Introduction to Discrete Math

Felipe P. Vista IV

Course Outline

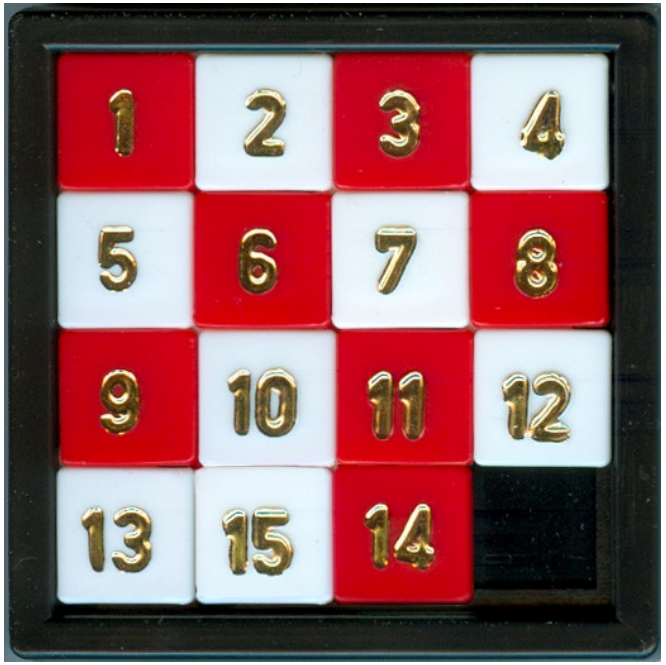
- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion, Logic, Invariants
- Probability & Combinatronics
  - Counting, Probability, Random Variables
- Graph Theory
  - Graphs (cycles, classes, parameters)
- Number Theory & Cryptography
  - Arithmetic in modular form
  - Intro to Cryptography

Mathematical Thinking – Invariants

THE 15-PUZZLE

- The game
- Permutations
- Proof: The Challenging Part
- Mission Impossible
- Classify a Permutation

The 15-Puzzle



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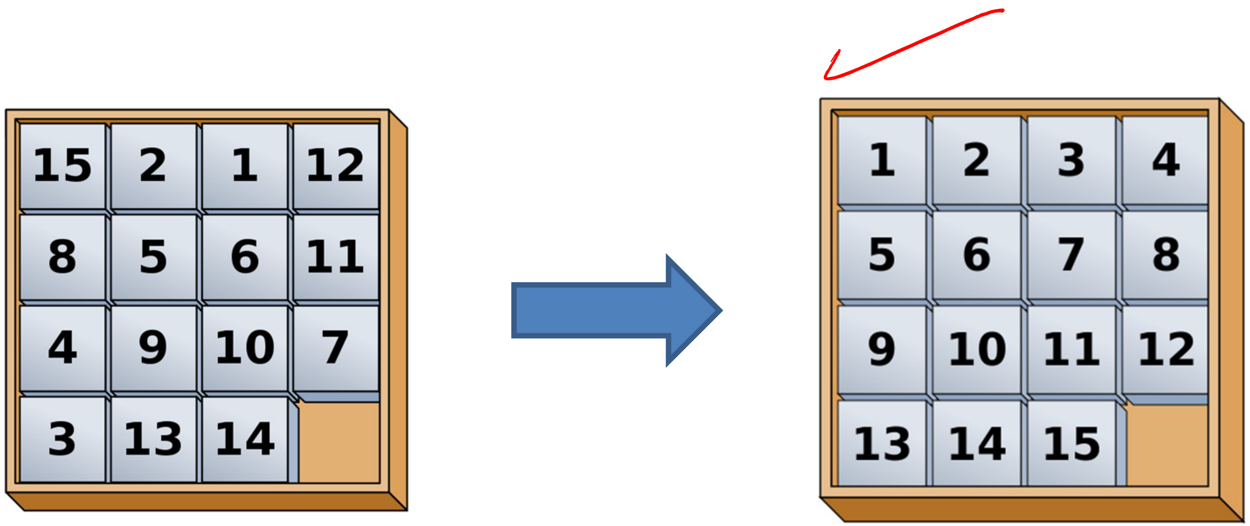
The Game

- move the pieces (into an empty neighbor square)
- goal → to obtain a particular configuration
- go back to starting configuration



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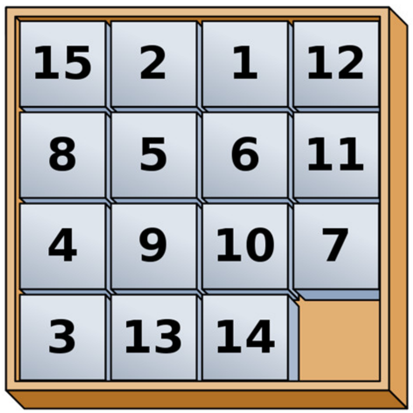

