11.2 Arbitrary Period. Even and Odd Functions. Half-Range Expansions

This section covers three topics:

1. Transition from period 2π to any period $2L$
2. Simplifications using odd/even property of a function
3. Half-range Expansion:

   Expansion of f for $0 \le x \le L$ in two Fourier series, one having only cosine terms and the other only sine terms.

1. Transition from period 2π to any period p=2L

Let f(x) have period p=2L.

Change of scale: $(1)\quad (a)\ x=\frac{p}{2\pi}v \qquad (b)\ v=\frac{2\pi}{p}x$

f(x) has period of 2π with respect to the variable v.

$$(2)\quad f(x)=f\left(\frac{L}{\pi}v\right)=a_0+\sum_{n=1}^{\infty}(a_n\cos nv+b_n\sin nv)$$

$$(3)\quad \begin{cases} a_0=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)dv, \\ a_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\cos nv\,dv \\ b_n=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\sin nv\,dv \end{cases}$$

(2) $$f(x) = f\left(\frac{L}{\pi}v\right) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv)$$

(3) $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right) dv$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\cos nv\, dv, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\sin nv\, dv$$

(4) $$\frac{L}{\pi}v = x, \quad v = \frac{\pi}{L}x, \quad dv = \frac{\pi}{L}dx$$

(5) $$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

(6)

(0) $$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$$

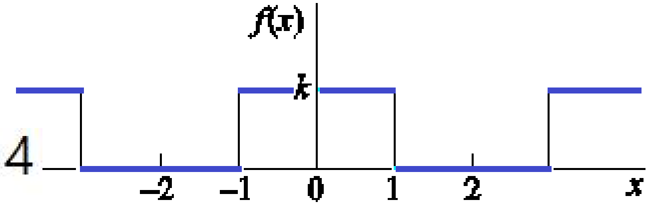
(a) $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L} dx$$

(b) $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L} dx$$

EX 1 Periodic Rectangular Wave

Find the Fourier series of the function with period of 4. (Fig. 263)

$$f(x)=\begin{cases}0, & if \quad -2<x<-1\\ k, & if \quad -1<x<1\\ 0, & if \quad 1<x<2\end{cases} \qquad p=2L=4$$



Sol.

(5) $$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$$

(0) $$a_0=\frac{1}{2L}\int_{-L}^{L}f(x)\,dx$$

(6) (a) $$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}\,dx$$

(b) $$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}\,dx$$

$$(6-0)\quad a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{4}\int_{-1}^{1} k\,dx = \frac{k}{2}$$

$$(6-a)\quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\cdot 2k\int_{0}^{1}\cos\frac{n\pi x}{2}\,dx = k\frac{\sin(n\pi x/2)}{n\pi/2}\Bigg|_0^1 = \frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

$$= \begin{cases} \dfrac{2k}{n\pi} & for\ \ n = 4k+1, \\ -\dfrac{2k}{n\pi} & for\ \ n = 4k+3, \\ 0 & for\ \ n = 4k,\ 4k+2 \end{cases}$$

$$(6-b)\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 0\,(\because odd\ \ function)$$

$$(5) \qquad f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$(6-0) \quad a_0 = \frac{k}{2}$$

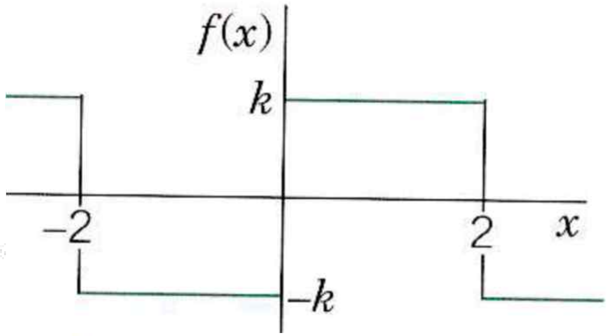
$$(6-a) \quad a_n = \begin{cases} \dfrac{2k}{n\pi} & for \;\; n = 4k+1, \\ -\dfrac{2k}{n\pi} & for \;\; n = 4k+3, \\ 0 & for \;\; n = 4k,\ 4k+2 \end{cases}$$

$$(6-b) \quad b_n = 0$$

$$= \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - + \cdots\right)$$

EX 2 **Periodic Rectangular Wave.** Change of Scale

Find the Fourier series of the function (Fig. 264)



$$f(x)=\begin{cases}-k, & if\ \ -2<x<0\\ k, & if\ \ 0<x<2\end{cases} \qquad p=2L=4$$

Sol.

$$(6-0)\ \ a_0=\frac{1}{2L}\int_{-L}^{L}f(x)\,dx=\frac{1}{4}\left[\int_{-2}^{0}-k\,dx+\int_{0}^{2}k\,dx\right]=0$$

$$(6-a)\ \ a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}\,dx=\frac{1}{2}\int_{-2}^{2}f(x)\cos\frac{n\pi x}{2}\,dx=0$$

$$(6-b)\ \ b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}\,dx=\frac{1}{2}\int_{-2}^{2}f(x)\sin\frac{n\pi x}{2}\,dx$$

$$=\frac{1}{2}\cdot 2\int_{0}^{2}k\sin\frac{n\pi x}{2}\,dx=k\left[\frac{-\cos(n\pi x/2)}{n\pi/2}\right]_{0}^{2}$$

$$=\frac{2k}{n\pi}(1-\cos n\pi)=\begin{cases}\dfrac{4k}{n\pi} & odd\ \ n\\ 0, & even\ \ n\end{cases}$$

$$(5) \qquad f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$(6-0)\ a_0 = 0$$

$$(6-a)\ a_n = 0$$

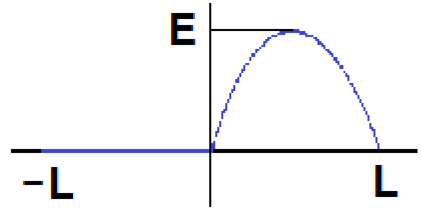
$$(6-b)\ b_n = \begin{cases} \dfrac{4k}{n\pi} & odd\ n \\ 0, & even\ n \end{cases}$$

$$= \frac{4k}{\pi}\left(\sin\frac{\pi}{2}x + \frac{1}{3}\sin\frac{3\pi}{2}x + \frac{1}{5}\sin\frac{5\pi}{2}x + \cdots\right)$$

