Introduction to Discrete Math

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Course Outline

- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion & Induction, Logic, Invariants
- **Probability & Combinatronics**
  - Basic Counting, Binomial Coeff, Advanced Counting, **Probability**, Random Variables

Probability & Combinatronics - Probability

CONDITIONAL PROBABILITY

- **What is Conditional Probability**
- How Reliable is the Test
- Bayes' Theorem
- Conditional Probability: A Paradox
- Past and Future
- Independence

Demographic Example

- age 65 and over: *about 14%* (Wikipedia)
- "for a random American the probability to be at least 65 years old is 0.14"
- random selection: *a difficult task for pollsters*
- 0.75 male/female ratio for age 65 and over (3 : 4)
  ≈ *43% males (3 : 7)*
- fraction of males of age 65 and over:
  $0.14 \times 0.43 \approx 0.06$ ≈ 6%

Probability Language

- probability to be at least 65 years old: ≈ **0.14**
- the conditional probability of being male **if** one is at least 65 years old: ≈ **0.43**
- probability to be at least 65 years old **and** male: 0.14 × 0.43 ≈ **0.06**

$$\Pr\left[\text{male } \textit{and} \geq 65\right] = \Pr\left[\geq 65\right] \times \Pr\left[\text{male} \mid \geq 65\right]$$

$$0.06\ldots \quad = \quad 0.14 \quad \times \quad 0.43$$

$$\Pr\left[\mathrm{A} \ \textit{and} \ B\right] = \Pr\left[B\right] \times \Pr\left[\mathrm{A} \mid B\right]$$

Dice Example

- probability space: 6 outcomes 1, 2, 3, 4, 5, 6
- *A* : "at least 3" (3, 4, 5, 6) ← ≥ 3
- *B* : "even number" (2, 4, 6)
- *A* and *B* : even number and at least 3 (4, 6)
- Pr [*A* and *B*] = Pr [*A*] × Pr [*B* | *A*] → 1/3 = (4/6) × (2/4)
- **Pr [*B* | *A*]**: the fraction of ***B***-outcomes among ***A***-outcomes (for equiprobable outcomes)

$A/B = \frac{2}{3}$

$B/A = \frac{2}{4}$

$$\Pr[B \mid A] = \Pr[A \text{ and } B] / \Pr[A]$$

* In General

Example

-Line

A*, *B*, *C forms a queue in random order wherein all orderings are equiprobable. What is the probability of the events ***X* = "*A* is the second"**, ***Y* = "*A* is before *B* in the queue"**? What are the probabilities **Pr [*X* | *Y*]** and **Pr [*Y* | *X*]**?

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Be Careful

- disease: *1% of population*
- test:
  false negative rate 10%
  (negative for 1/10 of people with the disease)
  false positive rate 10%
  (positive for 1/10 of people without the disease)
- fraction of having the disease among the *test-positive = ?*
- **Pr [D] = 0.01,** **Pr [T | D] = 0.9;**
  Pr [T | not D] = 0.1, **Pr [D | T] = ?**
- test is quite reliable, isn't it?

Keep Calm and Compute

- 90% positive among 1% ill = *0.9% positive and ill*
- 10% positive among 99% healthy = *9.9% positive and healthy*
- 0.9 : 9.9 = 1 : 11
- **only 1/12 of test-positives are ill**

The Same Computation

- $\Pr[\,D\,] = 0.01$, $\Pr[\,T \mid D\,] = 0.9$, $\Pr[\,T \mid \text{not } D\,] = 0.1$
  $\Pr[\,D \mid T\,] = ?$
  0.01 0.9
- $\Pr[\,T \text{ and } D\,] = \Pr[\,D\,] \Pr[\,T \mid D\,]$
  $= 0.01 * 0.9 = \mathbf{0.009}$
  0.99 0.10
- $\Pr[\,T \text{ and not } D\,] = \Pr[\,\text{not } D\,] \Pr[\,T \mid \text{not } D\,]$
  $= 0.99 * 0.1 = \mathbf{0.099}$
- $\Pr[\,T\,] = \Pr[\,T \text{ and } D\,] + \Pr[\,T \text{ and not } D\,]$
  $= 0.009 + 0.099 = \mathbf{0.108}$
- $\Pr[\,D \mid T\,] = \Pr[\,D \text{ and } T\,] / \Pr[\,T\,]$
  $= 0.009 / 0.108 = 9 / 108 = 1/12$
  $= 0.0833\ldots \approx$ **8.3%**

