Introduction to Discrete Math

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- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion, Logic, Invariants
- Probability & Combinatronics
  - Counting, Probability

Mathematical Thinking – Combinatronics & Probability

Probability

PROBABILITY DO’S & DON’T’S

- Not Equiprobable Outcomes
- More About Finite Spaces
- Not All Questions Make Sense

World is Not Perfect

- *Equiprobable* outcomes
  - More exercises than real life
- Real dice maybe *asymmetric*
- *Frequencies* of outcomes are different
- But *stabilize* around
  - $p_1, p_2, \ldots, p_6$
- That's OK:
  - probability of even number, $p_{even} = p_2 + p_4 + p_6$
  - In a long series, an even number appears w/ this frequency (approx)
- $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$

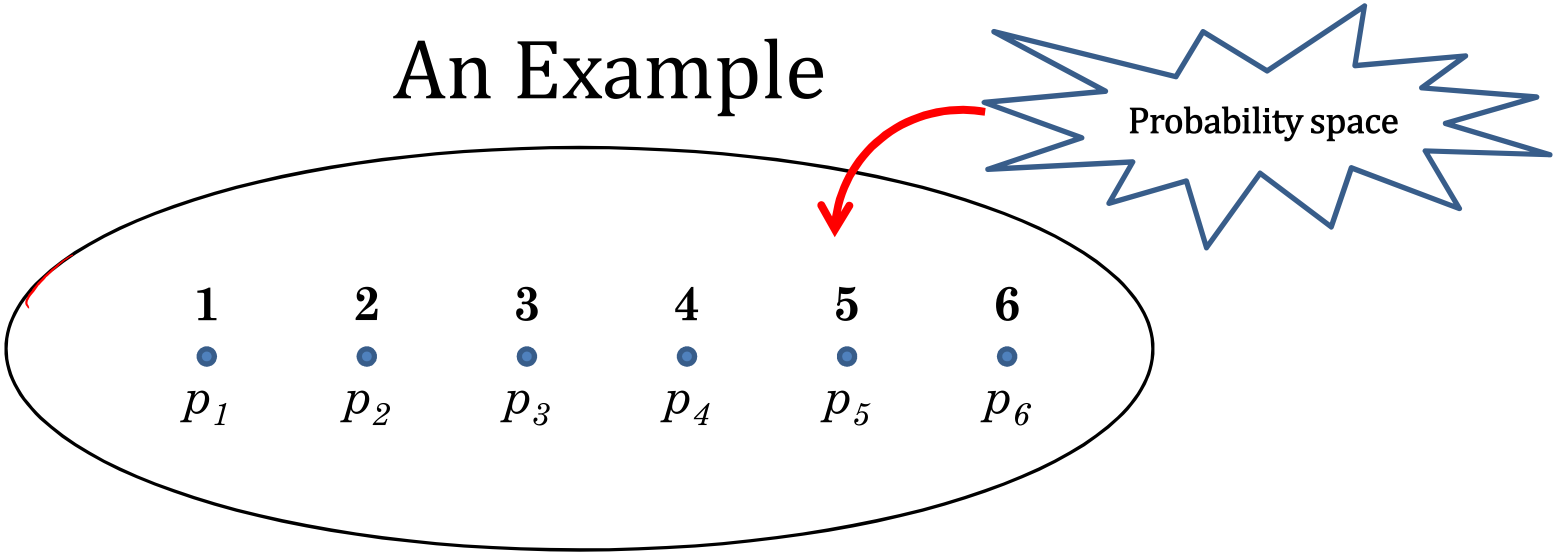
* asymmetric – not exactly same shape and size

Finite Probability Space

- A finite set X of *outcomes*
- Each outcome i has some probability, $p_i \geq 0$
- $\sum p_i = 1$
- *Event*: a set of outcomes
- *Probability* of the event: sum of outcome probabilities
- $p_i = 0$ possible?
  - Formally yes
  - But these outcomes do not matter

* finite – having limit or bounds

An Example



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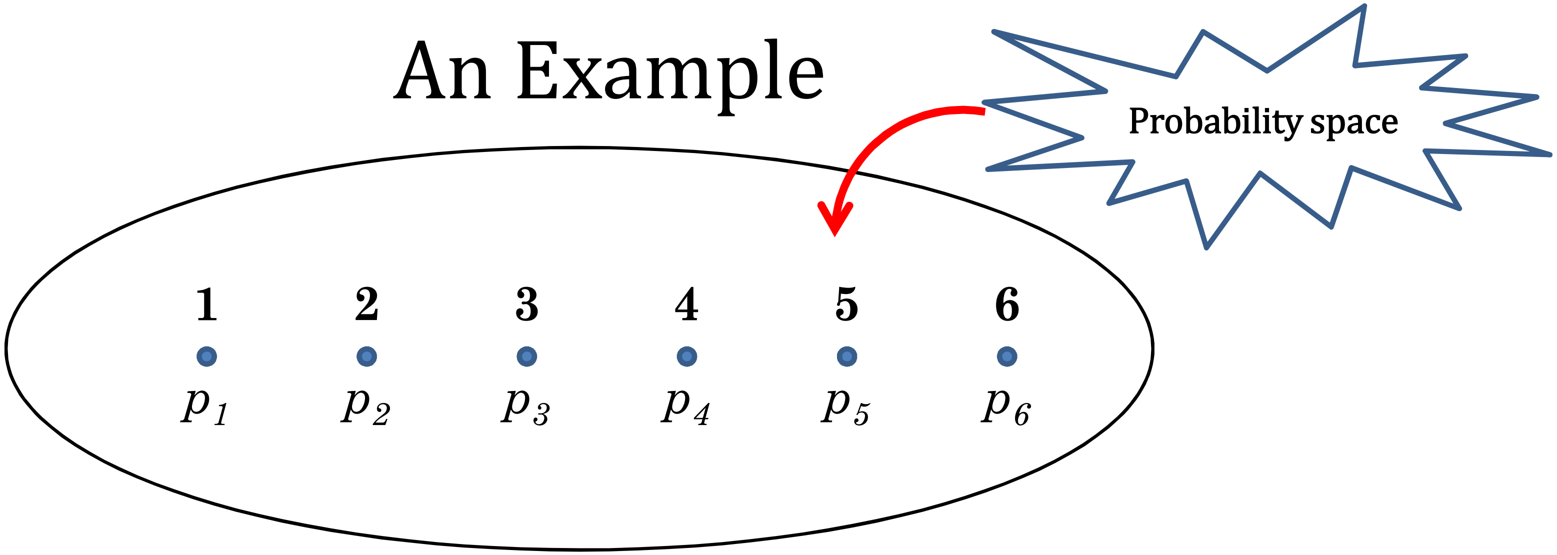

