11.8 Fourier Cosine and Sine Transforms

Integral Transform (적분변환) :

An integral transform is a transformation in the form of an integral that produces new functions depending on a different variable from given functions.

Examples:

· Laplace transform

· Fourier Cosine Transform

· Fourier Sine Transform

Fourier Cosine Transform

Fourier Cosine Integral: $A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos\omega v \, dv$ (f is even.)

$$f(x) = \int_0^\infty A(\omega)\cos\omega x \, d\omega$$

Fourier Cosine Transform:

(1a) $$\mathcal{F}_c(f) = \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x \, dx$$

Inverse Fourier Cosine Transform:

(1b) $$f(x) = \mathcal{F}_c^{-1}[\hat{f}_c(\omega)] = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\omega)\cos\omega x \, d\omega$$

Fourier Sine Transform

Fourier Sine Integral: $A(\omega) = \dfrac{2}{\pi}\displaystyle\int_0^\infty f(v)\sin\omega v\, dv$ (f is odd.)

$$f(x) = \int_0^\infty A(\omega)\sin\omega x\, d\omega$$

Fourier Sine Transform:

(2a) $$\mathcal{F}_s(f) = \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\omega x\, dx$$

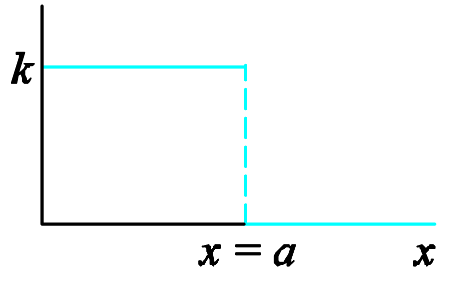
Inverse Fourier Sine Transform:

(2b) $$f(x) = \mathcal{F}_s^{-1}[\hat{f}_s(\omega)] = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(\omega)\sin\omega x\, d\omega$$

EXAMPLE 1 Fourier Cosine and Fourier Sine Transforms

Find the Fourier cosine and Fourier sine transforms of the function

$$f(x) = \begin{cases} k & (0 < x < a) \\ 0 & (x > a) \end{cases}$$



Sol.

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a k \cos \omega x \, dx = \sqrt{\frac{2}{\pi}} k \left(\frac{\sin a\omega}{\omega} \right)$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a k \sin \omega x \, dx = \sqrt{\frac{2}{\pi}} k \left(\frac{1 - \cos a\omega}{\omega} \right)$$

Ex. 2 Fourier Cosine Transform of the Exponential Function

Find $\mathcal{F}_c(e^{-x})$.

Sol.

$$\mathcal{F}_c(e^{-x}) = \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \cos \omega x \, dx$$:See Next slide

$$= \sqrt{\frac{2}{\pi}} \frac{e^{-x}}{1+\omega^2} (-\cos \omega x + \omega \sin \omega x) \Big|_0^{\infty}$$

$$= \frac{\sqrt{2/\pi}}{1+\omega^2}$$

Integration of $e^{-ax}\cos\omega x$

$$\int e^{-ax}\cos\omega x\,dx = \int e^{-ax}Re(e^{j\omega x})\,dx$$

$$= Re\left[\int e^{-ax}e^{j\omega x}dx\right] = Re\left[\int e^{(-a+j\omega)x}dx\right]$$

$$= Re\left[\frac{e^{(-a+j\omega)x}}{-a+j\omega}\right] = Re\left[\frac{e^{(-a+j\omega)x}(-a-j\omega)}{(-a+j\omega)(-a-j\omega)}\right]$$

$$= Re\left[\frac{e^{-ax}(\cos\omega x + j\sin\omega x)(-a-j\omega)}{a^2+\omega^2}\right]$$

$$= \frac{e^{-ax}}{a^2+\omega^2}(-a\cos\omega x + \omega\sin\omega x)$$

Linearity

(3a) $\mathcal{F}_c(af+bg) = a\mathcal{F}_c(f) + b\mathcal{F}_c(g)$

(3b) $\mathcal{F}_s(af+bg) = a\mathcal{F}_s(f) + b\mathcal{F}_s(g)$

$$\begin{aligned}
\mathcal{F}_c(af+bg) &= \sqrt{\frac{2}{\pi}}\int_0^\infty [af(x)+bg(x)]\cos\omega x\,dx \\
&= a\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\,dx \\
&\qquad + b\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos\omega x\,dx \\
&= a\,\mathcal{F}_c(f) + b\,\mathcal{F}_c(g)
\end{aligned}$$

