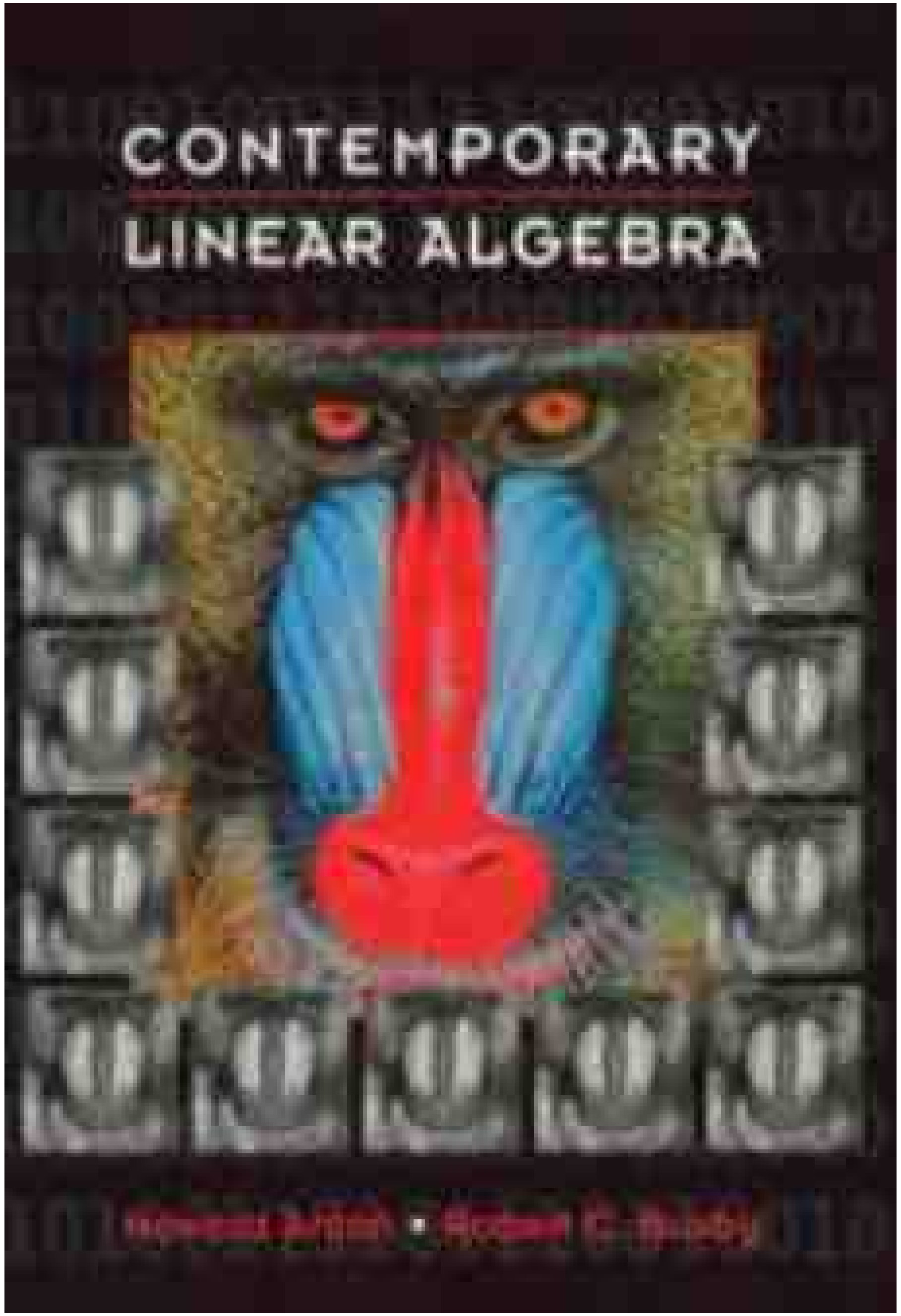


CHAPTER 3

Systems of Linear Equations

3.1 Operations on Matrices

Matrix Notation and Terminology

- Matrix: a rectangular array of numbers
- Entry: each number of a matrix
- Size of a matrix : mxn with m rows and n columns

Examples:

$$\underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}}, \quad \underset{2\times 3}{\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 4 \end{bmatrix}}, \quad \underset{1\times 4}{\begin{bmatrix} 2 & 1 & 0 & 3 \end{bmatrix}}$$

$$\underset{3\times 3}{\begin{bmatrix} \pi & -\sqrt{2} & \frac{1}{2} \\ 0.5 & 1 & 7 \\ 0 & 0 & 0 \end{bmatrix}}, \quad \underset{2\times 1}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}, \quad \underset{1\times 1}{\begin{bmatrix} 4 \end{bmatrix}}$$

3.1 Operations on Matrices-conti

A general mxn matrix: $$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (1)$$

$$\mathbf{A} = [a_{ij}] \qquad \mathbf{A} = [a_{ij}]_{m\times n}$$

The matrix A is square matrix of order n if m=n.

The entries of matrix A is usually denoted by the small letter matching the matrix.

$$(\mathbf{A})_{ij} = a_{ij}$$

$$\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 7 & 0 \end{bmatrix} \Rightarrow \begin{array}{ll} (\mathbf{A})_{11} = a_{11} = 2, & (\mathbf{A})_{12} = a_{12} = -3 \\ (\mathbf{A})_{21} = a_{21} = 7, & (\mathbf{A})_{22} = a_{22} = 0 \end{array}$$

Operations on Matrices

Definition 3.1.1 Two matrices are equal if they have the same size and their corresponding entries are equal.

Example 1 Determine x when A=B=C.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & x+1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix}$$

(a) $\mathbf{A} = \mathbf{B} \quad \Leftrightarrow \quad x = 4$

(b) $\mathbf{A} \neq \mathbf{C}$

Definition 3.1.2 The Sum and the Difference

Definition 3.1.2 If A and B are matrices with the same size, the *sum* A+B is defined to be the matrix obtained by adding the entries of B to the corresponding entries of A, and the *difference* A-B to be the matrix obtained by subtracting the entries of B from the corresponding entries of A.

If $A = [a_{ij}]$ and $B = [b_{ij}]$, then

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij} \tag{2}$$

$$(A-B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij} \tag{3}$$

Example 2 Adding and Subtracting Matrices

Find (a) A+B and A-B (b) A+C and B+C when

$$\mathbf{A}=\begin{bmatrix}2 & 1 & 0 & 3\\ -1 & 0 & 2 & 4\\ 4 & -2 & 7 & 0\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}-4 & 1 & 5 & 1\\ 2 & 2 & 0 & 1\\ 3 & 2 & -4 & 5\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$$

Sol.

(a)

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix}2 & 1 & 5 & 4\\ 1 & 2 & 2 & 3\\ 7 & 0 & 3 & 5\end{bmatrix},\ \mathbf{A}-\mathbf{B}=\begin{bmatrix}6 & -2 & -5 & 2\\ -3 & -2 & 2 & 5\\ 1 & -4 & 11 & -5\end{bmatrix}$$

(b) A+C and B+C are not defined because of different sizes.

Definition 3.1.3 Scalar Product of a Matrix

Definition 3.1.3 If A is any matrix and *c* is any scalar, then the *product* *c*A is defined to be the matrix obtained by multiplying each entry of A by *c*.

$$\mathbf{A} = [a_{ij}] \quad \longrightarrow \quad (c\mathbf{A})_{ij} = c(\mathbf{A})_{ij} = ca_{ij} \qquad (4)$$

Example 3 Determine 2A, (-1)B, and (1/3)C when

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

Sol.

$$2\mathbf{A} = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad -\mathbf{B} = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}, \quad \frac{1}{3}\mathbf{C} = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

$$-\mathbf{A} = (-1)\mathbf{A} = \begin{bmatrix} -2 & -3 & -4 \\ -1 & -3 & -1 \end{bmatrix}$$

Row and Column Vectors

Row Vector: 1xn matrix : $\mathbf{r} = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$

Column Vector: mx1 matrix $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \tag{5}$$

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4 \end{bmatrix} \tag{6}$$

$$\mathbf{A} = \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix} \tag{7}$$

Row and Column Vectors-conti

i th row vector of a matrix A: $\mathbf{r}_i(\mathbf{A})$

j th column vector of a matrix A: $\mathbf{c}_j(\mathbf{A})$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$\mathbf{r}_1(\mathbf{A}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}$$
$$\mathbf{r}_2(\mathbf{A}) = \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$
$$\mathbf{r}_3(\mathbf{A}) = \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$\mathbf{c}_1(\mathbf{A}) = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \ \mathbf{c}_2(\mathbf{A}) = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}, \ \mathbf{c}_3(\mathbf{A}) = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}, \ \mathbf{c}_4(\mathbf{A}) = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix}$$

The Product **Ax**

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array} \qquad (8)
$$

Define the product ***Ax*** such that the equation (8) can be written Ax=b.

$$A\mathbf{x} = \mathbf{b} \qquad (9)$$

$$
\mathbf{Ax} = \mathbf{b} : \begin{bmatrix}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n \\
\vdots & & \vdots & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n
\end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

The Product **Ax** -conti

$$\mathbf{Ax}=\mathbf{b}: \begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \mathbf{b}$$

$$\mathbf{Ax} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \mathbf{b}$$

Definition 3.1.4 Let **A**: mxn matrix, **x**: nx1 column vector, and $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ are column vectors of a matrix **A**. Then the product **Ax** is defined by

$$A\mathbf{x} = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n \qquad (10)$$

Example:

$$\begin{bmatrix} 1 & -3 & 2 \\ 4 & 0 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = 2\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + (-5)\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \\ 3 \end{bmatrix}$$

Example 4 Writing a Linear System as **Ax=b**

Example 4 Write the following system as **Ax=b.**

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 5x_2 & + & 3x_3 & = & 3 \\ x_1 & & & + & 8x_3 & = & 17 \end{array}$$

Sol.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

Theorem 3.1.5 Linearity Properties

Theorem 3.1.5 (Linearity Properties) *If A is mxn matrix, then the following relationships hold for all column vectors* **u** *and* **v** *in R^n and for every scalar c:*

(a) $\mathbf{A}(c\mathbf{u}) = c(\mathbf{A}\mathbf{u})$

(b) $\mathbf{A}(\mathbf{u}+\mathbf{v}) = \mathbf{A}\mathbf{u}+\mathbf{A}\mathbf{v}$

Proof

Let $\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]$, $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

Then

$$c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}, \quad \mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1+v_1 \\ u_1+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}$$

Theorem 3.1.5

$$\mathbf{A}(c\mathbf{u}) = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix} = (cu_1)\mathbf{a}_1 + (cu_2)\mathbf{a}_2 + \cdots + (cu_n)\mathbf{a}_n$$

$$= c(u_1\mathbf{a}_1) + c(u_2\mathbf{a}_2) + \cdots + c(u_n\mathbf{a}_n) = c(\mathbf{Au})$$

$$\mathbf{A}(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

$$= (u_1 + v_1)\mathbf{a}_1 + (u_2 + v_2)\mathbf{a}_2 + \cdots + (u_n + v_n)\mathbf{a}_n$$

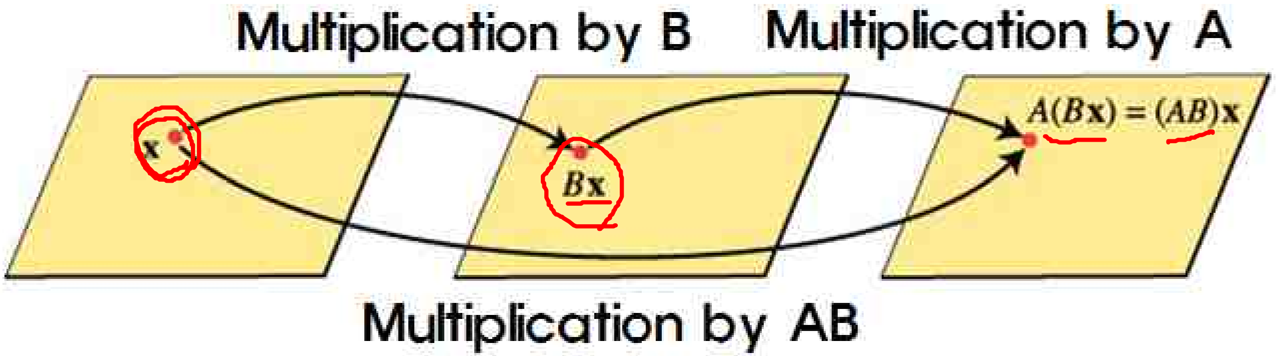
$$= (u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \cdots + u_n\mathbf{a}_n) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \cdots + v_n\mathbf{a}_n) = \mathbf{Au} + \mathbf{Av}$$

Remark

$$\mathbf{A}(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n) = c_1(\mathbf{Au}_1) + c_2(\mathbf{Au}_2) + \cdots + c_n(\mathbf{Au}_n) \quad (11)$$

The Product *AB*

$$\mathbf{A}(\mathbf{B}\mathbf{x}) = (\mathbf{A}\mathbf{B})\mathbf{x} \tag{12}$$



Let's define a matrix multiplication such that the equation (12) holds.

The Product *AB*

$$\mathbf{Bx} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n$$

Hence,
$$\begin{aligned} A(B\mathbf{x}) &= A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n) \\ &= x_1(A\mathbf{b}_1) + x_2(A\mathbf{b}_2) + \cdots + x_n(A\mathbf{b}_n) \end{aligned}$$

$$(\mathbf{AB})\mathbf{x} = [\mathbf{c}_1(\mathbf{AB}) \quad \mathbf{c}_2(\mathbf{AB}) \quad \cdots \quad \mathbf{c}_n(\mathbf{AB})] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\mathbf{c}_1(\mathbf{AB}) + x_2\mathbf{c}_2(\mathbf{AB}) + \cdots + x_n\mathbf{c}_n(\mathbf{AB})$$

The Product *AB*

$$A(B\mathbf{x}) = x_1(A\mathbf{b}_1) + x_2(A\mathbf{b}_2) + \cdots + x_n(A\mathbf{b}_n)$$

$$(\mathbf{AB})\mathbf{x} = x_1\mathbf{c}_1(\mathbf{AB}) + x_2\mathbf{c}_2(\mathbf{AB}) + \cdots + x_n\mathbf{c}_n(\mathbf{AB})$$

To satisfy the equation (12), the two equations should be same for all x_i.

