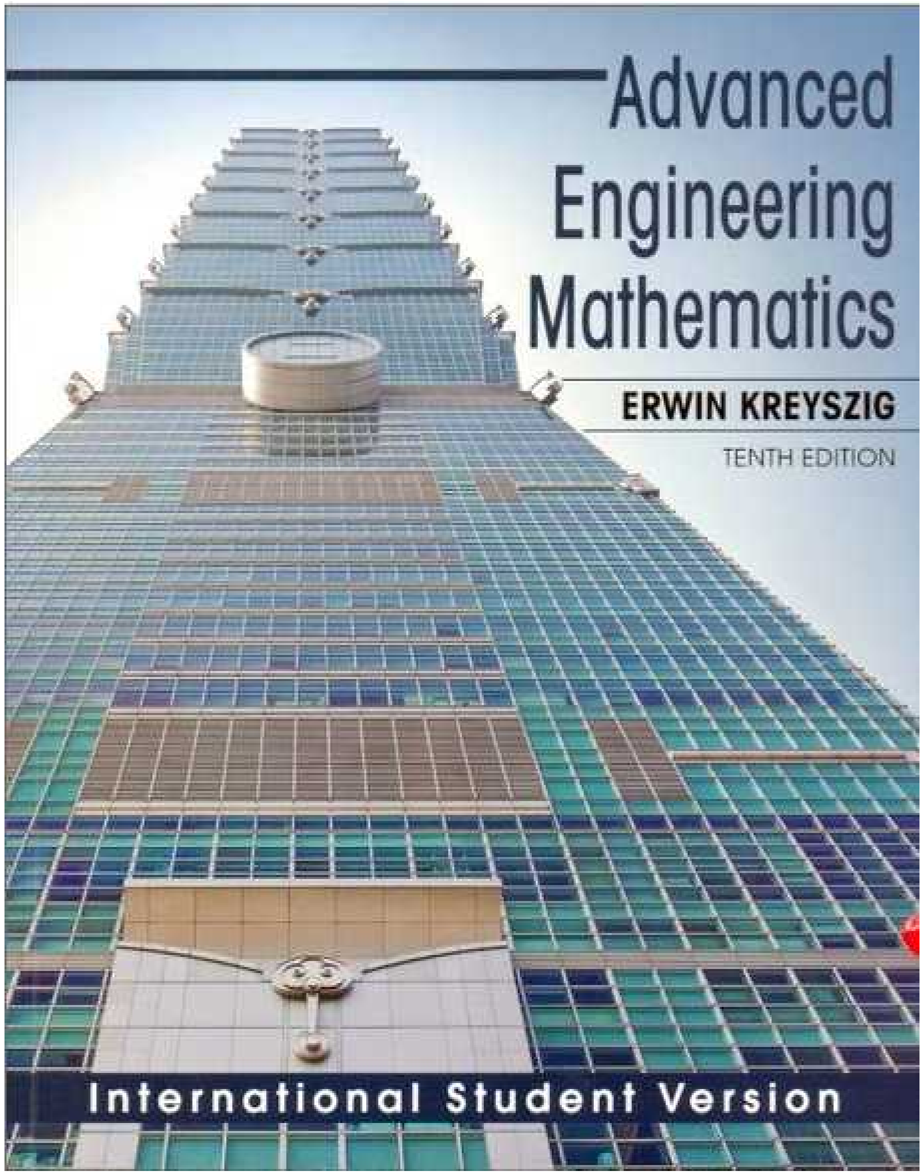
Final Exam(AEM2)

On Monday, 12.13, 09:00-10:30

Classroom: On-Line Test,

Test Problems will be uploaded on Week 15 at ieilmsold.jbnu.ac.kr

Scope: Chapter 14 - Chapter 16, Covered in the class



CHAPTER 16

Laurent Series, Residue Integration

16.1 Laurent Series (Laurent 급수)

Taylor Series:

If f(z) is analytic at z_0, then Taylor series can be used.

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

Laurent Series:

If f(z) is not analytic at z_0, then Taylor series cannot be used.

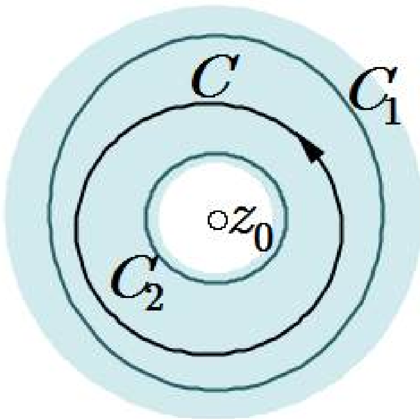
$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

16.1 Laurent Series (Laurent 급수)

THEOREM 1 Laurent's Theorem

Let $f(z)$ be analytic in a domain containing two concentric circles C_1 and C_2 with center z_0 and the annulus between them. Then $f(z)$ can be represented by the Laurent series

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$



If z_0 is the only singular point of $f(z)$ inside C_2, then the series of the negative powers is called the **principal part** of $f(z)$.

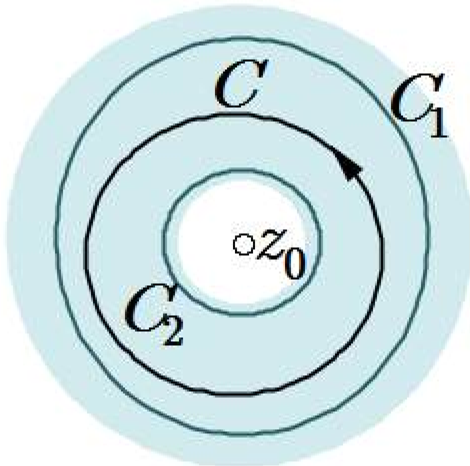
$$(1)\quad f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

$$=a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots$$

$$\cdots+\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\cdots$$

$$(2)\quad a_n=\frac{1}{2\pi i}\oint_C\frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*$$

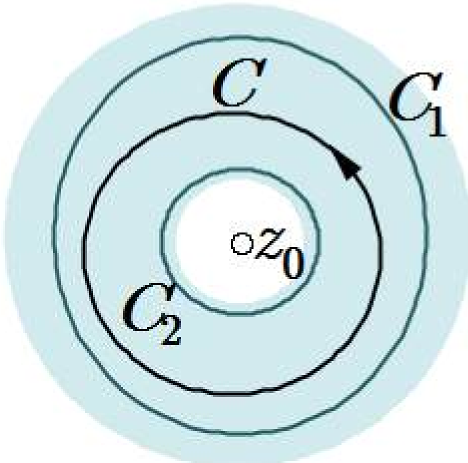
$$b_n=\frac{1}{2\pi i}\oint_C(z^*-z_0)^{n-1}f(z^*)dz^*$$



COMMENT: Simpler Form of Laurent series

$$(1') \quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$(2') \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*$$



PROOF: Omitted

Uniqueness:

The Laurent series of a given analytic function in its annulus of convergence is unique.

EXAMPLE 1 Use of Maclaurin Series

Find the Laurent series of $z^{-5}\sin z$ with center 0.

Sol.

$$z^{-5}\sin z = z^{-5}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n-4}$$

$$= \frac{1}{z^4} - \frac{1}{6z^2} + \frac{1}{120} - \frac{1}{540}z^2 + - \cdots \qquad (|z| > 0)$$

EXAMPLE 2 Substitution

Find the Laurent series of $z^2 e^{1/z}$ with center 0.

Sol.

$$\begin{aligned} z^2 e^{1/z} &= z^2\left(1+\frac{1/z}{1!}+\frac{(1/z)^2}{2!}+\frac{(1/z)^3}{3!}+\frac{(1/z)^4}{4!}+\cdots\right) \\ &= z^2\left(1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3!z^3}+\frac{1}{4!z^4}+\cdots\right) \\ &= z^2+z+\frac{1}{2}+\frac{1}{3!z}+\frac{1}{4!z^2}+\cdots \end{aligned}$$

EXAMPLE 3 Development of 1/(1-z)

Develop 1/(1-z)

(a) in nonnegative powers of z

(b) In negative powers of z.

Sol.

(a) in nonnegative powers of z

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n \qquad (valid\ \ if\ \ |z|<1)$$

(b) in negative powers of z.

$$\frac{1}{1-z}=\frac{1}{-z(1-z^{-1})}=\frac{-1}{z}\sum_{n=0}^{\infty}(z^{-1})^n$$

$$=-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}=-\frac{1}{z}-\frac{1}{z^2}-\cdots \qquad (valid\ \ if\ \ |z|>1)$$

EXAMPLE 4 Laurent Expansions in Different Concentric Annuli

Find all Laurent series of $1/(z^3-z^4)$ with center 0.

Sol.

(a) $|z|<1$:

$$\frac{1}{z^3-z^4}=\frac{1}{z^3}\cdot\frac{1}{1-z}=\sum_{n=0}^{\infty} z^{n-3}$$

$$=\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+z+z^2+\cdots$$

(b) $|z|>1$:

$$\frac{1}{z^3-z^4}=\frac{1}{-z^4(1-z^{-1})}=-z^{-4}\sum_{n=0}^{\infty}(z^{-1})^n$$

$$=-\sum_{n=0}^{\infty}\frac{1}{z^{n+4}}=-\frac{1}{z^4}-\frac{1}{z^5}-\cdots$$

EXAMPLE 5 Use of Partial Fraction

Find all Taylor and Laurent series of $f(z)$ with center 0.

$$f(z)=\frac{-2z+3}{z^2-3z+2}$$

Sol.