THEOREM 1 Cosine and Sine Transforms of Derivatives

Let f be continuous and absolutely integrable on the x-axis, let f' be piecewise continuous on every finite interval, and let $f(x) \to 0$ as $x \to \infty$. Then

(4)
$$(a)\quad \mathcal{F}_c[f'(x)] = \omega \mathcal{F}_s[f(x)] - \sqrt{\frac{2}{\pi}} f(0)$$
$$(b)\quad \mathcal{F}_s[f'(x)] = -\omega \mathcal{F}_c[f(x)]$$

(5)
$$(a)\quad \mathcal{F}_c[f''(x)] = -\omega^2 \mathcal{F}_c[f(x)] - \sqrt{\frac{2}{\pi}} f'(0)$$
$$(b)\quad \mathcal{F}_s[f''(x)] = -\omega^2 \mathcal{F}_s[f(x)] + \sqrt{\frac{2}{\pi}} f(0)$$

Proof

$$\mathcal{F}_c[f'(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x)\cos\omega x\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left[f(x)\cos\omega x\Big|_0^\infty + \omega \int_0^\infty f(x)\sin\omega x\,dx \right]$$

$$= -\sqrt{\frac{2}{\pi}} f(0) + \omega \mathcal{F}_s[f(x)]$$

$$\mathcal{F}_s[f'(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x)\sin\omega x\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left[f(x)\sin\omega x\Big|_0^\infty - \omega \int_0^\infty f(x)\cos\omega x\,dx \right]$$

$$= 0 - \omega \mathcal{F}_c[f(x)]$$

EX. 3 An Application of the Operational Formula (5)

Find the Fourier cosine transform $\mathcal{F}_c\left(e^{-ax}\right)$ of $f(x)=e^{-ax}$, where $a>0$.

Sol.

$$f''(x) = (e^{-ax})'' = a^2 e^{-ax} = a^2 f(x)$$

(5a) $$\mathcal{F}_c[f''(x)] = -\omega^2 \mathcal{F}_c(f) - \sqrt{\frac{2}{\pi}} f'(0) = a^2 \mathcal{F}_c(f)$$

$$(\omega^2 + a^2)\mathcal{F}_c(f) = -\sqrt{\frac{2}{\pi}} f'(0) = a\sqrt{\frac{2}{\pi}}$$

$$\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{\omega^2 + a^2}\right)$$

1. Find the cosine transform $\hat{f}_c(w)$ of $f(x) = 1$ if $0 < x < 1$, $f(x) = -1$ if $1 < x < 2$, $f(x) = 0$ if $x > 2$.

Sol.

$$\mathcal{F}_c(f) = \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 \cos wx dx + \int_1^2 - \cos wx dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin wx}{w}\Big|_0^1 - \frac{\sin wx}{w}\Big|_1^2 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin w}{w} - \frac{1}{w}(\sin 2w - \sin w) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{2\sin w - \sin 2w}{w} \right)$$

2. Find f in Prob. 1 when $\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}}\left(\frac{2\sin w - \sin 2w}{w}\right)$

Sol.

$$f(x) = \mathcal{F}_C^{-1}[\hat{f}_c(\omega)] = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\omega)\cos\omega x\, d\omega$$

$$= \sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\int_0^\infty \left(\frac{2\sin w - \sin 2w}{w}\right)\cos wx\, dw$$

$$= \underbrace{\frac{2}{\pi}\int_0^\infty \frac{2\sin w}{w}\cos wx\, dw}_{A} - \underbrace{\frac{2}{\pi}\int_0^\infty \frac{\sin 2w}{w}\cos wx\, dw}_{B}$$

$$A = \frac{2}{\pi}\int_0^\infty \frac{2\sin w}{w}\cos wx\, dw = \int_0^\infty A(\omega)\cos\omega x\, d\omega$$

$$A(\omega) = \frac{2}{\pi}\frac{2\sin w}{w} = \frac{2}{\pi}\int_0^\infty g(v)\cos\omega v\, dv$$

$$g(x) = \frac{4}{\pi}\int_0^\infty \frac{\sin w}{w}\cos wx\, dw = \begin{cases} 2 & (0 < x < 1) \\ 0 & (x > 1) \end{cases}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{2\sin w}{w}\cos wx\,dw - \underbrace{\frac{2}{\pi}\int_0^\infty \frac{\sin 2w}{w}\cos wx\,dw}_{\mathbf{B}}$$

