Introduction to Discrete Math

Felipe P. Vista IV

Course Outline

- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion, Logic, Invariants
- Probability & Combinatronics
  - Counting, Probability, Random Variables
- Graph Theory
  - Graphs (cycles, classes, parameters)
- Number Theory & Cryptography
  - Arithmetic in modular form
  - Intro to Cryptography

Mathematical Thinking – Invariants

INVARIANTS

- Invariants
- Coffee with milk
- More Coffee
- Debugging Problem

Invariants

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vary = changes

- Invariants → properties that do not change/ remain constant

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- Double counting is a special case

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There are two cups, one with coffee and another with milk. We take a spoon of coffee and add it to the cup of milk. We then take a spoonful from the cup of milk and put it into the cup of coffee. Which is larger, the amount of milk in the cup of coffee or the amount of coffee in the cup of milk?

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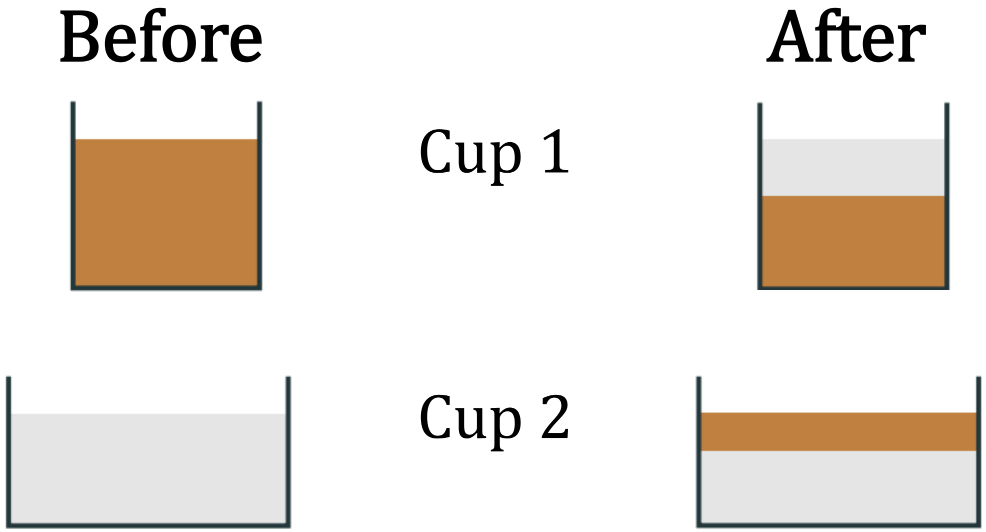
- We don’t know size of cups not size of the spoon
  - Maybe the answer depends on these parameters?
- Well, it turns out we do not need to do any calculation ☺!

Coffee with milk



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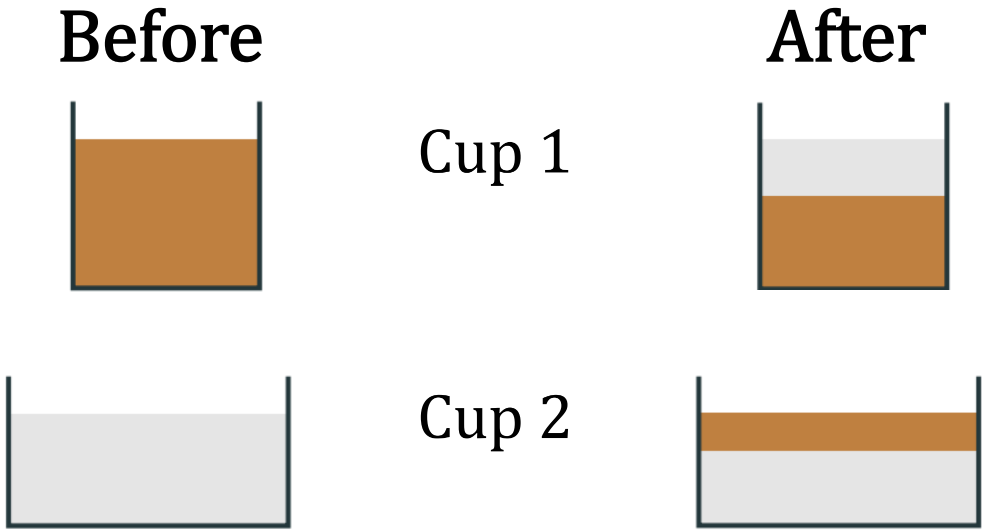
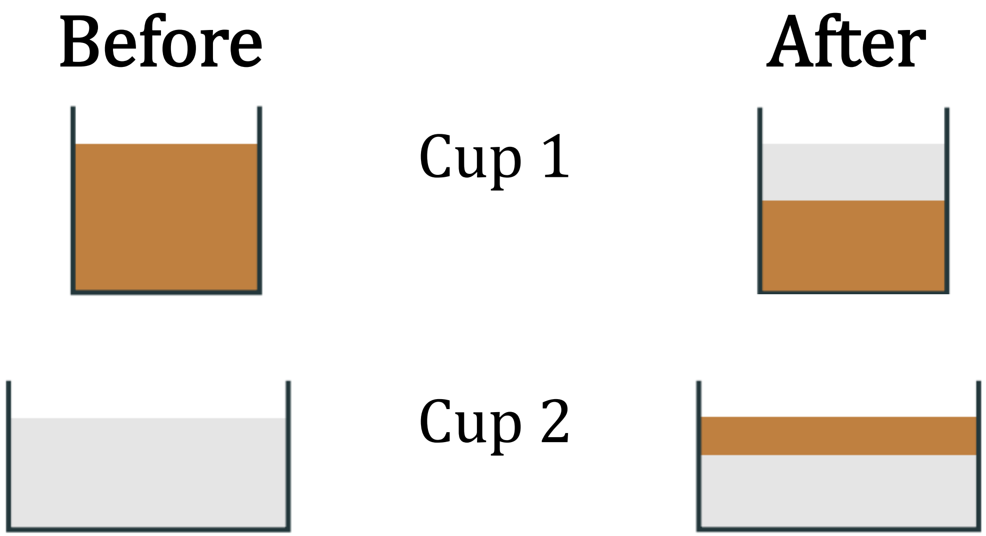


- The size of the drink in cup 1 is invariant!

Coffee with milk



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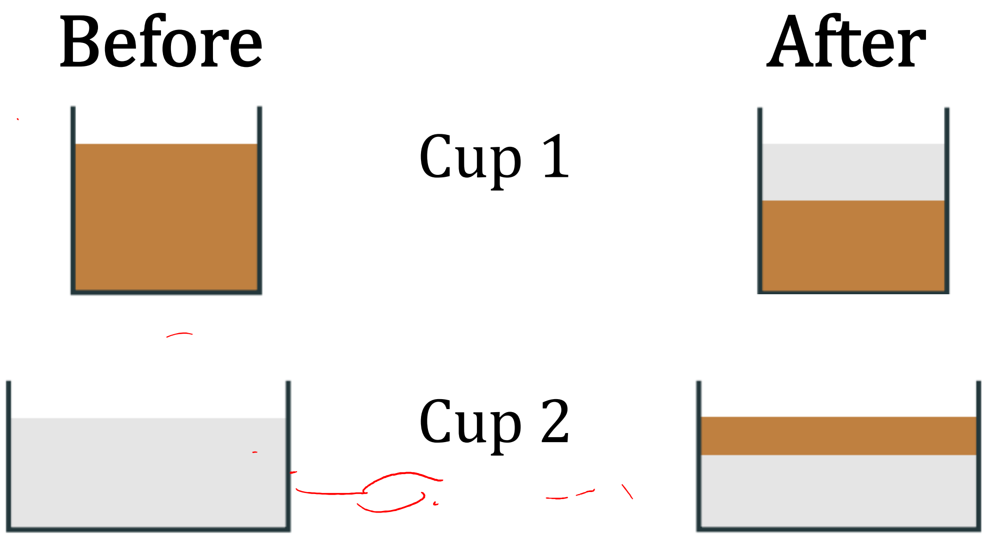

- The size of the drink in cup 1 is invariant!
- So the amount of coffee missing in cup 1 is the same as the amount of milk added into cup 1
- Conversely, amount of milk missing in cup 2 is the same as the amount of coffee added into cup 2

- Invariants
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There are two equally-sized cups, one with coffee & another milk. Both cups are half-full. We want coffee w/ lots of milk on the 1st cup: 1/3 coffee, 2/3 milk. We can pour from one cup to another back & forth. Can we get our favorite coffee mix into our favorite cup? Any amount would do, right proportion matters.

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  - It's good if second cup is mostly coffee
- Yet, invariants can help again
  - We just have to choose the right invariant

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Claim

The proportion of coffee in the 1^{st} cup is always greater than the 2^{nd} cup. That is, coffee in the 1^{st} cup is stronger than 2^{nd} cup.

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  - Else we have more milk than coffee in both cups
- But, total amount of coffee and milk are the same

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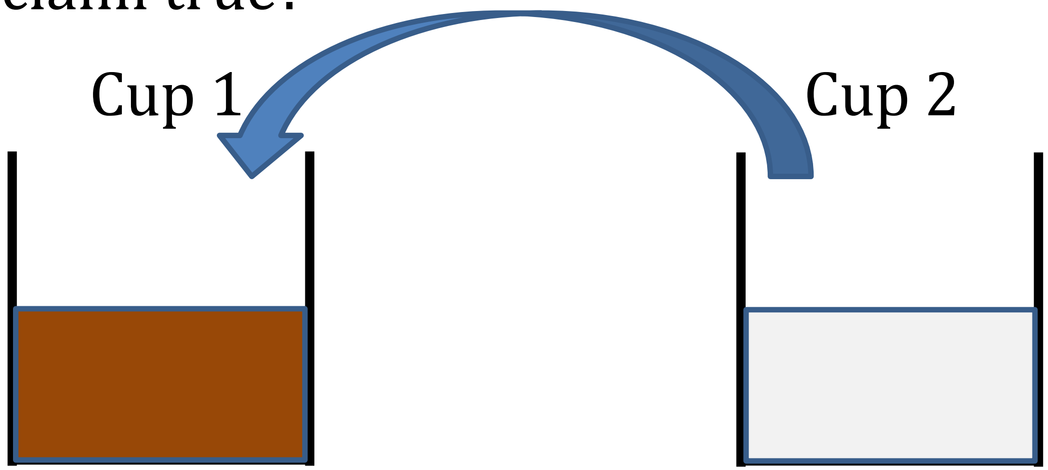

