APPENDIX A
AREA MOMENTS OF INERTIA


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APPENDIX OUTLINE

Article A/1 Introduction

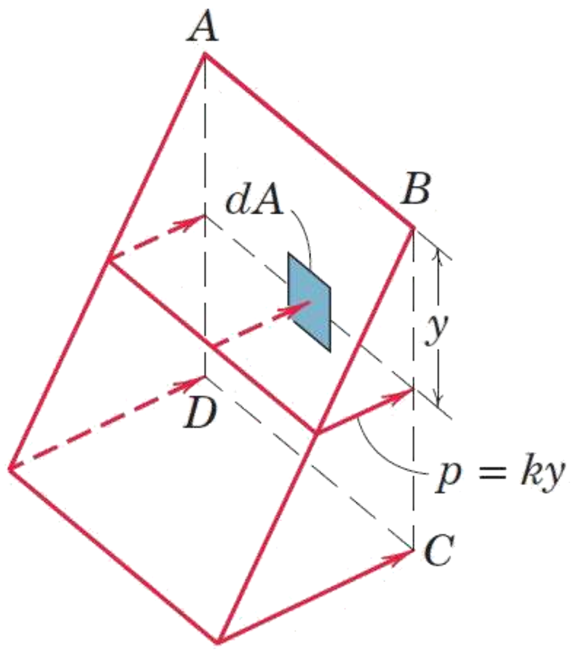
- Origin of the Area Moment of Inertia Calculation

  When forces are distributed continuously over an area on which they act, it is often necessary to calculate the moment of these forces about some axis either in or perpendicular to the plane of the area. When this occurs, an area moment of inertia calculation appears which has the form of an integral of a distance squared over an area.

- Form of the Calculation
  - $\int(\text{distance})^2\, d(\text{area})$

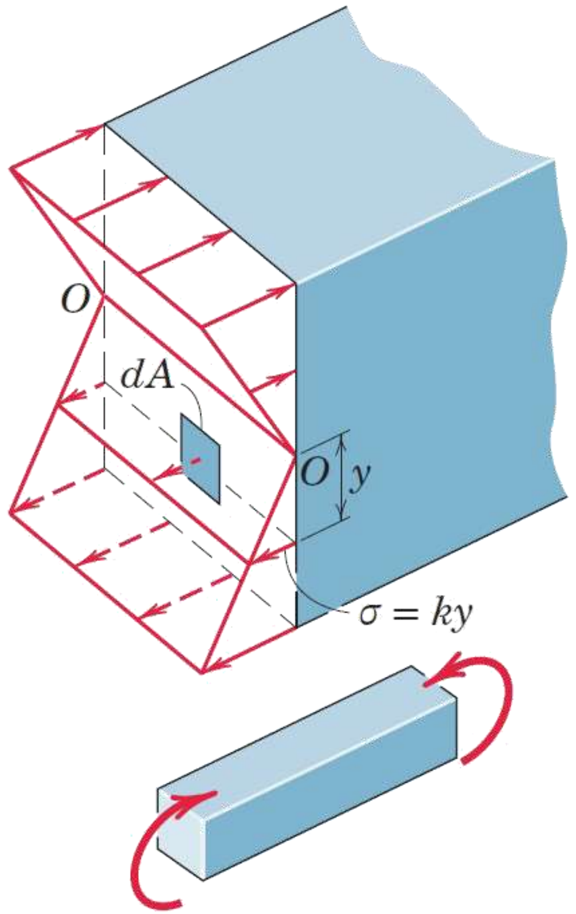
Article A/1 – Physical Origin of the Calculation (1 of 3)

- Example with Fluid Pressure
  - Consider the surface area *ABCD* which is subjected to a distributed pressure p whose intensity is proportional to the distance y from the axis *AB*. The pressure at depth in a fluid, $p = \rho g y$, where ρ is the density of the fluid and the pressure is considered relative to the atmospheric pressure.
  - The pressure at a depth y is equal to some constant k multiplied by the depth. Thus, $p = ky$.
  - The pressure p acting over a differential area dA will produce a differential force $dF = p\,dA = ky\,dA$.
  - The moment of this differential force about the axis *AB* can be written as $dM = y\,dF = yp\,dA = ky^2\,dA$.
  - The total moment $M = \int dM = \int ky^2\,dA$, which has the form of an integral of a distance squared over an area.



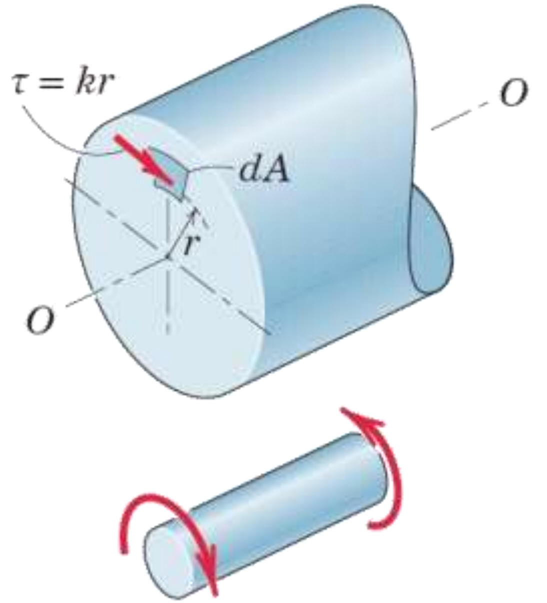
Article A/1 – Physical Origin of the Calculation (2 of 3)

- Example with Beams in Bending
  - Consider the normal stress distribution acting on a transverse section of a simple elastic beam bent by equal and opposite couples applied to its ends.
  - At any section of the beam, a linear distribution of force intensity or stress σ, given by $\sigma = ky$, is present.
  - The stress is positive (tensile) below the axis *O*-*O* and negative (compressive) above the axis. Axis *O*-*O* is termed the neutral axis.
  - The stress σ acting over a differential area dA will produce a differential force $dF = \sigma\, dA = ky\, dA$.
  - The moment of this differential force about the axis *O*-*O* can be written as $dM = y\, dF = y\sigma\, dA = ky^2\, dA$.
  - The total moment $M = \int dM = \int ky^2\, dA$, which has the form of an integral of a distance squared over an area.



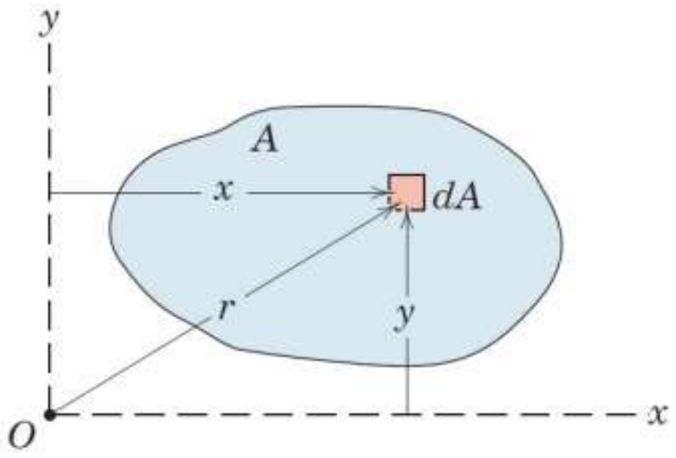
Article A/1 – Physical Origin of the Calculation (3 of 3)

- Example with Shafts in Torsion
  - Consider the shear stress distribution acting on a transverse section of a shaft subjected to equal and opposite torsional moments applied to its ends.
  - Within the elastic limit of the material, this moment is resisted at each cross section of the shaft by a distribution of tangential or shear stress τ, which is proportional to the radial distance r from the center of the shaft. Thus, $\tau = kr$.
  - The shear stress τ acting over a differential area dA will produce a differential force $dF = \tau\, dA = kr\, dA$.
  - The moment of this differential force about the shaft axis O-O can be written as $dM = r\, dF = r\tau\, dA = kr^2\, dA$.
  - The total moment $M = \int dM = \int kr^2\, dA$, which has the form of an integral of a distance squared over an area.



Article A/2 Definitions

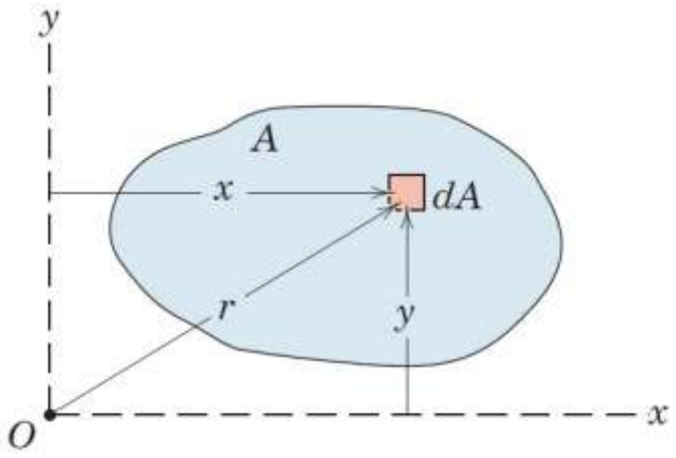
- Rectangular Moments of Inertia
  - $I_x = \int y^2 \, dA$
  - $I_y = \int x^2 \, dA$



- Polar Moments of Inertia
  - $I_z = \int r^2 \, dA = \int x^2 \, dA + \int y^2 \, dA = I_x + I_y$

Article A/2 – Comments about Moments of Inertia

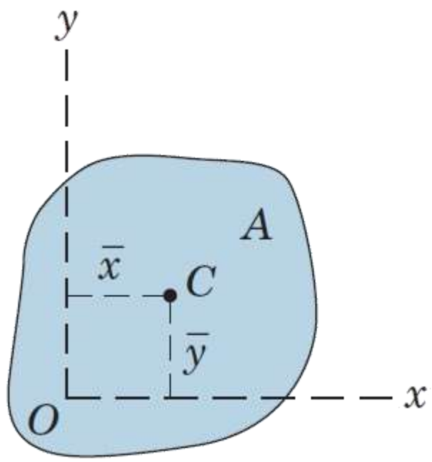
- Dimensions and Units
  - Length to the 4th Power, L^4
  - SI Units: mm^4 (most common)
  - U.S. Customary Units: $in.^4$ (most common)



- Things to Note
  - A moment of inertia is always positive since it involves the square of a distance from the inertia axis to the element of integration.
  - The choice of coordinates (rectangular or polar) to use for the calculation of moments of inertia is important.

Article A/2 – Radius of Gyration (1 of 3)

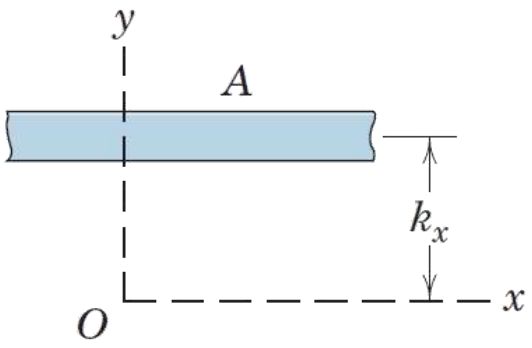
- Concept
  - The radius of gyration is a measure of the distribution of an area from a particular axis.



- Illustration with x-Axis
  - Consider the area A with moment of inertia I_x.
  - Concentrate the area into a thin strip which is parallel to the x-axis and located a distance k_x from the x-axis.
  - The moment of inertia of the strip about the x-axis is…

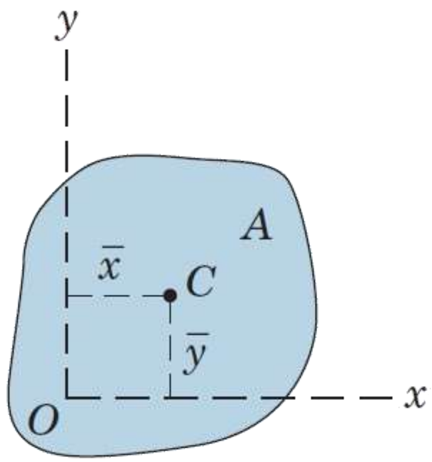
$$I_x = \int y^2 \, dA = \int k_x^2 \, dA = k_x^2 \int dA = k_x^2 A$$

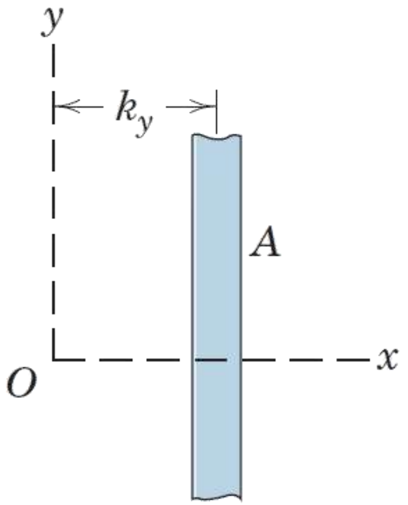
  - The radius of gyration $k_x = \sqrt{I_x / A}$



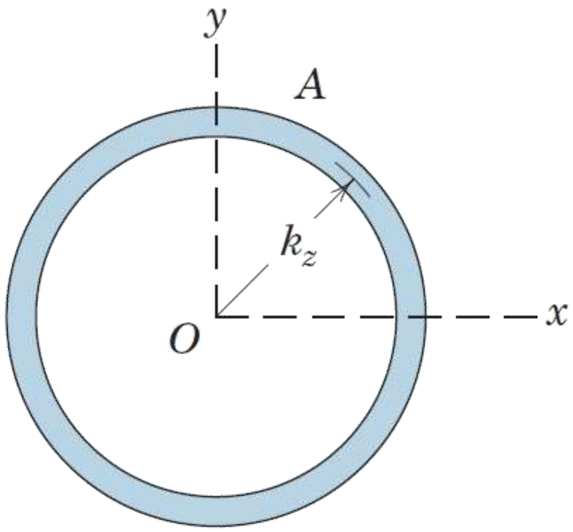
Article A/2 – Radius of Gyration (2 of 3)

- Illustration with *y*-Axis





- Illustration with the *z*-Axis



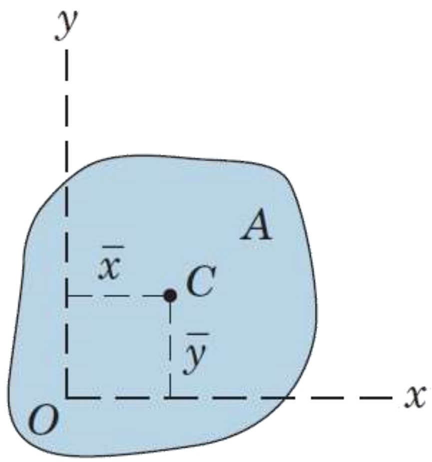
Article A/2 – Radius of Gyration (3 of 3)

- Summary of Equations

$$I_x = k_x^2 A \qquad I_y = k_y^2 A \qquad I_z = k_z^2 A$$

or

$$k_x = \sqrt{I_x/A} \qquad k_y = \sqrt{I_y/A} \qquad k_z = \sqrt{I_z/A}$$



- Alternative Equation for k_z

$$k_z^2 = k_x^2 + k_y^2$$

- Final Comment
  - Do not confuse the centroid coordinate C with the radii of gyration for the shape. They are not the same. For example, the moment of inertia about the x-axis is not equal to $A\bar{y}^2$.