$$\mathbf{B} = -\frac{2}{\pi}\int_0^\infty \frac{\sin 2w}{w}\cos wx\,dw = \int_0^\infty A(\omega)\cos\omega x\,d\omega$$

$$A(\omega) = -\frac{2}{\pi}\frac{\sin 2w}{w} = \frac{2}{\pi}(-1)\frac{\sin wx}{w}\Big|_0^2 = \frac{2}{\pi}\int_0^\infty h(v)\cos\omega v\,dv$$

$$h(x) = -\frac{2}{\pi}\int_0^\infty \frac{\sin 2w}{w}\cos wx\,dw = \int_0^\infty A(\omega)\cos\omega x\,d\omega = \begin{cases} -1 & (0<x<2) \\ 0 & (x>1) \end{cases}$$

$$f(x) = \mathcal{F}_c^{-1}[\hat{f}_c(\omega)] = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\omega)\cos\omega x\,d\omega$$

$$= \sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\int_0^\infty \left(\frac{2\sin w - \sin 2w}{w}\right)\cos wx\,dw = g(x) + h(x) = \begin{cases} 1 & (0<x<1) \\ -1 & (1<x<2) \\ 0 & (x>2) \end{cases}$$

3. Find $\hat{f}_c(w)$ for $f(x) = x$ if $0 < x < 2$, $f(x) = 0$ if $x > 2$.

Sol.

$$\mathcal{F}_c(f) = \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos\omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^2 x \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[x \frac{\sin wx}{w} \bigg|_0^2 - \int_0^2 \frac{\sin wx}{w} \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\sin 2w}{w} + \frac{\cos wx}{w^2} \bigg|_0^2 \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{\cos 2w + 2w \sin 2w - 1}{w^2}$$

4. Find $\hat{f}_c(w)$ for $f(x) = e^{-ax}$.

Sol.

$$\mathcal{F}_c(f) = \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos\omega x\,dx = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\cos wx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+w^2}(-a\cos wx + w\sin wx) \right]_0^\infty = \sqrt{\frac{2}{\pi}} \frac{a}{w^2+a^2}$$

$$\int_0^\infty e^{-ax}\cos wx\,dx = e^{-ax}\frac{\sin wx}{w}\Big|_0^\infty - \int_0^\infty -a\, e^{-ax}\frac{\sin wx}{w}\,dx$$

$$= a\left[e^{-ax}\frac{-\cos wx}{w^2}\Big|_0^\infty - \int_0^\infty -a\, e^{-ax}\frac{-\cos wx}{w^2}\,dx \right]$$

$$= \frac{a}{w^2} - \frac{a^2}{w^2}\int_0^\infty e^{-ax}\cos wx\,dx$$

$$\left[1 + \frac{a^2}{w^2}\right]\int_0^\infty e^{-ax}\cos wx\,dx = \frac{a}{w^2} \Rightarrow \int_0^\infty e^{-ax}\cos wx\,dx = \frac{a}{w^2+a^2}$$

5. Find $\hat{f}_c(w)$ for $f(x) = x^2$ if $0 < x < 1$, $\quad f(x) = 0$ if $x > 1$.

Sol.

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos wx\,dx = \sqrt{\frac{2}{\pi}}\underline{\int_0^1 x^2\cos wx\,dx}$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{x^2}{w}\sin wx + \frac{2x}{w^2}\cos wx - \frac{2}{w^3}\sin wx\right]_0^1 = \sqrt{\frac{2}{\pi}}\,\frac{2w\cos w + (w^2-2)\sin w}{w^3}$$

$$\underline{\int_0^1 x^2\cos wx\,dx} = x^2\frac{\sin wx}{w}\Big|_0^1 - \int_0^1 2x\frac{\sin wx}{w}\,dx$$

$$= \frac{\sin w}{w} - 2x\frac{-\cos wx}{w^2}\Big|_0^1 + 2\int_0^1 \frac{-\cos wx}{w^2}\,dx$$

$$= \frac{\sin w}{w} + 2\frac{\cos w}{w^2} - 2\frac{\sin w}{w^3} = \frac{2w\cos w + (w^2-2)\sin w}{w^3}$$

