Introduction to Discrete Math

Felipe P. Vista IV

Course Outline

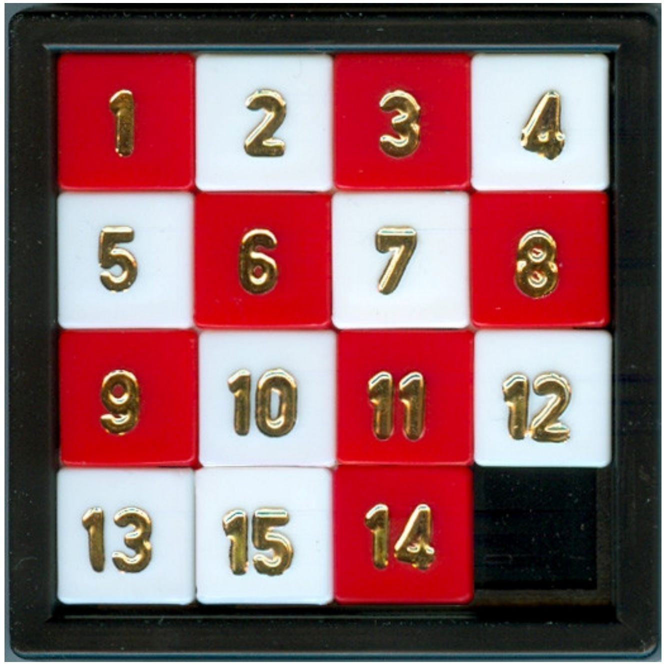
- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion, Logic, Invariants
- Probability & Combinatronics
  - Counting, Probability, Random Variables
- Graph Theory
  - Graphs (cycles, classes, parameters)
- Number Theory & Cryptography
  - Arithmetic in modular form
  - Intro to Cryptography

Mathematical Thinking – Invariants

THE 15-PUZZLE

- The game
- Permutations
- Proof: The Challenging Part
- Mission Impossible
- Classify a Permutation

The 15-Puzzle



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The Game

- move the pieces (into an empty neighbor square)
- goal → to obtain a particular configuration
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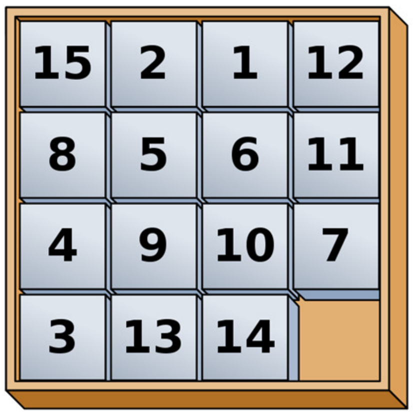

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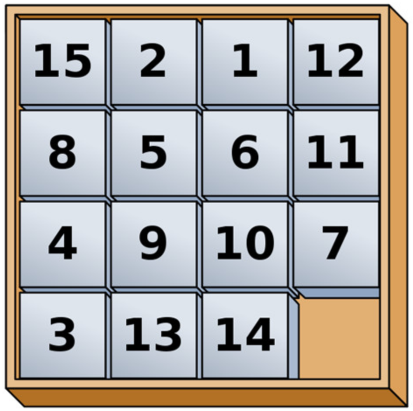

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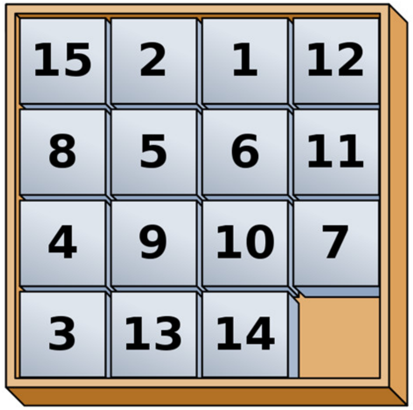

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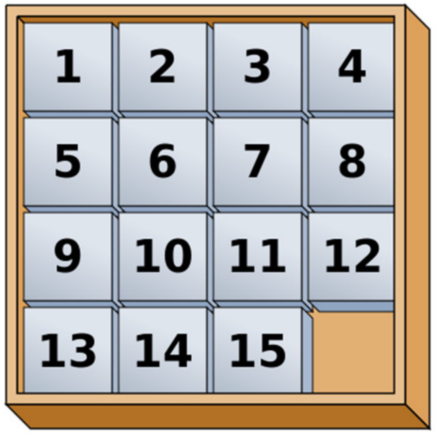