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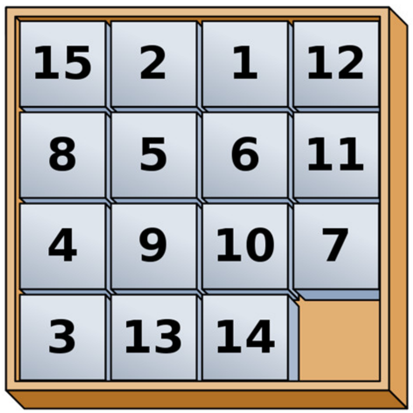

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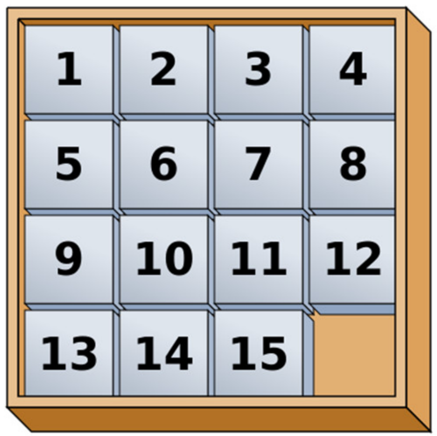

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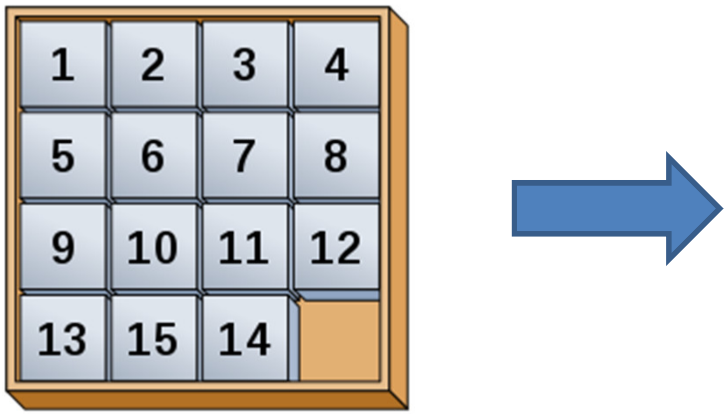

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$100 Dare!

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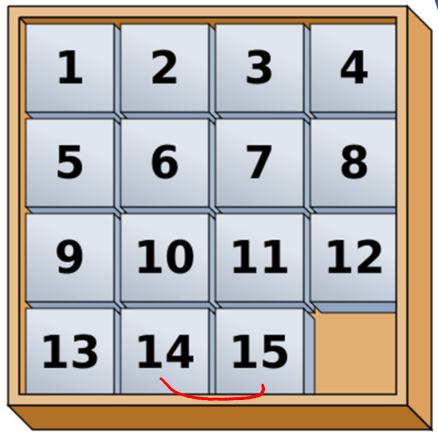

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STOP → SPOT

STOP → POST

steps?

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- Generally:
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  - STOP → SPOT:
    - one transposition enough
  - STOP →POST:
    - how many transposition?

STOP → POST
SPTOS
POTS
POST

Even and Odd Permutations

- STOP $\rightarrow$ SPOT: 1, 3, 5, 7, …
- STOP $\rightarrow$ POST: 3, 5, 7, …
- STOP $\rightarrow$ POTS: 2, 4, 6, …
- n + n transposition: twice nothing
- Conjecture: permutations are two types
  - Even
  - Odd

A Counterexample

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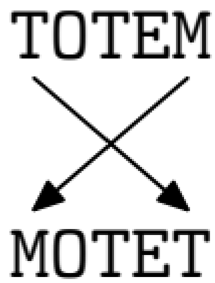

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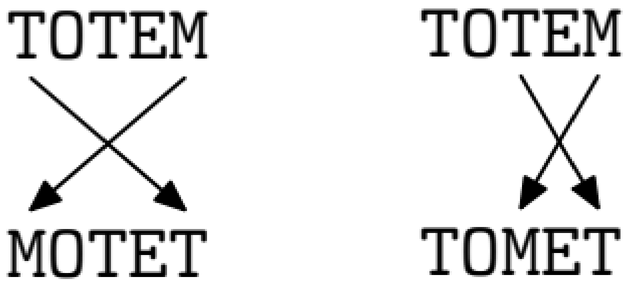

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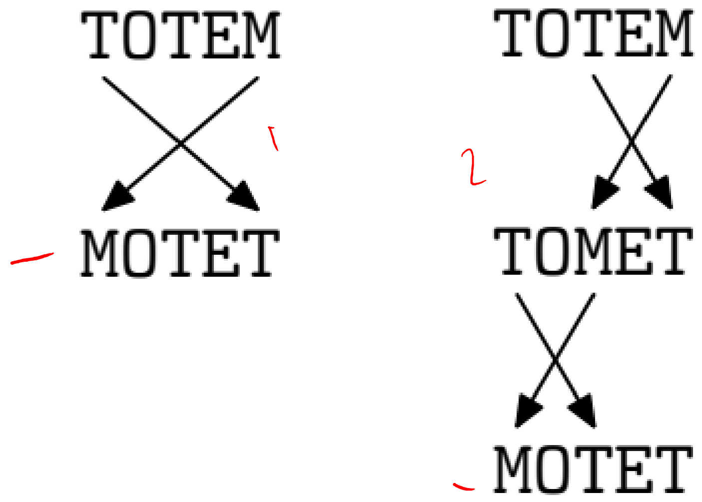