EX 3 Half-Wave Rectifier

Find the Fourier series of the output of a half-wave rectifier.

$$f(x) = \begin{cases} 0, & if \ -L < x < 0 \\ E\sin\omega t, & if \ 0 < x < L \end{cases} \qquad p = 2L = \frac{2\pi}{\omega}, \ L = \frac{\pi}{\omega}$$



Sol.

$$(6-0)\ a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2\pi/\omega}\int_0^{\pi/\omega} E\sin\omega t\,dt$$

$$= \frac{E\omega}{2\pi}\left[-\cos\omega t/\omega\right]_0^{\pi/\omega} = \frac{E}{\pi}$$

$$(6\text{-}a)\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{\pi/\omega}\int_0^{\pi/\omega} E\sin\omega t\cos\frac{n\pi t}{\pi/\omega}\,dt$$

$$= \frac{\omega E}{\pi}\int_0^{\pi/\omega} \sin\omega t\cos n\omega t\,dt$$

$$= \frac{\omega E}{2\pi}\int_0^{\pi/\omega} [\sin(1+n)\omega t + \sin(1-n)\omega t]\,dt$$

$$a_n = \frac{\omega E}{2\pi}\int_0^{\pi/\omega}[\sin(1+n)\omega t + \sin(1-n)\omega t]\,dt$$

$$\text{If } n=1,\; a_1 = \frac{\omega E}{2\pi}\int_0^{\pi/\omega}[\sin 2\omega t + 0]\,dt = 0$$

$$\begin{aligned}\text{If } n\neq 1,\; a_n &= \frac{\omega E}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega} \\ &= \frac{E}{2\pi}\left[\frac{-\cos(1+n)\pi + 1}{1+n} + \frac{-\cos(1-n)\pi + 1}{1-n}\right] \\ &= \begin{cases} 0, & odd\ n \\ \dfrac{E}{2\pi}\left[\dfrac{2}{1+n} + \dfrac{2}{1-n}\right] = -\dfrac{2E}{(n-1)(n+1)\pi}, & even\ n \end{cases}\end{aligned}$$

$$(6-b)\ \ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{\pi/\omega}\int_0^{\pi/\omega} E\sin\omega t\sin\frac{n\pi t}{\pi/\omega}\,dt$$

$$= \frac{\omega E}{\pi}\int_0^{\pi/\omega} \sin\omega t\sin n\omega t\,dt$$

$$= -\frac{\omega E}{2\pi}\int_0^{\pi/\omega} [\cos(1+n)\omega t - \cos(1-n)\omega t]\,dt$$

$$If\ \ n=1,\ \ b_1 = \frac{\omega E}{2\pi}\int_0^{\pi/\omega} [\cos 2\omega t + 1]\,dt = \frac{E}{2}$$

$$If\ \ n\neq 1,\ \ b_n = \frac{\omega E}{2\pi}\left[\frac{\sin(1+n)\omega t}{(1+n)\omega} - \frac{\sin(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega} = 0$$

$$(5) \qquad f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{E}{\pi}$$

$$\text{If } n = 1,\ a_n = 0$$

$$\text{If } n \neq 1,\ a_n = \begin{cases} 0, & \text{odd } n \\ -\dfrac{2E}{(n-1)(n+1)\pi}, & \text{even } n \end{cases}$$

$$\text{If } n = 1,\ b_n = \frac{E}{2} \qquad \text{If } n \neq 1,\ b_n = 0$$

$$= \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos 4\omega t + \cdots\right)$$

2. Simplifications: Even and Odd Functions

If $f(x)$ is an even function, $f(-x)=f(x)$.

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_{0}^{L} f(x)dx$$

$$\int_{-L}^{0} f(x)dx = \int_{L}^{0} f(-t)(-dt)[\because x=-t,\ dx=-dt]$$

$$=-\int_{L}^{0} f(t)dt = \int_{0}^{L} f(x)dx$$

$$\therefore \int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_{0}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$$

If $f(x)$ is an odd function, $f(-x) = -f(x)$.

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_{0}^{L} f(x)dx$$

$$\int_{-L}^{0} f(x)dx = \int_{L}^{0} f(-t)(-dt)[\because x = -t,\ dx = -dt]$$

$$= -\int_{L}^{0}[-f(t)]dt = -\int_{0}^{L} f(x)dx$$

$$\therefore \int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_{0}^{L} f(x)dx = 0$$

Case 1: $f(x)$ is an even function.

$$(6\text{-}0)\quad a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{2L}\cdot 2\int_0^L f(x)dx = \frac{1}{L}\int_0^L f(x)dx \qquad (5^*)$$

$$(6\text{-}a)\quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad (5^*)$$

$$(6\text{-}b)\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 0(\because odd\ \ function)$$

$$(5) \qquad f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

$$(6\text{-}0) \quad a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{2L}\cdot 2\int_0^L f(x)dx = \frac{1}{L}\int_0^L f(x)dx \qquad (5^*)$$

$$(6\text{-}a) \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad (5^*)$$

$$(6\text{-}b) \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 0\,(\because odd\ \ function)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

Case 2: $f(x)$ is an odd function.

$$(6-0)\quad a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = 0$$

$$(6-a)\quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = 0(\because odd\ \ function)$$

$$(6-b)\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

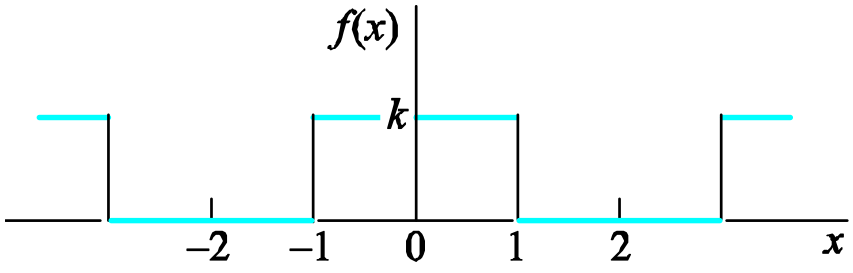
$$= \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad (6^{**})$$

$$(5) \qquad f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$a_0 = 0$$
$$a_n = 0$$
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

EX 4 Fourier Cosine and Sine Series



The rectangular wave is even.