Article A/2 – Transfer of Axes (1 of 2)

- Overview

- Illustration with the *x*-Axis

$$dI_x = (y_0 + d_x)^2 \, dA$$

Expanding and integrating give us

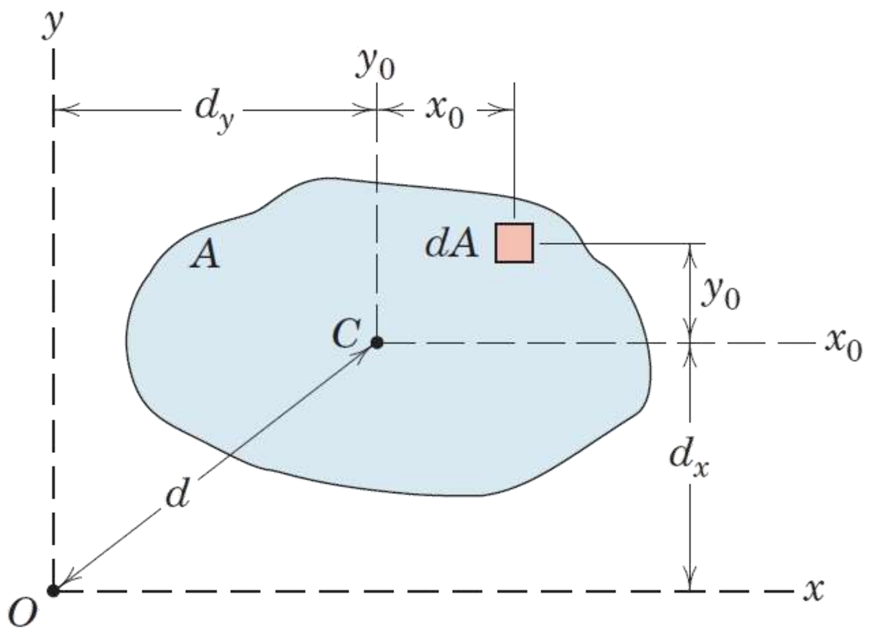
$$I_x = \int y_0^2 \, dA + 2d_x \int y_0 \, dA + d_x^2 \int dA$$

- Result

$$\boxed{\begin{aligned} I_x &= \bar{I}_x + Ad_x^2 \\ I_y &= \bar{I}_y + Ad_y^2 \end{aligned}}$$

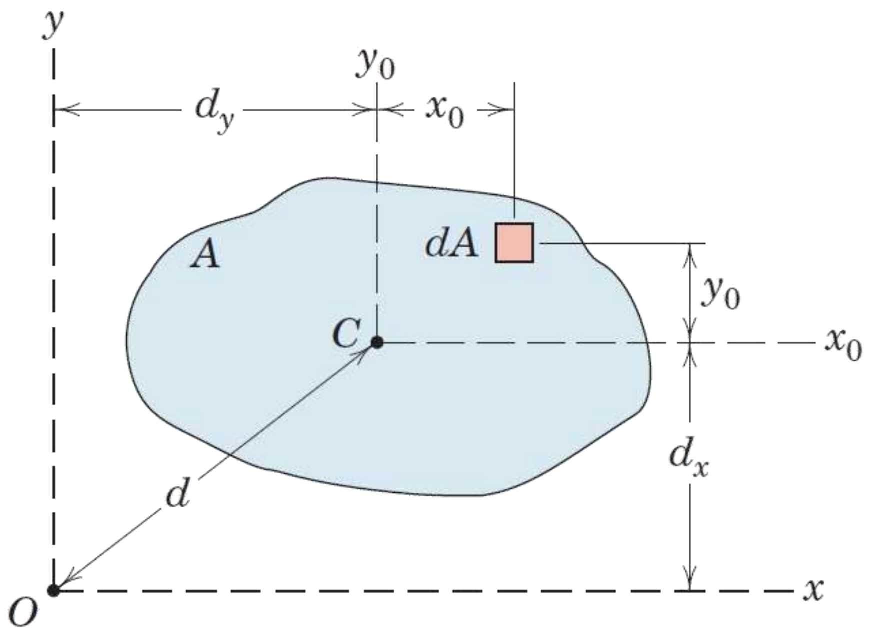
and

$$I_z = \bar{I}_z + Ad^2$$



Article A/2 – Transfer of Axes (2 of 2)

- Important Comments
  - The axes between which the transfer is made *must be parallel*.
  - One of the axes *must pass through the centroid of the area*.



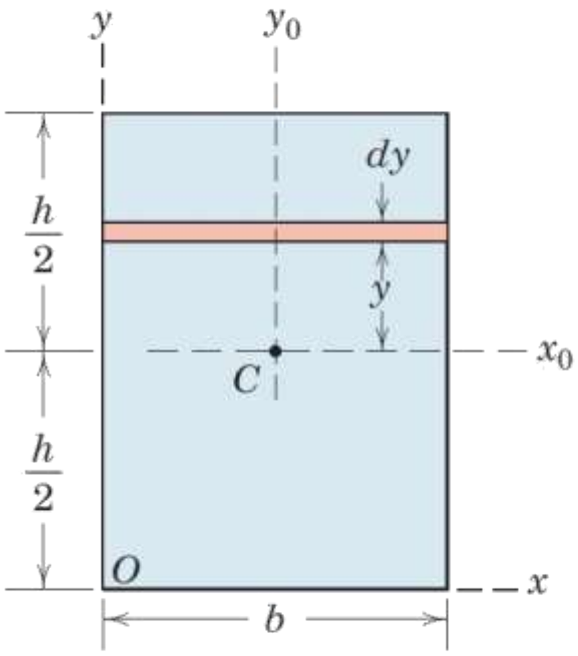
- Effect on the Radius of Gyration

$$k^2 = \bar{k}^2 + d^2$$

Article A/2 – Sample Problem A/1 (1 of 2)

- Problem Statement

  Determine the moments of inertia of the rectangular area about the centroidal x_0- and y_0-axes, the centroidal polar axis z_0 through C, the x-axis, and the polar axis z through O.



Article A/2 – Sample Problem A/1 (2 of 2)

- Horizontal Strip of Area $dA = b\,dy$

$$[I_x = \int y^2\,dA] \qquad \bar{I}_x = \int_{-h/2}^{h/2} y^2 b\,dy = \tfrac{1}{12}bh^3 \qquad \textit{Ans.}$$

By interchange of symbols, the moment of inertia about the centroidal y_0-axis is

$$\bar{I}_y = \tfrac{1}{12}hb^3 \qquad \textit{Ans.}$$

The centroidal polar moment of inertia is

$$[\bar{I}_z = \bar{I}_x + \bar{I}_y] \qquad \bar{I}_z = \tfrac{1}{12}(bh^3 + hb^3) = \tfrac{1}{12}A(b^2 + h^2) \qquad \textit{Ans.}$$

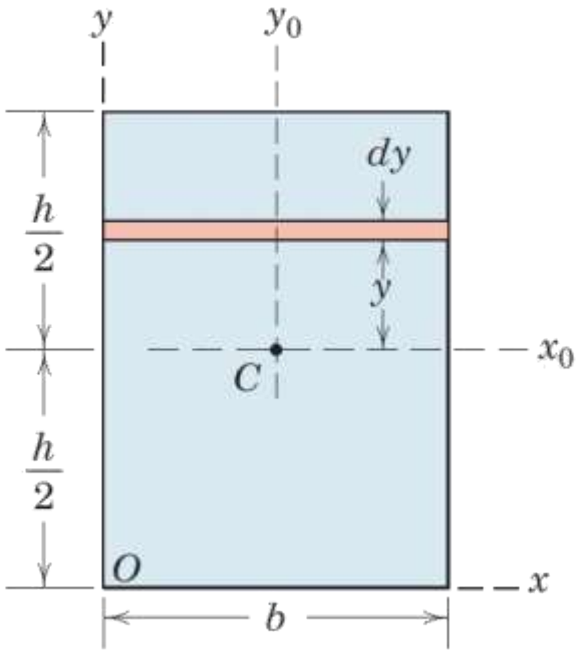
By the parallel-axis theorem, the moment of inertia about the x-axis is

$$[I_x = \bar{I}_x + Ad_x{}^2] \qquad I_x = \tfrac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \tfrac{1}{3}bh^3 = \tfrac{1}{3}Ah^2 \qquad \textit{Ans.}$$

We also obtain the polar moment of inertia about O by the parallel-axis theorem, which gives us

$$[I_z = \bar{I}_z + Ad^2] \qquad I_z = \tfrac{1}{12}A(b^2 + h^2) + A\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right]$$

$$I_z = \tfrac{1}{3}A(b^2 + h^2) \qquad \textit{Ans.}$$

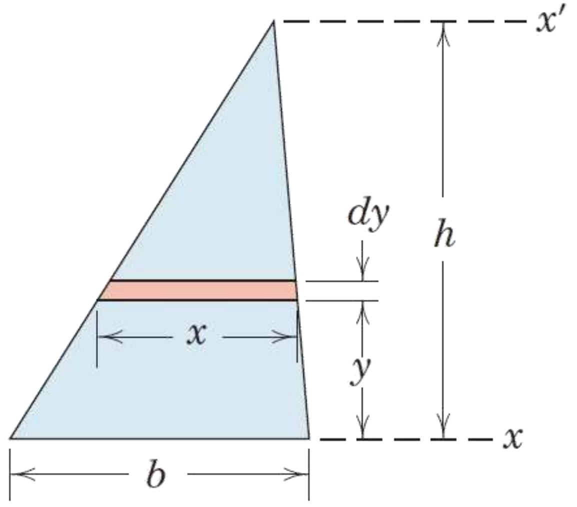


① If we had started with the second-order element $dA = dx\,dy$, integration with respect to x holding y constant amounts simply to multiplication by b and gives us the expression $y^2b\,dy$, which we chose at the outset.

Article A/2 – Sample Problem A/2 (1 of 2)

- Problem Statement

  Determine the moments of inertia of the triangular area about its base and about parallel axes through its centroid and vertex.



Article A/2 – Sample Problem A/2 (2 of 2)

- Solution

A strip of area parallel to the base is selected as shown in the figure, and it has the area $dA = x\,dy = [(h - y)b/h]\,dy$. ① ② By definition

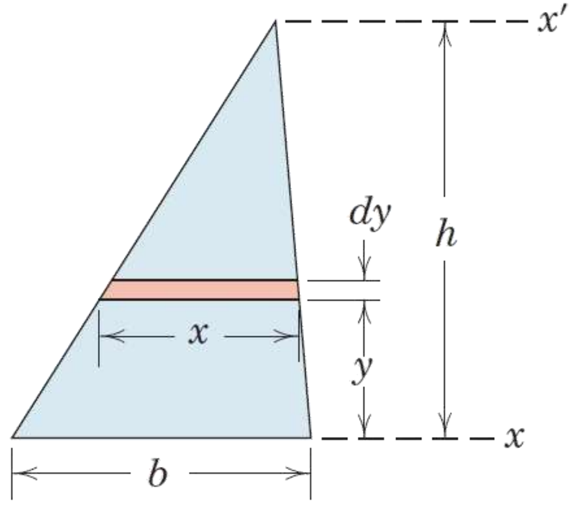
$$[I_x = \int y^2\,dA] \qquad I_x = \int_0^h y^2 \frac{h-y}{h} b\,dy = b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = \frac{bh^3}{12} \qquad \textit{Ans.}$$

By the parallel-axis theorem, the moment of inertia $\bar{I}$ about an axis through the centroid, a distance $h/3$ above the x-axis, is

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36} \qquad \textit{Ans.}$$

A transfer from the centroidal axis to the x'-axis through the vertex gives

$$[I = \bar{I} + Ad^2] \qquad I_{x'} = \frac{bh^3}{36} + \left(\frac{bh}{2}\right)\left(\frac{2h}{3}\right)^2 = \frac{bh^3}{4} \qquad \textit{Ans.}$$



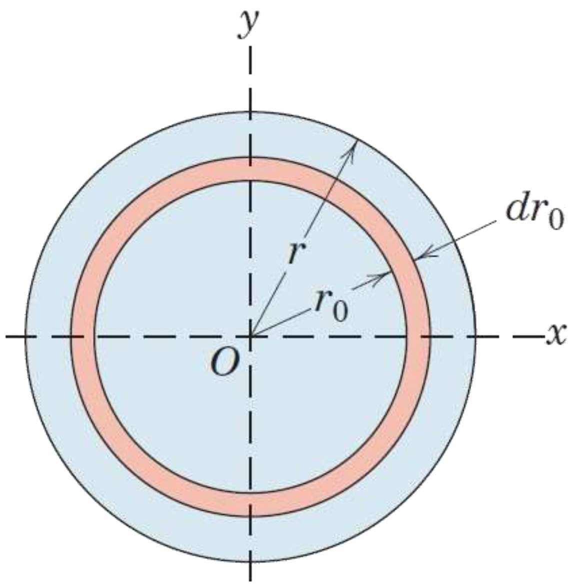
① Here again we choose the simplest possible element. If we had chosen $dA = dx\,dy$, we would have to integrate $y^2\,dx\,dy$ with respect to x first. This gives us $y^2x\,dy$, which is the expression we chose at the outset.

