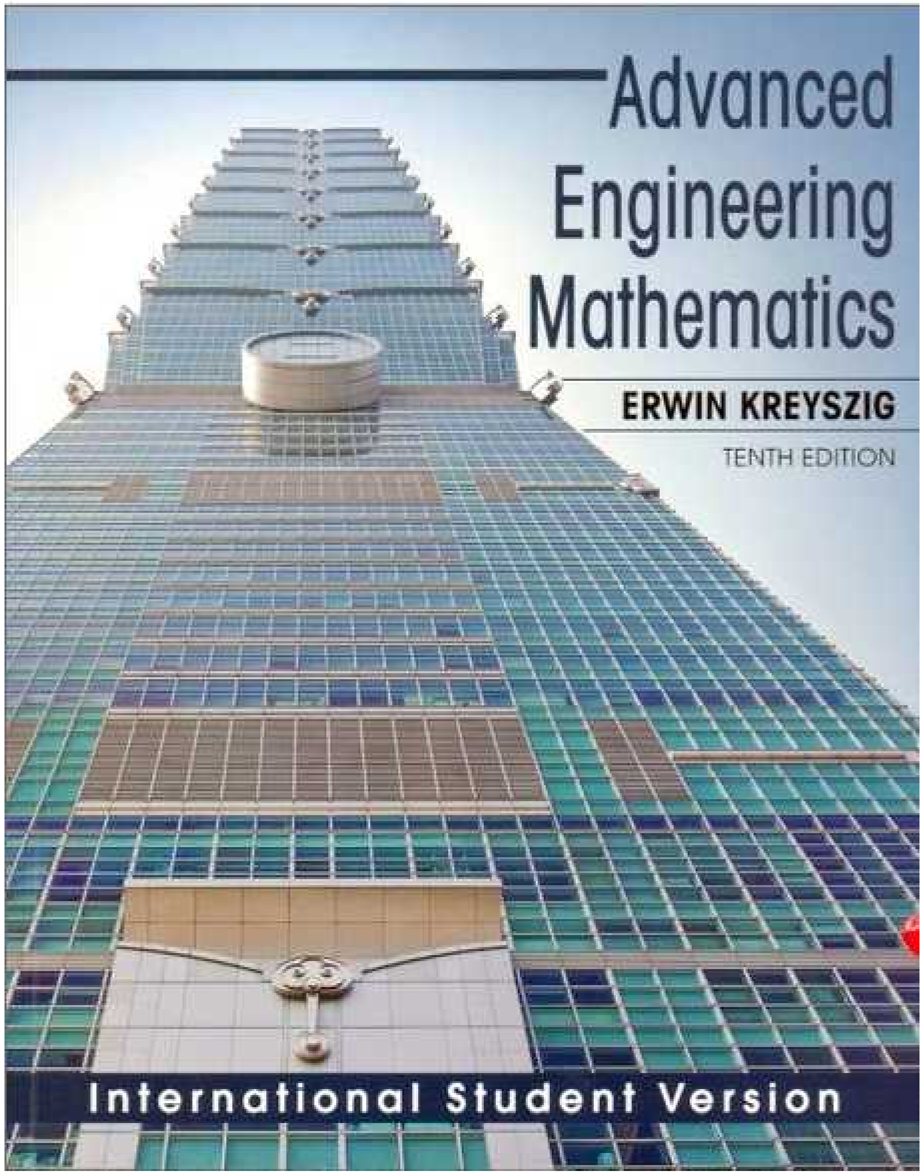


CHAPTER 13

Complex Numbers and Functions. Complex Differentiation

13.1 Complex Numbers and Their Geometric Representation

13.1 Complex Numbers. Complex Plane (복소수와 복소평면)

By definition, a **complex number** z is an ordered pair (x, y) of real numbers x and y, written

$$z = (x, y).$$

x is called the **real part** and y the **imaginary part** of z, written

$$x = \operatorname{Re} z, \quad y = \operatorname{Im} z.$$

By definition, two complex numbers are **equal** if and only if their real parts are equal and their imaginary parts are equal.

$(0, 1)$ is called the **imaginary unit** and is denoted by i,

$$i = (0, 1). \tag{1}$$

13.1 Complex Numbers and Their Geometric Representation

Addition, Multiplication. Notation $z = x + iy$

Let $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$.

Addition of two complex numbers **,** $z = z_1 + z_2$, is defined by

(2) $\quad z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i\,(y_1 + y_2)$

Multiplication, $z = z_1 z_2$, is defined by

(3) $\quad z = z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$.