The Law of Total Probability

- Probability space is split into mutually exclusive cases

$$X = B_1 \cup B_2 \cup \ldots \cup B_n$$

$$B_i \cap B_j = \varnothing$$

- A is split into mutually exclusive
  "A and B_1", "A and B_2", … , "A and B_n"
- Pr [A] = Pr [A and B_1] + … + Pr [A and B_n]
  = Pr [B_1] * Pr [A | B_1] + … + Pr [B_n] * Pr [A | B_n]
  = Pr [B] * Pr [A | B] + (1 − Pr [B]) * Pr [A | not B]

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Crime and Evidence

- e-mail: MESSAGE FROM BANK OF AMERICA
  - From: <*info4@water.ocn.ne.jp*> (Japan)
- probably a scam, I thought … why?
  - *many scam messages use foreign e-mail address*
  - *foreign e-mail address is rather unusual in general*
  - *scam messages are quite frequent now*
- three components of Bayes' reasoning

World is Not Perfect

Pr [H and E]
Pr [E and H]

- $\boldsymbol{H}$ for hypothesis, $\boldsymbol{E}$ for evidence
  (arbitrary two events having a common probability space)

$$\Pr[H \mid E] = \frac{\Pr[H \text{ and } E]}{\Pr[E]} = \frac{\Pr[E \mid H]\Pr[H]}{\Pr[E]}$$

$$\Pr[H \mid E] = \frac{\Pr[E \mid H]}{\Pr[E]}\Pr[H] \quad (Bayes)$$

Pr [H and E]
= P[H | E) P[E)
Pr [E and H]
= P[E | H] (P H]

- condition $\boldsymbol{E}$ multiplies the probability of $\boldsymbol{H}$ by factor
  Pr [E | H] / Pr [E]
  that measure how much condition $\boldsymbol{H}$ increases the probability of $\boldsymbol{E}$

Theory and Practice

$$\Pr[H \mid E] = \frac{\Pr[E \mid H]}{\Pr[E]} \Pr[H] \quad (Bayes)$$

- ***H*** : message is a scam, ***E*** : message uses foreign address
- **Pr [H | E]** is high, since:
  - **Pr [E | H]** is quite high;
  - **Pr [E]** is low;
  - **Pr [H]** is not low;
- "foreign address make the scam hypothesis much more probable, because it appears in scam messages much more often than in general"

Disease and Test Revisited

$fn = 0.8$

$$\Pr[D \mid T] = \frac{\Pr[T \mid D]}{\Pr[T]} \Pr[D]$$

$$Pr[D \mid T] = \frac{0.9}{0.108} * 0.01 = 0.0833\ldots$$

- Indeed, Pr [T | D] is much bigger than Pr [T], but Pr [D] is so small that Pr [D | T] is still rather low
- take-home message from Bayes:

*if condition **B** increases the probability of **A** by factor **k**, then condition **A** increases the probability of **B** by the same factor **k***

$BK = AK$; $AK = BK$

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Two Children

- "Mary has two children. At least one of them is a girl. What is the probability that she has two daughters?"
- "Mary tossed a fair coin twice. At least one of the outcomes is a tail. What is the probability that she got two tails?"
- *solution attempt:*
  - either first or the second bit was a tail
  - may assume w.l.o.g. it was the first
  - two outcomes: **TH**, **TT**
  - probability **1/2**

An Alternative Solution

"Mary tossed a fair coin twice. At least one of the outcomes is a tail. What is the probability that she got two tails?"