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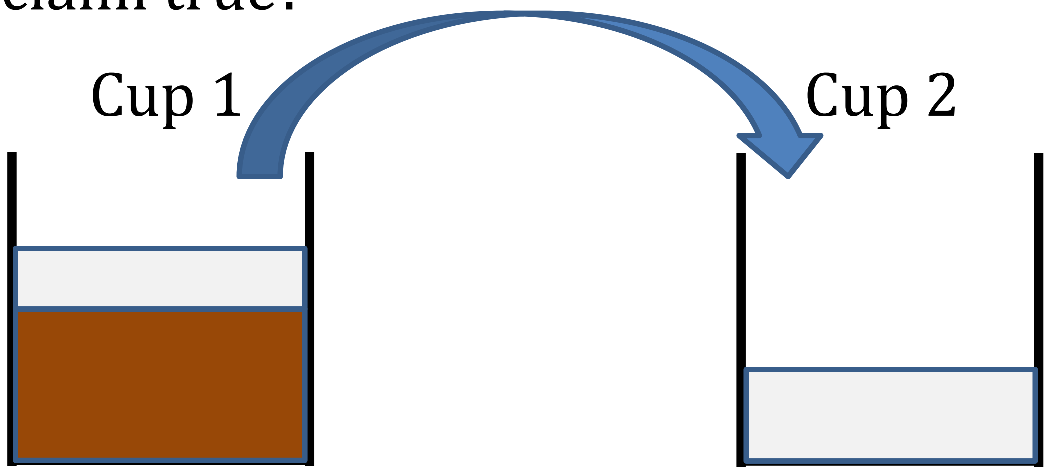

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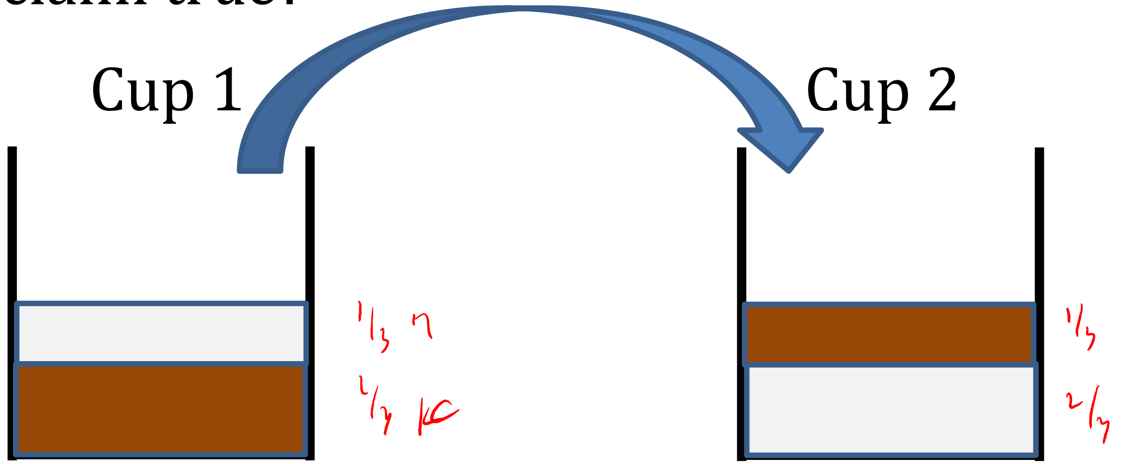

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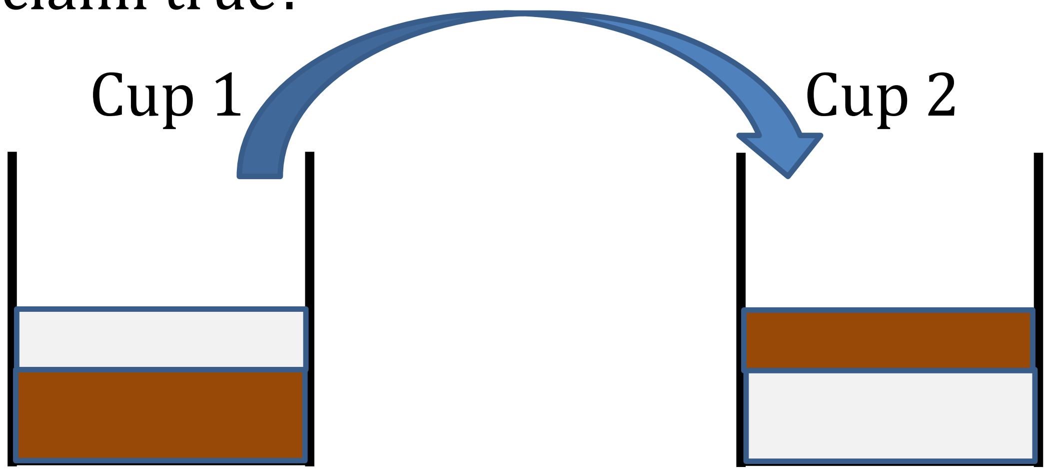

We "milk down" drink from Cup 1 into Cup 2

We "coffee down" drink from Cup 2 into Cup 1

- Invariants
- Coffee with milk
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Typical Debugging

$1 \rightarrow 3$

Problem

Bob is debugging his code. There is only one bug when he starts. But three new bugs appear once he fixes a bug. Bob fixed 15 bugs after several hours. How many pending bugs (to be fixed) does he have at this point?

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Pending:	1					

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Fixed:	0	1				
Pending:	1	3				

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Fixed:	0	1	2			
Pending:	1	3	5			

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Pending:	1	3	5	7	9	?

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- #*Pending* = *????*

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- $\#Pending = 1 + 2 \times \#Fixed$
  - This is our invariant!

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- Hence, if #Fixed = 15

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Mathematical Thinking – Invariants

TERMINATION

- Termination
- Football fans
- King Julien’s Books

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- In a more wider sense, invariant $\rightarrow$ properties that change in the right way
- Another standard use of an invariant is showing the termination of a process

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Football Fans

Problem

There are 2 football teams in a town. Each of the citizens support a team. If there are more fans of the other team among someone's friend other than his own, this person tends to switch to the other team. One such person switch teams every day. Is it possible that this switching process will go on forever? Assume the following: friendship is always mutual, population does not change, friendship does not change.

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- It seems natural that the process will stop
- How can we prove it though?
  - We need to look at the right value

Football Fans

Solution

Let us look at the number of opposite team's friendships, that is, the pairs of friends supporting opposite teams at the start of the 1st day.

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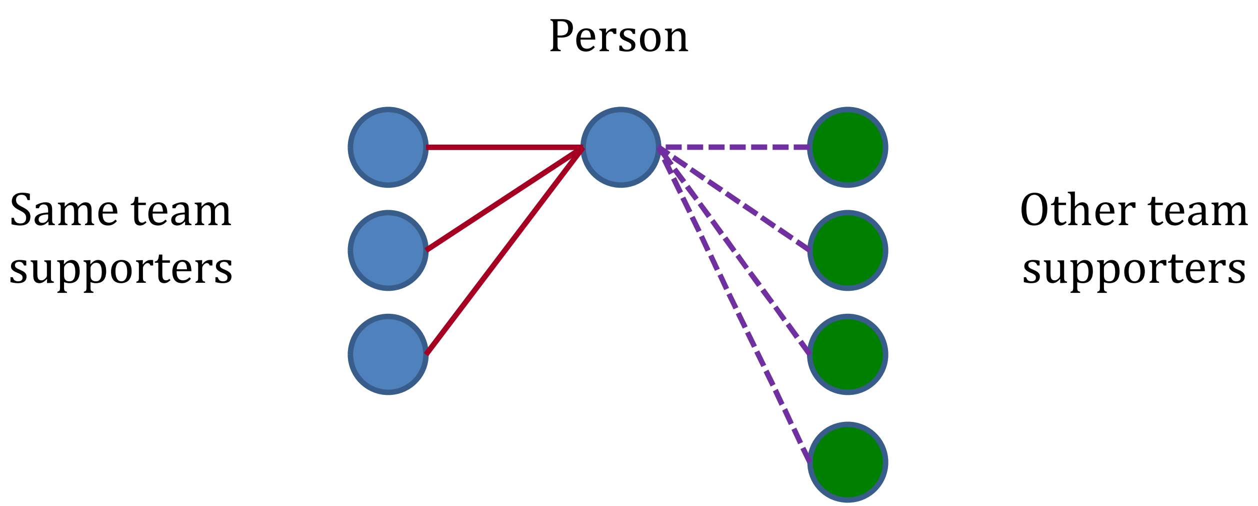

Football Fans

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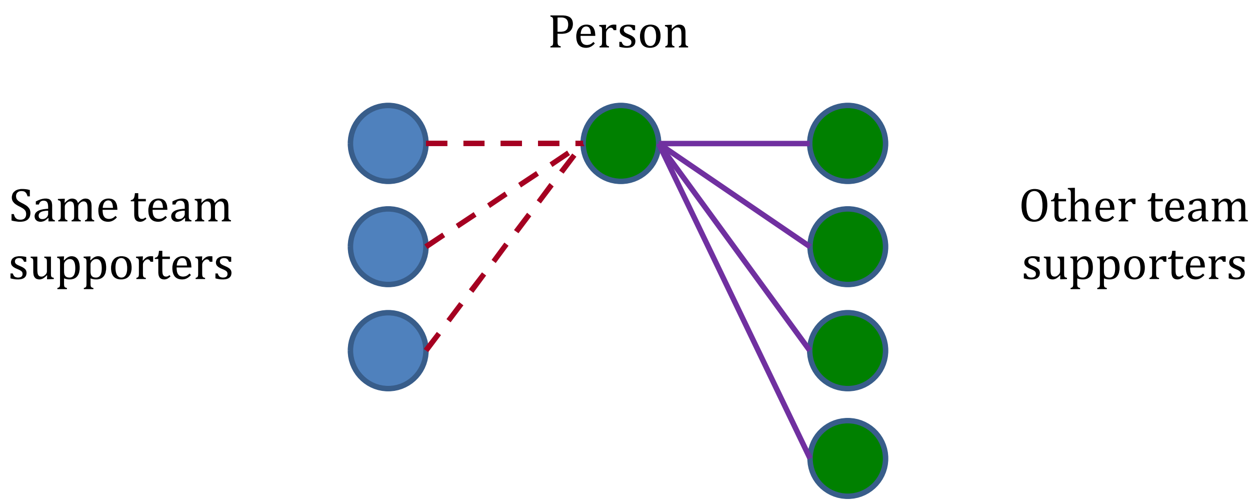
Let us look at the number of opposite team's friendships, that is, the pairs of friends supporting opposite teams. Let's see what happens with this value at **the end of day 1**.



