Introduction to Discrete Math

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Course Outline

- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion, Logic, Invariants
- Probability & Combinatronics
  - Counting, Probability, Random Variables
- Graph Theory
  - Graphs (cycles, classes, parameters)
- Number Theory & Cryptography
  - Arithmetic in modular form
  - Intro to Cryptography

Mathematical Thinking – Counting

TUPLES AND PERMUTATIONS

- Number of Tuples
- License Plates
- Tuples with Restrictions
- Permutations

Number of Passwords

Problem

How many 5-character passwords can we create using lower case Latin letters only? (the size of the alphabet is 26)

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Number of Passwords

Problem

How many 5-character passwords can we create using lower case Latin letters only? (the size of the alphabet is 26)

- It turns out that the rule of product is all we need to solve this problem
- But we need to do it step by step

Number of Passwords

- Let's start with a 1 letter password

26 = ??

*

Number of Passwords

- Let's start with a 1 letter password
  - Clearly, then there are 26 options

26 = 26

*

Number of Passwords

- Let’s start with a 1 letter password
  - Clearly, then there are 26 options
- What about two letters?
  - Then we can choose both letters in 26 ways

26 x 26 =

* *

Number of Passwords

- Let's start with a 1 letter password
  - Clearly, then there are 26 options
- What about two letters?
  - Then we can choose both letters in 26 ways
- Use the rule of product: the answer is 676

26 x 26 = 676

*   *

Number of Passwords

- Let’s move on to the case of 3-character password
- We already know that we can choose the first two letters in 676 ways

26 x 26 = ???

* * *

Number of Passwords

- Let's move on to the case of 3-character password
- We already know that we can choose the first two letters in 676 ways
- We apply rule of product again!

676 x 26 =

* * *

Number of Passwords

- Let’s move on to the case of 3-character password
- We already know that we can choose the first two letters in 676 ways
- We apply rule of product again!
- The answer is 17 576

676 x 26 = 17 576

* * *

Number of Passwords

- We proceed the same way for 4-character password
- We apply rule of product again!

$$\underbrace{26 \times 26 \times 26}^{17576} \times 26 = ???$$

$$* \quad * \quad * \quad *$$

Number of Passwords

- We proceed the same way for 4-character password
- We apply rule of product again!
- Answer is

26	x	26	x	26	x	26	=	456 976
*		*		*		*		

Number of Passwords

- And for a 5-character password

456 794

$$\left[\,26 \times 26 \times 26 \times 26\,\right] \times 26 = ???$$

*   *   *   *   *

Number of Passwords

- And for a 5-character password
- Applying rule of product, we get

26	x	26	x	26	x	26	x	26	=	11 881 376
*		*		*		*		*		

Number of Tuples

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Suppose we have a set of n symbols. How many different sequences of length k we can form out of these symbols?

Number of Tuples

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Suppose we have a set of n symbols. How many different sequences of length k we can form out of these symbols?

Set = [1, 2, .., n]

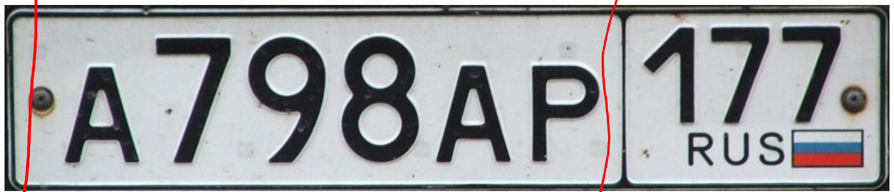
k, [1, 2, .. k]

These sequences are usually called tuples

- Can apply the same argument
- There are n possibilities to pick the first letter
- Each next letter multiplies the number of sequences by n
- Thus the answer is a product of n by itself k times,
  - that is n^k