Hence it follows without calculation that $b_n=0$.

THEOREM 1 Sum and Scalar Multiple

The Fourier coefficients of a sum f_1+f_2 are the sum of the corresponding Fourier coefficients of f_1 and f_2.

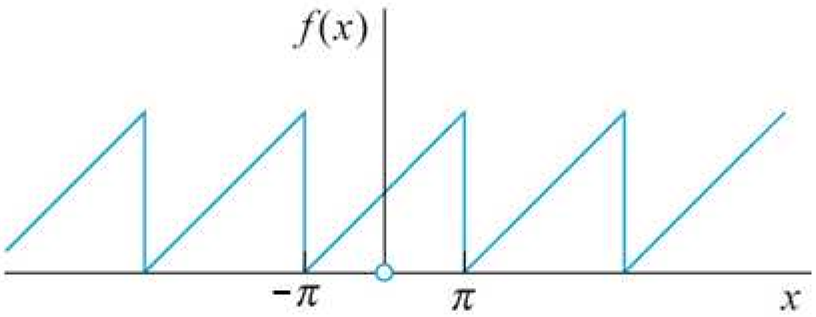
The Fourier coefficients of cf are c times the corresponding Fourier coefficients of f.

EX 5 Sawtooth Wave

Find the Fourier series of the function(Fig. 268).

$$f(x) = x + \pi \quad if \quad -\pi < x < \pi$$

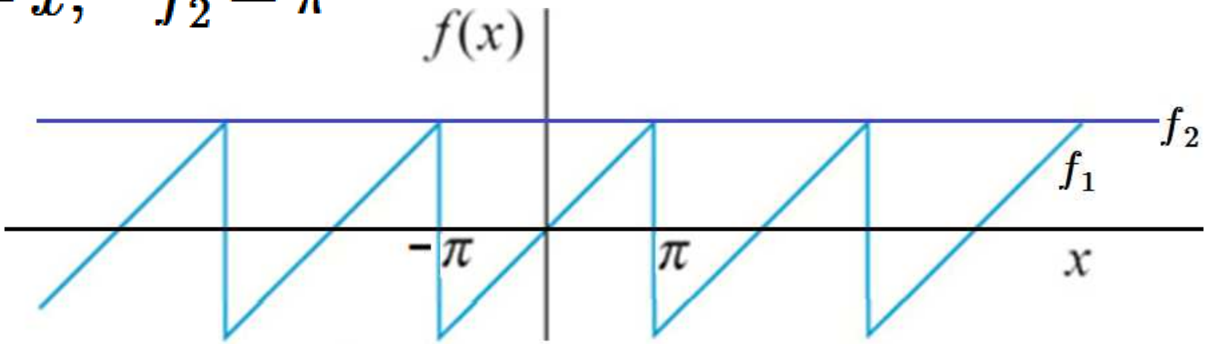
$$and \quad f(x + 2\pi) = f(x)$$



Sol.

$$f(x) = f_1 + f_2$$

$$where \quad f_1 = x, \quad f_2 = \pi$$



$$f_1(x) = x = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(x)\cos nx\, dx = 0(\because odd\ \ function)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)\Bigg|_0^{\pi} - \int_0^{\pi} 1 \cdot \frac{-\cos nx}{n} dx\right]$$

$$= -\frac{2}{n}\cos n\pi$$

$$f_1(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

$$a_0 = a_n = 0$$

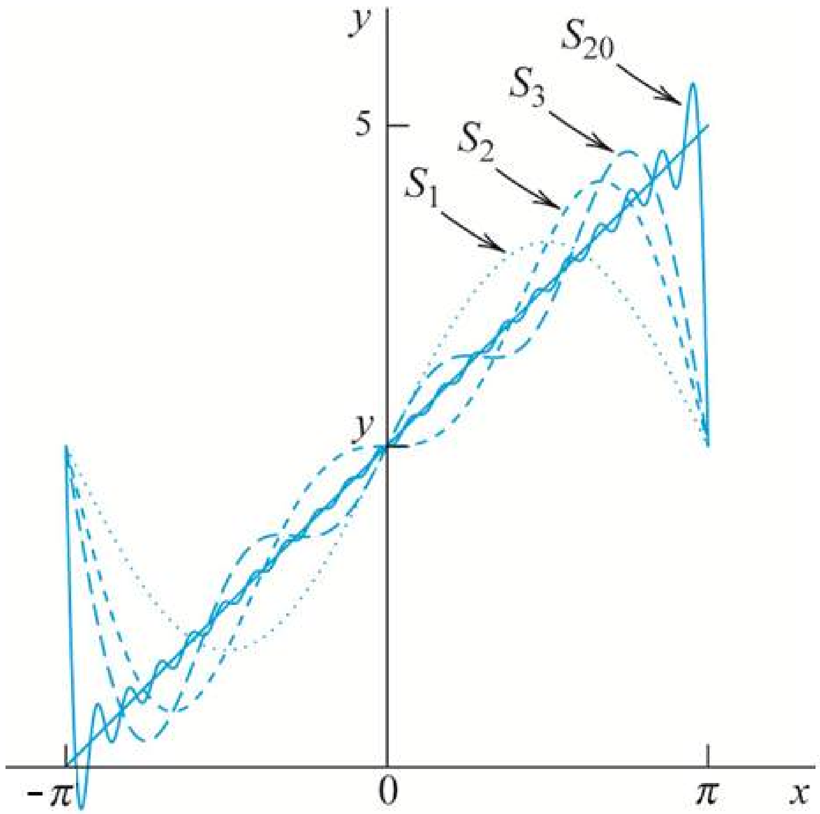
$$b_n = -\frac{2}{n}\cos n\pi$$

$$= 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \quad + \cdots\right)$$

$$f(x) = f_1(x) + f_2(x)$$

$$= \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \quad + \cdots\right)$$