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$$

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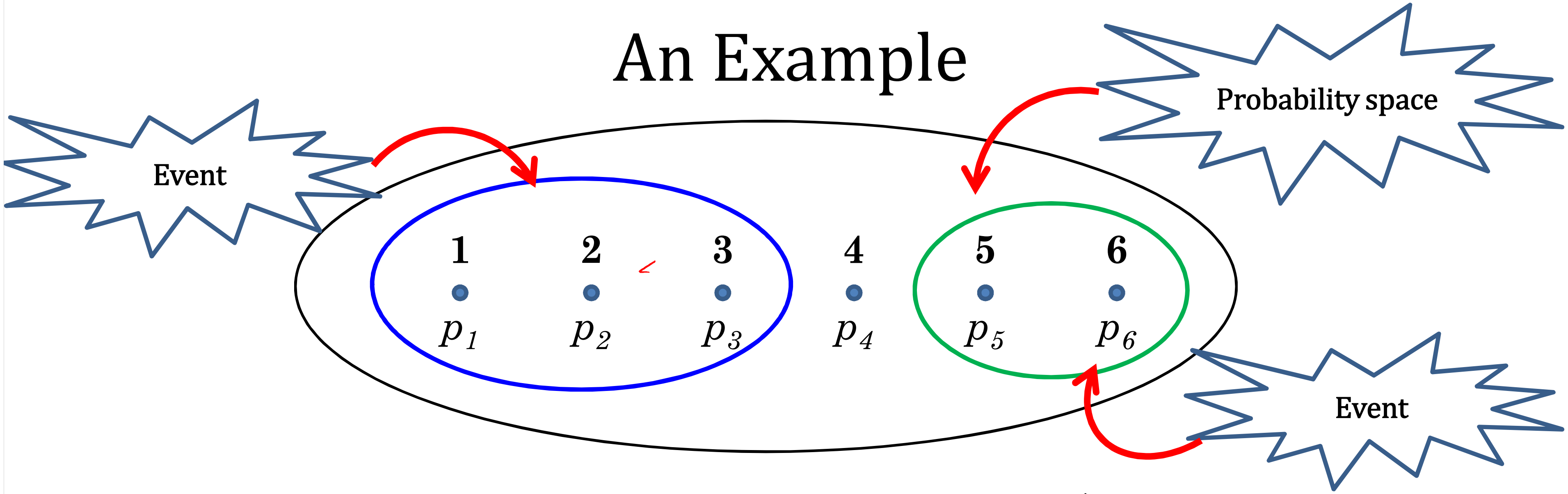

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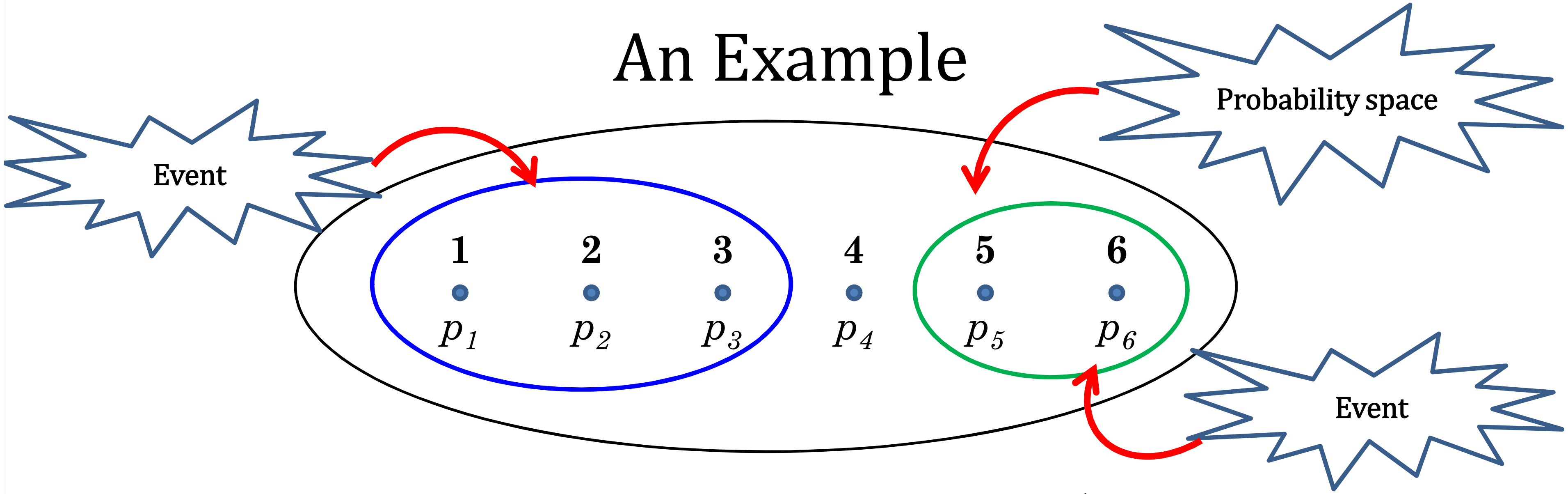

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$$event\ A = \{1,2,3\};$$

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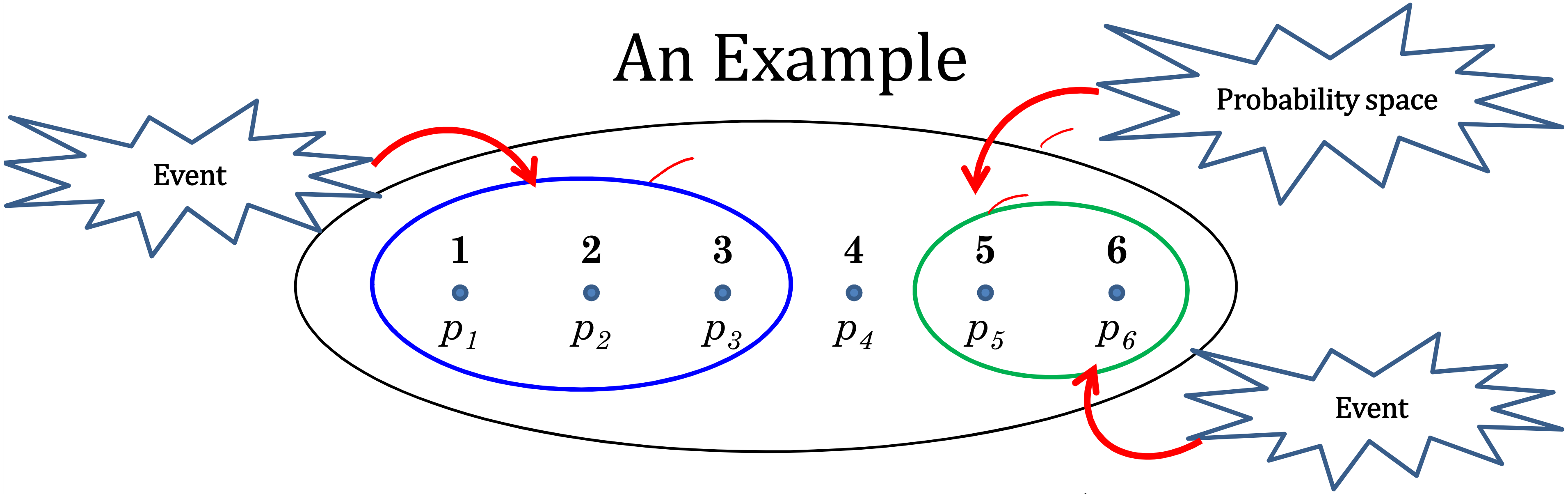