$$f(z)=\frac{-2z+3}{z^2-3z+2}=\frac{-2z+3}{(z-1)(z-2)}=\frac{A}{z-1}+\frac{B}{z-2}$$

$$-2z+3=A(z-2)+B(z-1)$$

$$By\ \ insertion: \begin{cases} z=1: & -2+3=-A \\ z=2: & -4+3=B \end{cases}$$

$$By\ \ comparing\ \ coefficients: \ \ -2z+3=(A+B)z+(-2A-B)$$

$$\begin{cases} -2=A+B \\ 3=-2A-B \end{cases}$$

$$f(z) = \frac{-2z+3}{z^2-3z+2} = -\frac{1}{z-1} - \frac{1}{z-2}$$

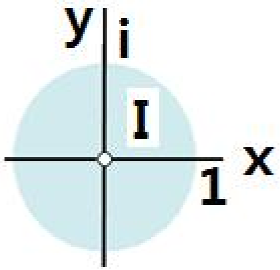
$$-\frac{1}{z-1} = \begin{cases} \sum_{n=0}^{\infty} z^n, & |z| < 1 \\ -\sum_{n=0}^{\infty} 1/z^{n+1}, & |z| > 1 \end{cases}$$

$$-\frac{1}{z-2} = \frac{-1}{-2[1-(z/2)]} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \qquad (|z/2| < 1) \quad (|z| < 2)$$

$$-\frac{1}{z-2} = \frac{-1}{z[1-(2/z)]} = \frac{-1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n \qquad (|2/z| < 1) \quad (|z| > 2)$$

$$f(z)=\frac{-2z+3}{z^2-3z+2}=-\frac{1}{z-1}-\frac{1}{z-2}$$

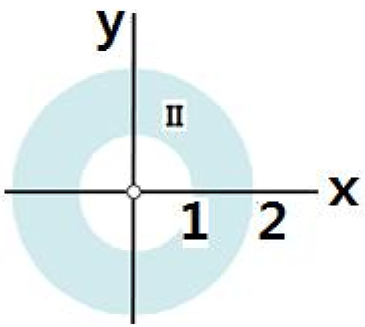
(a) |z|<1



$$f(z)=\sum_{n=0}^{\infty}z^n+\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n=\sum_{n=0}^{\infty}\left(1+\frac{1}{2^{n+1}}\right)z^n$$

$$=\frac{3}{2}+\frac{5}{4}z+\frac{9}{8}z^2+\cdots$$

(b) 1<|z|<2



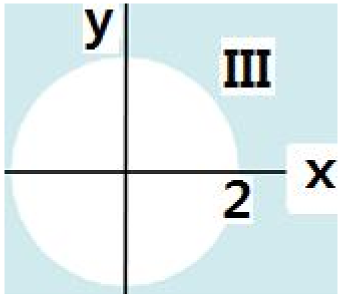
$$f(z)=-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}+\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$=\left(-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right)+\left(\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+\cdots\right)$$

$$f(z)=\frac{-2z+3}{z^2-3z+2}=-\frac{1}{z-1}-\frac{1}{z-2}$$

(c) 2<|z|

$$f(z)=-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n=-\sum_{n=0}^{\infty}(1+2^n)\frac{1}{z^{n+1}}$$



$$=-\frac{2}{z}-\frac{3}{z^2}-\frac{5}{z^3}-\frac{9}{z^4}-\cdots$$

PROBLEM Set 16.1

1. Expand the function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence.

$$\frac{\cos z}{z^4}$$

$$\frac{\cos z}{z^4} = \frac{1}{z^4}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + - \cdots\right)$$

$$= \frac{1}{z^4} - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!} - \frac{1}{6!}z^2 + \cdots$$

$$R = \infty$$

PROBLEM Set 16.1

2. Expand the function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence.

$$\frac{\exp(-1/z^2)}{z^2}$$

$$\begin{aligned}\frac{\exp(-1/z^2)}{z^2} &= \frac{1}{z^2}\left[1-\frac{1}{z^2}+\frac{1}{2!}\frac{1}{z^4}-\frac{1}{3!}\frac{1}{z^6}+-\cdots\right]\\ &= \frac{1}{z^2}-\frac{1}{z^4}+\frac{1}{2!}\frac{1}{z^6}-\frac{1}{3!}\frac{1}{z^8}+-\cdots\end{aligned}$$

$R=\infty$

PROBLEM Set 16.1

5. Expand the function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence.

$$\frac{1}{z^2 - z^3}$$

$$\begin{aligned}\frac{1}{z^2 - z^3} &= \frac{1}{z^2}\frac{1}{1-z} \\ &= \frac{1}{z^2}\left(1+z+z^2+z^3+\cdots\right) \\ &= \frac{1}{z^2}+\frac{1}{z}+1+z+z^2+z^3+\cdots\end{aligned}$$

$R=1$

PROBLEM Set 16.1

8. Expand the function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence.

$$\frac{e^z}{z^2 - z^3}$$

$$\frac{e^z}{z^2 - z^3} = \frac{1}{z^2}\frac{e^z}{1-z} = \frac{1}{z^2}\ e^z\ \frac{1}{1-z}$$

$$= \frac{1}{z^2}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)\left(1 + z + z^2 + z^3 + \cdots\right)$$

$$= \frac{1}{z^2}\left(1 + 2z + \frac{5}{2}z^2 + \frac{8}{3}z^3 + \cdots\right)$$

$$= \frac{1}{z^2} + \frac{2}{z} + \frac{5}{2} + \frac{8}{3}z + \cdots$$

$R = 1$

PROBLEM Set 16.1

15. Find the Laurent series that converges for $0 < |z - z_0| < R$ and determine the precise region of convergence. Show details.

$$\frac{\cos z}{(z-\pi)^2}, \quad z_0 = \pi$$

$$\cos z = \cos((z-\pi)+\pi) = \cos(z-\pi)\cos\pi + \sin(z-\pi)\sin\pi$$
$$= -\cos(z-\pi)$$

$$\cos w = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \frac{w^6}{6!} + - \cdots \qquad w = z - \pi$$

$$= -1 + \frac{(z-\pi)^2}{2!} - \frac{(z-\pi)^4}{4!} + \frac{(z-\pi)^6}{6!} - + \cdots$$

$$\frac{\cos z}{(z-\pi)^2} = -(z-\pi)^{-2} + \frac{1}{2} - \frac{1}{24}(z-\pi)^2 + \frac{1}{720}(z-\pi)^4 - + \cdots$$

The principal part is $-(z-\pi)^{-2}$

and the radius of convergence is $0 < |z-\pi| < \infty$ (converges for all $z \neq \pi$).

PROBLEM Set 16.1

19. Find all Taylor and Laurent series with center z_0. Determine the precise regions of convergence. Show details.

$$\frac{1}{1-z^2}, \quad z_0 = 0$$

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \qquad |w| < 1 \qquad \text{[by (11), p. 694].}$$

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} (z^2)^n \qquad |z^2| < 1$$

$$= \sum_{n=0}^{\infty} z^{2n} \qquad \text{or} \qquad |z^2| = |z|^2 < 1 \qquad \text{so that } |z| < 1$$

$$= 1 + z^2 + z^4 + z^6 + \cdots.$$

PROBLEM Set 16.1

Similarly, we obtain the Laurent series converging for $|z| > 1$ by the following trick, which you should remember:

$$\frac{1}{1-z^2} = \frac{1}{-z^2\left(1-\dfrac{1}{z^2}\right)} = \frac{1}{-z^2}\cdot\frac{1}{1-\left(\dfrac{1}{z}\right)^2}$$

$$= \frac{1}{-z^2}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{2n}$$

$$= \frac{1}{-z^2}\left(1+z^{-2}+z^{-4}+z^{-6}+\cdots\right)$$

$$= -\frac{1}{z^2}-\frac{1}{z^4}-\frac{1}{z^6}-\frac{1}{z^8}-\cdots$$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{2n+2}} \qquad |z| > 1.$$

PROBLEM Set 16.1

23. Find all Taylor and Laurent series with center z_0. Determine the precise regions of convergence. Show details.

$$\frac{z^8}{1 - z^4}, \quad z_0 = 0$$

Case I. $|z| < 1$

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \qquad |w| < 1 \qquad \text{[by (11), p. 694].}$$

$$\frac{z^8}{1-z^4} = z^8 \frac{1}{1-z^4} = z^8 \sum_{n=0}^{\infty} (z^4)^n = \sum_{n=0}^{\infty} z^{4n+8} \qquad |z| < 1$$

$$= z^8 + z^{12} + z^{16} + \cdots .$$

Case II. $|z| > 1$

From Prob. 19 we know that the Laurent series for

$$\frac{1}{1-w^2} = -\sum_{n=0}^{\infty} \frac{1}{w^{2n+2}} \qquad |w| > 1.$$

$$\frac{1}{1-z^4} = -\sum_{n=0}^{\infty} \frac{1}{(z^2)^{2n+2}} = -\sum_{n=0}^{\infty} \frac{1}{z^{4n+4}} \qquad |z^2| > 1$$

$$\frac{z^8}{1-z^4} = -z^8 \sum_{n=0}^{\infty} \frac{1}{z^{4n+4}} = -\sum_{n=0}^{\infty} \frac{z^8}{z^{4n+4}} = -\sum_{n=0}^{\infty} z^{4-4n}$$

$$= -z^4 - 1 - z^{-4} - z^{-8} - \cdots$$

The principal part is $-z^{-4} - z^{-8} - \cdots$

and the radius of convergence is $|z| > 1$.

16.2 Singularities and Zeros. Infinity (특이점과 영점. 무한대)

Singular point: a point where f(z) is not analytic

Zero: a point where f(z)=0

Isolated Singular point: a point z_0 where
- $f(z_0)$ is not analytic and
- z_0 has a neighborhood without further singularities

Example:
tanz has isolated singularities at $\mp\pi/2$, $\mp 3\pi/2$, etc

$$(1)\quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

z_0 : isolated Singular point

If $f(z)$ has only finite negative terms, then

$$(2)\quad f(z) = \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_m}{(z - z_0)^m}$$

z_0 : a pole of order m

If the order of a pole is infinity, then $f(z)$ has an **isolated essential singularity(진성특이점)**.

EXAMPLE 1 Poles, Essential Singularities

Find the poles and orders of the following functions.

$$\frac{1}{z(z-5)^5}+\frac{1}{(z-2)^2}, \qquad e^{1/z}, \qquad \sin(1/z)$$

Sol.

$$f(z)=\frac{1}{z(z-5)^5}+\frac{1}{(z-2)^2}$$

A simple pole at $z=0$
A second order pole at $z=2$
A fifth order pole at $z=5$

$$f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

An essential pole at $z = 0$

$$f(z) = \sin(1/z) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - + \cdots$$

An essential pole at $z = 0$

EXAMPLE 2 Behavior Near a Pole

Describe the behavior of $f(z) = 1/z^2$ near the pole.

Sol.

$|f(z)| \to \infty$ as $z \to 0$ in any manner.

THEOREM 1 Poles

If $f(z)$ is analytic and has a pole at $z = z_0$
$|f(z)| \to \infty$ as $z \to z_0$ in any manner.

PROOF: Omitted(see Prob. 24)

EXAMPLE 3 Behavior Near an Essential Singularity

The function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$

Describe the behavior of $f(z)$ in an arbitrarily small ε-neighborhood of $z = 0$

Sol.

No limit as $z \to 0$:

Real positive axis: $z = x \to 0^+$: $f(z) \to \infty$
Real negative axis: $z = x \to 0^-$: $f(z) \to 0$

$f(z) \to ?$ as $z = re^{i\theta} \to 0$:

$f(z) \to ?$ as $z = re^{i\theta} \to 0$:

$$f(z) = e^{1/z} = e^{1/(re^{i\theta})} = e^{(\cos\theta - i\sin\theta)/r} = c_0 e^{i\alpha} = c$$

$$\textit{where}\ c_0 = e^{\cos\theta/r},\ \alpha = -\sin\theta/r$$

$$\cos\theta = r\ln c_0, \quad -\sin\theta = \alpha r$$

$$\cos^2\theta + \sin^2\theta = r^2(\ln c_0)^2 + \alpha^2 r^2 = 1$$

$$r^2 = \frac{1}{(\ln c_0)^2 + \alpha^2}, \quad \tan\theta = -\frac{\alpha}{\ln c_0}$$

$r \to 0$ by adding multiples of 2π to α leaving c unaltered.

→ Picard's Theorem

THEOREM 2 Picard's Theorem

If $f(z)$ is analytic and has an isolated essential singularity at a point z_0, it takes on every value, with at most one exceptional value, in an arbitrarily small ε-neighborhood of z_0.