$$f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \quad + \cdots\right)$$



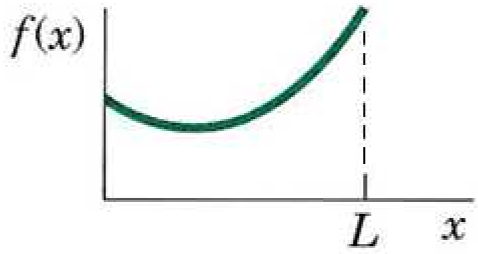
Partial Sum

3. Half-Range Expansions

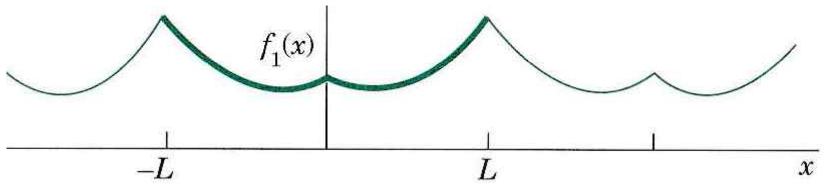
Consider a function $f(x)$ given 0<x<L.

For example,

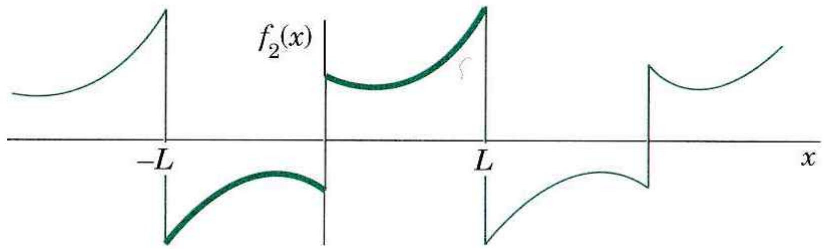
- shape of a distorted violin string
- the temperature in a metal bar of length L



given function $f(x)$



f(x) continued as an **even** periodic function

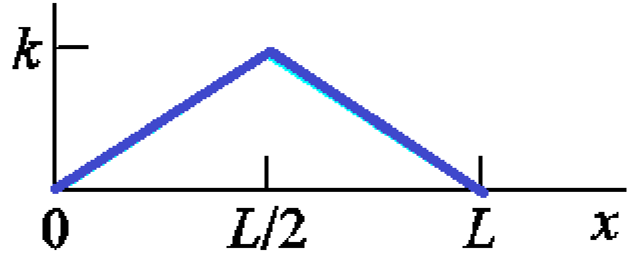


f(x) continued as an **odd** periodic function

EX 6 "Triangle" and Its Half-Range Expansions

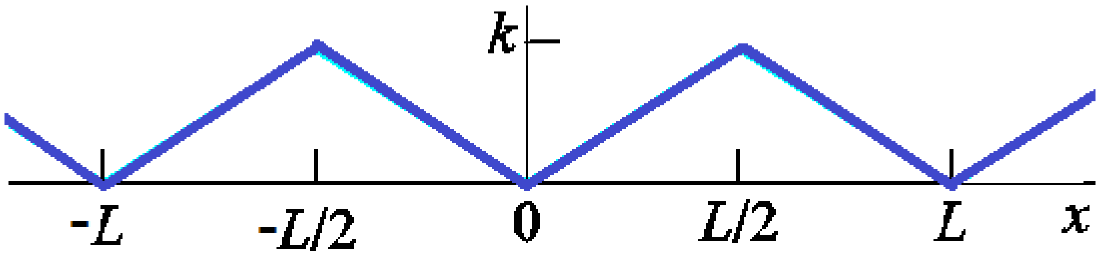
Find the two half-range expansions of the following function.

$$f(x) = \begin{cases} (2k/L)x & \text{if } 0 < x < L/2 \\ (2k/L)(L-x) & \text{if } L/2 < x < L \end{cases}$$



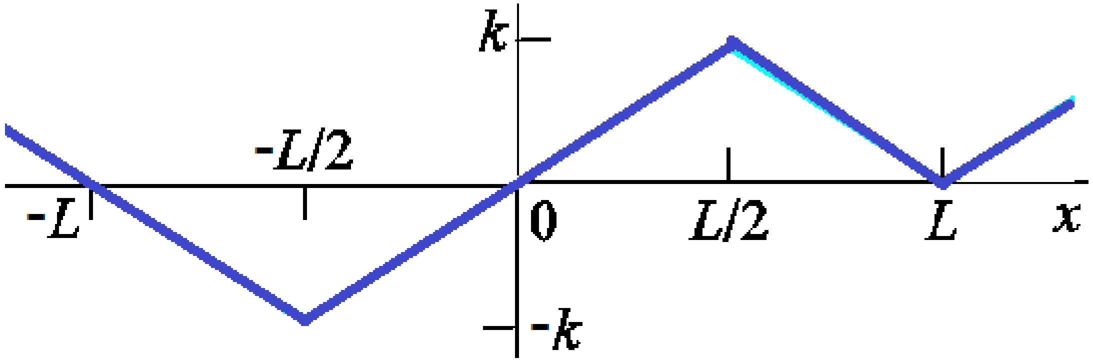
Sol.

(a) Even periodic expansion



$$(6^*)\begin{cases} a_0 = \dfrac{1}{L}\displaystyle\int_0^L f(x)\,dx \\ a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\dfrac{n\pi}{L}x\,dx \end{cases}$$

(b) Odd periodic expansion



$$(6^{**}) \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

Problems at the End of Sec 11.2

1-7. Are the following functions even or odd or neither even nor odd?