- Number of Tuples
- License Plates
- Tuples with Restrictions
- Permutations

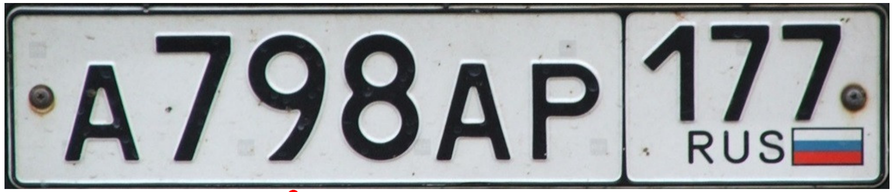
License Plates



Now we are ready to get back to our motivating example

- Russian license plate:
  - 3 digits, 3 letters; ~~78~~ 177 is a regional code

License Plates



0 - 9

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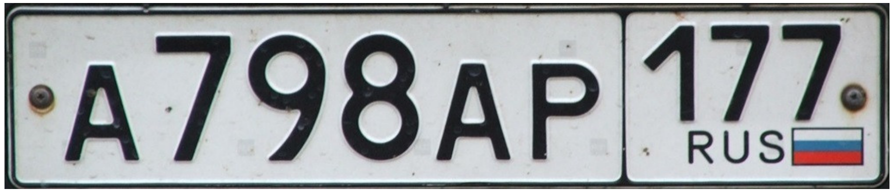
- Russian license plate:
  - *3* digits, *3* letters; *177* is a regional code
- *10* options for digits, *12* options for letters
  - only Cyrillic letters that are similar to Latin ones are used

License Plates



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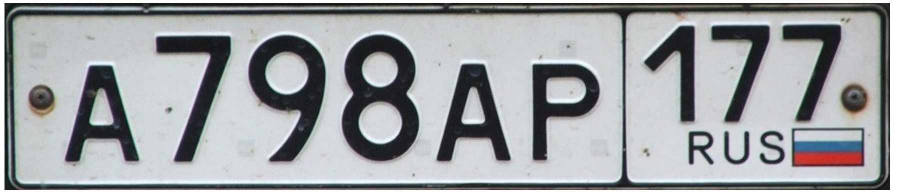
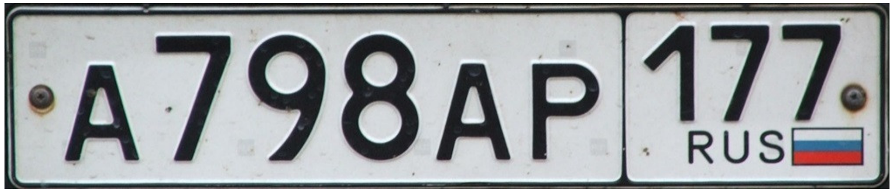
- Russian license plate:
  - *3* digits, *3* letters; *177* is a regional code
- *10* options for digits, *12* options for letters
  - only Cyrillic letters that are similar to Latin ones are used
- How many plates are there for one region?

License Plates



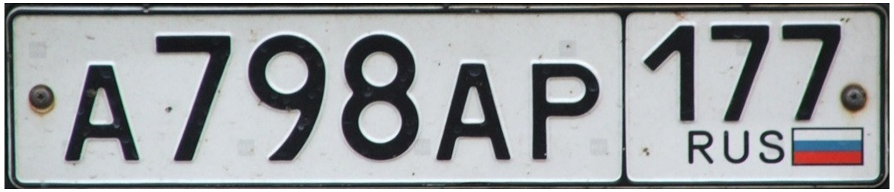
- Each digit can be chosen in *10* ways [0 - 9]

License Plates



- Each digit can be chosen in *10* ways
- Thus a sequence of digits can be chosen in
  - $10 \times 10 \times 10 = 1\ 000$ ways

License Plates



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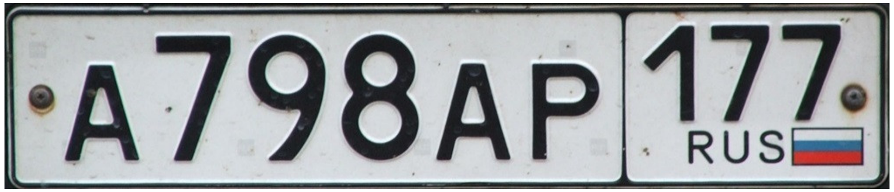

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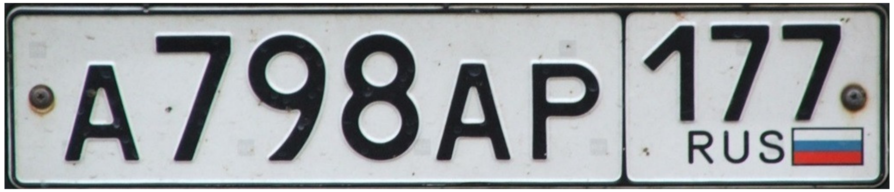
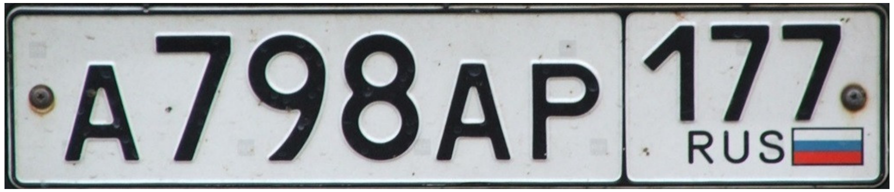
- Overall, there are *1 728 000* license plates for a region

License Plates



- Is *1 728 000* license plates enough for a region?