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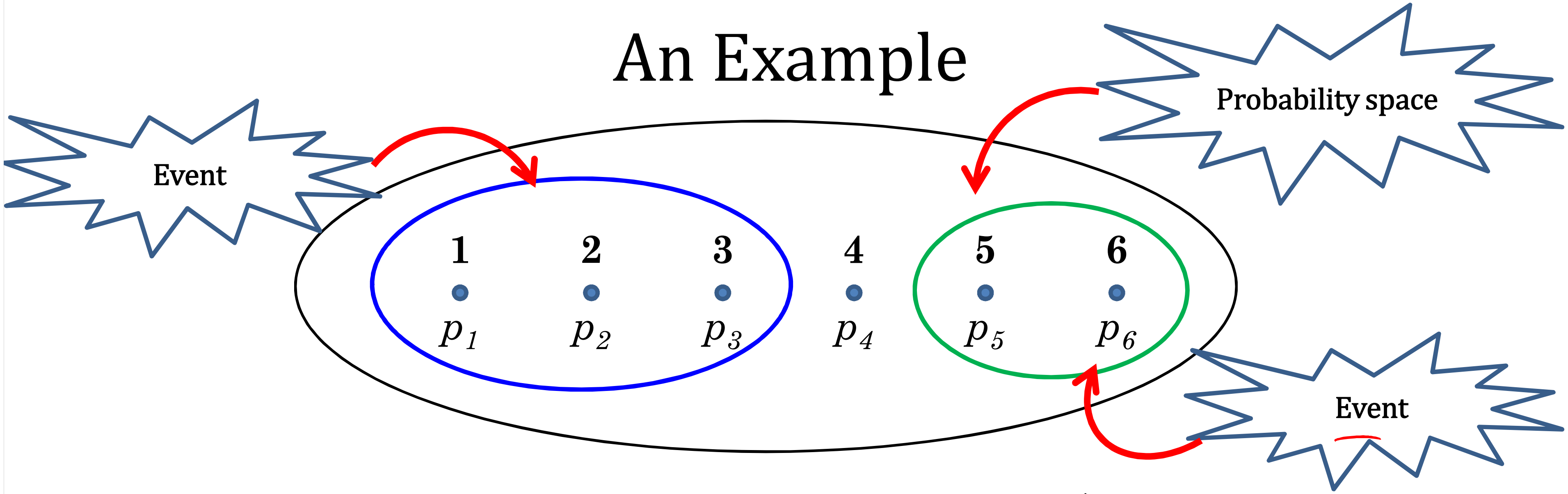

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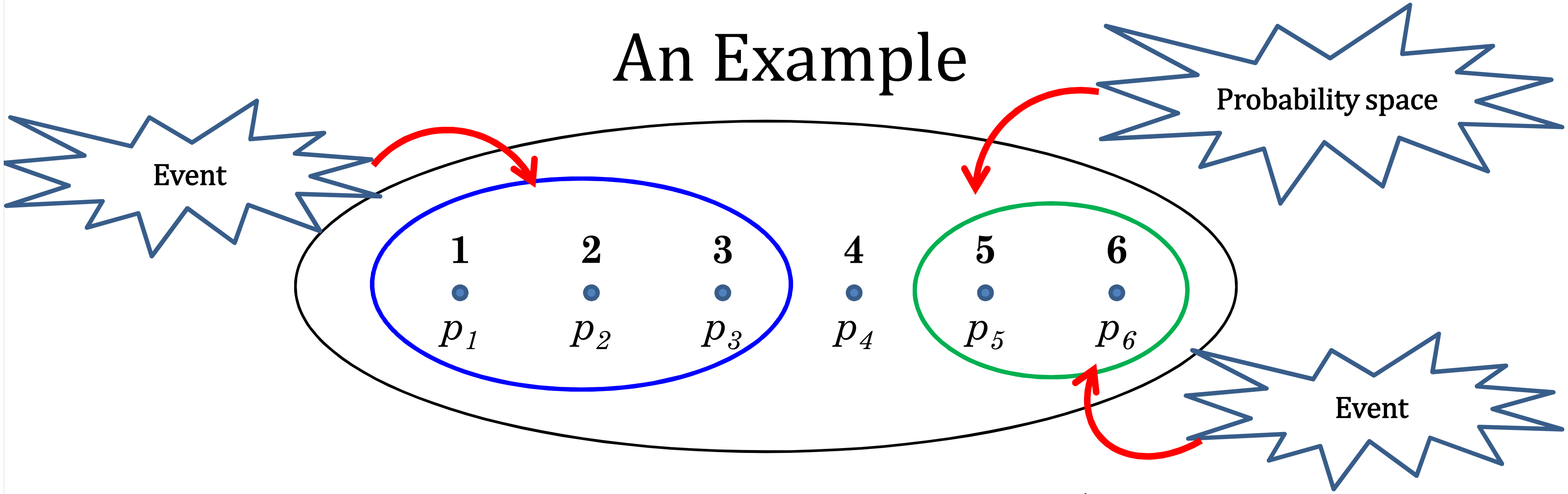

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$event\ A = \{1,2,3\};$ $\Pr[A] = p_1 + p_2 + p_3$

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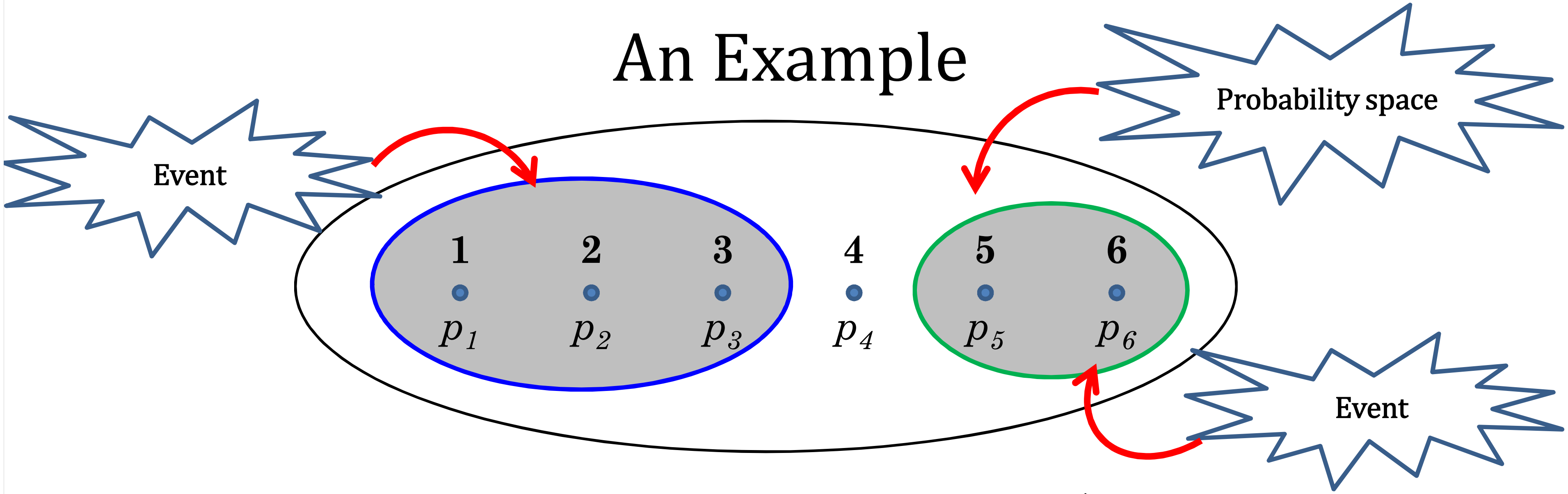