PROOF See Ref. [D4], vol. 2. p.258

[D4] Hille, E., Analytic Function Theory. 2vols. 2nd ed. Providence. RI: American Mathematical Society, Reprint V1 1983, V2 2005

Removable Singularities

f(z) has a removable singularity at $z = z_0$ if

- f(z) is not analytic at $z = z_0$, but
- can be made analytic there by assigning a suitable value $f(z_0)$

Example of Removable Singularities

$$\begin{cases} f(z) = \dfrac{\sin z}{z}, & z \neq 0 \\ f(0) = 1 \end{cases}$$

Zeros of Analytic Functions

Zero: z_0 is a zero if $f(z_0) = 0$.

Zero of order n: z_0 is a zero of order n if

$$f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(n-1)}(z_0) = 0$$

Simple zero: Zero of order 1

Example: Zero of order 3

$$f(z) = 3(z-1)^3$$

$$f'(z) = 9(z-1)^2$$

$$f''(z) = 18(z-1)$$

EXAMPLE 4 Zeros

Find zeros of the following functions.

$(a)\ f(z) = 1 + z^2$

$(b)\ f(z) = (1 - z^4)^2$

$(c)\ f(z) = (z - a)^3$

$(d)\ f(z) = e^z$

$(e)\ f(z) = \sin z$

$(f)\ f(z) = \sin^2 z$

$(g)\ f(z) = 1 - \cos z$

$(h)\ f(z) = (1 - \cos z)^2$

Sol.

$$(a)\ f(z) = 1 + z^2 = (z+i)(z-i) = 0$$

Simple zeros at $z = \pm i$.

$$(b)\ f(z) = (1 - z^4)^2 = (z-1)^2(z+1)^2(z-i)^2(z+i)^2,$$

Double zeros at $z = \pm 1,\ \pm i$.

$(c)\ \ f(z) = (z-a)^3$

A triple zero at $z = a$.

$(d)\ \ f(z) = e^z \neq 0$

No zero at all.

$(e)\ \ f(z) = \sin z = 0$

Zeros at $z = n\pi,\ n = 0,\ \pm 1,\ \pm 2, \cdots$.

$(f)\ f(z) = \sin^2 z$

Double zeros at $z = n\pi,\ n = 0, \pm 1, \pm 2, \cdots$.

$(g)\ f(z) = 1 - \cos z$

Simple zeros at $z = 2n\pi,\ n = 0, \pm 1, \pm 2, \cdots$

$(h)\ f(z) = (1 - \cos z)^2$

Double zeros at $z = 2n\pi,\ n = 0, \pm 1, \pm 2, \cdots$

Taylor Series of $f(z)$ with an n-th Order Zero at z_0

$$\text{(3)} \quad \begin{aligned} f(z) &= a_n(z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \cdots \\ &= (z-z_0)^n[a_n + a_{n+1}(z-z_0) + a_{n+2}(z-z_0)^2 \\ &\qquad + \cdots] \qquad (a_n \neq 0) \end{aligned}$$

THEOREM 3 Zeros

The zeros of an analytic function $f(z)$ ($\neq 0$) are isolated; that is, each of them has a neighborhood that contains no further zeros of $f(z)$.

PROOF Omitted!

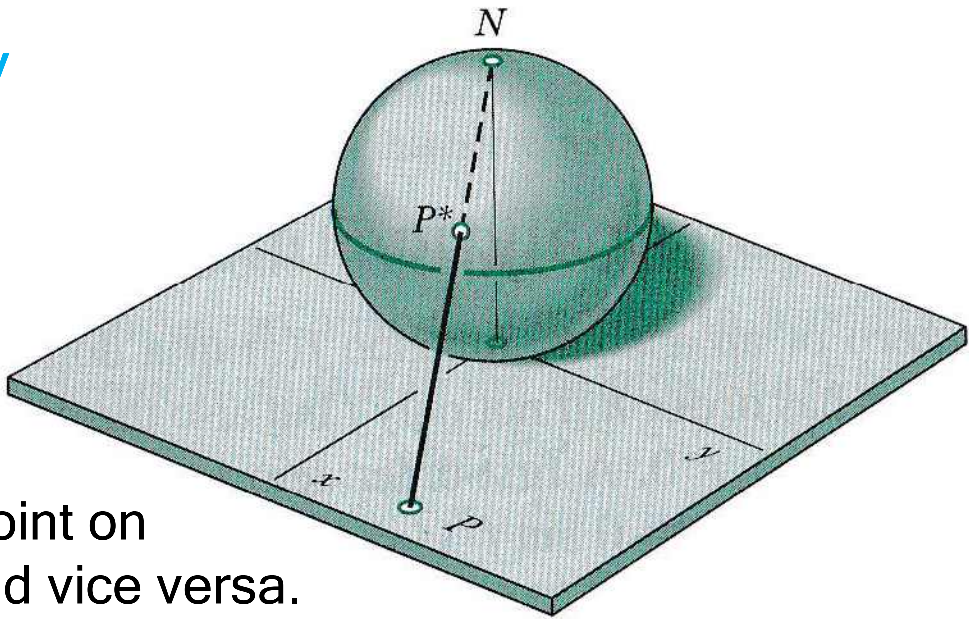
THEOREM 4 Poles and Zeros

Let $f(z)$ be analytic at $z = z_0$ and have a zero of n-th order at $z = z_0$. Then

- $1/f(z)$ has a pole of n-th order at $z = z_0$; and
- so does $h(z)/f(z)$ provided $h(z)$ is analytic at $z = z_0$ and $h(z_0) \neq 0$

PROOF Omitted!

Riemann Sphere. Point at Infinity



Riemann Sphere:

A sphere S of diameter 1 touching the complex plane at z=0.

Each point on S represents a point on the complex plane except N, and vice versa.

Let the image for the **point at infinity**, denoted ∞, be N. Then

-The **finite complex plane**=The complex plane
-The **extended complex plane**

The mapping of the extended complex plane onto the sphere is called a **stereographic projection**.

Analytic or Singular at Infinity

Investigate a function $f(z)$ for large $|z|$

$$z = 1/w$$

$$f(z) = f(1/w) = g(w)$$

$$(4) \quad g(0) = \lim_{w \to 0} g(w)$$

Investigate a function $g(w)$ near $w = 0$

$f(z)$ has an n-th order zero at infinity

$\longrightarrow$ $f(1/w)$ has an n-th order zero at $w = 0$

EXAMPLE 5 Functions Analytic or Singular at Infinity Entire and Meromorphic Functions

Find zeros and poles of the following functions.

$(a)\ f(z) = 1/z^2$

$(b)\ f(z) = z^3$

$(c)\ f(z) = e^z$

$(d)\ f(z) = \cos z$

$(e)\ f(z) = \sin z$

Sol.

$(a)\ f(z) = 1/z^2$

$f(z)$ has a double pole at $z = \infty$

since $g(w) = f(1/w) = w^2$ has a double zero at $w = 0$.

(b) $f(z) = z^3$

$f(z)$ has a triple pole at $z = \infty$

Since $g(w) = f(1/w) = 1/w^3$ has a triple zero at $w = 0$.

(c) $f(z) = e^z$

$f(z)$ has an essential singularity at $z = \infty$

Since $g(w) = f(1/w) = e^{1/w}$ has an essential singularity at $w = 0$.

$(d)\ \ f(z) = \cos z$

$f(z)$ has an essential singularity at $z = \infty$

Since $g(w) = f(1/w) = \cos(1/w)$ has an essential singularity at $w = 0$.

$$\cos\frac{1}{w} = 1 - \frac{1}{2!}\left(\frac{1}{w}\right)^2 + \frac{1}{4!}\left(\frac{1}{w}\right)^4 - + \cdots$$

$(e)\ \ f(z) = \sin z$

$f(z)$ has an essential singularity at $z = \infty$

Since $g(w) = f(1/w) = \sin(1/w)$ has an essential singularity at $w = 0$.

$$\sin\frac{1}{w} = \left(\frac{1}{w}\right) - \frac{1}{3!}\left(\frac{1}{w}\right)^3 + \frac{1}{5!}\left(\frac{1}{w}\right)^5 - + \cdots$$

PROBLEM Set 16.2

3. Determine the location and order of the zeros.

$(z + 81i)^4$

We claim that $f(z) = (z + 81i)^4$ has a fourth-order zero at $z = -81i$.

$$f(z) = (z + 81i)^4 = 0 \qquad \text{gives} \qquad z = z_0 = -81i.$$

To determine the order of that zero we differentiate until $f^{(n)}(z_0) \neq 0$.

$$\begin{aligned} f(z) &= (z + 81i)^4, & f(-81i) &= f(z_0) = 0; \\ f'(z) &= 4(z + 81i)^3, & f'(-81i) &= 0; \\ f''(z) &= 12(z + 81i)^2, & f''(-81i) &= 0; \\ f'''(z) &= 24(z + 81i), & f'''(-81i) &= 0; \\ f^{iv}(z) &= 24, & f^{iv}(-81i) &\neq 0. \end{aligned}$$

Hence, by definition of order of a zero, we conclude that the order at z_0 is 4.

PROBLEM Set 16.2

5. Determine the location and order of the zeros.

$z^{-2}\sin^2 \pi z$

The point of this, and similar problems, is that we have to be cautious. In the present case, $z = 0$ is not a zero of the given function because

$$z^{-2}\sin^2 \pi z = z^{-2}\left((\pi z)^2 + \cdots\right) = \pi^2 + \cdots .$$

$\sin \pi z = 0 \quad \pi z = \pm n\pi \quad z = \pm n$: double zero

16.3 Residue Integration Method (유수적분법)

Cauchy's residue integration is to calculate

$$\oint_C f(z)\, dz \qquad \text{C: Simple Closed Path}$$

If $f(z)$ is analytic everywhere on C and inside C:

$$\oint_C f(z)\, dz = 0$$

If $f(z)$ is singular at a point z= z_0 inside C but is otherwise analytic on C and inside C as before. $\Rightarrow$?

If $f(z)$ is singular at a point $z = z_0$ inside C but is otherwise analytic on C and inside C as before.

Laurent series, for $0 < |z - z_0| < R$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

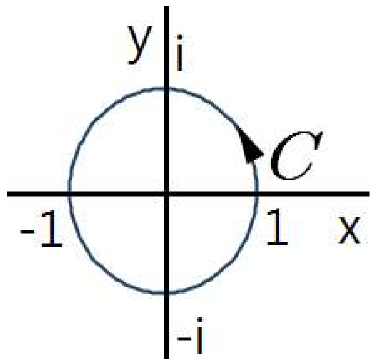
$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

$\Rightarrow$ (1) $\displaystyle \oint_C f(z)\, dz = 2\pi i b_1$

(2) $\displaystyle b_1 = \operatorname*{Res}_{z = z_0} f(z)$: Residue of $f(z)$ at $z = z_0$

EXAMPLE 1 Integration of an Integral by Means of a Residue

Determine $\oint_C f(z)dz = \oint_C \frac{\sin z}{z^4} dz$



Sol.