License Plates



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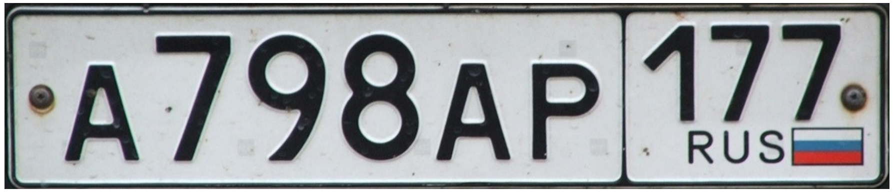
  - No, it's not: for example, there are about *5 600 000* vehicles in Moscow (as of *2016*)

License Plates



- Is *1 728 000* license plates enough for a region?
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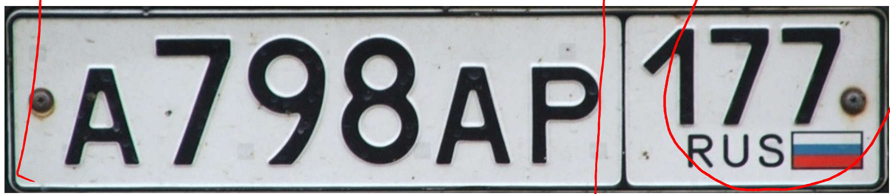
- Does it mean that by the Pigeonhole principle there are vehicles with identical license plates?

License Plates



- Is *1 728 000* license plates enough for a region?
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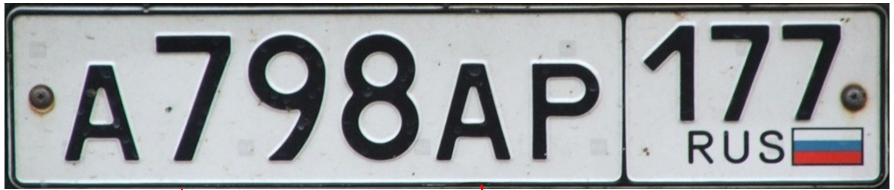
  - No, several regional codes were introduced for same region

n^k

26 + 26 + 26 + 26 × 26

License Plates



- Is *1 728 000* license plates enough for a region?
  - No, it's not: for example, there are about *5 600 000* vehicles in Moscow (as of *2016*)
- Does it mean that by the Pigeonhole principle there are vehicles with identical license plates?
  - No, several regional codes were introduced for same region
- But this required intro of three-digit regional codes

- Number of Tuples
- License Plates
- Tuples with Restrictions
- Permutations

Tuples with Restrictions

- We have shown how using the rule of product we can compute the number of tuples or a certain length of a fixed set of symbols

- But the rule of product can also give us other things too

Numbers with exactly one *7*-digit

Problem

How many integer numbers between *0* and *9999* are there that have exactly one *7* digit?

[0 ... 9999]
0007 - 1237 -
0070 - 7635.

Numbers with exactly one *7*-digit

Problem

How many integer numbers between *0* and *9999* are there that have exactly one *7* digit?

- Numbers between *0* and *9999* are sequences of digits of length *4*
- For numbers below *1000* for length *4* → 0000 → 0999
  - Three digital numbers correspond to sequences starting with *0*

Numbers with exactly one *7*-digit

$$* \quad * \quad * \quad *$$

- We can place the unique *7* at any of the four positions

Numbers with exactly one *7*-digit

y x x 7
x y 7 x
* * * *
7 x y y
x 7 y x

- We can place the unique *7* at any of the four positions
- This gives us *4* cases;
  - if we compute the number of sequences in all four cases, we can get the answer by the rule of sum

Numbers with exactly one *7*-digit

* * * *

- We can place the unique *7* at any of the four positions
- This gives us *4* cases;
  - if we compute the number of sequences in all four cases, we can get the answer by the rule of sum
- Consider one of the cases
  - Each of other three digits can be picked out in *9* options!