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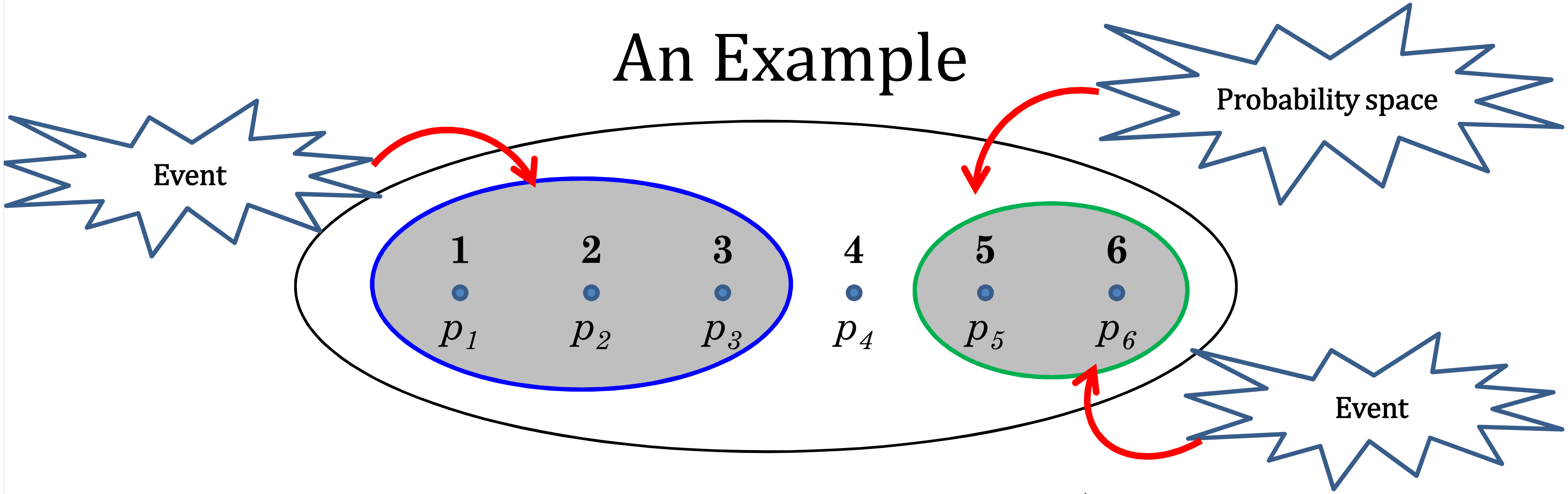

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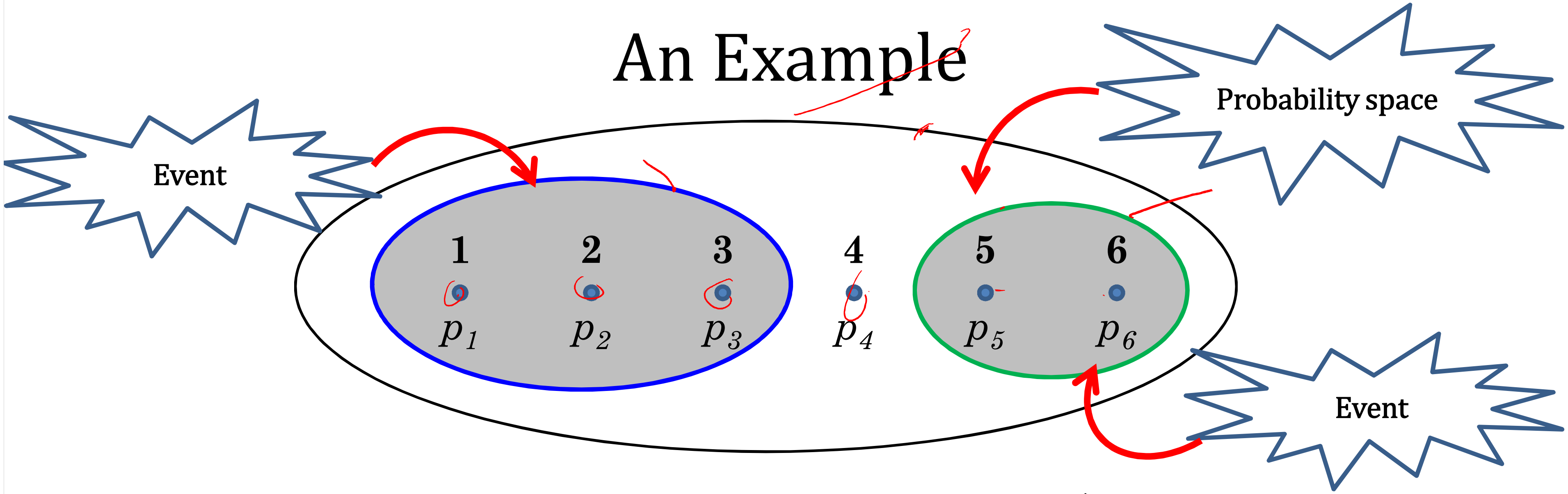
$A\ or\ B = \{1,2,3,5,6\};$

$\Pr[A\ or\ B] = \Pr[A] + \Pr[B]$

An Example



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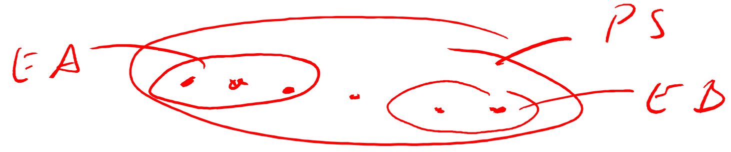
$event\ B = \{5,6\};$ $\Pr[B] = p_5 + p_6$

$A\ or\ B = \{1,2,3,5,6\};$

$$\Pr[A\ or\ B] = \Pr[A] + \Pr[B] = p_1 + p_2 + p_3 + p_5 + p_6$$

- Not Equiprobable Outcomes
- More About Finite Spaces
- Not All Questions Make Sense

Mutually Exclusive Events



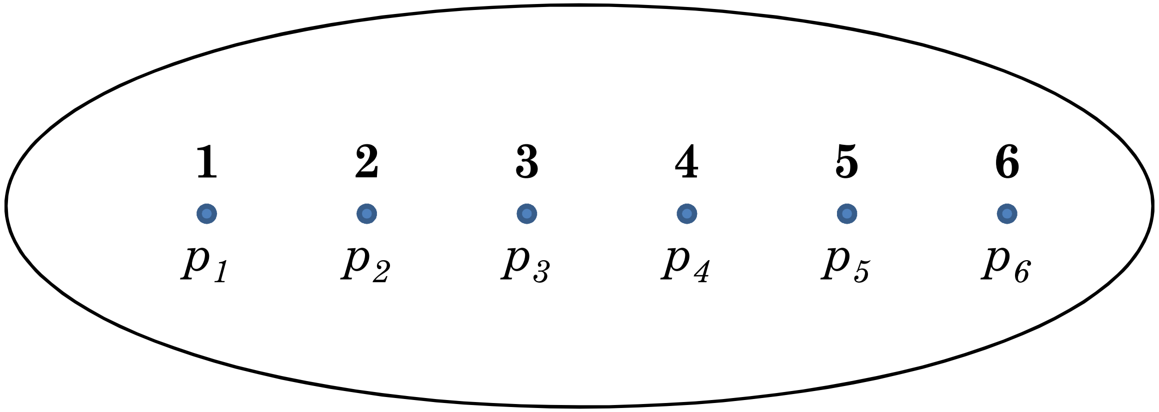
- Disjoint sets of outcomes
  - In the same probability space
- *Pr[A or B] = Pr[A] + Pr[B]*
  - If *A* and *B* are mutually exclusive
- *Pr[not A] = 1 – Pr[A]*
  - "*A*" and "*not A*" are mutually exclusive & fill the entire space
- What if *A* and *B* are not mutually exclusive?
  - *Pr[A or B] <> Pr[A] + Pr[B]?*