1. e^x, $e^{-|x|}$, $x^3 \cos nx$, $x^2 \tan \pi x$, $\sinh x - \cosh x$

2. $\sin^2 x$, $\sin (x^2)$, $\ln x$, $x/(x^2 + 1)$, $x \cot x$

3. Sums and products of even functions

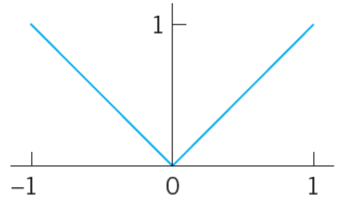
4. Sums and products of odd functions

5. Absolute values of odd functions

6. Product of an odd times an even function

7. Find all functions that are both even and odd.

8. Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.



$$f(x) = |x| \quad (-1 < x < 1)\,, \quad L = 1$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} |x|\,dx = \int_{0}^{1} x\,dx = \frac{1}{2}$$

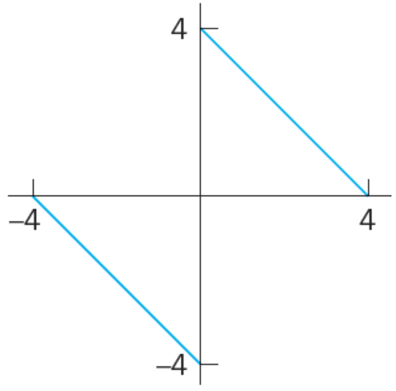
$$a_n = \int_{-1}^{1} |x| \cos n\pi x\,dx = 2\int_{0}^{1} x \cos n\pi x\,dx$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} 0 & (n:\text{ even}) \\ \dfrac{-4}{n^2\pi^2} & (n:\text{ odd}) \end{cases}$$

$$b_n = \int_{-1}^{1} |x| \sin(n\pi x)\,dx = 0$$

$$\therefore\ f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left[\cos\pi x + \frac{1}{9}\cos 3\pi x + \frac{1}{25}\cos 5\pi x + \cdots\right]$$

10. Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.



$$f(x)=\begin{cases}-x-4 & (-4<x<0)\\ -x+4 & (0<x<4)\end{cases},\quad L=4$$

$$a_0=\frac{1}{8}\int_{-4}^{0}(-x-4)\,dx+\frac{1}{8}\int_{0}^{4}(-x+4)\,dx=0$$

$$a_n=\frac{1}{4}\int_{-4}^{0}(-x-4)\cos\frac{n\pi x}{4}dx+\frac{1}{4}\int_{0}^{4}(-x+4)\cos\frac{n\pi x}{4}dx$$

$$=\frac{1}{4}\int_{-4}^{4}-x\cos\frac{n\pi x}{4}dx-\int_{-4}^{0}\cos\frac{n\pi x}{4}dx+\int_{0}^{4}\cos\frac{n\pi x}{4}dx=0$$

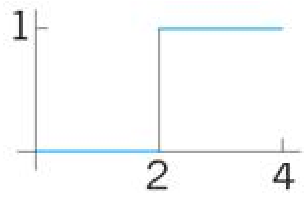
$$b_n=\frac{1}{4}\int_{-4}^{0}(-x-4)\sin\frac{n\pi x}{4}dx+\frac{1}{4}\int_{0}^{4}(-x+4)\sin\frac{n\pi x}{4}dx$$

$$=\frac{1}{4}\left[\frac{4(x+4)}{n\pi}\cos\frac{n\pi x}{4}-\frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}\right]_{-4}^{0}+\frac{1}{4}\left[\frac{4(x-4)}{n\pi}\cos\frac{n\pi x}{4}-\frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}\right]_{0}^{4}$$

$$=\frac{8}{n\pi}$$

$$\therefore\ f(x)=\frac{8}{\pi}\left(\sin\frac{\pi x}{4}+\frac{1}{2}\sin\frac{\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{4}+\cdots\right)$$

24. Find **(a)** the Fourier cosine series, **(b)** the Fourier sine series. Sketch and its two periodic extensions. Show the details.



$$f(x)=\begin{cases}0 & (0<x<2)\\ 1 & (2<x<4)\end{cases},\quad p=8,\quad L=4$$

(a) Fourier cosine series

$$a_0=\frac{1}{4}\int_2^4 1dx=\frac{1}{2},\qquad a_n=\frac{1}{2}\int_2^4\cos\frac{n\pi x}{4}dx=-\frac{2}{n\pi}\sin\frac{n\pi}{2}$$

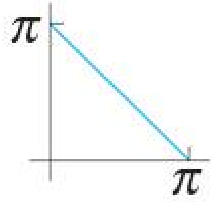
$$f(x)=\frac{1}{2}-\frac{2}{\pi}\left(\cos\frac{\pi x}{4}-\frac{1}{3}\cos\frac{3\pi x}{4}+\frac{1}{5}\cos\frac{5\pi x}{4}-\cdots\right)$$

(b) Fourier sine series

$$b_n=\frac{1}{2}\int_2^4\sin\frac{n\pi x}{4}dx=\frac{2}{n\pi}\left(\cos\frac{n\pi}{2}-\cos n\pi\right)=\begin{cases}0 & (n=4k)\\ \dfrac{2}{n\pi} & (n=4k+1)\\ \dfrac{-4}{n\pi} & (n=4k+2)\\ \dfrac{2}{n\pi} & (n=4k+3)\end{cases}$$

$$\therefore\ f(x)=\frac{2}{\pi}\left(\sin\frac{\pi x}{4}-\sin\frac{\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{4}+\frac{1}{5}\sin\frac{5\pi x}{4}-\frac{1}{3}\sin\frac{3\pi x}{2}+\frac{1}{7}\sin\frac{7\pi x}{4}\right.$$
$$\left.+\frac{1}{9}\sin\frac{9\pi x}{4}-\frac{1}{5}\sin\frac{5\pi x}{2}+\cdots\right)$$

25. Find **(a)** the Fourier cosine series, **(b)** the Fourier sine series. Sketch and its two periodic extensions. Show the details.