Football Fans

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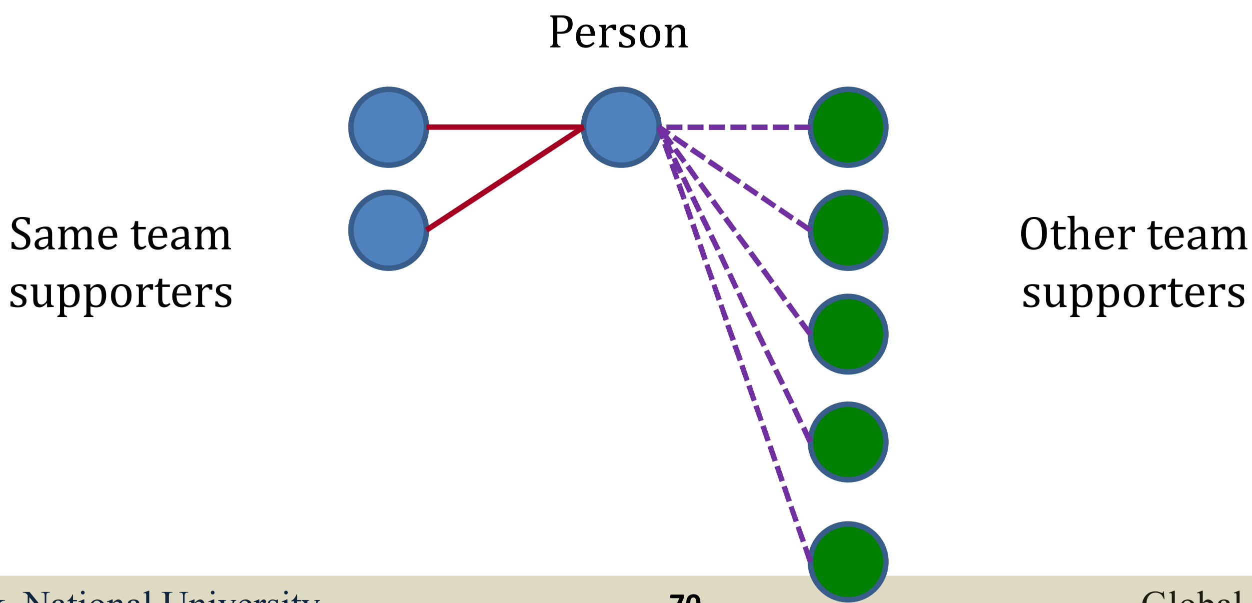

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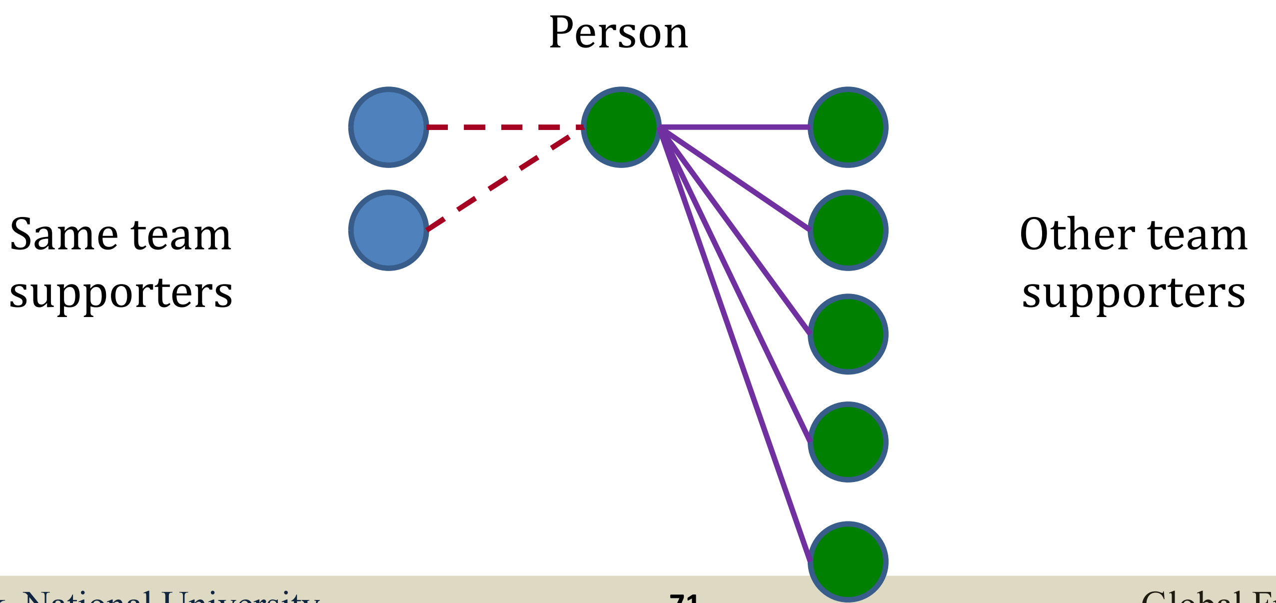

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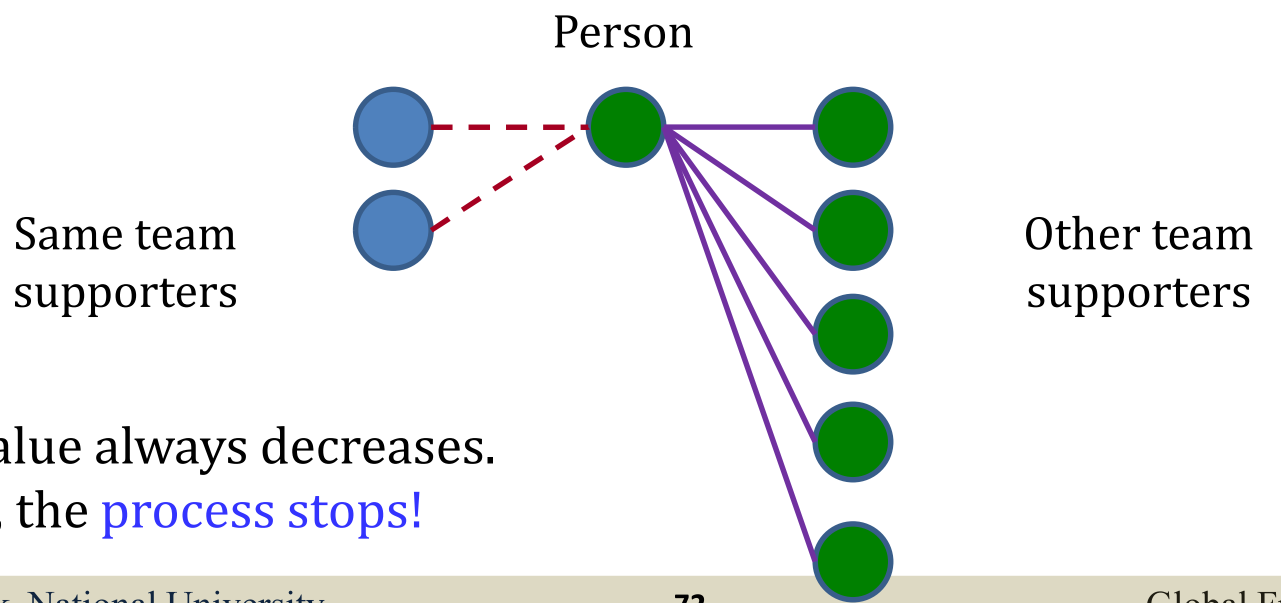


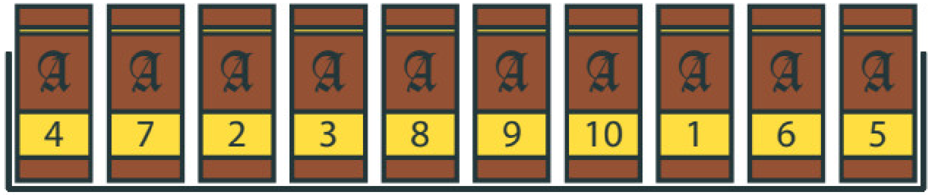
This value always decreases.
Hence, the process stops!

- Termination
- Football fans
- King Julien’s Books

King Julien's Books

Problem

King Julien has a shelf of his works consisting of 10 volumes, labeled 1, 2, …,10. These got jumbled & out of order over the years. King J asked Maurice to sort the collection but he can only take 2 books at once. The books are heavy hence he can switch only two per day. In how many days can Maurice guarantee that the volumes are sorted?



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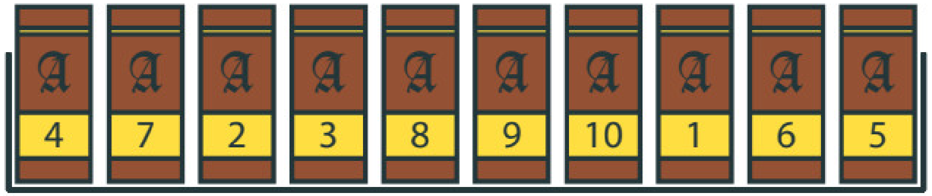
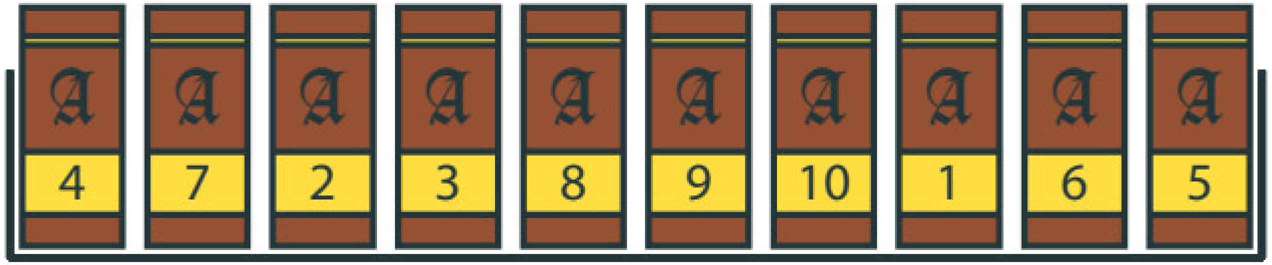


King Julien's Books

Let's check

We can always place books in the right order in at most 9 days?



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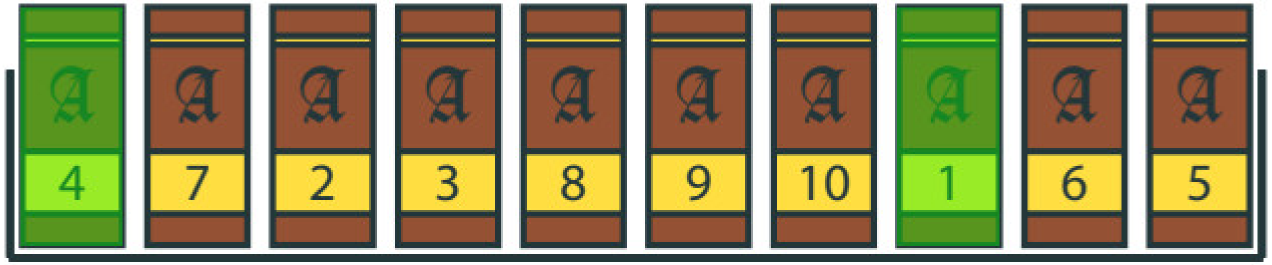

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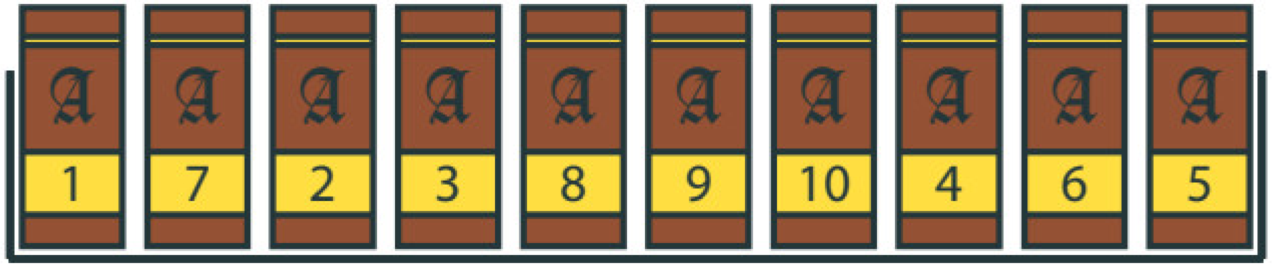