Get A   {not[A]}

$[P_1 + P_2 + P_3] + P_4 + P_5 + P_6 = 1$

$\therefore$ [not A] = 1 − Pr[A]

Inclusion & Exclusion



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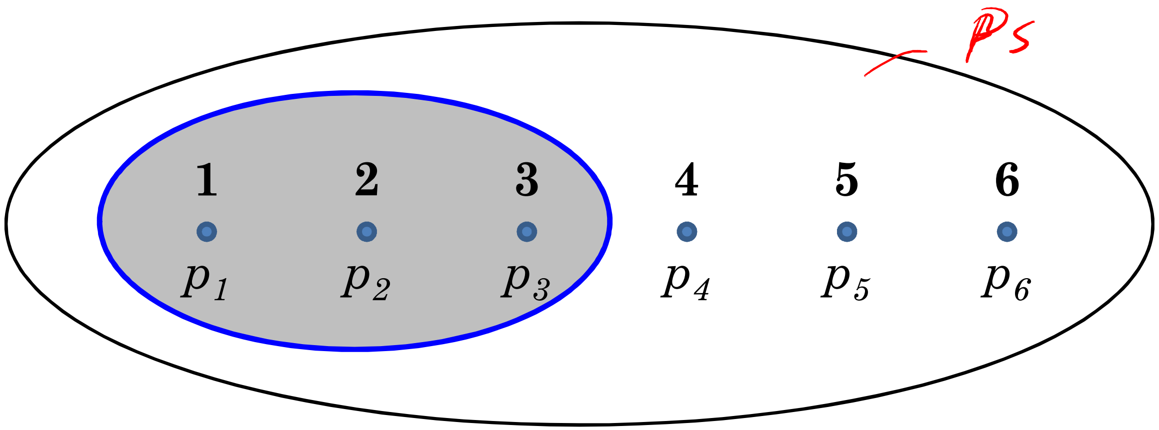

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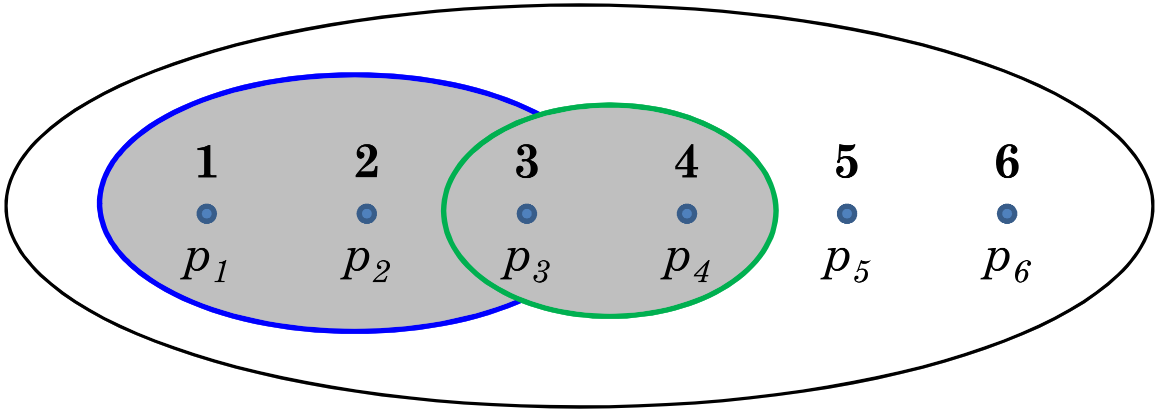

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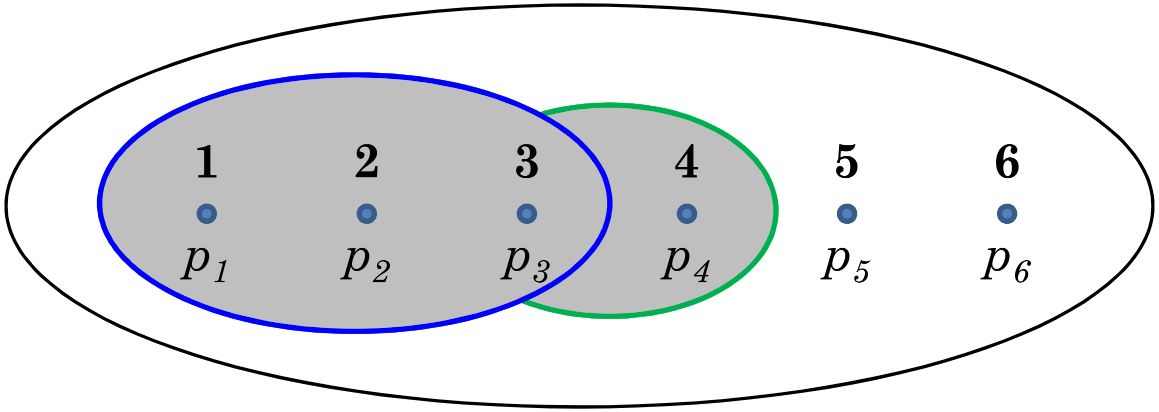
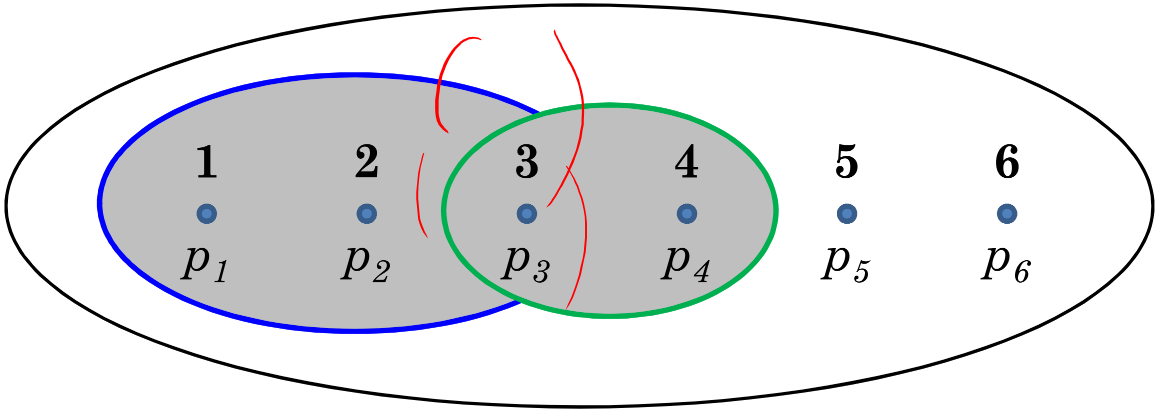


