Introduction to Discrete Math

Felipe P. Vista IV

Reminder

- Everybody, make sure that your name in ZOOM is in the following format:
  - Ex: 202054321 Juan Dela Cruz

Not changing your name to this format

* you **will** be marked Absent * → absent?

Course Outline

- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion, Logic, Invariants
- Probability & Combinatronics
  - Counting, Probability, Random Variables
- Graph Theory
  - Graphs (cycles, classes, parameters)
- Number Theory & Cryptography
  - Arithmetic in modular form
  - Intro to Cryptography

Mathematical Thinking – Binomial Coefficients

PASCAL'S TRIANGLE

- Pascal’s Triangle
- Symmetries
- Row Sums
- Binomial Theorem

Combinations

Question

There are n students. What is the number of ways of forming a team with k students from the group?

Answer

$$\binom{n}{k}$$

"n choose k"

Forming a Team

- Fix one of the students, let's call her Janin.
- There will be two team types
  1. Team(s) with Janin $\binom{n-1}{k-1}$
  2. Team(s) without Janin $\binom{n-1}{k}$

Answer $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Jan

$$\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$$

Pascal's Triangle

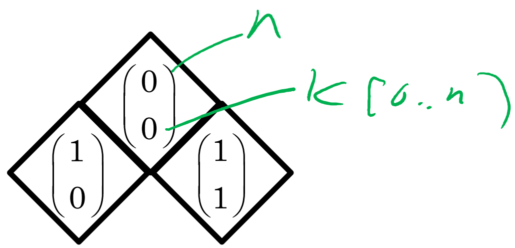
$n = 0$

$$\binom{0}{0}$$

Pascal's Triangle

$n = 0$

$n = 1$



Pascal's Triangle

$n = 0$

$n = 1$

$n = 2$

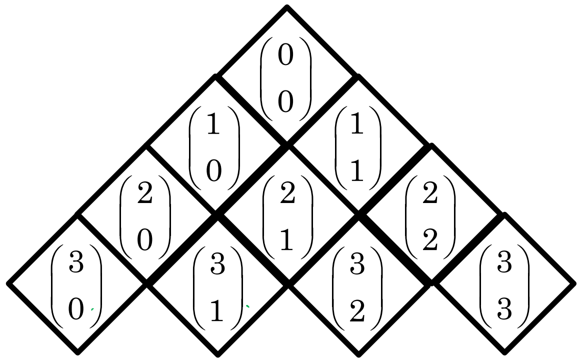
$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

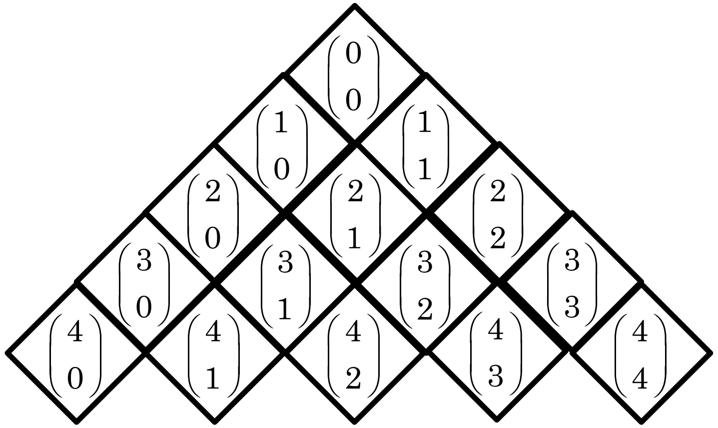
Pascal's Triangle

$$n = 0$$
$$n = 1$$
$$n = 2$$
$$n = 3$$



Pascal's Triangle

$n = 0$
$n = 1$
$n = 2$
$n = 3$
$n = 4$



Pascal’s Triangle

$n = 0$ $$\binom{0}{0}$$

$n = 1$ $$\binom{1}{0} \quad \binom{1}{1}$$

$n = 2$ $$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$n = 3$ $$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$n = 4$ $$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$n = 5$ $$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Pascal's Triangle