Numbers with exactly one *7*-digit

*   *   *   *

7   x   x   x

x = [0 - 9], exc 7

- We can place the unique *7* at any of the four positions
- This gives us *4* cases;
  - if we compute the number of sequences in all four cases, we can get the answer by the rule of sum
- Consider one of the cases
  - Each of other three digits can be picked out in *9* options!
  - digit *7* is forbidden

Numbers with exactly one 7-digit

$$* \quad * \quad 7 \quad *$$

9(x) 9(x) 1 9(x)

- Thus there are $9 \times 9 \times 9 = 729$ sequences in this case

Numbers with exactly one *7*-digit

0 - 9, exc 7

7 9(7) 9(x) 9(x) 7 729

* * 7 * = 729

9(x) 7 9(x) 9(x) 729

7 9(x) 9(x) 9( ) 729

- Thus there are $9\times9\times9=729$ sequences in this case
  - And in all other cases as well!

Numbers with exactly one 7-digit

$$* \quad * \quad 7 \quad *$$

- Thus there are $9 \times 9 \times 9 = 729$ sequences in this case
  - And in all other cases as well!
- There are 4 cases, so there are $4 \times 729 = 2\,916$ 4-digit numbers below 10 000 with exactly one digit 7 present

Numbers with exactly one *7*-digit

$$* \quad * \quad 7 \quad *$$

100 $\overline{)2916}$

29.16%

- Thus there are $9 \times 9 \times 9 = 729$ sequences in this case
  - And in all other cases as well!
- There are 4 cases, so there are $4 \times 729 = 2\,916$ 4-digit numbers below 10 000 with exactly one digit 7 present
  - This is below $1/3$ (3333), but above $1/4$ (2500) of all four digit numbers
  - This is an estimation of the probability to get exactly one digit *7* if we pick a number below *10 000* "randomly"

- Number of Tuples
- License Plates
- Tuples with Restrictions
- Permutations

Permutations

- We have discussed how to count the number of tuples
- Now we are ready to proceed to the second standard combinatorial setting: permutations

Permutations

$n = 3$; 1 2 3

$k = 2$; 12, 13, 23, 21, 31, 32

Set = size(n)

Seq = siz(k)

Problem

Suppose we have a set of n symbols. How many different sequences of length k we can form out of these symbols if we are not allowed to use the same symbol twice?

Permutations

Problem

Suppose we have a set of n symbols. How many different sequences of length k we can form out of these symbols if we are not allowed to use the same symbol twice?

12, 23, 31, ~~11~~, ~~22~~, ~~33~~
121, 122, 313

- *k*-permutations
  - Tuples of length k without repetitions

Permutations

Problem

Suppose we have a set of n symbols. How many different sequences of length k we can form out of these symbols if we are not allowed to use the same symbol twice?

- k-permutations
  - Tuples of length k without repetitions
- Observe that if $n < k$, then there are no k-permutations: there are simply not enough different letters

Permutations

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Suppose we have a set of n symbols. How many different sequences of length k we can form out of these symbols if we are not allowed to use the same symbol twice?

- k-permutations
  - Tuples of length k without repetitions
- Observe that if $n < k$, then there are no k-permutations: there are simply not enough different letters
- So it is enough to solve the problem for the case $k \leq n$

Permutations

$$
\begin{array}{ccccc}
1 & 2 & 3 & \ldots & k \\
* & * & * & \ldots & *
\end{array}
$$

- Let us apply rule of product

Permutations

if n = 10
[0, 9]

$$1 \quad 2 \quad 3 \quad \ldots \quad k$$

$$* \quad * \quad * \quad \ldots \quad *$$

$$n \qquad (n-1)$$

n = 6
[0, 5]

- Let us apply rule of product
- The first symbol can be picked in n ways

Permutations

$$1 \quad * \quad 2 \quad * \quad 3 \quad * \quad \ldots \quad \ldots \quad k \quad *$$

$$n$$

- Let us apply rule of product
- The first symbol can be picked in n ways
- How many choices there are for the second symbol?

Permutations

1 2 3 $\ldots$ k

$*$ $*$ $*$ $\ldots$ $*$

n

- Let us apply rule of product
- The first symbol can be picked in n ways
- How many choices there are for the second symbol?
  - We can place anything there, except for the symbol on the first place

Permutations

1	2	3	…	k
*	*	*	…	*
n	n-1			

- Let us apply rule of product
- The first symbol can be picked in n ways
- How many choices there are for the second symbol?
  - We can place anything there, except for the symbol on the first place
- Symbol on the first place might be arbitrary, but whatever it is there are $n-1$ choices for the second symbol!

