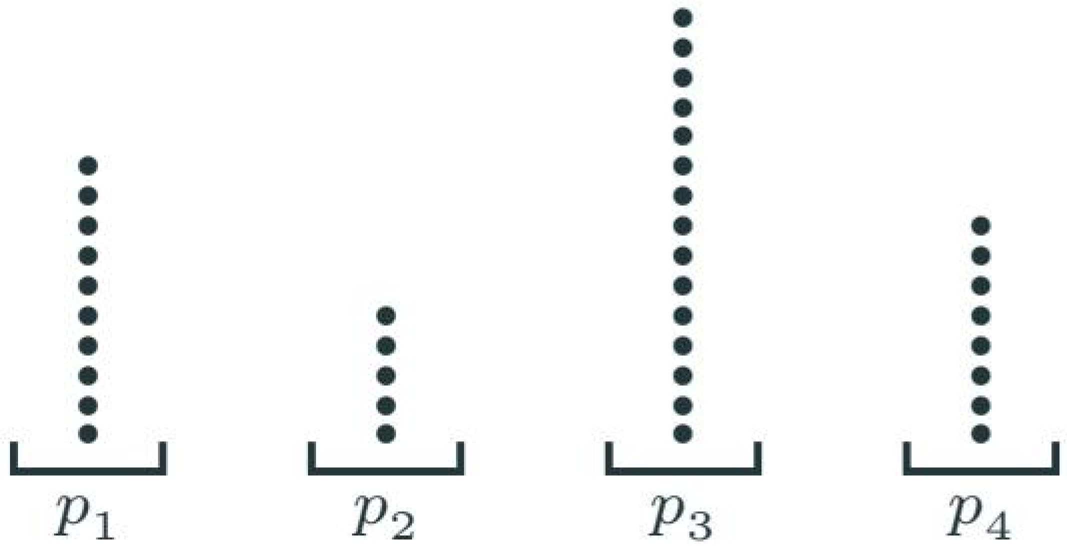
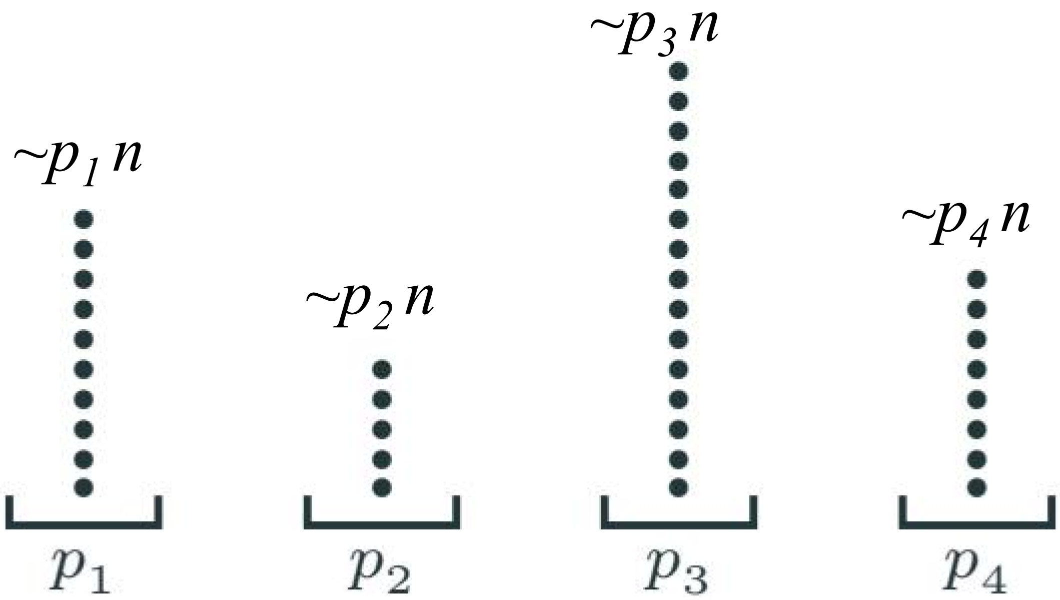


- Repeat n times, where n is large

Expectation



- Repeat n times, where n is large
- What is the average value of f on these outcomes?