These two definitions imply that

$$(x_1\ 0) + (x_2\ 0) = (x_1 + x_2, 0)$$

and

$$(x_1\ 0)\ (x_2\ 0) = (x_1 x_2, 0)$$

as for real numbers x_1, x_2. Hence the complex numbers "***extend***" the real numbers.

13.1 Complex Numbers and Their Geometric Representation

We can thus write

$(x, 0) = x$. Similarly, $(0, y) = iy$

because by (1), and the definition of multiplication, we have

$iy = (0, 1)y = (0, 1)(y, 0) = (0 \cdot y - 1 \cdot 0, 0 \cdot 0 + 1 \cdot y) = (0, y)$.

In practice, complex numbers $z = (x, y)$ ***are written***

(4) $$z = (x, y) = x + iy = x + yi.$$

Electrical engineers often write j instead of i because they need i for the current.

If $x = 0$, then $z = (0, y) = iy$ and is called **pure imaginary**. Also, (1) and (3) give

(5) $$i^2 = -1$$

because, by the definition of multiplication, $i^2 = ii = (0, 1)(0, 1) = (-1, 0) = -1$.

13.1 Complex Numbers and Their Geometric Representation

For **addition** the standard notation (4) gives [see (2)]

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

For **multiplication** the standard notation gives the following very simple recipe. Multiply each term by each other term and use $i^2 = -1$ when it occurs [see (3)]:

$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).\end{aligned}$$

13.1 Complex Numbers and Their Geometric Representation

Ex1 Real Part, Imaginary Part, Sum and Product of Complex Numbers

Let $z_1=8+3i$ and $z_2=9-2i$. Find
(1) $\text{Re } z_1$, $\text{Im } z_1$, $\text{Re } z_2$, and $\text{Im } z_2$.
(2) z_1+z_2 and z_1z_2.

Sol.

(1) $\text{Re } z_1 = \text{Re}(8+3i) = 8$, $\quad \text{Im } z_1 = \text{Im}(8+3i)=3$

$\text{Re } z_2 = \text{Re}(9-2i)=9$, $\quad \text{Im } z_2 = \text{Im}(9-2i)=-2$

(2) $z_1+z_2=(8+3i)+(9-2i)=17+i$

$z_1z_2=(8+3i)(9-2i)=(8\cdot9+3\cdot2)+i(8\cdot(-2)+3\cdot9)=78+i11$

13.1 Complex Numbers and Their Geometric Representation

Subtraction, Division

Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$.

(6) $z_1-z_2=(x_1+iy_1)-(x_2+iy_2)=(x_1-x_2)+i(y_1-y_2)$

(7) $$z=\frac{z_1}{z_2}=\frac{x_1+iy_1}{x_2+iy_2}=\frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}$$
$$=\frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2}$$
$$=\frac{(x_1x_2+y_1y_2)}{x_2^2+y_2^2}+i\frac{(x_1x_2+y_1y_2)}{x_2^2+y_2^2}$$

13.1 Complex Numbers and Their Geometric Representation

Ex 2 Difference and Quotient of Complex Numbers

For $z_1=8+i3$ and $z_2=9-i2$, find z_1-z_2 and z_1/z_2.

Sol.

(a) $z_1-z_2=(8+i3)-(9-i2)=-1+i5$

(b) $$z=\frac{z_1}{z_2}=\frac{8+3i}{9-2i}=\frac{(8+3i)(9+2i)}{(9-2i)(9+2i)}$$

$$=\frac{72-6}{9^2+2^2}+i\frac{27+16}{9^2+2^2}$$

$$=\frac{66}{85}+i\frac{43}{85}$$

13.1 Complex Numbers and Their Geometric Representation

- Complex Plane

Consider the geometric representation of complex numbers, which is of great practical importance. We choose two perpendicular coordinate axes, the horizontal *x*-axis, called the **real axis**, and the vertical *y*-axis, called the **imaginary axis**.
On both axes we choose the same unit of length (Fig. 318). This is called a **Cartesian coordinate system**.

We now plot a given complex number as the point *P* with coordinates
x, *y*. The *xy*-plane in which the complex numbers are represented in this way is called the **complex plane**.2 Figure 319 shows an example.