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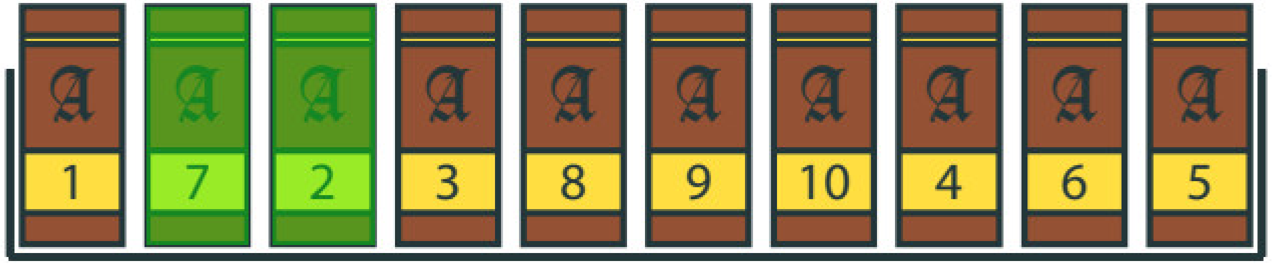

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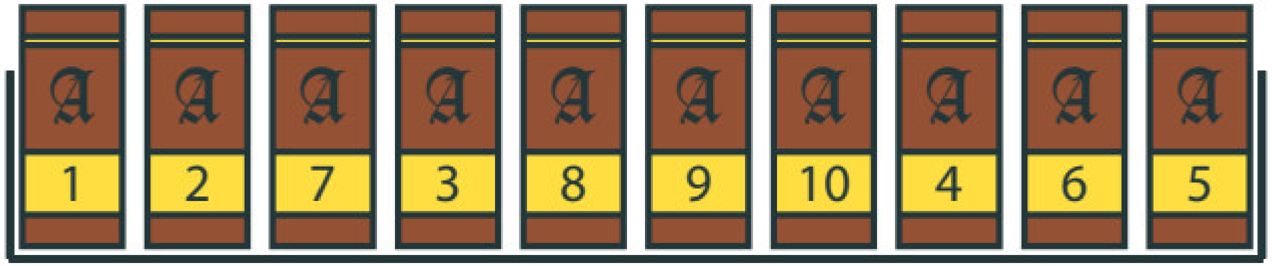

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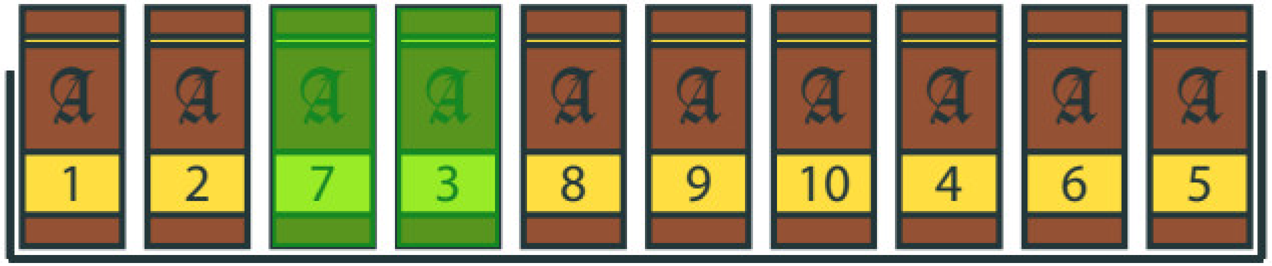

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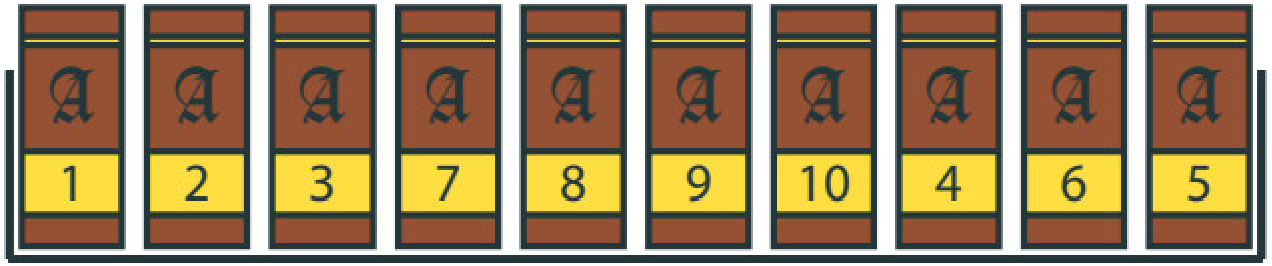

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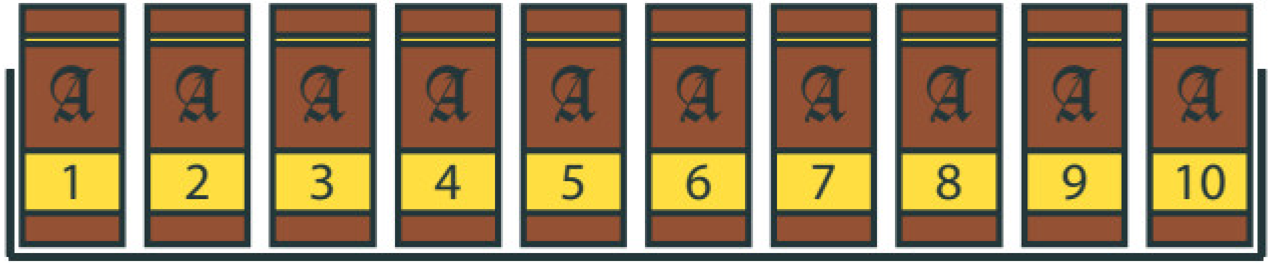

- Day 1: place Vol 1 on it's place
- Day 2: place Vol 2 on it's place
- Day 3: place Vol 3 on it's place
- And so on…
- On Day 9, the first 9 volumes are on their proper places. The 10^{th} volumes must also be in it's proper place since it's only one left.

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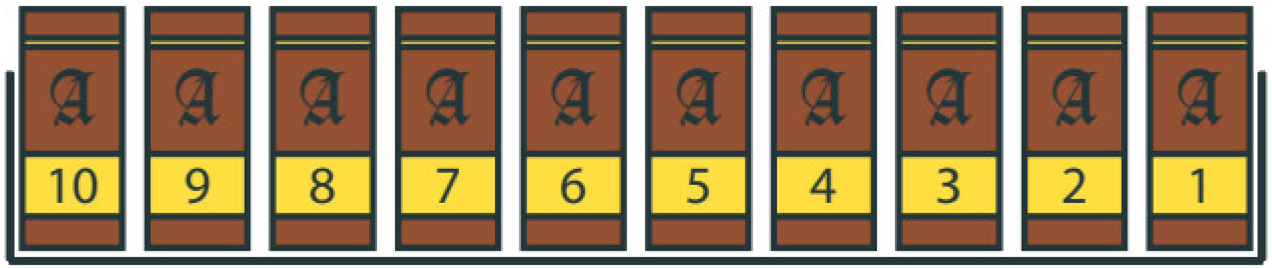
Recall:
- Permutation– order of selection is a factor
- Combination – order of selection not a factor

Ex: Permutation/Combination pairs from the set {A, B, C, D, E}
Permutation: AB AC AD AE BA BC BD BE CA CB CD CE DA DB DC DE EA EB EC ED
Combination: AB AC AD AE BC BE CD CE DE

King Julien's Books

Let's check

- Are 9 days the optimal?
- What could be the hardest permutation of the books?
  - It seems that the hardest is when the books are in opposite order



Recall:

- Permutation– order of selection is a factor
- Combination – order of selection not a factor

Ex: Permutation/Combination pairs from the set {A, B, C, D, E}

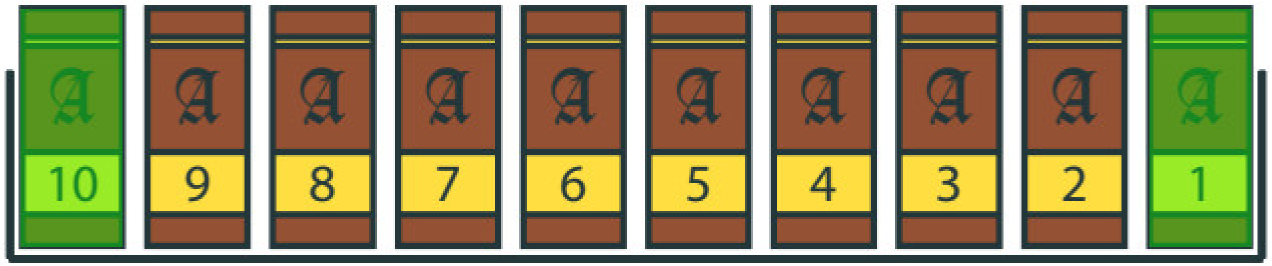
Permutation: AB AC AD AE BA BC BD BE CA CB CD CE DA DB DC DE EA EB EC ED

Combination: AB AC AD AE BC BE CD CE DE

King Julien's Books

Let's check

- Are 9 days the optimal?
- What could be the hardest permutation of the books?
  - It seems that the hardest is when the books are in opposite order



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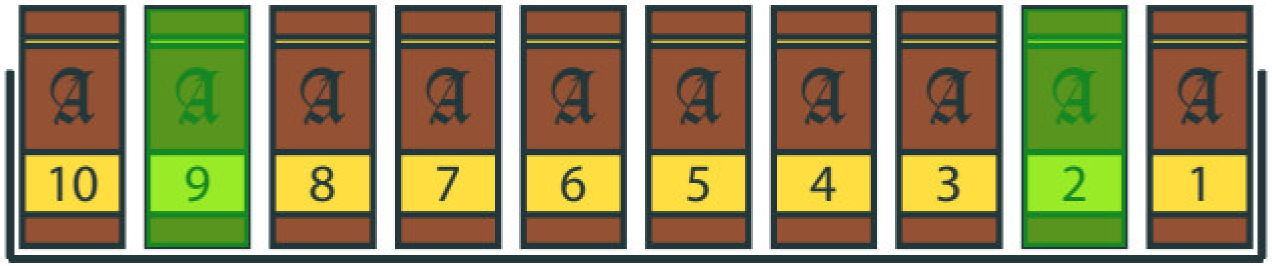

