15.3 Functions Given by Power Series

$$\sum_{n=0}^{\infty} a_n(\hat{z}-z_0)^n$$

$$\Big\downarrow \quad \hat{z}-z_0=z$$

(1) $$\sum_{n=0}^{\infty} a_n z^n$$

(2) $$f(z)=\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots \qquad (|z|<R)$$

$f(z)$ is represented by the power series.

$f(z)$ is developed in the power series

15.3 THEOREM 1 Continuity of the Sum of a Power Series

THEOREM 1 Continuity of the Sum of a Power Series

If a function $f(z)$ can be represented by a power series (2) with radius of convergence R, then it is continuous at z=0.

$$(2) \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots \qquad (|z| < R)$$

PROOF

THEOREM 2 Identity Theorem for Power Series. Uniqueness

Let $a_0 + a_1 z + a_2 z^2 + \cdots$ and $b_0 + b_1 z + b_2 z^2 + \cdots$ be convergent for $|z| < R$ and let them have the same sum for all $|z| < R$. Then the series are identical, that is, $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2, \cdots$

Hence, if a function $f(z)$ can be represented by a power series with any center z_0, then this representation is unique.

PROOF

15.3 Operations on Power Series

Operations on Power Series

Let $f(z)$ and $g(z)$ converge for $|z| < R_1$ and $|z| < R_2$, respectively

Then, termwise operations of two power series are possible for $|z| < \min(R_1,\ R_2)$.

- Termwise addition or subtraction
- Termwise multiplication
- Termwise differentiation and integration

15.3 Operations on Power Series-conti

Cauchy Product

$$f(z)=\sum_{m=0}^{\infty} a_m z^m = a_0 + a_1 z + a_2 z^2 + \cdots$$

$$g(z)=\sum_{m=0}^{\infty} b_m z^m = b_0 + b_1 z + b_2 z^2 + \cdots$$

$$\begin{aligned} f(z)g(z) &= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a b_1 + a_2 b_0) z^2 + \cdots \\ &= \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) z^n \end{aligned}$$

15.3 THEOREM 3 Termwise Differentiation

THEOREM 3 Termwise Differentiation of a Power Series

The derived series (3) of a power series has the same radius of convergence as the original one.

PROOF

$$f(z)=\sum_{m=0}^{\infty} a_m z^m = a_0 + a_1 z + a_2 z^2 + \cdots$$

$$(3)\qquad f'(z)=\sum_{n=1}^{\infty} n a_n z^{n-1} = a_1 + 2a_2 z + 3a_3 z^2 + \cdots$$

$$R=\lim_{n\to\infty}\left|\frac{n a_n}{(n+1)a_{n+1}}\right|$$

$$=\lim_{n\to\infty}\frac{n}{n+1}\cdot\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

15.3 EXAMPLE 1 Application of Theorem 3

EXAMPLE 1 Application of Theorem 3

Find the radius of convergence R of the following series by applying Theorem 3.

$$\sum_{n=2}^{\infty} \binom{n}{2} z^n = z^2 + 3z^3 + 6z^4 + 10z^5 + \cdots$$

Sol.

$$\sum_{n=2}^{\infty} \binom{n}{2} z^n = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} z^n = \frac{z^2}{2} \sum_{n=2}^{\infty} n(n-1) z^{n-2}$$

$$= \frac{z^2}{2} \sum_{n=2}^{\infty} (z^n)''$$

$\Rightarrow$ $R = 1$

15.3 THEOREM 4 Termwise Integration

THEOREM 4 Termwise Integration of Power Series

The power series

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \cdots$$

obtained by integrating the series $a_0 + a_1 z + a_2 z^2 + \cdots$ term by term has the same radius of convergence as the original series.