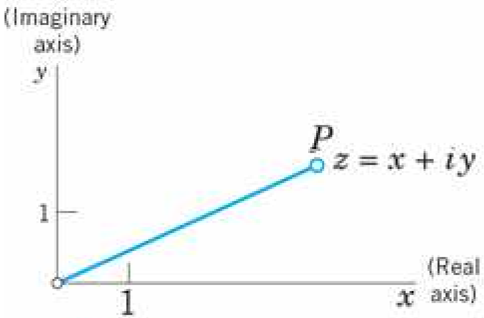


Fig. 318. The complex plane

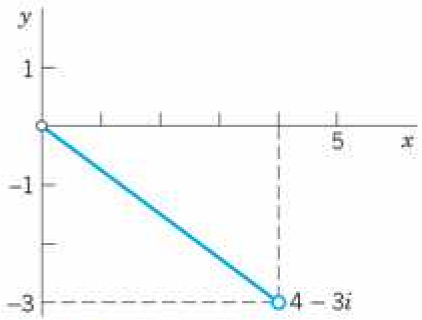


Fig. 319. The number 4 − 3*i* in the complex plane

13.1 Complex Numbers and Their Geometric Representation

Addition and subtraction in the complex plane

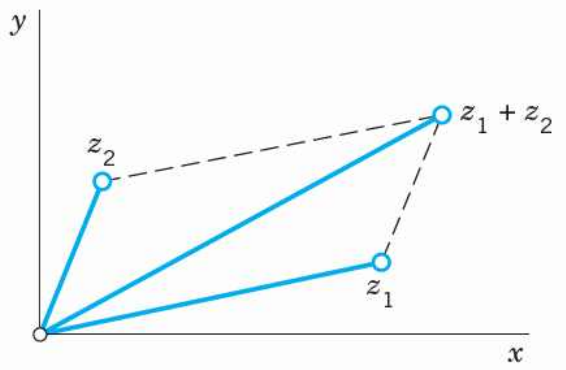


Fig. 320. Addition of complex numbers

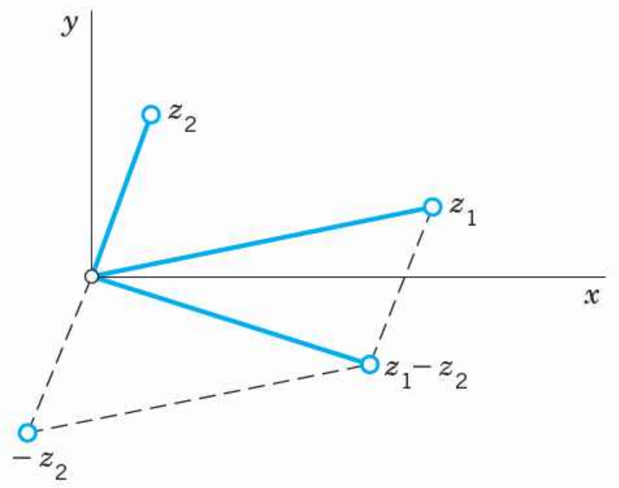


Fig. 321. Subtraction of complex numbers

13.1 Complex Numbers and Their Geometric Representation

Complex Conjugate Numbers

The complex conjugate $\bar{z}$ of a complex number $z = x + iy$ is defined by

$$\bar{z} = x - iy.$$

It is obtained geometrically by reflecting the point z in the real axis. Figure 322 shows this for $z = 5 + 2i$ and its conjugate $\bar{z} = 5 - 2i$.

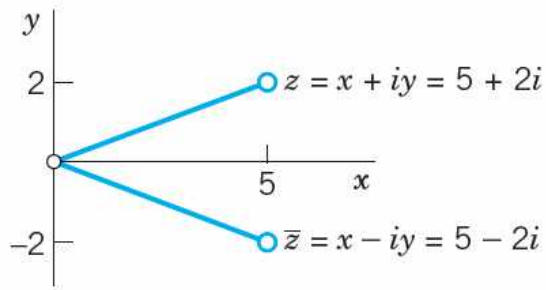


Fig. 322. Complex conjugate numbers

13.1 Complex Numbers and Their Geometric Representation

The complex conjugate is important because it permits us to switch from complex to real. Indeed, by multiplication, $z\bar{z} = x^2 + y^2$ (verify!).

By addition and subtraction, $z + \bar{z} = 2x$, $z - \bar{z} = 2iy$.

