Introduction to Discrete Math

Felipe P. Vista IV

Course Outline

- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion, Logic, Invariants
- Probability & Combinatronics
  - Counting, Probability, Random Variables
- Graph Theory
  - Graphs (cycles, classes, parameters)
- Number Theory & Cryptography
  - Arithmetic in modular form
  - Intro to Cryptography

Mathematical Thinking – Invariants

INVARIANTS

- Invariants
- Coffee with milk
- More Coffee
- Debugging Problem

Invariants

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vary = change

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- It could be a number, or it could be something else
- Double counting is a special case

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There are two cups, one with coffee and another with milk. We take a spoon of coffee and add it to the cup of milk. We then take a spoonful from the cup of milk and put it into the cup of coffee. Which is larger, the amount of milk in the cup of coffee or the amount of coffee in the cup of milk?

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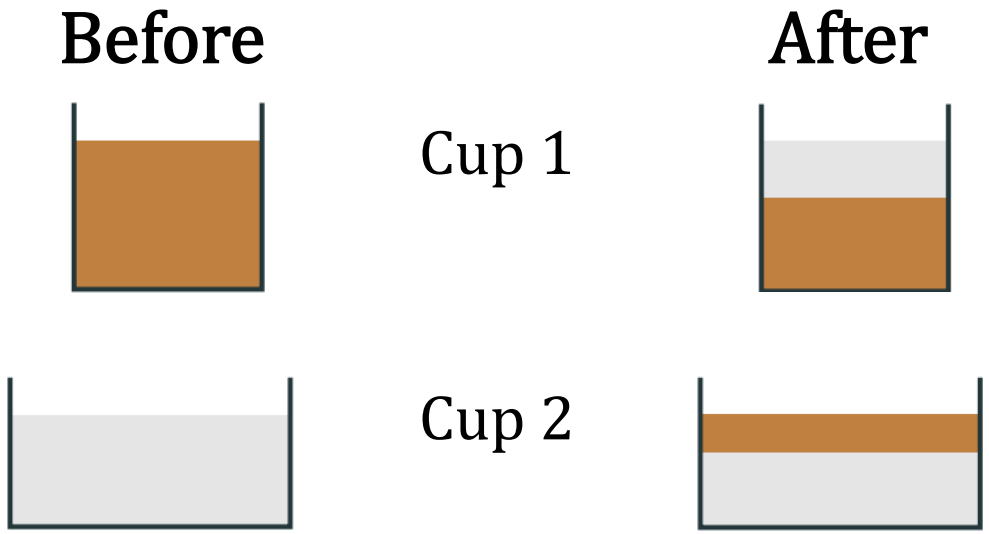
- It seems we need to do some serious calculation
- We don't know size of cups not size of the spoon
  - Maybe the answer depends on these parameters?
- Well, it turns out we do not need to do any calculation ☺!

Invariants – Coffee with milk

Coffee with milk



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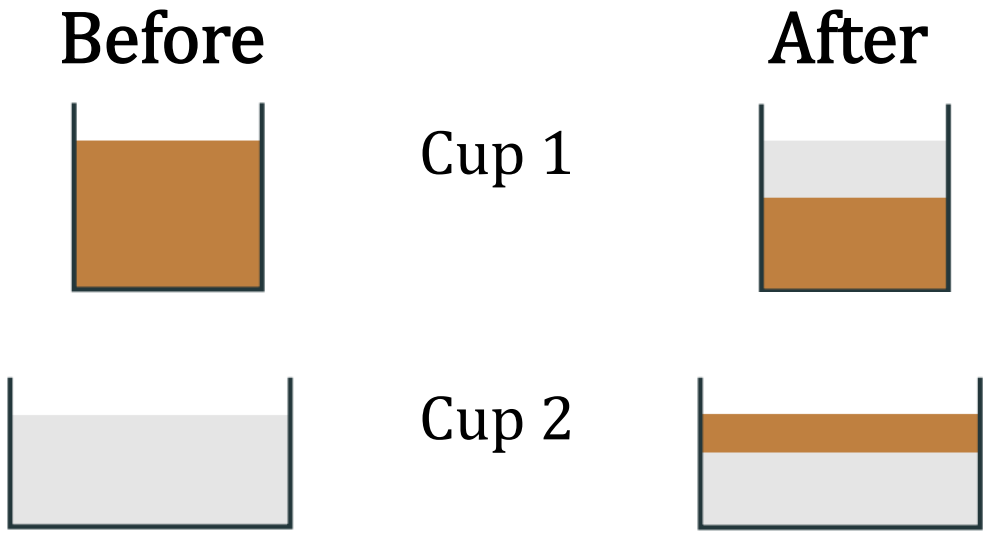

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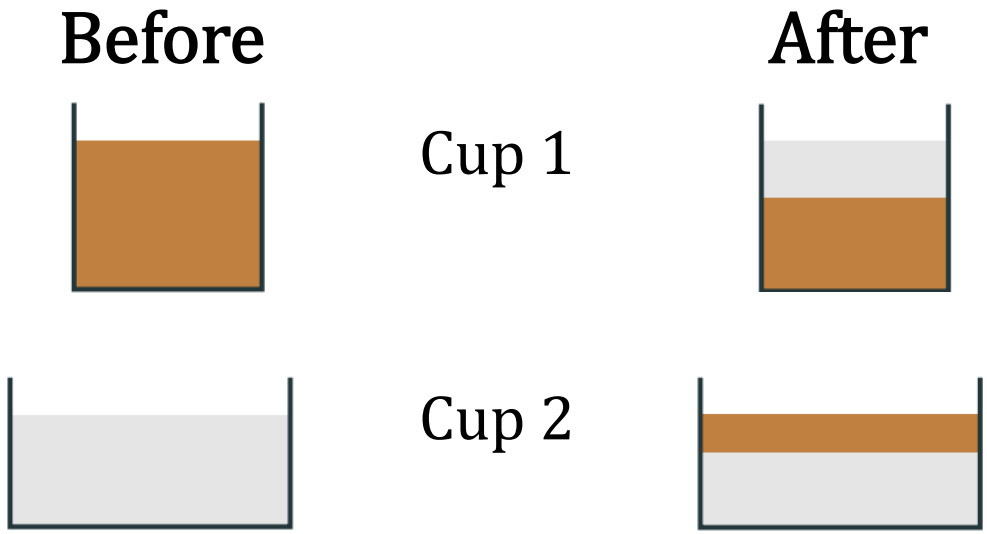