PROOF

$$R = \lim_{n\to\infty} \left| \frac{a_n/(n+1)}{a_{n+1}/(n+2)} \right|$$

$$= \lim_{n\to\infty} \frac{n+2}{n+1} \cdot \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

15.3 THEOREM 5 Analytic Functions

THEOREM 5 Analytic Functions. Their Derivatives

A power series with a nonzero radius of convergence R represents an analytic function at every point interior to its circle of convergence. The derivatives of this function are obtained by differentiating the original series term by term. All the series thus obtained have the same radius of convergence as the original series. Hence by the first statement, each of them represents an analytic function.

PROOF

Problem Set for Section 15.3

5. Find the radius of convergence in two ways:

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}(z-2i)^n$$

(a) directly by the Cauchy–Hadamard formula in Sec. 15.2

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{n(n-1)/2^n}{(n+1)n/2^{n+1}}\right| = 2\left|\frac{n-1}{n+1}\right| \to 2 = R$$

(b) from a series of simpler terms by using Theorem 3 or Theorem 4.

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}(z-2i)^n = \left[\sum_{n=0}^{\infty}\left(\frac{z-2i}{2}\right)^n\right]''$$

ROC of $\sum_{n=0}^{\infty}\left(\frac{z-2i}{2}\right)^n$: $\left|\frac{z-2i}{2}\right| < 1$, $\quad |z-2i| < 2$

Thus, ROC of $\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}(z-2i)^n = 2$

Problem Set for Section 15.3

6. Find the radius of convergence in two ways:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{z}{2\pi}\right)^{2n+1}$$

(a) directly by the Cauchy–Hadamard formula in Sec. 15.2

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{(-1)^n/[(2n+1)(2\pi)^{2n+1}]}{(-1)^{n+1}/[(2n+3)(2\pi)^{2n+3}]}\right| = \frac{(2n+3)(2\pi)^2}{2n+1} \rightarrow (2\pi)^2$$

(b) from a series of simpler terms by using Theorem 3 or Theorem 4.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{z}{2\pi}\right)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\pi)^{2n+1}}\left(\frac{z^{2n+1}}{2n+1}\right) = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\pi)^{2n+1}} z^{2n}\, dz$$

$$= \frac{1}{2\pi} \int \sum_{n=0}^{\infty} \left(\frac{iz}{2\pi}\right)^{2n}$$

ROC of $\int \sum_{n=0}^{\infty} \left(\frac{iz}{2\pi}\right)^{2n} dz \ : \ \sqrt{2\pi}$

Thus, ROC of $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{z}{2\pi}\right)^{2n+1} \ : \ \sqrt{2\pi}$

Problem Set for Section 15.3

12. Find the radius of convergence in two ways:

$$\sum_{n=1}^{\infty} \frac{2n(2n-1)}{n^n} z^{2n-2}$$

(a) directly by the Cauchy–Hadamard formula in Sec. 15.2

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{2n(2n-1)/n^n}{(2n+2)(2n+1)/(n+1)^{n+1}}\right| = \frac{2n(2n-1)}{(2n+2)(2n+1)}\left(\frac{n+1}{n}\right)^n (n+1) \rightarrow \infty = R$$

(b) from a series of simpler terms by using Theorem 3 or Theorem 4.

$$\sum_{n=1}^{\infty} \frac{2n(2n-1)}{n^n} z^{2n-2} = \left[\sum_{n=1}^{\infty} n^n z^{2n}\right]''$$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1/n^n}{1/(n+1)^{n+1}}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n (n+1) = \infty$$

ROC of $\sum_{n=1}^{\infty} \frac{2n(2n-1)}{n^n} z^{2n-2}$: ∞

Problem Set for Section 15.3

15. Find the radius of convergence in two ways:

$$\sum_{n=2}^{\infty} \frac{4^n n(n-1)}{3^n}(z-i)^n$$

(a) directly by the Cauchy–Hadamard formula in Sec. 15.2

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{4^n n(n-1)/3^n}{4^{n+1}(n+1)n/3^{n+1}}\right| = \frac{3(n-1)}{4(n+1)} \rightarrow \frac{3}{4} = R$$

(b) from a series of simpler terms by using Theorem 3 or Theorem 4.