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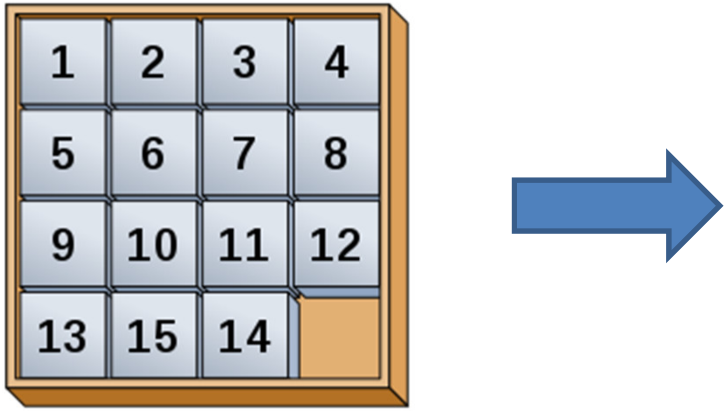

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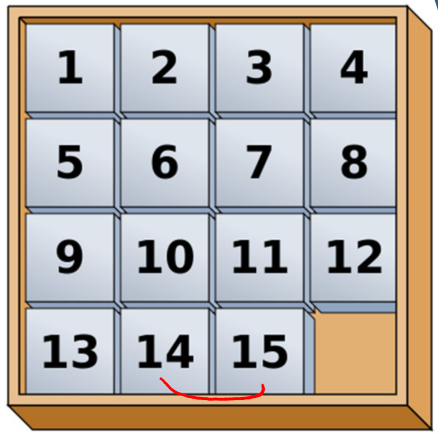

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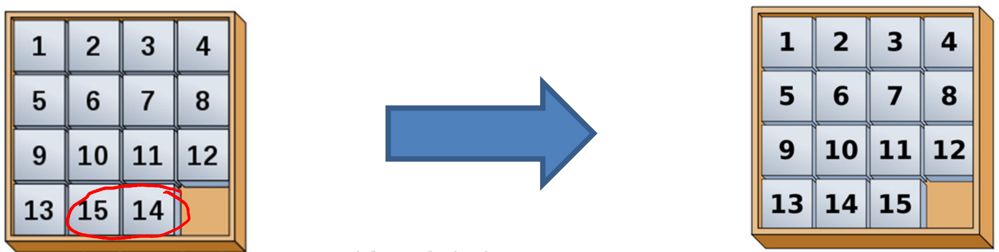

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  - STOP → SPOT:
    - one transposition enough
  - STOP →POST:
    - how many transposition?

STOP → POST
PTOS
POTS
POST

Even and Odd Permutations

- STOP $\to$ SPOT: 1, 3, 5, 7, …
- STOP $\to$ POST: 3, 5, 7, …
- STOP $\to$ POTS: 2, 4, 6, …
- $n \to n+2$ transposition: twice nothing
- Conjecture: permutations are two types
  - Even
  - Odd

A Counterexample

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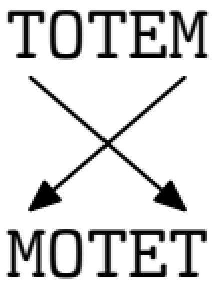

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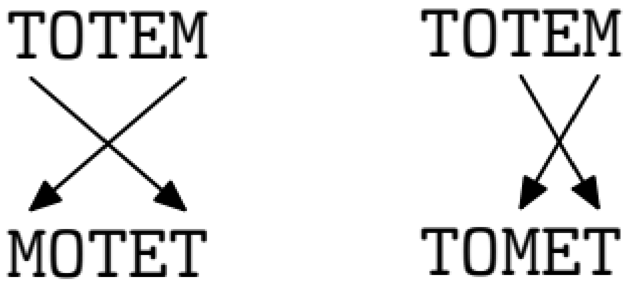