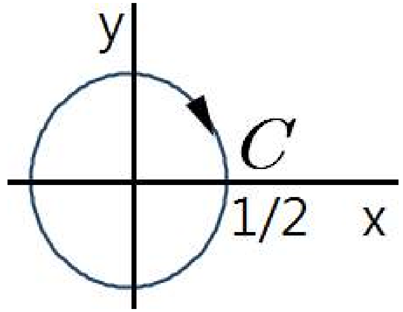
$$f(z) = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + - \cdots$$

$f(z)$: has a pole of third order at $z = 0$.

$$\oint_C \frac{\sin z}{z^4} dz = 2\pi i b_1 = 2\pi i\left(-\frac{1}{3!}\right) = -\frac{\pi i}{3}$$

EXAMPLE 2 CAUTION! Use the Right Laurent Series

Determine $\displaystyle\oint_C f(z)\,dz = \oint_C \frac{1}{z^3 - z^4}\,dz$



Sol.

$$f(z) = \frac{1}{z^3} \cdot \frac{1}{1-z} = \frac{1}{z^3}(1 + z + z^2 + \cdots) \qquad |z| < 1$$

$$= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots$$

$$\oint_C \frac{1}{z^3 - z^4}\,dz = -2\pi i\, b_1 = -2\pi i \cdot 1 = -2\pi i$$

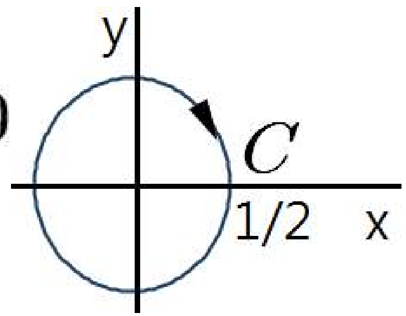
CAUTION! Use the Right Laurent Series

$$f(z)=\frac{1}{z^3-z^4}=-\frac{1}{z^4}\cdot\frac{1}{1-z^{-1}}$$

$$=-\frac{1}{z^4}\left(1+z^{-1}+z^{-2}+\cdots\right) \qquad |z^{-1}|<1$$

$$=-\frac{1}{z^4}-\frac{1}{z^5}-\frac{1}{z^6}-\cdots$$

$$\oint_C \frac{1}{z^3-z^4}\,dz=-2\pi i\, b_1=-2\pi i\cdot 0=0$$



CAUTION! Incorrect Integration!

Formulas for Residues

Simple pole at z_0 :

$$(3) \quad \operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z - z_0) f(z)$$

$$(4) \quad \operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z \to z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

PROOF

Simple pole at z_0:

$$(3) \qquad \operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z - z_0) f(z)$$

PROOF

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$
$$(0 < |z - z_0| < R)$$

$$\lim_{z \to z_0} f(z)(z - z_0) = \lim_{z \to z_0} \left[b_1 + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots \right]$$
$$= b_1$$

Simple pole at z_0:

$$(4) \qquad \operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z\to z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

PROOF

$$b_1 = \operatorname*{Res}_{z\to z_0} (z-z_0) f(z) = \operatorname*{Res}_{z\to z_0} (z-z_0)\frac{p(z)}{q(z)}$$

$q(z)$ has a simple zero z_0:

$$\Rightarrow \quad q(z) = (z-z_0)q'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \frac{(z-z_0)^3}{3!} f'''(z_0) + \cdots$$

$$b_1 = \underset{z \to z_0}{Res} \ (z - z_0)\frac{p(z)}{q(z)}$$

$$= \underset{z \to z_0}{Res} \ \frac{p(z)}{q'(z_0) + \frac{z - z_0}{2!} f''(z_0) + \frac{(z - z_0)^2}{3!} f'''(z_0) + \cdots}$$

$$= \underset{z \to z_0}{Res} \ \frac{p(z)}{q'(z_0) + \frac{z - z_0}{2!} f''(z_0) + \frac{(z - z_0)^2}{3!} f'''(z_0) + \cdots}$$

$$= \frac{p(z_0)}{q'(z_0) + 0 + 0 + \cdots} = \frac{p(z_0)}{q'(z_0)}$$

EXAMPLE 3 Residue at a Simple Pole

Determine the residue of $f(z)$ at z=i where

$$f(z)=\frac{9z+i}{z^3+z}$$

Sol.

$$f(z)=\frac{9z+i}{z^3+z}=\frac{9z+i}{z(z+i)(z-i)}$$

$$\underset{z=i}{Res}\, f(z)=\lim_{z\to i}(z-i)f(z)$$

$$=\lim_{z\to i}(z-i)\cdot\frac{9z+i}{z(z+i)(z-i)}$$

$$=\lim_{z\to i}\frac{9z+i}{z(z+i)}=\frac{10i}{i\cdot 2i}=-5i$$

Poles of any order at z_0:

The residue of $f(z)$ at m-th order pole at z_0 is

$$(5) \qquad \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \right\}$$

For a second-order pole $(m=2)$:

$$(5^*) \qquad \operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \{ [(z-z_0)^2 f(z)]' \}$$

PROOF Omitted!

EXAMPLE 4 Residue at a Pole of Higher Order

Determine the residue of $f(z)$ at z=1 where

$$f(z) = 50z/\left(z^3+2z^2-7z+4\right)$$

Sol.

$$f(z) = \frac{50z}{z^3+2z^2-7z+4} = \frac{50z}{(z-1)^2(z+4)}$$

$$\begin{aligned}
\operatorname*{Res}_{z=1} f(z) &= \lim_{z\to 1}\left[(z-1)^2 f(z)\right]' \\
&= \lim_{z\to 1}\left[(z-1)^2\cdot 50z/\{(z-1)^2(z+4)\}\right]' \\
&= \lim_{z\to 1}\left[\frac{50z}{(z+4)}\right]' = 50\left[\frac{(z+4)-z}{(z+4)^2}\right]_{z=1} \\
&= 8
\end{aligned}$$

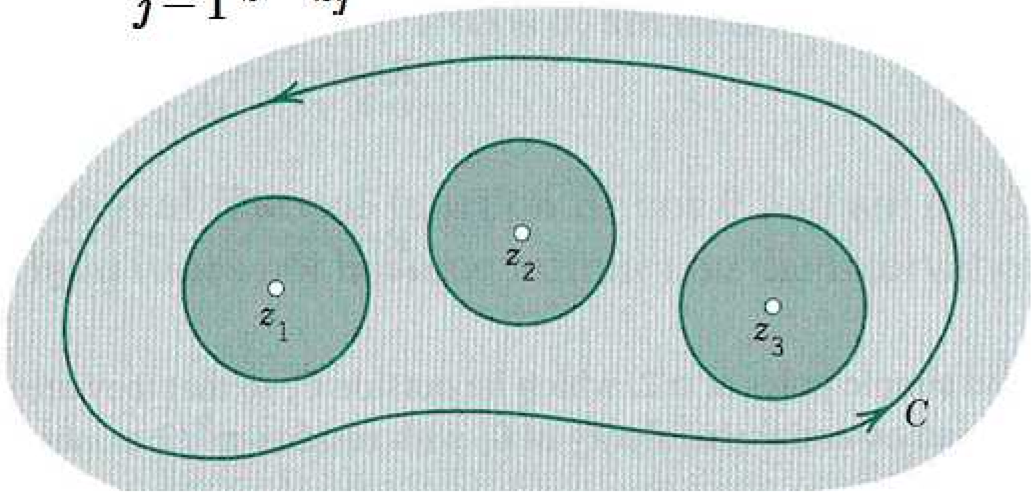
Several Singularities Inside the Contour. Residue Theorem

THEOREM 1 Residue Theorem

$f(z)$: analytic inside a simple closed path C and on C, except for finitely many points, $z_1, z_2, \cdots, z_k$ inside C.

Then

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z) \tag{6}$$



PROOF

EXAMPLE 5 Integration by the Residue Theorem. Several Contours

Evaluate the following integral CCW around any simple closed path such that

$$J = \oint_C \frac{4-3z}{z^2 - z}\,dz$$

(a) 0 and 1 are inside C
(b) 0 is inside, 1 outside
(c) 1 is inside, 0 outside
(d) 0 and 1 are outside

Sol.

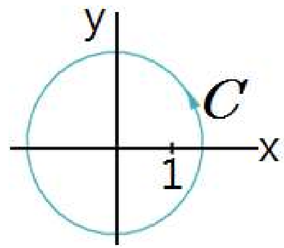
$$\oint_C \frac{4-3z}{z^2 - z}\,dz = \oint_C \frac{4-3z}{z(z-1)}\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$

$$\operatorname*{Res}_{z=0} f(z) = \left[z f(z) \right]_{z=0} = \left[z \frac{4-3z}{z(z-1)} \right]_{z=0} = -4$$

$$\operatorname*{Res}_{z=1} f(z) = \left[(z-1) f(z) \right]_{z=1} = \left[(z-1) \frac{4-3z}{z(z-1)} \right]_{z=1} = 1$$

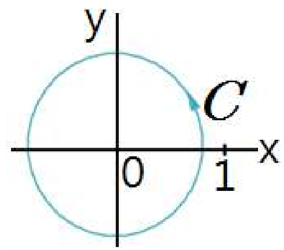
$$\operatorname*{Res}_{z=0} f(z) = -4 \qquad \operatorname*{Res}_{z=1} f(z) = 1$$

(a) 0 and 1 are inside C



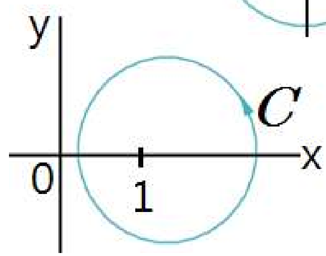
$$J = 2\pi i \left[\operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=1} f(z)\right]$$
$$= -6\pi i$$

(b) 0 is inside, 1 outside



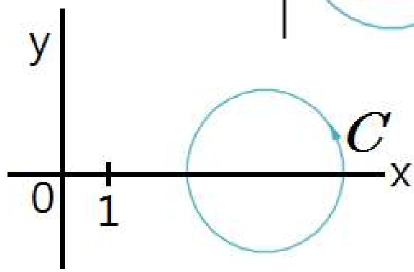
$$J = 2\pi i \left[\operatorname{Res}_{z=0}\right] = -8\pi i$$

(c) 1 is inside, 0 outside



$$J = 2\pi i \left[\operatorname{Res}_{z=1} f(z)\right] = 2\pi i$$

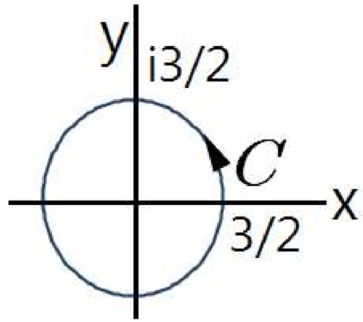
(d) 0 and 1 are outside



$$J = 0 \left(\because analytic\right)$$

EXAMPLE 6 Another Application of the Residue Theorem

Calculate $\displaystyle\oint_C \frac{\tan z}{z^2-1}\,dz$



Sol.