$n = 0$

$n = 1$

$n = 2$

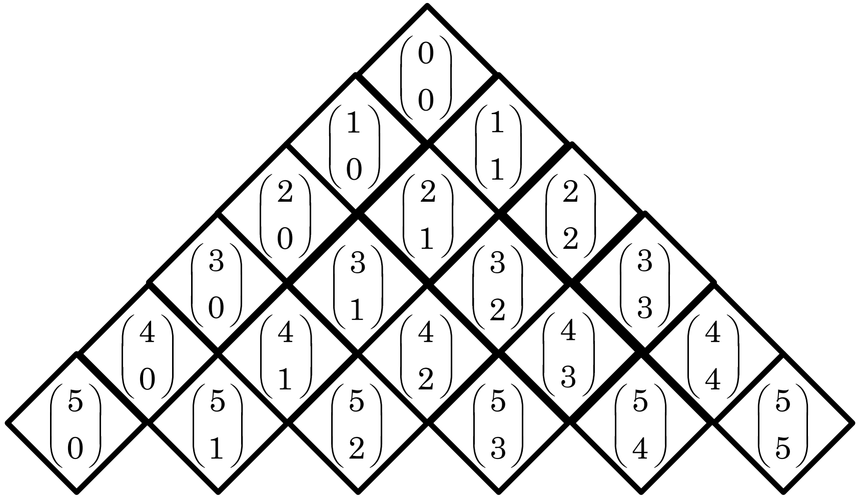
$n = 3$

$n = 4$

$n = 5$

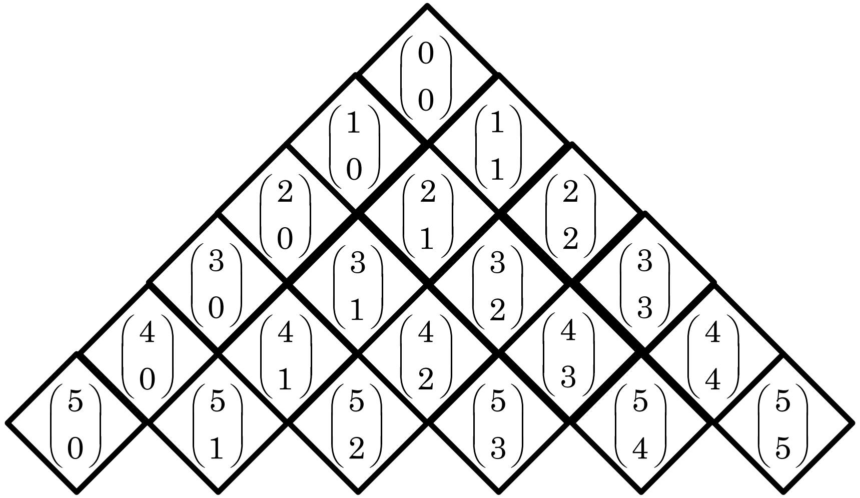
w/5 w/o 5

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



Pascal's Triangle

$n = 0$
$n = 1$
$n = 2$
$n = 3$
$n = 4$
$n = 5$



$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad \binom{4}{3} = \binom{4-1}{3-1} + \binom{4-1}{3} = \binom{3}{2} + \binom{3}{3}$$

– if $n=4$, $k=3$

Total number n *choose* k teams possible

Possible n *choose* k teams with Janin

Possible n *choose* k teams without Janin

Pascal's Triangle

$n = 0$ $\quad \binom{0}{0}$

$n = 1$ $\quad \binom{1}{0} \; \binom{1}{1}$

$n = 2$ $\quad \binom{2}{0} \; \binom{2}{1} \; \binom{2}{2}$

$n = 3$ $\quad \binom{3}{0} \; \binom{3}{1} \; \binom{3}{2} \; \binom{3}{3}$

$n = 4$ $\quad \binom{4}{0} \; \binom{4}{1} \; \binom{4}{2} \; \binom{4}{3} \; \binom{4}{4}$