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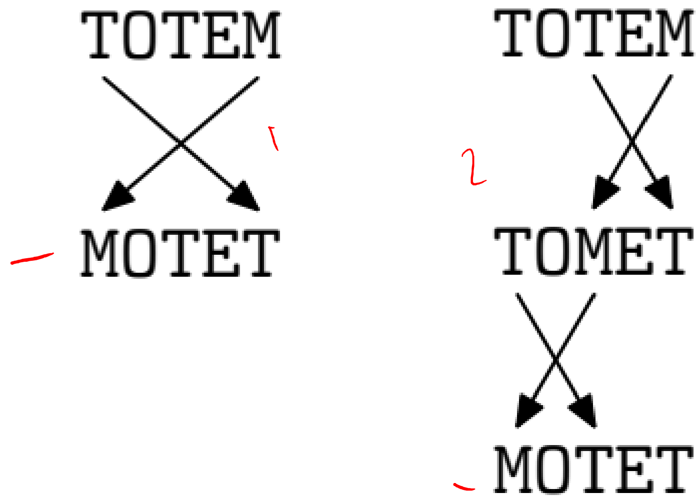

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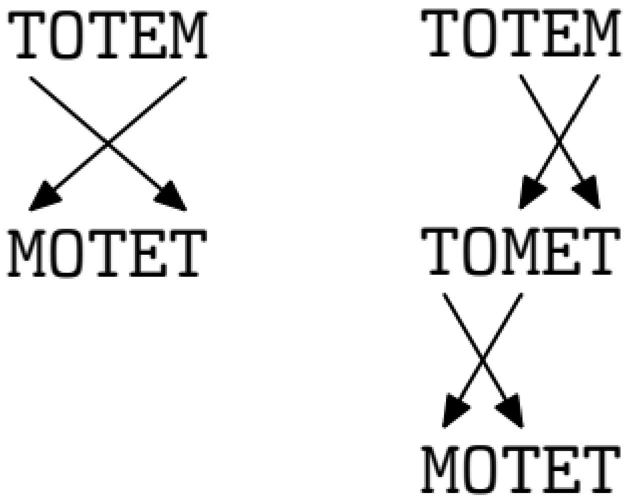

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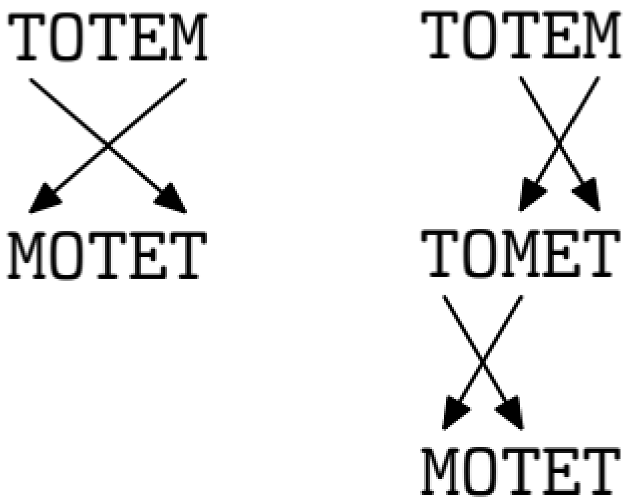

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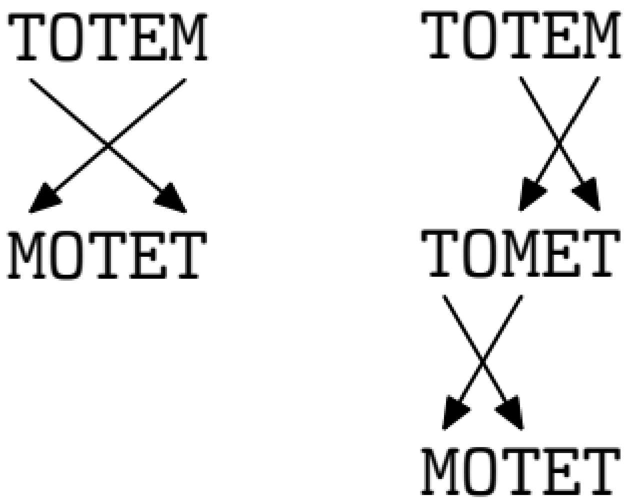