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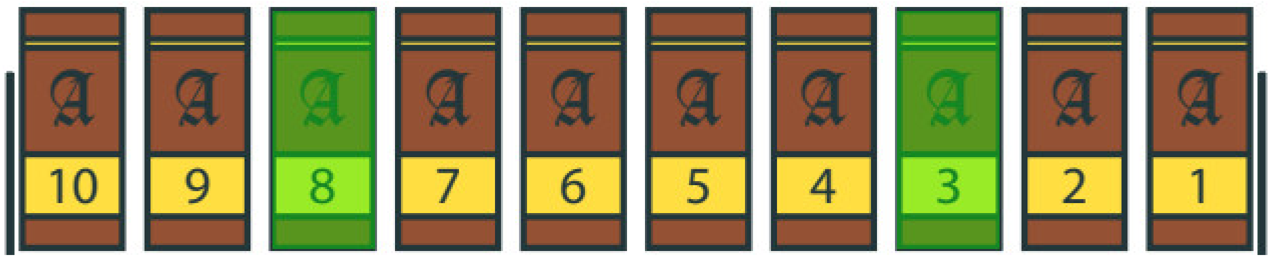

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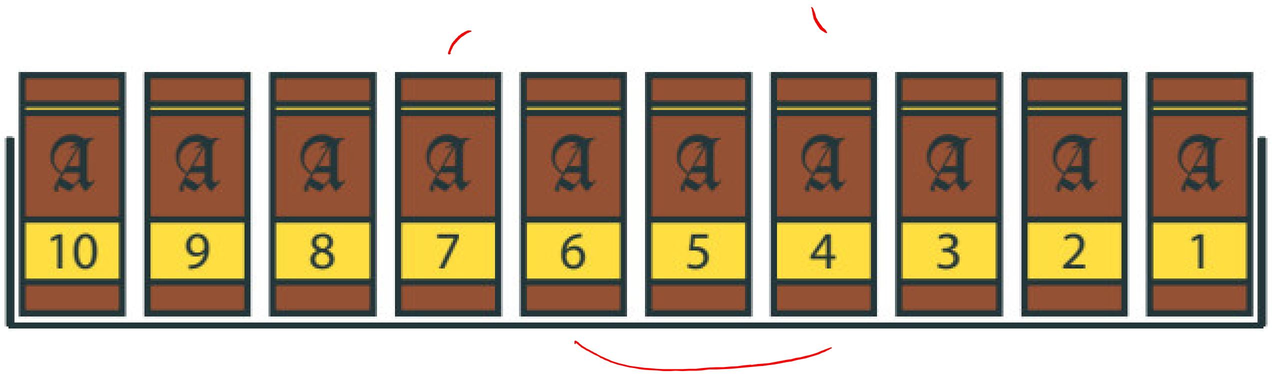

➢ All the books can be switched and sorted in just 5 days!

King Julien’s Books

Let’s check

- So what is the right number of days?

King Julien's Books

Let's check

- So what is the right number of days?
- And how to prove that it is the correct answer?

King Julien's Books

Let's check

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- To answer, we need to find some invariant that:

King Julien's Books

Let's check

- So what is the right number of days?
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- To answer, we need to find some invariant that:
  - Do not change fast

King Julien's Books

Let's check

- So what is the right number of days?
- And how to prove that it is the correct answer?
- To answer, we need to find some invariant that:
  - Do not change fast
  - Should change substantially while ordering the book

King Julien’s Books

Recall this:

Puzzle

- There is a sequence of 10 cells. The leftmost contains “1” while the rightmost has “30”. Is it possible to fill other cells with consecutive numbers in such a way that they differ by 3 at the most?

1									30

King Julien's Books

The Invariant:

King Julien’s Books

The Invariant: The number of books staying to the right of their intended place

King Julien’s Books

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- Small in the end: equals 0

King Julien's Books

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King Julien's Books

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- Large in the beginning? **Yes!**

King Julien's Books

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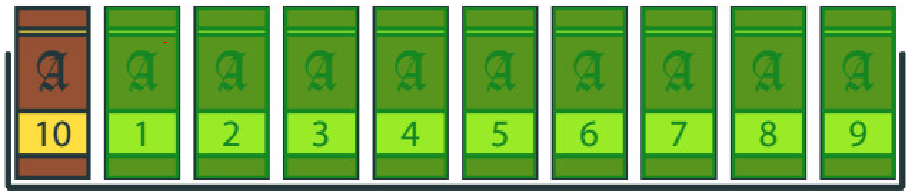

- The invariant is 9 in the beginning, we need at least 9 days.

Reminder

- Everybody, make sure that your name in ZOOM is in the following format:
  - Ex: 202054321 Juan Dela Cruz

Not changing your name to this format

* you **will** be marked Absent * → absent?

Mathematical Thinking – Invariants

EVEN AND ODD NUMBERS

- Even and Odd Numbers
- Piece on a Chessboard
- Summing up Digits
- Switching Signs
- Advanced Signs Switching

Even and Odd Numbers

- Even numbers - integer numbers divisible by 2

Even and Odd Numbers

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- Odd numbers - all others

Even and Odd Numbers

- Even numbers - integer numbers divisible by 2
  - 2, 4, 6, 8, … are even
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  - 1, 3, 5, 7, … are even

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- How about 0?
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…	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	…

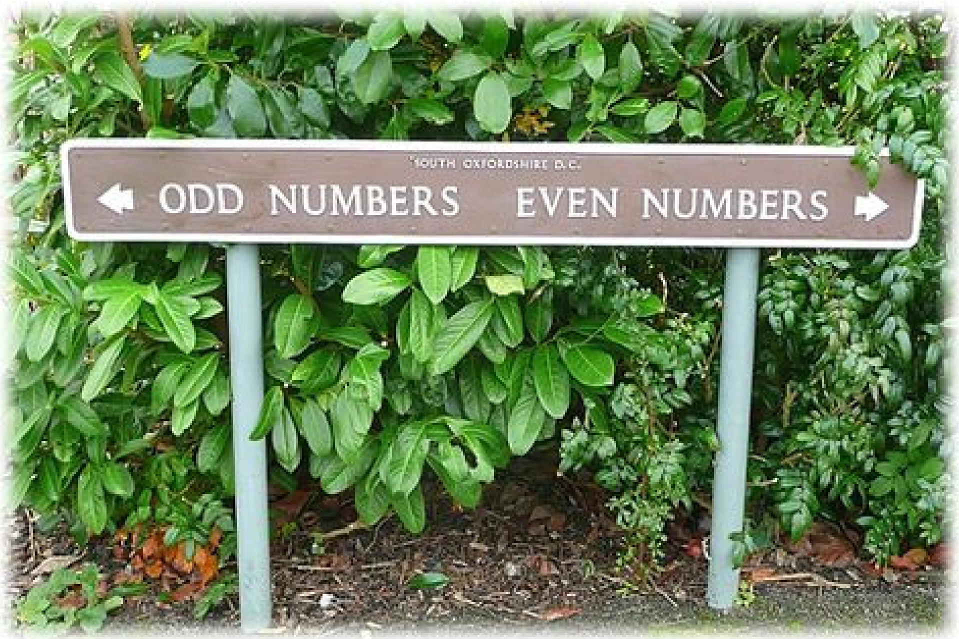
Even and Odd Numbers

- Even numbers - integer numbers divisible by 2
  - 2, 4, 6, 8, … are even
- Odd numbers - all others
  - 1, 3, 5, 7, … are even
- How about 0?
  - Even! Recall section on “Connecting Points”

…	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	…

Even and Odd Numbers

- The property of a number to be either even or odd is an important variant.



https://i.pinimg.com/474x/0e/0f/54/0e0f54737e21ff81d07ce97ab90587ef.jpg

- Even and Odd Numbers
- Piece on a Chessboard
- Summing up Digits
- Switching Signs
- Advanced Signs Switching

Piece on a Chessboard

Puzzle

- A piece on a chessboard can move to any cell adjacent by edge to the current one. Can it return to the original position after 17 moves? What about after 18 moves?

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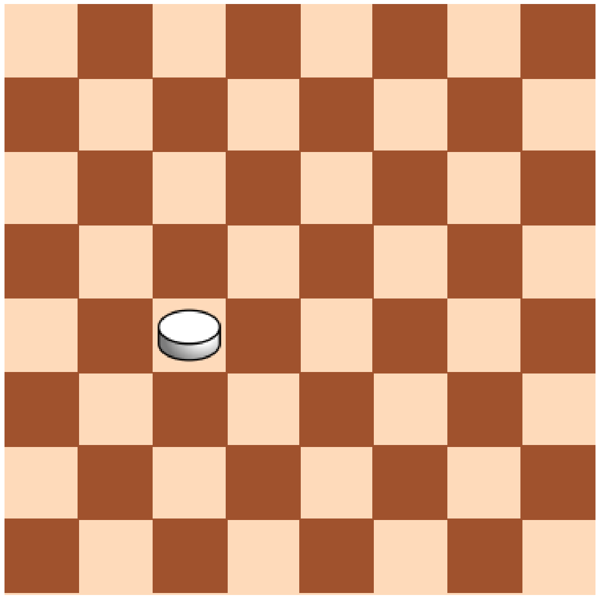

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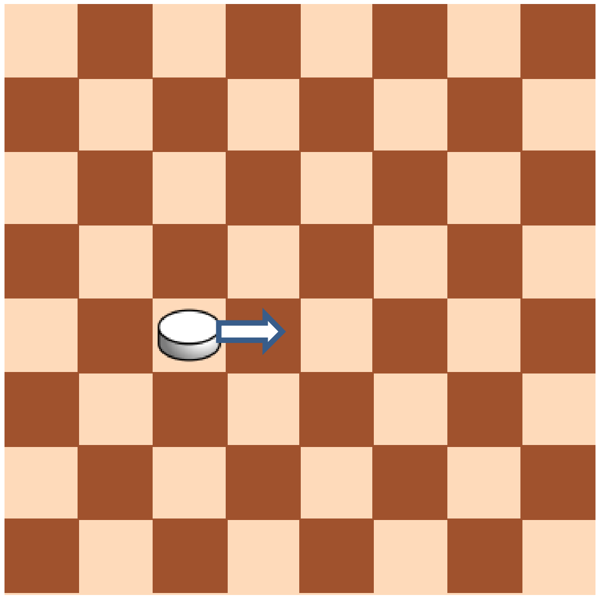

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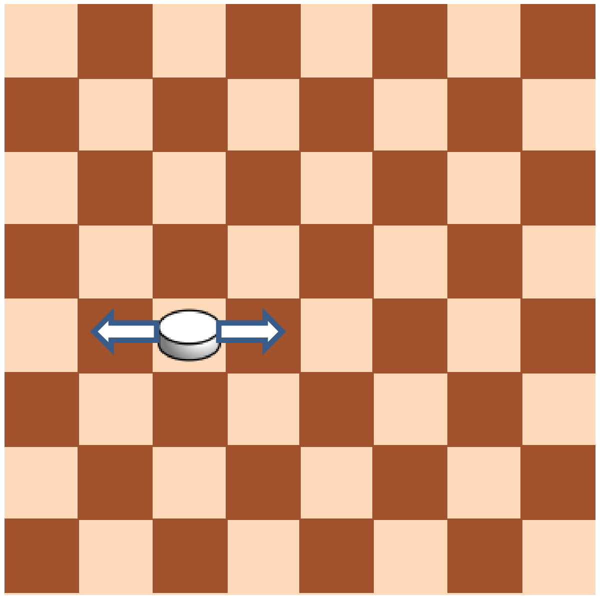

Piece on a Chessboard

Puzzle

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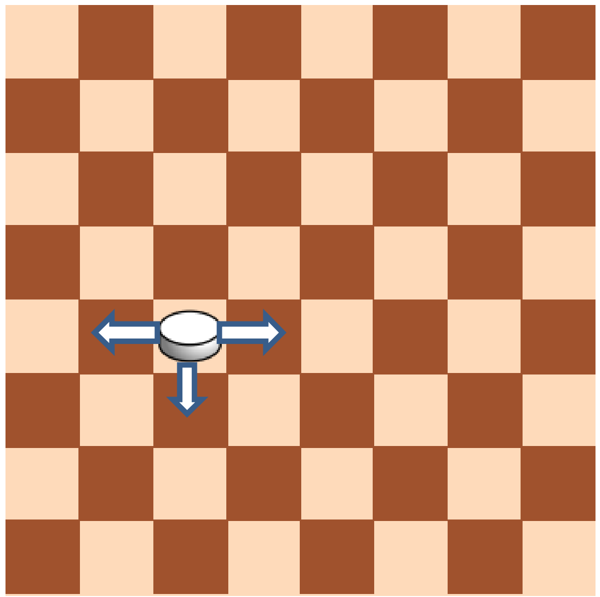
17 moves, possible?