Thus,

$$A\mathbf{b}_1 = \mathbf{c}_1(\mathbf{AB}), \quad A\mathbf{b}_2 = \mathbf{c}_2(\mathbf{AB}), \quad \cdots, \quad A\mathbf{b}_n = \mathbf{c}_n(\mathbf{AB})$$

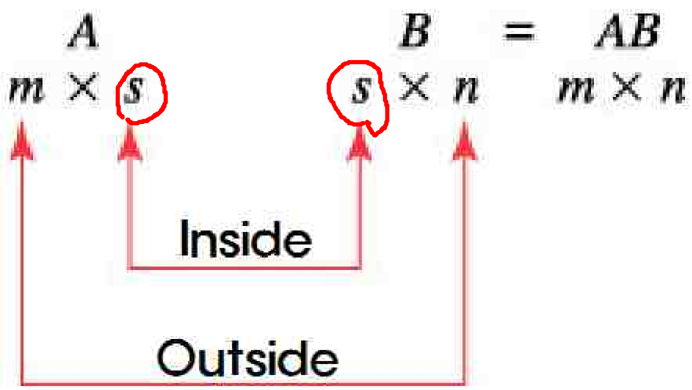
$$\Downarrow$$

$$\mathbf{AB} = [\mathbf{Ab}_1 \quad \mathbf{Ab}_2 \quad \cdots \quad \mathbf{Ab}_n]$$

Definition 3.1.6 The Product of Matrices

Definition 3.1.6 *If A is an mxs matrix and B is an sxn matrix, and if the column vectors of B are* $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$*, then the product AB is the mxn matrix defined as*

$$\mathbf{AB} = [\mathbf{Ab}_1 \quad \mathbf{Ab}_2 \quad \cdots \quad \mathbf{Ab}_n]$$



Example 5 Computing a Matrix AB

Find the product AB for $\mathbf{A}=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}$

2x3 3x4

Sol.

$$\mathbf{Ab}_1=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}=(4)\begin{bmatrix}1\\ 2\end{bmatrix}+(0)\begin{bmatrix}2\\ 6\end{bmatrix}+(2)\begin{bmatrix}4\\ 0\end{bmatrix}=\begin{bmatrix}12\\ 8\end{bmatrix}$$

$$\mathbf{Ab}_2=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\begin{bmatrix}1\\ -1\\ 7\end{bmatrix}=(1)\begin{bmatrix}1\\ 2\end{bmatrix}+(-1)\begin{bmatrix}2\\ 6\end{bmatrix}+(7)\begin{bmatrix}4\\ 0\end{bmatrix}=\begin{bmatrix}27\\ -4\end{bmatrix}$$

$$\mathbf{Ab}_3=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\begin{bmatrix}4\\ 3\\ 5\end{bmatrix}=(4)\begin{bmatrix}1\\ 2\end{bmatrix}+(3)\begin{bmatrix}2\\ 6\end{bmatrix}+(5)\begin{bmatrix}4\\ 0\end{bmatrix}=\begin{bmatrix}30\\ 26\end{bmatrix}$$

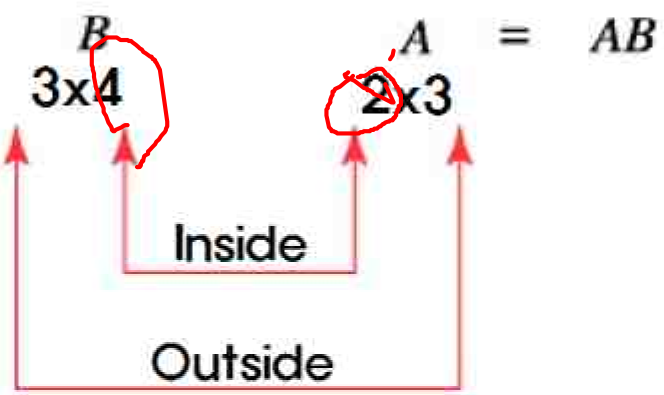
$$\mathbf{Ab}_4=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\begin{bmatrix}3\\ 1\\ 2\end{bmatrix}=(3)\begin{bmatrix}1\\ 2\end{bmatrix}+(1)\begin{bmatrix}2\\ 6\end{bmatrix}+(2)\begin{bmatrix}4\\ 0\end{bmatrix}=\begin{bmatrix}13\\ 12\end{bmatrix}$$

Thus, $\mathbf{AB}=\begin{bmatrix}12 & 27 & 30 & 13\\ 8 & -4 & 26 & 12\end{bmatrix}$

Example 6

Find the product BA for $\mathbf{A}=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}$

Sol.



The product BA is not defined.

Theorem 3.1.7 The Row-Column Rule or Dot Product Rule

The entry (AB)ij is the product of *i*-th row vector of A and *j*-th column vector of B, or equivalently, the dot product of the *i*-th row vector and the *j*-th column vector.

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{is} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{ms} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots\vdots\vdots & \vdots \\ b_{s1} & b_{s2} & \cdots & b_{sj} & \cdots & b_{sn} \end{bmatrix}$$

$$(\mathbf{AB})_{ij} = \mathbf{r}_i(\mathbf{A})\mathbf{c}_j(\mathbf{B}) = \mathbf{r}_i(\mathbf{A}) \cdot \mathbf{c}_j(\mathbf{B}) \qquad (17)$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{is}b_{sj} \qquad (16)$$

Example 7

Use the dot product rule to compute each entries of AB.

$$\mathbf{A}=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}$$

Sol.

$$AB=\begin{bmatrix}\mathbf{r}_1(A)\cdot\mathbf{c}_1(B) & \mathbf{r}_1(A)\cdot\mathbf{c}_2(B) & \mathbf{r}_1(A)\cdot\mathbf{c}_3(B) & \mathbf{r}_1(A)\cdot\mathbf{c}_4(B)\\ \mathbf{r}_2(A)\cdot\mathbf{c}_1(B) & \mathbf{r}_2(A)\cdot\mathbf{c}_2(B) & \mathbf{r}_2(A)\cdot\mathbf{c}_3(B) & \mathbf{r}_2(A)\cdot\mathbf{c}_4(B)\end{bmatrix}$$

$$(AB)_{23}=\mathbf{r}_2(A)\cdot\mathbf{c}_3(A)=2\cdot4+6\cdot3+0\cdot5=26$$

$$\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}=\begin{bmatrix}\boxed{} & \boxed{} & \boxed{} & \boxed{}\\ \boxed{} & \boxed{} & \boxed{26} & \boxed{}\end{bmatrix}$$

$$(AB)_{13}=\mathbf{r}_1(A)\cdot\mathbf{c}_3(A)=1\cdot3+2\cdot1+4\cdot2=13$$

$$\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix}\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}=\begin{bmatrix}\boxed{} & \boxed{} & \boxed{} & \boxed{13}\\ \boxed{} & \boxed{} & \boxed{} & \boxed{}\end{bmatrix}$$

Finding Specific Rows and Columns o르 a Matrix Product

Column rule for matrix multiplication:

$$\mathbf{c}_j(\mathbf{AB}) = \mathbf{Ab}_j = \mathbf{Ac}_j(\mathbf{B}) \tag{18}$$

Row rule for matrix multiplication:

$$\mathbf{r}_i(\mathbf{AB}) = \mathbf{r}_i(\mathbf{A})\mathbf{B} \tag{19}$$

$$\mathbf{AB} = \mathbf{A}\left[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n\right] = \left[\mathbf{Ab}_1 \quad \mathbf{Ab}_2 \quad \cdots \quad \mathbf{Ab}_n\right]$$

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{is} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{ms} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{s1} & b_{s2} & \cdots & b_{sj} & \cdots & b_{sn} \end{bmatrix}$$

Example 8 Finding a Specific Row and Column of AB

Find the second column and the first row of AB by using the column rule and the row rule.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Sol.

$$\mathbf{c}_2(\mathbf{AB}) = \mathbf{A}\mathbf{c}_2(\mathbf{B}) = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -7 \end{bmatrix} = \begin{bmatrix} 27 \\ -4 \end{bmatrix}$$

$$\mathbf{r}_1(\mathbf{AB}) = \mathbf{r}_1(\mathbf{A})\mathbf{B} = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \end{bmatrix}$$

Theorem 3.1.8 Matrix Products as Linear Combinations

Let $\mathbf{A}: m \times s$

$\mathbf{x} = (x_1, x_2, \cdots, x_s)'$: Column vector

Then, by Definition 3.1.4,

$$\mathbf{Ax} = x_1\mathbf{c}_1(\mathbf{A}) + x_2\mathbf{c}_2(\mathbf{A}) + \cdot\cdot\cdot + x_s\mathbf{c}_s(\mathbf{A}) \qquad (20)$$

Let $\mathbf{B}: s \times n$

$\mathbf{y} = (y_1, y_2, \cdots, y_s)$: row vector

Then, by Definition 3.1.4,

$$\mathbf{yB} = y_1\mathbf{r}_1(\mathbf{B}) + y_2\mathbf{r}_2(\mathbf{B}) + \cdot\cdot\cdot + y_s\mathbf{r}_s(\mathbf{B}) \qquad (21)$$

Example 9 Rows and Columns of AB as a Linear Combinations

Find the second column and the first row of AB by linear combinations.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Sol.

$$\mathbf{c}_2(\mathbf{AB}) = \mathbf{A}\mathbf{c}_2(\mathbf{B}) = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -7 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 27 \\ -4 \end{bmatrix}$$

$$\mathbf{r}_1(\mathbf{AB}) = \mathbf{r}_1(\mathbf{A})\mathbf{B} = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

$$= (1)\begin{bmatrix} 4 & 1 & 4 & 3 \end{bmatrix} + (2)\begin{bmatrix} 0 & -1 & 3 & 1 \end{bmatrix} + (4)\begin{bmatrix} 2 & 7 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 27 & 30 & 13 \end{bmatrix}$$

Definition 3.1.9 Transpose of a Matrix(전치행렬)

Definition 3.1.9 The transpose of A

Notation: A' or A^T

$(A^T)_{ij}=(A)_{ji}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \quad \Longleftrightarrow \quad \mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}$$

Example 10

Find the transpose matrices of the following matrices.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 3 & -5 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 4 \end{bmatrix}$$

Sol.

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix} \quad \mathbf{B}^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix} \quad \mathbf{C}^T = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} \quad \mathbf{D}^T = \begin{bmatrix} 4 \end{bmatrix}$$

Transpose of a Square Matrix

Interchange entries that are symmetrically positioned about the main diagonal.

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 7 & 0 \\ 5 & 8 & -6 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 7 & 8 \\ 4 & 0 & -6 \end{bmatrix}$$

Interchange entries that are
symmetrically positioned
about the main diagonal.

Trace(대각합)

Definition 3.1.10 If A is a square matrix, then the trace of A, denoted by tr(A), is defined to be the sum of the entries on the main diagonal of A.

Example:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & -8 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$\operatorname{tr}(\mathbf{A}) = 3 + (-8) = -5$$

$$\operatorname{tr}(\mathbf{B}) = b_{11} + b_{22} + b_{33}$$

Inner and Outer Matrix Products

Definition 3.1.11 If **u** and **v** are column vectors with the same size, then the product $\mathbf{u}^T\mathbf{v}$ is called the matrix inner product of **u** with **v**, and if **u** and **v** are column vectors of any size, then the product $\mathbf{u}\mathbf{v}^T$ is called the matrix outer product of **u** with **v**.

Example 11. Matrix Inner and Outer Products

Find matrix inner and outer products of **u** with **v**.

$$\mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Sol.

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 13 \end{bmatrix} = \mathbf{u}\cdot\mathbf{v}$$

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} -1 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 6 & 15 \end{bmatrix}$$

Matrix Inner and Outer Products of **u** with **v**

In general, if

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Then,

$$\mathbf{u}^T \mathbf{v} = [u_1 \ u_2 \cdots \ u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = [u_1 v_1 + u_2 v_2 + \cdots + u_n v_n] = \mathbf{u} \cdot \mathbf{v} \qquad (23)$$

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} [v_1 \ v_2 \cdots \ v_n] = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & \vdots & & \vdots \\ u_n v_1 & u_n v_2 & \cdots & u_n v_n \end{bmatrix} \qquad (24)$$

Matrix Inner and Outer Products of **u** with **v**-conti

$$\mathbf{u}^T\mathbf{v} = [u_1\ u_2 \cdots u_n]\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = [u_1 v_1 + u_2 v_2 + \cdots + u_n v_n] = \mathbf{u}\cdot\mathbf{v} \qquad (23)$$

$$\mathbf{u}^T\mathbf{v} = \mathrm{tr}(\mathbf{u}\mathbf{v}^T) \qquad (25)$$

$$\mathbf{u}^T\mathbf{v} = \mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u} = \mathbf{v}^T\mathbf{u} \qquad (26)$$

$$\mathrm{tr}(\mathbf{u}\mathbf{v}^T) = \mathrm{tr}(\mathbf{v}\mathbf{u}^T) = \mathbf{u}\cdot\mathbf{v} \qquad (27)$$

These formulas apply only when **u** and **v** are column vectors.

3.2 Inverses; Algebraic Properties of Matrices

Theorem 3.2.1 If a and b are scalars, and if the sizes of the matrices A, B, and C are such that the indicated operations can be performed, then;

$$(a)\quad A+B=B+A \qquad \textbf{[Commutative law for addition]}$$

$$(b)\quad A+(B+C)=(A+B)+C \qquad \textbf{[Associative law for addition]}$$

$$(c)\quad (ab)A=a(bA)$$

$$(d)\quad (a+b)A=aA+bA$$

$$(e)\quad (a-b)A=aA-bA$$

$$(f)\quad a(A+B)=aA+aB$$

$$(g)\quad a(A-B)=aA-aB$$

Proof. Omitted

Properties of Matrix Multiplication

In the arithmetic of real numbers, it is always true that

$$ab = ba$$

However, the commutative law does not hold for matrix multiplication.

1. AB may be defined and BA may not(for example, if A:2x4, B:3x4)
2. AB and BA may both defined, but they may have different sizes (for example A:2x3, B:3x2).
3. AB and BA may both defined and have the same size, but the two matrices may be different.(Example1)

Example 1

Show that AB≠BA for the given matrices.

$$\mathbf{A}=\begin{bmatrix}-1 & 0\\ 2 & 3\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}$$

Sol.

$$\mathbf{AB}=\begin{bmatrix}-1 & 0\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}=\begin{bmatrix}-1\text{x}1+0\text{x}3 & -1\text{x}2+0\text{x}0\\ 2\text{x}1+3\text{x}0 & 2\text{x}2+3\text{x}0\end{bmatrix}=\begin{bmatrix}-1 & -2\\ 11 & 4\end{bmatrix}$$

$$\mathbf{BA}=\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}\begin{bmatrix}-1 & 0\\ 2 & 3\end{bmatrix}=\begin{bmatrix}1\text{x}(-1)+2\text{x}2 & 1\text{x}0+2\text{x}3\\ 3\text{x}(-1)+0\text{x}2 & 3\text{x}0+0\text{x}3\end{bmatrix}=\begin{bmatrix}3 & 6\\ -3 & 0\end{bmatrix}$$

$$\Rightarrow \quad \mathbf{AB}\neq\mathbf{BA}$$

Theorem 3.2.2

Theorem 3.2.1 If a is a scalar, and if the sizes of the matrices A, B, and C are such that the indicated operations can be performed, then;

(a) $A(BC) = (AB)C$ **[Associative law for multiplication]**

(b) $A(B + C) = AB + AC$ **[Left distributive law]**

(c) $(B + C)A = BA + CA$ **[Right distributive law]**

(d) $A(B - C) = AB - AC$

(e) $(B - C)A = BA - CA$

(f) $a(BC) = (aB)C = B(aC)$

Proof. Omitted

Zero Matrices

A matrix whose entries are all zero is called a ***zero matrix***.

Some examples are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [0]$$

Notation:

$$\mathbf{0}, \quad \mathbf{0}_{m\times n}$$

Theorem 3.2.3

Theorem 3.2.3 If c is a scalar, and if the sizes of the matrices are such that the indicated operations can be performed, then;

(a) $A + \mathbf{0} = \mathbf{0} + A = A$

(b) $A - \mathbf{0} = A$

(c) $A - A = A + (-A) = \mathbf{0}$

(d) $0A = \mathbf{0}$

(e) *If* $cA = \mathbf{0}$, *then* $c = 0$ *or* $A = \mathbf{0}$.

Proof. Omitted

Example 2 Cancellation Law

Cancellation law for real numbers:

If $ab = ac$ and $a \neq 0$, then $b = c$.

For matrix multiplication, the cancellation does not hold in general.

If $\mathbf{AB} = \mathbf{AC}$ and $\mathbf{A} \neq \mathbf{0}$, then $\mathbf{B}$ may not be equal to $\mathbf{C}$.