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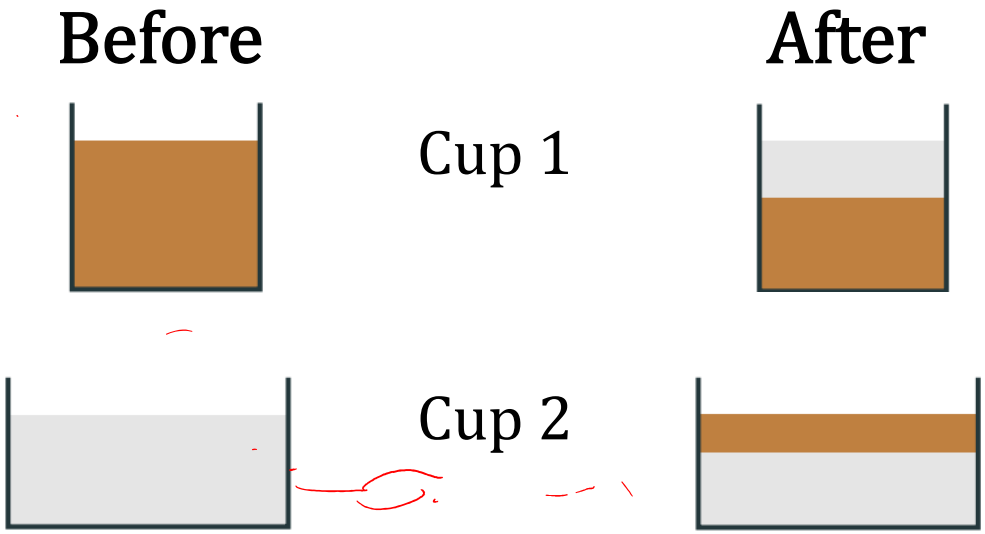

- The size of the drink in cup 1 is invariant!
- So the amount of coffee missing in cup 1 is the same as the amount of milk added into cup 1
- Conversely, amount of milk missing in cup 2 is the same as the amount of coffee added into cup 2

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There are two equally-sized cups, one with coffee & another milk. Both cups are half-full. We want coffee w/ lots of milk on the 1st cup: 1/3 coffee, 2/3 milk. We can pour from one cup to another back & forth. Can we get our favorite coffee mix into our favorite cup? Any amount would do, right proportion matters.

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  - It's good if second cup is mostly coffee
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  - We just have to choose the right invariant

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The proportion of coffee in the 1st cup is always greater than the 2nd cup. That is, coffee in the 1st cup is stronger than 2nd cup.

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- But, total amount of coffee and milk are the same

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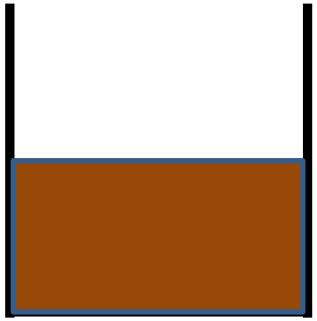

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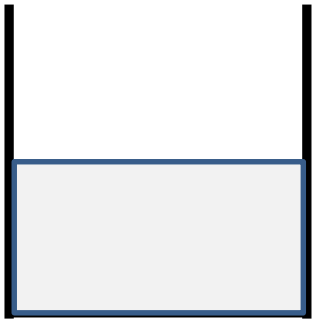

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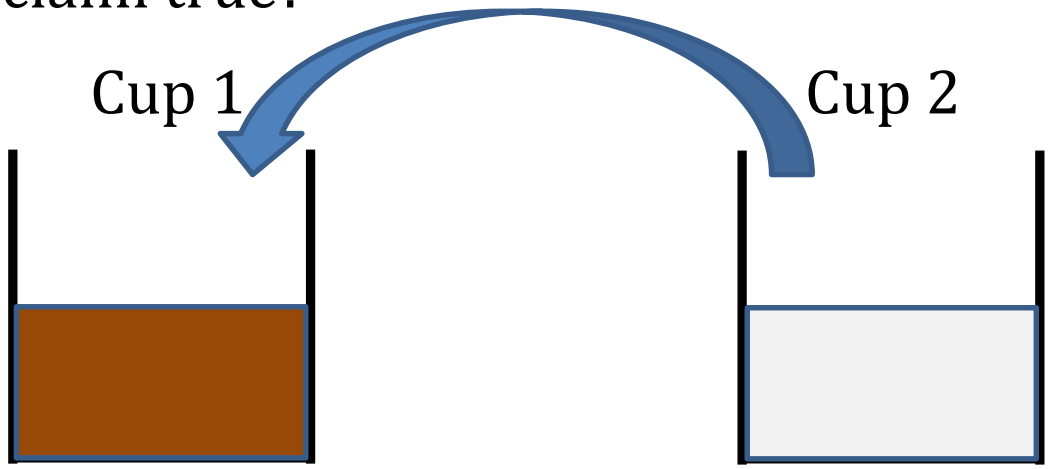