$f(x) = \pi - x, \quad (0 < x < \pi)$

(a) Fourier cosine series

$$a_0 = \frac{1}{\pi}\int_0^{\pi}(\pi - x)dx = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx = \frac{-2}{n^2\pi}(\cos n\pi - 1)$$

$$\therefore\ f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{-2}{n^2\pi}((-1)^2 - 1)\cos nx = \frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$$

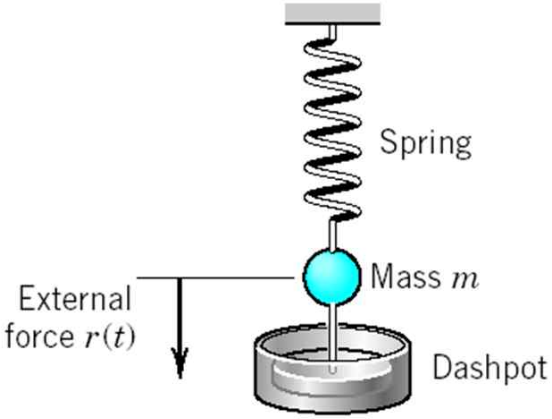
(b) Fourier sine series

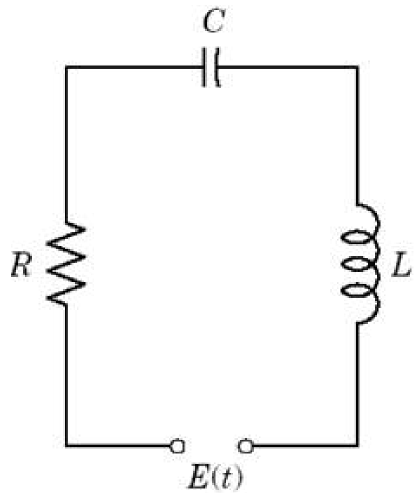
$$b_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx = \frac{2}{n}$$

$$\therefore\ f(x) = \sum_{n=1}^{\infty}\frac{2}{n}\sin nx = 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)$$

11.3 Forced Oscillations (강제진동)

(1) $my'' + cy' + ky = r(t)$ (1*) $LI'' + RI' + \frac{1}{C}I = E(t)$





Steady-state solution:

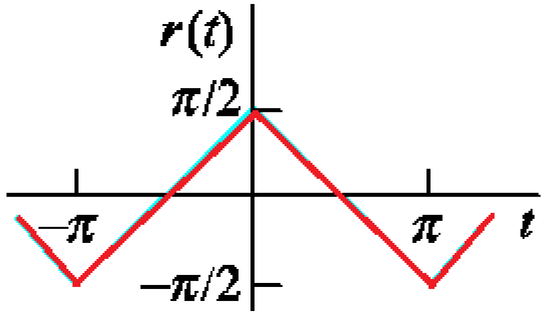
a harmonic oscillation with frequency equal to that of r(t)

Ex. 1 Forced Oscillations under a Nonsinusoidal Periodic Driving Force

$$y'' + 0.05y' + 25y = r(t),$$

$$r(t) = \begin{cases} t + \dfrac{\pi}{2} & (-\pi < t < 0) \\ -t + \dfrac{\pi}{2} & (0 < t < \pi) \end{cases}$$

$$r(t + 2\pi) = r(t)$$

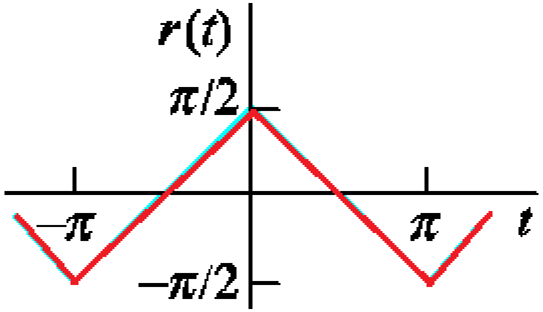


Find the steady-state solution y(t).

Sol.

1. Represent r(t) by a Fourier series.
2. Find the steady-state solution y(t) by solving the ODE.

1. Represent $r(t)$ by a Fourier series.



$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = 0$$

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt dt = \frac{2}{\pi}\int_{0}^{\pi}\left(-t+\frac{\pi}{2}\right)\cos nt\,dt \\ &= \frac{2}{\pi}\left(-\int_{0}^{\pi} t\cos nt\,dt + 0\right) = \frac{2}{\pi}\left(-\int_{0}^{\pi} t\cos nt\,dt\right) \end{aligned}$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\left(-\int_0^\pi t\cos nt\,dt\right) \\
&= \frac{2}{\pi}\left(-t\frac{\sin nt}{n}\Big|_0^\pi + \int_0^\pi \frac{\sin nt}{n}dt\right) \\
&= \frac{2}{\pi}\left[-\frac{\cos nt}{n^2}\right]_0^\pi = \frac{2}{\pi}\left[\frac{1-\cos n\pi}{n^2}\right] \\
&= \begin{cases} 0 \quad for \ \ even \ \ n \\ \dfrac{4}{n^2\pi} for \ \ odd \ \ n \end{cases}
\end{aligned}
$$

$$
b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt = 0
$$

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right), \quad L = \pi$$

$$a_0 = b_n = 0$$

$$a_n = \frac{4}{n^2\pi} \quad for \quad odd \quad n$$

$$= \sum_{odd\ n}^{\infty} \frac{4}{n^2\pi}\cos nt$$

2. Find the steady-state solution $y(t)$ by solving the ODE.

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \qquad (n = 1, 3, 5, \cdots)$$