Example 2 Show that $\mathbf{AB} = \mathbf{AC}$ does not tell $\mathbf{B} = \mathbf{C}$ even when $\mathbf{A} \neq \mathbf{0}$.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

Sol.

$\mathbf{AB} = \mathbf{AC} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$ and $\mathbf{A} \neq \mathbf{0}$ $\Rightarrow$ However, $\mathbf{B} \neq \mathbf{C}$

Example 3 Nonzero Matrices Can Have a Zero Product

In the arithmetic for real numbers:

If $ca = 0$, then $c = 0$ or $a = 0$.

For matrix multiplication, this does not hold in general.

Example 3 Show that $\mathbf{CA} = \mathbf{0}$ when $\mathbf{C} \neq \mathbf{0}$ and $\mathbf{A} \neq \mathbf{0}$.

$$\mathbf{C} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$$

Sol.

$\mathbf{A} \neq \mathbf{0}$ and $\mathbf{C} \neq \mathbf{0}$ $\Rightarrow$ However,

$$\mathbf{CA} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

Identity Matrices(항등행렬)

A square matrix with 1's on the main diagonal and zeros elsewhere is called an *identity matrix*.

Some examples are

$$[1], \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notation: $\mathbf{I}, \ \ \mathbf{I}_n$

Property:

$\mathbf{A} : m \times n$

$$\mathbf{A}\mathbf{I}_n = \mathbf{A}, \quad \mathbf{I}_m\mathbf{A} = \mathbf{A}$$

Theorem 3.2.4

Theorem 3.2.4 If $\mathbf{R}$ is the reduced row echelon form of an $n \times n$ matrix A, then either $\mathbf{R}$ has a row of zeros or $\mathbf{R}$ is the identity matrix $\mathbf{I}_n$.

Inverse of a matrix

In ordinary arithmetic, if $a \cdot a^{-1} = a^{-1} \cdot a = 1$, then a^{-1} is called the multiplicative inverse of a.

Definition 3.2.5 $\mathbf{A}, \mathbf{B}$: square matrices of the same size.
If $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, then $\mathbf{A}$ is said to be invertible(or *nonsingular*), and $\mathbf{B}$ is called an inverse of $\mathbf{A}$. If There is no matrix $\mathbf{B}$ with this property, then $\mathbf{A}$ is said to be *singular*.

Example 4 An Invertible Matrix

Example 4 An Invertible Matrix

(a) Calculate $\mathbf{AB}$.

(b) Show that $\mathbf{A}$ and $\mathbf{B}$ are invertible.

$$\mathbf{A}=\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix},\quad \mathbf{B}=\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Sol.

(a) Calculate $\mathbf{AB}$.

$$\mathbf{AB}=\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\mathbf{I}$$

$$\mathbf{BA}=\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\mathbf{I}$$

(b) Show that $\mathbf{A}$ and $\mathbf{B}$ are invertible.

by (a)

Example 5 Nonsingular Matrix

Example 4 Show that the following matrix is singular.

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

Sol.

Let's show that there is no $\mathbf{B}$ which satisfies $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix} = [\mathbf{c}_1 \ \mathbf{c}_2 \ \mathbf{0}]$$

$$\mathbf{BA} = \mathbf{B}[\mathbf{c}_1 \ \mathbf{c}_2 \ \mathbf{0}] = [\mathbf{Bc}_1 \ \ \mathbf{Bc}_2 \ \mathbf{0}] \neq \mathbf{I}$$

Thus, there is no $\mathbf{B}$ which satisfies $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$.

Generally, a matrix with a row or column of zeros is singular.

Theorem 3.2.6 Uniqueness of the Inverse of a Matrix

If $\mathbf{A}$ is an invertible matrix, and if $\mathbf{B}$ and $\mathbf{C}$ are both inverses of $\mathbf{A}$, then $\mathbf{B} = \mathbf{C}$; that is, an invertible matrix has a unique inverse.

Proof

B is an inverse of A:

$$(\mathbf{BA})\mathbf{C} = \mathbf{IC} = \mathbf{C}$$

C is an inverse of A:

$$\mathbf{AC} = \mathbf{I}$$

$$(\mathbf{BA})\mathbf{C} = \mathbf{B}(\mathbf{AC})$$

$$\mathbf{B}(\mathbf{AC}) = \mathbf{BI} = \mathbf{B}$$

$$\therefore \mathbf{B} = \mathbf{C}$$

Theorem 3.2.7 Matrix Invertibility

Theorem 3.2.7 *The matrix* $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is invertible if and only if* $ad - bc \neq 0$, *in which case the inverse is given by the formula*

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

Proof:

Show that $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$ by direct multiplication.

$$\mathbf{A}\,\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Determinant(행렬식)

The quantity $ad - bc$ is called the determinant of the 2x2 matrix A and is denoted by $\det(\mathbf{A})$ or, alternatively, by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 6

Example 6 Find inverses if possible.

$$\mathbf{A} = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Sol.

$$\det(\mathbf{A}) = (6)(2) - (1)(5) = 7 \neq 0$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} 2/7 & -1/7 \\ -5/7 & 6/7 \end{bmatrix}$$

$$\det(\mathbf{B}) = (-1)(-6) - (2)(3) = 0$$

Thus, the matrix $\mathbf{B}$ is not invertible.

Example 7 Solution of a Linear System by Matrix Inversion

Express x and y in terms of u and v when $ad-bc\neq 0$.

$$u = ax + by$$
$$v = cx + dy$$

Sol.

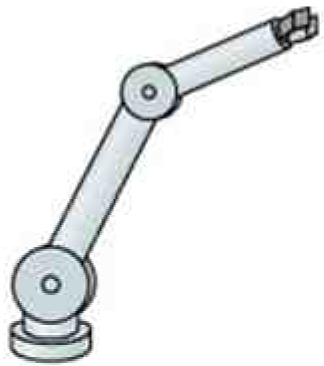
$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

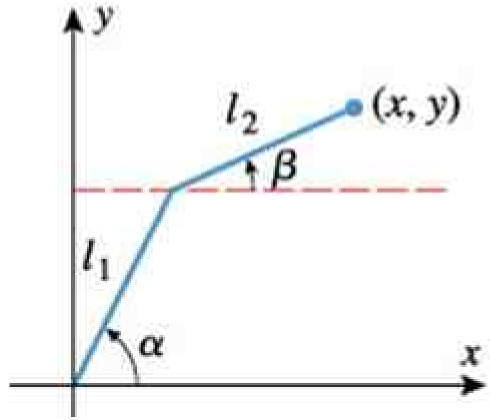
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$x = \frac{du-bv}{ad-bc} \qquad y = \frac{av-cu}{ad-bc}$$

Example 8

Find the lengths l_1 and l_2 when the angles α, β , and tip of the working arm is known.





Sol.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$x = l_1\cos\alpha + l_2\cos\beta$$

$$y = l_1\sin\alpha + l_2\sin\beta$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\alpha & \cos\beta \\ \sin\alpha & \sin\beta \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} \cos\alpha & \cos\beta \\ \sin\alpha & \sin\beta \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \frac{1}{\sin(\beta-\alpha)} \begin{bmatrix} \sin\beta & -\cos\beta \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 8

$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \frac{1}{\sin(\beta-\alpha)} \begin{bmatrix} \sin\beta & -\cos\beta \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$l_1 = \frac{\sin\beta}{\sin(\beta-\alpha)} x - \frac{\cos\beta}{\sin(\beta-\alpha)} y$$

$$l_2 = -\frac{\sin\alpha}{\sin(\beta-\alpha)} x + \frac{\cos\alpha}{\sin(\beta-\alpha)} y$$

Theorem 3.2.8 Invertibility of AB

Theorem 3.2.8 If $\mathbf{A}$ and $\mathbf{B}$ are invertible matrices with the same size, then $\mathbf{AB}$ is invertible and

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Proof

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$$

$$(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{B}^{-1}\mathbf{IB} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$$

May be extended to three or more matrices.

$$(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$$

Example 9

Find matrices $\mathbf{AB}$, $(\mathbf{AB})^{-1}$, $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$, and $\mathbf{B}^{-1}\mathbf{A}^{-1}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

Sol.

$$\mathbf{AB} = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix} \qquad (\mathbf{AB})^{-1} = \begin{bmatrix} 4 & -3 \\ -9/2 & 7/2 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{B}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 3/2 \end{bmatrix}$$

$$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2/3 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -9/2 & 7/2 \end{bmatrix}$$

$$\therefore (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Powers of a Matrix(행렬의 거듭제곱)

For a square matrix $\mathbf{A}$,

$$\mathbf{A}^0 = \mathbf{I} \quad \mathbf{A}^2 = \mathbf{A}\mathbf{A}, \quad \mathbf{A}^3 = \mathbf{A}\mathbf{A}\mathbf{A}, \quad \mathbf{A}^n = \mathbf{A}\mathbf{A}\cdots\mathbf{A}$$

If $\mathbf{A}$ is invertible,

$$\mathbf{A}^{-n} = (\mathbf{A}^{-1})^n = \mathbf{A}^{-1}\mathbf{A}^{-1}\cdots\mathbf{A}^{-1}$$

$$\mathbf{A}^r\mathbf{A}^s = \mathbf{A}^{r+s}, \quad (\mathbf{A}^r)^s = \mathbf{A}^{rs}$$

Theorem 3.2.9 *If* $\mathbf{A}$ *is invertible, and* n *is a nonnegative integer, then;*

(a) $\mathbf{A}^{-1}$ is invertible and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$

(b) $\mathbf{A}^n$ is invertible and $(\mathbf{A}^n)^{-1} = \mathbf{A}^{-n} = (\mathbf{A}^{-1})^n$

(c) $k\mathbf{A}$ is invertible for any nonnegative k, and $(k\mathbf{A})^{-1} = k^{-1}\mathbf{A}^{-1}$.

Example 10

Show that $(\mathbf{A}^3)^{-1} = (\mathbf{A}^{-1})^3$. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, $\mathbf{A}^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

Sol.

$$\mathbf{A}^{-3} = (\mathbf{A}^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

$$\mathbf{A}^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

$$(\mathbf{A}^3)^{-1} = \frac{1}{(11)(41)-(30)(15)}\begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

$$= (\mathbf{A}^{-1})^3$$

Example 11 The Square of a Matrix Sum

In the arithmetic of real numbers,

$$(a+b)^2 = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$$

However, in the arithmetic of matrices,

$$(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2 \neq \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$$

If $\mathbf{AB} = \mathbf{BA}$ (Commutative Law)

$$(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$$

Matrix Polynomials

Polynomial: $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$

Matrix polynomial: $\mathbf{A}$ is an $n \times n$ square matrix

$$p(\mathbf{A}) = a_0 \mathbf{I} + a_1 \mathbf{A} + a_2 \mathbf{A}^2 + \cdots + a_m \mathbf{A}^m$$

: Matrix polynomial in $\mathbf{A}$

If $p(x) = p_1(x)\ p_2(x)$, then

$$P(\mathbf{A}) = p_1(\mathbf{A})\ p_2(\mathbf{A})$$

Example 12 A Matrix Polynomial

Find $p(\mathbf{A})$ for

$$p(x) = x^2 - 2x - 3 \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

Sol.

$$p(\mathbf{A}) = \mathbf{A}^2 - 2\mathbf{A} - 3\mathbf{I}$$

$$= \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Properties of the Transpose

Theorem 3.2.10 *If the sizes of the matrices are such that the stated operations can be performed, then:*

(a) $(\mathbf{A}^T)^T = \mathbf{A}$

(b) $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

(c) $(\mathbf{A}-\mathbf{B})^T = \mathbf{A}^T - \mathbf{B}^T$

(d) $(k\mathbf{A})^T = k\mathbf{A}^T$

(e) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

may be extended to three or more matrices

$$(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$$

Properties of the Transpose-conti

The key to the proof of Theorem 3.2.10(e) is the following relationship.

$$\mathbf{r}_i(A)\mathbf{c}_j(B) = \mathbf{r}_j(B^T)\mathbf{c}_i(A^T) \tag{10}$$

$$\begin{aligned}\mathbf{r}_i(A)\mathbf{c}_j(B) &= [a_{i1}\ a_{i2} \cdots\ a_{is}]\ [b_{1j}\ \ b_{2j} \cdots b_{sj}]^{\mathsf{T}} \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{is}b_{sj}\end{aligned}$$

$$\begin{aligned}\mathbf{r}_j(B^T)\mathbf{c}_i(A^T) &= [b_{1j}\ b_{2j} \cdots\ b_{sj}]\,[a_{i1}\ \ a_{i2} \cdots a_{is}]^{\mathsf{T}} \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{is}b_{sj}\end{aligned}$$

$$\begin{aligned}\left((AB)^T\right)_{ji} &= (AB)_{ij} && \text{[Formula (22), Section 3.1]} \\ &= \mathbf{r}_i(A)\mathbf{c}_j(B) && \text{[Row-Column Rule]} \\ &= \mathbf{r}_j(B^T)\mathbf{c}_i(A^T) && \text{Formula (10)} \\ &= (B^TA^T)_{ji} && \text{[Row-Column Rule]}\end{aligned}$$

Theorem 3.2.11

Theorem 3.2.11 *If* $\mathbf{A}$ *is an invertible matrix, then* $\mathbf{A}^T$ *is also invertible and*

$$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$$

Proof

$$\mathbf{A}^T(\mathbf{A}^{-1})^T = (\mathbf{A}^{-1}\mathbf{A})^T \quad [\because (AB)^T = B^T A^T]$$

$$= \mathbf{I}^T = \mathbf{I}$$

$$(\mathbf{A}^{-1})^T\mathbf{A}^T = (\mathbf{A}\mathbf{A}^{-1})^T \quad [\because (AB)^T = B^T A^T]$$

$$= \mathbf{I}^T = \mathbf{I}$$

$$\rightarrow (\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$$

Example 13 Inverse of a Transpose

Find the inverse of invertible matrix $\mathbf{A}$ and show $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Sol.

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

$$\mathbf{A}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$(A^T)^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{c}{ad-bc} \\ -\dfrac{b}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

Properties of the Trace

Theorem 3.2.12 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are square matrices with the same size, then*:

(a) $\operatorname{tr}(\mathbf{A}^T) = \operatorname{tr}(\mathbf{A})$

(b) $\operatorname{tr}(c\mathbf{A}^T) = c \operatorname{tr}(\mathbf{A})$

(c) $\operatorname{tr}(\mathbf{A} + \mathbf{B}) = \operatorname{tr}(\mathbf{A}) + \operatorname{tr}(\mathbf{B})$

(d) $\operatorname{tr}(\mathbf{A} - \mathbf{B}) = \operatorname{tr}(\mathbf{A}) - \operatorname{tr}(\mathbf{B})$

(e) $\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA})$

Example 14 Trace of a Product

Show that $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$ even though $\mathbf{AB} \neq \mathbf{BA}$.