$$\sum_{n=2}^{\infty} \frac{4^n n(n-1)}{3^n}(z-i)^n = \sum_{n=2}^{\infty} \frac{4^n}{3^n}(z-i)^2\, n(n-1)\,(z-i)^{n-2}$$

$$= \sum_{n=2}^{\infty} \frac{4^n}{3^n}(z-i)^2\,[(z-i)^n]'' = \left[\sum_{n=2}^{\infty} (z-i)^2 \left(\frac{z-i}{4/3}\right)^n\right]''$$

ROC of $\sum_{n=2}^{\infty} (z-i)^2 \left(\frac{z-i}{4/3}\right)^n$: $4/3$

Thus, ROC of $\sum_{n=2}^{\infty} \frac{4^n n(n-1)}{3^n}(z-i)^n$: $4/3$

15.4 Taylor and Maclaurin **Series**

Taylor Series

(1) $$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad where \;\; a_n = \frac{1}{n!} f^{(n)}(z_0)$$

(2) $$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*$$

Remainder after the term $a_n (z - z_0)^n$:

(3) $$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1} (z^* - z)} dz^*$$

: Taylor's formula *with remainder*

A **Maclaurin series** is a Taylor series with center $z_0 = 0$.

15.4 THEOREM 1 Taylor's Theorem

THEOREM 1 Taylor's Theorem

$f(z)$: analytic in D $\quad z = z_0 \epsilon D$

Then there exists precisely one Taylor series:

$$(1) \quad f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n \quad where \;\; a_n = \frac{1}{n!} f^{(n)}(z_0)$$

$$(3) \quad R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)} dz^*$$

$$(5) \quad |a_n| \leq \frac{M}{r^n}$$

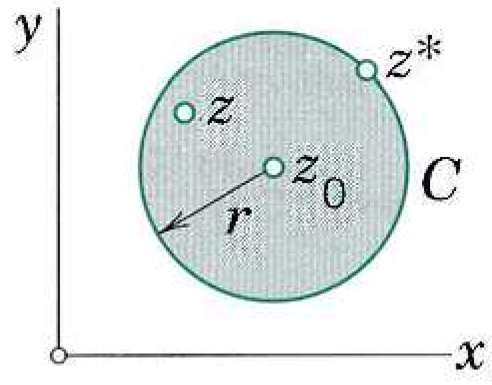
where M: Maximum of $|f(z)|$ on a circle $|z - z_0| = r$

Whose interior is also in D.

15.4 THEOREM 1 Taylor's Theorem-proof

PROOF

Cauchy's Integral Formula(Sec. 14.3):



$$(1) \quad f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

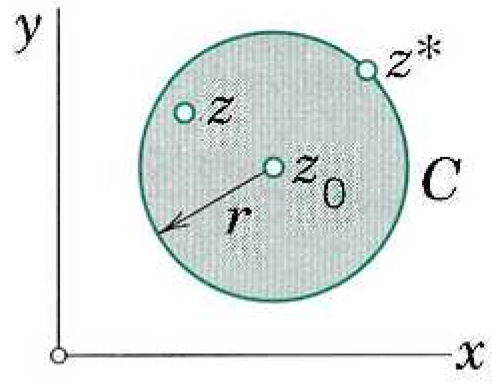
(Sec. 14.3)

$$(6) \quad f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{z^* - z} dz^*$$

$$(7) \quad \frac{1}{z^* - z} = \frac{1}{z^* - z_0 - (z - z_0)} = \frac{1}{(z^* - z_0)\left(1 - \dfrac{z - z_0}{z^* - z_0}\right)}$$

15.4 THEOREM 1 Taylor's Theorem-proof

(7) $$\frac{1}{z^*-z}=\frac{1}{(z^*-z_0)\left(1-\dfrac{z-z_0}{z^*-z_0}\right)}$$



(7*) $$\left|\frac{z-z_0}{z^*-z_0}\right|<1 \quad (\because Fig.\ 367)$$

(8*) $$1+q+\cdots+q^n \equiv \frac{1-q^{n+1}}{1-q}=\frac{1}{1-q}-\frac{q^{n+1}}{1-q}$$