- 4 equiprobable outcomes HH, HT, TH, TT
- event $\boldsymbol{C}$: *at least one tail*
  - contains HT, TH, TT: probability 3/4
- event $\boldsymbol{E}$: *two tails*
  - contains TT: probability 1/4
- conditional probability:

$$\Pr[E \mid C] = \frac{\Pr[E \text{ and } C]}{\Pr[C]} = \frac{\Pr[E]}{\Pr[C]} = \frac{1}{3} = \frac{1/\cancel{4}}{3/\cancel{4}} = \frac{1}{3}$$

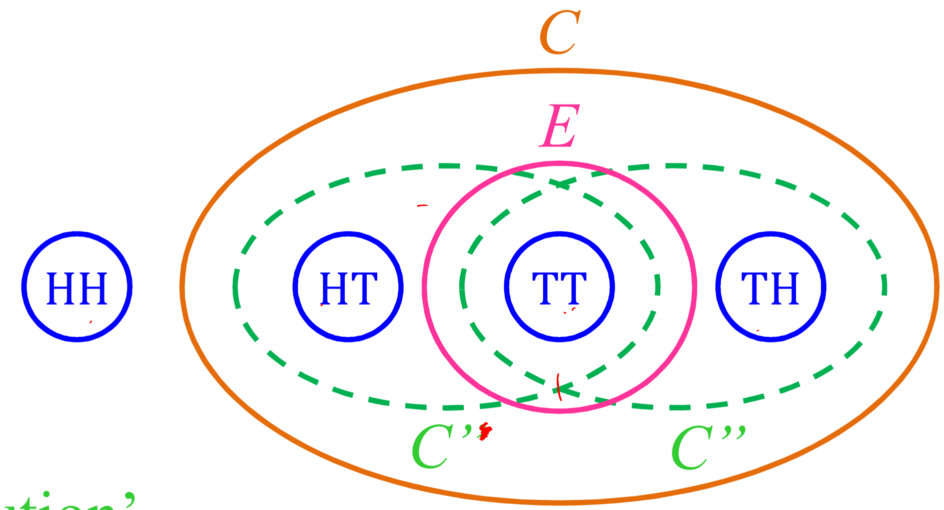
Reading the Problem Statement

"Mary tossed a fair coin twice. At least one of the outcomes is a tail. What is the probability that she got two tails?"

- probability space: HH, HT, TH, TT
- event ***C*** : *at least one tail* [condition]
  - contains HT, TH, TT: probability 3/4
- event ***E*** : *two tails* (part of C)
  - contains TT: probability 1/4
- conditional probability:

$$\Pr[E \mid C] = \frac{\Pr[E \text{ and } C]}{\Pr[C]} = \frac{\Pr[E]}{\Pr[C]} = \frac{1}{3}$$

What the First "Solution" Says



- space, condition, event
- two events in the first 'solution'
- together they make C, but
  $\Pr[\,E \mid C'\,] = \Pr[\,E \mid C''\,] = 1/2$ while
  $\Pr[\,E \mid C\,] = 1/3$.
- information $\neq$ condition !!!

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Does the History Matter

A dice is rolled twice. Does the probability of six in the second experiment increase/ decrease/ remain unchanged if we get six the first time we roll?

$f_{roll\,1} = 6$

$f_{roll\,2} = P_6 = ?$

D_1 D_2

- $Pr\,[\,y = 6 \mid x = 6\,] = Pr\,[\,x = 6\,]\,?$
- Model of 36 equiprobable outcomes:

$$\Pr[y=6 \mid x=6] = \frac{\Pr[x=6, y=6]}{\Pr[x=6]} = \frac{1/3\cancel{6}}{6/\cancel{3}6} = \frac{1}{6} = \Pr[y=6].$$

11
12
13
14
15
16

(61)

Models and Real World

- If we get 999 heads while tossing a coin, what about the next coin tossing?
  - the model says "still 1 / 2" ($2^{-1000} = 2^{-999} = 1 / 2$)
- would you agree?
  - probably not: head looks "more probable"
- for a good reason: probably the model does not fit this experiment
- statistics: finding a good model (two heads coin?)