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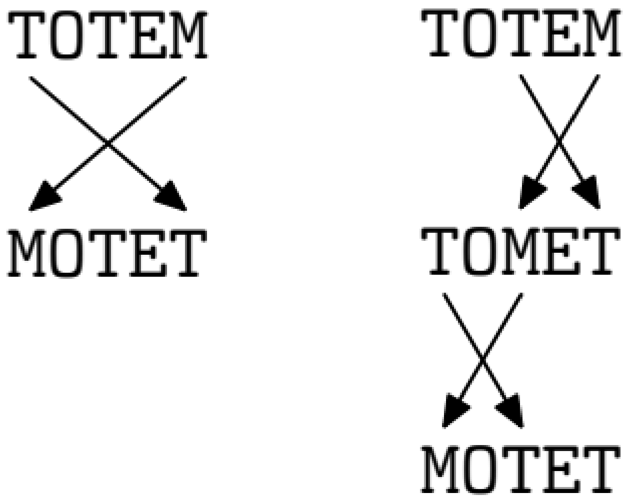

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- some permutations can be derived only through an even number of transpositions, while others can be derived only through odd number of transpositions

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permutation

STOP → POST

STOP
PTOS
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Reminder

- Everybody, make sure that your name in ZOOM is in the following format:
  - Ex: 202054321 Juan Dela Cruz

Not changing your name to this format

* you **will** be marked Absent * → absent?

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  – Not possible if we believe in the special case

A B C D A C B D A B D C A C B D

odd 1 even 2

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- note that neighbor transposition does not change order in other pairs

a b = 2k +1 → odd

b a = 2k +1 → odd

even

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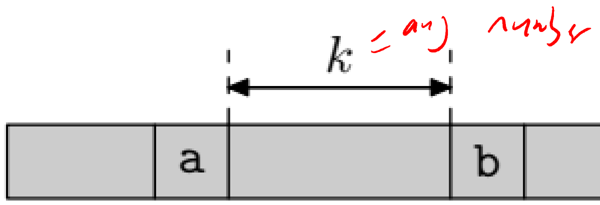

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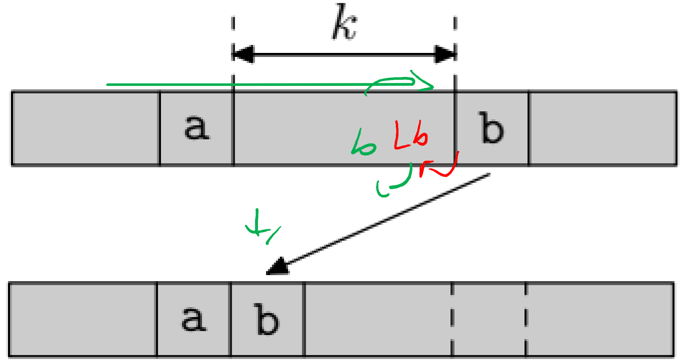

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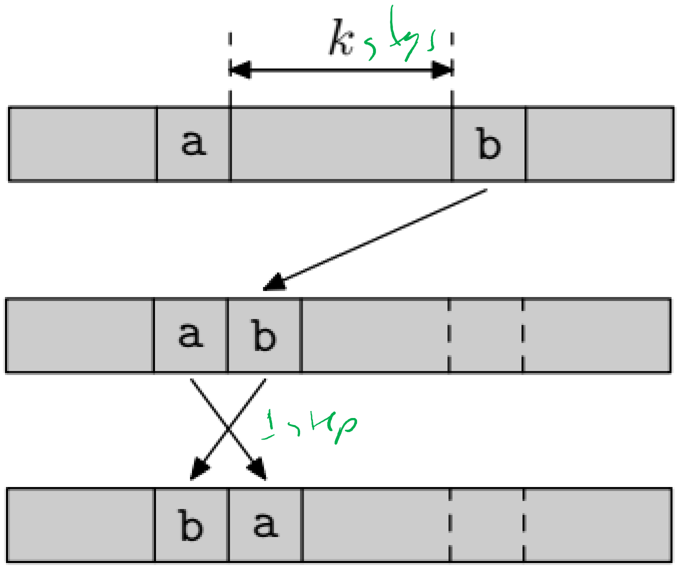