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

Sol.

$$\mathbf{AB} = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} \Rightarrow \mathrm{tr}(\mathbf{AB}) = 3$$

$$\mathbf{BA} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix} \Rightarrow \mathrm{tr}(\mathbf{BA}) = 3$$

Concept Problem

What is the relationship between tr(**A**) and tr(**A**$^{-1}$) for a 2x2 invertible matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Sol.

$$\text{tr}(\mathbf{A}) = \text{tr}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$$

$$\text{tr}(\mathbf{A}^{-1}) = \text{tr}\begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

$$= \frac{d}{ad-bc} + \frac{a}{ad-bc} = \frac{a+d}{ad-bc} = \frac{\text{tr}(\mathbf{A})}{|\mathbf{A}|}$$

Theorem 3.2.13

Theorem 3.2.13 *If* $\mathbf{r}$ *is a 1xn row vector and* $\mathbf{c}$ *is an nx1 column vector, then*

$$\mathbf{rc} = \mathrm{tr}(\mathbf{cr}) \qquad (11)$$

Proof

The above statement (11) is a restatement of (25), in Section 3.1, which is

$$\mathbf{u}^T\mathbf{v} = \mathrm{tr}(\mathbf{u}\mathbf{v}^T) \qquad (25)\ \text{In Section 3.1}$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{r} = \mathbf{u}^T$$

$$\mathbf{rc} = \mathbf{u}^T\mathbf{c} = \mathrm{tr}(\mathbf{u}\mathbf{c}^T) = \mathrm{tr}(\mathbf{r}^T\mathbf{c}^T) = \mathrm{tr}((\mathbf{cr})^T) = \mathrm{tr}(\mathbf{cr})$$

Example 15 Trace of a Column Vector times a Row Vector

Show that the Theorem 3.2.13 holds for

$$\mathbf{r} = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Sol.

Theorem 3.2.13 : **rc=tr(cr)** (11)

$$\mathbf{rc} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = (1)(3) + (2)(4) = 11$$

$$\mathbf{cr} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix} \Rightarrow \operatorname{tr}(\mathbf{cr}) = 3 + 8 = 11$$

$$\therefore \operatorname{tr}(\mathbf{cr}) = \mathbf{rc}$$

Transpose and Dot Product

If **u** and **v** are column vectors, then their dot product can be expressed as the matrix product

$$\mathbf{u}^T\mathbf{v} = \mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u} = \mathbf{v}^T\mathbf{u} \qquad \text{(26) In Sec 3.1}$$

If $\mathbf{A}$ is an nxn matrix, then

$$\begin{aligned}(\mathbf{Au})\cdot\mathbf{v} &= \mathbf{v}^T(\mathbf{Au}) = (\mathbf{v}^T\mathbf{A})\mathbf{u} \\ &= (\mathbf{A}^T\mathbf{v})^T\mathbf{u} = \mathbf{u}\cdot(\mathbf{A}^T\mathbf{v})\end{aligned} \Rightarrow (\mathbf{Au})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{A}^T\mathbf{v})$$

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{Av}) &= (\mathbf{Av})^T\mathbf{u} = (\mathbf{v}^T\mathbf{A}^T)\mathbf{u} \\ &= \mathbf{v}^T(\mathbf{A}^T\mathbf{u}) = (\mathbf{A}^T\mathbf{u})\cdot\mathbf{v}\end{aligned} \Rightarrow \mathbf{u}\cdot(\mathbf{Av}) = (\mathbf{A}^T\mathbf{u})\cdot\mathbf{v}$$

$$\Rightarrow \quad \mathbf{u}\cdot(\mathbf{Av}) = (\mathbf{A}^T\mathbf{u})\cdot\mathbf{v} \qquad (12)$$

$$(\mathbf{Au})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{A}^T\mathbf{v}) \qquad (13)$$

The matrix A can be moved across the dot product sign by transposing A.

3.3 Elementary Matrices; A Method for Finding A^{-1}

Elementary Matrices

Elementary row operations(**ERO**)

1. Interchange two rows
2. Multiply a row by a nonzero constant
3. Add a multiple of one row to another

Definition: An elementary matrix is a matrix that results from a single elementary row operation to an identity matrix.

Example of elementary matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$(-3)\times R_2 \qquad R_2 \longleftrightarrow R_4 \qquad R_1 \longleftarrow R_1 + 3R_3$

Theorem 3.3.1

Theorem 3.3.1 *If A is an mxn matrix and if the elementary matrix E results by a certain row operation on the mxm identity matrix, then the product EA is the matrix that results when the same row operation is performed on A.*

Example 1 Find an elementary matrix E such that EA is the matrix that results by adding 4 times the first row A to the third row.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

$$\mathbf{EA} : 4R_1 + R_3 \to R_3$$

Sol.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \; 4R_1 + R_3 \to R_3$$

Check:

$$\mathbf{EA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 5 & 4 & 12 & 12 \end{bmatrix}$$

Inverse Operations

An elementary operation: An operation that transforms an identity matrix I to an elementary matrix E.

An inverse operation: An operation that transforms an elementary matrix E to an identity matrix I.

$$\text{Elementary matrix } I \quad \underset{\text{Inverse operation}}{\overset{\text{Elementary operation}}{\rightleftarrows}} \quad \text{Identity matrix } E$$

$I \Rightarrow E$: **Row Operation on I That Produces E**	$E \Rightarrow I$: **Row Operation on E That Produces I**
Multiply row i by $c \neq 0$	Multiply row i by $1/c$
Interchange rows i and j	Interchange rows i and j
Add c times row i to row j	Add $-c$ times row i to row j

Example 2 Recovering I_n from Elementary Matrices

Example 2 Find row operations and inverse row operations for the following elementary matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

Sol.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[7R_2]{} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \xrightarrow[R_2/7]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow[5R_2 + R_1 \rightarrow R_1]{} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \xrightarrow[-5R_2 + R_1 \rightarrow R_1]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem 3.3.2

Theorem 3.3.2 *An elementary matrix is invertible, and the inverse is also an elementary matrix.*

Proof

Let E : an elementary matrix obtained by an elementary row operation to an identity matrix I

E_0 : the elementary matrix performing the inverse row operation of that used to obtain E.

Then, by Theorem 3.3.1,

$E_0E=I$ and $EE_0=I$

Characterizations of Invertibility(가역성)

Theorem 3.3.3 *If A is an nxm matrix, then the following statements are equivalent; that is, they are all true or all false.*

(a) The reduced row echelon form of A is I_n.
(b) A is expressible as a product of elementary matrices.
(c) A is invertible.

Proof : prove by showing (a) ⇒ (b) ⇒ (c) ⇒ (a)

(a) ⇒ (b) :

Since the RREF of A is I_n, there is a sequence of elementary row operations $E_k \cdots E_2 E_1$ that reduces A to I_n.

$$E_k \cdots E_2 E_1 A = I_n \qquad (1)$$

By Theorem 3.3.2, each E_k is invertible.

$$(1): \; (E_1^{-1} E_2^{-1} \cdots E_k^{-1})(E_k \cdots E_2 E_1 A) = (E_1^{-1} E_2^{-1} \cdots E_k^{-1}) I_n$$

$$A = (E_1^{-1} E_2^{-1} \cdots E_k^{-1}) I_n = (E_1^{-1} E_2^{-1} \cdots E_k^{-1}) I_n \qquad (2)$$

Characterizations of Invertibility-cont

(b) ⇒ (c) :

Suppose that A is expressible as a product of elementary matrices. Since a product of invertible matrices is invertible, and since elementary matrices are invertible, it follows that A is invertible.

(c) ⇒ (a) :

Suppose that A is invertible and its RREF is R. Since R is obtained from A by a sequence of elementary operations, it follows that there exists a sequence of elementary matrices $E_1, E_2, \cdots, E_k$ *such that*

$$E_k \cdots E_2 E_1 A = R$$

Since each E_k is invertible, R is invertible.

By Theorem 3.2.4, Either R is I_n or R has a row of zeros. Since R is invertible, R must be I_n.

Row Equivalence(행 동치)

If a matrix B can be obtained from a matrix A by performing a finite sequence of elementary row operations, then there exists a sequence of elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \cdots, \mathbf{E}_k$ such that

$$\mathbf{E}_k \cdots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{B}$$

Since elementary matrices are invertible,

$$\mathbf{A} = \mathbf{E}_1^{-1} \mathbf{E}_2^{-1} \cdots \mathbf{E}_k^{-1} \mathbf{B}$$

Two matrices A and B are said to be ***row equivalent*** if A is obtained by performing a finite sequence of elementary row operations on B.

A square matrix is invertible if and only if it is row equivalent to the identity matrix of the same size.

Theorem 3.3.4

Theorem 3.3.3 If A and B are square matrix of the same size, then the following are equivalent;

(a) A and B are row equivalent.
(b) There is an invertible matrix E such that B=EA.
(c) There is an invertible matrix F such that A=FB.

An Algorithm for Inverting Matrices(반전 알고리즘)

Suppose that A is reduced to In by a sequence of elementary operations corresponding to $E_k \cdots E_2 E_1$.

Then, $E_k \cdots E_2 E_1 A = I_n$ $\quad$ $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$

$$A^{-1} = E_k \cdots E_2 E_1 = E_k \cdots E_2 E_1 I_n$$

$\Rightarrow$ The same sequence $E_k \cdots E_2 E_1$ also produces A^{-1} from I_n.

The Inversion Algorithm *To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to I, and then perform the same sequence of operation on I to obtain A^{-1}.*

Example 3 Applying the Inversion Algorithm

Example 3 Find the inverse of $A=\begin{bmatrix}1&2&3\\2&5&3\\1&0&8\end{bmatrix}$

Sol. $[A \mid I] \Rightarrow [I \mid A^{-1}]$

$$\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\2&5&3&0&1&0\\1&0&8&0&0&1\end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&-3&-2&1&0\\0&-2&5&-1&0&1\end{array}\right] \begin{array}{l}-2R_1+R_2 \to R_2\\ -R_1+R_3 \to R_3\end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&-3&-2&1&0\\0&0&-1&-5&2&1\end{array}\right] 2R_1+R_3 \to R_3 \Rightarrow \left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&-3&-2&1&0\\0&0&1&5&-2&-1\end{array}\right] -R_3$$

$$\Rightarrow \left[\begin{array}{ccc|ccc}1&2&0&-14&6&3\\0&1&0&13&-5&-3\\0&0&1&5&-2&-1\end{array}\right] \begin{array}{l}-3R_3+R_1 \to R_1\\ 3R_3+R_2 \to R_2\end{array} \Rightarrow \left[\begin{array}{ccc|ccc}1&0&0&-40&16&9\\0&1&0&13&-5&-3\\0&0&1&5&-2&-1\end{array}\right] -2R_2+R_1 \to R_1$$

Example 3 Applying the Inversion Algorithm-conti

$[A \mid I] \Rightarrow [I \mid A^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Example 4 The inversion Algorithm Will Reveal for Singular A

Example 4 Find the inverse of $$A=\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Sol.

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} -2R_1+R_2 \rightarrow R_2 \\ R_1+R_3 \rightarrow R_3 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] R_2+R_3 \rightarrow R_3$$

Thus, A is not invertible since there is a row of zeros on the left side.

Solving Linear Systems by Matrix Inversion

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

$$
\begin{bmatrix}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\vdots \qquad \vdots \qquad\qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Theorem 3.3.5 *If $A\mathbf{x}=\mathbf{b}$ is a linear system of n equations in n unknowns, and if the coefficient matrix A is invertible, then the system has a unique solution, namely* $\mathbf{x}=A^{-1}\mathbf{b}$.

$$A\mathbf{x}=\mathbf{b}$$

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$$

$$(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Example 5

Solve the linear system.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\quad + 8x_3 &= 17 \end{aligned}$$

Sol.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

(by Example 3)

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = 2$$

Theorem 3.3.6

Theorem 3.3.6 If A**x**=**0** is a homogeneous linear system of n equations in n unknowns, then the system has only the trivial solution if and only if the coefficient matrix is invertible.

Proof.

If A is invertible, then the unique solution is **x**=A^{-1}**0**=**0**.

If the system has only the trivial solution, then the linear system

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{matrix} \qquad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Theorem 3.3.7

Theorem 3.3.7 *If A is an nxn matrix, then the following statements are equivalent .*

(a) *The reduced row echelon form of A is I_n.*

(b) *A is expressible as a product of elementary matrices.*

(c) *A is invertible.*

(d) *$A\mathbf{x}=\mathbf{0}$ has only the trivial solution.*

Example 6 Homogeneous System with an Invertible A

Example 6 Show that the following linear system has only the trivial solution.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 \qquad\quad + 8x_3 &= 0 \end{aligned}$$

Sol.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Example 3 showed that A is invertible. Thus, the linear system has only the trivial solution.

,

Theorem 3.3.8

(a) If A and B are square matrices such that $\mathbf{AB} = \mathbf{I}$ *or* $\mathbf{BA} = \mathbf{I}$*, then A and B are both invertible, and each is the inverse of the other.*

(b) If A and B are square matrices whose product is invertible, then A and B are invertible.

Proof

(a) Suppose that $\mathbf{BA} = \mathbf{I}$.

(b) If $\mathbf{AB}$ is invertible, then

$$(\mathbf{AB})(\mathbf{AB})^{-1} = \mathbf{A}(\mathbf{B}(\mathbf{AB})^{-1}) = \mathbf{I}$$

$$(\mathbf{AB})^{-1}(\mathbf{AB}) = ((\mathbf{AB})^{-1}\mathbf{A})\mathbf{B}$$

A Unifying Theorem

Theorem 3.3.9 *If A is an nxn matrix, then the following statements are equivalent.*

(a) *The RREF of A is I_n.*
(b) *A is expressible as a product of elementary matrices.*
(c) *A is invertible.*
(d) $A\mathbf{x}=\mathbf{0}$ *has only the trivial solution.*
(e) $A\mathbf{x}=\mathbf{b}$ *is consistent for every vector* $\mathbf{b}$ *in R^n.*
(f) $A\mathbf{x}=\mathbf{b}$ *has exactly one solution for every vector* $\mathbf{b}$ *in R^n.*

Proof

Solving Multiple Linear Systems with a Common Coeff. Matrix

Let's solve k linear systems.

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \ldots, \quad A\mathbf{x} = \mathbf{b}_k \tag{8}$$

A **poor** method: solve each linear system separately

Better procedure:

$$\mathbf{x}_1 = A^{-1}\mathbf{b}_1, \quad \mathbf{x}_2 = A^{-1}\mathbf{b}_2, \ldots, \quad \mathbf{x}_k = A^{-1}\mathbf{b}_k$$

$$[A \mid I] \quad \Big\downarrow \quad [I \mid A^{-1}]$$

$$[A \mid \mathbf{b}_1 \mid \mathbf{b}_2 \mid \cdots \mid \mathbf{b}_k] \quad \Big\downarrow \quad [I \mid A^{-1}\mathbf{b}_1 \mid A^{-1}\mathbf{b}_2 \mid \cdots \mid A^{-1}\mathbf{b}_k]$$

By Gauss-Jordan elimination

Example 7

Solve the linear systems.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_2 + 3x_3 &= 5 \\ x_1 \qquad\quad + 8x_3 &= 9 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 6 \\ x_1 \qquad\quad + 8x_3 &= -6 \end{aligned}$$

Sol.

$$\left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

Consistency of Linear Systems(해 존재성 문제)

The Consistency Problem For a given matrix A, find all vectors **b** for which the linear system A**x**=**b** is consistent.

▲If A is an invertible nxn matrix, then A**x**=**b** is consistent for every $\mathbf{b} \in R^n$.

$\mathbf{x}=A^{-1}\mathbf{b}$ is the solution.

▲If A is not square, or if A is square but not invertible, then the system will typically consistent for some vectors but not others.

REMARK

A linear system A**x**=**b** is always consistent for at least one vector. Why?

A**x**=**0** has a solution **x**=**0** for any A.

Example 8 Consistency Problem

What conditions must b_1, b_2, and b_3 satisfy for the following linear system to be consistent?

$$\begin{aligned} x_1 + x_2 + 2x_3 &= b_1 \\ x_1 \qquad + \ x_3 &= b_2 \\ 2x_1 + x_2 + 3x_3 &= b_3 \end{aligned}$$

Sol.