9. Find $\mathcal{F}_s(e^{-ax}), \quad a > 0$, by integration.

Sol.

$$\mathcal{F}_S(f) = \hat{f}_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx$$: Def. of Fourier Sine Transform

$$\mathcal{F}_s(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin wx \, dx$$ (Integration by parts)

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2 + w^2} (-a \sin wx - w \cos wx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{w}{a^2 + w^2}$$

11. Find $f_s(w)$ for $f(x) = x^2$ if $0 < x < 1$, $f(x) = 0$ if $x > 1$.

Sol.

$$\mathcal{F}_s(f) = \hat{f}_S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x \, dx$$: Def. of Fourier Sine Transform

$$\mathcal{F}_s(e^{-ax}) = \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^1 x^2 \sin wx dx$$ (Integration by parts)

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{x^2}{w} \cos wx + \frac{2x}{w^2} \sin wx + \frac{2}{w^3} \cos wx \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \frac{2w \sin w + (-w^2 + 2) \cos w - 2}{w^3}$$

12. Find $\mathcal{F}_s(xe^{-x^2/2})$ from (4b) and a suitable formula in Table I of Sec. 11.10.

Sol.

Let $f(x) = e^{-x^2/2}$,

then $f'(x) = -xe^{-x^2/2} = -xf(x)$

$\Downarrow$ (4) (b) $\mathcal{F}_s[f'(x)] = -\omega\, \mathcal{F}_c[f(x)]$

$$\mathcal{F}_s(xe^{-x^2/2}) = \mathcal{F}_s[-f'(x)] = \omega\, \mathcal{F}_c[f(x)]$$

$$= \omega\, \mathcal{F}_c[e^{-x^2/2}] = \omega \cdot we^{-w^2/2} = \omega^2 e^{-w^2/2}$$

Table I. 4: $\mathcal{F}_c[e^{-x^2/2}] = we^{-w^2/2}$

13. Find $\mathcal{F}_s(e^{-x})$ from (4a) and formula 3 of Table I in Sec. 11.10.

Sol.

(4) $(a)\quad \mathcal{F}_c[f'(x)] = \omega\mathcal{F}_s[f(x)] - \sqrt{\frac{2}{\pi}}\,f(0)$

Table I. 3: $\mathcal{F}_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+\omega^2}\right)$

Let $f(x) = e^{-x}$, then $f'(x) = -f(x)$, $f(0) = 1$.

(4) $(a)\quad \mathcal{F}_c[f'(x)] = -\mathcal{F}_c[f(x)] = \omega\mathcal{F}_s[f(x)] - \sqrt{\frac{2}{\pi}}\,f(0)$

$$-\sqrt{\frac{2}{\pi}}\left(\frac{1}{1+w^2}\right) = \omega\,\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}$$

$$\mathcal{F}_s(e^{-x}) = \frac{1}{w}\sqrt{\frac{2}{\pi}}\left(1 - \frac{1}{1+w^2}\right) = \sqrt{\frac{2}{\pi}}\,\frac{w}{1+w^2}$$

14. Gamma function. Using formulas 2 and 4 in Table II of Sec. 11.10, prove $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ [(30) in App. A3.1], a value needed for Bessel functions and other applications.

Sol.

Table II 2: $\mathcal{F}_s(1/\sqrt{x}) = 1/\sqrt{w}$

Table II 4: $\mathcal{F}_s(x^{a-1}) = \sqrt{\frac{2}{\pi}}\frac{\Gamma(a)}{w^a}\sin\frac{a\pi}{2}$

$\xrightarrow{a=1/2}$

$$\mathcal{F}_s\{x^{-1/2}\} = 1/\sqrt{w} = \sqrt{\frac{2}{\pi}}\frac{1}{w^{1/2}}\Gamma\left(\frac{1}{2}\right)\sin\frac{\pi}{4}$$

$\downarrow a=1/2$

$$\mathcal{F}_s\{x^{-1/2}\} = 1/\sqrt{w} = \sqrt{\frac{2}{\pi}}\frac{1}{w^{1/2}}\Gamma\left(\frac{1}{2}\right)\sin\frac{\pi}{4} = \sqrt{\frac{2}{\pi}}\frac{1}{w^{1/2}}\Gamma\left(\frac{1}{2}\right)\frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{w}} = \sqrt{\frac{1}{\pi}}\frac{1}{w^{1/2}}\Gamma\left(\frac{1}{2}\right) \rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$