Introduction to Discrete Math

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Course Outline

- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion, Logic, Invariants
- Probability & Combinatronics
  - Counting, Probability, Random Variables
- Graph Theory
  - Graphs (cycles, classes, parameters)
- Number Theory & Cryptography
  - Arithmetic in modular form
  - Intro to Cryptography

Mathematical Thinking – Logic

EXAMPLES, COUNTEREXAMPLES, LOGIC

- Examples
- Counterexamples
- Logic

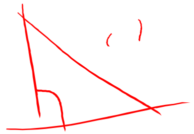
Examples

- Sometimes, one example is already enough
- If we want to prove that white lions exists
  - It is enough to show just one white lion
  - such examples are not always easy to come up with
    - 13 in the wild and approximately 300 in captivity
    - https://whitelions.org/white-lion/faqs/

Examples

Problem I

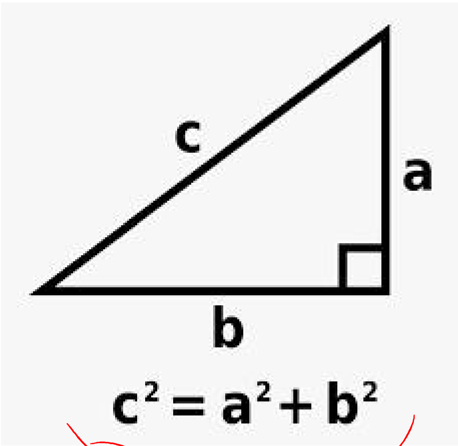
- Is it possible that for three positive integer numbers a, b, and c so that $a^2+b^2 = c^2$?



Examples

Solution

- To come up with this example, recall
  - Pythagorean Theorem with right angle, and sides 3, 4, and 5



$$h = \sqrt{(a^2 + b^2)}; h = c$$

$$c^2 = a^2 + b^2$$

$$25 = 16 + 9$$

$$5^2 = 4^2 + 3^2$$

Examples

Problem II

- Is it possible that for three positive integer numbers a, b, and c so that $a^3+b^3 = c^3$?

Examples

Solution

* conjecture – conclusion assume to be true due to initial supporting evidence but no proof or disproof has yet been found

- This is in fact impossible, so there is no example
- Fermat's Last Theorem (1637), a famous mathematical conjecture
  - for any integer $n > 2$, there are no such integers a, b, & c such that $a^n + b^n = c^n$.
  - Mathematicians failed tried to prove it for hundreds of years
  - Andrew Wiles was able to prove it in 1995

Examples

Problem III

- Is it possible that for positive integer numbers a, b, c, and d so that $a^4+b^4+c^4=d^4$?

Examples

Solution

- We could initially assume that it is impossible due to Fermat's Last Theorem
- But the theorem only focus on equations with the form

$$a^n + b^n = c^n$$

- Hence, it is not applicable for this problem a, b c d

Examples

Solution

- It is actually possible, the smallest number though is very big

$$95800^4 + 217519^4 + 414560^4 = 422481^4$$

- Computers used to derive the examples
- But the possible number of values are so huge that it is hard to find examples that satisfy the equation, even with the help of computers

Examples

Problem IV

- Is there a power of 2 that starts with 65?

$x^2 = 65yyyy\ldots$

Examples

Problem IV

- Is there a power of 2 that starts with 65?

Solution

- The answer is $2^{16} = 65536$ 1 (1) - 2 4 - 8 - 16 - 32 - 64
- This is the complete solution
- In fact, there is a power of 2 that starts w/ any integer n, $n > 0$
  - But much more difficult to prove

- Examples
- Counterexamples
- Logic

Counterexamples

- Just one counterexample is enough to disprove a statement
- If we want to prove that all swans are white
  - Just one instance of black swan is enough to disprove it
  - However, it is often difficult to find such counterexamples

Counterexamples

Theorem I

- *All rectangles are squares*   F

Counterexamples

Theorem I

- *All rectangles are squares*

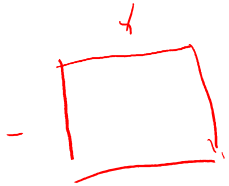
Solution

- A rectangle with sides of sizes 1 and 2, respectively, is not a square.
- This is a counterexample for the theorem, hence the theorem is wrong

Counterexamples

Theorem II

- *All square are rectangles*



Solution

- There is no counterexample for this case, hence the theorem is true
- Since square is a rectangle with equal sides.

Counterexamples

Theorem II

- *All square are rectangles*

Counterexamples

Theorem III

- Euler came up w/ a generalization of Fermat's Last Theorem
- *For any $n > 2$, it is impossible for an n-th power of a positive integer to be represented as a sum of $n - 1$ numbers w/c are the n-th powers of positive integers*
- For $n = 3$, it is the same as Fermat's Last Theorem: It is impossible that $a^3 + b^3 = c^3$.

Counterexamples

Solution

- Lander came up with a counterexample in 1966 for $n=5$:

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

- Elkies found another counterexample in 1986 for $n=4$:

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$$

- Frye found the smallest counterexample for $n=4$ in1988:
  - Which is an example for one statement & counterexample for another

$$95800^4 + 217519^4 + 414560^4 = 422481^4$$

- Examples
- Counterexamples
- Logic

Logic

Logical Operators

- Negation
- Logical AND
- Logical OR
- If-then

Negation

Statement

- All swans are white.