(8) $$\frac{1}{1-q}=1+q+\cdots+q^n+\frac{q^{n+1}}{1-q}$$

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15.4 THEOREM 1 Taylor's Theorem-proof

$$(7)\quad \frac{1}{z^*-z}=\frac{1}{(z^*-z_0)\left(1-\dfrac{z-z_0}{z^*-z_0}\right)}$$

$$=\frac{1}{(z^*-z_0)}\left[1+\frac{z-z_0}{z^*-z_0}+\left(\frac{z-z_0}{z^*-z_0}\right)^2+\cdots+\left(\frac{z-z_0}{z^*-z_0}\right)^n\right]$$

$$+\frac{1}{(z^*-z_0)}\left(\frac{z-z_0}{z^*-z_0}\right)^{n+1}\Bigg/\left[1-\left(\frac{z-z_0}{z^*-z_0}\right)\right]$$

$$=\frac{1}{(z^*-z_0)}\left[1+\frac{z-z_0}{z^*-z_0}+\left(\frac{z-z_0}{z^*-z_0}\right)^2+\cdots+\left(\frac{z-z_0}{z^*-z_0}\right)^n\right]$$

$$+\frac{1}{z^*-z}\left(\frac{z-z_0}{z^*-z_0}\right)^{n+1}$$

15.4 THEOREM 1 Taylor's Theorem-proof

Insertion of (7) into (6):

$$(6)\quad f(z)=\frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^*-z}dz^*$$

$$=\frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^*-z_0}dz^*+\frac{z-z_0}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^2}dz^*+\cdots$$

$$\cdots+\frac{(z-z_0)^n}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*$$

$$+\frac{(z-z_0)^{n+1}}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)}dz^*$$

$$=\frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^*-z_0}dz^*+\frac{z-z_0}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^2}dz^*+\cdots$$

$$\cdots+\frac{(z-z_0)^n}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*+R_n(z)$$

15.4 THEOREM 1 Taylor's Theorem-proof

where

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)} dz^*$$

Question: $R_n(z) \to 0$ as $n \to \infty$?

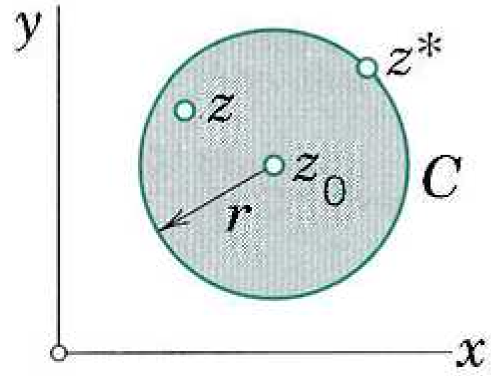
Proof of (9) $\lim_{n\to\infty} R_n(z) = 0$

Since z^* lies on C, whereas z lies inside C, $|z^*-z| > 0$

$f(z)$ is analytic inside and on C, it is bounded, and so is the function $f(z^*)/(z^*-z)$.

Thus, for all z^* on C, $\left|\frac{f(z^*)}{z^*-z}\right| \le \tilde{M}$

15.4 THEOREM 1 Taylor's Theorem-proof



$$|z^* - z_0| = r$$

By ML-inequality,

$$(10) \quad |R_n| = \frac{|z - z_0|^{n+1}}{2\pi} \left| \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)} dz^* \right|$$

$$\leq \frac{|z - z_0|^{n+1}}{2\pi} \tilde{M} \frac{1}{r^{n+1}} 2\pi r = \tilde{M} \left| \frac{z - z_0}{r} \right|^{n+1} \to 0$$

$as \ n \to \infty$

because z lies inside C

15.4 THEOREM 2 Relation to the Previous Section

THEOREM 2 Relation to the Previous Section

A power series with a nonzero radius of convergence is the Taylor series of its sum.