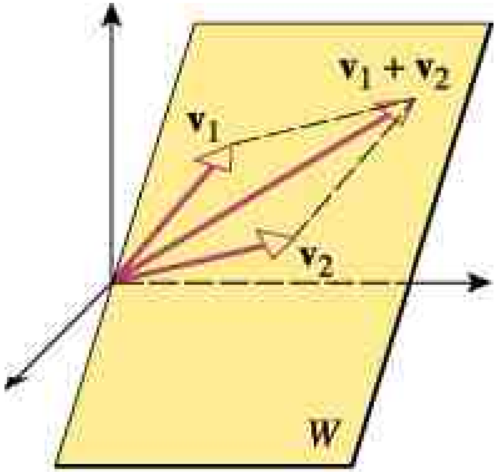
$$\begin{array}{r} \\ -R_1+R_2 \\ -2R_1+R_3 \end{array} \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix} \qquad \begin{array}{r} \\ -R_1+R_2 \\ -2R_1+R_3 \end{array} \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix}$$

$$\begin{array}{r} \\ -R_2 \\ \\ \end{array} \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix} \qquad \begin{array}{r} \\ \\ R_2+R_3 \end{array} \begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

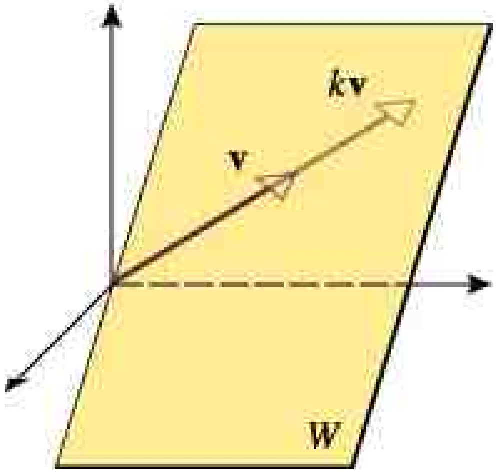
(b_1, b_2, b_3): on the plane passing through the origin, (1, 0, 1) and (0, 1, 1).

3.4 subspaces and Linear Independence



Closed under addition: If $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors that lie in a plane W, then $\mathbf{v}_1 + \mathbf{v}_2$ is also in W.

$$\mathbf{v}_1 \in W, \ \mathbf{v}_2 \in W \longrightarrow \mathbf{v}_1 + \mathbf{v}_2 \in W$$



Closed under scalar multiplication: If $\mathbf{v}$ is a vector that lie in a plane W and k is a scalar, then $k\mathbf{v}$ is also in W.

$$\mathbf{v} \in W, \ k : \text{scalar} \longrightarrow k\mathbf{v} \in W$$

Definition 3.4.1

Definition 3.4.1 A nonempty set of vectors in R^n is called a **subspace** of R^n if it is closed under scalar multiplication and addition.

Trivial subspaces of R^n

Zero subspace: {0}

R^n : a subspace of R^n

Concept Problem

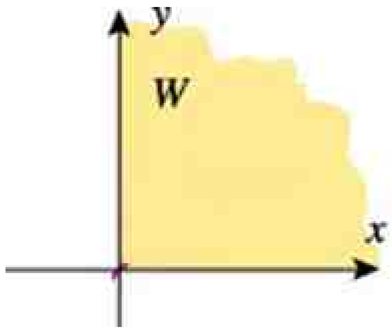
Every subspace must contain the vector **0**. Why?

closed under scalar multiplication:

0**v**=**0**. For closedness, 0 should be an element of every subspace.

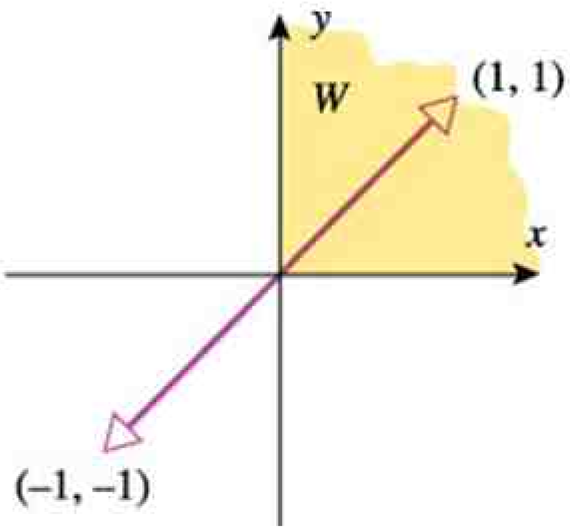
Example 1 A Subset of R^2 That Is Not a Subspace

Show that the set of all points (x, y) in R^2 with $x>0$ and $y>0$ is not a subspace.



Sol.

- *Closed Under addition*
- *Not closed under scalar multiplication for non-positive scalar*



Closed under Linear Combinations

W is a subspace if and only if

Closed under addition :

$$\mathbf{v}_1 \in W,\ \mathbf{v}_2 \in W \longrightarrow \mathbf{v}_1 + \mathbf{v}_2 \in W$$

Closed under scalar multiplication :

$$\mathbf{v} \in W,\ k : \text{scalar} \longrightarrow k\mathbf{v} \in W$$

$\Longleftrightarrow$ *Closed under linear combinations*

$$t_1,\ t_2 : \text{scalar},\quad \mathbf{v}_1 \in W,\ \mathbf{v}_2 \in W \longrightarrow t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \in W$$

Theorem 3.4.2

Theorem 3.4.2 *If* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ *are vectors in* R^n*, then the set of all linear combinations*

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_s\mathbf{v}_s \tag{3}$$

is a subspace of R^n.

The subspace W of R^n whose vectors satisfy (3) is called the **span** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ *and is denoted by*

$$W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\} \tag{4}$$

Example 2 Spanning the Trivial Subspaces

Show that

(a) the zero subspace {**0**} is spanned by **0**

(b) R^n is spanned by the standard unit vectors.

Sol.

(a) the zero subspace {**0**} is spanned by **0**

$$t_1\mathbf{0} + t_2\mathbf{0} = \mathbf{0} \in \{\mathbf{0}\}$$

(b) R^n is spanned by the standard unit vectors.

$$\begin{aligned}\mathbf{x} &= (x_1, x_2, \cdots, x_n) \\ &= x_1(1,0,\cdots,0) + x_2(0,1,\cdots,0) + \cdots + x_n(0,0,\cdots,1) \\ &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n\end{aligned}$$

Example 3 Spanning Lines and Planes Through the Origin

Express the following line by a span notation.

$$(x_1,\ x_2,\ x_3,\ x_4) = t(1,\ 3,\ -2,\ 5)$$

Sol.

$$\mathbf{x} = t\mathbf{v} \qquad \text{span}\{\mathbf{v}\}$$

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \qquad \text{span}\{\mathbf{v}_1,\ \mathbf{v}_2\}$$

$$t(1,\ 3,\ -2,\ 5) \qquad \text{span}\{(1, 3, -2, 5)\}$$

Example 4 A Complete List of Subspaces

Categorize all subspaces in R^2 and R^3.

Sol.

(a) in R^2

1. The zero subspace, $\{\mathbf{0}\}$
2. Lines through the origin
3. All of R^2

(b) in R^3.

1. The zero subspace, $\{\mathbf{0}\}$
2. Lines through the origin
3. Planes through the origin
4. All of R^3

Solution Space of a Linear System

Theorem 3.4.3 *If $A\mathbf{x}=0$ is a homogeneous linear system with n unknowns, then its solution set is a subspace of R^n.*

Proof

The solution set is nonempty since $\mathbf{x}=\mathbf{0}$ is a solution.

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solutions of the system.

Then, for scalars t_1 and t_2, for $\mathbf{x}=t_1\mathbf{x}_1+t_2\mathbf{x}_2$,

$$A\mathbf{x}=A(t_1\mathbf{x}_1+t_2\mathbf{x}_2)=t_1A\mathbf{x}_1+t_2A\mathbf{x}_2)= t_1\mathbf{0}+t_2\mathbf{0}=\mathbf{0}$$

Thus, $\mathbf{x}=t_1\mathbf{x}_1+t_2\mathbf{x}_2$ is also a solution for the system.

In general, the solution space must be expressible in the form

$$\mathbf{x}=t_1\mathbf{v}_1+t_2\mathbf{v}_2 + \cdots +t_s\mathbf{v}_s \qquad (5)$$

which is called a *general solution* of the system.

Example 5 A General Solution of a Homo. Linear System

Express the general solution by a span notation.

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Sol.

1. Comma-delimited form:

   $(x_1, \cdots, x_6)=r(-3, \cdots, 0)^T+s(-4,0,-2,1,0,0)^T+t(-2,0,0,0,1,0)^T$

2. Parametric form: $x_1=-3r-4s-2t$, $x_2=r$, $\cdots$, $x_6=t$

3. Span notation:

   $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

   where $\mathbf{v}_1=(-3,1,0,0,0,0)'$, $\mathbf{v}_2=(-4,0,-2,1,0,0)'$,

   $\mathbf{v}_3=(-2,0,0,0,1,0)'$

Example 6 Geometry of Homo. Systems in Two Unknowns

The solution space of a linear system in two unknowns is a subspace of R^2. The space must either be the origin **0**, a line through the origin, or all of R^2.

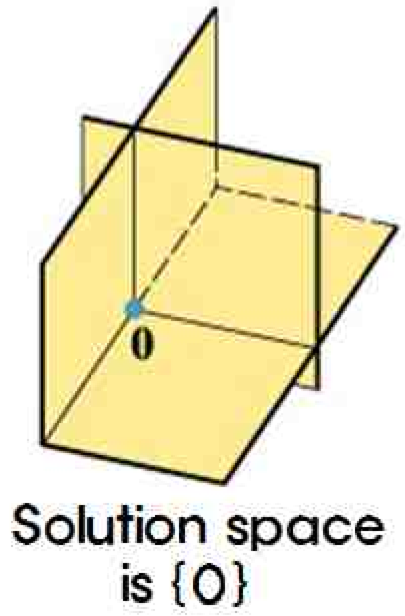
Show that the solution space of the linear system is R^2.

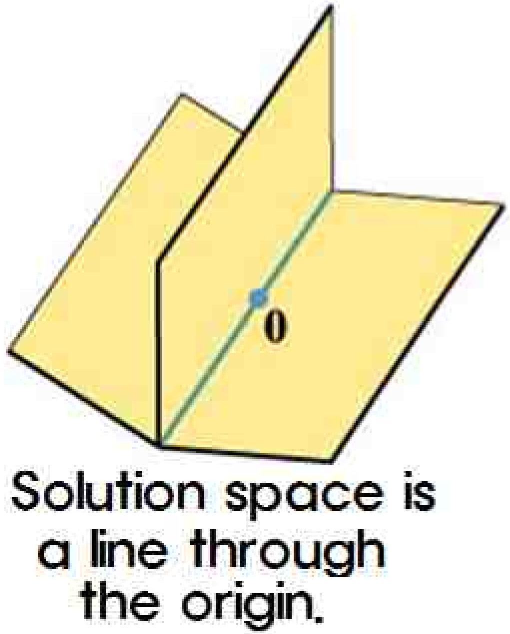
$$\begin{aligned} 0x + 0y &= 0 \\ 0x + 0y &= 0 \end{aligned}$$

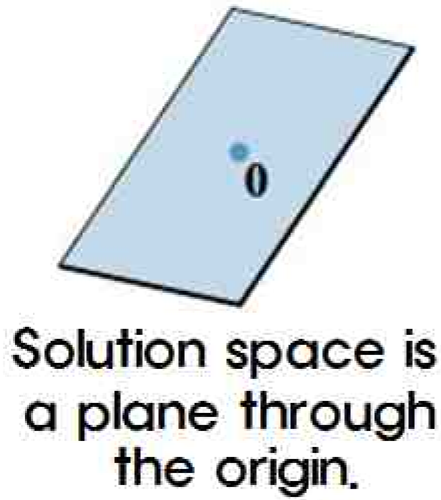
Sol.

Example 7 Geometry of Homogeneous Systems

The solution space of a linear system in three unknowns is a subspace of R^3. The space must either be the origin **0**, a line through the origin, , a plane through the origin, or all of R^3.







Theorem 3.4.4

(a) *If A is a matrix with n columns, then the solution space of the homogeneous system $A\mathbf{x}=\mathbf{0}$ is all of R^n if and only if $A=\mathbf{0}$.*

(b) *If A and B are matrices with n columns, then $A=B$ if and only if $A\mathbf{x}=B\mathbf{x}$ for every $\mathbf{x}$ in R^n.*

Proof

(a) $A\mathbf{x}=\mathbf{0}$ *for every x in* R^n *if and only if* $A=\mathbf{0}$

$A\mathbf{x}=\mathbf{0}$ *for every* $\mathbf{x}$ *in* R^n $\longrightarrow$ $A=\mathbf{0}$:

Let $\mathbf{I}=[\mathbf{e}_1 \mid \mathbf{e}_2 \mid \cdots \mid \mathbf{e}_n]$: nxn identity matrix

Then, $\mathbf{A}=\mathbf{AI}=\mathbf{A}[\mathbf{e}_1 \mid \mathbf{e}_2 \mid \cdots \mid \mathbf{e}_n]$

$$=[\mathbf{Ae}_1 \quad \mathbf{Ae}_2 \quad \cdots \quad \mathbf{Ae}_n]=[\mathbf{0} \quad \mathbf{0} \quad \cdots \quad \mathbf{0}]$$

$A\mathbf{x}=\mathbf{0}$ *for every* $\mathbf{x}$ *in* R^n $\longleftarrow$ $A=\mathbf{0}$:

$$\mathbf{Ax}=\mathbf{0x}=\mathbf{0}$$

Theorem 3.4.4-conti

(b) $A = B$ if and only if $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x}$ in R^n.

$A = B \longrightarrow A\mathbf{x} = B\mathbf{x}$ *for every* $\mathbf{x}$ *in* R^n.

$$\mathbf{Ax} = \mathbf{Bx}$$

$A = B \longleftarrow A\mathbf{x} = B\mathbf{x}$ *for every* $\mathbf{x}$ *in* R^n.

$$\mathbf{Ax} - \mathbf{Bx} = (\mathbf{A} - \mathbf{B})\mathbf{x} = \mathbf{0}$$

$\therefore\ \mathbf{A} = \mathbf{B}$ [by part (a)]

Linear Independence

Equation (8) presents the plane passing through the origin and parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ if

- $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero vectors, and
- $\mathbf{v}_1$ is not scalar multiple of $\mathbf{v}_2$.

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \tag{8}$$

If $\mathbf{v}_2 = c\mathbf{v}_1$, then

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 = t_1\mathbf{v}_1 + t_2(c\mathbf{v}_1) = (t_1 + ct_2)\mathbf{v}_1$$

A line rather than a plane

Definition 3.4.5 Linearly Independent

Definition 3.4.5 A nonempty set of vectors $S=\{\mathbf{v}_1,\ \mathbf{v}_2,\cdots,\ \mathbf{v}_s\}$ in R^n is linearly independent if the only scalars $c_1,\ c_2,\cdots,c_s$ that satisfy the equation

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_s\mathbf{v}_s=\mathbf{0} \qquad (9)$$

are $c_1=c_2=\cdots=c_s=0$.

If there are scalars, not all zero, that satisfy this equation, then the set is *linearly dependent.*

Example 8 Linear independency of a single vector

A single vector v≠0 is linearly independent since $c\mathbf{v}=\mathbf{0}$ iff $c=0$.

A zero vector v=0 is linearly dependent since $c\mathbf{v}=\mathbf{0}$ for c≠0.

Example 9 Sets Containing Zero Are Linearly Dependent

Show that the set $S = \{\mathbf{0}, \mathbf{v}_2, \cdots, \mathbf{v}_s\}$ is linearly dependent.