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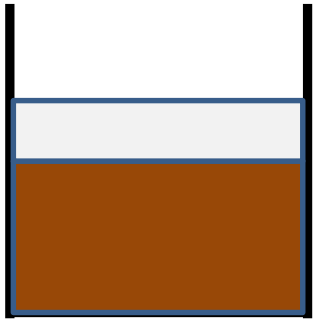
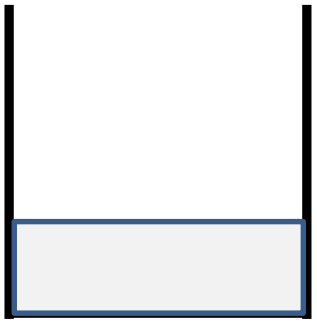


Cup 2



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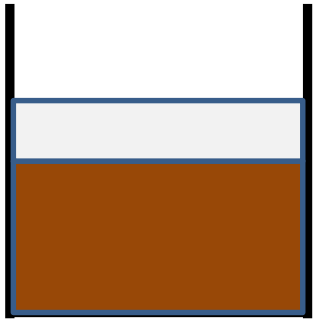

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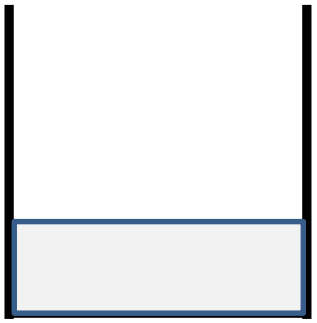

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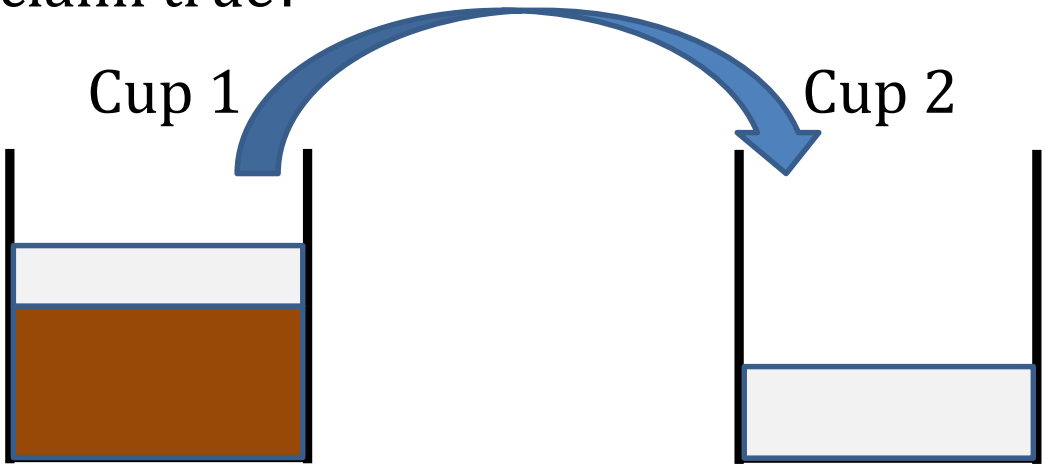

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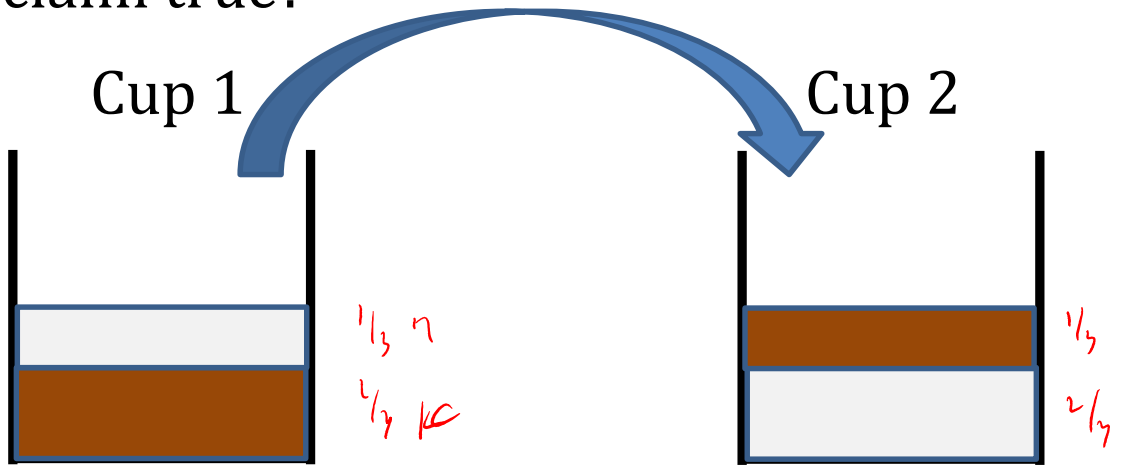

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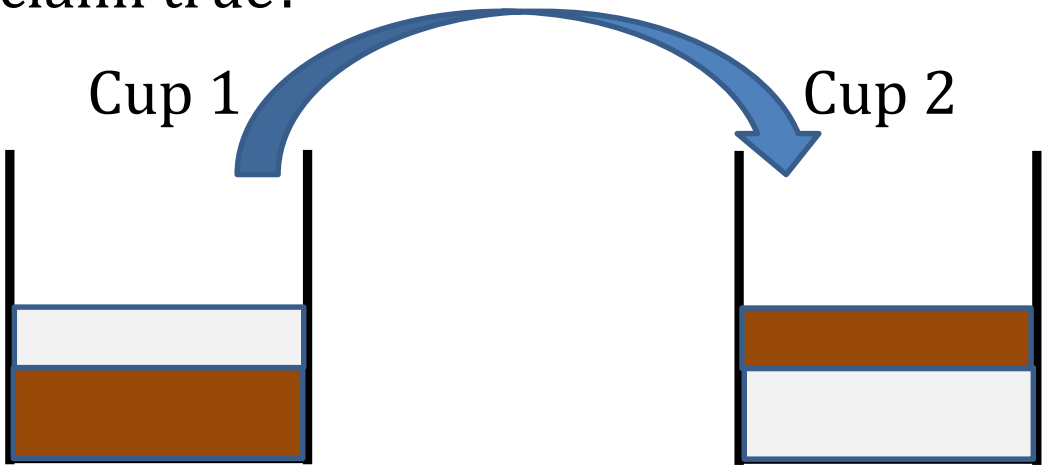

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Invariants

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Typical Debugging

Problem

Bob is debugging his code. There is only one bug when he starts. But three new bugs appear once he fixes a bug. Bob fixed 15 bugs after several hours. How many pending bugs (to be fixed) does he have at this point?