② Expressing x in terms of y should cause no difficulty if we observe the proportional relationship between the similar triangles.

Article A/2 – Sample Problem A/3 (1 of 3)

- Problem Statement

  Calculate the moments of inertia of the area of a circle about a diametral axis and about the polar axis through the center. Specify the radii of gyration.



Article A/2 – Sample Problem A/3 (2 of 3)

- Circular Ring of Area $dA = 2\pi r_0 dr_0$

$$\left[I_z = \int r^2\, dA\right] \qquad I_z = \int_0^r r_0^{\,2}(2\pi r_0\, dr_0) = \frac{\pi r^4}{2} = \frac{1}{2}Ar^2 \qquad Ans.$$

The polar radius of gyration is

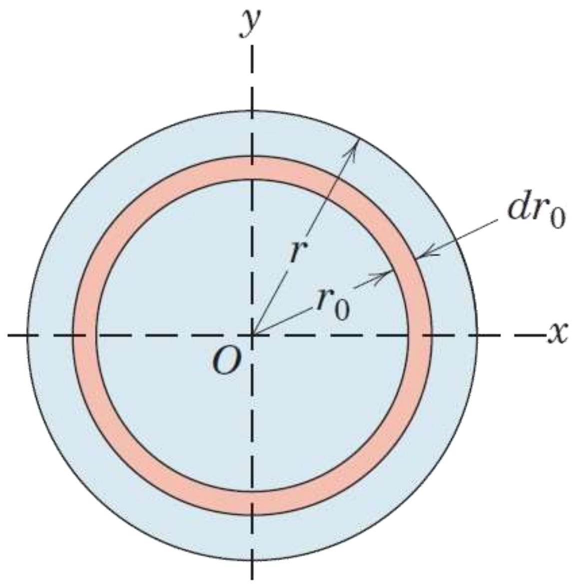
$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_z = \frac{r}{\sqrt{2}} \qquad Ans.$$

By symmetry $I_x = I_y$, so that from Eq. A/3

$$[I_z = I_x + I_y] \qquad I_x = \frac{1}{2}I_z = \frac{\pi r^4}{4} = \frac{1}{4}Ar^2 \qquad Ans.$$

The radius of gyration about the diametral axis is

$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_x = \frac{r}{2} \qquad Ans.$$



① Polar coordinates are certainly indicated here. Also, as before, we choose the simplest and lowest-order element possible, which is the differential ring. It should be evident immediately from the definition that the polar moment of inertia of the ring is its area $2\pi r_0\, dr_0$ times $r_0^{\,2}$.

Article A/2 – Sample Problem A/3 (3 of 3)

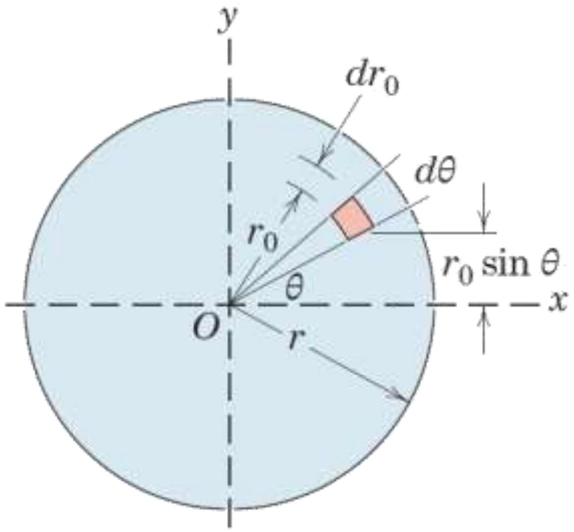
- Alternative Solution for Inertia about *x*-Axis

  Element of Area $dA = \pi r_0 dr_0 d\theta$

$$[I_x = \int y^2 \, dA] \qquad I_x = \int_0^{2\pi} \int_0^r (r_0 \sin\theta)^2 r_0 \, dr_0 \, d\theta$$

$$= \int_0^{2\pi} \frac{r^4 \sin^2\theta}{4} \, d\theta$$

$$= \frac{r^4}{4} \frac{1}{2} \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{\pi r^4}{4} \quad ② \qquad \textit{Ans.}$$

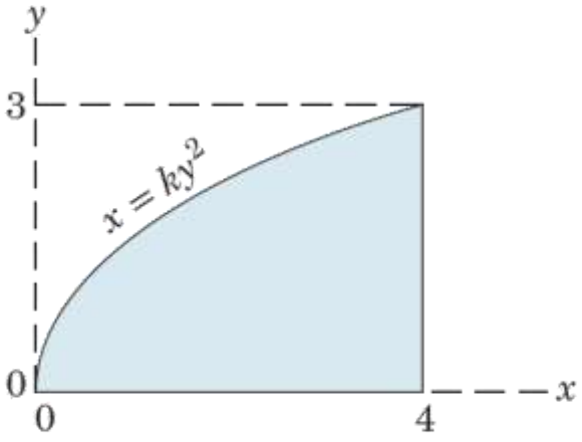


② This integration is straightforward, but the use of Eq. A/3 along with the result for I_z is certainly simpler.

Article A/2 – Sample Problem A/4 (1 of 3)

- Problem Statement

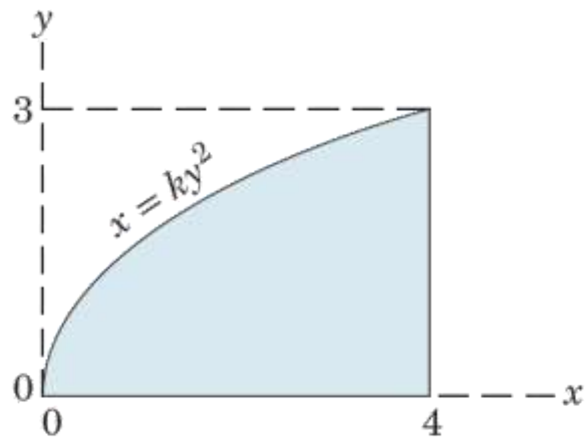
  Determine the moment of inertia of the area under the parabola about the x-axis. Solve by using (a) a horizontal strip of area and (b) a vertical strip of area.



Article A/2 – Sample Problem A/4 (2 of 3)

- Equation of the Parabola

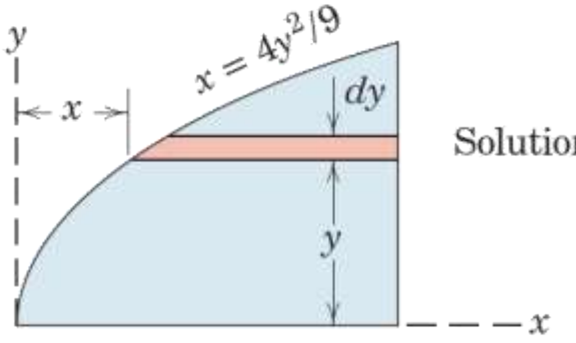
  The constant $k = 4/9$ is obtained first by substituting $x = 4$ and $y = 3$ into the equation for the parabola.



- (a) Solution using a Horizontal Strip

Since all parts of the horizontal strip are the same distance from the x-axis, the moment of inertia of the strip about the x-axis is $y^2\,dA$ where $dA = (4 - x)\,dy = 4(1 - y^2/9)\,dy$. Integrating with respect to y gives us

$$\left[I_x = \int y^2\,dA\right] \qquad I_x = \int_0^3 4y^2\left(1 - \frac{y^2}{9}\right)dy = \frac{72}{5} = 14.4\ (\text{units})^4 \qquad \textit{Ans.}$$



Solution (a)

Article A/2 – Sample Problem A/4 (3 of 3)

- (*b*) Solution using a Vertical Strip

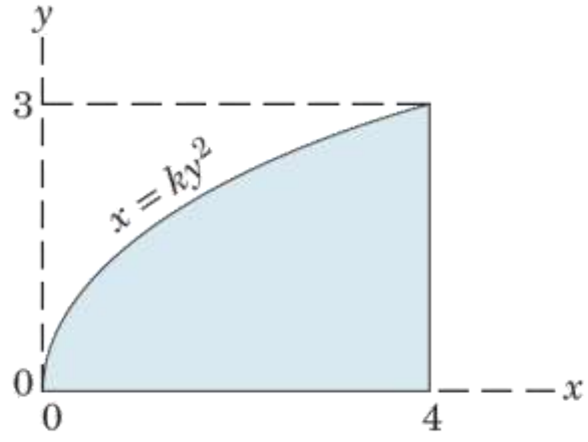
Here all parts of the element are at different distances from the x-axis, so we must use the correct expressions for the moment of inertia of the elemental rectangle about its base, which, from Sample Problem A/1, is $bh^3/3$. For the width dx and the height y the expression becomes

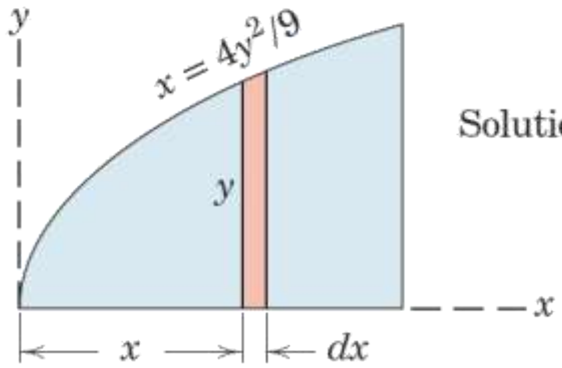
$$dI_x = \frac{1}{3}(dx)y^3$$

To integrate with respect to x, we must express y in terms of x, which gives $y = 3\sqrt{x}/2$, and the integral becomes

$$I_x = \frac{1}{3}\int_0^4 \left(\frac{3\sqrt{x}}{2}\right)^3 dx = \frac{72}{5} = 14.4 \text{ (units)}^4 \quad ① \qquad \textit{Ans.}$$

① There is little preference between Solutions (*a*) and (*b*). Solution (*b*) requires knowing the moment of inertia for a rectangular area about its base.



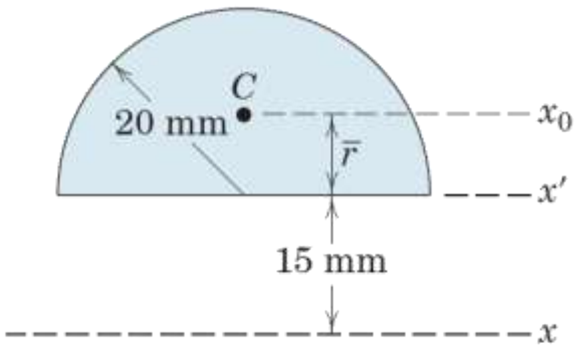


Solution (*b*)

Article A/2 – Sample Problem A/5 (1 of 2)

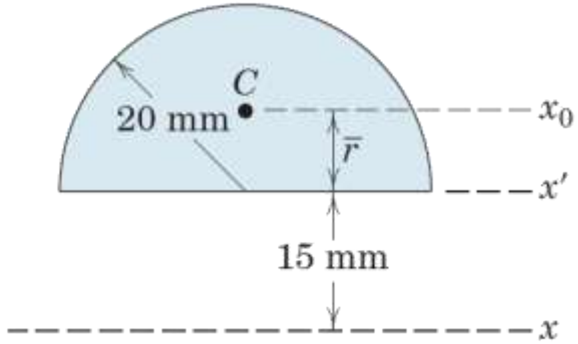
- Problem Statement

  Find the moment of inertia about the *x*-axis of the semicircular area.



Article A/2 – Sample Problem A/5 (2 of 2)

- Solution



The moment of inertia of the semicircular area about the x'-axis is one-half of that for a complete circle about the same axis. Thus, from the results of Sample Problem A/3,

$$I_{x'} = \frac{1}{2}\frac{\pi r^4}{4} = \frac{20^4\pi}{8} = 2\pi(10^4)\ \text{mm}^4$$

We obtain the moment of inertia $\bar{I}$ about the parallel centroidal axis x_0 next. Transfer is made through the distance $\bar{r} = 4r/(3\pi) = (4)(20)/(3\pi) = 80/(3\pi)$ mm by the parallel-axis theorem. Hence,

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = 2(10^4)\pi - \left(\frac{20^2\pi}{2}\right)\left(\frac{80}{3\pi}\right)^2 = 1.755(10^4)\ \text{mm}^4$$

Finally, we transfer from the centroidal x_0-axis to the x-axis. ① Thus,

$$[I = \bar{I} + Ad^2] \qquad I_x = 1.755(10^4) + \left(\frac{20^2\pi}{2}\right)\left(15 + \frac{80}{3\pi}\right)^2$$

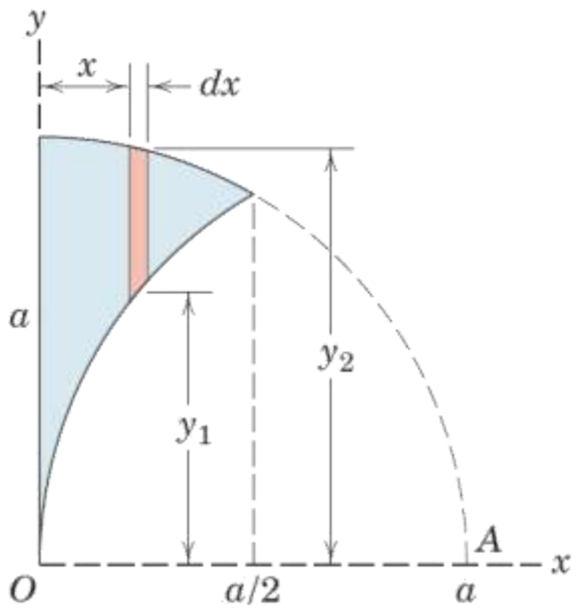
$$= 1.755(10^4) + 34.7(10^4) = 36.4(10^4)\ \text{mm}^4 \qquad \textit{Ans.}$$

① This problem illustrates the caution we should observe in using a double transfer of axes since neither the x'- nor the x-axis passes through the centroid C of the area. If the circle were complete with the centroid on the x' axis, only one transfer would be needed.