Sol.

$$c_1\mathbf{0} + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_s = \mathbf{0}$$

Thus, (9) is satisfied when there are scalars, not all zero.
Therefore, the set is linearly dependent.

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_s\mathbf{v}_s = \mathbf{0} \qquad (9)$$

Theorem 3.4.6

Theorem 3.4.6 A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_s\}$ in R^n with one or more vectors is linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of other vectors in S.

Proof

Linearly dependent → linear combination of other vectors in S

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_s\mathbf{v}_s = 0 \qquad (10)$$

Linearly dependence means there is at least one $c_i \neq 0$ that satisfies (10).

Without loss of generality, assume that $c_1 \neq 0$. Then

$$\mathbf{v}_1 = \left(-\frac{c_2}{c_1}\right)\mathbf{v}_2 + \cdots + \left(-\frac{c_s}{c_1}\right)\mathbf{v}_s$$

Theorem 3.4.6-conti

linearly dependent $\leftarrow$ linear combination of other vectors in S

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_s\mathbf{v}_s$$

$$\mathbf{v}_1 + (-c_2)\mathbf{v}_2 + (-c_3)\mathbf{v}_3 + \cdots + (-c_s)\mathbf{v}_s = \mathbf{0}$$

The equation shows that there are scalars, not all zero, which satisfy (10).

Example 10

Show that two vectors are linearly dependent if and only if one vector is a scalar multiple of the other.

Geometrically, two vectors are linearly dependent if they are collinear and linearly independent if they are not.

Sol.

linearly dependent → one vector is a scalar multiple of the other

There exist c_1 and c_2, not both zero, that satisfy

$$c_1\mathbf{v}_2 + c_2\mathbf{v}_2 = 0$$

Let's assume, without loss of generality, $c_1 \neq 0$, then

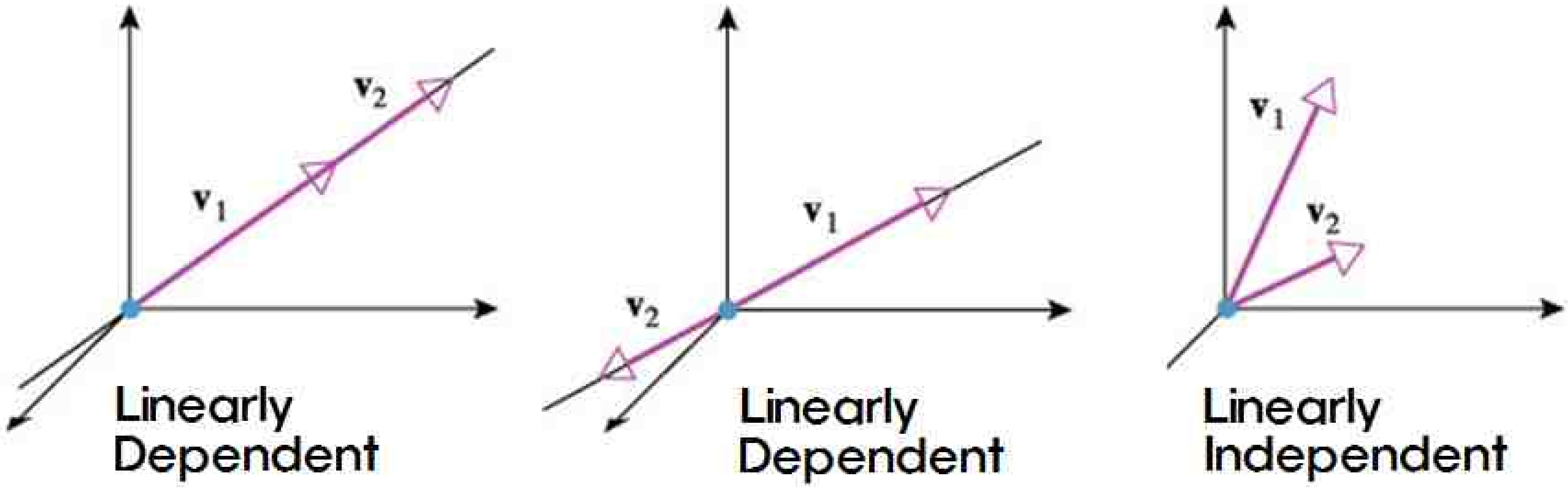
$$\mathbf{v}_1 = -(c_2 / c_1)\mathbf{v}_2$$

linearly dependent ← one vector is a scalar multiple of the other

$$\mathbf{v}_2 = c\,\mathbf{v}_1, \quad c\,\mathbf{v}_1 - \mathbf{v}_2 = 0$$

Example 10-conti

$\mathbf{v}_2 = c\,\mathbf{v}_1$: Linearly Dependent



Example 11 Linear Independence of Three Vectors

By Theorem 3.4.6, three vectors in R^n are linearly dependent iff at least one of them is a linear combination of the other two.

Show that if one of them is a linear combination of the other two, then the three vectors must lie in a common plane through the origin.

Thus, three vectors in R^n are linearly dependent if they line in a plane through the origin and are linearly independent if they do not.

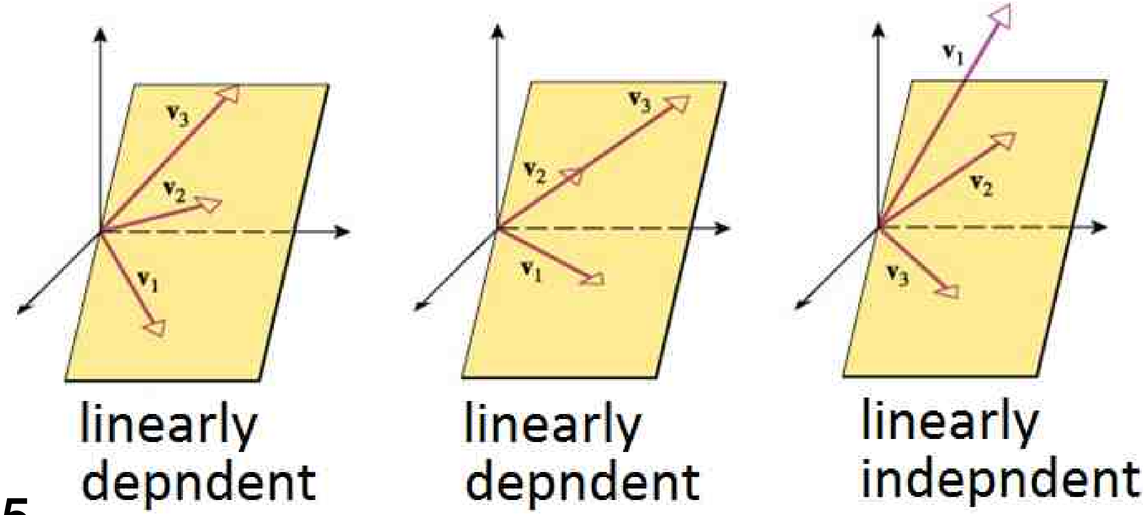


Fig. 3.4.5

Linear Independence and Homogeneous Linear Systems

Consider the nxs matrix A

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_s \end{bmatrix}$$

Then, by (10) of Section 3.1,

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_s \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (11)$$

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_s\mathbf{v}_s = \mathbf{0}$$

Thus, the problem of determining whether $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_s$ are linearly independent reduces to determining whether (11) has nontrivial solutions.

If nontrivial solutions, then linearly dependent.
If only the trivial solution, then linearly independent.

Theorem 3.4.7

Theorem 3.4.7 *A homogeneous linear system $A\mathbf{x}=\mathbf{0}$ has only the trivial solution if and only if the column vectors of A are linearly independent.*

Proof

$$A\mathbf{x} = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Example 12

Determine whether the following vectors are linearly independent.

$$\mathbf{v}_1 = (1,2,1),\ \mathbf{v}_2 = (2,5,0),\ \mathbf{v}_3 = (3,3,8)$$

$$\mathbf{v}_1 = (1,2,-1),\ \mathbf{v}_2 = (6,4,2),\ \mathbf{v}_3 = (4,-1,5)$$

$$\mathbf{v}_1 = (2,-4,6),\ \mathbf{v}_2 = (0,7,-5),\ \mathbf{v}_3 = (6,9,8),\ \mathbf{v}_4 = (5,0,1)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Ex 6 Sec 3.3

$$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Ex 4 Sec 3.5

$$\begin{bmatrix} 2 & 0 & 6 & 5 \\ -4 & 7 & 9 & 0 \\ 6 & -5 & 8 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

More variables than eq.s

Theorem 3.4.8

Theorem 3.4.8 *A set with more than n vectors in R^n is linearly dependent.*

Proof

Translated Subspaces(평행이동된 부분공간)

In R^2, $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ is a line passing the origin and parallel to $\mathbf{x} = t\mathbf{v}$

In R^3, $\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$ is a plane passing the origin and parallel to $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$

More generally, in R^n, $\mathbf{v} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_s\mathbf{v}_s$ is a plane passing the origin

$$W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_s\}$$

$$\mathbf{x}_0 + W$$

$$\mathbf{x}_0 + \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_s\}$$

Translation of subspaces:

Linear manifolds, flats, affine spaces

A Unifying Theorem(통합 정리)

Theorem 3.4.9 *If A is an nxn matrix, then the following statements are equivalent.*

(a) The RREF of A is I_n.
(b) A is expressible as a product of elementary matrices.
(c) A is invertible.
(d) $A\mathbf{x}=\mathbf{0}$ has only the trivial solution.
(e) $A\mathbf{x}=\mathbf{b}$ is consistent for every vector $\mathbf{b}$ in R^n.
(f) $A\mathbf{x}=\mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in R^n.
(g) The column vectors of A are linearly independent.
(h) The row vectors of A are linearly independent.

3.5 The Geometry of Linear Systems

$\mathbf{Ax} = \mathbf{0}$ is associated with $\mathbf{Ax} = \mathbf{b}$.

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

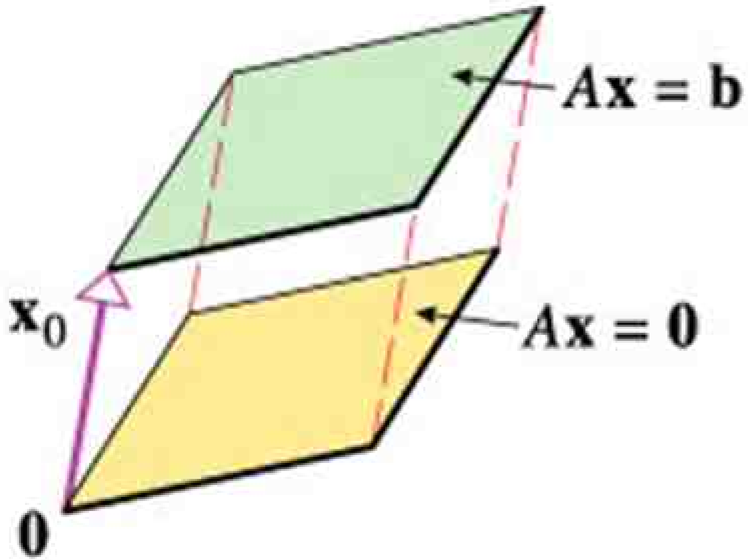
$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1/3 \end{bmatrix}^T$$

Theorem 3.5.1

Theorem 3.5.1 *If* $A\mathbf{x} = \mathbf{b}$ *is a consistent nonhomogeneous linear system, and if* W *is the solution space of the associated homogeneous system* $A\mathbf{x} = \mathbf{0}$, *then the solution set of* $A\mathbf{x} = \mathbf{b}$ *is the translated subspace* $\mathbf{x}_0 + W$, *where* $\mathbf{x}_0$ *is any solution of the nonhomogeneous system* $A\mathbf{x} = \mathbf{b}$ (*Figure* 3.5.1).

The solution set of $A\mathbf{x} = \mathbf{b}$ is a traslation of of the solution space of $A\mathbf{x} = \mathbf{0}$.



Theorem 3.5.2

Theorem 3.5.2 *A general solution of a consistent linear system* $A\mathbf{x} = \mathbf{b}$ *can be obtained by adding a particular solution of* $A\mathbf{x} = \mathbf{b}$ *to a general solution of* $A\mathbf{x} = \mathbf{0}$.

Particular solution $\mathbf{x}_0$

any solution satisfying $\mathbf{A}\mathbf{x} = \mathbf{b}$

General solution for $\mathbf{A}\mathbf{x} = \mathbf{0}$

$$\mathbf{x}_h = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_s\mathbf{v}_s$$

General solution for $\mathbf{A}\mathbf{x} = \mathbf{b}$

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_h = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_s\mathbf{v}_s$$

Example 1 The Geometry of Nonhomogeneous Linear Systems

The solution set of a consistent nonhomogeneous linear system is the translation of the solution space of the associated homogeneous system.

Solution Sets in R^2	Solution Sets in R^3
A point A line All of R^2	A point A line A Plane All of R^3

Theorem 3.5.3, 3.5.4

Theorem 3.5.3 *If A is an $m \times n$ matrix, then the following statements are equivalent.*

(a) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(b) $A\mathbf{x} = \mathbf{b}$ *has at most one solution for every* $\mathbf{b}$ *in* R^m (*i.e., is inconsistent or has a unique solution*).

Theorem 3.5.4 *A nonhomogeneous linear system with more unknowns than equations is either inconsistent or has infinitely many solutions.*

Consistency of a Linear System from the Point of View

Consistency of $A\mathbf{x} = \mathbf{b}$ is determined by the relationship between the vector b and the column vectors of A.

$$A\mathbf{x} = \mathbf{b}$$

$$A\mathbf{x} = [\mathbf{a}_1, \ \mathbf{a}_2, \ \cdots \ , \ \mathbf{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \qquad (5)$$

The system is consistent if and only if b can be expressed as a linear combinations of the column vectors of A.

If so, the solutions of the system are given by the coefficients in (5).

Theorem 3.5.5

Theorem 3.5.5 *A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.*

$$A\mathbf{x} = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Example 2 Linear Combinations Revisited

Express the vector $\mathbf{w} = (9, 1, 0)$ as a linear combination of the following vectors, if possible.

$$\mathbf{v}_1 = (1, 2, 3),\ \mathbf{v}_2 = (1, 4, 6),\ \mathbf{v}_3 = (2, -3, -5)$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix} \Rightarrow c_1 = 1,\ c_2 = 2,\ c_3 = 3$$

$$\mathbf{w} = \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3$$

Hyperplanes(초평면)

Hyperplane: The set of points $(x_1, x_2, \cdots, x_n)$ in R^n that satisfy the linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{8}$$

$(a_1, a_2, \cdots, a_n$ not all zero)

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \tag{9}$$

$$\mathbf{a} \cdot \mathbf{x} = b \quad (\mathbf{a} \neq \mathbf{0}) \tag{10}$$

Hyperplane through the origin with normal $\mathbf{a}$

or Orthogonal complement of $\mathbf{a}$

Symbol : $\mathbf{a}^{\perp}$ (read, $\mathbf{a}$ perp)

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

$$\mathbf{a} \cdot \mathbf{x} = 0 \quad (\mathbf{a} \neq \mathbf{0}) \tag{11}$$

Example 3 Finding an Equation for a Hyperplane

Find a hyperplane $\mathbf{a}^{\perp}$ when $\mathbf{a} = (1, -2, 4)$.

Sol.