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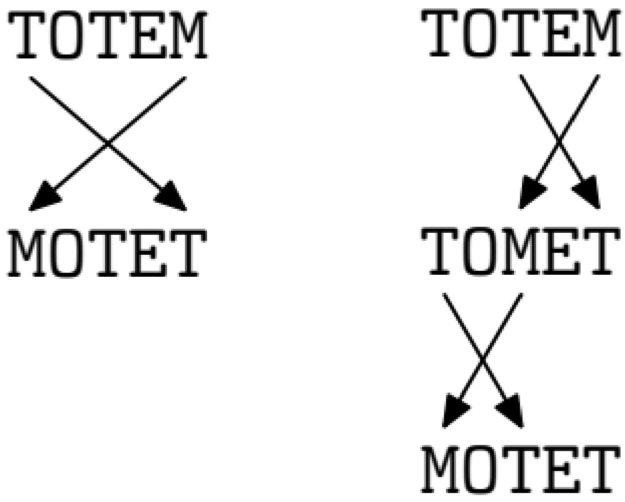

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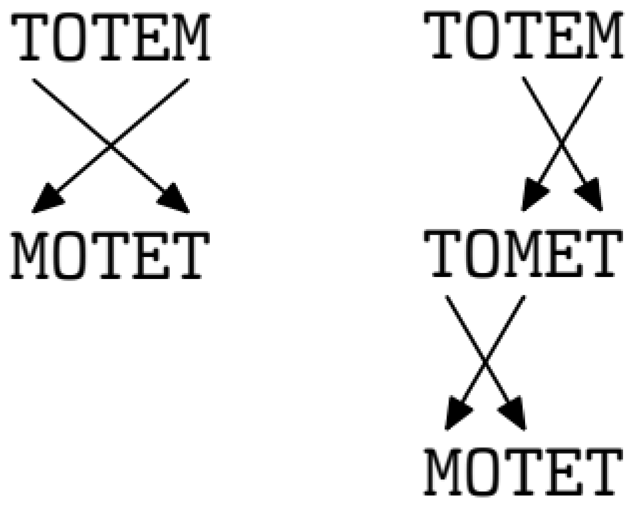

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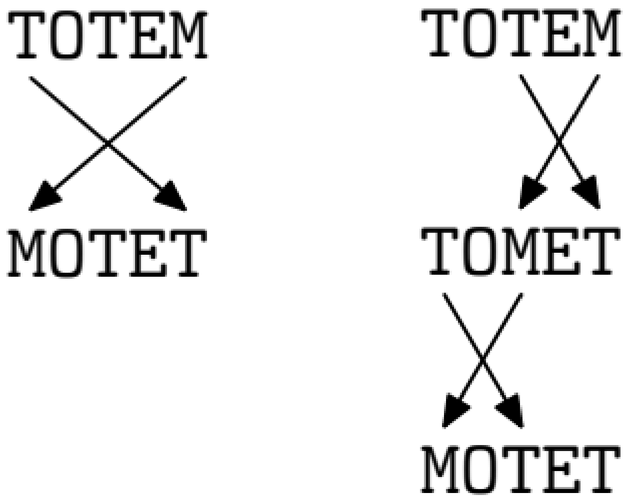

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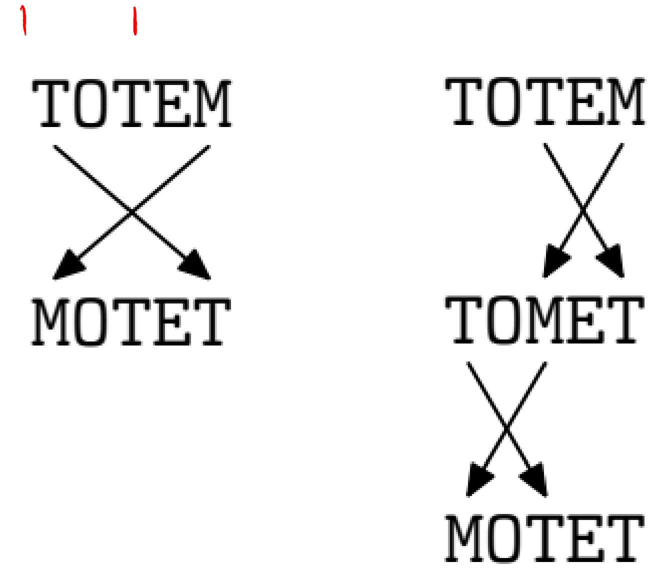

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- each permutation can be obtained through transpositions
- some permutations can be derived only through an even number of transpositions, while others can be derived only through odd number of transpositions

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$$\mathrm{STOP} \rightarrow \mathrm{POST}$$

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STOP → POST
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permutation

STOP → POST

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Thank you.