Introduction to Discrete Math

Felipe P. Vista IV

Course Outline

- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion, Logic, Invariants
- Probability & Combinatronics
  - Counting, Probability, Random Variables
- Graph Theory
  - Graphs (cycles, classes, parameters)
- Number Theory & Cryptography
  - Arithmetic in modular form
  - Intro to Cryptography

Mathematical Thinking – Combinatronics & Probability

Advanced Counting

COMBINATIONS W/ REPETITIONS

- Review
- Salad
- Combinations w/ Repetitions

Review

Let us consider selection of *k-items* out of *n* possible options. Let k=2 & n=3 options where {a,b,c}.

	With Repetitions	Without Repetitions
Ordered		
Unordered		

Review

Let us consider selection of *k-items* out of *n* possible options. Let k=2 & n=3 options where {a,b,c}.

	With Repetitions	Without Repetitions
Ordered	(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (b,c),	
Unordered		

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Let us consider selection of *k-items* out of *n* possible options. Let k=2 & n=3 options where {a,b,c}.

	With Repetitions	Without Repetitions
Ordered	(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (b,c),	(a,b), (a,c), (b,a), (b,c), (c,a), (c,b)
Unordered		

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Let us consider selection of *k-items* out of *n* possible options. Let k=2 & n=3 options where {a,b,c}.

	With Repetitions	Without Repetitions
Ordered	(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (b,c),	(a,b), (a,c), (b,a), (b,c), (c,a), (c,b)
Unordered	{a, b}, {a, c}, {b, c} {a, a}, {b, b}, {c, c}	

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Let us consider selection of *k-items* out of *n* possible options. Let k=2 & n=3 options where {a,b,c}.

	With Repetitions	Without Repetitions
Ordered	(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (b,c),	(a,b), (a,c), (b,a), (b,c), (c,a), (c,b)
Unordered	{a, b}, {a, c}, {b, c} {a, a}, {b, b}, {c, c}	{a, b}, {a, c}, {b, c}

Review

Let us consider selection of *k-items* out of n possible options. Current state of things:

	With Repetitions	Without Repetitions
Ordered		
Unordered		

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Let us consider selection of *k-items* out of n possible options. Current state of things:

	With Repetitions	Without Repetitions
Ordered	Tuples	
Unordered		

Review

Let us consider selection of *k-items* out of n possible options. Current state of things:

	With Repetitions	Without Repetitions
Ordered	Tuples n^k	
Unordered		

Review

Let us consider selection of *k-items* out of n possible options. Current state of things:

	With Repetitions	Without Repetitions
Ordered	Tuples n^k	Permutations
Unordered		

Review

Let us consider selection of k*-items* out of n possible options. Current state of things:

	With Repetitions	Without Repetitions
Ordered	Tuples n^k	Permutations $\frac{n!}{(n-k)!}$
Unordered		

Review

Let us consider selection of *k-items* out of n possible options. Current state of things:

	With Repetitions	Without Repetitions
Ordered	Tuples n^k	Permutations $\frac{n!}{(n-k)!}$
Unordered		Combinations

Review

Let us consider selection of k*-items* out of n possible options. Current state of things:

	With Repetitions	Without Repetitions
Ordered	Tuples n^k	Permutations $\frac{n!}{(n-k)!}$
Unordered		Combinations $\binom{n}{k}$ $\Rightarrow \frac{n!}{k!(n-k)!}$

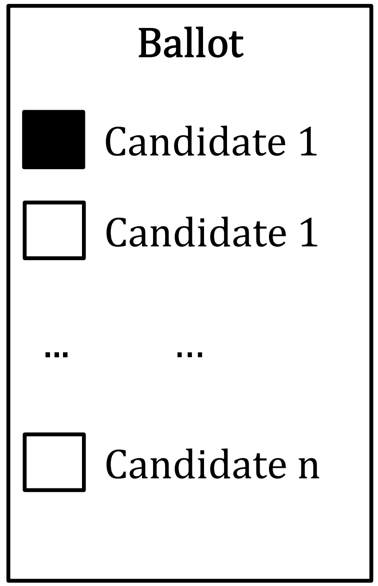
Review

Let us consider selection of *k-items* out of n possible options. Current state of things:

	With Repetitions	Without Repetitions
Ordered	Tuples n^k	Permutations $\frac{n!}{(n-k)!}$
Unordered	???	Combinations $\binom{n}{k}$

Example: Voting

There are *k-voters* that vote for n candidates.



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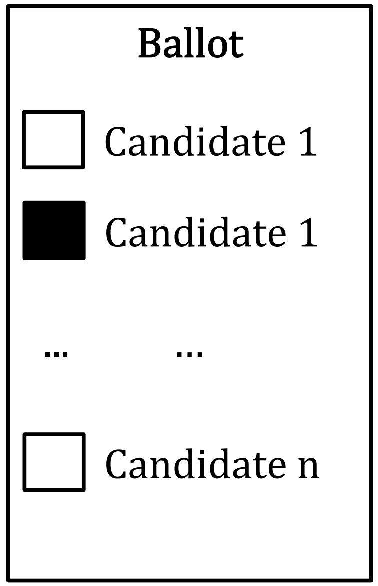
There are *k-voters* that vote for *n* candidates.