$$\Pr[A] = p_1 + p_2 + p_3;$$

Inclusion & Exclusion



$$\Pr[A] = p_1 + p_2 + p_3; \qquad \Pr[B] = p_3 + p_4;$$

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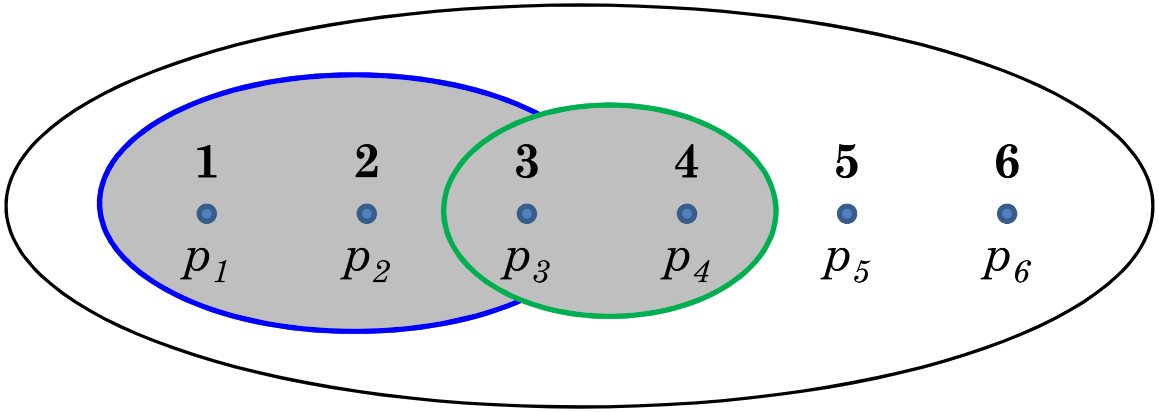

$$\Pr[A] = p_1 + p_2 + p_3; \qquad \Pr[B] = p_3 + p_4;$$

$$\Pr[A\, or\, B] =$$

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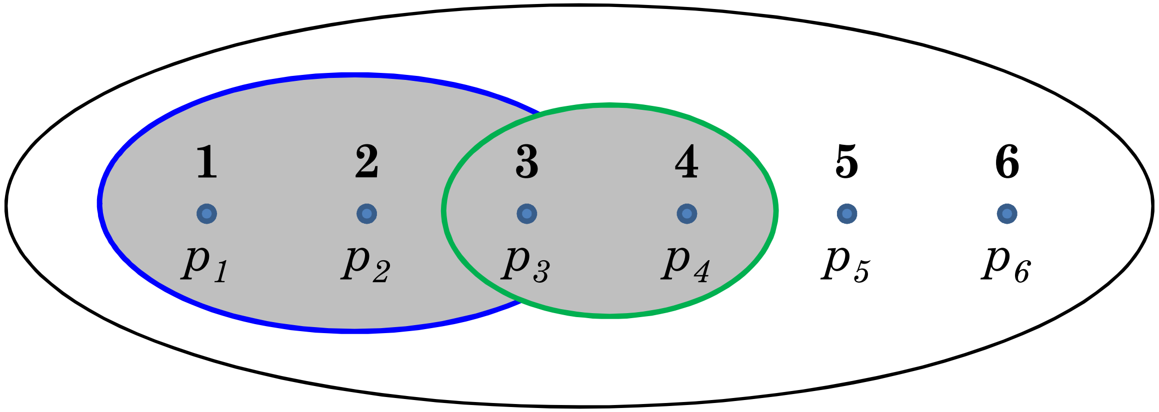

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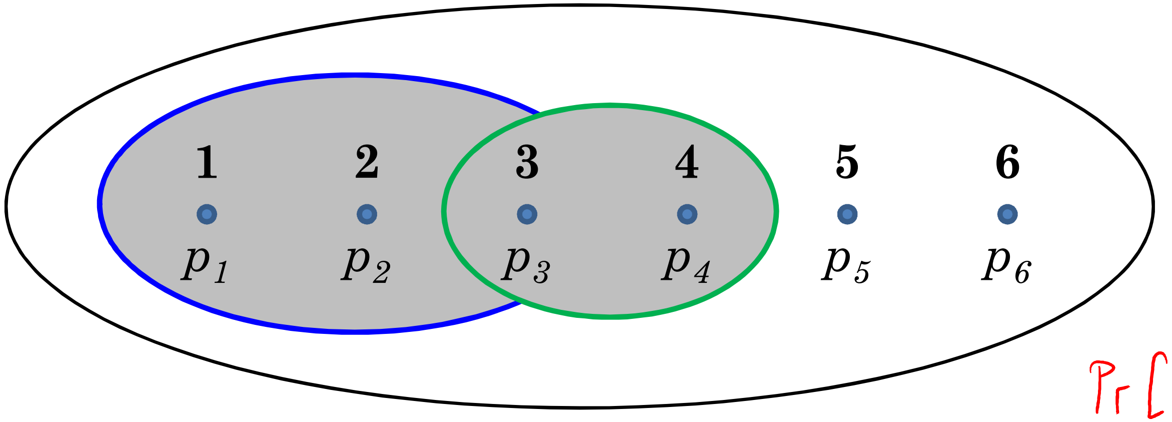


$A = \{1, 2, 3\}$
$B = \{3, 4\}$

$\Pr[A \text{ or } B]$
$= \Pr[A] + \Pr[B]$

$$\Pr[A] = p_1 + p_2 + p_3; \qquad \Pr[B] = p_3 + p_4;$$

$$\Pr[A\,or\,B] = p_1 + p_2 + p_3 + p_4$$

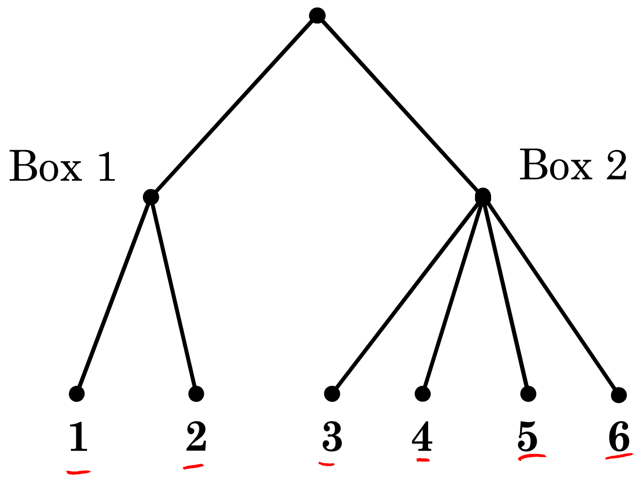
$$= \Pr[A] + \Pr[B] - \Pr[A\,and\,B]$$

$= \{p_1 + p_2\ p_3\} + \{\cancel{p_3} + p_4\} - \cancel{Pr} -$

Sequential Choice

- Selecting a reasonable model
- Six balls labeled *1, 2, 3, 4, 5, 6*
- In two boxes: *{1, 2,}* and *{3, 4, 5, 6}*
- "choose a random box & then choose a random ball"
- Two boxes are *equiprobable*
- All balls in each box are *equiprobable*
- What are the p_1, p_2, p_3, p_4, p_5, & p_6?