Hyperplane:

$$(1, -2, 4) \cdot (x, y, z) = x - 2y + 4z = 0 \tag{12}$$

Parametric eq.:

$$x = 2t_1 - 4t_2, \quad y = t_1, \quad z = t_2$$

Geometric Interpretations of Solution Spaces

Intersection of hyperplanes

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array} \qquad (13)$$

$\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$: row vectors

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= 0 \\ \mathbf{a}_2 \cdot \mathbf{x} &= 0 \\ &\cdots \\ \mathbf{a}_n \cdot \mathbf{x} &= 0 \end{aligned} \qquad (14)$$

Theorem 3.5.6

Theorem 3.5.6 *If A is an $m \times n$ matrix, then the solution space of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in R^n that are orthogonal to every row vector of A.*

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array} \tag{13}$$

$\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$: row vectors of A

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= 0 \\ \mathbf{a}_2 \cdot \mathbf{x} &= 0 \\ &\cdots \\ \mathbf{a}_n \cdot \mathbf{x} &= 0 \end{aligned} \tag{14}$$

Example 4

Show that dot products of each row of A and a solution for A**x**=**0** is zero.

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

Sol.

Example 7 of Sec. 2.2 :

$$x_1 = 3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

$$\mathbf{x} = (-3r - 4s - 2t,\ r,\ -2s,\ s,\ t,\ 0)$$

$$\mathbf{a}_1 = (1,\ 3,\ -2,\ 0,\ 2,\ 0)$$

$$\mathbf{a}_1 \cdot \mathbf{x} = 1(-3r - 4s - 2t) + 3(r) + (-2)(-2s) + 0(s) + 2(t) + 0(0) = 0$$

Similarly,

$$\mathbf{a}_2 \cdot \mathbf{x} = 0, \quad \mathbf{a}_3 \cdot \mathbf{x} = 0, \quad \mathbf{a}_4 \cdot \mathbf{x} = 0$$

3.6 Matrices with Special Forms

Diagonal Matrix

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}, \text{ or } (\mathbf{D})_{ij} = 0 \quad \text{for } i \neq j \tag{1}$$

$$\mathbf{D}^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix} \quad \text{If all } d_i \neq 0 \tag{2}$$

$$\mathbf{D}^{k} = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix} \tag{3}$$

For positive k

For negative k and $d_i \neq 0$

Example 1

Find A^{-1}, A^5, and A^{-5} where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol.

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$\mathbf{A}^{5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix}$$

$$\mathbf{A}^{-5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/243 & 0 \\ 0 & 0 & 1/32 \end{bmatrix}$$

Matrix Products Involving Diagonal Matrices

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & d_1 a_{13} & d_1 a_{14} \\ d_2 a_{21} & d_2 a_{22} & d_2 a_{23} & d_2 a_{24} \\ d_3 a_{31} & d_3 a_{32} & d_3 a_{33} & d_3 a_{34} \end{bmatrix}$$

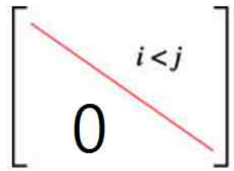
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_2 a_{12} & d_3 a_{13} \\ d_1 a_{21} & d_2 a_{22} & d_3 a_{23} \\ d_1 a_{31} & d_2 a_{32} & d_3 a_{33} \\ d_1 a_{41} & d_2 a_{42} & d_3 a_{43} \end{bmatrix}$$

Triangular Matrices

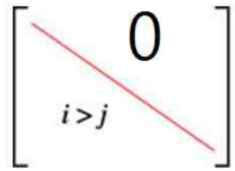
Triangular Matrix : A square matrix

Lower Triangular Matrix(하부 삼각행렬):
all the entries above the main diagonal are zero

Upper Triangular Matrix(상부 삼각행렬):
all the entries below the main diagonal are zero



Upper Triangular Matrix



Lower Triangular Matrix

Example 2

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Upper Triangular

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Lower Triangular

Example 3 Triangular Matrices

Row Echelon Form: Upper Triangular Matrix

Reduced Row Echelon Form: Diagonal Matrix

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Upper Triangular Matrices

Properties of Triangular Matrices

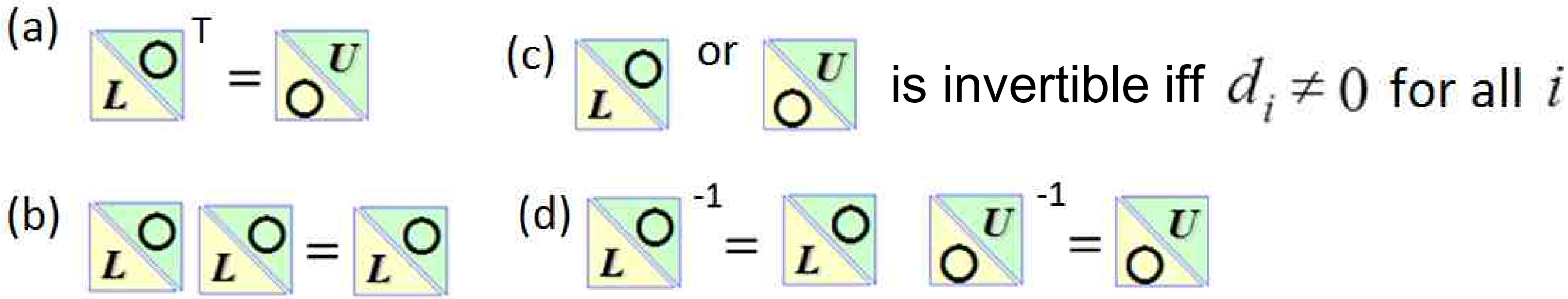
Theorem 3.6.1

(*a*) *The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.*

(*b*) *A product of lower triangular matrices is lower triangular, and a product of upper triangular matrices is upper triangular.*

(*c*) *A triangular matrix is invertible if and only if its diagonal entries are all nonzero.*

(*d*) *The inverse of an invertible lower triangular matrix is lower triangular, triangular, and the inverse of an invertible upper triangular matrix is upper triangular.*



Example 4

Consider the upper triangular matrices 와 $\mathbf{A}=\begin{bmatrix}1&3&-1\\0&2&4\\0&0&5\end{bmatrix}$ $\mathbf{B}=\begin{bmatrix}3&-2&2\\0&0&-1\\0&0&1\end{bmatrix}$

Show that, by direct calculation, A is invertible but B is not.

Sol.

By Theorem 3.6.4, A is invertible but B is not.

By direct calculation,

$$\mathbf{A}^{-1}=\begin{bmatrix}1&-3/2&7/5\\0&1/2&-2/5\\0&0&1/5\end{bmatrix},\quad \mathbf{AB}=\begin{bmatrix}3&-2&-2\\0&0&2\\0&0&5\end{bmatrix},\quad \mathbf{BA}=\begin{bmatrix}3&5&-1\\0&0&-5\\0&0&5\end{bmatrix}$$

Symmetric and Skew-Symmetric Matrices

A square matrix A is

Symmetric(대칭행렬) if $\mathbf{A}^T = \mathbf{A}$ or $(\mathbf{A})_{ij} = (\mathbf{A})_{ji}$

Skew-symmetric(반대칭행렬) if $\mathbf{A}^T = -\mathbf{A}$ or $(\mathbf{A})_{ij} = -(\mathbf{A})_{ji}$

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & -6 \\ 5 & -6 & 7 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

Symmetric Matrices

$$\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -4 & 5 \\ 4 & 0 & -9 \\ -5 & 9 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Skew-symmetric Matrices

Theorem 3.6.2

Theorem 3.6.2 *If A and B are symmetric matrices with the same size, and if k is any scalar, then:*

(a) A^T *is symmetric.*

(b) $A + B$ *and* $A - B$ *are symmetric.*

(c) kA *is symmetric.*

Theorem 3.6.3 *The product of two symmetric matrices is symmetric if and only if the matrices commute.*

AB is symmetric iff AB=BA.

Example 5

Show that the product AB is not symmetric where A and B are symmetric given as

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}$$

Sol.

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{BA} \neq \mathbf{AB}$$

Invertibility of Symmetric Matrices

Theorem 3.6.4 *If A is an invertible symmetric matrix, then* A^{-1} *is symmetric.*

Proof

$$(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} = \mathbf{A}^{-1}$$

Matrices of the Form AA^T and A^TA

AA^T and A^TA is always symmetric, since

$$(\mathbf{A}\mathbf{A}^T)^T = (\mathbf{A}^T)^T\mathbf{A}^T = \mathbf{A}\mathbf{A}^T \tag{7}$$

$$A^TA = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} \mathbf{a}_1^T\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{a}_2 & \cdots & \mathbf{a}_1^T\mathbf{a}_n \\ \mathbf{a}_2^T\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{a}_2 & \cdots & \mathbf{a}_2^T\mathbf{a}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_n^T\mathbf{a}_1 & \mathbf{a}_n^T\mathbf{a}_2 & \cdots & \mathbf{a}_n^T\mathbf{a}_n \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{a}_1 & \mathbf{a}_1 \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_1 \cdot \mathbf{a}_n \\ \mathbf{a}_2 \cdot \mathbf{a}_1 & \mathbf{a}_2 \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_2 \cdot \mathbf{a}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_n \cdot \mathbf{a}_1 & \mathbf{a}_n \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_n \cdot \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \|\mathbf{a}_1\|^2 & \mathbf{a}_1 \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_1 \cdot \mathbf{a}_n \\ \mathbf{a}_1 \cdot \mathbf{a}_2 & \|\mathbf{a}_2\|^2 & \cdots & \mathbf{a}_2 \cdot \mathbf{a}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_1 \cdot \mathbf{a}_n & \mathbf{a}_2 \cdot \mathbf{a}_n & \cdots & \|\mathbf{a}_n\|^2 \end{bmatrix} \tag{9}$$

by (23) of Section 3.1

Theorem 3.6.5

Theorem 3.6.5 *If A is a square matrix, then the matrices* A, AA^T *and* A^TA *are either all invertible or singular.*

Proof

P5

Fixed Points of a Matrix

For square matrix A, the solutions of $\mathbf{Ax} = \mathbf{x}$ are called *fixed points of A*, since they remain unchanged when multiplied by A.

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0} \tag{10}$$

EXAMPLE 6 Find the fixed points of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Sol.

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0} \qquad \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = 0, \quad x_2 = t \qquad \mathbf{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

$$\mathbf{Ax} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = \mathbf{x}$$

A Technique for Inverting I-A When A is Nilpotent

In the ordinary algebra,

$$(1-x)(1+x+x^2+\cdots+x^{k-1})=1-x^k$$

Similarly, in matrix algebra for square A,

$$(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{k-1})=\mathbf{I}-\mathbf{A}^k \qquad (11)$$

If $\mathbf{A}^k=\mathbf{0}$, then

$$(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^{k-1})=\mathbf{I}$$

$\mathbf{I}-\mathbf{A}$ is invertible.

A square matrix A is called *nilpotent* if $\mathbf{A}^k=\mathbf{0}$ for some positive integer k, and the smallest k is called the *index of nilpotency* (멱영 지표).

Theorem 3.6.6

Theorem 3.6.6 *If A is a square matrix, and if there is a positive integer k such that $A^k = 0$, A is invertible and then the matrix $I - A$ is invertible and*

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1} \tag{12}$$

,

EXAMPLE 7 Show that

(a) A is nilpotent, and

(b) find the inverse of $\mathbf{I} - \mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol.

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{A}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{I} - \mathbf{A} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverting I-A by Power series

For $0 < x < 1$,

$$(1-x)(1+x+x^2+\cdots+x^{k-1}) = 1-x^k \approx 1 \qquad (13)\ (14)$$

$$(1-x)(1+x+x^2+x^3+\cdots) = 1$$

Similarly,

$$(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\cdots) = \mathbf{I} \qquad (15)$$

If $\mathbf{I}-\mathbf{A}$ is invertible, then

$$(\mathbf{I}-\mathbf{A})^{-1} = \mathbf{I}+\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\cdots \qquad (16)$$

$$\approx \mathbf{I}+\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\cdots+\mathbf{A}^k \qquad (17)$$

Theorem 3.6.7 Power Series Representation of $(I\text{-}A)^{-1}$

Theorem 3.6.7 *If A is an $n \times n$ matrix for which the sum of the absolute values of the entries in each column (or each row) is less than 1, then $I - A$ is invertible and can be expressed as*

$$(I - A)^{-1} = I + A + A^2 + A^3 + \cdots \tag{18}$$

EXAMPLE 8 Find the inverse of I-A for the matrix given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1/4 & 1/8 \\ 1/4 & 1/5 & 1/6 \\ 1/7 & 1/8 & 1/9 \end{bmatrix}$$

Sol.

The sum of absolute values of the entries in each column(or row) is less than 1. Thus, the condition in Theorem 3.6.7 is satisfied.

Example 8 Power Series Representation of $(I\text{-}A)^{-1}$ -cont.

$$(\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} 1 & -1/4 & -1/8 \\ -1/4 & 4/5 & -1/6 \\ -1/7 & -1/8 & 8/9 \end{bmatrix}^{-1}$$

$$\approx \mathbf{I}+\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\cdots+\mathbf{A}^k$$

$$\approx \begin{bmatrix} 1.1305 & 0.3895 & 0.2320 \\ 0.4029 & 1.4266 & 0.3241 \\ 0.2384 & 0.2632 & 1.2079 \end{bmatrix} \tag{19}$$

$k=2$	$k=5$	$k=10$	$k=12$
$\begin{bmatrix} 1.0804 & 0.3156 & 0.1806 \\ 0.3238 & 1.3233 & 0.2498 \\ 0.1900 & 0.1996 & 1.1621 \end{bmatrix}$	$\begin{bmatrix} 1.1248 & 0.3819 & 0.2265 \\ 0.3947 & 1.4154 & 0.3162 \\ 0.2333 & 0.2564 & 1.2030 \end{bmatrix}$	$\begin{bmatrix} 1.1304 & 0.3894 & 0.2319 \\ 0.4027 & 1.4263 & 0.3240 \\ 0.2382 & 0.2631 & 1.2078 \end{bmatrix}$	$\begin{bmatrix} 1.1305 & 0.3895 & 0.2320 \\ 0.4029 & 1.4265 & 0.3241 \\ 0.2383 & 0.2632 & 1.2078 \end{bmatrix}$

3.7 Matrix Factorizations; LU-Decomposition

SOLVING LINEAR SYSTEMS BY FACTORIZATION

Primary Goal in this section: factoring a square matrix in the form

$$A=LU \tag{1}$$

where L: Lower triangular matrix
U: Upper triangular matrix

Solving a Linear System: LU decomposition

Step 1. Rewrite the system $A\mathbf{x}=\mathbf{b}$ as

$$LU\mathbf{x}=\mathbf{b} \tag{2}$$

Step 2. Rewrite the system $A\mathbf{x}=\mathbf{b}$ as

$$L\mathbf{y}=\mathbf{b} \quad \text{where } \mathbf{y}=U\mathbf{x} \tag{3}$$

Step 3. Solve the system $L\mathbf{y}=\mathbf{b}$ for $\mathbf{y}$.

Step 4. Solve the system $U\mathbf{x}=\mathbf{y}$ for $\mathbf{x}$.

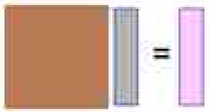
Example 1 Solving A**x**=**b** by LU-Decomposition

Solve A**x**=**b** by LU-decomposition where

$$A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

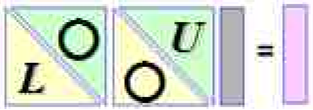
$$\mathbf{b} = \begin{bmatrix} 2 & 2 & 3 \end{bmatrix}^{\mathrm{T}}$$

Sol.