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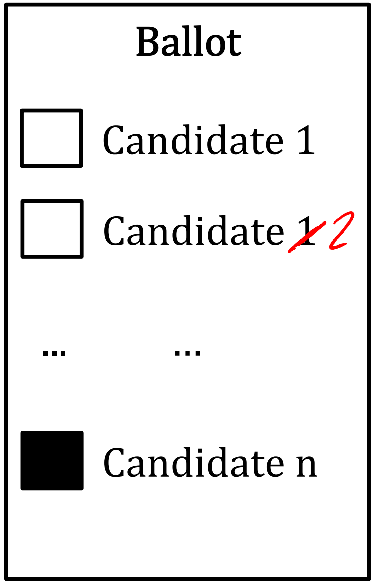
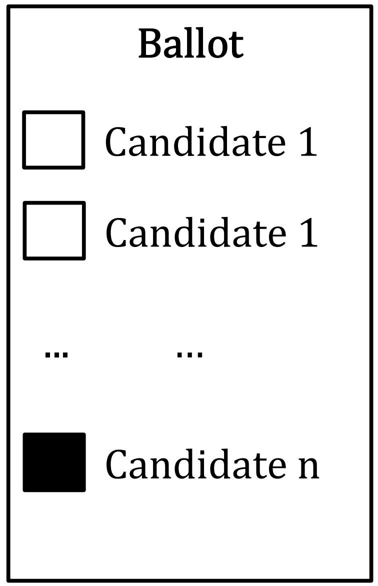


Example: Voting

There are *k-voters* that vote for n candidates.

- All votes equally matter



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There are *k-voters* that vote for n candidates.

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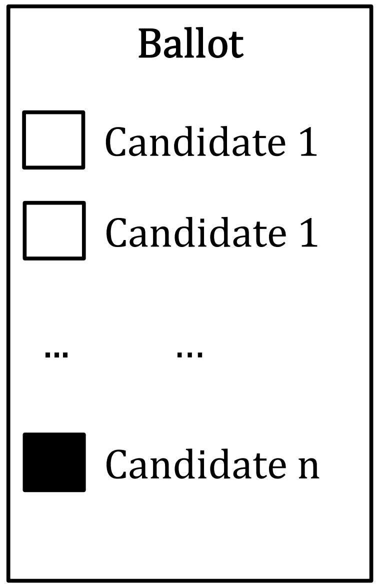

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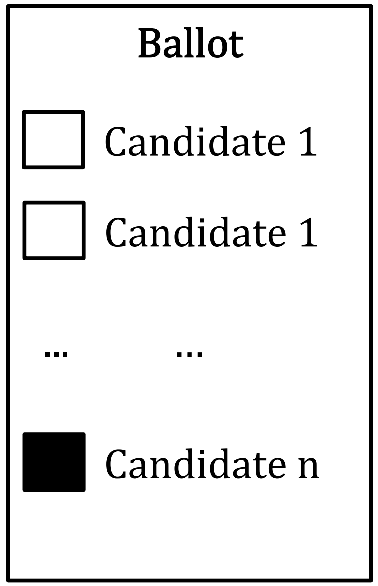
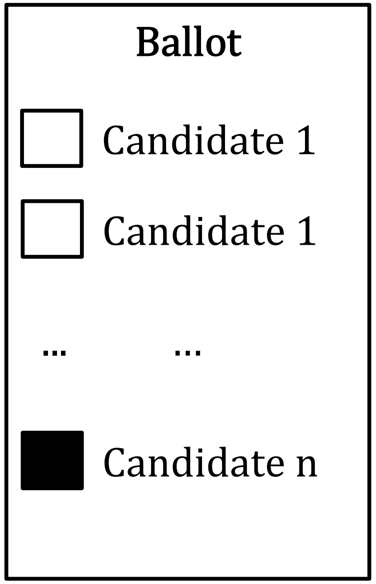


Example: Voting

There are *k-voters* that vote for n candidates.

- All votes equally matter
- So votes are unordered
- Candidates can be voted several times
- So, voters as a group pick k people out of n with repetitions



- Review
- Salad
- Combinations w/ Repetitions

Salad

Problem

There is unlimited supply of tomatoes, bell peppers, and lettuce. We want to make a salad out of 4 units among the three ingredients (we don't need to use all three). How many different salads can we make?

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- How do we count the total?
    - List all possible salads then count them

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- We pick 4 items out of the 3 ingredients with repetitions
- Order we pick do not matter
- This will be the setup
- How do we count the total?
  - List all possible salads then count them
  - We want to do it wisely