Article A/2 – Sample Problem A/6 (1 of 4)

- Problem Statement

  Calculate the moment of inertia about the *x*-axis of the area enclosed between the *y*-axis and the circular arcs of radius *a* whose centers are at *O* and *A*.



Article A/2 – Sample Problem A/6 (2 of 4)

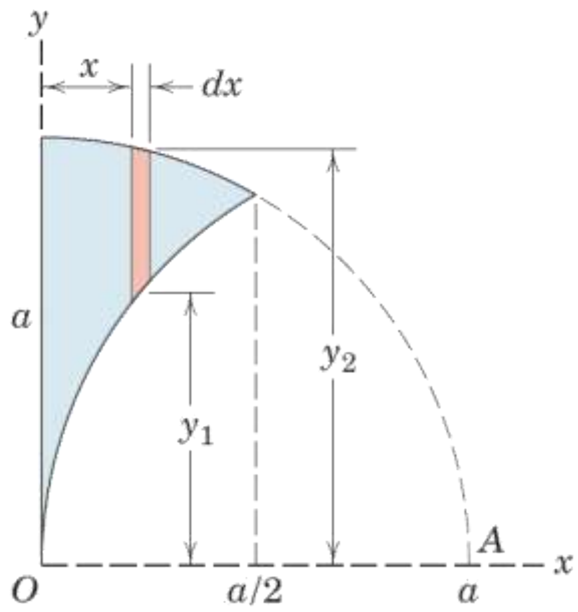
- Solution

The choice of a vertical differential strip of area permits one integration to cover the entire area. A horizontal strip would require two integrations with respect to y by virtue of the discontinuity. The moment of inertia of the strip about the x-axis is that of a strip of height y_2 minus that of a strip of height y_1. Thus, from the results of Sample Problem A/1 we write

$$dI_x = \frac{1}{3}(y_2\,dx)y_2^{\,2} - \frac{1}{3}(y_1\,dx)y_1^{\,2} = \frac{1}{3}(y_2^{\,3} - y_1^{\,3})\,dx$$

The values of y_2 and y_1 are obtained from the equations of the two curves, which are $x^2 + y_2^{\,2} = a^2$ and $(x-a)^2 + y_1^{\,2} = a^2$, and which give $y_2 = \sqrt{a^2 - x^2}$ and $y_1 = \sqrt{a^2 - (x-a)^2}$. ① Thus,

$$I_x = \frac{1}{3}\int_0^{a/2} \left\{(a^2 - x^2)\sqrt{a^2 - x^2} - [a^2 - (x-a)^2]\sqrt{a^2 - (x-a)^2}\right\} dx$$



① We choose the positive signs for the radicals here since both y_1 and y_2 lie above the x-axis.

Article A/2 – Sample Problem A/6 (3 of 4)

- Solution (cont.)

Simultaneous solution of the two equations which define the two circles gives the x-coordinate of the intersection of the two curves, which, by inspection, is $a/2$. Evaluation of the integrals gives

$$\int_0^{a/2} a^2\sqrt{a^2 - x^2}\,dx = \frac{a^4}{4}\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right)$$

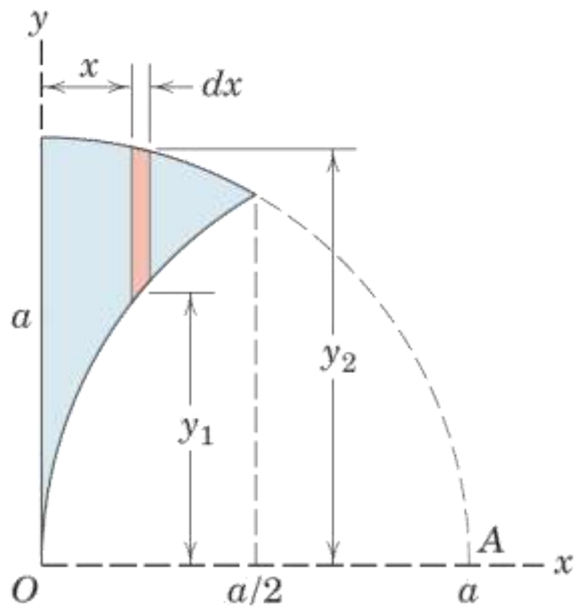
$$-\int_0^{a/2} x^2\sqrt{a^2 - x^2}\,dx = \frac{a^4}{16}\left(\frac{\sqrt{3}}{4} + \frac{\pi}{3}\right)$$

$$-\int_0^{a/2} a^2\sqrt{a^2 - (x-a)^2}\,dx = \frac{a^4}{4}\left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3}\right)$$

$$\int_0^{a/2} (x-a)^2\sqrt{a^2 - (x-a)^2}\,dx = \frac{a^4}{8}\left(\frac{\sqrt{3}}{8} + \frac{\pi}{3}\right)$$

Collection of the integrals with the factor of $\frac{1}{3}$ gives

$$I_x = \frac{a^4}{96}(9\sqrt{3} - 2\pi) = 0.0969a^4 \qquad \textit{Ans.}$$



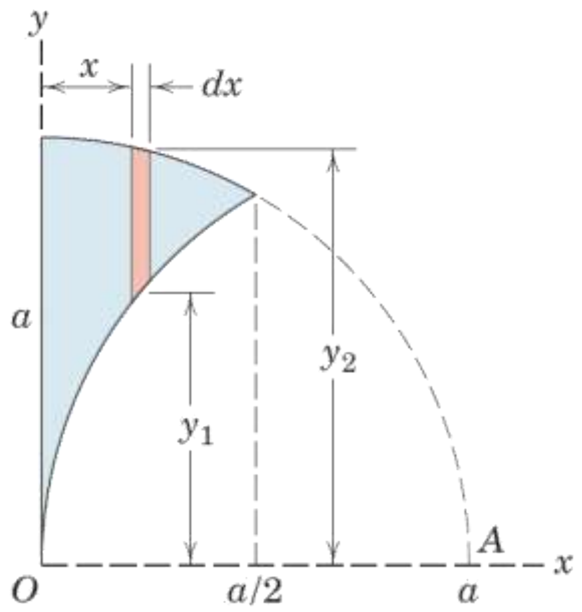
Article A/2 – Sample Problem A/6 (4 of 4)

- Alternative Element Form

If we had started from a second-order element $dA = dx\,dy$, we would write $y^2\,dx\,dy$ for the moment of inertia of the element about the x-axis. Integrating from y_1 to y_2 holding x constant produces for the vertical strip

$$dI_x = \left[\int_{y_1}^{y_2} y^2\,dy\right] dx = \frac{1}{3}({y_2}^3 - {y_1}^3)\,dx$$

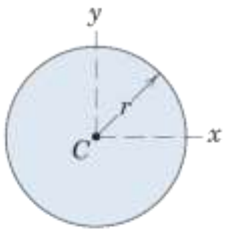
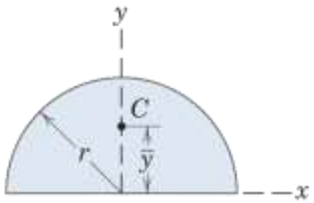
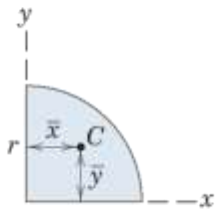
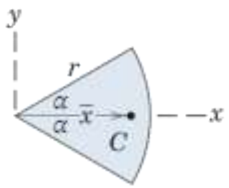
which is the expression we started with by having the moment-of-inertia result for a rectangle in mind.



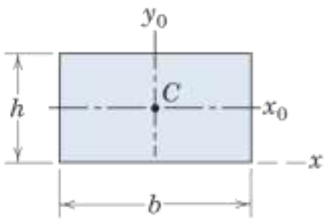
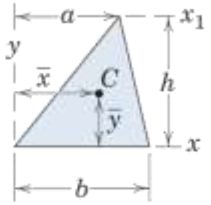
Article A/3 Composite Areas

- Introduction
  - It is frequently necessary to calculate the moment of inertia of an area composed of a number of distinct parts of simple and calculable geometric shape.
  - The moment of inertia of a composite area about a particular axis is simply the sum of the moments of inertia of its component parts about the same axis.
  - It is convenient to regard a composite area as being composed of positive and negative parts. We may then treat the moment of inertia of a negative area (hole or cutout) as a negative quantity.
  - Tabulated inertias and a systematic approach prove quite useful for these calculations.

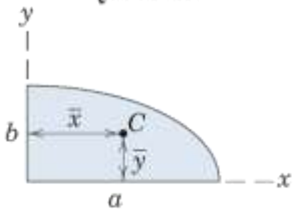
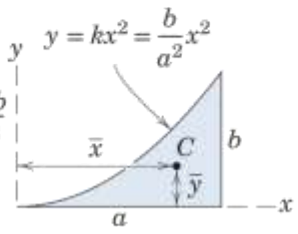
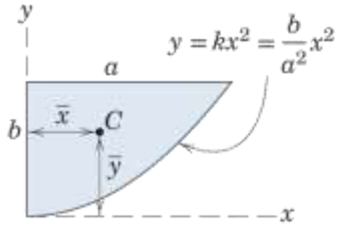
Article A/3 – Inertias of Common Shapes (1 of 3)

Figure	Centroid	Area Moments of Inertia
Circular Area 	—	$I_x = I_y = \dfrac{\pi r^4}{4}$ $I_z = \dfrac{\pi r^4}{2}$
Semicircular Area 	$\bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{8}$ $\bar{I}_x = \left(\dfrac{\pi}{8} - \dfrac{8}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{4}$
Quarter-Circular Area 	$\bar{x} = \bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{8}$
Area of Circular Sector 	$\bar{x} = \dfrac{2}{3}\dfrac{r \sin \alpha}{\alpha}$	$I_x = \dfrac{r^4}{4}\left(\alpha - \dfrac{1}{2}\sin 2\alpha\right)$ $I_y = \dfrac{r^4}{4}\left(\alpha + \dfrac{1}{2}\sin 2\alpha\right)$ $I_z = \dfrac{1}{2} r^4 \alpha$

Article A/3 – Inertias of Common Shapes (2 of 3)

Figure	Centroid	Area Moments of Inertia
Rectangular Area 	—	$I_x = \frac{bh^3}{3}$ $\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_z = \frac{bh}{12}(b^2 + h^2)$
Triangular Area 	$\bar{x} = \frac{a+b}{3}$ $\bar{y} = \frac{h}{3}$	$I_x = \frac{bh^3}{12}$ $\bar{I}_x = \frac{bh^3}{36}$ $I_{x_1} = \frac{bh^3}{4}$

Article A/3 – Inertias of Common Shapes (3 of 3)

Figure	Centroid	Area Moments of Inertia
Area of Elliptical Quadrant 	$\bar{x} = \frac{4a}{3\pi}$ $\bar{y} = \frac{4b}{3\pi}$	$I_x = \frac{\pi ab^3}{16}, \bar{I}_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)ab^3$ $I_y = \frac{\pi a^3 b}{16}, \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)a^3 b$ $I_z = \frac{\pi ab}{16}(a^2 + b^2)$
Subparabolic Area Area $A = \frac{ab}{3}$ 	$\bar{x} = \frac{3a}{4}$ $\bar{y} = \frac{3b}{10}$	$I_x = \frac{ab^3}{21}$ $I_y = \frac{a^3 b}{5}$ $I_z = ab\left(\frac{a^2}{5} + \frac{b^2}{21}\right)$
Parabolic Area Area $A = \frac{2ab}{3}$ 	$\bar{x} = \frac{3a}{8}$ $\bar{y} = \frac{3b}{5}$	$I_x = \frac{2ab^3}{7}$ $I_y = \frac{2a^3 b}{15}$ $I_z = 2ab\left(\frac{a^2}{15} + \frac{b^2}{7}\right)$

Article A/3 – Typical Solution Process

- Tabulated Approach

Part	Area, A	d_x	d_y	Ad_x^2	Ad_y^2	$\bar{I}_x$	$\bar{I}_y$
Sums	ΣA			ΣAd_x^2	ΣAd_y^2	$\Sigma\bar{I}_x$	$\Sigma\bar{I}_y$

From the sums of the four columns, then, the moments of inertia for the composite area about the x- and y-axes become

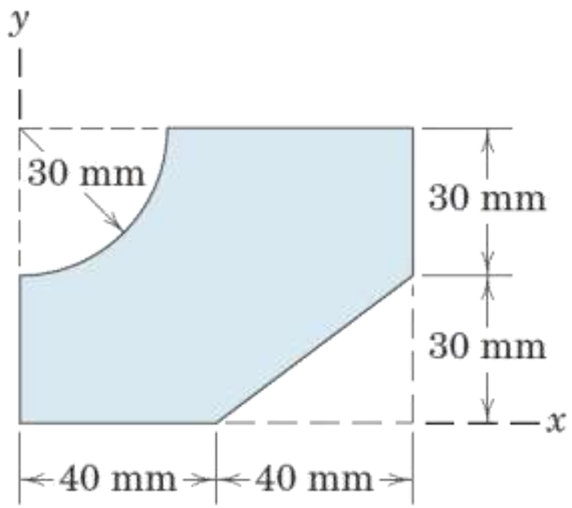
$$I_x = \Sigma\bar{I}_x + \Sigma Ad_x^2$$

$$I_y = \Sigma\bar{I}_y + \Sigma Ad_y^2$$

Article A/3 – Sample Problem A/7 (1 of 2)

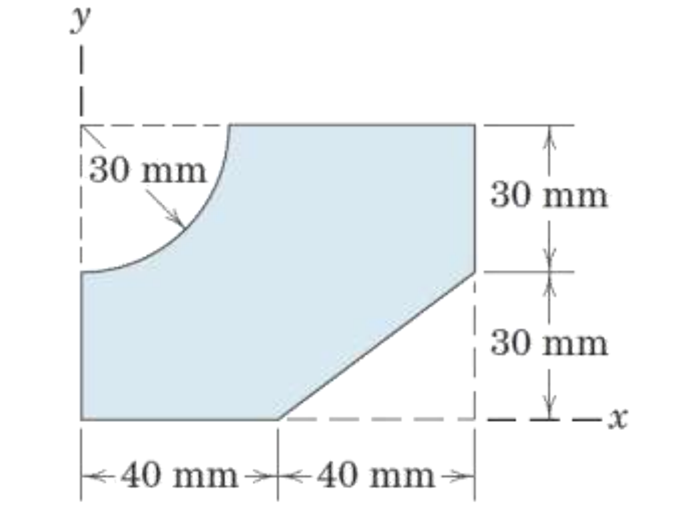
- Problem Statement

  Determine the moments of inertia about the x- and y-axes for the shaded area. Make direct use of the expressions given in Table D/3 for the centroidal moments of inertia of the constituent parts.