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Fixed:	0	1	2	3	4	15
Pending:	1	3	5	7	9	?

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- #*Pending* = ????

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- *#Pending = 1 + 2 x #Fixed*
  - This is our invariant!

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  - *#Pending = 1 + 2 x 15*
  - *#Pending = 31*

Typical Debugging

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Pending:	1	3	5	7	9	…	31

- #*Pending* = *1* + *2 x* #*Fixed*
  - This is our invariant!
- Hence, if #Fixed = 15
  - #*Pending* = *1* + *2 x 15*
  - #*Pending* = *31*

Mathematical Thinking – Invariants

TERMINATION

Invariants

- Termination
- Football fans
- King Julien's Books

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- For us, invariants are properties that do not change
- In a more wider sense, invariant → properties that change in the right way
- Another standard use of an invariant is showing the termination of a process

Invariants

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Football Fans

Problem

There are 2 football teams in a town. Each of the citizens support a team. If there are more fans of the other team among someone's friend other than his own, this person tends to switch to the other team. One such person switch teams every day. Is it possible that this switching process will go on forever? Assume the following: friendship is always mutual, population does not change, friendship does not change.

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- It seems natural that the process will stop
- How can we prove it though?
  - We need to look at the right value

Football Fans

Solution

Let us look at the number of opposite team's friendships, that is, the pairs of friends supporting opposite teams at the start of the 1^{st} day.

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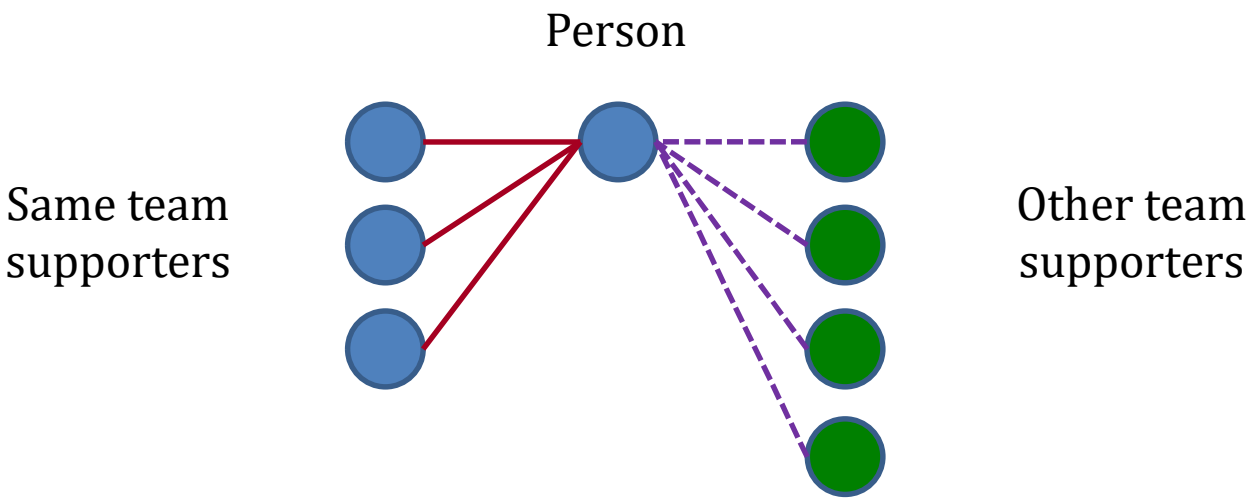

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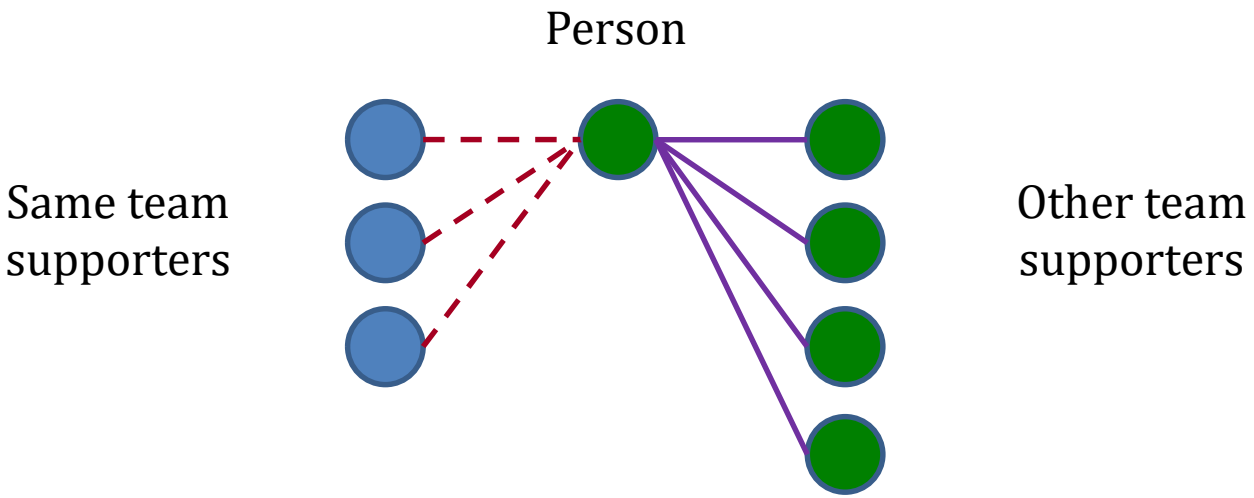