$n = 5$ $\quad \binom{5}{0} \; \binom{5}{1} \; \binom{5}{2} \; \binom{5}{3} \; \binom{5}{4} \; \binom{5}{5}$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad \binom{4}{3} = \binom{4-1}{3-1} + \binom{4-1}{3} = \binom{3}{2} + \binom{3}{3}$$

Total number n ***choose*** k teams possible

Possible n *choose* k teams with Janin

Possible n *choose* k teams without Janin

Pascal's Triangle

Replacing the numbers with actual values…

$n = 0$	1
$n = 1$	1 1
$n = 2$	1 2 1
$n = 3$	1 3 3 1
$n = 4$	1 4 6 4 1
$n = 5$	1 5 10 10 5 1

$n=4$
$k=3$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$4 = 3 + 1$

Python Code

```
C = dict() # C([n,k]) is  equal to n choose k

for n in range(8):
    C[n,0] = 1
    C[n,n] = 1
    for k in range(1,n):
        C[n,k] = C[n-1,k-1] + C[n-1,k]

print (C[7,4])
```

```
// OUTPUT
```

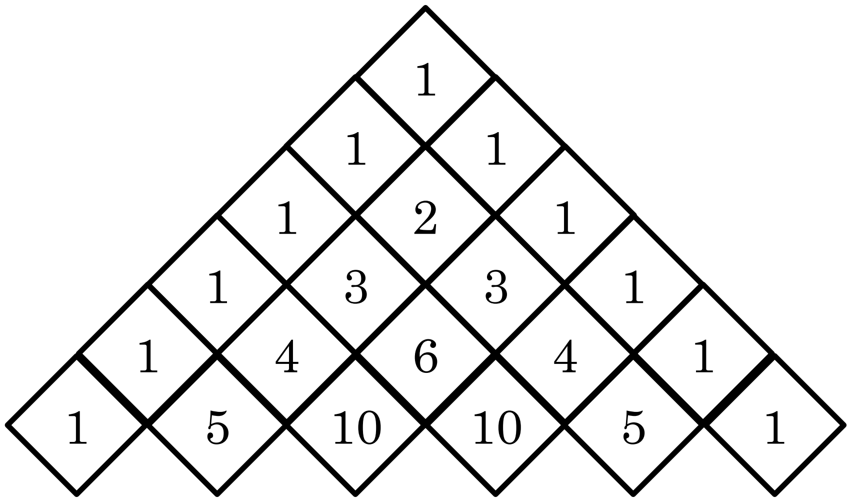
Python Code

```
C = dict() # C([n,k]) is  equal to n choose k

for n in range(8):
    C[n,0] = 1
    C[n,n] = 1
    for k in range(1,n):
        C[n,k] = C[n-1,k-1] + C[n-1,k]

print (C[7,4])
```

```
// OUTPUT
38
```