Article A/3 – Sample Problem A/7 (2 of 2)

- Composite Shapes
  - Rectangular Area (1)
  - Quarter-Circular Cutout (2)
  - Triangular Cutout (3)

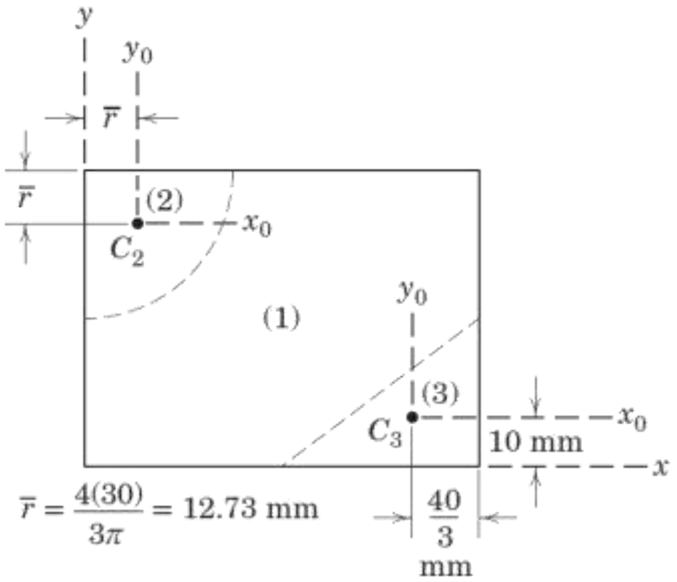


- Tabulated Values

PART	A mm^2	d_x mm	d_y mm	Ad_x^2 mm^4	Ad_y^2 mm^4	$\bar{I}_x$ mm^4	$\bar{I}_y$ mm^4
1	$80(60)$	30	40	$4.32(10^6)$	$7.68(10^6)$	$\frac{1}{12}(80)(60)^3$	$\frac{1}{12}(60)(80)^3$
2	$-\frac{1}{4}\pi(30)^2$	$(60-12.73)$	12.73	$-1.579(10^6)$	$-0.1146(10^6)$	$-\left(\frac{\pi}{16}-\frac{4}{9\pi}\right)30^4$	$-\left(\frac{\pi}{16}-\frac{4}{9\pi}\right)30^4$
3	$-\frac{1}{2}(40)(30)$	$\frac{30}{3}$	$\left(80-\frac{40}{3}\right)$	$-0.06(10^6)$	$-2.67(10^6)$	$-\frac{1}{36}\,40(30)^3$	$-\frac{1}{36}\,(30)(40)^3$
TOTALS	3490			$2.68(10^6)$	$4.90(10^6)$	$1.366(10^6)$	$2.46(10^6)$

$$[I_x = \Sigma \bar{I}_x + \Sigma A d_x^2] \quad I_x = 1.366(10^6) + 2.68(10^6) = 4.05(10^6)\ \text{mm}^4 \quad \textit{Ans.}$$

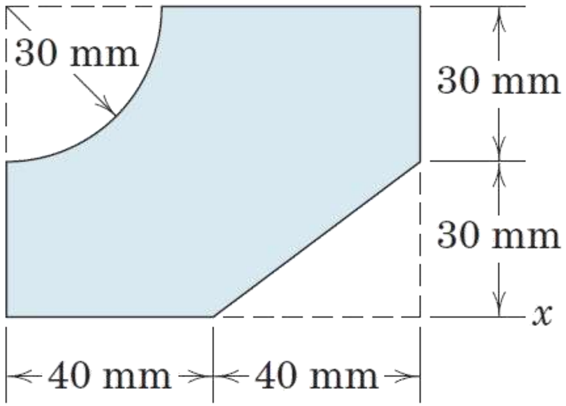
$$[I_y = \Sigma \bar{I}_y + \Sigma A d_y^2] \quad I_y = 2.46(10^6) + 4.90(10^6) = 7.36(10^6)\ \text{mm}^4 \quad \textit{Ans.}$$



Article A/3 – Sample Problem A/8 (1 of 4)

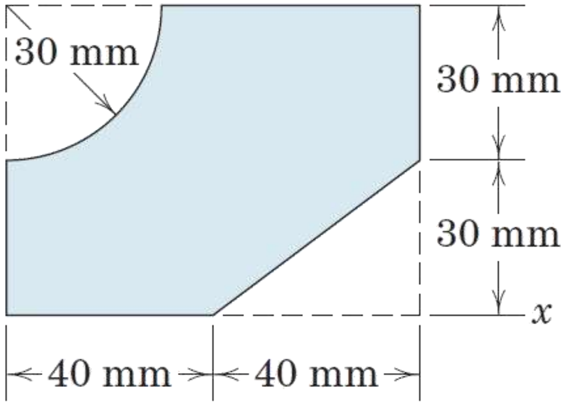
- Problem Statement

  Calculate the moment of inertia and radius of gyration about the *x*-axis for the shaded area shown. Wherever possible, make expedient use of tabulated moments of inertia.



Article A/3 – Sample Problem A/8 (2 of 4)

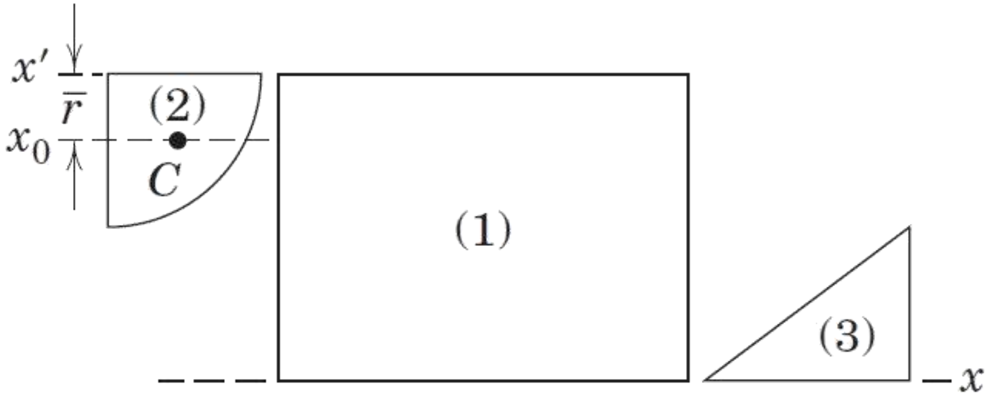
- Composite Shapes
  - Rectangular Area (1)
  - Quarter-Circular Cutout (2)
  - Triangular Cutout (3)



- Inertia for the Rectangular Area

The composite area is composed of the positive area of the rectangle (1) and the negative areas of the quarter circle (2) and triangle (3). For the rectangle the moment of inertia about the x-axis, from Sample Problem A/1 (or Table D/3), is

$$I_x = \frac{1}{3}Ah^2 = \frac{1}{3}(80)(60)(60)^2 = 5.76(10^6)\ \text{mm}^4$$



Article A/3 – Sample Problem A/8 (3 of 4)

- Inertia for the Quarter-Circular Cutout

From Sample Problem A/3 (or Table D/3), the moment of inertia of the negative quarter-circular area about its base axis x' is

$$I_{x'} = -\frac{1}{4}\left(\frac{\pi r^4}{4}\right) = -\frac{\pi}{16}(30)^4 = -0.1590(10^6)\ \text{mm}^4$$

We now transfer this result through the distance $\bar{r} = 4r/(3\pi) = 4(30)/(3\pi) = 12.73$ mm by the transfer-of-axis theorem to get the centroidal moment of inertia of part (2) (or use Table D/3 directly).

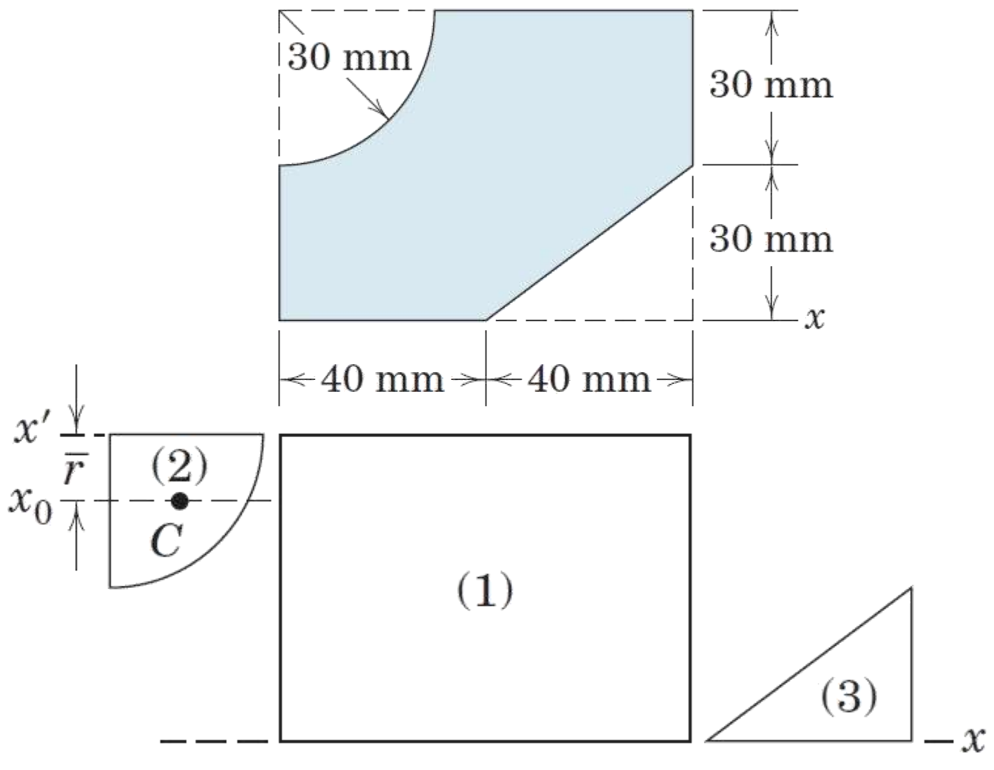
$$[\bar{I} = I - Ad^2] \qquad \bar{I}_x = -0.1590(10^6) - \left[-\frac{\pi(30)^2}{4}(12.73)^2\right] \quad ①$$

$$= -0.0445(10^6)\ \text{mm}^4$$

The moment of inertia of the quarter-circular part about the x-axis is now

$$[I = \bar{I} + Ad^2] \quad I_x = -0.0445(10^6) + \left[-\frac{\pi(30)^2}{4}\right](60 - 12.73)^2 \quad ②$$

$$= -1.624(10^6)\ \text{mm}^4$$



① Note that we must transfer the moment of inertia for the quarter-circular area to its centroidal axis x_0 before we can transfer it to the x-axis, as was done in Sample Problem A/5.

② We watch our signs carefully here. Since the area is negative, both $\bar{I}$ and A carry negative signs.

Article A/3 – Sample Problem A/8 (4 of 4)

- Inertia for the Triangular Cutout

Finally, the moment of inertia of the negative triangular area (3) about its base, from Sample Problem A/2 (or Table D/3), is

$$I_x = -\frac{1}{12}bh^3 = -\frac{1}{12}(40)(30)^3 = -0.90(10^6)\ \text{mm}^4$$

- Total Inertia and Radius of Gyration

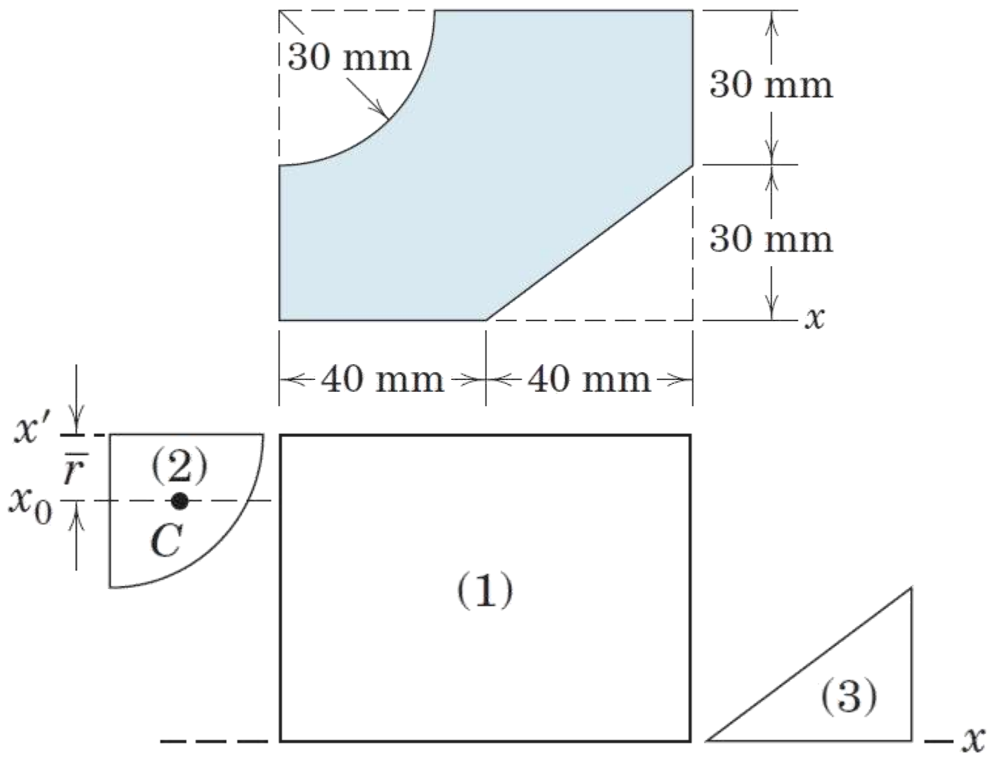
The total moment of inertia about the x-axis of the composite area is, consequently,

$$I_x = 5.76(10^6) - 1.624(10^6) - 0.09(10^6) = 4.05(10^6)\ \text{mm}^4 \quad ③ \quad \textit{Ans.}$$

This result agrees with that of Sample Problem A/7.