Choice Tree



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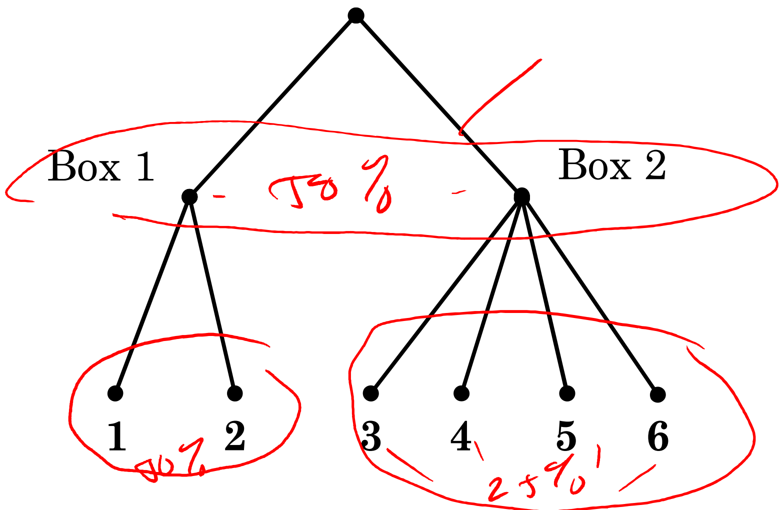
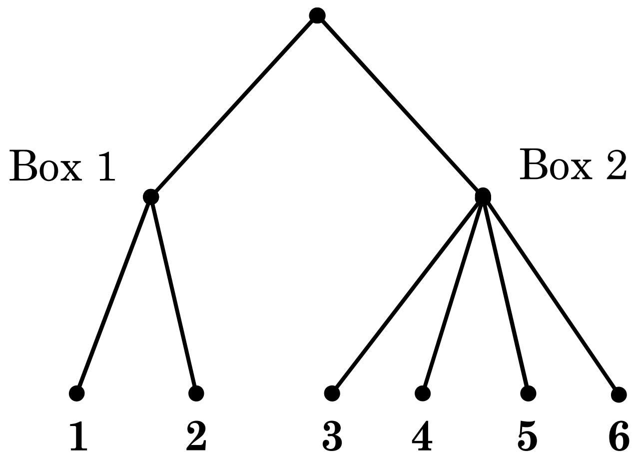


– At each point, all choices are *equiprobable*

Choice Tree



$$p_1 + p_2 = 1/2,$$

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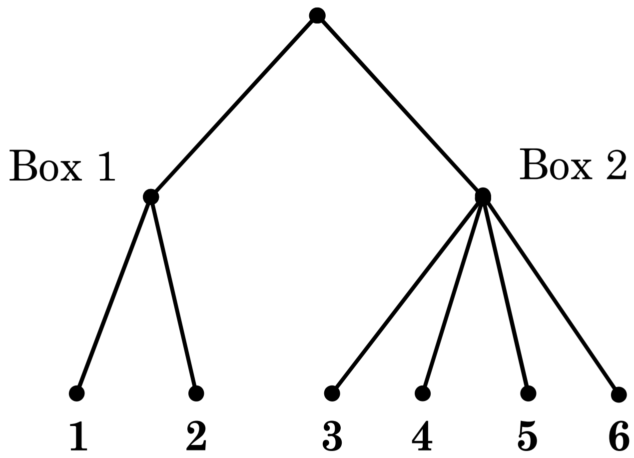

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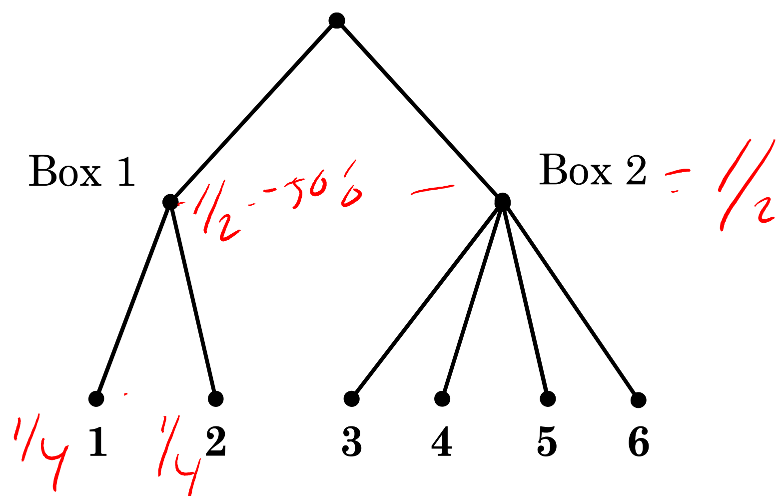

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$$p_1 = p_2,$$

$$\underline{p_1 = p_2 = 1/4},$$

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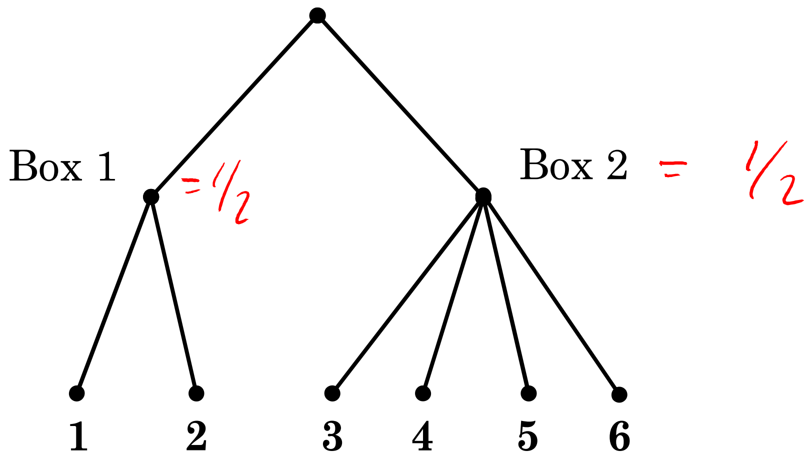

$$p_1 + p_2 = 1/2, \quad p_3 + p_4 + p_5 + p_6 = 1/2$$

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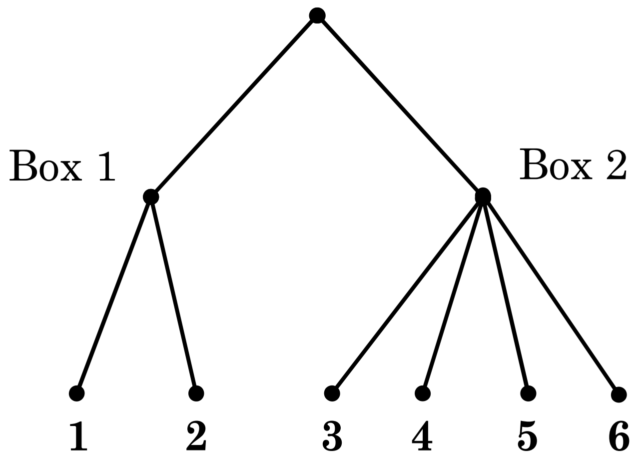
Choice Tree