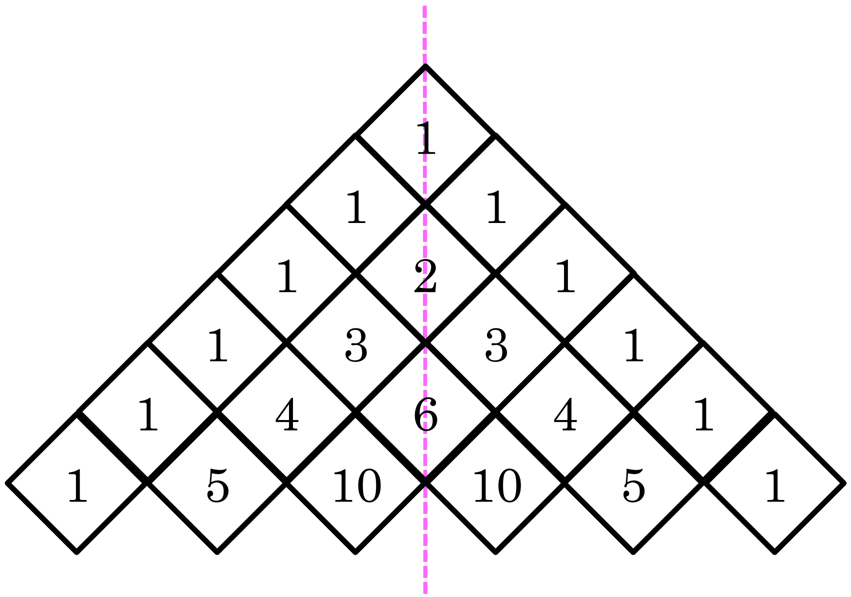
- Pascal's Triangle
- Symmetries
- Row Sums
- Binomial Theorem

Pascal’s Triangle is Symmetric



$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle is Symmetric



$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's Triangle

Theorem

$$\binom{n}{k} = \binom{n}{n-k}$$

Pascal's Triangle

Theorem

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof

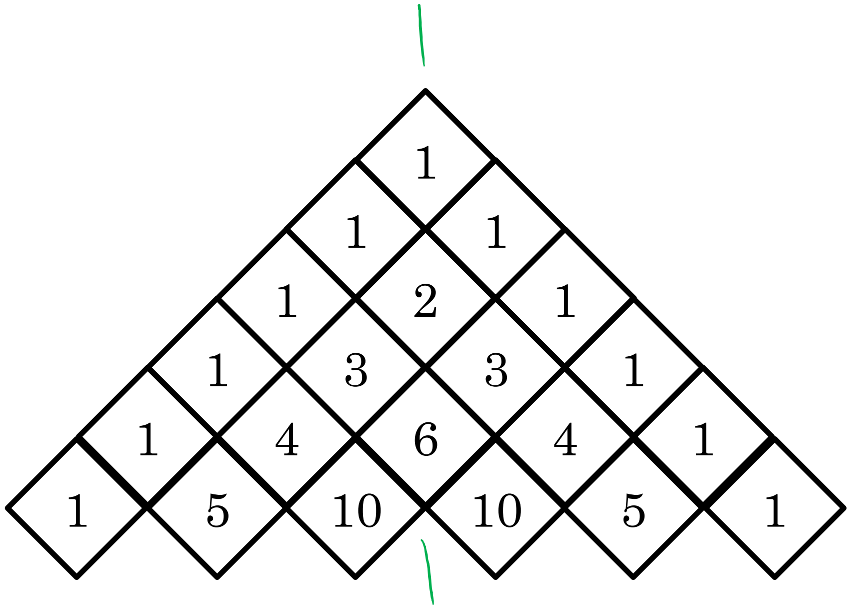
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$$

Combinatorial Proof

- $\binom{n}{k}$ is the number of ways we can select a team of size k from n number of students
- $\binom{n}{n-k}$ is the number of ways we can select a team with size n-k from n number of students
- this is just the number of ways we can divide n students into two teams with sizes k and n-k.

- Pascal's Triangle
- Symmetries
- Row Sums
- Binomial Theorem

Row Sums



Row Sums

$$
\begin{array}{rcl}
1 & = & ? \\
1 + 1 & = & ? \\
1 + 2 + 1 & = & ? \\
1 + 3 + 3 + 1 & = & ? \\
1 + 4 + 6 + 4 + 1 & = & ? \\
1 + 5 + 10 + 10 + 5 + 1 & = & ?
\end{array}
$$

Row Sums

$$
\begin{array}{ccc}
1 & = 1 & 2^0 \\
1 + 1 & = 2 & 2^1 \\
1 + 2 + 1 & = 4 & 2^2 \\
1 + 3 + 3 + 1 & = 8 & 2^3 \\
1 + 4 + 6 + 4 + 1 & = 16 & 2^4 \\
1 + 5 + 10 + 10 + 5 + 1 & = 32 & 2^5
\end{array}
$$