PROOF

15.4 Comparison with Real Functions

Comparison with Real Functions

Complex analytic functions have

- Derivatives of all orders
- Always be represented by power series of the form (1)

$$(1) \qquad f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n$$

Real analytic functions have

- Derivatives of all orders
- But, may not be represented by a power series

Example:

$$f(x) = \begin{cases} \exp(-1/x^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

15.4 Important Special Taylor Series

Important Special Taylor Series

EXAMPLE 1 Geometric Series

Find the Taylor series of the following function with center=0.

$$f(z) = \frac{1}{1-z}.$$

Sol.

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}, \quad f^{(n)}(0) = n!$$

$$(11) \qquad \frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, z^n = 1 + z + z^2 + \cdots \qquad (|z| < 1)$$

$f(z)$ *is singular at* $z = 1.$

15.4 EXAMPLE 2 Exponential Function

EXAMPLE 2 Exponential Function

Find the Taylor series of the following function with center=0.

$$f(z) = e^z$$

Sol.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n$$

$$f^{(n)}(z) = e^z, \qquad f^{(n)}(0) = 1$$

$$(11) \quad e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

15.4 Euler's Formula

$$(11) \quad e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\Big\downarrow \; z = iy$$

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{k=0}^{\infty} \frac{(iy)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(iy)^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} (-1)^K \frac{y^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!}$$

$$= \cos y + i \sin y$$

$$(13) \quad e^{iy} = \cos y + i \sin y$$

EXAMPLE 3 Trigonometric and Hyperbolic Functions

Find the Taylor series of $\cos z$, $\sin z$, $\cosh z$ and $\sinh z$

Sol.

$$\cos z = (1/2)(e^{iz} + e^{-iz})$$

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} + \sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(iz)^n + (-iz)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}$$

$$= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

15.4 EXAMPLE 3 Trigonometric and Hyperbolic Functions-sinz

$$\sin z = [1/(2i)](e^{iz} - e^{-iz})$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2i}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} - \sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}\right]$$

$$= \frac{1}{2i}\sum_{n=0}^{\infty}\frac{(iz)^n - (-iz)^n}{n!} = \frac{1}{2i}\sum_{k=0}^{\infty}\frac{(iz)^{2k+1} - (-iz)^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots$$

15.4 EXAMPLE 3 Trigonometric and Hyperbolic Functions-coshz

$$\cosh z = (1/2)(e^z + e^{-z}) \qquad \sinh z = (1/2)(e^z - e^{-z})$$

$$\cosh z = \frac{1}{2}(e^z + e^{-z}) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{z^n}{n!} + \sum_{n=0}^{\infty}\frac{(-z)^n}{n!}\right]$$

$$= \sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

$$\sinh z = \frac{1}{2}(e^z - e^{-z}) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{z^n}{n!} - \sum_{n=0}^{\infty}\frac{(-z)^n}{n!}\right]$$

$$= \sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

15.4 EXAMPLE 4 Log(1+z)

EXAMPLE 4 Logarithm

Find the Taylor series of $f(z) = \mathrm{Ln}\,(1+z)$ with center=0.

Sol.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n$$

$$f' = (1+z)^{-1} \qquad f'' = -(1+z)^{-2}$$

$$f''' = 2!(1+z)^{-3} \qquad f^{(4)} = -3!(1+z)^{-4}$$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{n}(0) z^n = 0 + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + - + \cdots$$

$$(16) \qquad \mathrm{Ln}\,(1+z) = 0 + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + - + \cdots \qquad (|z| < 1)$$

15.4 EXAMPLE 4 Log(1+z)/(1-z)

Find the Taylor series of $\mathrm{Ln}\,(1+z)/(1-z)$ with center=0.

Sol.