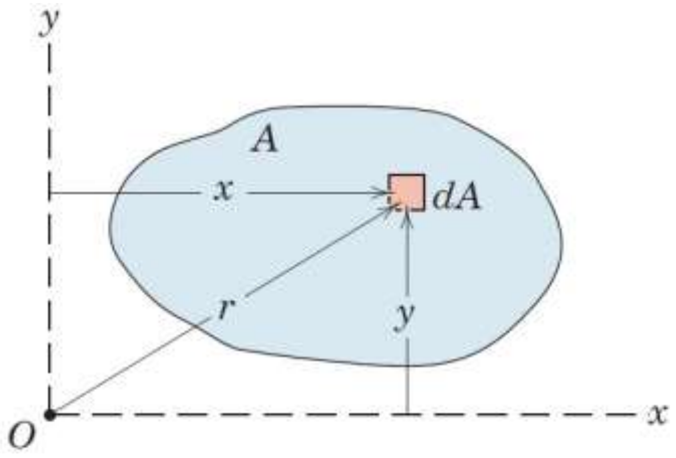
The net area of the figure is $A = 60(80) - \frac{1}{4}\pi(30)^2 - \frac{1}{2}(40)(30) = 3490\ \text{mm}^2$ so that the radius of gyration about the x-axis is

$$k_x = \sqrt{I_x/A} = \sqrt{4.05(10^6)/3490} = 34.0\ \text{mm} \qquad \textit{Ans.}$$



③ Always use common sense at key points such as this. The two minus signs are consistent with the fact that subareas (2) and (3) reduce the numerical value of the moment of inertia of the basic rectangular area.

Article A/4 Products of Inertia and Rotation of Axes



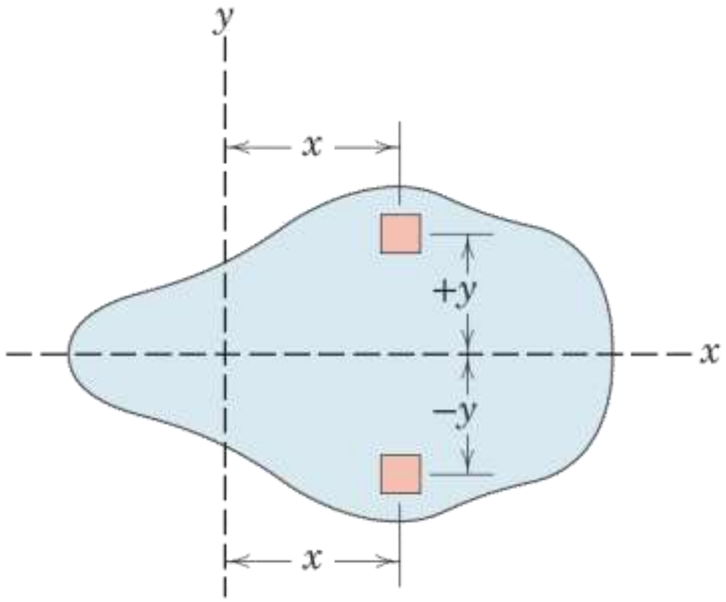
- Definition
  - $I_{xy} = \int xy \, dA$

- Dimensions and Units
  - Length to the 4th Power, L^4
  - SI Units: mm^4 (most common)
  - U.S. Customary Units: $in.^4$ (most common)

Article A/4 – Comments About Products of Inertia

- A product of inertia can be positive, negative, or zero.

- The product of inertia is zero whenever either of the reference axes is an axis of symmetry. This is illustrated in the figure at right where the area is symmetric about the *x*-axis.



Article A/4 – Transfer of Axes

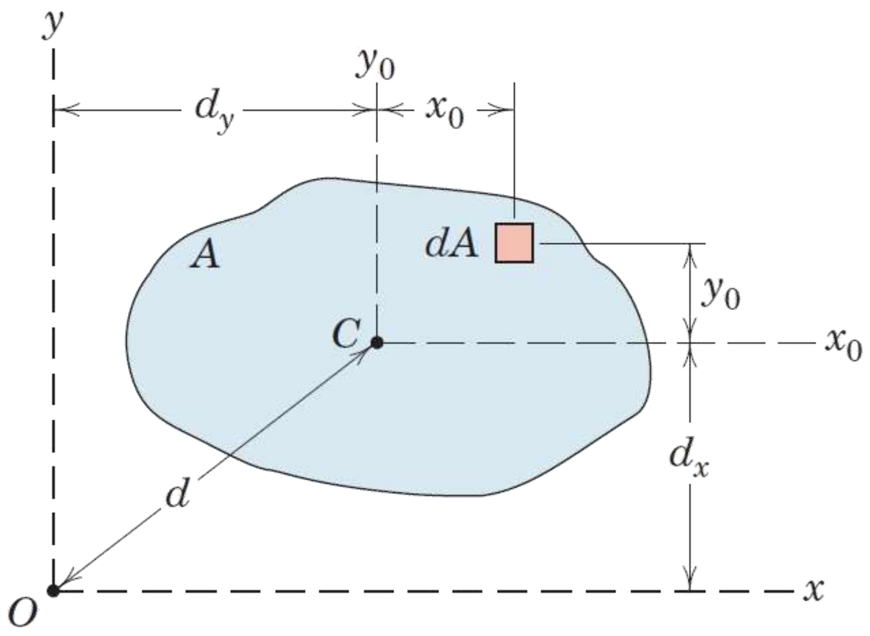
- Overview

- Mathematics

$$I_{xy} = \int (x_0 + d_y)(y_0 + d_x)\, dA$$
$$= \int x_0 y_0\, dA + d_x \int x_0\, dA + d_y \int y_0\, dA + d_x d_y \int dA$$

- Result

$$\boxed{I_{xy} = \bar{I}_{xy} + d_x d_y A}$$



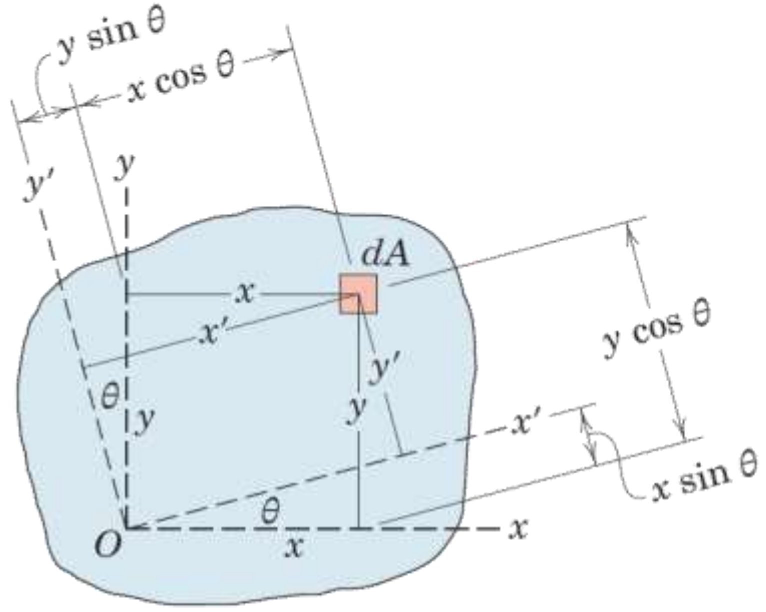
Article A/4 – Rotation of Axes (1 of 4)

- Introduction

- Moments of Inertia about Tilted Axes

$$I_{x'} = \int y'^2\, dA = \int (y \cos\theta - x \sin\theta)^2\, dA$$

$$I_{y'} = \int x'^2\, dA = \int (y \sin\theta + x \cos\theta)^2\, dA$$



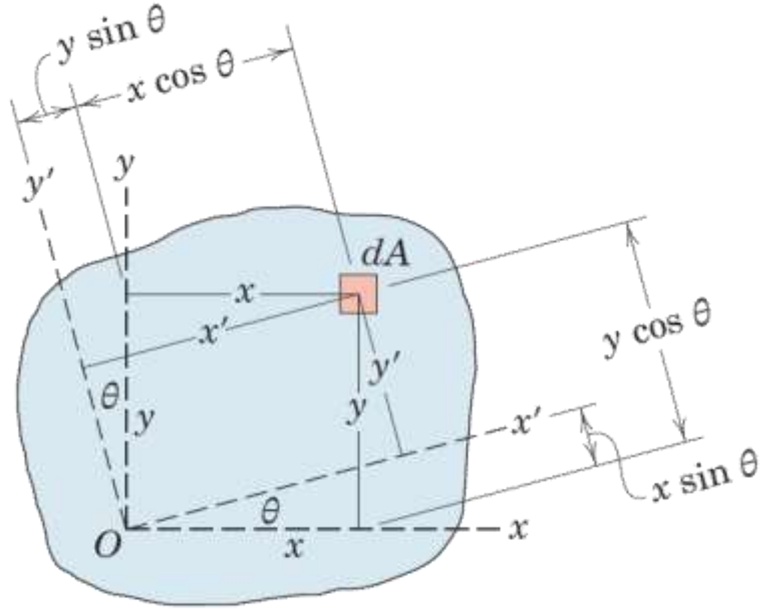
Article A/4 – Rotation of Axes (2 of 4)

- Trigonometric Identities of Importance

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

- Substitution and Simplification

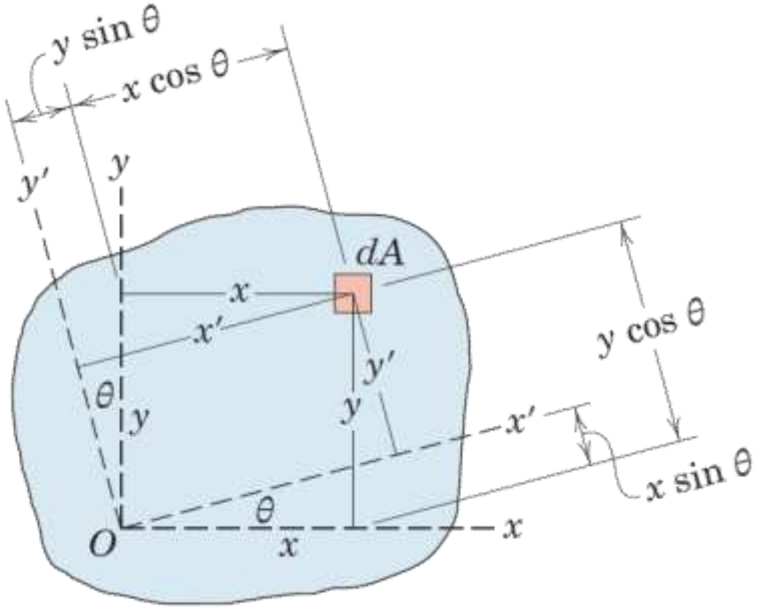
$$\boxed{\begin{aligned} I_{x'} &= \frac{I_x+I_y}{2} + \frac{I_x-I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta \\ I_{y'} &= \frac{I_x+I_y}{2} - \frac{I_x-I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta \end{aligned}}$$



Article A/4 – Rotation of Axes (3 of 4)

- Product of Inertia about Tilted Axes

$$I_{x'y'} = \int x'y' \, dA = \int (y \sin\theta + x \cos\theta)(y \cos\theta - x \sin\theta) \, dA$$



- Trigonometric Identities of Importance

$$\sin\theta \cos\theta = \frac{1}{2}\sin 2\theta \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta$$

- Substitution and Simplification

$$\boxed{I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta}$$

Article A/4 – Rotation of Axes (4 of 4)