Row Sums

Theorem

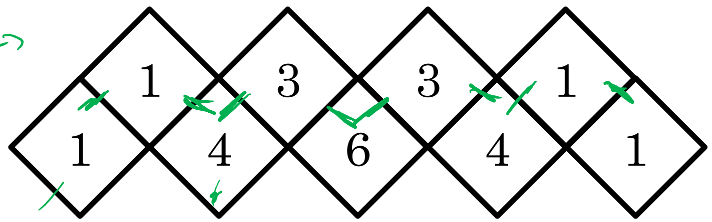
The sum of all the numbers in the *n-th* row of Pascal's triangle is equal to 2^n:

$$\binom{n}{0} = \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Proof by Induction

- The base case (*0-th* row) holds
- We will show that the sum of each row is twice the sum of the previous row:

$k = 0$
$2^0 = 1$



$$
\begin{array}{lccccccc}
 & 1 & 4 & 6 & 4 & 1 & & \\
 & 1 & (1 + 3) & (3 + 3) & (3 + 1) & 1 & & \\
\text{Rearranging:} & (1 + 1) & (3 + 3) & (3 + 3) & (1 + 1) & & & = 16 \\
\text{Multiply by 2:} & 1 & 3 & 3 & 1 & & = 8 \times 2 & = 16
\end{array}
$$

- **Proved!**

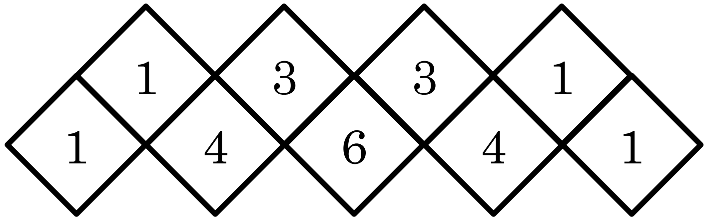
Combinatorial Proof

- $\binom{n}{k}$ is the number k subsets from set of size n
- the sum of $\binom{n}{k}$ for all k (*from 0 to n*) is the number of all subsets of an n element set
- this is 2^n by the product rule:
  - each of the n elements is either included or not

$$\binom{n}{k} = \underset{w/}{\binom{n-1}{k-1}} + \underset{w/o}{\binom{n-1}{k}}$$

Combinatorial Proof

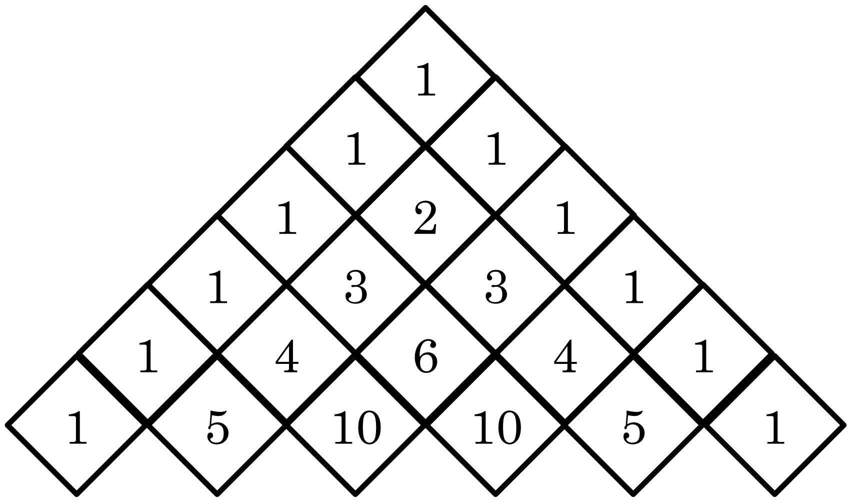
- The base case (*0-th* row) holds
- We will show that the sum of each row is twice the sum of the previous row:



	1	4	6	4	1	
	1	$(1+3)$	$(3+3)$	$(3+1)$	1	
Rearranging:	$(1+1)$	$(3+3)$	$(3+3)$	$(1+1)$		$= 16$
Multiply by 2:	1	3	3	1		$= 8 \times 2 = 16$

- **Proved!**

Alternating Row Sums



Alternating Row Sums

$$
\begin{array}{c}
1 \\
1 - 1 \\
1 - 2 + 1 \\
1 - 3 + 3 - 1 \\
1 - 4 + 6 - 4 + 1 \\
1 - 5 + 10 - 10 + 5 - 1
\end{array}
$$

$a^2 + 2ab + b^2$
if $a = 1$, $b = -1$
$(1)^2 + 2(1)(-1) + (-1)^2$
$1 - 2 + 1$

Alternating Row Sums

$$
\begin{array}{ccccccccccccl}
 & & & & & 1 & & & & & & & = 1 \\
 & & & & 1 & - & 1 & & & & & & = 0 \\
 & & & 1 & - & 2 & + & 1 & & & & & = 0 \\
 & & 1 & - & 3 & + & 3 & - & 1 & & & & = 0 \\
 & 1 & - & 4 & + & 6 & - & 4 & + & 1 & & & = 0 \\
1 & - & 5 & + & 10 & - & 10 & + & 5 & - & 1 & & = 0
\end{array}
$$

Alternating Row Sums

Theorem

$$for\ \ n > 0, \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

- for odd n, follows the symmetry property
- in general, can be shown by using the sum pattern of the triangle
  - each internal element is equal to the sum of the two elements above it

Combinatorial Proof

- we need to show that:

$$\binom{n}{1}+\binom{n}{3}+\cdots=\binom{n}{0}+\binom{n}{2}+\cdots$$

- Combinatorial meaning:
  - the number of **odd** size subsets is the same as the number of **even** size subsets
  - To prove this, we'll construct a one-to-one correspondence between odd size subsets and even size subsets

One-to-One Correspondence

- Fix any element x (can do this since $n > 0$)
- Partition all subsets into pairs (A, B) where $A = B + x$ (more formally, $A = B \cup \{x\}$)
- One of pair (A, B) has odd size, the other one has even size

Example

S = {a, b, c, d}

Even size subjects	Odd size subjects
null → "0 = even"	{a}
{a, b}	{b}
{a ,c}	{c}
{a, b}	{d}
{a, d}	{a, b, c}
{b, c}	{a, b, d}
{b, d}	{a, c, d}
{a, b, c, d}	{b, c, d}