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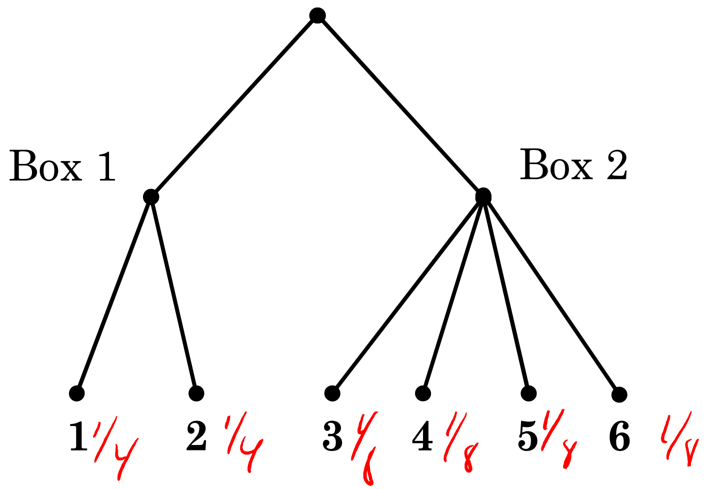

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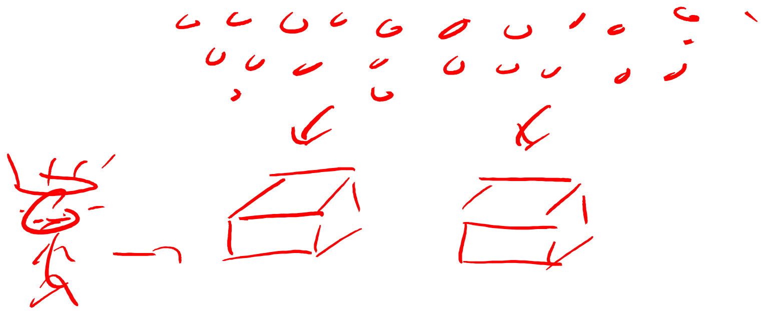
$$p_1 = p_2, \quad p_3 = p_4 = p_5 = p_6$$

$$p_1 = p_2 = 1/4, \quad p_3 = p_4 = p_5 = p_6 = 1/8$$

$\Sigma \rightarrow \frac{1}{4} \ \frac{1}{4} \ \frac{1}{8} \ \frac{1}{8} \ \frac{1}{8} \ \frac{1}{8} \Rightarrow 1$

Prisoner & King

- King:
  - Here are *15* white balls, *15* black balls, & *2* boxes
  - You may put balls in the *boxes as you wish*
  - Each box should contain *at least one ball*
  - I will choose a *random box* and then pick a *random ball* from it
  - If it is *white*, you will be *set free*



15 B
15 W

Prisoner & King

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- What should the prisoner do?
  - Yes, the prisoner wants to be free; and no cheating!

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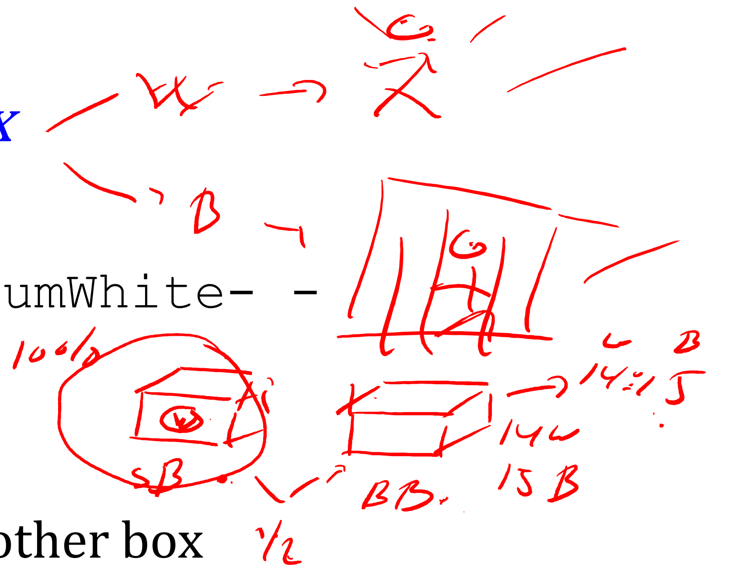
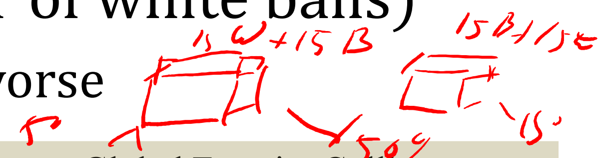
  - Yes, the prisoner wants to be free; and no cheating!
- Answer: Box 1 Box 2
  - *[1 white, 0 black], [14 white, 15, black]*
  - Why is this optimal?

Mathematics for the Prisoners

- Two *parameters*
  - *30 balls* = *box$_1$* + *box$_2$*; Colors: *15* black and *15* white
- Fix the *proportion* between boxes
- Balls's color *matters* more in *small box*
- Can *improve* situation by (C notation):
  - `smallBox.numWhite ++, bigBox.numWhite- -`
  - Until *small box = all white*



- More than one white ball *not needed*
  - Small box has one white, all other balls at other box
- In case of boxes of equal size (same number of white balls)
  - Doesn't make things better, nor does it make it worse



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Probability of the Past Events

- You may read something like:
  - "*It is quite probable that, in the 12th century, Arabian alchemists have obtained elementary phosphorus by distilling urine*"
  - It is perfectly OK, but far from being a mathematical probability theory
    - But if a person asks if the probability is greater than 2/3
    - no meaning, since it means that if we repeat experiment several times, 2/3 of the time we get something
    - But we can't repeat the experiment
- What is the probability that I have a dollar bill in my pocket?
  - You might guess but it's not correct to thing to do
  - I can come w/out the bill then it's not clear what the probability will be
  - Is it when I have the bill or when I didn't bring it with me

Probability of the Past Events

- Large prime numbers useful for cryptography
- Fast randomized algorithms
- Give different answers depending on "internal coin"
- Good algorithm
  - For every X, the probability of wrong answer < 10^{-2}
  - Perfectly good statement(OK)
- "The number $(2^{5240707} - 1)$ / $75{,}392{,}810{,}903$ is a prime with probability of > 0.99"
  - Not a good statement! Number is either prime or not, not probability of it being a prime!

Good News

Not in our course, but let us touch briefly

- Infinite probability spaces:
  - Random n in $0, 1, 2, \ldots$; with probabilities n [# *heads before tail*]
    - 0 [T]: ½, 1 [HT]: ¼, 2 [HHT]: 1/8, ...
  - Take random integer ($0, 1, 2, \ldots$) & make them equiprobable
    - This has no meaning, since if they are equiprobable, what is their probability p_i? If it is positive then $\sum p_i > 1$ $\left[\sum p_i = 0\right]$
    - Should be p_i=0, but then if $p_{i,\ i=1 \text{ to } n}$=0, therefore not possible
  - Random point in a square

Good News

Not in our course, but let us touch briefly

- Market interpretation
  - '*probability* that politician X will be reelected is *80%*'
  - Let's assume there are some people who don't want X re-elected, they paid for insurance as a guarantee (sounds silly and stupid but let's just go with it :D)
    - the obligation "to pay *\$1* if X is reelected"
    - This guarantee is now traded at around *\$0.8*
    - We can see that probability of X being reelected is *80%*
  - Problem in this scenario, there is no market for it
    - But it is possible and legal in some countries to bet in political events similar to this :D

Thank you.