$$\oint_C \frac{\tan z}{z^2-1}\,dz = \oint_C \frac{\tan z}{(z+1)(z-1)}\,dz$$
$$= 2\pi i\left[\operatorname*{Res}_{z=-1} f(z) + \operatorname*{Res}_{z=1} f(z)\right]$$
$$= 2\pi i\left\{[\tan z/(z-1)]_{z=-1} + [\tan z/(z+1)]_{z=1}\right\}$$
$$= 2\pi i \cdot \tan 1 = 9.7855i$$

EXAMPLE 7 Poles and Essential Singularities

Calculate $\displaystyle\oint_C \left(\frac{ze^{\pi z}}{z^4-16} + ze^{\pi/z}\right) dz$ $\quad C: 9x^2 + y^2 = 9$, CCW

Sol.

$$I_1 = \oint_C \frac{ze^{\pi z}}{z^4-16} dz = 2\pi i \sum \mathrm{Res}$$

$$\begin{aligned} z^4 - 16 &= (z^2+4)(z^2-4) \\ &= (z-2i)(z+2i)(z-2)(z+2) \end{aligned}$$

$$(4) \quad \operatorname*{Res}_{z \to z_0} f(z) = \frac{p(z_0)}{q'(z_0)}$$

$$I_1 = \oint_C \frac{ze^{\pi z}}{z^4 - 16} dz = 2\pi i \sum \text{Res} \frac{ze^{\pi z}}{z^4 - 16}$$

$$= 2\pi i \left[\frac{ze^{\pi z}}{4z^3} \bigg|_{z=2i} + \frac{ze^{\pi z}}{4z^3} \bigg|_{z=-2i} \right]$$

$$= 2\pi i \left(-\frac{1}{16} - \frac{1}{16} \right) = -\frac{\pi i}{4}$$

$$I_2 = \oint_C ze^{\pi/z} \, dz = 2\pi i b_1$$

$$I_2 = \oint_C z e^{\pi/z}\, dz = 2\pi i b_1$$

$$\begin{aligned} z e^{\pi/z} &= z\left(1 + \frac{\pi}{z} + \frac{\pi^2}{2!z^2} + \frac{\pi^3}{3!z^3} + \cdots\right) \\ &= z + \pi + \frac{\pi^2}{2!z} + \frac{\pi^3}{3!z^2} + \cdots \qquad (|z| > 0) \end{aligned}$$

$$I_2 = \oint_C z e^{\pi/z}\, dz = 2\pi i b_1 = 2\pi i \cdot \frac{\pi^2}{2!} = \pi^3 i$$

$$I = I_1 + I_2 = -\frac{\pi}{4} i + \pi^3 i = 30.221 i$$

PROBLEM Set 16.3

3. Find all the singularities in the finite plane and the corresponding residues. Show the details. $(\sin 2z)/z^6$

$$z^{-6}\sin 2z = z^{-6}\left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \frac{(2z)^7}{7!} + - \cdots\right)$$

$$= \frac{2}{z^5} - \frac{8}{3!}\frac{1}{z^3} + \frac{32}{5!}\frac{1}{z} - \frac{128}{7!}z + - \cdots .$$

$f(z) = \dfrac{\sin 2z}{z^6}$ has a pole of fifth order at $z = z_0 = 0$

The principal part of (B) is $\dfrac{2}{z^5} - \dfrac{8}{3!}\dfrac{1}{z^3} + \dfrac{32}{5!}\dfrac{1}{z}$.

The coefficient of z^{-1} in the Laurent series (C) is

$$b_1 = \frac{32}{5!} = \frac{32}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{4}{15}.$$

Hence the desired residue at 0 is $\frac{4}{15}$.

PROBLEM Set 16.3

$$\operatorname*{Res}_{z=z_0=0} \frac{\sin 2z}{z^6} = \frac{1}{(6-1)!} \lim_{z\to 0} \left\{ \frac{d^{6-1}}{dz^{6-1}} \left[(z-0)^6 f(z) \right] \right\}$$

$$= \frac{1}{5!} \lim_{z\to 0} \left\{ \frac{d^5}{dz^5} \left[z^6 \frac{\sin 2z}{z^6} \right] \right\}$$

$$= \frac{1}{5!} \lim_{z\to 0} \left\{ \frac{d^5}{dz^5} \sin 2z \right\} = \frac{1}{5!} \lim_{z\to 0} \{32 \cos 2z\}$$

$$= \frac{4}{15}, \qquad \text{as before.}$$

PROBLEM Set 16.3

5. Find all the singularities in the finite plane and the corresponding residues. Show the details. $8/(1+z^2)$

singularities at $z_0 = i$ and $z_0 = -i$

Solution 1. By (3), *p. 721*, we have

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z \to i} \left\{ (z-i) \cdot \frac{8}{1+z^2} \right\} = \lim_{z \to i} \left\{ (z-i) \cdot \frac{8}{(z-i)(z+i)} \right\}$$

$$= \lim_{z \to i} \left\{ \frac{8}{z+i} \right\} = \frac{8}{2i} = \frac{4}{i} = -4i.$$

$$\operatorname*{Res}_{z=-i} f(z) = \lim_{z \to -i} \left\{ (z-(-i)) \cdot \frac{8}{(z-i)(z+i)} \right\}$$

$$= \lim_{z \to -i} \left\{ (z+i) \cdot \frac{8}{(z-i)(z+i)} \right\} = \frac{8}{-2i} = 4i.$$

PROBLEM Set 16.3

Solution 2. By (4), *p. 721*, we have

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)} = \left.\frac{8}{(1+z^2)'}\right|_{z=z_0} = \left.\frac{8}{2z}\right|_{z=z_0} = \frac{8}{2z_0}$$

$$\operatorname*{Res}_{z_0=i} f(z) = \frac{8}{2i} = -4i,$$

$$\operatorname*{Res}_{z_0=-i} f(z) = \frac{8}{-2i} = 4i,$$

PROBLEM Set 16.3

15. Evaluate (counterclockwise). Show the details.

$$\oint_C \tan 2\pi z\, dz, \quad C: |z - 0.2| = 0.2$$

$$f(z) = \tan 2\pi z = \frac{\sin 2\pi z}{\cos 2\pi z} = \frac{p(z)}{q(z)} \qquad \begin{aligned} p(z) &= \sin 2\pi z, \\ q(z) &= \cos 2\pi z \end{aligned}$$

singularities at $\cos 2\pi z = 0$.

$$2\pi z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \cdots, \qquad z = \pm\tfrac{1}{4}, \pm\tfrac{3}{4}, \pm\tfrac{5}{4} \cdots .$$

$$\operatorname*{Res}_{z_0 = \frac{1}{4}} f(z) = \frac{p(\frac{1}{4})}{q'(\frac{1}{4})} = \frac{1}{-2\pi} = -\frac{1}{2\pi}.$$

$$\oint_C f(z)\, dz = \oint_{C:|z-0.2|=0.2:} \tan 2\pi z\, dz \quad = 2\pi i \cdot \operatorname*{Res}_{z_0 = \frac{1}{4}} f(z)$$

$$= 2\pi i \left(-\frac{1}{2\pi}\right) = -i$$

PROBLEM Set 16.3

17. Evaluate (counterclockwise). Show the details.

$$\oint_C \frac{e^z}{\cos z}\, dz, \quad C: |z - \pi i/2| = 4.5$$

$$\cos z = 0 \qquad \text{at} \qquad z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \cdots .$$

$$\text{only} \qquad z = \frac{\pi}{2} \;\text{ and }\; z = -\frac{\pi}{2} \quad \text{lie within } C.$$

$$\operatorname*{Res}_{z=\pi/2} f(z) = \frac{e^{\pi/2}}{-\sin \pi/2} = -e^{\pi/2},$$

$$\operatorname*{Res}_{z=-\frac{\pi}{2}} f(z) = \frac{e^{-\pi/2}}{-\sin(-\pi/2)} = \frac{e^{-\pi/2}}{\sin \pi/2} = e^{-\pi/2}.$$

$$\oint_C f(z)\, dz = \oint_{C:|z-\pi i/2|=4.5} \frac{e^z}{\cos z}\, dz = 2\pi i \left[\operatorname*{Res}_{z=\pi/2} f(z) + \operatorname*{Res}_{z=-\pi/2} f(z)\right]$$

$$= -28.919i$$

PROBLEM Set 16.3

24. Evaluate (counterclockwise). Show the details.

$$\oint_C \frac{\exp(-z^2)}{\sin 4z}\,dz, \quad C: |z| = 1.5$$

singularities inside C: $z = -\dfrac{\pi}{4},\ 0,\ \dfrac{\pi}{4}$

$$\operatorname*{Res}_{z=-\pi/4} f(z) = \lim_{z\to -\pi/4} \frac{\exp(-z^2)}{4\cos 4z} = \frac{\exp(-\pi^2/16)}{-4}$$

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{\exp(-z^2)}{4\cos 4z} = \frac{1}{4}$$

$$\operatorname*{Res}_{z=\pi/4} f(z) = \lim_{z\to \pi/4} \frac{\exp(-z^2)}{4\cos 4z} = \frac{\exp(-\pi^2/16)}{-4}$$

$$\oint_C \frac{\exp(-z^2)}{\sin 4z}\,dz = 2\pi i\left[\operatorname*{Res}_{z=-\pi/4} f(z) + \operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=\pi/4} f(z)\right]$$

16.4 Residue Integration of Real Integrals (실적분의 유수적분)

Integrals of Rational Functions of cosθ and sinθ

$$(1) \quad J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

where

$F(\cos\theta, \sin\theta)$:

- real rational function of cosθ and sinθ, and
- finite on the interval of integration

(1) $$J = \int_0^{2\pi} F(\cos\theta,\ \sin\theta)\, d\theta$$

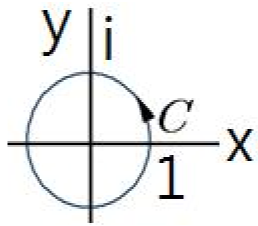
$$e^{i\theta} = z$$

$$dz/d\theta = ie^{i\theta} = iz \qquad d\theta = dz/(iz)$$

(2)
$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

(3) $$J = \oint_C f(z) \frac{dz}{iz}$$



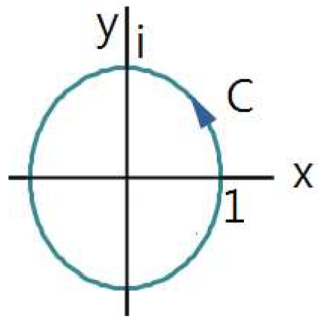
EXAMPLE 1 An Integral of the Type (1)

Show that $$\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = 2\pi$$

Sol.