Salad

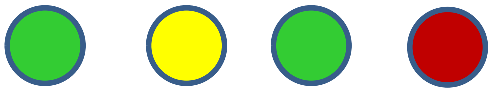
Tomato

Bell Pepper

Lettuce

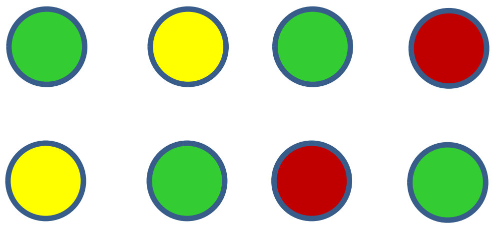
Salad

Tomato Bell Pepper Lettuce



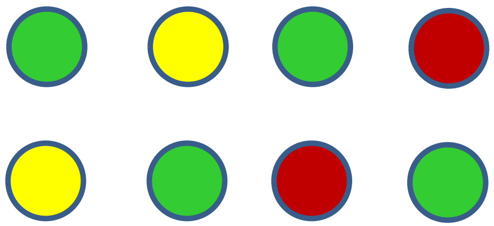
Salad

Tomato Bell Pepper Lettuce



Salad

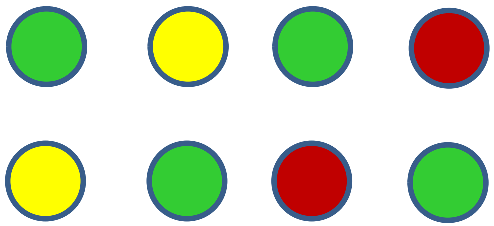
Tomato Bell Pepper Lettuce



Same salad

Salad

Tomato   Bell Pepper   Lettuce

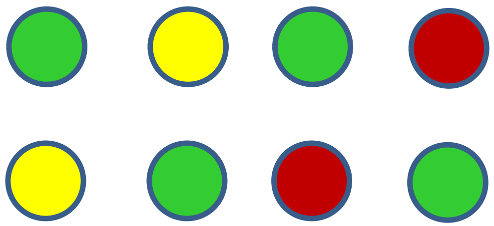


Same salad

- The order does not matter

Salad

Tomato Bell Pepper Lettuce



Same salad

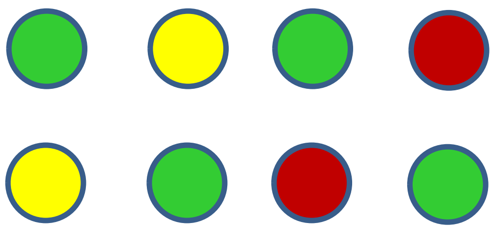
- The order does not matter
- So we will draw tomatoes first, then bell peppers, then lettuce

Salad

Tomato

Bell Pepper

Lettuce



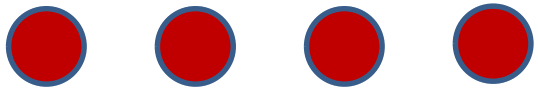
Same salad

- The order does not matter
- So we will draw tomatoes first, then bell peppers, then lettuce
- Let us consider all possible numbers of tomatoes in the salad and count each case separately,

Combinations W/ Repetitions

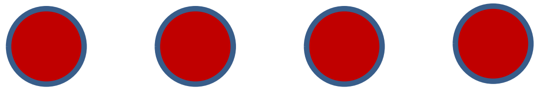
Salad

Salad



Case 1: 4 tomatoes

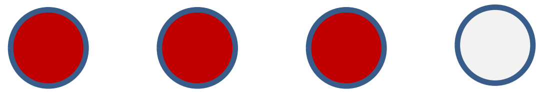
Salad



Case 1: 4 tomatoes

- 4 tomatoes: 1 salad

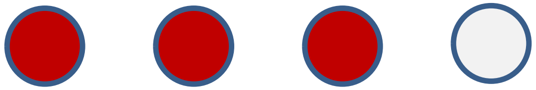
Salad



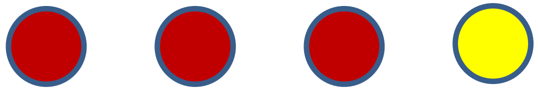
Case 2: 3 tomatoes

- 4 tomatoes: 1 salad

Salad

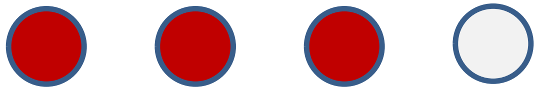


Case 2: 3 tomatoes



- 4 tomatoes: 1 salad

Salad



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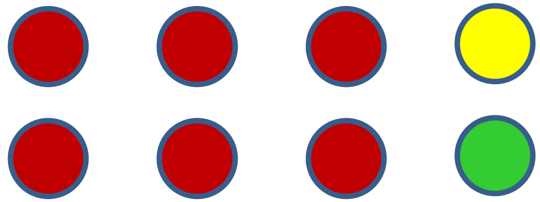
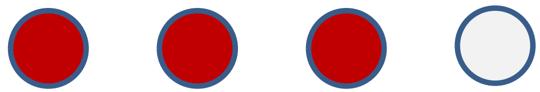
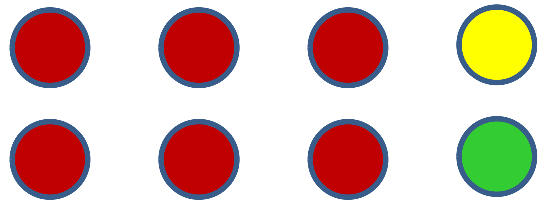


- 4 tomatoes: 1 salad

Salad

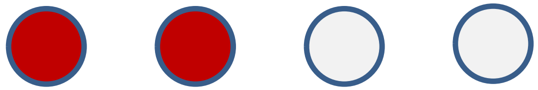


Case 2: 3 tomatoes



- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads

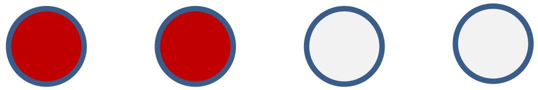
Salad



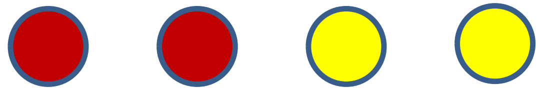
Case 3: 2 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads

Salad

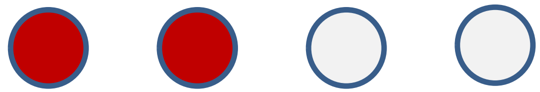


Case 3: 2 tomatoes

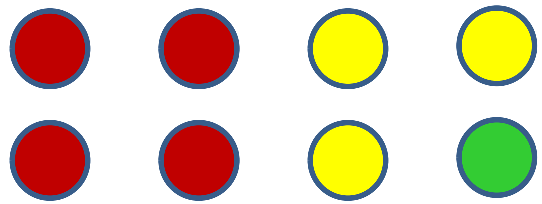


- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads

Salad

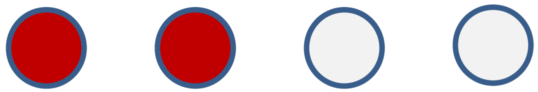


Case 3: 2 tomatoes



- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads

Salad



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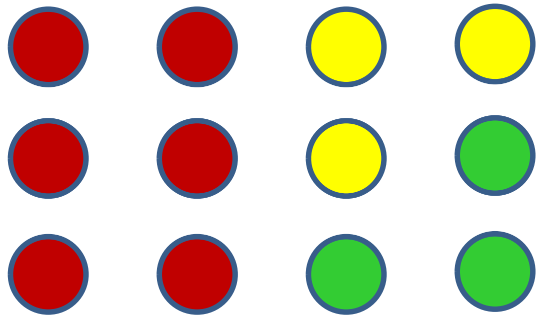

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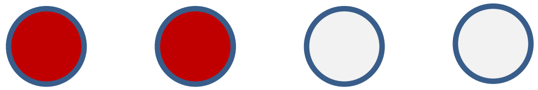
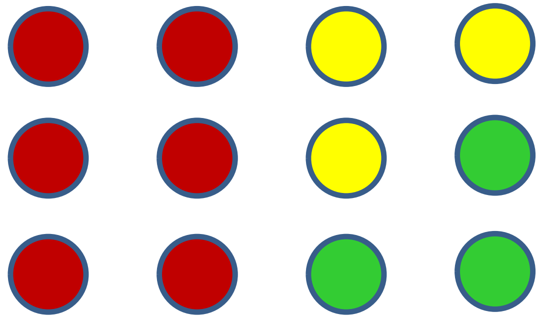
Salad



Case 3: 2 tomatoes



- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads

Salad

Case 4: 1 tomato

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads

Salad

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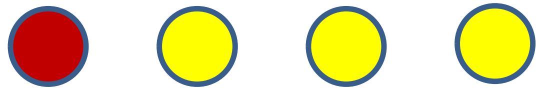

- 4 tomatoes: 1 salad
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Salad

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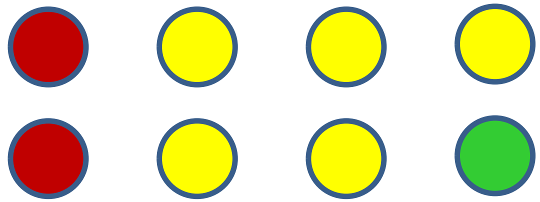

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Salad

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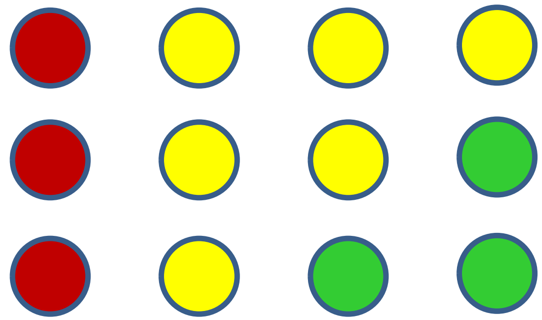

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Salad

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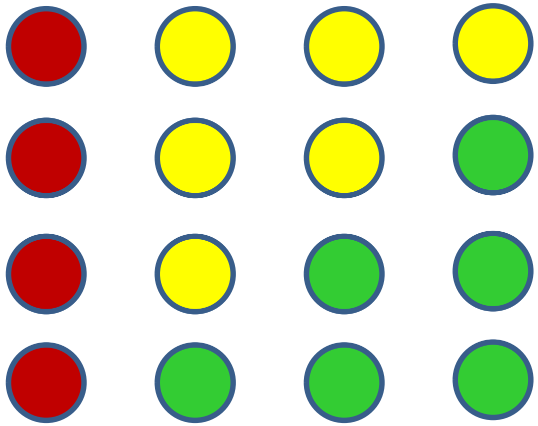

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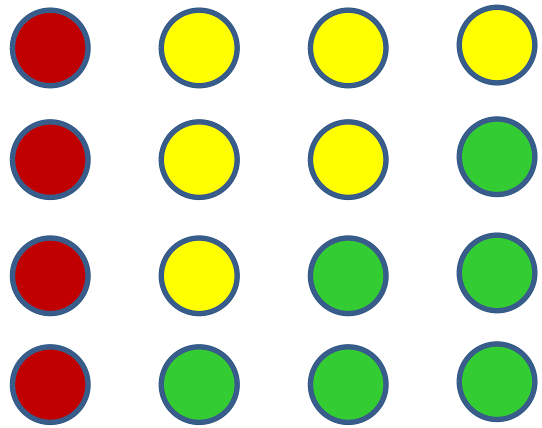
- 2 tomatoes: 3 salads

Salad

Case 4: 1 tomato



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- 1 tomato   : 4 salads

Salad

Case 5: 0 tomatoes

- 4 tomatoes: 1 salad
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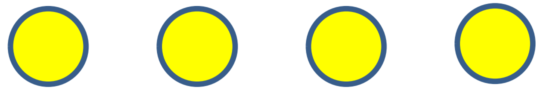
- 2 tomatoes: 3 salads
- 1 tomato   : 4 salads