- Angle for Principal Axes of Inertia

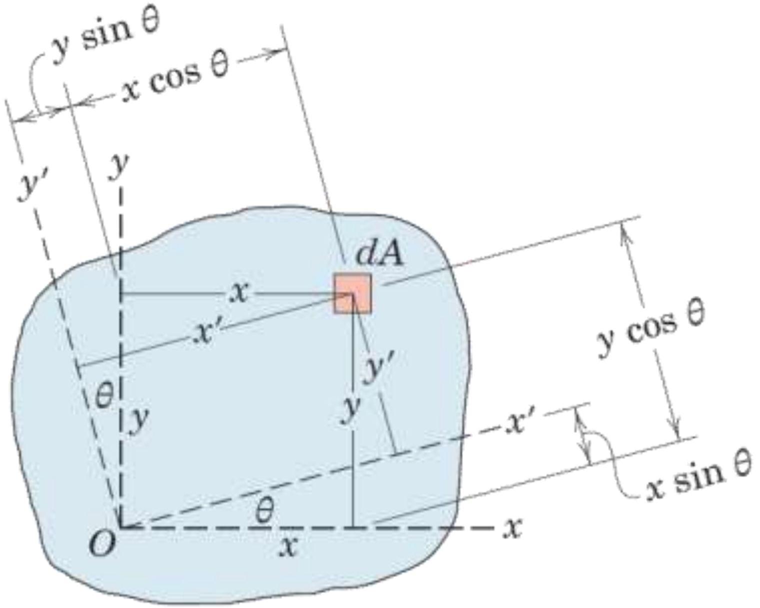
$$\frac{dI_{x'}}{d\theta} = (I_y - I_x)\sin 2\theta - 2I_{xy}\cos 2\theta = 0$$

$$\boxed{\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}}$$

- Principal Moments of Inertia

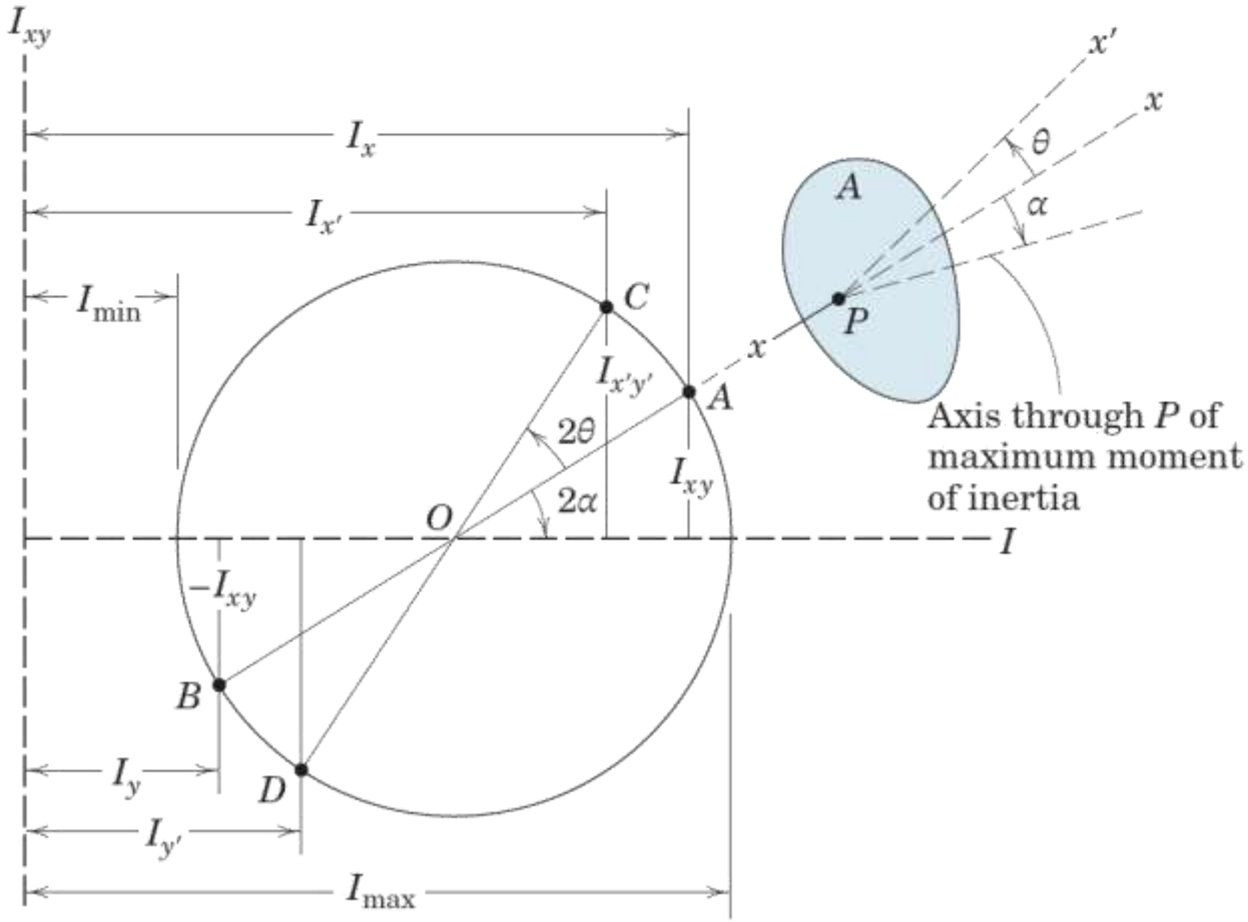
$$I_{\max} = \frac{I_x + I_y}{2} + \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}{}^2}$$

$$I_{\min} = \frac{I_x + I_y}{2} - \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}{}^2}$$



Article A/4 – Mohr's Circle of Inertia (1 of 2)

- Illustration



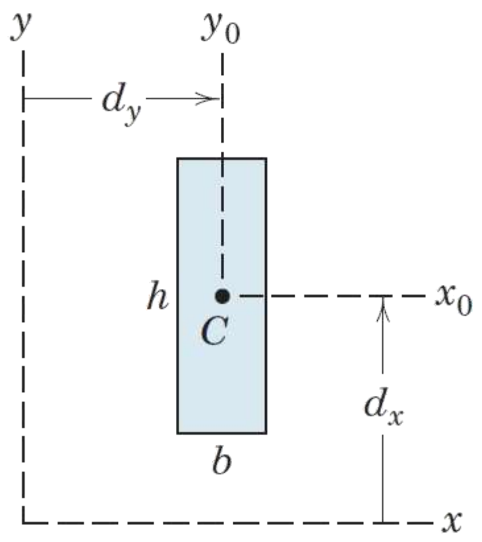
Article A/4 – Mohr’s Circle of Inertia (2 of 2)

- Procedure
  1. Sketch a horizontal axis for the measurement of moments of inertia and a vertical axis for the measurement of products of inertia.
  2. Plot point *A* which has coordinates (I_x, I_{xy}) and point *B* which has coordinates (I_y, $-I_{xy}$).
  3. Sketch a line between points *A* and *B* and then draw a circle with these two points as the extremities of a diameter.
- Calculated Quantities
  1. The angle from radius *OA* to the horizontal axis is 2α or twice the angle from the *x*-axis of the area in question to the axis of maximum moment of inertia. The angle on the diagram and the angle on the area are both measured in the same sense.
  2. The coordinates of any point *C* are ($I_{x'}$, $I_{x'y'}$), and those of corresponding point *D* are ($I_{y'}$, $-I_{x'y'}$), and can be found by rotating the diametral line sketched previously through the angle 2θ in the correct sense.
  3. The radius of the circle, maximum and minimum moments of inertia, and intermediate moments of inertia can be calculated using the geometry of the sketch.

Article A/4 – Sample Problem A/9 (1 of 2)

- Problem Statement

  Determine the product of inertia of the rectangular area with centroid at *C* with respect to the *x*-*y* axes parallel to its sides.



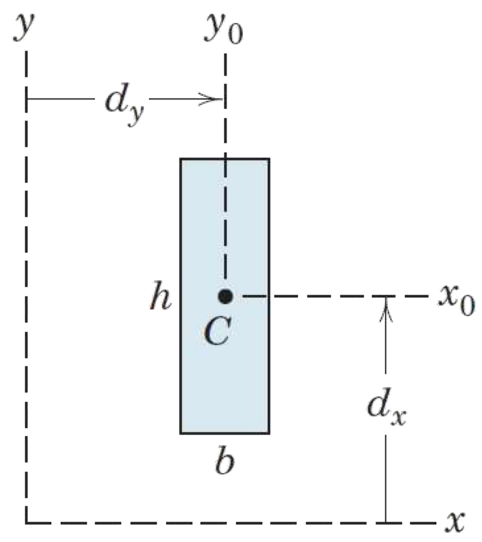
Article A/4 – Sample Problem A/9 (2 of 2)

- Solution

Since the product of inertia $\bar{I}_{xy}$ about the axes x_0-y_0 is zero by symmetry, the transfer-of-axis theorem gives us

$[I_{xy} = \bar{I}_{xy} + d_x d_y A] \qquad I_{xy} = d_x d_y bh \qquad$ *Ans.*

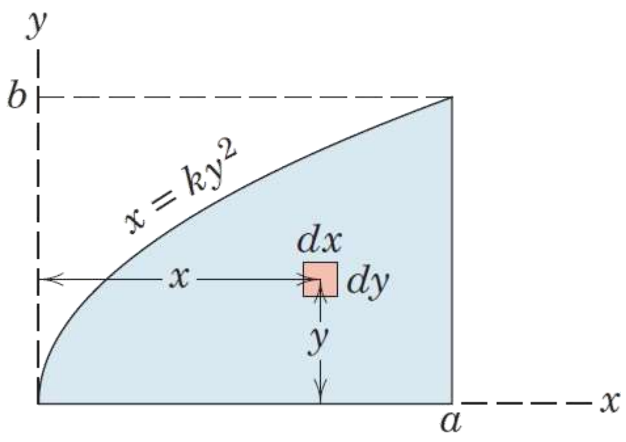
In this example both d_x and d_y are shown positive. We must be careful to be consistent with the positive directions of d_x and d_y as defined, so that their proper signs are observed.



Article A/4 – Sample Problem A/10 (1 of 3)

- Problem Statement

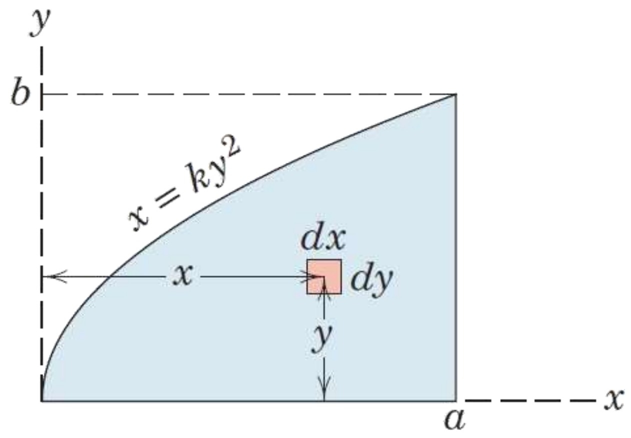
  Determine the product of inertia about the *x*-*y* axes for the area under the parabola.



Article A/4 – Sample Problem A/10 (2 of 3)

- Equation of the Parabola

With the substitution of $x = a$ when $y = b$, the equation of the curve becomes $x = ay^2/b^2$.



- Second-Order Element *dA* = *dxdy*

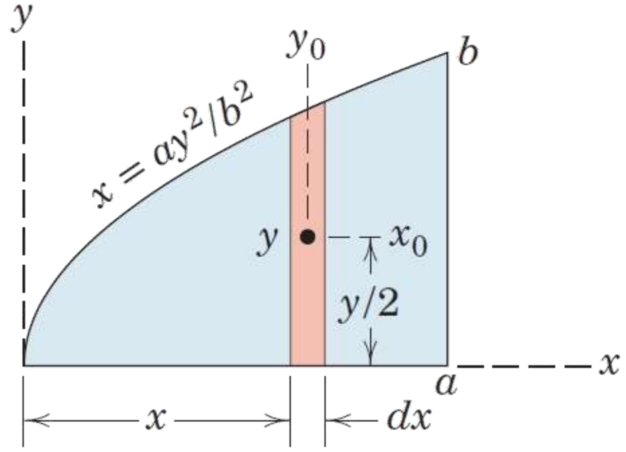
$$I_{xy} = \int_0^b \int_{ay^2/b^2}^a xy\, dx\, dy = \int_0^b \frac{1}{2}\left(a^2 - \frac{a^2y^4}{b^4}\right) y\, dy = \frac{1}{6}a^2b^2 \qquad Ans.$$

Article A/4 – Sample Problem A/10 (3 of 3)

- First Order Element, Vertical Strip

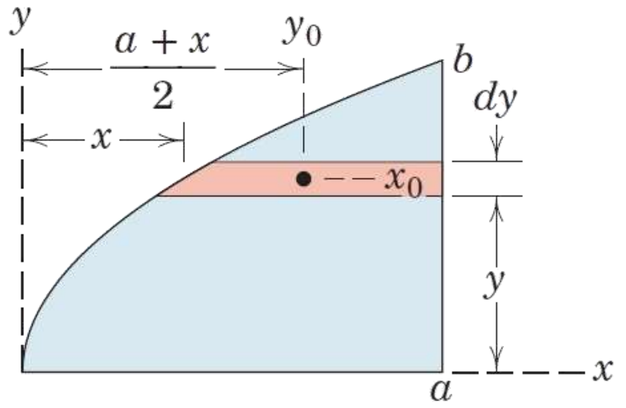
Alternatively, we can start with a first-order elemental strip and save one integration by using the results of Sample Problem A/9. Taking a vertical strip $dA = y\ dx$ gives $dI_{xy} = 0 + (\frac{1}{2}y)(x)(y\ dx)$, where the distances to the centroidal axes of the elemental rectangle are $d_x = y/2$ and $d_y = x$. ① Now we have

$$I_{xy} = \int_0^a \frac{y^2}{2} x\ dx = \int_0^a \frac{xb^2}{2a} x\ dx = \frac{b^2}{6a} x^3 \Big|_0^a = \frac{1}{6}a^2b^2 \qquad \textit{Ans.}$$



- First Order Element, Horizontal Strip

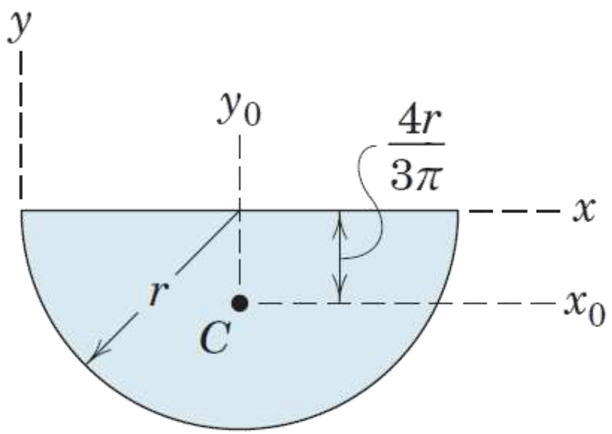
① If we had chosen a horizontal strip, our expression would have become $dI_{xy} = y\frac{1}{2}(a + x)[(a - x)\ dy]$, which when integrated, of course, gives us the same result as before.



Article A/4 – Sample Problem A/11 (1 of 2)

- Problem Statement

  Determine the product of inertia of the semicircular area with respect to the *x*-*y* axes.