$$z = e^{i\theta} \quad dz/d\theta = ie^{i\theta} = iz \quad d\theta = dz/(iz)$$

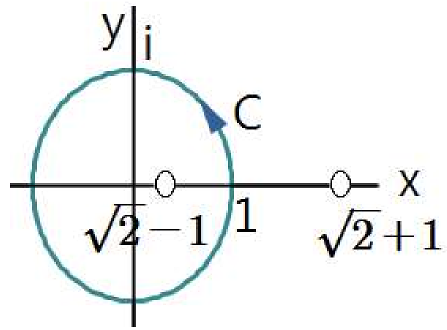
$$\cos\theta = (1/2)(e^{i\theta} + e^{-i\theta}) = (z + z^{-1})/2$$



$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = \oint_C \frac{1}{\sqrt{2} - \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta} = \oint_C \frac{1}{\sqrt{2}-\frac{1}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz}$$

$$= \oint_C \frac{dz}{-\frac{i}{2}(z^2-2\sqrt{2}\,z+1)} = -\frac{2}{i}\oint_C \frac{dz}{(z^2-2\sqrt{2}\,z+1)}$$

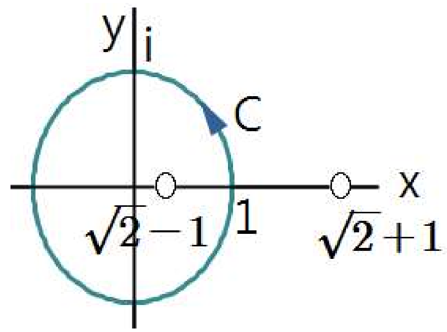


Poles:

$$z^2-2\sqrt{2}\,z+1=0$$

$$z=\sqrt{2}\pm\sqrt{(\sqrt{2})^2-1}=\sqrt{2}\pm 1$$

$$z^2-2\sqrt{2}\,z+1=(z-\sqrt{2}-1)(z-\sqrt{2}+1)=0$$



$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2}-\cos\theta}$$

$$= -\frac{2}{i}\oint_C \frac{dz}{(z^2-2\sqrt{2}\,z+1)}$$

$$= 2\pi i\left(-\frac{2}{i}\right)\operatorname*{Res}_{z=\sqrt{2}-1}\frac{1}{(z^2-2\sqrt{2}\,z+1)}$$

$$= -4\pi\left.\frac{1}{z-\sqrt{2}-1}\right|_{z=\sqrt{2}-1} = 2\pi$$

Improper Integral

$$(4) \quad \int_{-\infty}^{\infty} f(x)dx$$: Improper Integral

Same meaning

$$(5') \quad \int_{-\infty}^{\infty} f(x)dz = \lim_{a \to -\infty} \int_{a}^{0} f(x)dx + \lim_{b \to \infty} \int_{0}^{b} f(x)dx$$

If both limit exist, then

$$(5) \quad \int_{-\infty}^{\infty} f(x)dx = \lim_{R \to \infty} \int_{-R}^{R} f(x)dx$$

Cauchy principal value of the integral:

$$\text{pr.v.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

Cauchy principal value may exist even if the limits in (5') do not.

$$(5') \quad \int_{-\infty}^{\infty} f(x)\,dz = \lim_{a\to-\infty} \int_{a}^{0} f(x)\,dx + \lim_{b\to\infty} \int_{0}^{b} f(x)\,dx$$

Example:

$$\lim_{R\to\infty} \int_{-R}^{R} x\,dx = \lim_{R\to\infty} \left(\frac{R^2}{2} - \frac{R^2}{2} \right) = 0$$

$$\lim_{b\to\infty} \int_{0}^{b} x\,dx = \infty$$

Calculation of Improper Integral

$$(4) \quad \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{N(x)}{D(x)}\,dx$$

Assumption:

- $f(x)$ is a real rational function

  ⟶ $f(z)$ has finitely many poles in the UHP.

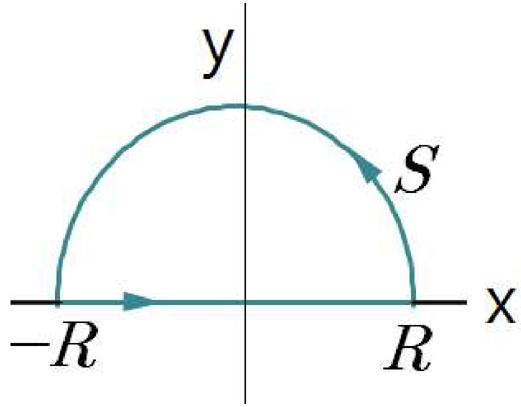
- Order of D(x) ≥ Order of N(x)+2

then the limits in (5') exist.

Therefore, start from (5).

$$(5) \quad \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

$$\oint_C f(z)dz = \int_S f(z)dz + \int_{-R}^{R} f(x)dx$$
$$= 2\pi i \sum_{IHC} \operatorname{Res} f(z)$$



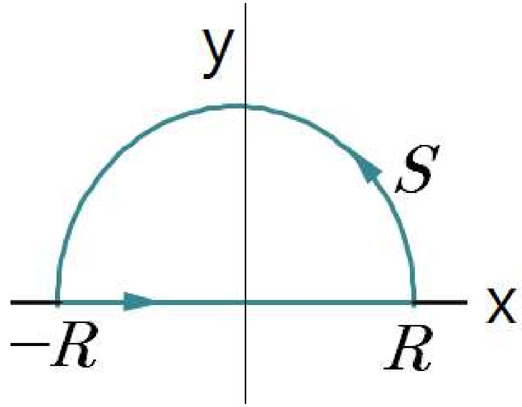
(6) $$\int_{-R}^{R} f(x)dx = 2\pi i \sum_{IHC} \operatorname{Res} f(z) - \int_S f(z)dz$$

IHC: Inside of the Half-Circle

Since $\int_S f(z)dz \to 0 \;\; as \;\; R \to \infty$ (Proof is at the next page.)

(7) $$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{UHP} \operatorname{Res} f(z)$$

PROOF of $\int_S f(z)dz \to 0 \;\; as \;\; R \to \infty$



For sufficiently large constants k and R_0,

$$|f(z)| < \frac{k}{|z|^2} \qquad (|z| = R > R_0)$$

$$\left| \int_S f(z)\,dz \right| \le \int_S |f(z)|\,dz < \frac{k}{R^2}\pi R = \frac{k\pi}{R}$$

Thus,

$$\int_S f(z)dz \to 0 \;\; as \;\; R \to \infty$$

EXAMPLE 2 An Integral from 0 to ∞

Show that $$\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

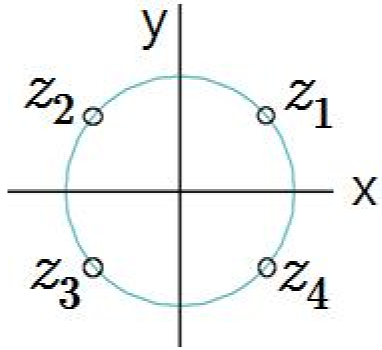
Sol.

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^4}$$

$$= \frac{1}{2}\cdot 2\pi i \sum_{UHP} \mathrm{Res} f(z) = \pi i \sum_{UHP} \mathrm{Res} f(z)$$

$$1+z^4=0, \quad z^4=-1$$

$$z = re^{i\theta} = e^{i[\pi/4+(2\pi/4)n]}$$



$$\operatorname*{Res}_{z=z_1} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_1} = \frac{1}{4z^3}\bigg|_{z=e^{i\pi/4}}$$

$$= \frac{1}{4e^{i3\pi/4}} = \frac{1}{4}e^{-i3\pi/4} = -\frac{1}{4}e^{i\pi/4}$$

$$\operatorname*{Res}_{z=z_2} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_2} = \frac{1}{4z^3}\bigg|_{z=e^{i3\pi/4}}$$

$$= \frac{1}{4e^{i9\pi/4}} = \frac{1}{4}e^{-i\pi/4}$$

$$\sum_{UHP} \operatorname{Res} f(z) = -\frac{1}{4}e^{i\pi/4} + \frac{1}{4}e^{-i\pi/4} = \frac{1}{4}\cdot\left(-\frac{2i}{\sqrt{2}}\right)$$
$$= -\frac{i}{2\sqrt{2}}$$

$$\int_0^\infty \frac{dx}{1+x^4} = \pi i \sum_{UHP} \operatorname{Res} f(z)$$
$$= \pi i \cdot \left(-\frac{i}{2\sqrt{2}}\right) = \frac{\pi}{2\sqrt{2}}$$

Fourier Integrals

$$(8) \quad \int_{-\infty}^{\infty} f(x)\cos sx\,dx \qquad \int_{-\infty}^{\infty} f(x)\sin sx\,dx \qquad (s:\ real)$$

$s: real\ and\ positive$

$$\int_{-\infty}^{\infty} f(x)\cos sx\,dx = \mathrm{Re}\left[\int_{-\infty}^{\infty} f(x)\,e^{isx}dx\right]$$

$$\int_{-\infty}^{\infty} f(x)\sin sx\,dx = \mathrm{Im}\left[\int_{-\infty}^{\infty} f(x)\,e^{isx}dx\right]$$

$$(9) \quad \int_{-\infty}^{\infty} f(x)\, e^{isx} dx = 2\pi i \sum_{UHP} \operatorname{Res}\left[f(z) e^{isz}\right]$$

Proof of (9) is shown in the next page.

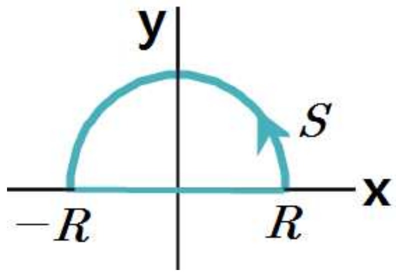
$$\int_{-\infty}^{\infty} f(x) \cos sx \, dx = \operatorname{Re}\left[\int_{-\infty}^{\infty} f(x)\, e^{isx} dx\right]$$

By (9)

$$(10) \quad \begin{cases} \displaystyle\int_{-\infty}^{\infty} f(x) \cos sx \, dx = -2\pi \sum_{UHP} \operatorname{Im} \operatorname{Res}\left[f(z) e^{isz}\right] \\ \displaystyle\int_{-\infty}^{\infty} f(x) \sin sx \, dx = 2\pi \sum_{UHP} \operatorname{Re} \operatorname{Res}\left[f(z) e^{isz}\right] \end{cases} \quad (s > 0)$$

PROOF of (9) $$\int_{-\infty}^{\infty} f(x)e^{isx}dx = 2\pi i \sum_{UHP} \operatorname{Res}[f(z)e^{isz}]$$

$$\int_C f(z)e^{isz}dz = \int_S f(z)e^{isz}dz + \int_{-R}^{R} f(x)e^{isx}dx$$



$$\int_{-R}^{R} f(x)e^{isx}dx = \int_C f(z)e^{isz}dz - \int_S f(z)e^{isz}dz$$

$$\left|\int_S f(z)e^{isz}dz\right| \le \int_S |f(z)e^{isz}|dz < \int \frac{L}{|z|^2}|e^{isz}|dz$$ By degree assumption

$$< \frac{k}{R^2} \cdot e^{-y} \cdot \pi R \to 0 \quad as \quad R \to \infty$$

Thus,

(9) $$\int_{-\infty}^{\infty} f(x)e^{isx}dx = 2\pi i \sum_{UHP} \operatorname{Res}[f(z)e^{isz}]$$