(16) $$\mathrm{Ln}\,(1+z) = 0 + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + - + \cdots \qquad (|z| < 1)$$

$z \to -z$

(17) $$-\mathrm{Ln}\,(1-z) = 0 + z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots \qquad (|z| < 1)$$

(18) $$\mathrm{Ln}\,\frac{1+z}{1-z} = \mathrm{Ln}\,(1+z) - \mathrm{Ln}\,(1-z)$$
$$= 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right) \qquad (|z| < 1)$$

15.4 Practical Methods-Example 5

Practical Methods

EXAMPLE 5 Substitution

Find the Maclaurin series of $f(z) = 1/(1+z^2)$.

Sol.

$$(11) \quad \frac{1}{1-z} = 1 + z + z^2 + \cdots \qquad (|z| < 1)$$

$$z \to -z^2$$

$$(19) \quad \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots \qquad (|z| < 1)$$

15.4 EXAMPLE 6 Integration

EXAMPLE 6 Integration

Find the Maclaurin series of $f(z) = \tan^{-1} z$

Sol.

$$f'(z) = \frac{1}{1+z^2}$$

$$(19) \quad \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots \qquad (|z| < 1)$$

$$\tan^{-1} z = \int \sum_{n=0}^{\infty} (-1)^n z^{2n} \, dz = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1} + C$$

$$= C + z - \frac{z^3}{3} + \frac{z^5}{5} - + \cdots \qquad (|z| < 1)$$

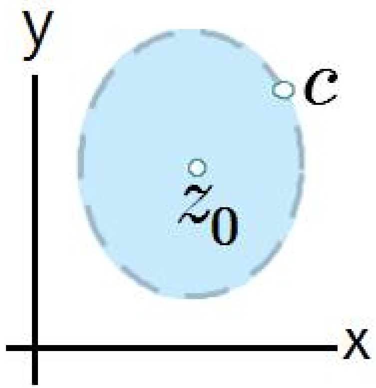
$$\tan^{-1} 0 = C = 0$$

15.4 EXAMPLE 7 Development by Using the Geometric Series

EXAMPLE 7 Development by Using the Geometric Series

Develop $1/(c-z)$ in powers of $z-z_0$ where $z-z_0 \neq 0$

Sol.
$$\frac{1}{c-z}=\frac{1}{c-z_0-(z-z_0)}=\frac{1}{(c-z_0)\left(1-\dfrac{z-z_0}{c-z_0}\right)}$$
$$=\frac{1}{c-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{c-z_0}\right)^n$$
$$=\frac{1}{c-z_0}\left[1+\frac{z-z_0}{c-z_0}+\left(\frac{z-z_0}{c-z_0}\right)^2+\cdots\right]$$



The series converges for

$$\left|\frac{z-z_0}{c-z_0}\right|<1, \text{ that is, } |z-z_0|<|c-z_0|.$$

EXAMPLE 8 Binomial Series, Reduction by Partial Fractions

Find the Taylor series of the following function with center $z_0 = 1$.

$$f(z) = \frac{2z^2 + 9z + 5}{z^3 + z^2 - 8z - 12}$$

Sol.

$$(20) \quad \frac{1}{(1+z)^m} = (1+z)^{-m} = \sum_{n=0}^{\infty} \binom{-m}{n} z^n$$

$$= 1 - mz + \frac{(-m)(-m-1)}{2} z^2 + \frac{(-m)(-m-1)(-m-2)}{3!} z^3 + \cdots$$

$$= 1 - mz + \frac{m(m+1)}{2} z^2 - \frac{m(m+1)(m+2)}{3!} z^3 + \cdots$$

EXAMPLE 8 Binomial Series, Reduction by Partial Fractions-conti

$$
\begin{aligned}
f(z) &= \frac{2z^2+9z+5}{z^3+z^2-8z-12} = \frac{2z^2+9z+5}{(z+2)^2(z-3)} \\
&= \frac{1}{(z+2)^2} + \frac{2}{z-3} \\
&= \frac{1}{[3+(z-1)]^2} - \frac{2}{2-(z-1)} \\
&= \frac{1}{9} \cdot \frac{1}{\left[1+\left(\frac{z-1}{3}\right)\right]^2} - \frac{1}{1-\frac{z-1}{2}} \\
&= \frac{1}{9}\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{z-1}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n
\end{aligned}
$$