Negation

Statement

- All swans are white.

Negation

- **Not** all swans are white. Or, there are swans that are **not** white.

Negation

Statement

- There exists three positive integers a, b, & c, such that

  $a^3 + b^3 = c^3$.

Negation

Statement

- There exists three positive integers a, b, & c, such that

  $$a^3 + b^3 = c^3.$$

Negation

- There are no such positive integer numbers a, b, & c, such that $a^3 + b^3 = c^3$.
- Or for any positive integers a, b, & c, such that $a^3 + b^3 \neq c^3$.

Negation

Statement

- $4 = 2 + 2$
- $5 = 2 + 2$

Negation

Statement

- $4 = 2 + 2$
- $5 = 2 + 2$

Negation

- $4 \neq 2 + 2$
- $5 \neq 2 + 2$

Negation

Negation

iff

- Is true, if and only if the initial statement is wrong
- Is false, if and only if the initial statement is correct

$4 = 2+2$ ✓ $\xrightarrow{N}$ $4 \neq 2+2$ ✗

$5 = 2+2$ ✗ $\qquad$ $5 \neq 2+2$ ✓

Logical AND

Statement

- $4 = 2 + 2$ **AND** $4 = 2 \times 2$
- The logical AND of two statements is true if and only if both statements are true.

- $4 = 2 + 2$ AND $4 = 2 \times 2 \rightarrow$ TRUE
- $4 = 2 + 2$ AND $5 = 2 \times 2 \rightarrow$ FALSE
- $5 = 2 + 2$ AND $4 = 2 \times 2 \rightarrow$ FALSE
- $5 = 2 + 2$ AND $5 = 2 \times 2 \rightarrow$ FALSE

Logical OR

Statement

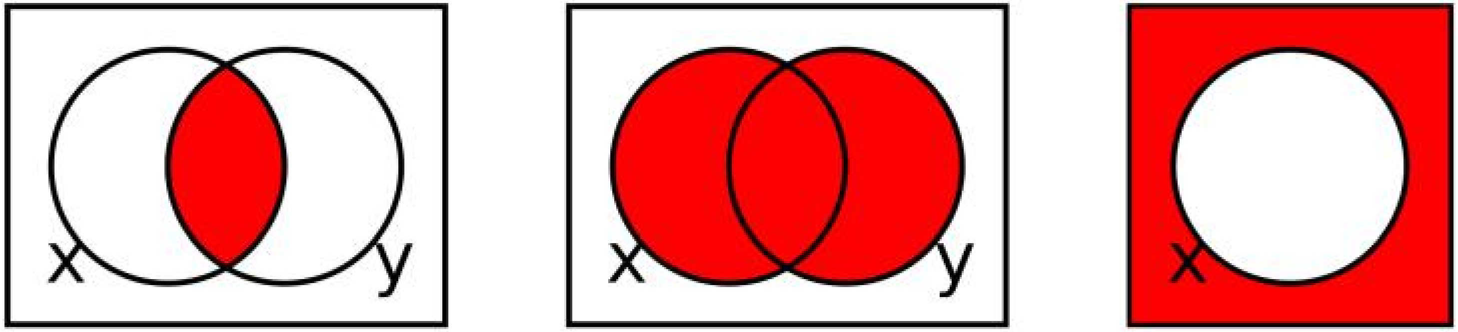
- $4 = 2 + 2$ **OR** $4 = 2 \times 2$
- The logical OR of two statements is true if and only if at least one of the statements is true.

- $4 = 2 + 2$ OR $4 = 2 \times 2 \rightarrow$ TRUE
- $4 = 2 + 2$ OR $5 = 2 \times 2 \rightarrow$ TRUE
- $5 = 2 + 2$ OR $4 = 2 \times 2 \rightarrow$ TRUE
- $5 = 2 + 2$ OR $5 = 2 \times 2 \rightarrow$ FALSE

Venn Diagram

Symbols:

- Logical NOT (¬), logical AND (∧), and logical OR (∨)



$x \wedge y$ $x \vee y$ $\neg x$

https://en.wikipedia.org/wiki/Logical_connective

Negation of AND

Statement

Negation of AND is OR of negations:

- Negation of "A AND B" is "Not A OR Not B"

$A \quad \neg A \quad \vee \quad \neg B$

Negation of AND

Statement

Negation of AND is OR of negations:

- Negation of “A AND B” is “Not A OR Not B”

A T AND B T

Negation of (4 = 2 + 2) AND (4 = 2 x 2). T

- 4 ≠ 2 + 2 OR 4 ≠ 2 x 2. Negated AND

F F → F.

Negation of OR

Statement

Negation of OR is AND of negations:

- Negation of “A OR B” is “Not A AND Not B”

$\neg A \quad \wedge \quad \neg B$

Negation of OR

Statement

Negation of OR is AND of negations:

- Negation of “A OR B” is “Not A AND Not B”

A T B F $\Rightarrow$ T

Negation of $(4 = 2 + 2)$ OR $(5 = 2 \times 2)$:

- $(4 \neq 2 + 2)$ AND $(5 \neq 2 \times 2.)$ F $\to$ $\neg$ T

F T

Thank you.