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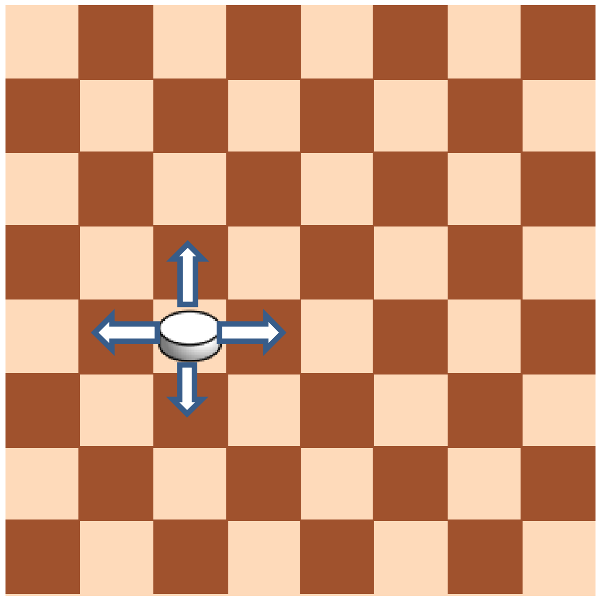
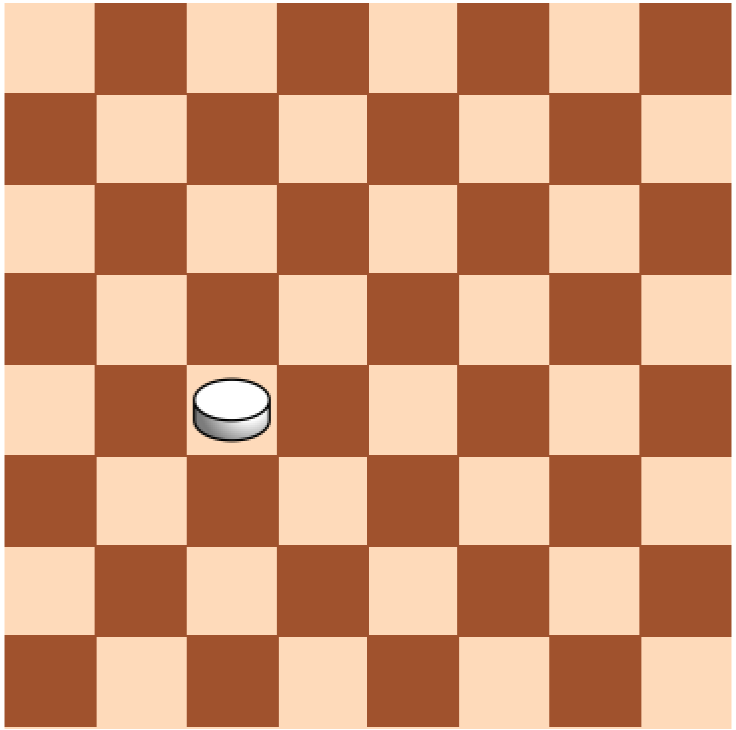


Piece on a Chessboard

Let's check

- Start with the more simpler case of 18 moves



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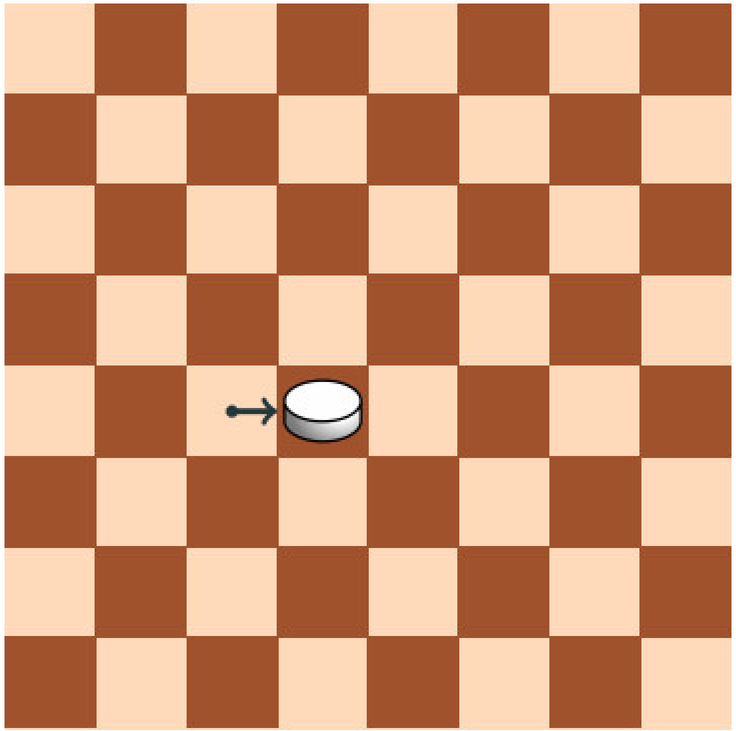

- We can return after 2 steps
  - 1^{st} step

Piece on a Chessboard

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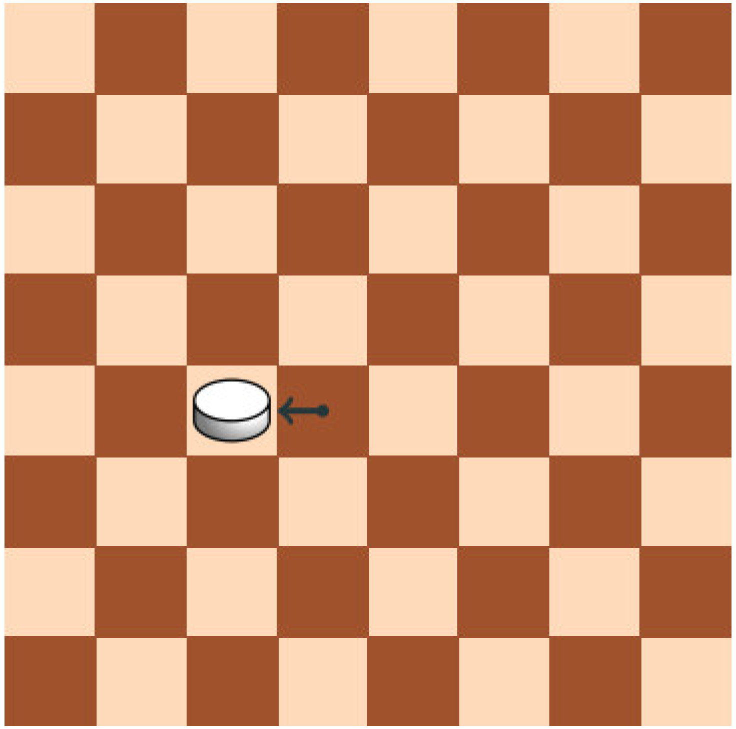

- We can return after 2 steps
  - 2nd step

Piece on a Chessboard

Let's check

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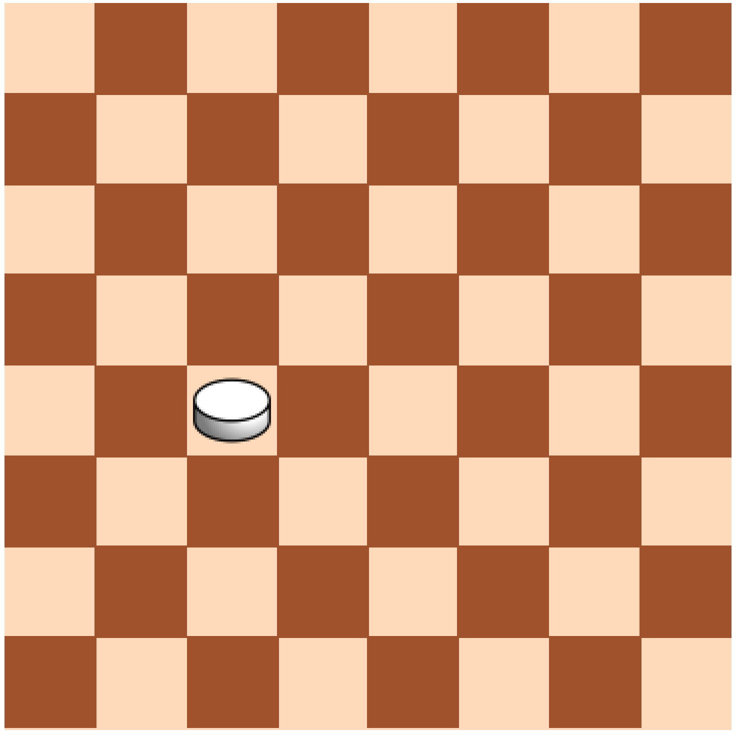