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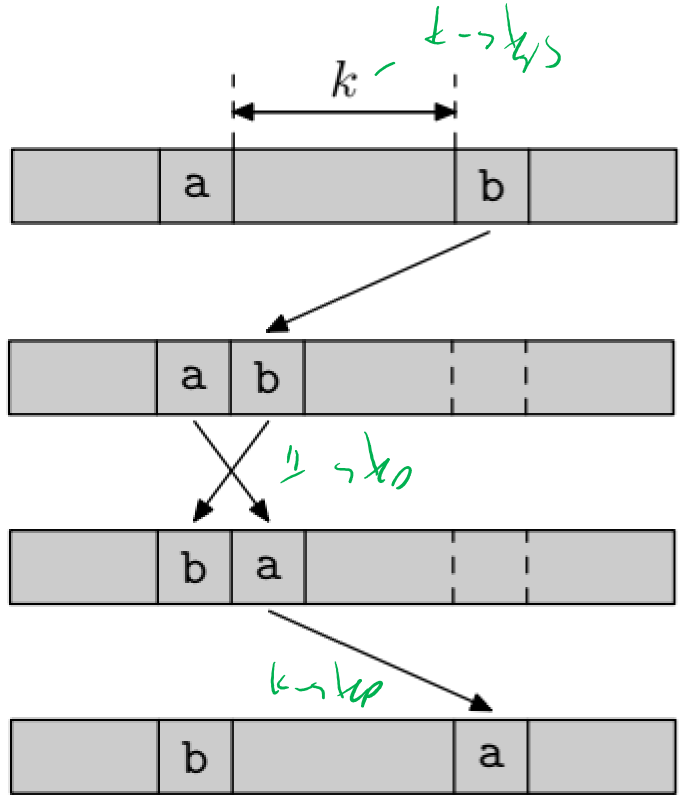

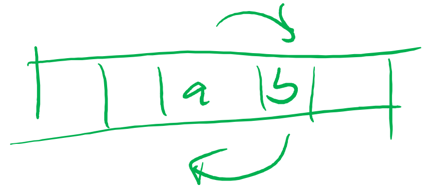

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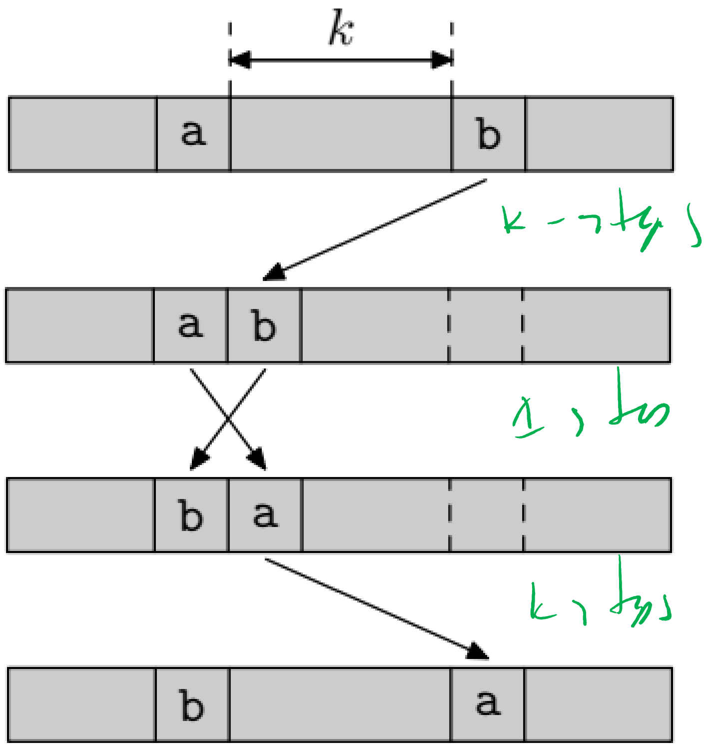