Football Fans

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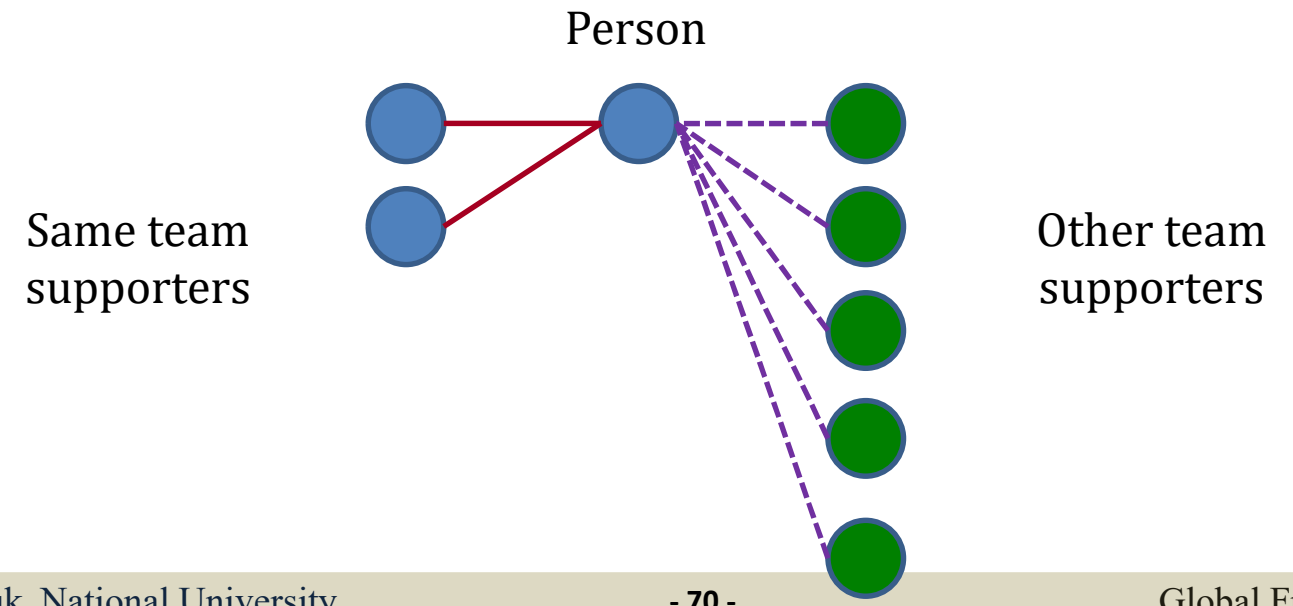

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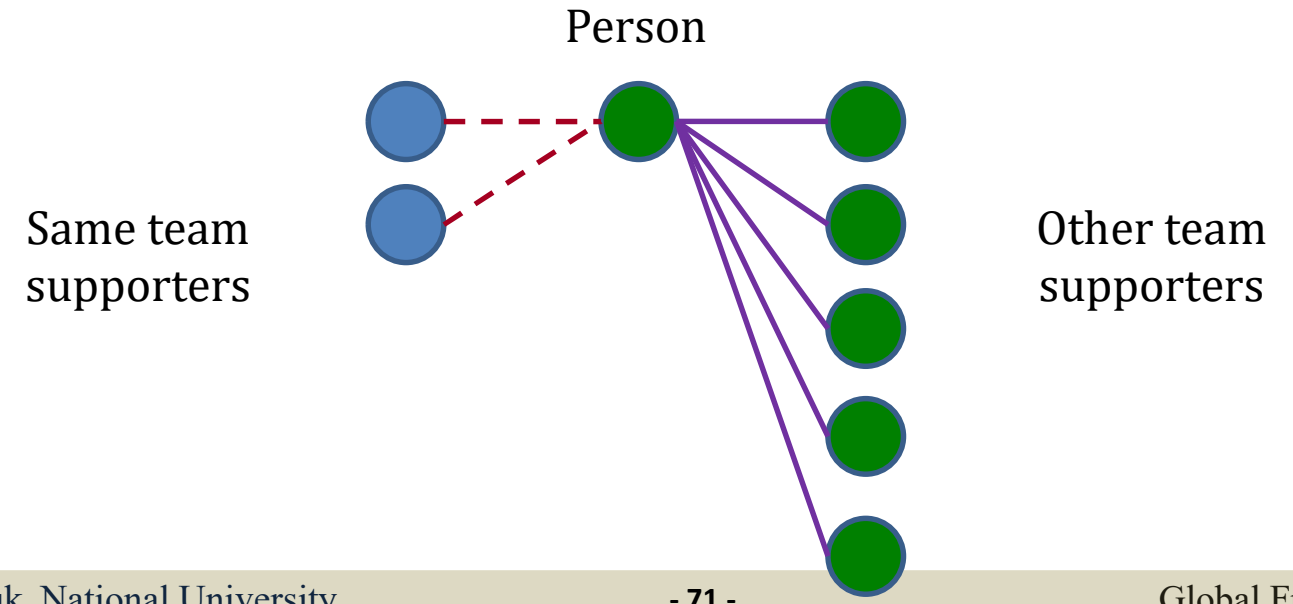