Permutations

1	*2*	*3*	…	*k*
*	*	*	…	*
n	*n-1*			

- So we can pick the first and the second symbol in $n \times (n-1)$ ways

Permutations

1	*2*	*3*	…	*k*
*	*	*	…	*
n	*n-1*	*n-2*		
10	9	n=10		

- So we can pick the first and the second symbol in $n \times (n-1)$ ways
- Now, the third symbol can be picked among $n-2$ options: all except the symbols in the first and the second position

Permutations

if $n = k$; $= 1$?

1	2	3	…	k
*	*	*	…	*
n	n-1	n-2	…	$(n-k+1)$

- So we can pick the first and the second symbol in $n \times (n-1)$ ways
- Now, the third symbol can be picked among $n-2$ options: all except the symbols in the first and the second position
- And so on; for each next symbol we have one less option

Permutations

$$\begin{array}{ccccc} 1 & 2 & 3 & \ldots & k \\ * & * & * & \ldots & * \\ n & n\text{-}1 & n\text{-}2 & \ldots & n\text{-}k+1 \end{array}$$

$k \leq n$

- So we can pick the first and the second symbol in $n \times (n-1)$ ways
- Now, the third symbol can be picked among $n-2$ options: all except the symbols in the first and the second position
- And so on; for each next symbol we have one less option
- In the end for the last object we have $n-k+1$ options

Permutations

$$\begin{array}{ccccc} 1 & 2 & 3 & \dots & k \\ * & * & * & \dots & * \\ n & n\text{-}1 & n\text{-}2 & \dots & n\text{-}k\text{+}1 \end{array}$$

- Overall we have $n \times (n-1) \times \dots \times (n-k+1)$ k-permutations

Permutations

1	2	3	…	k
*	*	*	…	*
n	n-1	n-2	…	n-k+1

- Overall we have $n\times(n-1)\times\ldots\times(n-k+1)$ k-permutations
- Convenient notation: $n!=1\times 2\times\ldots\times n$; this number is called **factorial** of n

Permutations

$n=6$, $k=3$ = $\frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot 4\cdot 5\cdot 6}{\cancel{1\cdot 2\cdot 3}} = 120$

1	2	3	…	k
*	*	*	…	*
n	n-1	n-2	…	n-k+1

- Overall we have $n \times (n-1) \times \ldots \times (n-k+1)$ k-permutations
- Convenient notation: $n! = 1 \times 2 \times \ldots \times n$; this number is called **factorial** of n
- In this notation the number of k-permutations of n symbols of length k looks nicer: it is $n!/(n-k)!$ k-permutation

if $n=3$, $k=2$ $\Rightarrow \frac{n!}{(n-k)!} = \frac{1\times 2\times 3}{(3-2)!} = \frac{\cancel{1}\times 2\times 3}{\cancel{1!}} = 6$

Permutations

1	2	3	…	k
*	*	*	…	*
n	n-1	n-2	…	n-k+1

- Overall we have $n\times(n-1)\times\ldots\times(n-k+1)$ k-permutations
- Convenient notation: $n!=1\times 2\times\ldots\times n$; this number is called **factorial** of n
- In this notation the number of k-permutations of n symbols of length k looks nicer: it is $n!/(n-k)!$
- What if $n-k=0$? Convention: $0!=1$

$n=3,\ k=3$

$\frac{3!}{(3-3)!} = \frac{1\cdot 2\cdot 3}{1} = 6$

Permutations

Problem

In how many orders can we place n books on the shelf?

Permutations

Problem

In how many orders can we place n books on the shelf?

- Each book is a symbol

Permutations

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In how many orders can we place n books on the shelf?

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- We need to count n-permutations of n symbols; these are called permutations

Permutations

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- We need to count n-permutations of n symbols; these are called permutations
- By the previous result there are $n!$ of them $\frac{n!}{(n-k)!}$

Permutations

Problem

In how many orders can we place n books on the shelf?

- Each book is a symbol
- We need to count n-permutations of n symbols; these are called permutations
- By the previous result there are $n!$ of them
- This is the formula that was used in the discussion of magic square in the course "What is a Proof?"

Conclusion

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- We have started the discussion of Combinatorics
- Important for Probability Theory, estimations on running time of algorithms, mathematics in general
- We have discussed two standard settings: tuples and permutations; they help in many cases
- Recursive counting is also useful, especially for computational applications
- Still are not ready to count, say, what are the chances to get **two aces** in a **6 card hand** :D

Thank you.