EXAMPLE 8 Binomial Series, Reduction by Partial Fractions-conti

$$f(z)=\frac{1}{9}\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{z-1}{3}\right)^n-\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

$$\binom{-2}{n}=\frac{(-2)(-3)\cdots(-2-n+1)}{n!}$$

$$=\frac{(-1)^n 2\cdot 3\cdots(n+1)}{n!}=(-1)^n(n+1)$$

$$=\sum_{n=0}^{\infty}\left[\frac{(-1)^n(n+1)}{3^{n+2}}-\frac{1}{2^n}\right](z-1)^n$$

$$=-\frac{8}{9}-\frac{31}{54}(z-1)-\frac{23}{108}(z-1)^2-\frac{275}{1944}(z-1)^3-\cdots$$

EXAMPLE 8 Binomial Series, Reduction by Partial Fractions-conti

$$f(z) = \frac{1}{9}\sum_{n=0}^{\infty}\binom{-2}{n}\left(\frac{z-1}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

converges for $|z-1| < 2$

converges for $|z-1| < 3$

Hence $f(z)$ is convergent for $|z-1| < 2$.

Problem Set for Section 15.4

3. Find the Maclaurin series and its radius of convergence for $\sin 2z^2$.

$$\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots$$

$$\sin 2z^2 = \sum_{k=0}^{\infty} (-1)^k \frac{(2z^2)^{2k+1}}{(2k+1)!} = (2z^2) - \frac{(2z^2)^3}{3!} + \frac{(2z^2)^5}{5!} - + \cdots$$

$$= (2z^2) - \frac{8z^6}{3!} + \frac{32z^{10}}{5!} - + \cdots$$

$\text{ROC} = \infty$

Problem Set for Section 15.4

4. Find the Maclaurin series and its radius of convergence for $\dfrac{z+2}{1-z^2}$.

$$\frac{z+2}{1-z^2} = (z+2)\left(\frac{1}{1-z^2}\right) = (z+2)\left(1+z^2+z^4+z^6+\cdots\right)$$
$$= 2+z+2z^2+z^3+2z^4+z^5+2z^6+z^7+\cdots$$

$\mathrm{ROC} = 1$

Problem Set for Section 15.4

5. Find the Maclaurin series and its radius of convergence for $\dfrac{1}{2+z^4}$.

$$(11) \quad \frac{1}{1-z} = 1 + z + z^2 + \cdots \qquad (|z| < 1)$$

$$\frac{1}{2+z^4} = \frac{1}{2}\frac{1}{1-(-z^4/2)} = \frac{1}{2}\left(1 - \frac{z^4}{2} + \frac{z^8}{2^2} - \frac{z^{12}}{2^3} + \cdots\right)$$

ROC: $|z^4/2| < 1$, $\quad$ ROC $= \sqrt[4]{2}$

Problem Set for Section 15.4

6. Find the Maclaurin series and its radius of convergence for $\dfrac{1}{1 + 3iz}$.

$$(11)\qquad \frac{1}{1-z} = 1 + z + z^2 + \cdots \qquad (|z| < 1)$$

$$\frac{1}{1+3iz} = 1 + (-3iz) + (-3iz)^2 + (-3iz)^3 + \cdots$$
$$= 1 - 3iz - 9z^2 + 27iz^3 + \cdots$$

ROC: $|3iz| < 1$, ROC = 1/3

Problem Set for Section 15.4

7. Find the Maclaurin series and its radius of convergence for $\cos^2 \frac{1}{2}z$.

$$\cos^2 \tfrac{1}{2}z = \frac{1+\cos z}{2}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

$$= \frac{1}{2}\left[1 + 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right]$$

Problem Set for Section 15.4

8. Find the Maclaurin series and its radius of convergence for $\sin^2 z$.

$$\sin^2 z = \frac{1-\cos 2z}{2}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$$

$$= \frac{1}{2}\left[1 - \left(1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} + \cdots\right)\right]$$