EXAMPLE 3 An Integral of Fourier Integral

Show that

$$(11)\qquad \int_{-\infty}^{\infty} \frac{\cos sx}{k^2+x^2}\,dx = \frac{\pi}{k}e^{-ks}, \qquad \int_{-\infty}^{\infty} \frac{\sin sx}{k^2+x^2}\,dx = 0 \qquad (s>0,\ k>0)$$

Sol.

$$(10)\qquad \begin{cases} \displaystyle\int_{-\infty}^{\infty} f(x)\cos sx\,dx = -2\pi \sum_{UHP} \operatorname{Im} \operatorname{Res}\,[f(z)e^{isz}] \\ \displaystyle\int_{-\infty}^{\infty} f(x)\sin sx\,dx = 2\pi \sum_{UHP} \operatorname{Re} \operatorname{Res}\,[f(z)e^{isz}] \end{cases} \qquad (s>0)$$

$f(x)=1/(k^2+x^2)$

$f(z)$ *has poles at* $z=\pm ik$.

$$\operatorname*{Res}_{z=ik}[f(z)e^{isz}]=\operatorname*{Res}_{z=ik}\frac{e^{isz}}{k^2+z^2}=\left[\frac{e^{isz}}{2z}\right]_{z=ik}=\frac{e^{-ks}}{i2k}$$

$$(a)\ \int_{-\infty}^{\infty}\frac{\cos sx}{k^2+x^2}dx=-2\pi\operatorname{Im}\operatorname*{Res}_{z=ik}[f(z)e^{isz}]$$

$$=-2\pi\cdot\left[-\frac{e^{-ks}}{2k}\right]=\frac{\pi}{k}e^{-ks}$$

$$(b)\ \int_{-\infty}^{\infty}\frac{\sin sx}{k^2+x^2}dx=2\pi\operatorname{Re}\operatorname*{Res}_{z=ik}[f(z)e^{isz}]=-2\pi\cdot 0=0$$

Another Kind of Improper Integral

Consider:

$$(11) \quad \int_A^B f(x)\,dx \qquad \textit{where} \ \lim_{x\to a}|f(x)| = \infty$$

By definition, (11) means

$$(12) \quad \int_A^B f(x)dx = \lim_{\epsilon\to 0^+}\int_A^{a-\epsilon} f(x)dx + \lim_{\eta\to 0^+}\int_{a+\eta}^B f(x)dx$$

Neither of two limits may not exist if ε and η go to 0 independently, but the limit exists.

$$(13) \quad \lim_{\eta\to 0^+}\left[\int_A^{a-\eta} f(x)dx + \int_{a+\eta}^B f(x)dx\right] = \mathrm{pr.v.}\int_A^B f(x)dx$$

Cauchy Principal Value of the integral:

$$(13)\quad \lim_{\eta\to 0^+}\left[\int_A^{a-\eta} f(x)dx+\int_{a+\eta}^{B} f(x)dx\right]=\text{pr.v.}\int_A^B f(x)dx$$

Example: $f(x)=1/x^3$

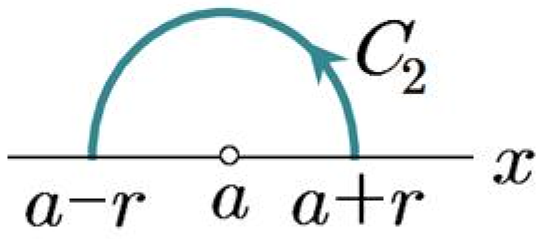
Not exist: $\lim_{\eta\to 0^+}\int_{-1}^{-\epsilon}\frac{dx}{x^3}$ and $\lim_{\eta\to 0^+}\int_{\epsilon}^{1}\frac{dx}{x^3}$

Exist: $\text{pr.v.}\int_{-1}^{1}\frac{dx}{x^2}=\lim_{\eta\to 0^+}\left[\int_{-1}^{-\epsilon}\frac{dx}{x^3}+\int_{\epsilon}^{1}\frac{dx}{x^3}\right]=0$

THEOREM 1 Simple Poles on the Real Axis

If $f(z)$ has a simple pole at $z = a$ on the real axis, then

$$\lim_{r\to 0}\int_{C_2} f(z)\,dz = \pi i \operatorname*{Res}_{z=a} f(z)$$



PROOF

$$f(z) = \frac{b_1}{z-a} + g(z) \qquad b_1 = \operatorname*{Res}_{z=a} f(z)$$

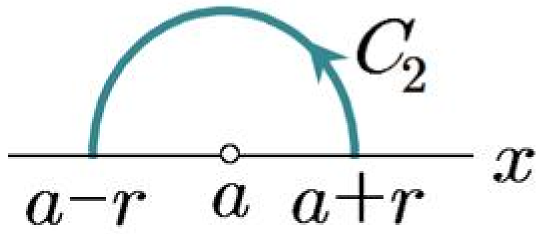
where $g(z)$ is analytic on the semicircle of integration

$$C_2: \quad z = a + re^{i\theta}, \quad 0 \le \theta \le \pi$$

and for all z between C_2 and the x-axis,
and thus bounded on C_2, or

$$|g(z)| \le M$$

$$\int_{C_2} f(z)\,dz = \int_0^{\pi} \frac{b_1}{re^{i\theta}}\, ire^{i\theta}\,d\theta + \int_{C_2} g(z)\,dz$$
$$= b_1\pi i + \int_{C_2} g(z)\,dz$$



The second integration:

$$\left|\int_{C_2} g(z)\,dz\right| \le \int_{C_2} |g(z)|\,dz \le M\pi r \to 0 \;\; as \;\; r \to 0$$

Thus,

$$\int_{C_2} f(z)\,dz = b_1\pi i = \pi i \operatorname*{Res}_{z=a} f(z)$$

Principal Value of an Integral from -Infinity to +Infinity

$$\text{(14)} \qquad \text{pr.v.} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{UHP} \operatorname{Res} f(z) + \pi i \sum_{real\ Axis} \operatorname{Res} f(z)$$

$$\text{pr.v.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

Assumption:

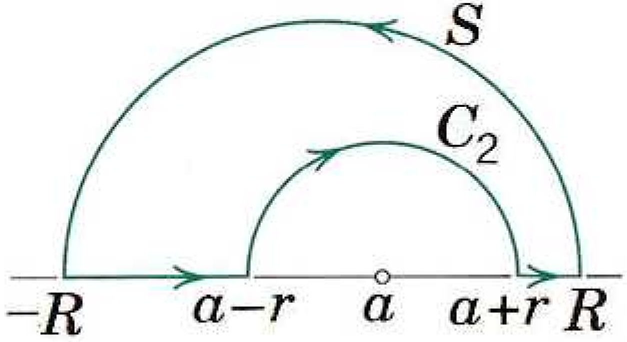
$f(x) = N(x)/D(x)$:

- $f(x)$ is a real rational function

  $\Longrightarrow$ $f(z)$ has finitely many poles in the UHP.

- Order of D(x) ≥ Order of N(x)+2

Integration from $-\infty$ to ∞



$$\lim_{R\to\infty}\lim_{r\to 0}\left[\int_{-R}^{a-r}+\int_{C_2}+\int_{a+r}^{R}+\int_{S}\right]=2\pi i \underset{UHP}{\operatorname{Res}} f(z)$$

$$\lim_{R\to\infty}\lim_{r\to 0}\left[\int_{-R}^{a-r}f(z)\,dz+\int_{a+r}^{R}f(z)\,dz\right]=\text{pr.v.}\int_{-\infty}^{\infty}f(z)dz$$

$$\lim_{r\to 0}\int_{C_2}f(z)dz=-\pi i \underset{z=a}{\operatorname{Res}} f(z)$$

$$(\because \textit{Theorem } 1 \textit{ and } CW \textit{ direction})$$

$$\left|\int_S f(z)dz\right| \le \int_S |f(z)|dz < \int_S \frac{k}{|Z|^2}dz = \frac{k}{R^2}\cdot\pi R$$

$$= \frac{\pi k}{R} \to 0 \quad as \quad R \to \infty$$

$$\therefore \lim_{R\to\infty}\int_S f(z)dz = 0$$

$$\lim_{R\to\infty}\lim_{r\to 0}\left[\int_{-R}^{a-r} + \int_{C_2} + \int_{a+r}^{R} + \int_S\right] = 2\pi i \operatorname*{Res}_{UHP} f(z)$$

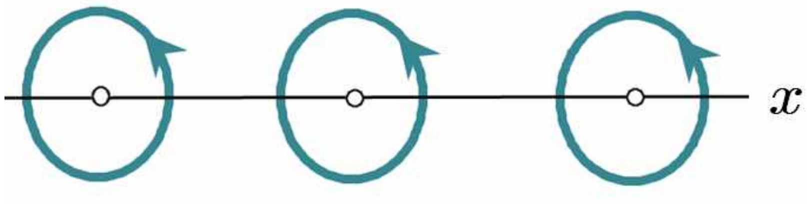
0

pr.v.

$$-\pi i \operatorname*{Res}_{z=a} f(z)$$

$$\text{pr.v.}\int_{-\infty}^{\infty} f(z)\,dz - \pi i \operatorname*{Res}_{z=a} f(z) = 2\pi i \sum_{UHP} \operatorname{Res} f(z)$$

$$\text{pr.v.}\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \sum_{UHP} \operatorname{Res} f(z) + \pi i \operatorname*{Res}_{z=a} f(z)$$



$$\text{(14)} \quad \text{pr.v.}\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \sum_{UHP} \operatorname{Res} f(z) + \pi i \sum_{real\ Axis} \operatorname{Res} f(z)$$

EXAMPLE 4 Poles on the Real Axis

Find the principal value $\text{pr.v.}\int_{-\infty}^{\infty}\frac{dx}{(x^2-3x+2)(x^2+1)}$

Sol.

$$\text{pr.v.}\int_{-\infty}^{\infty}\frac{dx}{(x^2-3x+2)(x^2+1)}$$

$$=2\pi i\sum_{UHP}\text{Res}\, f(z)+\pi i\sum_{real\ Axis}\text{Res}\, f(z)$$

$$(z^2-3z+2)(z^2+1)=(z-1)(z-2)(z+i)(z-i)$$

UHP: $z=i$

Real axis: $z=1,\ 2$

$$z=1,\ \operatorname*{Res}_{z=1} f(z) = \left[(z-1)\frac{1}{(z^2-3z+2)(z^2+1)}\right]_{z=1}$$
$$= \left[\frac{1}{(z-2)(z^2+1)}\right]_{z=1} = -\frac{1}{2}$$

$$z=2,\ \operatorname*{Res}_{z=2} f(z) = \left[(z-2)\frac{1}{(z^2-3z+2)(z^2+1)}\right]_{z=2}$$
$$= \left[\frac{1}{(z-1)(z^2+1)}\right]_{z=2} = \frac{1}{5}$$

$$z = i,\ \operatorname*{Res}_{z=i} f(z) = \left[(z-i)\frac{1}{(z^2-3z+2)(z^2+1)}\right]_{z=i}$$

$$= \left[\frac{1}{(z^2-3z+2)(z+i)}\right]_{z=i} = \frac{1}{(1-3i)(2i)}$$

$$= \frac{1}{6+2i} = \frac{6-2i}{40} = \frac{3-i}{20}$$

$$\text{pr.v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)}$$

$$= 2\pi i \sum_{UHP} \operatorname{Res} f(z) + \pi i \sum_{real\ Axis} \operatorname{Res} f(z)$$

$$= 2\pi i\left(\frac{3-i}{20}\right) + \pi i\left(-\frac{1}{2}+\frac{1}{5}\right) = \frac{\pi}{10}$$