- We can return after 2 steps
  - Repeat 9 times

Piece on a Chessboard

Let's check

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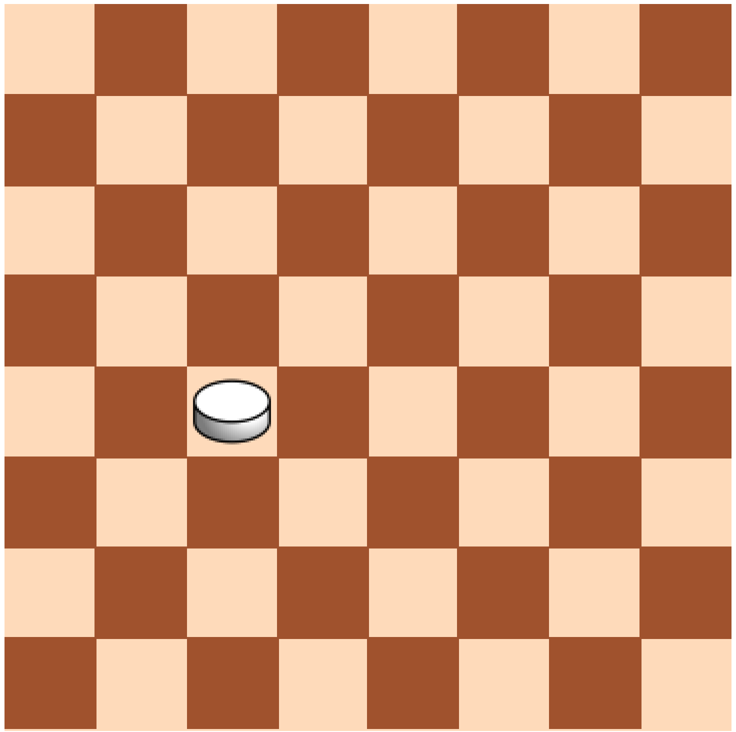
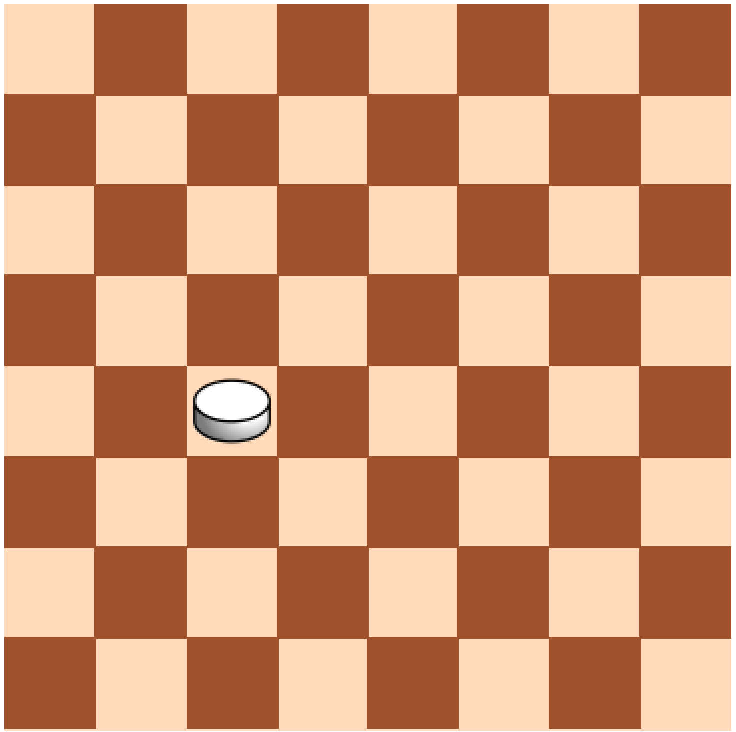


- We can return after 2 steps
  - Repeat 9 times
- The piece would be back to its original place!

Piece on a Chessboard

Let's check

- The same argument does not work for 17 steps



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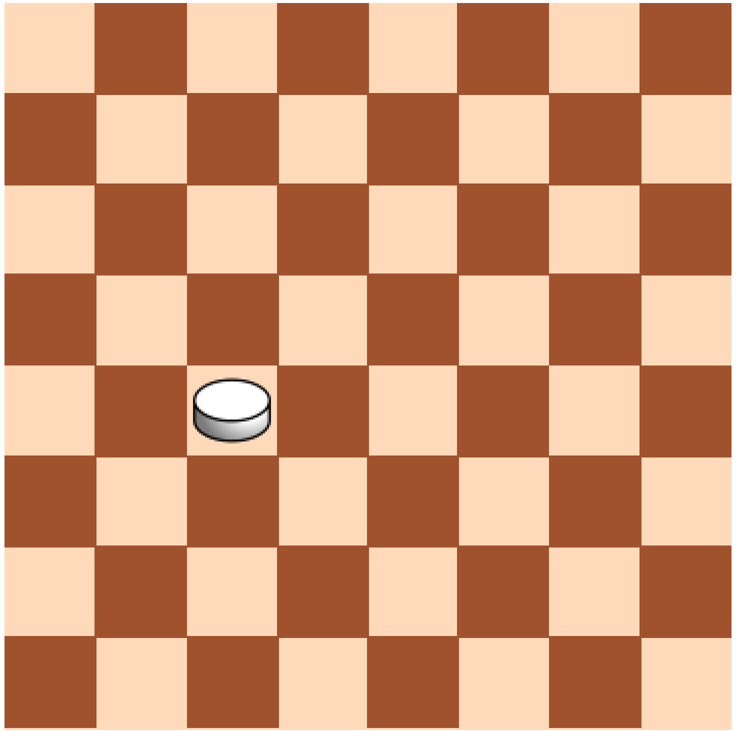

- Indeed, 17 is odd

Piece on a Chessboard

Let’s check

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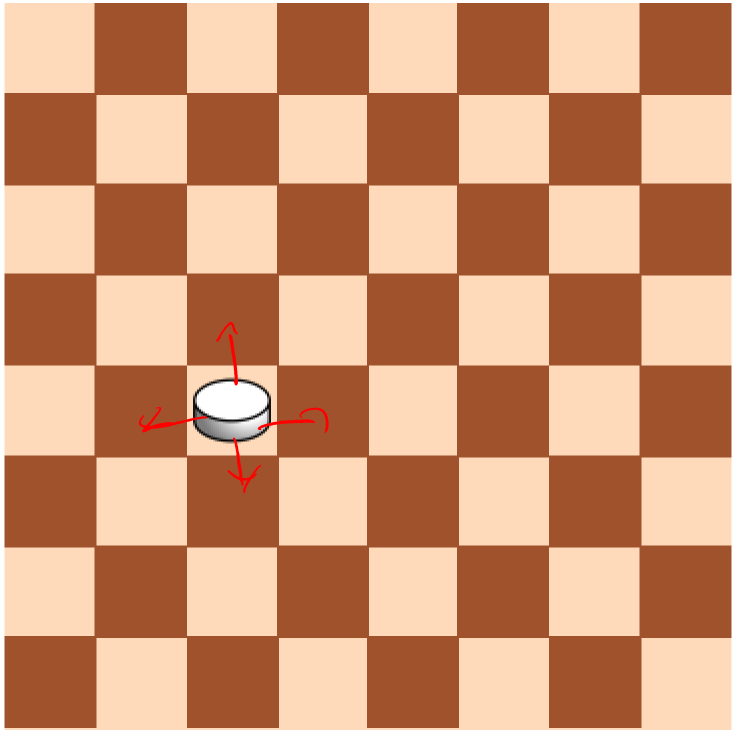

- Indeed, 17 is odd
- Observation: after even number of steps, a piece is on the lighter square

Piece on a Chessboard

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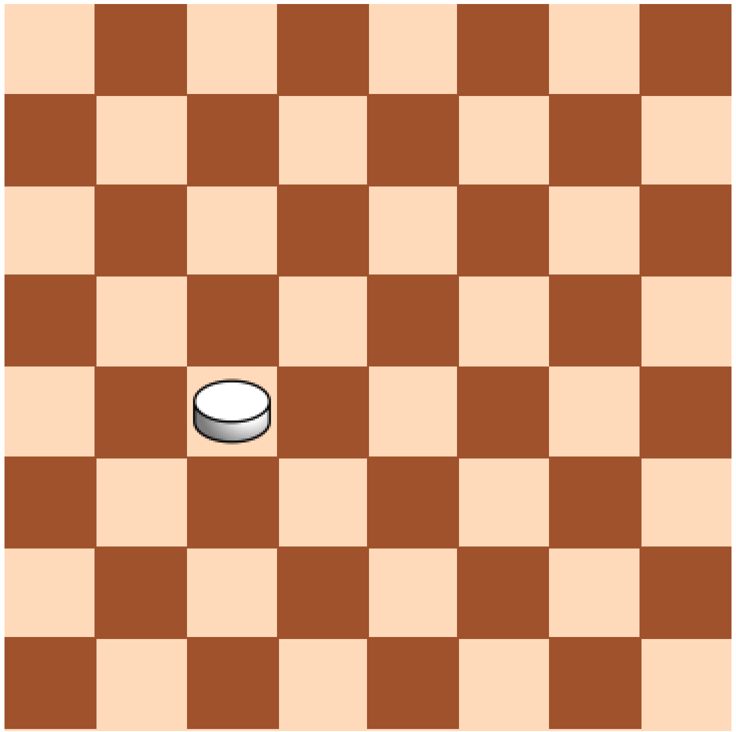

- Indeed, 17 is odd
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- After odd number, it is on a dark square

Piece on a Chessboard

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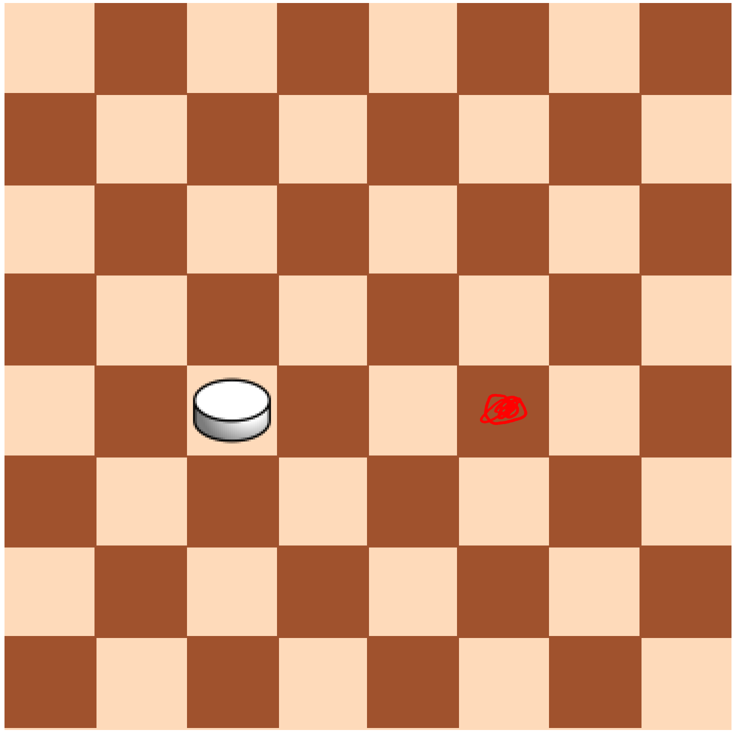

- Indeed, 17 is odd
- Observation: after even number of steps, a piece is on the lighter square
- After odd number, it is on a dark square
- No way to get back after odd number of steps

Mathematical Thinking – Invariants

SUMMING UP DIGITS

- Even and Odd Numbers
- Piece on a Chessboard
- Summing up Digits
- Switching Signs
- Advanced Signs Switching

Summing up Digits

Problem I

- Is it possible to place signs in the form of the expression $\pm 1 \pm 2 \pm 3 \pm \ldots \pm 9$ to get as a result the sum 100? Can we get for 2?

Summing up Digits

Problem I

- Is it possible to place signs in the form of the expression $\pm 1\ \pm 2\ \pm 3\ \pm\ \ldots\ \pm 9$ to get as a result the sum 100? Can we get for 2?

2=?