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \quad (n = 1,3,5,\cdots)$$

$$y_n = A_n \cos nt + B_n \sin nt$$
$$y_n' = -nA_n \sin nt + nB_n \cos nt$$
$$y_n'' = -n^2(A_n \cos nt + B_n \sin nt)$$

$$(-n^2 A_n + 0.05nB_n + 25A_n)\cos nt + (-n^2 B_n - 0.05nA_n + 25B_n)\sin nt = \frac{4}{n^2\pi}\cos nt$$

$$\begin{cases} -n^2 A_n + 0.05nB_n + 25A_n = \dfrac{4}{n^2\pi} \\ -n^2 B_n - 0.05nA_n + 25B_n = 0 \end{cases}$$

$$\begin{cases} -n^2 A_n + 0.05 n B_n + 25 A_n = \dfrac{4}{n^2 \pi} \\ -n^2 B_n - 0.05 n A_n + 25 B_n = 0 \end{cases}$$

$$\begin{bmatrix} 25-n^2 & 0.05n \\ -0.05n & 25-n^2 \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 4/(n^2\pi) \\ 0 \end{bmatrix}$$

$$A_n = \frac{\begin{vmatrix} 4/(n^2\pi) & 0.05n \\ 0 & 25-n^2 \end{vmatrix}}{\begin{vmatrix} 25-n^2 & 0.05n \\ -0.05n & 25-n^2 \end{vmatrix}} = \frac{[4/(n^2\pi)](25-n^2)}{(25-n^2)^2+(0.05n)^2}$$

$$= \frac{4(25-n^2)}{(n^2\pi)D_n}, \quad where \;\; D_n = (25-n^2)^2 + (0.05n)^2$$

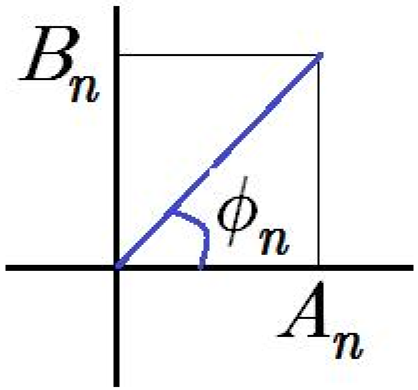
$$\begin{bmatrix} 25-n^2 & 0.05n \\ -0.05n & 25-n^2 \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 4/(n^2\pi) \\ 0 \end{bmatrix}$$

$$B_n = \frac{\begin{vmatrix} 25-n^2 & 4/(n^2\pi) \\ -0.05n & 0 \end{vmatrix}}{\begin{vmatrix} 25-n^2 & 0.05n \\ -0.05n & 25-n^2 \end{vmatrix}} = \frac{0.2n/(n^2\pi)}{D_n}$$

$$= \frac{0.2}{n\pi D_n}$$

$$y_n = A_n \cos nt + B_n \sin nt$$
$$= \sqrt{A_n^2 + B_n^2}\cos(nt - \phi_n)$$

$$where$$
$$\phi_n = \tan^{-1}(B_n / A_n)$$



$$C_n = \sqrt{A_n^2 + B_n^2} \qquad A_n = \frac{4(25 - n^2)}{(n^2\pi)D_n}, \quad B_n = \frac{0.2}{n\pi D_n}$$
$$= \sqrt{\left[\frac{4(25 - n^2)}{(n^2\pi)D_n}\right]^2 + \left[\frac{0.2}{n\pi D_n}\right]^2}$$

$$C_n = \sqrt{\left[\frac{4(25-n^2)}{(n^2\pi)D_n}\right]^2 + \left[\frac{0.2}{n\pi D_n}\right]^2}$$

$$= \sqrt{\frac{[4(25-n^2)]^2 + (0.2n)^2}{(n^2\pi D_n)^2}}$$

$$= \sqrt{\frac{16[(25-n^2)^2 + (0.05n)^2]}{(n^2\pi D_n)^2}} = \sqrt{\frac{16D_n}{(n^2\pi D_n)^2}}$$