Expectation

- We have n experiments
- Value a_i occurs about $p_i n$ times
- On average we have:

$$\simeq \frac{a_1 p_1 n + a_2 p_2 n + a_3 p_3 n + a_4 p_4 n}{n}$$

$$= a_1 p_1 + a_2 p_2 + a_3 p_3 + a_4 p_4$$

- This is denoted by **E*f*** and called the expectation of *f*
- Does not depend on n
- An approximation to what we would expect as an average outcome of an experiment repeated many times

Expectation

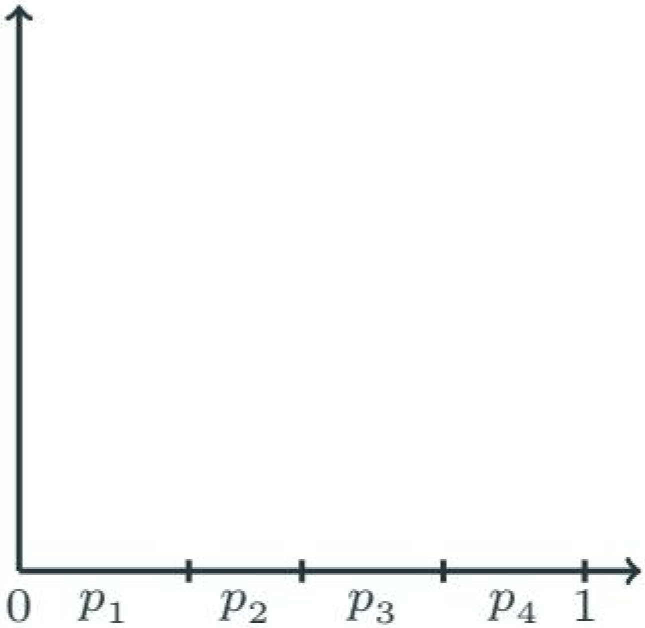
- In the general case:
  outcomes $a_1, \ldots, a_k$ with probabilities $p_1, \ldots, p_k$
- To compute the expectation:
  - multiply $a_i \times p_i$ over all i
  - And add up through 1 to k
- Expectation is a number!
- An important characteristic of a random variable

Geometric Representation of Expectation

Suppose f:

- obtains values a_1, a_2, a_3, a_4 with probabilities p_1, p_2, p_3, p_4

- $\mathbf{E}f = a_1p_1 + a_2p_2 + a_3p_3 + a_4p_4$



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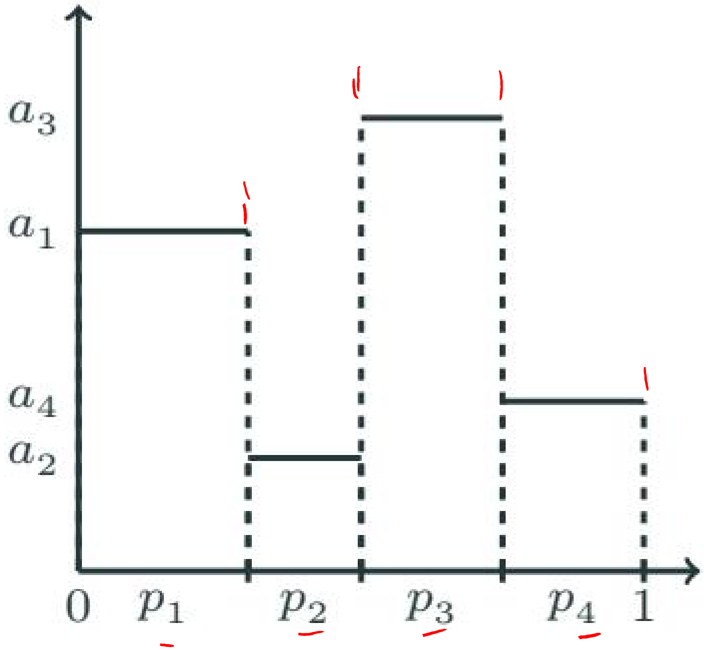