Salad

Case 5: 0 tomatoes



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- 2 tomatoes: 3 salads
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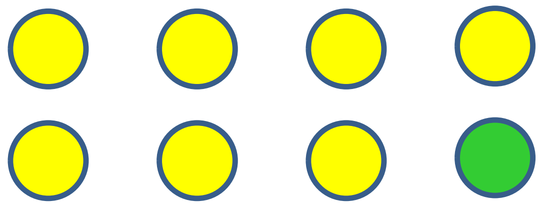
Salad

Case 5: 0 tomatoes



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Salad

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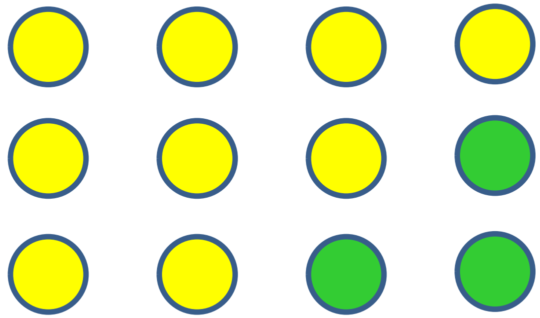

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Salad

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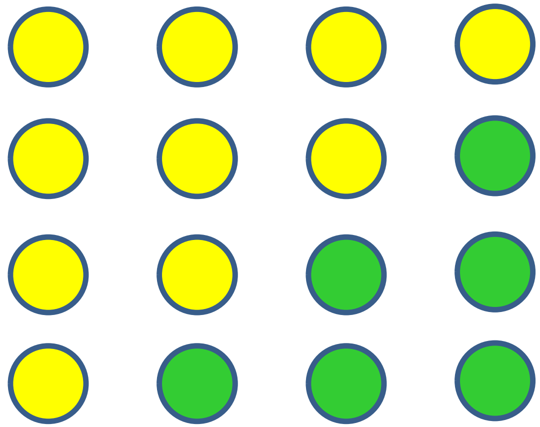

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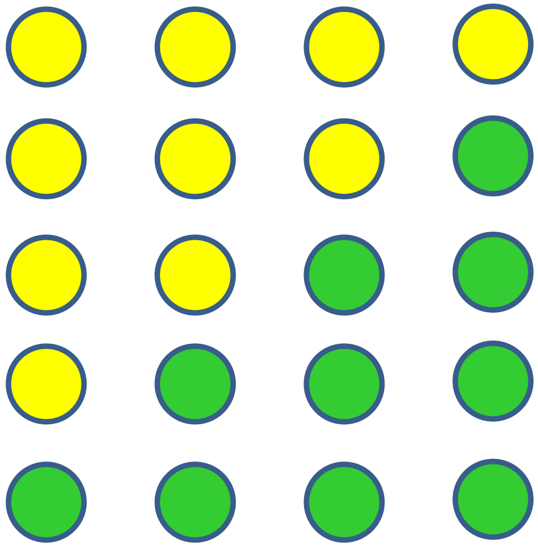
- 1 tomato : 4 salads

Salad

Case 5: 0 tomatoes



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- 1 tomato   : 4 salads

Salad

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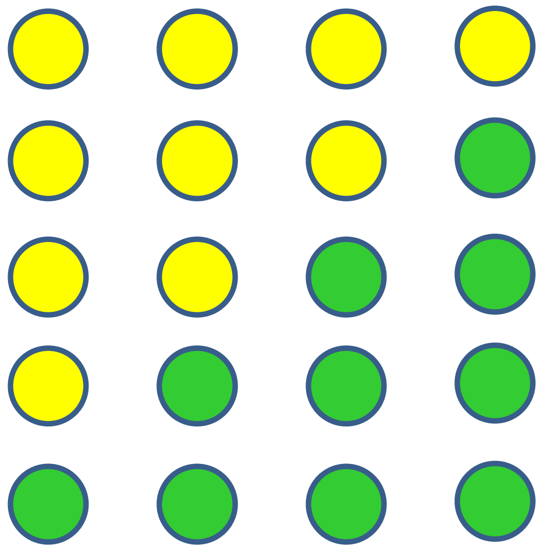
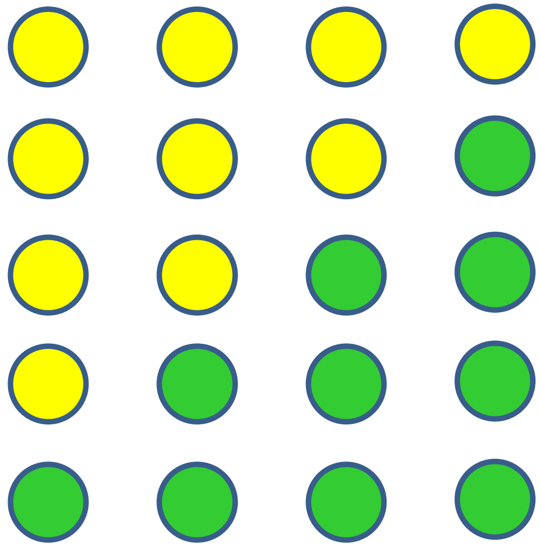


- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads
- 1 tomato   : 4 salads
- 0 tomatoes: 5 salads