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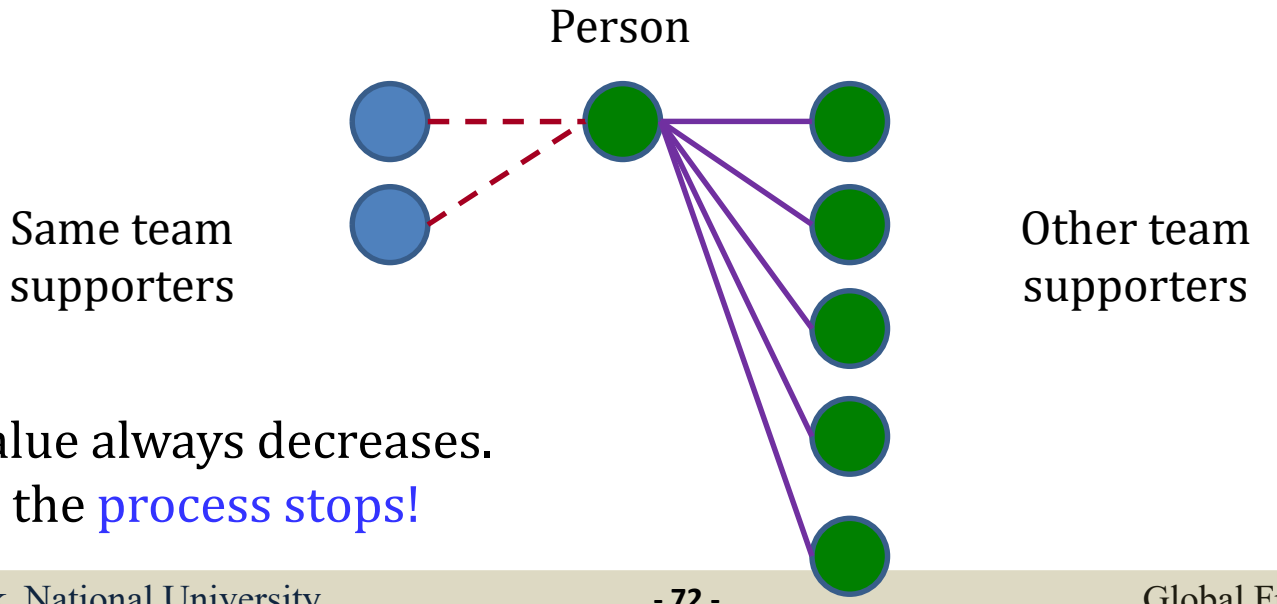

This value always decreases.
Hence, the process stops!

Invariants

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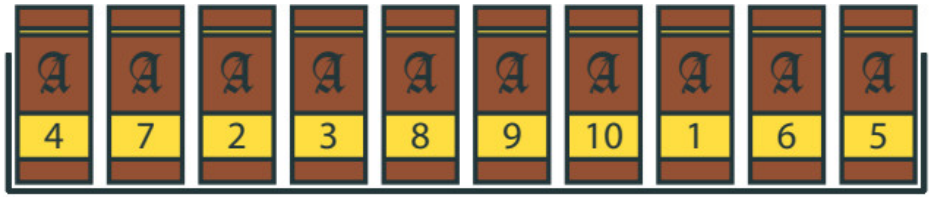
King Julien's Books

Problem

King Julien has a shelf of his works consisting of 10 volumes, labeled 1, 2, …,10. These got jumbled & out of order over the years. King J asked Maurice to sort the collection but he can only take 2 books at once. The books are heavy hence he can switch only two per day. In how many days can Maurice guarantee that the volumes are sorted?



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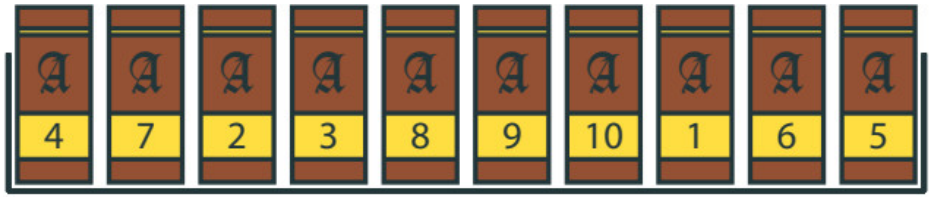

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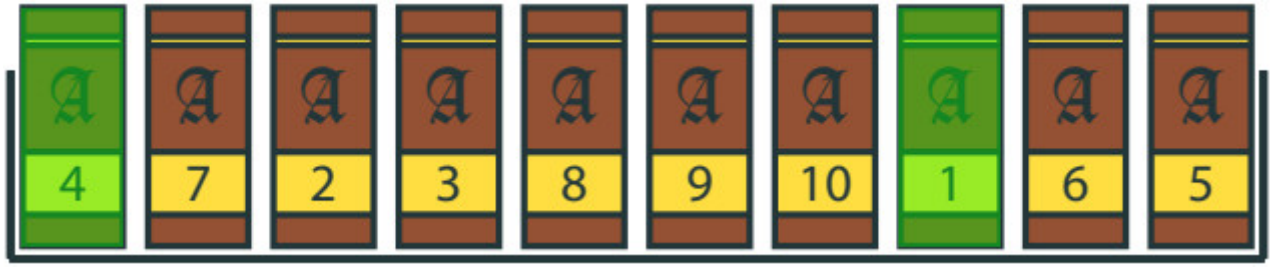

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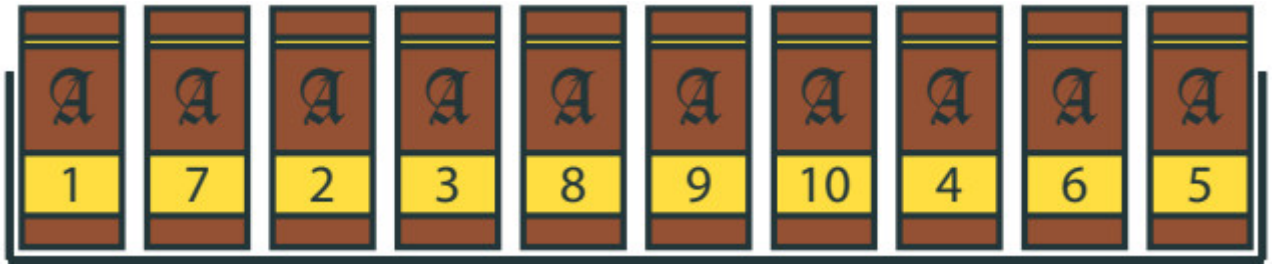