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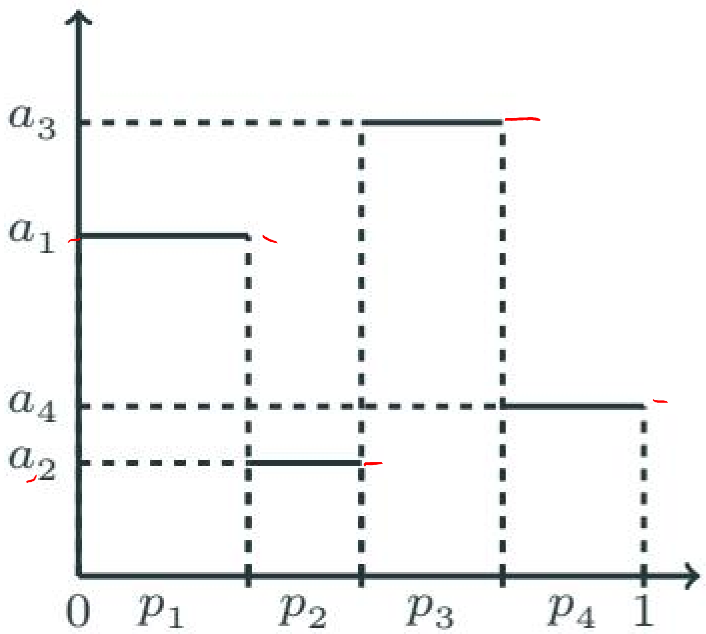

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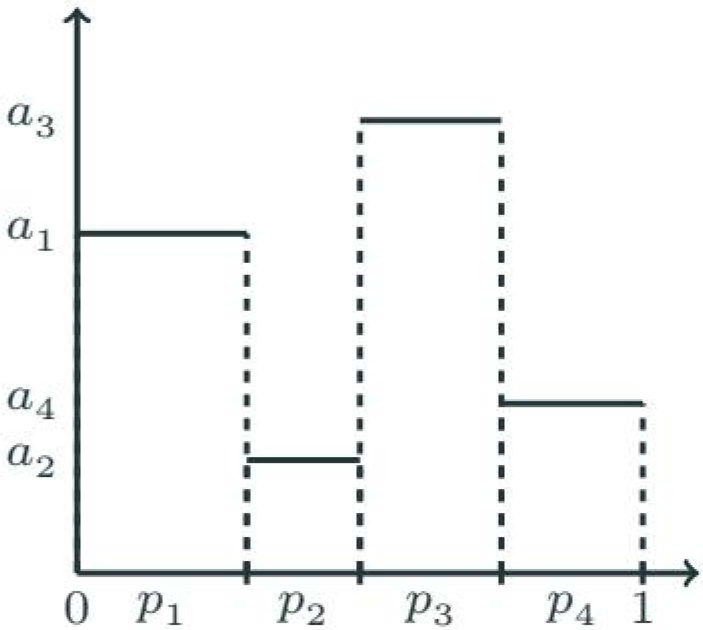

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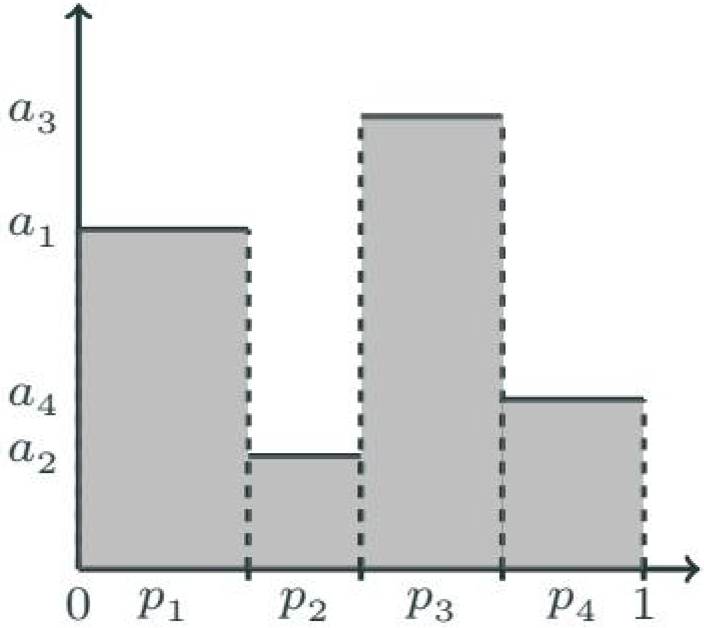

- $\mathbf{E}f$ is the area of the gray region

Real Life Examples

- Expectations occur everywhere in statistics and sociology
- Average age
- Life expectancy
- Average grades and evaluations

Thank you.