Salad

Case 5: 0 tomatoes



- 4 tomatoes: 1 salad
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- 2 tomatoes: 3 salads
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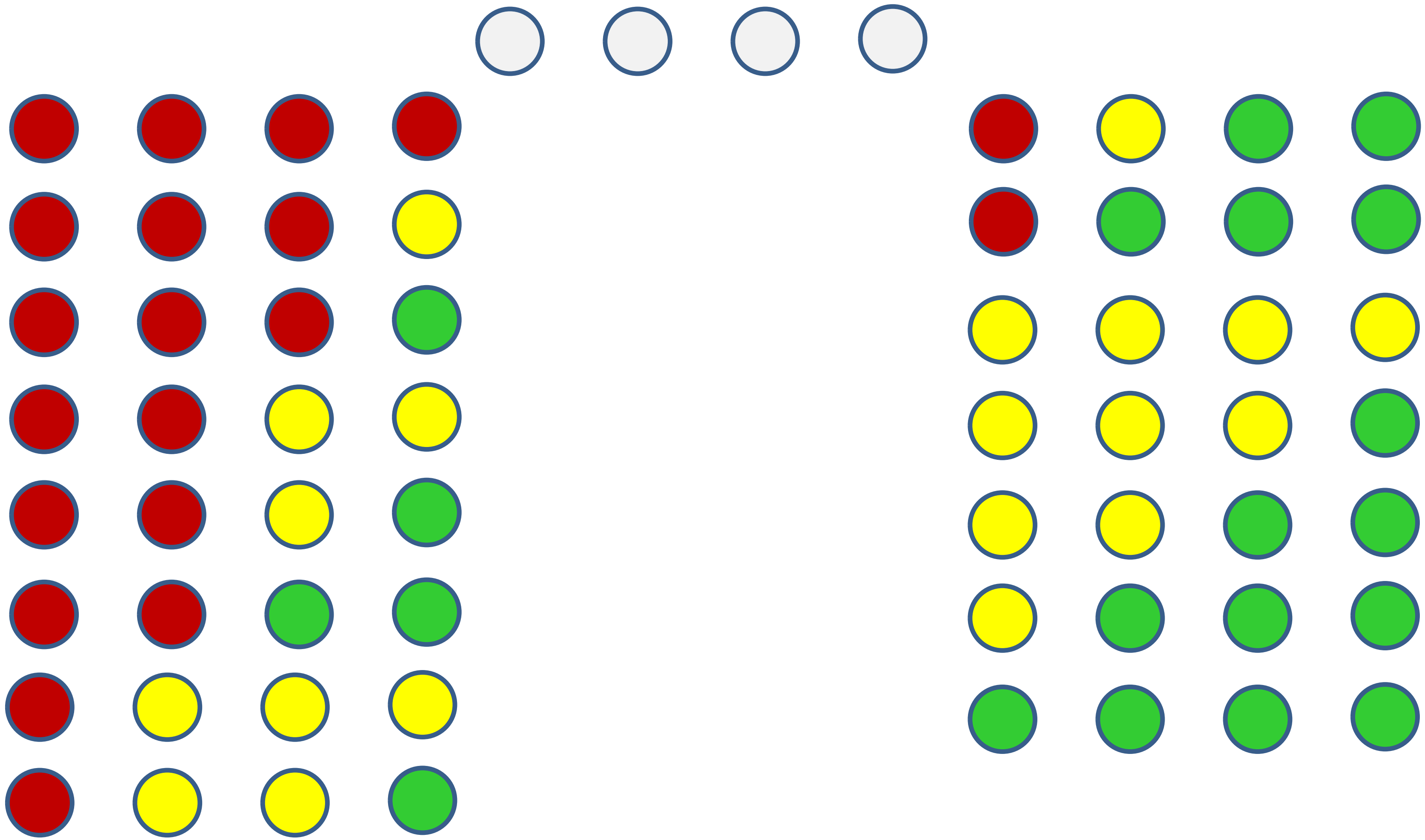
For a total of 15 salad varieties

Salad



Summary

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- But more complicated for more ingredients

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- The solution looks very structured
- Same structure for larger salad
- But more complicated for more ingredients
- Yet, the same strategy works for recursive counting of any salad size with any number of ingredients

- Review
- Salad
- Combinations w/ Repetitions

Large Salad

Problem

We now have an unlimited supply of tomatoes, bell peppers, lettuce, and eggplant. We want to make a salad out of 7 units among the four ingredients (we don't need to use all ~~three~~). How many different salads can we make?

four

t b l e

Large Salad

Problem

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- We can use recursive counting here as well

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- This will be a general solution