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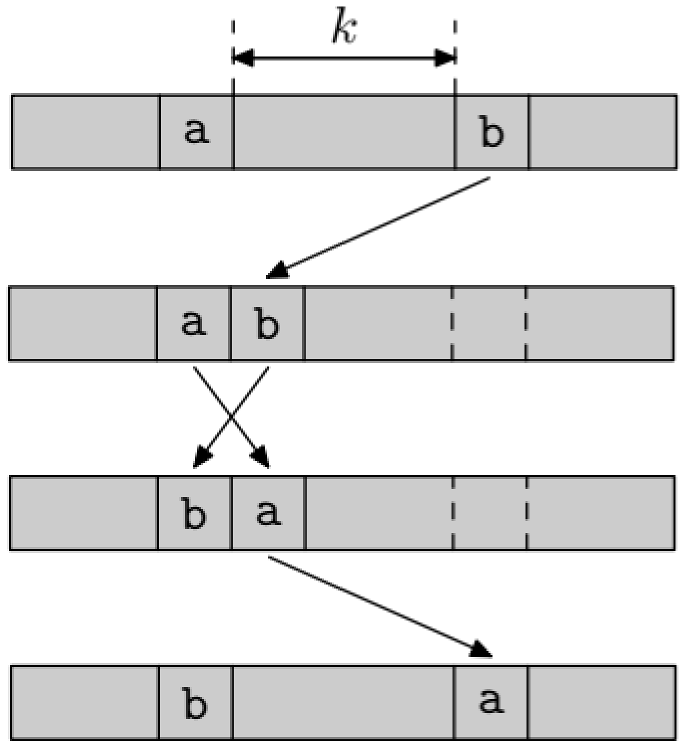

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$= 1 \pmod 2$

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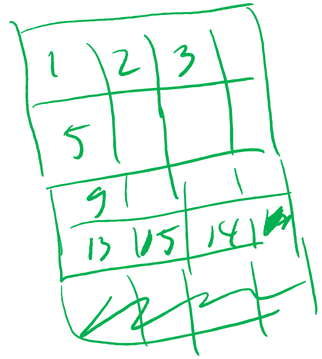

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- OK… so what ?

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- the required exchange requires an odd number of moves
- but for other reasons, it requires an even number of moves, why is this?
- so the required exchange is **impossible!**

odd ≠ even

$2k+1$ = odd

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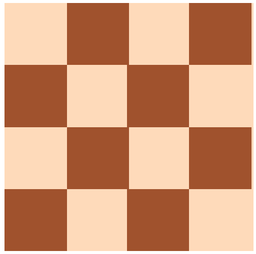

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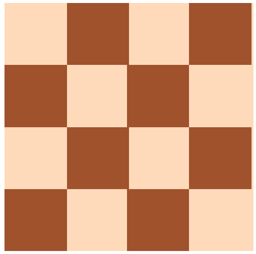

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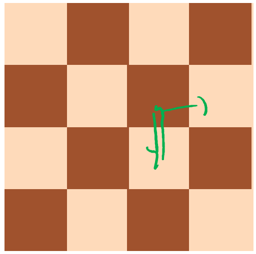

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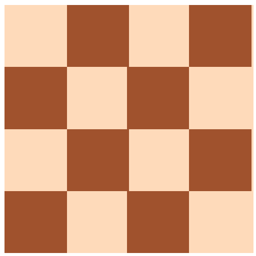

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bringing it back requires an even number of moves. No risk to lose the $100 prize :D…

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Classify a Permutation

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array starts @ 1
index

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How to?:

A Possible Approach

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STOP

SOTP –

SOPT –

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- recall situation when permutation is obtained through transpositions
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- hint: sort a and count the number of exchanges
  - To see if this number is even or odd

Assignment

Implementation

```
// sorting a[1]…a[n]
sign=0 // sign = number of transpositions mod 2
s=0    // first s elements at the right places
while (s < 2) {
   u=s+1; t=u; // a[t] is minimal among a[s+1]…a[u]
   while (u<n) {
      u=u+1;
      if a[u]<a[u] {t = u;}
   }
   // a[t] is minimal among s[t+1]…a[n]
   tmp=a[s+1]; a[s+1]=a[t]; a[t]=tmp; sign=1-sign;
}
```

What is wrong with this code?

Bonus

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- Hint: use $O(n \log n)$ sorting to get $O(n \log n)$ algorithm

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A more efficient algorithm?

- Hint: use $O(n \log n)$ sorting to get $O(n \log n)$ algorithm
  - mergeSort (worst case), heapSort (worst case), quickSort (worst case $\rightarrow O(n^2)$, depending on pivot point)

Bonus

- Running time (number of steps) for this approach?
  - $O(n^2)$, since two nested loops

A more efficient algorithm?

- Hint: use $O(n \log n)$ sorting to get $O(n \log n)$ algorithm
  - mergeSort (worst case), heapSort (worst case), quickSort (best ~~worst~~ case → $O(n^2)$, depending on pivot point)

- Challenge: can you think of an $O(n)$ algorithm?

Thank you.