If f(x) is a function of period 2π, then the function can be represented by the following series (5).

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

The coefficients, called Fourier coefficients of f(x), are given by the Euler formula.

$$(0)\ \ a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$(a)\ \ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad n = 1, 2, \cdots$$

$$(b)\ \ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad n = 1, 2, \cdots$$

THEOREM 1 Orthogonality of the Trigonometric System (3)

The trigonometric system (3) is orthogonal on the interval -π≤x ≤ π(hence on any other interval of length 2π); that is

$$(9)\quad \begin{aligned} &(a) \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \quad n \neq m \\ &(b) \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \quad n \neq m \\ &(c) \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \quad n \neq m \;\; or \;\; n = m \end{aligned}$$

Trigonometric system:

(3) 1, cos*x*, sin*x*, cos*2x*, sin*2x*, …., cos*nx*, sin*nx*, …

THEOREM 2 Representation by a Fourier Series

Let f(x) be periodic with period 2π and piecewise continuous in the interval $-\pi \le x \le \pi$. Furthermore, let f(x) have a left- and right-hand derivatives at each point of that interval. Then the Fourier series (5) of f(x) with the coefficients in (6) converges. Its sum is f(x), except at point x_0 where f(x) is discontinuous. There the sum of the series is the average of left- and right-hand limits of f(x) at x_0.

If f(x) is a function of period $2L$, then the function can be represented by the following series.

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

The coefficients, called Fourier coefficients of f(x), are given by the Euler formula.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

2. Simplifications: Even and Odd Functions

Case 1: $f(x)$ is an even function, $f(-x)=f(x)$.

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{0}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

Case 2: $f(x)$ is an odd function.

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

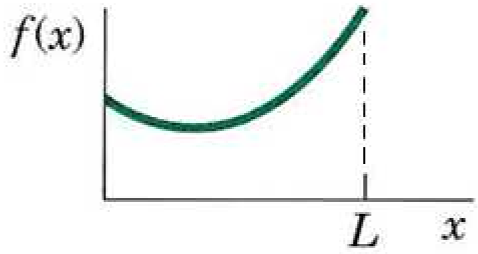
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Half-Range Expansions

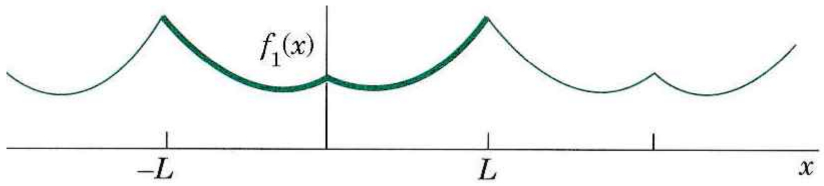
Consider a function $f(x)$ given 0<x<L.

For example,

- shape of a distorted violin string
- the temperature in a metal bar of length L



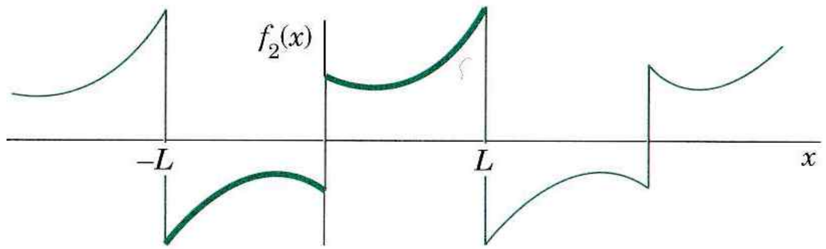
given function $f(x)$



f(x) continued as an **even** periodic function

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx$$



f(x) continued as an **odd** periodic function

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

Orthogonal Functions

Functions $y_1(x)$, $y_2(x)$, … defined on some interval $a \leqq x \leqq b$ are called **orthogonal** on this interval with respect to the **weight function** $r(x)>0$ if for all m and n different from m,

$$(4) \quad (y_m,\ y_n) = \int_a^b r(x)y_m(x)\,y_n(x)\,dx = 0 \quad (m \neq n)$$

The Norm $\|y_m\|$ of y_m is defined by

$$(5) \quad \|y_m\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x)y_m^2(x)\,dx}$$

Normal/Orthonormal Functions

y_m is a normal function if and only if $\|y_m\| = 1$.

Functions $y_1(x)$, $y_2(x)$, …are orthonormal(정규직교) if and only if

$$(y_m,\ y_n) = \int_a^b r(x)\,y_m(x)\,y_n(x)\,dx$$

$$= \delta_{mn} = \begin{cases} 0 & (m \neq n) \\ 1 & (m = n) \end{cases}$$

Kronecker symbol, Kronecker delta function

Orthogonal Functions when r(x)=1

If the **weight function** r(x)=1, the term orthogonal is more briefly used than orthogonal with respect to r(x)=1.

$$(y_m,\ y_n) = \int_a^b y_m(x)\, y_n(x)\, dx = 0 \quad (m \neq n)$$

$$\|y_m\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b y_m^2(x)\, dx}$$

11.6 Ortthogonal Series, Generalized Fourier Series

Let y_0, y_1, y_2,… be orthogonal with respect to a weight function r(x) on an interval a≦x≦b, and let f(x) be a function that can be represented by a convergent series

$$(1) \quad f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots$$

(1) is called an **orthogonal series**(직교급수), **orthogonal expansion**(직교전개), or **generalized Fourier series**(일반화된 Fourier 급수)라 부른다.

If y_m are eigenfunctions of a Sturm-Liouville problem, (1) is called an **eigenfunction expansion**(고유함수 전개).

Examples of Generalized Fourier Series

- Fourier-Legendre Series
- Fourier-Bessel Series

Legendre Polynomials(Sec 5.2)

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = (1/2)(3x^2 - 1)$$
$$P_3(x) = (1/2)(5x^3 - 3x)$$
$$P_4(x) = (1/8)(35x^4 - 30x^2 + 3)$$
$$P_5(x) = (1/8)(63x^5 - 70x^3 + 15x)$$

THEOREM 1 Fourier integral

If f(x)

- is piecewise continuous in every finite interval and
- has a right-hand derivative and a left-hand derivative at every point, and
- is absolutely integrable,

then f(x) can be represented by a Fourier integral (5) with A and B given by (4).

At a discontinuous point the value of the Fourier integral is the average of the left- and right-hand limits of f(x) at that point.

(5) $$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega$$

(4) *where* $$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv$$

f(x) is absolutely integrable if the following integral exists.

(2) $$\int_{-\infty}^{\infty} |f(x)|dx = \lim_{a \to -\infty} \int_{a}^{0} |f(x)|dx + \lim_{b \to \infty} \int_{0}^{b} |f(x)|dx$$

(5) $$f(x) = \int_{0}^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$$

(4) $$where \quad A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\, dv$$

Fourier Cosine Integral

Fourier Integral:

$$f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega$$

$$where \quad A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\, dv$$

If f(x) is even, B(ω)=0,

Fourier Cosine Integral:

$$f(x) = \int_0^{\infty} A(\omega)\cos\omega x\, d\omega \qquad A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v)\cos\omega v\, dv$$

Fourier Sine Integral

Fourier Integral:

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$$

$$where \quad A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\, dv$$

If f(x) is odd, A(ω)=0,

Fourier Sine Integral:

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\, d\omega \qquad B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin\omega v\, dv$$