$\boxed{\pm 1}\quad \boxed{\pm 2}\quad \boxed{\pm 3}\quad \boxed{\pm 4}\quad \boxed{\pm 5}\quad \boxed{\pm 6}\quad \boxed{\pm 7}\quad \boxed{\pm 8}\quad \boxed{\pm 9} = \boxed{???}$

100=?

Summing up Digits

Problem I

- Is it possible to place signs in the form of the expression $\pm 1 \ \pm 2 \ \pm 3 \ \pm \ \ldots \ \pm 9$ to get as a result the sum 100? Can we get for 2?
  - Let's start with 100

0		1		2		3		4		5		6		7		8		9	=	

  - We get largest sum if we place '+' everywhere

Summing up Digits

Problem I

- Is it possible to place signs in the form of the expression $\pm 1\ \pm 2\ \pm 3\ \pm \ldots\ \pm 9$ to get as a result the sum 100? Can we get for 2?
  - Let's start with 100

0	+	1	+	2	+	3	+	4	+	5	+	6	+	7	+	8	+	9	=	45

  - We get largest sum if we place '+' everywhere
  - But we cannot get any value greater than 45, hence it is not possible have an expression that will give a sum of 100

Summing up Digits

Problem I

- Is it possible to place signs in the form of the expression $\pm 1\ \pm 2\ \pm 3\ \pm\ \ldots\ \pm 9$ to get as a result the sum 100? Can we get for 2?
    - Let's check for sum of 2

$$\boxed{\pm 1} \quad \boxed{\pm 2} \quad \boxed{\pm 3} \quad \boxed{\pm 4} \quad \boxed{\pm 5} \quad \boxed{\pm 6} \quad \boxed{\pm 7} \quad \boxed{\pm 8} \quad \boxed{\pm 9} = \boxed{???}$$

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± 1	± 2	± 3	± 4	± 5	± 6	± 7	± 8	± 9	

  - Consider properties of the number sequence
    – Note that there are 5 odd and 4 even numbers

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± 1	± 2	± 3	± 4	± 5	± 6	± 7	± 8	± 9	

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    - Note that there are 5 odd and 4 even numbers
    - O+ O = E, O + E = O, E + E = E

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    - No matter how we put the signs, the sum will always be ODD

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Summing up Digits

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  - Let's check for sum of 2

$\boxed{\pm 1}$ $\boxed{\pm 2}$ $\boxed{\pm 3}$ $\boxed{\pm 4}$ $\boxed{\pm 5}$ $\boxed{\pm 6}$ $\boxed{\pm 7}$ $\boxed{\pm 8}$ $\boxed{\pm 9}$ $\boxed{}$

  - Consider properties of the number
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Problem II

- Which sums is it possible to get by varying the signs in the expression $\pm 1 \pm 2 \pm 3 \pm \ldots \pm 9$?

Summing up Digits

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- Which sums is it possible to get by varying the signs in the expression $\pm 1\ \pm 2\ \pm 3\ \pm \ldots\ \pm 9$?
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Summing up Digits

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- Which sums is it possible to get by varying the signs in the expression $\pm 1\ \pm 2\ \pm 3\ \pm \ldots\ \pm 9$?
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    - Sum is ODD and sum is at most 45 (symmetrically, at least -45)

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  - How about the other sums?

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    - Turns out, all sums satisfying the found restrictions are possible!

Summing up Digits

Problem II

- Which sums is it possible to get by varying the signs in the expression $\pm 1\ \pm 2\ \pm 3\ \pm \ldots\ \pm 9$?   4E + 50 = 0
  - We have seen two obstacles   all (+)   all (-)s
    - Sum is ODD and sum is at most 45 (symmetrically, at least -45)
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  - You can check this out yourself.

Summing up Digits

Problem II

- Which sums is it possible to get by varying the signs in the expression $\pm 1 \pm 2 \pm 3 \pm \ldots \pm 9$?

$[-45, -43, \ldots, 43, 45]$

  - We have seen two obstacles
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    - Seems like this are the only restrictions
  - How about the other sums?
    - Turns out, all sums satisfying the found restrictions are possible!
  - You can check this out yourself.
  - Try this "greedy algorithm", start from left to right and place the signs greedily through:

Summing up Digits

Problem II

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Summing up Digits

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  - How about the other sums?
    - Turns out, all sums satisfying the found restrictions are possible!
  - You can check this out yourself.
  - Try this "greedy algorithm", start from left to right and place the signs greedily through:
    - If current sum is less than goal, increase the sum
    - Else decrease it.

Mathematical Thinking – Invariants

SWITCHING SIGNS

- Even and Odd Numbers
- Piece on a Chessboard
- Summing up Digits
- **Switching Signs**
- Advanced Signs Switching

Switching Signs

Puzzle

- Given a 4 x 4 table. The top-left corner has the '-' sign and all other cells has "+" sign. For each step, we are allowed to switch all the signs in a column or row. Is it possible to switch all the signs to "+"?

Switching Signs

Puzzle

- Given a 4 x 4 table. The top-left corner has the '-' sign and all other cells has "+" sign. For each step, we are allowed to switch all the signs in a column or row. Is it possible to switch all the signs to "+"?

-	+	+	+
+	+	+	+
+	+	+	+
+	+	+	+

Switching Signs

Let’s try (Case I)

+	+	+	+

- Let’s take the case of switching a row

Switching Signs

Let’s try (Case I)

(-)			
-	-	-	-

- Let’s take the case of switching a row
- If we try to do it, we will notice that the number of minus signs is always ODD
  - Take note that we started with a “-” sign in the upper-left cell

Switching Signs

Let's try (Case II)

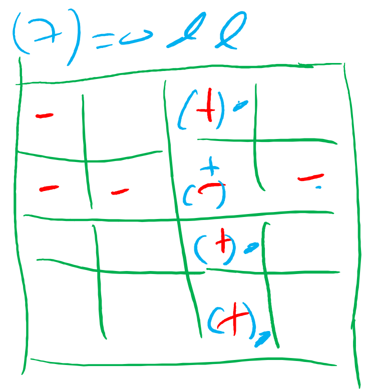
+	+	-	+

- Let's take another case

Switching Signs

Let's try (Case II)

-	-	+	-



- Let's take another case
- If we try to do it, we will notice that the number of minus signs is still ODD

Switching Signs

Let’s try (Case III)

-	+	-	+

- Let’s try another case

Switching Signs

Let’s try (Case III)

+	-	+	-

- Let’s take another case
- If we try to do it, we will notice that the number of minus signs is still ODD
  - recall that we started with one “-” sign

Switching Signs

Let’s try

-	+	+	+
+	+	+	+
+	+	+	+
+	+	+	+

- After trying it, we will notice that the number of minus signs is always ODD

Switching Signs

Let’s try

-	+	+	+
+	+	+	+
+	+	+	+
+	+	+	+

= 16 = even

#(-) = odd

- After trying it, we will notice that the number of minus signs is always ODD
- This is an invariant!

Switching Signs

Let's try

-	+	+	+
+	+	+	+
+	+	+	+
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- After trying it, we will notice that the number of minus signs is always ODD
- This is an invariant!
- Hence, we cannot switch all to "+", since the number of "-" signs would then be zero, which is even and violates this invariant

Mathematical Thinking – Invariants

ADVANCED SIGNS SWITCHING

- Even and Odd Numbers
- Piece on a Chessboard
- Summing up Digits
- Switching Signs
- Advanced Signs Switching

Advanced Signs Switching

Puzzle

- Given a 4 x 4 table. All corner cells contain the '-' sign and all other cells has "+" sign. For each step, we are allowed to switch all the signs in a column or row. Is it possible to switch all the signs to "+"?

-	+	+	-
+	+	+	+
+	+	+	+
-	+	+	-

Advanced Signs Switching

Take note

-	+	+	-
+	+	+	+
+	+	+	+
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- This looks more tricky. The previous solution won't work.

Advanced Signs Switching

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-	+	+	-
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- This looks more tricky. The previous solution won't work.
- If you know about residues modulo 4 (we will learn later), you can use them but it also does not help.

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- This looks more tricky. The previous solution won't work.
- If you know about residues modulo 4 (we will learn later), you can use them but it also does not help.
- There is a very simple solution
  - Before proceeding, try to analyze more and you might find about it ☺

Advanced Signs Switching

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Advanced Signs Switching

Let’s try

-	+	+	-
+	+	+	+
+	+	+	+
-	+	+	-

- The idea is simple. Let’s look at the small part of the problem

Advanced Signs Switching

Let's try

-	+	+	-
+	+	+	+
+	+	+	+
-	+	+	-

- The idea is simple. Let's look at the small part of the problem
- If we switch a row or column in the large table, we switch a row or column in the smaller one (or do nothing)

Advanced Signs Switching

Let’s try

-	+	+	-
+	+	+	+
+	+	+	+
-	+	+	-

- To solve the big problem, we need to solve this small 2 x 2 problem

Advanced Signs Switching

Let’s try

-	+	+	-
+	+	+	+
+	+	+	+
-	+	+	-

- To solve the big problem, we need to solve this small 2 x 2 problem
- But this 2 x 2 problem cannot be solved using the previous argument!

Advanced Signs Switching

Let's try

-	+	+	-
+	+	+	+
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- We have a very strong obstacle – each 2 x 2 square should have even number of '-' signs for the puzzle to be solved

Advanced Signs Switching

Let's try

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+	+	+	+
+	+	+	+
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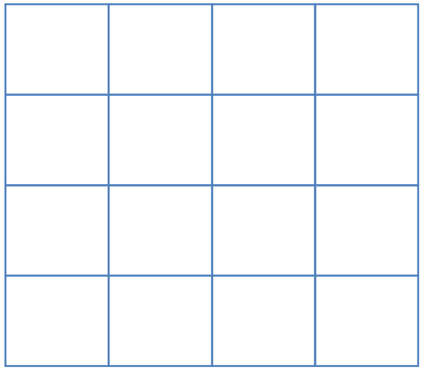
- We have a very strong obstacle – each 2 x 2 square should have even number of '-' signs for the puzzle to be solved
- In fact, these are the only obstacles. That is, if all 2 x 2 squares has even number of '-' signs, then it is possible to switch all the signs into "+"

Advanced Signs Switching

Hint



- Assume that our invariant holds: each 2 x 2 square has even number of "-" signs

Advanced Signs Switching

Hint

- Assume that our invariant holds: each 2 x 2 square has even number of “-” signs
- Switch the 1^{st} row and 1^{st} column to all “+”

Advanced Signs Switching

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+			
+			
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Advanced Signs Switching

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+			
+			
+			

- Assume that our invariant holds: each 2 x 2 square has even number of "-" signs
- Switch the 1st row and 1st column to all "+"
- Use invariant to show that the rest of the table is filled with "+"

Advanced Signs Switching

Hint

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+			
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- Assume that our invariant holds: each 2 x 2 square has even number of "-" signs
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  - Zero is even

Advanced Signs Switching

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- Also to establish termination and running time bounds (very important)
- Can take many forms
  - numbers, “even or odd”, equations, inequalities, etc…
- Double counting is a special case
- Yet, this is not all about invariants.
  - Still other forms (ex: residues); we will see later on

Thank you.