- Pascal’s Triangle
- Symmetries
- Row Sums
- Binomial Theorem

Binomial Theorem

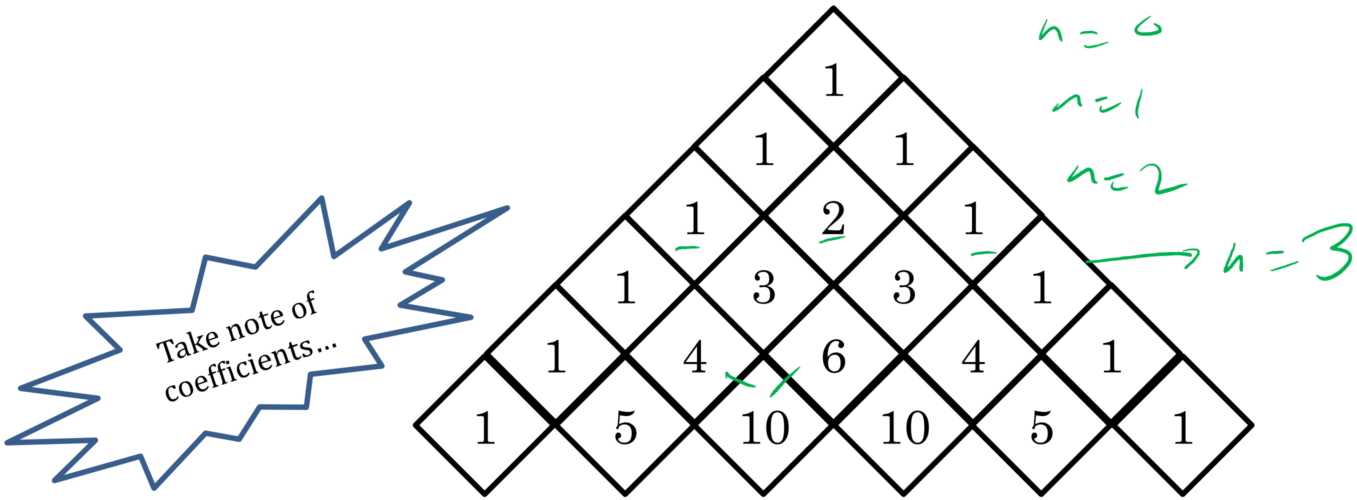
$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Binomial Theorem



$$(a+b)^2 = (1)a^2 + (2)ab + (1)b^2$$

$$(a+b)^3 = (1)a^3 + (3)a^2b + (3)ab^2 + (1)b^3$$

$$(a+b)^4 = (1)a^4 + (4)a^3b + (6)a^2b^2 + (4)ab^3 + (1)b^4$$

Binomial Theorem

coeff

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1} + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + \binom{n}{n}b^n$$

Equivalently,

coeff

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

Can be demonstrated by expanding the expression:

$$(a+b)^3 = (a+b)(a+b)\cdots(a+b)$$

Proof by Induction

$$
\begin{aligned}
(a+b)^4 &= \\
&= (a+b)^3(a+b) \\
&= (a^3+3a^2b+3ab^2+b^3)(a+b) \\
&= a^4+3a^3b+3a^2b^2+ab^3 \\
&\qquad +a^3b\ +3a^2b^2+3ab^3+b^4 \\
&= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \leftarrow n=4
\end{aligned}
$$

Example

Evaluate:

$$(2a-b)^4 = ????$$

$(a+b)^n$ $\quad (3x-4y)^5 = ?$

Set values:

$$set\,(2a)=a,\ (-b)=b;\ hence$$

set $2a = x$; $-b = y$

$$(2a-b)^4 = ((2a)+(-b))^4$$

Recall:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4;\ substituting$$

$$(a+b)^4 = \quad a^4 \quad + 4 \quad a^3 \quad b \quad + 6 \quad a^2 \quad b^2 \quad + 4 \quad a \quad b^3 \quad + \quad b^4$$

$$((2a)+(-b))^4 = (2a)^4 + 4(2a)^3(-b) + 6(2a)^2(-b)^2 + 4(2a)(-b^3) + (-b)^4$$

$$= 16a^4 - 24a^3b - 24a^2b - 8ab^3 + b^4$$

Consequence(“Result”)

- Set $a = b = 1$. Number of subsets is ~~2n~~. 2^n

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Alternate Rule of Sum

- Set $a = 1,\ b = -1$. Number of odd size subsets is the same as number of even size subsets.

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

$\#_{ODD\_S} = \#_{even\_S}$

Consequence("Result")

- Set $a = 1,\ b = 2$.

$$3^n = \binom{n}{0} + \binom{n}{1}2 + \binom{n}{2}2^2 + \cdots + \binom{n}{n}2^2$$

Combinatorial Proof

- $3n$ - number of words of length n over the alphabet $\{x, y, z\}$
- $\binom{n}{0}$ - number of words consisting of the letter x only
- $\binom{n}{1}2$ - number of words with exactly $n - 1$ letters x
- $\binom{n}{1}2^2$ - number of words with exactly $n - 2$ letters x

Thank you.