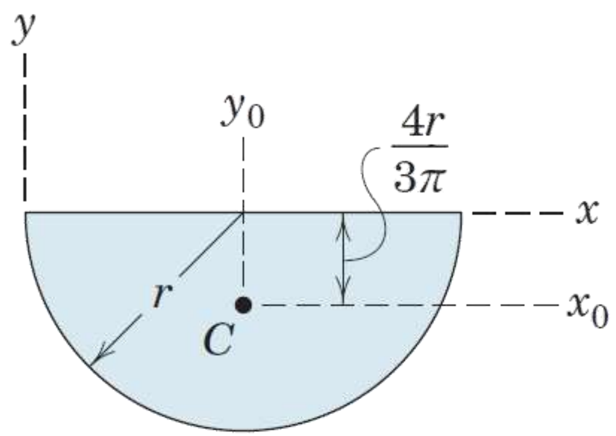
Article A/4 – Sample Problem A/11 (2 of 2)

- Solution

$[I_{xy} = \bar{I}_{xy} + d_x d_y A] \qquad I_{xy} = 0 + \left(-\dfrac{4r}{3\pi}\right)(r)\left(\dfrac{\pi r^2}{2}\right) = -\dfrac{2r^4}{3}$ *Ans.*

where the x- and y-coordinates of the centroid C are $d_y = +r$ and $d_x = -4r/(3\pi)$. Because y_0 is an axis of symmetry, $\bar{I}_{xy} = 0$.

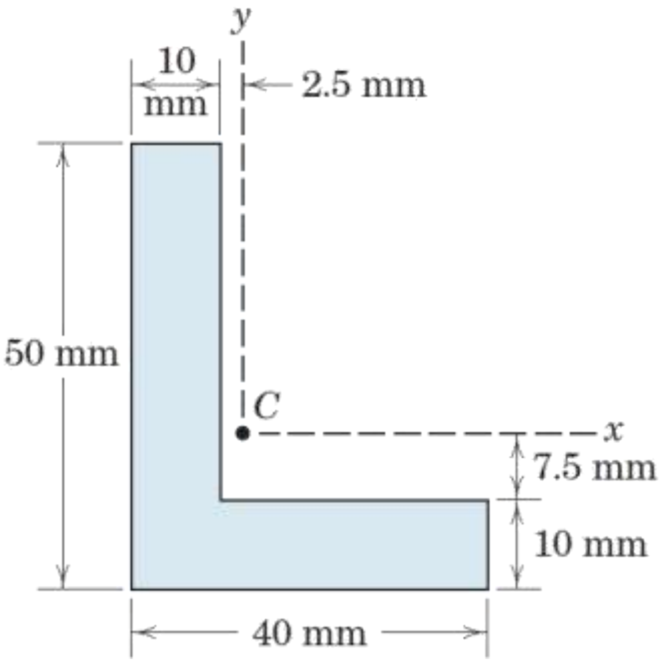
① Proper use of the transfer-of-axis theorem saves a great deal of labor in computing products of inertia.



Article A/4 – Sample Problem A/12 (1 of 5)

- Problem Statement

  Determine the orientation of the principal axes of inertia through the centroid of the angle section and determine the corresponding maximum and minimum moments of inertia.



Article A/4 – Sample Problem A/12 (2 of 5)

- ## Products of Inertia

Products of Inertia for the rectangles about their own centroidal axes are zero.

$[I_{xy} = \bar{I}_{xy} + d_x d_y A]$ $\qquad I_{xy} = 0 + (-12.5)(+7.5)(400) = -3.75(10^4)\ \text{mm}^4$

where $\qquad d_x = -(7.5 + 5) = -12.5\ \text{mm}$

and $\qquad d_y = +(20 - 10 - 2.5) = 7.5\ \text{mm}$

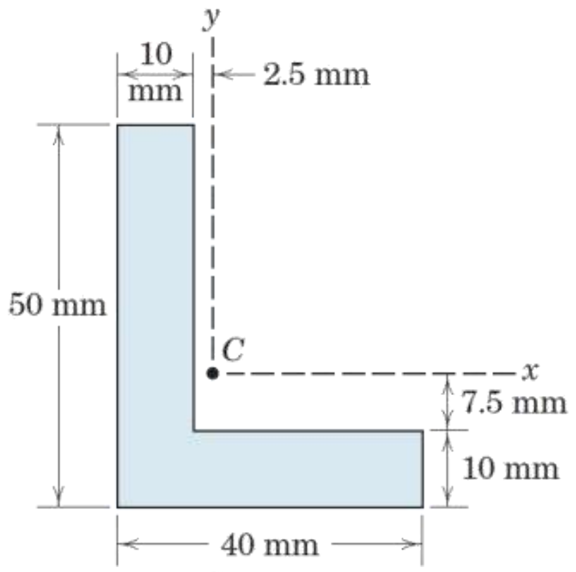
Likewise for part II,

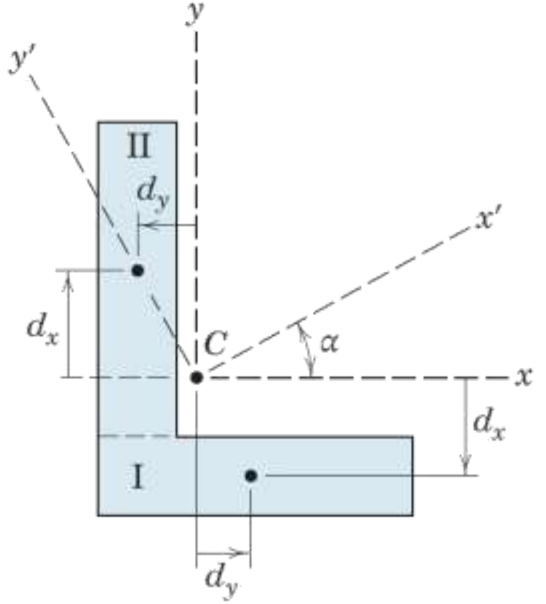
$[I_{xy} = \bar{I}_{xy} + d_x d_y A]$ $\qquad I_{xy} = 0 + (12.5)(-7.5)(400) = -3.75(10^4)\ \text{mm}^4$

where $\quad d_x = +(20 - 7.5) = 12.5\ \text{mm}, \qquad d_y = -(5 + 2.5) = -7.5\ \text{mm}$

For the complete angle,

$$I_{xy} = -3.75(10^4) - 3.75(10^4) = -7.5(10^4)\ \text{mm}^4$$





Article A/4 – Sample Problem A/12 (3 of 5)

- Moments of Inertia

$[I = \bar{I} + Ad^2]$ $\qquad I_x = \frac{1}{12}(40)(10)^3 + (400)(12.5)^2 = 6.58(10^4)\text{ mm}^4$

$$I_y = \frac{1}{12}(10)(40)^3 + (400)(7.5)^2 = 7.58(10^4)\text{ mm}^4$$

and the moments of inertia for part II about these same axes are

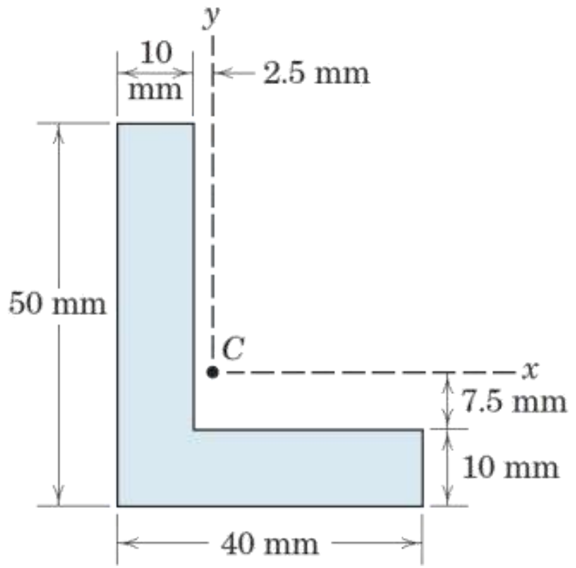
$[I = \bar{I} + Ad^2]$ $\qquad I_x = \frac{1}{12}(10)(40)^3 + (400)(12.5)^2 = 11.58(10^4)\text{ mm}^4$

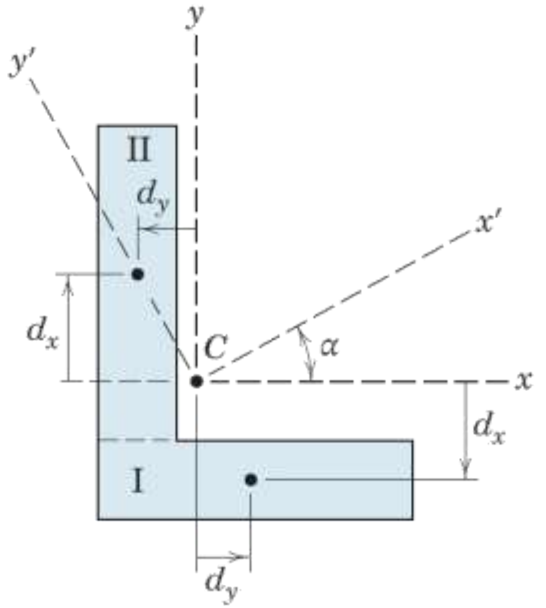
$$I_y = \frac{1}{12}(40)(10)^3 + (400)(7.5)^2 = 2.58(10^4)\text{ mm}^4$$

Thus, for the entire section we have

$$I_x = 6.58(10^4) + 11.58(10^4) = 18.17(10^4)\text{ mm}^4$$

$$I_y = 7.58(10^4) + 2.58(10^4) = 10.17(10^4)\text{ mm}^4$$





Article A/4 – Sample Problem A/12 (4 of 5)

- Principal Axes

$$\left[\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}\right] \qquad \tan 2\alpha = \frac{2(-7.50)}{10.17 - 18.17} = 1.875$$

$$2\alpha = 61.9^\circ \qquad \alpha = 31.0^\circ \qquad \textit{Ans.}$$

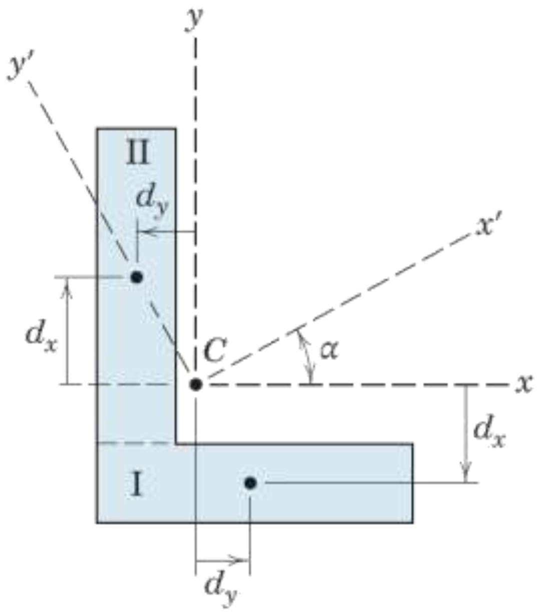
We now compute the principal moments of inertia from Eqs. A/9 using α for θ and get I_{max} from $I_{x'}$ and I_{min} from $I_{y'}$. Thus,

$$I_{max} = \left[\frac{18.17 + 10.17}{2} + \frac{18.17 - 10.17}{2}(0.471) + (7.50)(0.882)\right](10^4)$$

$$= 22.7(10^4)\ \text{mm}^4 \qquad \textit{Ans.}$$

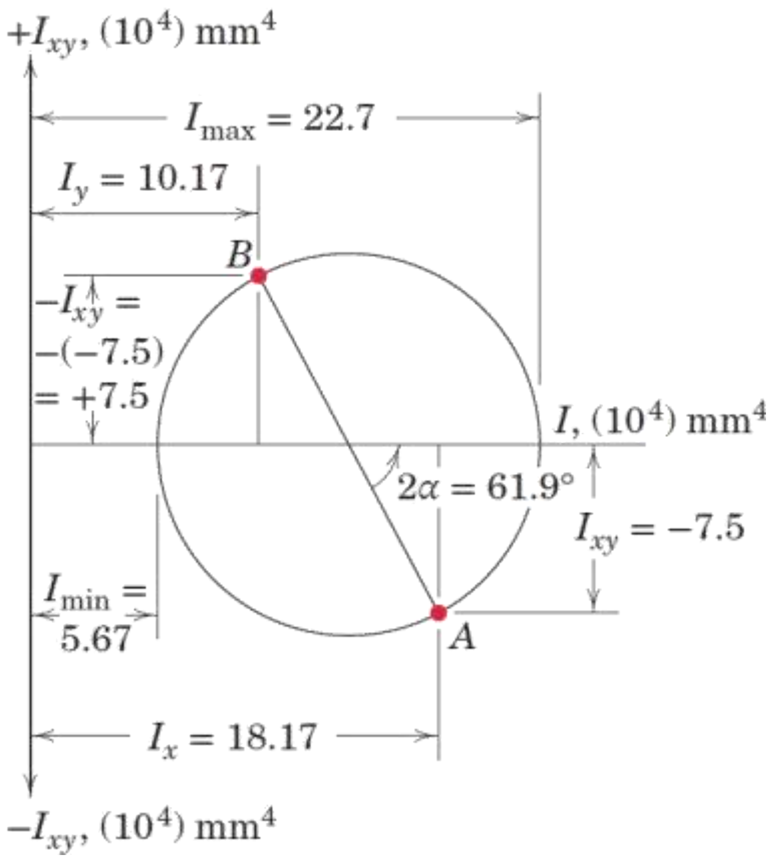
$$I_{min} = \left[\frac{18.17 + 10.17}{2} - \frac{18.17 - 10.17}{2}(0.471) - (7.50)(0.882)\right](10^4)$$

$$= 5.67(10^4)\ \text{mm}^4 \qquad \textit{Ans.}$$



Article A/4 – Sample Problem A/12 (5 of 5)

- Mohr's Circle of Inertia Plot



Mohr's circle. Alternatively, we could use Eqs. A/11 to obtain the results for I_{max} and I_{min}, or we could construct the Mohr's circle from the calculated values of I_x, I_y, and I_{xy}. These values are spotted on the diagram to locate points A and B, which are the extremities of the diameter of the circle. The angle 2α and I_{max} and I_{min} are obtained from the figure, as shown.