Large Salad

- Tomato
- Bell Pepper
- Lettuce
- Eggplant

→ salad

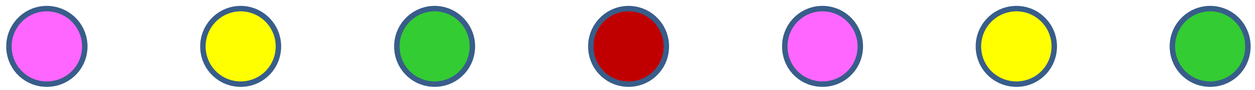
Large Salad

Tomato

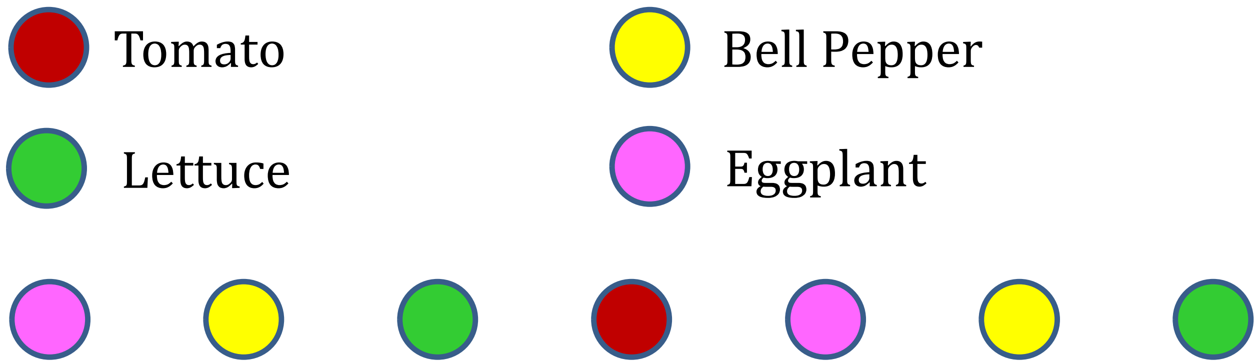
Bell Pepper

Lettuce

Eggplant



Large Salad



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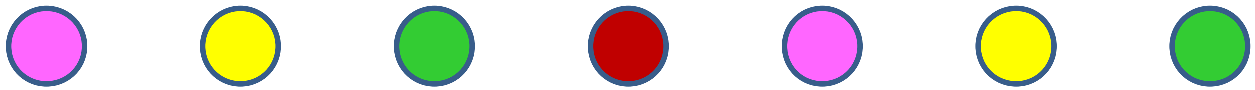
Large Salad

Tomato

Bell Pepper

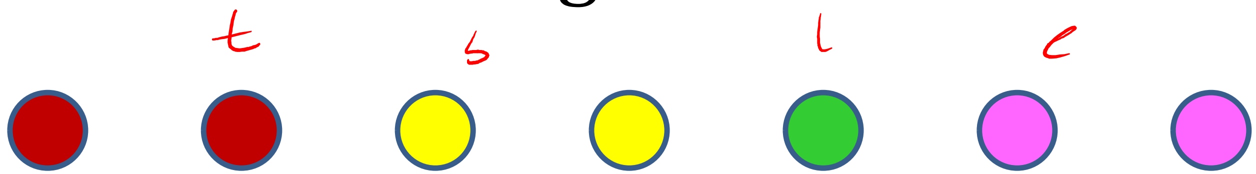
Lettuce

Eggplant



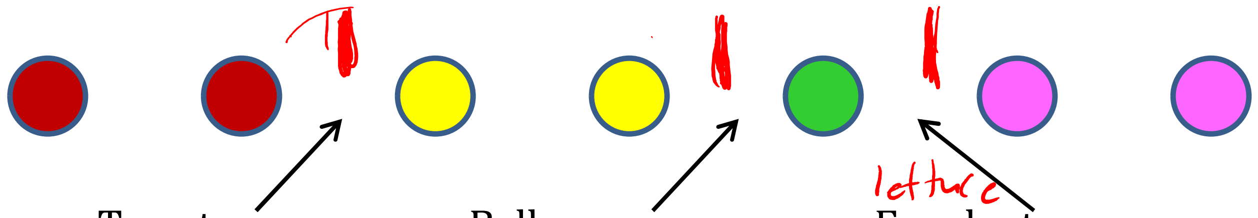
- The order does not matter
- For the next part:
  - So we will draw tomatoes first,
  - then bell peppers,
  - then lettuce, and
  - then the eggplant

Large Salad



- Idea 1: To specify the list, it is enough to indicate where the ingredients switch

Large Salad



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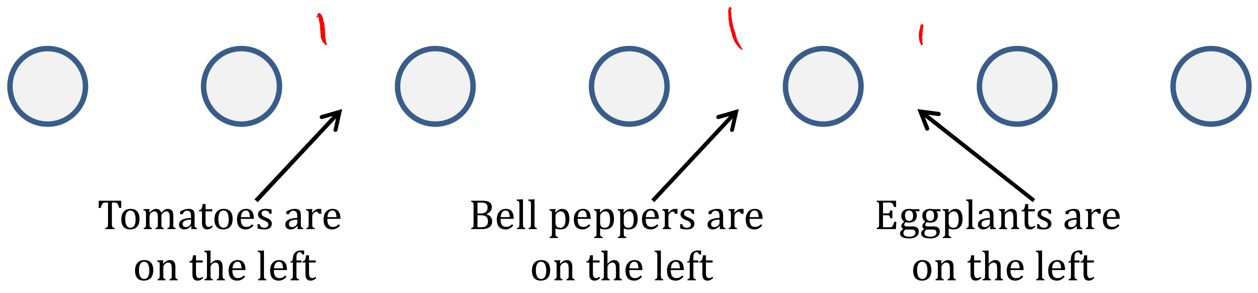
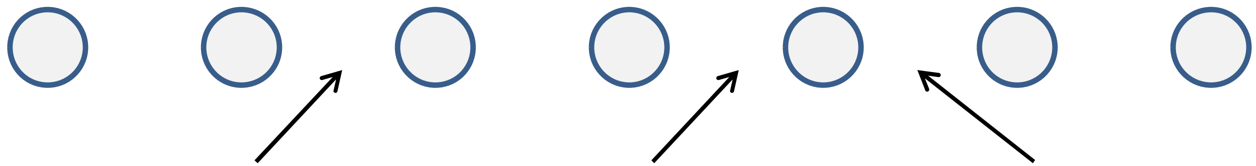


- Idea 1: To specify the list, it is enough to indicate where the ingredients switch
- Idea 2: We do not even need the text descriptions