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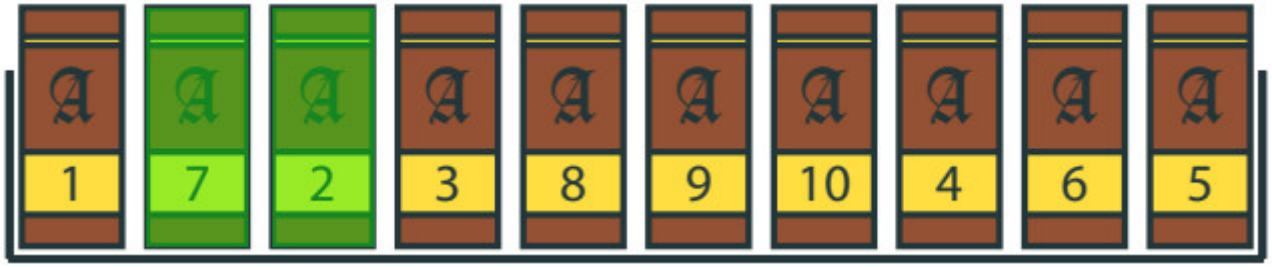

- Day 1: place Vol 1 on it's place
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King Julien's Books

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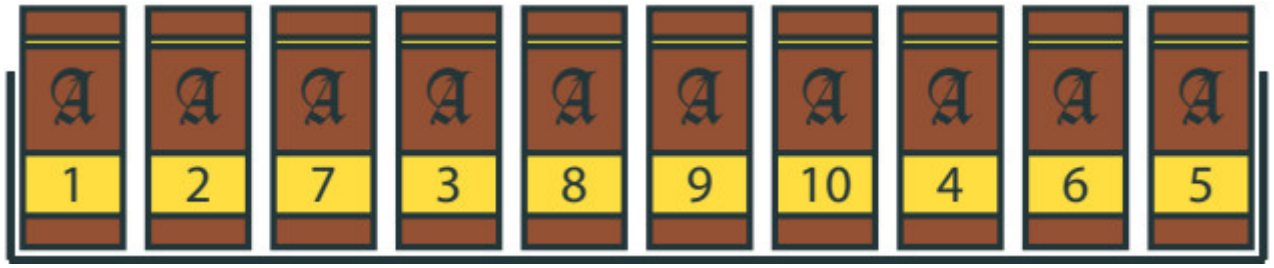

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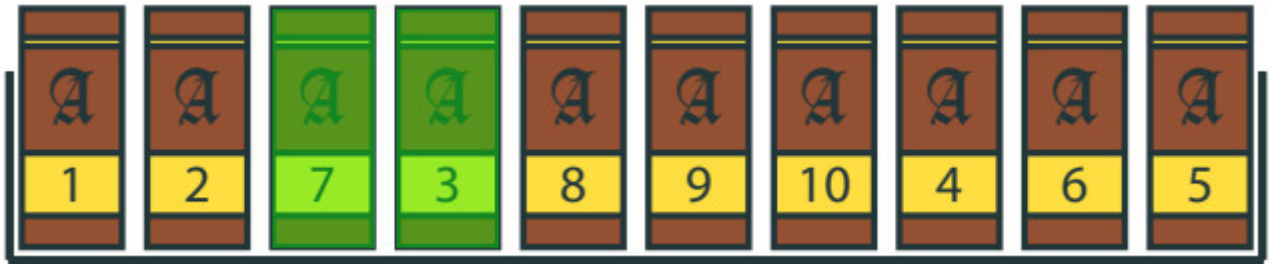

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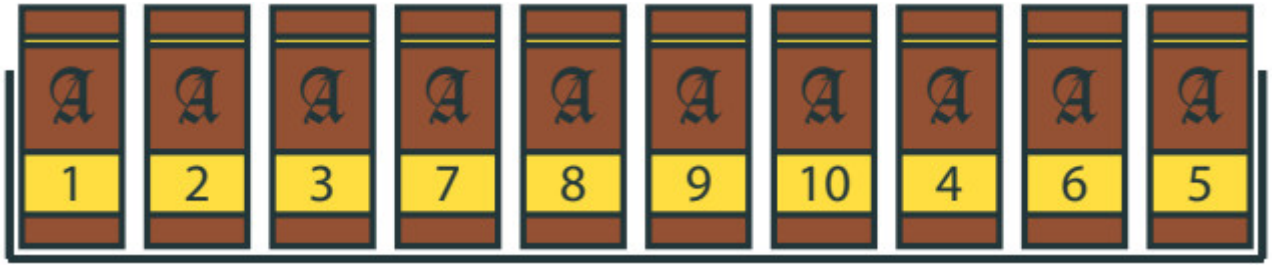

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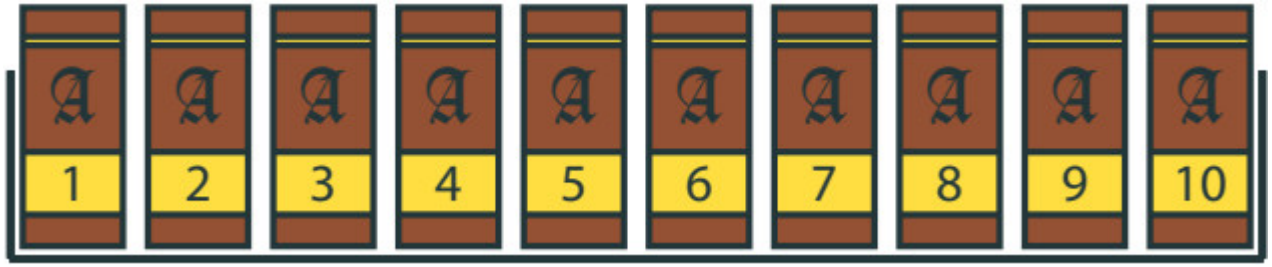

- Day 1: place Vol 1 on it's place
- Day 2: place Vol 2 on it's place
- Day 3: place Vol 3 on it's place
- And so on…
- On Day 9, the first 9 volumes are on their proper places. The 10^{th} volumes must also be in it's proper place since it's only one left.