SUMMARY OF CHAPTER 16

A Laurent series:

$$(1)\quad f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

$$(1^*)\quad f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$$

$$a_n=\frac{1}{2\pi i}\oint_C\frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*$$

Residue:

$$(2)\quad b_1=\operatorname*{Res}_{z\to z_0}f(z)=\frac{1}{2\pi i}\oint_C f(z^*)dz^*$$

$$\therefore \oint_C f(z^*)dz^*=2\pi i\operatorname{Res}_{z\to z_0}f(z)$$

Residue at a pole of order m:

$$(3) \quad \operatorname*{Res}_{z \to z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left(\frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] \right)$$

Residue at a simple pole:

$$\operatorname*{Res}_{z = z_0} f(z) = \lim_{z \to z_0} [(z-z_0) f(z)]$$

$$\operatorname*{Res}_{z = z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

PROBLEM SET 16.4

1. Evaluate the following integrals and show the details of your work.

$$\int_0^{\pi} \frac{2\,d\theta}{k - \cos\theta}$$

$$\int_0^{\pi} \frac{2d\theta}{k-\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{2d\theta}{k-\cos\theta} \qquad e^{i\theta} = z \quad d\theta = dz/(iz)$$

$$\cos\theta = (z+1/z)/2$$

$$= \oint_{C:|z|=1} \frac{dz/iz}{k-(z+1/z)/2} = \oint_{C:|z|=1} \frac{2idz}{z^2-2kz+1}$$

$$z^2-2kz+1=0 \quad z = k \pm \sqrt{k^2-1}$$

$$\int_0^{\pi} \frac{2d\theta}{k-\cos\theta} = 2\pi i \operatorname{Res}_{z=k-\sqrt{k^2-1}} \frac{2i}{z^2-2kz+1}$$

$$= 2\pi i \left.\frac{2i}{2z-2k}\right|_{z=k-\sqrt{k^2-1}} = \frac{2\pi}{\sqrt{k^2-1}}$$

PROBLEM SET 16.4

3. Evaluate the following integrals and show the details of your work.

$$\int_0^{2\pi} \frac{1+\sin\theta}{3+\cos\theta}\,d\theta \qquad e^{i\theta}=z \quad d\theta = dz/(iz)$$

$$\cos\theta=(z+1/z)/2 \quad \sin\theta=(z+1/z)/(2i)$$

$$=\oint_{C:|z|=1} \frac{1+(z-1/z)/(2i)}{3+(z+1/z)/2}\frac{dz}{iz} = \oint_{C:|z|=1} \frac{-z^2-2iz+1}{z(z^2+6z+1)}dz$$

$$z(z^2+6z+1)=0 \quad z=0,\ z=-3\pm 2\sqrt{2}$$

$$\int_0^{2\pi} \frac{1+\sin\theta}{3+\cos\theta}d\theta = \oint_{C:|z|=1} \frac{-z^2-2iz+1}{z(z^2+6z+1)}dz$$

$$=2\pi i\left[\operatorname{Res}_{z=0}\frac{-z^2-2iz+1}{z(z^2+6z+1)} + \operatorname{Res}_{z=-3+2\sqrt{2}}\frac{-z^2-2iz+1}{z(z^2+6z+1)}\right]$$

$$=2\pi i\left[\left.\frac{-z^2-2iz+1}{3z^2+12z+1}\right|_{z=1} + \left.\frac{-z^2-2iz+1}{3z^2+12z+1}\right|_{z=-3+2\sqrt{2}}\right] = \frac{\pi}{\sqrt{2}}$$

PROBLEM SET 16.4

9. Evaluate the following integrals and show the details of your work.

$$\int_0^{2\pi} \frac{\cos\theta}{13 - 12\cos 2\theta}\, d\theta \qquad e^{i\theta} = z \qquad d\theta = dz/(iz)$$

$$\cos\theta = (z + 1/z)/2 \qquad \sin\theta = (z - 1/z)/(2i)$$

$$= \int_0^{2\pi} \frac{\cos\theta}{13 - 12[\cos^2\theta - \sin^2\theta]}\, d\theta = \oint_{C:|z|=1} \frac{(z+1/z)/2}{13 - 3[(z+1/z)^2 + (z-1/z)^2]} \frac{dz}{iz}$$

$$= \frac{i}{2} \oint_{C:|z|=1} \frac{z^2+1}{6z^4 - 13z^2 + 6}\, dz \qquad 6z^4 - 13z^2 + 6 = 0 \quad z = \pm\sqrt{\frac{3}{2}}, \pm\sqrt{\frac{2}{3}}$$

$$\int_0^{2\pi} \frac{\cos\theta}{13 - 12\cos 2\theta}\, d\theta = \frac{i}{2} \oint_{C:|z|=1} \frac{z^2+1}{6z^4 - 13z^2 + 6}\, dz$$

$$= \frac{i}{2} 2\pi i \left[\operatorname*{Res}_{z=\sqrt{\frac{2}{3}}} + \operatorname*{Res}_{z=-\sqrt{\frac{2}{3}}} \right]$$

$$= \pi \left[\left. \frac{z^2+1}{24z^3 - 26z} \right|_{z=\sqrt{\frac{2}{3}}} + \left. \frac{z^2+1}{24z^3 - 26z} \right|_{z=-\sqrt{\frac{2}{3}}} \right] = 0$$

11. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$

$$(7) \quad \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{UHP} \operatorname{Res} f(z)$$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx = 2\pi i \sum_{UHP} \operatorname{Res} f(z) = 2\pi i \operatorname*{Res}_{z=i} f(z)$$

$$= 2\pi i \left[\frac{d}{dz}\left[(z-i)^2 f(z)\right]\right]_{z=i} = 2\pi i \left[\frac{d}{dz}\frac{1}{(z+i)^2}\right]_{z=i}$$

$$= 2\pi i \left[\frac{-2}{(z+i)^3}\right]_{z=i} = 2\pi i\left(-\frac{i}{4}\right) = -\frac{\pi i^2}{2} = \frac{\pi}{2}$$

13. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+4)}\,dx$.

$(x^2+1)(x^2+4) = (x+i)(x-i)(x+2i)(x-2i) = 0,\ x = \pm i,\ \pm 2i$

$$(7) \qquad \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{UHP} \operatorname{Res} f(z)$$

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+4)}\,dx = 2\pi i \sum_{UHP} \operatorname{Res} f(z)$$

$$= 2\pi i \left[\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=2i} f(z) \right] \qquad \frac{p(z)}{q'(z)} = \frac{z}{4z^3+10z}$$

$$= 2\pi i \left[\frac{p(i)}{q'(i)} + \frac{p(2i)}{q'(2i)} \right] = 2\pi i \left[\frac{i}{6i} + \frac{2i}{-12i} \right] = 0$$

23. Find the Cauchy principal value $\text{pr.v.} \int_{-\infty}^{\infty} \frac{dx}{x^4 - 1}$.

$$x^4 - 1 = (x-1)(x+1)(x-i)(x+i) = 0, \quad x = \pm 1, \pm i$$

$$\text{pr.v.} \int_{-\infty}^{\infty} \frac{dx}{x^4 - 1} = 2\pi i \sum_{UHP} \operatorname{Res} f(z) + \pi i \sum_{real\ Axis} \operatorname{Res} f(z)$$

$$= 2\pi i \operatorname*{Res}_{z=i} f(z) + \pi i \left[\operatorname*{Res}_{z=-1} f(z) + \operatorname*{Res}_{z=1} f(z) \right]$$

$$= 2\pi i \frac{p(i)}{q'(i)} + \pi i \left[\frac{p(1)}{q'(1)} + \frac{p(-1)}{q'(-1)} \right] \qquad \frac{p(z)}{q'(z)} = \frac{1}{4z^3}$$

$$= 2\pi i \frac{1}{-4i} + \pi i \left[\frac{1}{4} + \frac{1}{-4} \right] = -\frac{\pi}{2}$$

24. Find the Cauchy principal value (showing details): $\int_{-\infty}^{\infty} \frac{dx}{x^4 + 3x^2 - 4}$

$$(14) \quad \text{pr.v.} \int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \sum_{UHP} \text{Res} f(z) + \pi i \sum_{real\ Axis} \text{Res} f(z)$$

$$x^4 + 3x^2 - 4 = (x^2+4)(x^2-1) = (x+2i)(x-2i)(x-1)(x+1) = 0,\ x = \pm 2i,\ \pm 1$$

$$\text{pr.v.} \int_{-\infty}^{\infty} \frac{dx}{x^4 + 3x^2 - 4} = 2\pi i \sum_{UHP} \text{Res} f(z) + \pi i \sum_{real\ Axis} \text{Res} f(z)$$

$$= 2\pi i \operatorname*{Res}_{z=2i} f(z) + \pi i \left[\operatorname*{Res}_{z=-1} f(z) + \operatorname*{Res}_{z=1} f(z) \right]$$

$$= 2\pi i \frac{p(2i)}{q'(2i)} + \pi i \left[\frac{p(-1)}{q'(-1)} + \frac{p(1)}{q'(1)} \right] \qquad \frac{p(z)}{q'(z)} = \frac{1}{4z^3 + 6z}$$

$$= 2\pi i \frac{1}{-20i} + \pi i \left[\frac{1}{-10} + \frac{1}{10} \right] = -\frac{\pi}{10}$$

PROBLEM SET 16.4

25. Find the Cauchy principal value (showing details): $\displaystyle\int_{-\infty}^{\infty} \frac{x+5}{x^3-x}\,dx$

$$(14) \quad \text{pr.v.} \int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \sum_{UHP} \operatorname{Res} f(z) + \pi i \sum_{real\ Axis} \operatorname{Res} f(z)$$

$$x^3 - x = x\left(x^2 - 1\right) = x(x-1)(x+1)$$

$$\text{pr.v.} \int_{-\infty}^{\infty} \frac{x+5}{x^3-x}\,dx = 2\pi i \sum_{UHP} \operatorname{Res} f(z) + \pi i \sum_{real\ Axis} \operatorname{Res} f(z)$$

$$= \pi i \left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=-1} f(z) \right]$$

$$= \pi i \left[\frac{p(0)}{q'(0)} + \frac{p(1)}{q'(1)} + \frac{p(-1)}{q'(-1)} \right] \qquad \frac{p(z)}{q'(z)} = \frac{z+5}{3z^2-1}$$

$$= \pi i \; (-5 + 3 + 2) = 0$$