Large Salad



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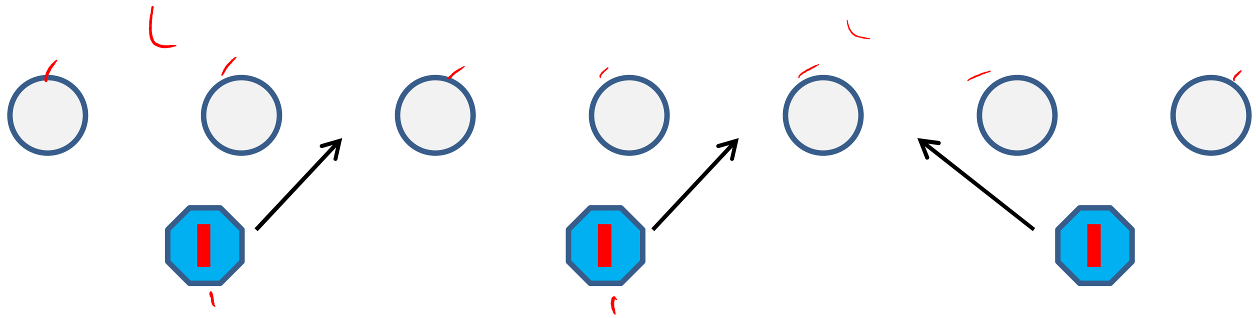

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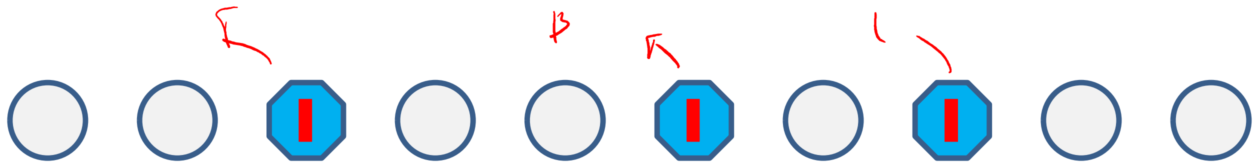

- Idea 1: To specify the list, it is enough to indicate where the ingredients switch
- Idea 2: We do not even need the text descriptions
- Idea 3: We can represent places of switch as delimiter signs
- The salad can still be restored:
  - Just color the indicators
  - Tomatoes on the left of the first delimiter
  - Bell peppers to the right of the first delimiter
  - Etc…

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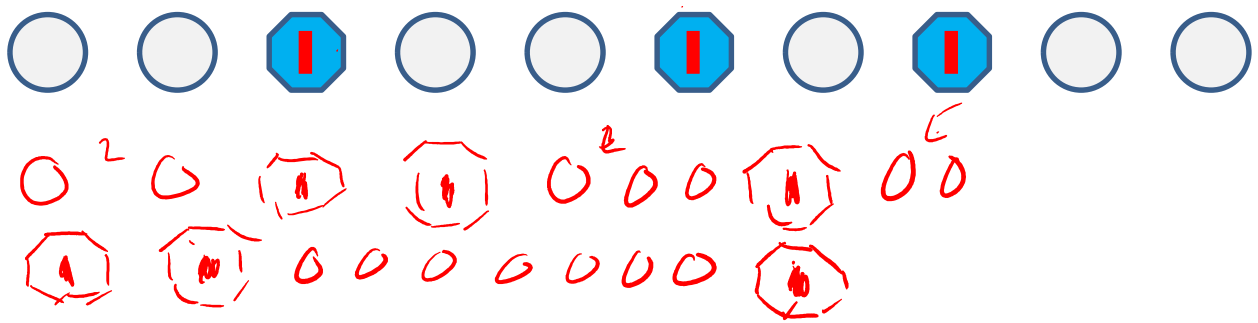

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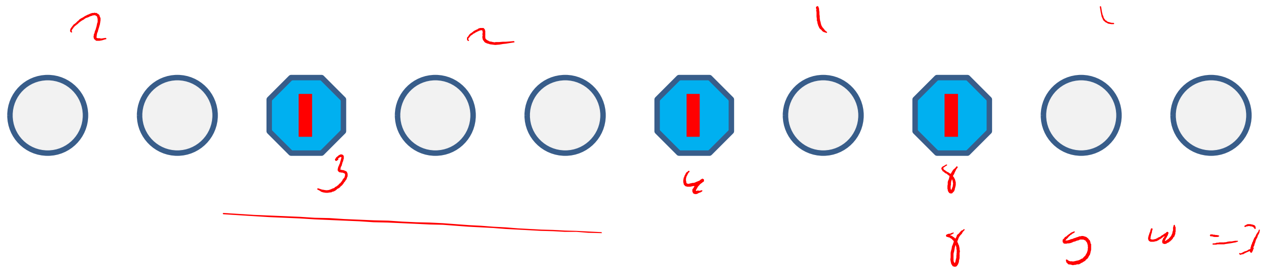

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$$\binom{10}{3} = 120$$

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- Choose place for delimiters in the line

General Case

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c2

C1: k = 3 , n = 4

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10 → 4+3-1 = $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

3 3-1 = 2

n = 3

k = 4

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- Number of objects (n) = number of ingredients
- The same general argument works
- Why k+n-1 and n-1?
  - n ingredients means there will be n-1 delimiter = n-1 => 3
  - Choosing (n-1) element in the line of k+(n-1) elements

↳ 3

7 4(-1) = 10

n = 4

k = 7

Standard Settings

Let us consider selection of k*-items* out of n possible options.

	With Repetitions	Without Repetitions
Ordered	Tuples n^k	Permutations $\cancel{\frac{n!}{(n-k)!}}$
Unordered	???	Combinations $\binom{n}{k}$

$\frac{n!}{k!(n-k)!}$

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Thank you.