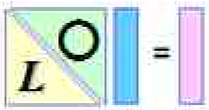
$$A\mathbf{x} = \mathbf{b} \qquad \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

Step 1. LUx=b



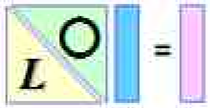
$$LU\mathbf{x} = \mathbf{b} \qquad \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \tag{5}$$

Step 2. Ly=b



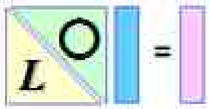
$$L\mathbf{y} = \mathbf{b} \qquad \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{6}$$

Example 1 Solving Ax=b by LU-Decomposition-conti



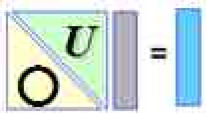
$$Ly = \mathbf{b} \qquad \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \tag{7}$$

Step 3. Solve the system *L* **y**=**b** for **y**.



$$Ly = \mathbf{b} \qquad \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$$

Step 4. Solve the system *U* **x**=**y** for **x**.



$$U\mathbf{x} = \mathbf{y} \qquad \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Definition 3.7.1

Definition 3.7.1 A factorization of a square matrix A as $A=LU$, where L is lower triangular and U is upper triangular, is called an ***LU-decomposition*** or ***LU-factorization*** of A.

In general, not every matrix A has an LU-decomposition, nor is an LU-decomposition is unique if it exists.

If A can be reduced to row echelon form by Gaussian elimination ***without row interchanges***, then A must have an LU-decomposition.

There exists a sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \ldots E_2 E_1 A = U \tag{8}$$
$$A = (E_k \ldots E_2 E_1)^{-1} U = E_1^{-1} E_2^{-1} \ldots E_k^{-1} U = LU$$

Theorem 3.7.2 LU-decomposition

There exists a sequence of elementary matrices E_1, E_2, …, E_k such that

$$E_k \dots E_2 E_1 A = U \tag{8}$$
$$A = (E_k \dots E_2 E_1)^{-1} U = E_1^{-1} E_2^{-1} \dots E_k^{-1} U = LU$$

where

$$L = E_1^{-1} E_2^{-1} \dots E_k^{-1} \tag{10}$$

Theorem 3.7.2 *If a square matrix A can be reduced to row echelon form by Gaussian elimination with no row interchanges, then A has an LU-decomposition.*

A Procedure for an LU-decomposition

A procedure for an LU-decomposition:

- Reduce A to a reduced row echelon form U.
- Find elementary matrices such that

$$E_k \dots E_2 E_1 A = U$$

- $L = E_1^{-1} E_2^{-1} \dots E_k^{-1}$
- $A = LU$

Step 1. A → REF U without row interchanges

Step 2. For main diagonal elements, placed the reciprocal of the multipliers in the position of leading 1 in U.

Step 3. In each position below the main diagonal of L, place the negative of the multiplier in that position in U.

Step 4. A=LU.

Example 2 Constructing an LU-Decomposition

Find an *LU*-decomposition of $A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}$

Sol.

$$A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} \qquad \begin{bmatrix} \bullet & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$$

• : unknown entries

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} \text{x } \frac{1}{6} \qquad \begin{bmatrix} 6 & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 2 & 1 \\ 0 & 8 & 5 \end{bmatrix} \begin{matrix} \\ \text{x } (-9) \\ \text{x } (-3) \end{matrix} \qquad \begin{bmatrix} 6 & 0 & 0 \\ 9 & \bullet & 0 \\ 3 & \bullet & \bullet \end{bmatrix}$$

Example 2-conti

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 8 & 5 \end{bmatrix} \times \frac{1}{2} \qquad \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \times (-8) \qquad \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & \bullet \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \times 1 \qquad L = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix}$$

$$E_k \cdots E_2 E_1 A = U \qquad L = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U = LU \qquad (E_k \cdots E_2 E_1) L = I$$

Gaussian Elimination and LU-Decomposition

Assume $E_k \cdots E_2 E_1 A = U$ (REF)

Then,

$$
\begin{aligned}
A\mathbf{x} &= \mathbf{b} \\
E_k \cdots E_2 E_1 A\mathbf{x} &= E_k \cdots E_2 E_1 \mathbf{b} \\
U\mathbf{x} &= E_k \cdots E_2 E_1 \mathbf{b} = \mathbf{y}
\end{aligned}
$$

Thus, the same sequence of matrices $E_k \cdots E_2 E_1$ produces **y** from **b**.

Example 3 Gaussian Elimination and LU-Decomposition

Solve the linear system
$$\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \tag{12}$$

Sol.

$$A\mathbf{x} = \mathbf{b} \qquad \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

$$[A \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 2 & 6 & 2 & 2 \\ -3 & -8 & 0 & 2 \\ 4 & 9 & 2 & 3 \end{array}\right] \quad \begin{bmatrix} \bullet & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ -3 & -8 & 0 & 2 \\ 4 & 9 & 2 & 3 \end{array}\right] \quad \begin{bmatrix} 2 & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & -3 & -2 & -1 \end{array}\right] \quad \begin{bmatrix} 2 & 0 & 0 \\ -3 & \bullet & 0 \\ 4 & \bullet & \bullet \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 7 & 14 \end{array}\right] \quad \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & \bullet \end{bmatrix}$$

Example 3-Conti

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 7 & 14 \end{array}\right] \quad \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & \bullet \end{bmatrix}$$

$$[U|\mathbf{y}] = \left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix} = L$$

$$U\mathbf{x} = \mathbf{y}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$$

$$x_1 = 2, \;\; x_2 = -1, \;\; x_3 = 2$$

Matrix Inversion by LU-Decomposition

Let $A^{-1}=[\mathbf{x}_1\ \mathbf{x}_2\ \dots\ \mathbf{x}_n]$ and $I=[\mathbf{e}_1\ \mathbf{e}_2\ \dots\ \mathbf{e}_n]$.

Then, $AA^{-1}=A[\mathbf{x}_1\ \mathbf{x}_2\ \dots\ \mathbf{x}_n]=[A\mathbf{x}_1\ A\mathbf{x}_2\ \dots\ A\mathbf{x}_n]=[\mathbf{e}_1\ \mathbf{e}_2\ \dots\ \mathbf{e}_n]$.

$$A\mathbf{x}=\mathbf{e}_1,\ A\mathbf{x}=\mathbf{e}_2,\ \dots,\ A\mathbf{x}=\mathbf{e}_n \tag{13}$$

Thus, the inverse of A is determined by solving a set of linear systems in (13).

LDU-Decompositions

A=LU

The matrix U has 1's on the main diagonal but L need not.

If it is preferred to have 1's on the main diagonal od L, then we can "shift" the diagonal entries to a diagonal matrix D.

$$L = L'D$$

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_{21}/a_{11} & 1 & 0 \\ a_{31}/a_{11} & a_{32}/a_{22} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Example:

$$\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Using Permutation Matrices to Deal with Row Interchanges

SKIP

Flops and the Cost of Solving a Linear System

SKIP

Cost Estimates for Solving Large Linear Systems

Table 3.7.1

Approximate Cost for an $n \times n$ Matrix A with Large n	
Algorithm	**Cost in Flops**
Gauss-Jordan elimination(forward phase) [Gaussian elimination]	$\approx \frac{2}{3}n^3$
Gauss-Jordan elimination(backward phase) [back substitution]	$\approx n^2$
LU-decomposition of A	$\approx \frac{2}{3}n^3$
Forward substitution to solve L**y**=**b**	$\approx n^2$
Backward substitution to solve U**x**=**y**	$\approx n^2$
A^{-1} by reducing [A | I] to [I | A^{-1}]	$\approx 2n^3$
Compute A^{-1}**b**	$\approx 2n^3$

Example 4 Cost of Solving a Large Linear System

Approximate the time required to execute the forward and backward phases of Gauss-Jordan elimination for a system of 10,000 unknowns using a computer that can execute 10 gigaflops per second.

Sol.

$n = 10{,}000$

10 gigaflops per sec=10-1 gigaflops per sec

gigaflops for forward phase≈ $\frac{2}{3}n^3 \times 10^{-9} = \frac{2}{3}(10^4)^3 \times 10^{-9} = \frac{2}{3} \times 10^3$

➡ time for forward phase ≈$(\frac{2}{3} \times 10^3) \times 10^{-1}$s ≈66.67s

gigaflops for backward phase≈ $n^2 \times 10^{-9} = (10^4)^2 \times 10^{-9} = 10^{-1}$

➡ time for backward phase ≈$(10^{-1}) \times 10^{-1}$s =0.01s

Considerations in Choosing Algorithm for Solving a Linear System

SKIP

3.8 Partitioned Matrices and Parallel Processing

GENERAL PARTITIOING

$$\mathbf{A} = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

$$\mathbf{A}_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad \mathbf{A}_{12} = \begin{bmatrix} a_{14} \\ a_{24} \end{bmatrix}$$

$$\mathbf{A}_{21} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{A}_{22} = \begin{bmatrix} a_{34} \end{bmatrix}$$

If partitioned appropriately, block multiplication is allowed.

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \\ \mathbf{A}_{31} & \mathbf{A}_{32} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{AB} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \\ \mathbf{A}_{31} & \mathbf{A}_{32} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} & \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} & \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \\ \mathbf{A}_{31}\mathbf{B}_{11} + \mathbf{A}_{32}\mathbf{B}_{21} & \mathbf{A}_{31}\mathbf{B}_{12} + \mathbf{A}_{32}\mathbf{B}_{22} \end{bmatrix}$$

Example 1 Block Multiplication

Show that block multiplication gives the correct result.

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}\begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{21} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} \end{bmatrix}$$

where $\mathbf{A} = \left[\begin{array}{ccc|cc} 3 & -4 & 1 & 0 & 2 \\ -1 & 5 & -3 & 1 & 4 \\ \hline 2 & 0 & -2 & 1 & 6 \end{array}\right] = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$ $\qquad \mathbf{B} = \left[\begin{array}{cc} 2 & -1 \\ 3 & 0 \\ -5 & 1 \\ \hline 4 & -3 \\ 0 & 2 \end{array}\right] = \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{21} \end{bmatrix}$

Sol.

$$\mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} = \begin{bmatrix} 3 & -4 & 1 \\ -1 & 5 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \\ -5 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -11 & 2 \\ 32 & 3 \end{bmatrix}$$

$$\mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} = \begin{bmatrix} 2 & 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \\ -5 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 6 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 18 & 5 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 3 & -4 & 1 & 0 & 2 \\ -1 & 5 & -3 & 1 & 4 \\ 2 & 0 & -2 & 1 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 0 \\ -5 & 1 \\ 4 & -3 \\ 0 & 2 \end{bmatrix} = \left[\begin{array}{cc} -11 & 2 \\ 32 & 3 \\ \hline 18 & 5 \end{array}\right]$$

Theorem 3.8.1 Column-Row Rule

Theorem 3.8.1 **(*Column-Row Rule*)** *If A has size $m \times s$ and B has size $s \times n$, and if these matrices are partitioned into column and row vectors as*

$$A = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \cdots \quad \mathbf{c}_s] \quad \textit{and} \quad B = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_s \end{bmatrix}$$

then

$$AB = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_s\mathbf{r}_s \tag{2}$$

Remark Formula (2) is sometimes called the **outer product rule** because it expresses AB as a sum of column vector times row vectors(outer products).

Example 2 tr(AB)=tr(BA)

Show that tr(AB)=tr(BA).

Sol.

$$\begin{aligned}
\mathrm{tr}(\mathbf{BA}) &= \mathbf{r}_1\mathbf{c}_1 + \mathbf{r}_2\mathbf{c}_2 + \cdots + \mathbf{r}_s\mathbf{c}_s \\
&= \mathrm{tr}(\mathbf{c}_1\mathbf{r}_1) + \mathrm{tr}(\mathbf{c}_2\mathbf{r}_2) + \cdots + \mathrm{tr}(\mathbf{c}_s\mathbf{r}_s) \\
&= \mathrm{tr}(\mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_s\mathbf{r}_s) \\
&= \mathrm{tr}(\mathbf{AB})
\end{aligned}$$

Block Diagonal Matrices

A partitioned matrix A is ***block diagonal*** if the matrices on the main diagonal are square and all matrices off the main diagonal are zero; that is, the matrix is partitioned as

$$\mathbf{A} = \begin{bmatrix} \mathbf{D}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{D}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{D}_k \end{bmatrix} \quad \mathbf{D}_1, \mathbf{D}_2, \cdots, \mathbf{D}_k \text{ : Square Matrices}$$

then

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{D}_1^{-1} & 0 & \cdots & 0 \\ 0 & \mathbf{D}_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{D}_k^{-1} \end{bmatrix}$$

Example 3

Find the inverse of the matrix

$$\mathbf{A} = \left[\begin{array}{cc|cc|c} 8 & -7 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 5 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 4 \end{array}\right]$$

Sol.

$$\begin{bmatrix} 8 & -7 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -7 \\ 1 & -8 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \left[\begin{array}{cc|cc|c} 1 & -7 & 0 & 0 & 0 \\ 1 & -8 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -5 & 3 & 0 \\ \hline 0 & 0 & 0 & 0 & 1/4 \end{array}\right]$$

Block Upper Triangular Matrices

A partitioned square matrix *A* is ***block upper triangular*** if the matrices on the main diagonal are square and all matrices below the main diagonal are zero; that is, the matrix is partitioned as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1k} \\ \mathbf{0} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2k} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_{kk} \end{bmatrix}$$

Block upper triangular matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} \Longrightarrow \mathbf{A}^{-1} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0} & \mathbf{A}_{22}^{-1} \end{bmatrix}$$

Block Lower Triangular Matrices

A partitioned square matrix *A* is ***block lower triangular*** if the matrices on the main diagonal are square and all matrices above the main diagonal are zero; that is, the matrix is partitioned as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}_{k1} & \mathbf{A}_{k2} & \cdots & \mathbf{A}_{kk} \end{bmatrix}$$

Block lower triangular matrix

Example 4

Confirm that the given matrix is invertible block upper triangular matrix, and find its inverse by using Formula (6).

$$\mathbf{A} = \begin{bmatrix} 4 & 7 & -5 & 3 \\ 3 & 5 & 3 & -2 \\ 0 & 0 & 7 & 2 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Sol.

$$\mathbf{A} = \left[\begin{array}{cc|cc} 4 & 7 & -5 & 3 \\ 3 & 5 & 3 & -2 \\ \hline 0 & 0 & 7 & 2 \\ 0 & 0 & 3 & 1 \end{array}\right] = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} \mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0} & \mathbf{A}_{22}^{-1} \end{bmatrix}$$

$$\mathbf{A}_{11} = \begin{bmatrix} 4 & 7 \\ 3 & 5 \end{bmatrix}, \ \mathbf{A}_{12} = \begin{bmatrix} -5 & 3 \\ 3 & -2 \end{bmatrix}, \ \mathbf{A}_{22} = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$$

$$\mathbf{A}_{11}^{-1} = \begin{bmatrix} -5 & 7 \\ 3 & -4 \end{bmatrix} \qquad \mathbf{A}_{22}^{-1} = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}$$

$$-\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1} = -\begin{bmatrix} -5 & 7 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} -5 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} -133 & 295 \\ 78 & -173 \end{bmatrix}$$

Example 4-conti

$$\mathbf{A}_{11}^{-1} = \begin{bmatrix} -5 & 7 \\ 3 & -4 \end{bmatrix} \qquad -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1} = \begin{bmatrix} -133 & 295 \\ 78 & -173 \end{bmatrix}$$

$$\mathbf{A}_{22}^{-1} = \begin{bmatrix} 1 & -2 \\ -3 & 7 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0} & \mathbf{A}_{22}^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 7 & -133 & 295 \\ 3 & -4 & 78 & -173 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -3 & 7 \end{bmatrix}$$