$$= \frac{4}{n^2\pi\sqrt{D_n}}$$

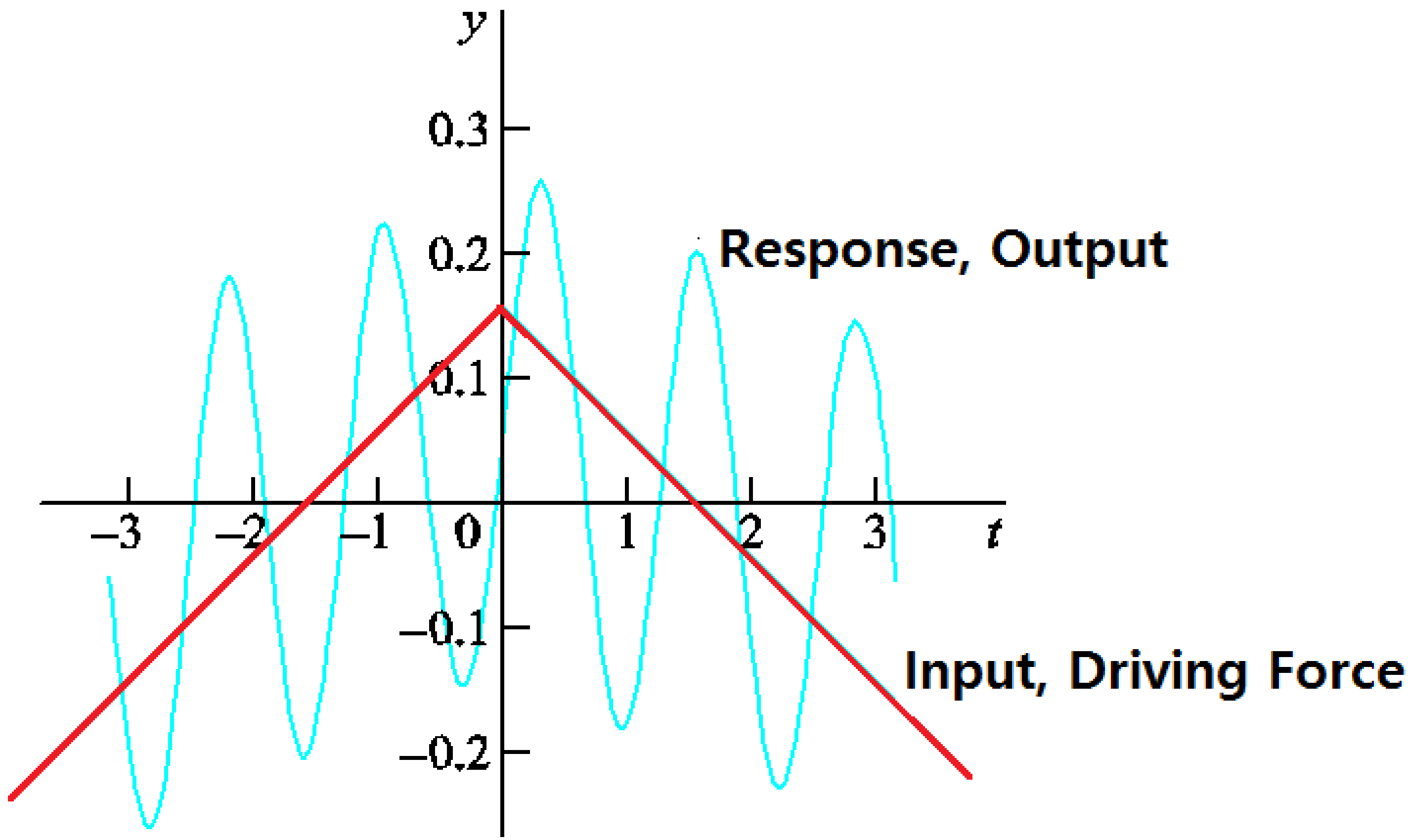
$$C_n = \frac{4}{n^2\pi\sqrt{D_n}}$$

$$C_1 = 0.0531, \quad C_3 = 0.0088, \quad C_5 = 0.2037,$$
$$C_7 = 0.0011, \quad C_9 = 0.0003$$

$$\begin{aligned} y &= y_1 + y_3 + y_5 + \cdots \\ &= C_1\cos(t-\phi_1) + C_3\cos(3t-\phi_3) \\ &\quad + C_5\cos(5t-\phi_5) + \cdots \end{aligned}$$

y_5 is the dominating term.

The steady-state solution $y(t)$:



7. Find a general solution of the ODE $y'' + \omega^2 y = r(t)$ with $r(t)$ as given. Show the details of your work.

$r(t) = \sin t,\ \omega = 0.5, 0.9, 1.1, 1.5, 10$

$r(t) = \sin t \Rightarrow y_p = A\cos t + B\sin t$

$y'' + \omega^2 y = r(t) \Rightarrow -(A\cos t + B\sin t) + \omega^2(A\cos t + B\sin t) = \sin t$

$(\omega^2 - 1)A = 0,\quad (\omega^2 - 1)B = 1 \Rightarrow A = 0,\ B = \dfrac{1}{\omega^2 - 1}$

$\therefore y_p = \dfrac{1}{\omega^2 - 1}\sin t \qquad y = c_1\cos\omega t + c_2\sin\omega t + \dfrac{1}{\omega^2 - 1}\sin t$

$\omega = 0.5: \quad y = c_1\cos 0.5t + c_2\sin 0.5t - \dfrac{4}{3}\sin t$

$\omega = 0.9: \quad y = c_1\cos 0.9t + c_2\sin 0.9t - \dfrac{100}{19}\sin t$

$\omega = 1.1: \quad y = c_1\cos 1.1t + c_2\sin 1.1t + \dfrac{100}{21}\sin t$

$\omega = 1.5: \quad y = c_1\cos 1.5t + c_2\sin 1.5t + \dfrac{4}{5}\sin t$

$\omega = 10: \quad y = c_1\cos 10t + c_2\sin 10t + \dfrac{1}{99}\sin t$

11. Find a general solution of the ODE $y'' + \omega^2 y = r(t)$ with $r(t)$ as given.

$$r(t) = \begin{cases} -1 & \text{if } -\pi < t < 0 \\ 1 & \text{if } \quad 0 < t < \pi, \end{cases} \qquad |\omega| \neq 1, 3, 5, \cdots$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nt\,dt = \frac{2}{n\pi}\left[1-(-1)^n\right] = \begin{cases} 0 & (n:\text{even}) \\ \dfrac{4}{n\pi} & (n:\text{odd}) \end{cases}$$

$$r(t) = \frac{4}{\pi}\left[\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right]$$

$$r(t) = \frac{4}{n\pi}\sin nt \qquad y_{pn} = A_n \cos nt + B_n \sin nt$$

$$A_n = 0, \quad B_n = \frac{4}{(\omega^2 - n^2)n\pi}$$

$$= \frac{4\sin nt}{(\omega^2 - n^2)n\pi}$$

$$y = c_1 \cos\omega t + c_2 \sin\omega t + \frac{4\sin t}{(\omega^2 - 1)\pi} + \frac{4\sin 3t}{3(\omega^2 - 9)\pi} + \cdots$$

13. Find the steady-state oscillations of $y'' + cy' + y = r(t)$ with $c>0$ and $r(t)$ as given.

$$r(t) = \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt)$$

$$y'' + cy' + y = r(t) = a_n \cos nt + b_n \sin nt$$

$$y_n = A_n \cos nt + B_n \sin nt$$

$$-n^2(A_n \cos nt + B_n \sin nt) + cn(-A_n \sin nt + B_n \cos nt)$$
$$+ (A_n \cos nt + B_n \sin nt) = a_n \cos nt + b_n \sin nt$$

$$\begin{cases} (-n^2+1)A_n + cnB_n = a_n \\ (-n^2+1)B_n - cnA_n = b_n \end{cases} \Rightarrow A_n = \frac{(1-n^2)a_n - cnb_n}{c^2n^2 + (1-n^2)^2}, \quad B_n = \frac{cna_n + (1-n^2)b_n}{c^2n^2 + (1-n^2)^2}$$

$$y = \sum_{n=1}^{N} [A_n \cos nt + B_n \sin nt]$$