Problem Set for Section 15.4

9. Find the Maclaurin series and its radius of convergence for

$$\int_0^z \exp\left(\frac{-t^2}{2}\right) dt.$$

$$(11) \quad e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\int_0^z \exp \frac{-t^2}{2}\, dt = \int_0^z \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-t^2}{2}\right)^n dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 2^n} \int_0^z t^{2n}\, dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 2^n} \frac{z^{2n+1}}{2n+1} = z - \frac{1}{6} z^3 + \frac{1}{40} z^5 - \frac{1}{336} z^7 + \cdots$$

$$R^2 = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n}{n!\, 2^n} \frac{1}{2n+1} \cdot \frac{(n+1)!\, 2^{n+1}}{(-1)^{n+1}} \frac{2n+3}{1} \right| = \infty$$

Problem Set for Section 15.4

10. Find the Maclaurin series and its radius of convergence

$$\text{for} \quad \exp(z^2)\int_0^z \exp(-t^2)\,dt\,.$$

$$(11) \quad e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\int_0^z \exp(-t^2)dt = \int_0^z \sum_{n=0}^{\infty} \frac{1}{n!} (-t^2)^n \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^z t^{2n} \, dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{z^{2n+1}}{2n+1} \quad = z - \frac{1}{3}z^3 + \frac{1}{10}z^5 - \frac{1}{42}z^7 + \cdots$$

$$\exp(z^2)\int_0^z \exp(-t^2)\,dt = \sum_{n=0}^{\infty} \frac{1}{n!} z^{2n} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{z^{2n+1}}{2n+1}$$

$$= \left(1 + \frac{z^2}{1} + \frac{z^4}{2} + \frac{z^6}{6} + \cdots\right)\left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \cdots\right)$$

$$= z + \frac{2}{3}z^3 + \frac{2^2}{1 \cdot 3 \cdot 5} z^5 + \frac{2^3}{1 \cdot 3 \cdot 5 \cdot 7} z^7 + \cdots$$

$$R = \lim\left|\frac{a_n}{a_{n+1}}\right| = \lim\left|\frac{2^n}{1 \cdot 3 \cdots (2n+1)} \cdot \frac{1 \cdot 3 \cdots (2n+1)(2n+3)}{2^{n+1}}\right| = \infty$$

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Problem Set for Section 15.4

18. Find the Taylor series with center z_0 and its radius of convergence for $1/z, \quad z_0 = i$.

$$\frac{1}{z} = \frac{1}{i+(z-i)} = \frac{1}{i}\frac{1}{1+(z-i)/i}$$
$$= \frac{1}{i}\left[1-\frac{z-i}{i}+\left(\frac{z-i}{i}\right)^2-\left(\frac{z-i}{i}\right)^3+\cdots\right]$$
$$=-i+(z-i)+i(z-i)^2-(z-i)^4+\cdots$$

$$R=\lim\left|\frac{a_n}{a_{n+1}}\right|= 1$$

Problem Set for Section 15.4

19. Find the Taylor series with center z_0 and its radius of convergence for $1/(1-z), \quad z_0 = i$.

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i}\,\frac{1}{1-(z-i)/(1-i)}$$

$$= \frac{1}{1-i}\left[1 + [(z-i)/(1-i)] + [(z-i)/(1-i)]^2 + [(z-i)/(1-i)]^3 + \cdots\right]$$

$$= \frac{1}{1-i} + \frac{z-i}{(1-i)^2} + \frac{(z-i)^2}{(1-i)^3} + \cdots = \frac{1+i}{2} + \frac{i}{2}(z-i) - \frac{1-i}{4}(z-i)^2 + \cdots$$

$$R = \lim\left|\frac{a_n}{a_{n+1}}\right| = \lim\left|\frac{1/(1-i)^n}{1/(1-i)^{n+1}}\right| = \sqrt{2}$$