King Julien's Books

Let's check

- Are 9 days the optimal?

King Julien's Books

Let's check

- Are 9 days the optimal?
- What could be the hardest permutation of the books?

King Julien's Books

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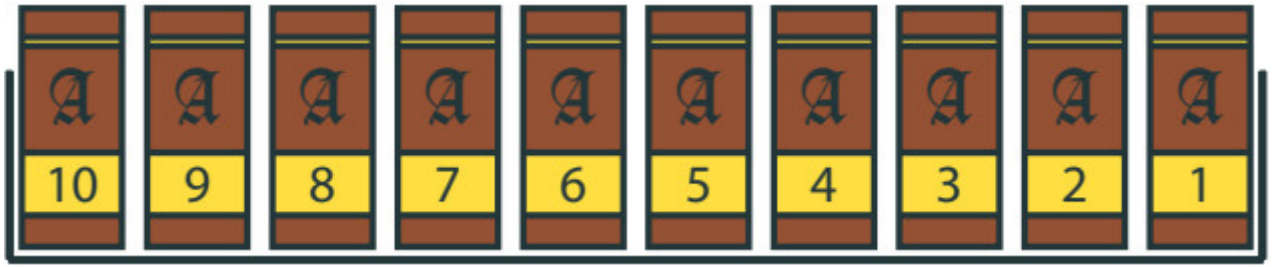
Recall:
- Permutation– order of selection is a factor
- Combination – order of selection not a factor

Ex: Permutation/Combination pairs from the set {A, B, C, D, E}
Permutation: AB AC AD AE BA BC BD BE CA CB CD CE DA DB DC DE EA EB EC ED
Combination: AB AC AD AE BC BE CD CE DE

King Julien's Books

Let's check

- Are 9 days the optimal?
- What could be the hardest permutation of the books?
  - It seems that the hardest is when the books are in opposite order



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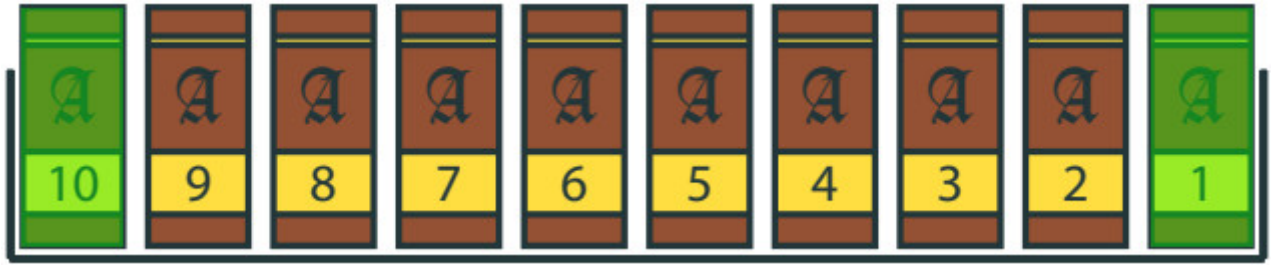

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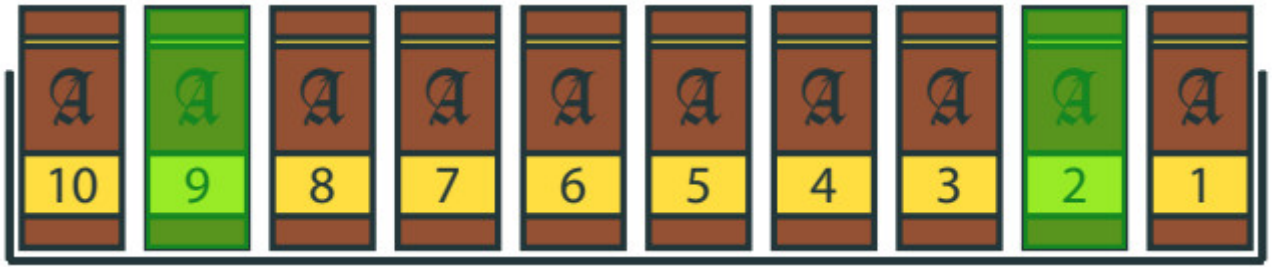

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➢ All the books can be switched and sorted in just 5 days!

King Julien's Books

Let's check

- So what is the right number of days?

King Julien's Books

Let's check

- So what is the right number of days?
- And how to prove that it is the correct answer?

King Julien's Books

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King Julien's Books

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- So what is the right number of days?
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- To answer, we need to find some invariant that:
  - Do not change fast

King Julien's Books

Let's check

- So what is the right number of days?
- And how to prove that it is the correct answer?
- To answer, we need to find some invariant that:
  - Do not change fast
  - Should change substantially while ordering the book

King Julien's Books

Recall this:

Puzzle

- There is a sequence of 10 cells. The leftmost contains "1" while the rightmost has "30". Is it possible to fill other cells with consecutive numbers in such a way that they differ by 3 at the most?

1									30

King Julien's Books

The Invariant:

King Julien's Books

The Invariant: The number of books staying to the right of their intended place

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➢ The invariant is 9 in the beginning, we need at least 9 days.

Thank you.