999 1

H = T

999 : 1

Edgar Poe Says

Nothing… is more difficult than to convince the merely general reader that the fact of sixes having been thrown twice in succession…is sufficient cause for betting the largest odds that sixes will not be thrown in the third attempt. A suggestion to this effect is usually rejected by the intellect at once. It does not appear that the two throws which have been completed, and which lie now absolutely in the Past, can have influence upon the throw which exists only in the Future. The chance for throwing sixes seems to be precisely as it was at any ordinary time… And this is a reflection which appears so exceedingly obvious that attempts to controvert it are received more frequently with a derisive smile than with any thing like respectful attention. The error here involved – a gross error redolent of mischief – I cannot pretend to expose within the limits assigned me at present…
(Edgar Poe, 'The Mystery of Marie Roget', 1850)

The Mystery of Edgar Allan

- Edgar Allan Poe (1809–1849):
  famous author (short stories, poems)
  - believed that probability of six decreases if we already had six
- and more:
  - he believed (?) that this is what probability theory says
  - and complained that 'general readers' do not agree with that

- crazy statistician brings the bomb to a plane because he thinks two bombs on the same plane are quite improbable

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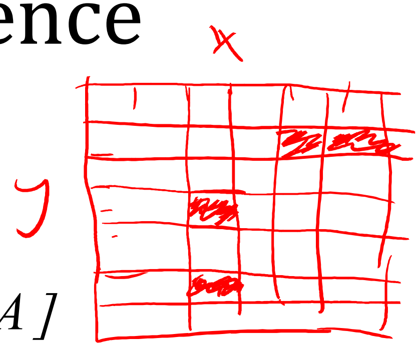
Magical Notion of Independence



- A and B are *independent* if Pr [A | B] = Pr [A]
  - B as a condition doesn't change the Pr [A]
- Recall cond' prob fraction: *Pr [A and B] / Pr [B] = Pr [A]*
- Can be rewritten as: Pr [A and B] = Pr [A] * Pr [B]
  - Symmetric & 'product rule' → actually not rule but def of independence

- Imagine two dice: A = "the first… ***x***"; B = "the second… ***y***"→ see as 2x2 table
  - X events are select columns, Y events

- Also zero probability case,
  - if Pr [A] or Pr [B] is zero then Pr [A and B] = 0, satisfy formal def of independence

Recalling Bayes

- dependent events:
  Pr [A and B] ≠ Pr [A] * Pr [B]
- say, Pr [A and B] > Pr [A] * Pr [B]
  then Pr [A | B] > Pr [A]
- "condition B makes A more probable"
- then condition A makes B more probable (symmetry)
- Bayes' formula: the factor is the same:

$$\Pr[B \mid A] = \frac{\Pr[A \mid B]}{\Pr[A]} \Pr[B]$$

Mathematics ≠ Common Sense

- two dice (x, y): A = 'x is a multiple of 2', B = 'y is a multiple of 3';
  - *they are independent*
- Now just one dice: 'x is a multiple of 2', 'x is a multiple of 3':
  - $\{ 2, 4, 6 \} \cap \{ 3, 6 \} = \{ 6 \}$ $= \frac{1}{6}$
- Therefore, mathematical independence:
  - $1 / 6 = (1 / 2) \times (1 / 3)$ $\Rightarrow \{2,4,6\} = \frac{3}{6}$; $\{3,6\} = \frac{2}{6}$

    $\left(\frac{1}{2}\right) \times \left(\frac{1}{3}\right) = \frac{1}{6}$
- Example for real-life independence:
  - an independent committee repeats the test, but no independence in mathematical sense : correlation ≠ causation
  - Pr [ill | visiting a doctor] > Pr [ill]

Thank you.