We thus obtain for the real part x and the imaginary part y (not iy!) of the important formulas

$$\textbf{(8)} \qquad \operatorname{Re} z = x = \tfrac{1}{2}(z + \bar{z}), \qquad \operatorname{Im} z = y = \frac{1}{2i}(z - \bar{z}).$$

If z is real, $z = x$, then $\bar{z} = z$ by the definition of $\bar{z}$, and conversely. Working with conjugates is easy, since we have

$$\text{(9)} \qquad \begin{aligned} &\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2, \qquad &&\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2, \\ &\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \qquad &&\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}. \end{aligned}$$

Ex 3 Illustration of (8) and (9)

Let z_1=4+3i and z_2= 2+5i. Find
(a) Im z_1= by (8).
(b) The complex conjugate of $z_1 z_2$ by (9).

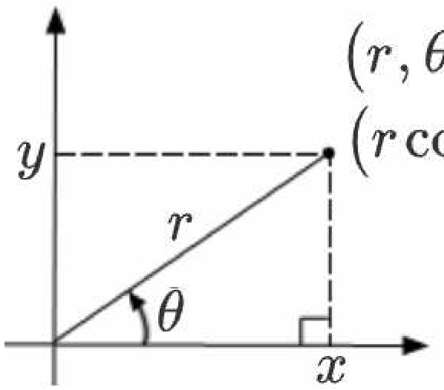
Sol

$$(a)\, Im(z_1) = \frac{1}{2i}(z - \bar{z})$$
$$= \frac{1}{2i}[(4+3i) - (\overline{4+3i})] = \frac{1}{2i}(3i+3i) = 3$$

$$(b)\, \overline{z_1 z_2} = \overline{(4+3i)(2+5i)} = \overline{-7+26i} = -7-26i$$
$$\bar{z}_1 \bar{z}_2 = (4-3i)(2-5i) = -7-26i$$

13.2 Polar Form of Complex Numbers. Powers and Roots

13.2 Polar Form of Complex Numbers. Powers and Roots



(r, θ) in polar coordinate

$(r\cos\theta, r\sin\theta)$ in rectangular coordinate

(1) $$x = r\cos\theta, \qquad y = r\sin\theta.$$

(2) $$z = r(\cos\theta + i\sin\theta)$$ **: polar form**

r is called the **absolute value** or **modulus** of z and is denoted by $|z|$. Hence

(3) $$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

(4) $$\tan\theta = \frac{y}{x} \qquad (z \neq 0)$$

13.2 Polar Form of Complex Numbers. Powers and Roots

Geometrically, $|z|$ is the distance of the point z from the origin (Fig. 323).
Similarly, $|z_1 - z_2|$ is the distance between z_1 and z_2(Fig. 324).

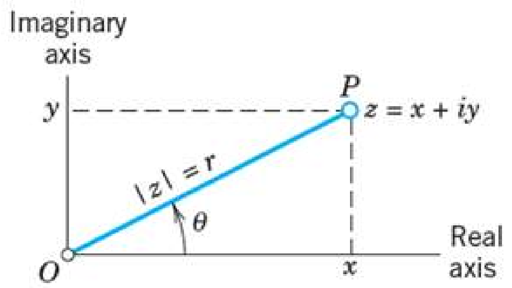


Fig. 323. Complex plane, polar form of a complex number

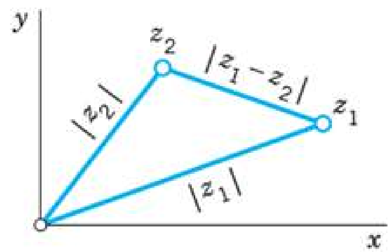


Fig. 324. Distance between two points in the complex plane

Geometrically, θ is the directed angle from the positive x-axis to OP in Fig. 323. Here, as in calculus, all ***angles are measured in radians and positive in the counterclockwise sense***.

arg $z = 2n\pi + \theta$

(5) $-\pi < \text{Arg } z = \theta \le \pi$: the **principal value** Arg z

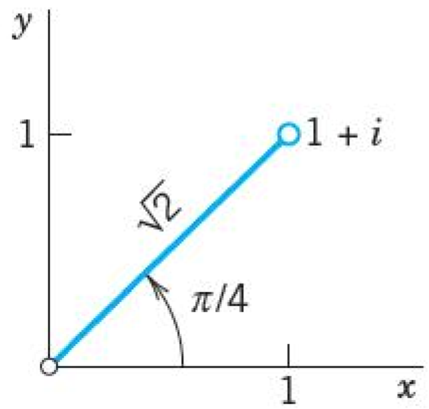
13.2 Polar Form of Complex Numbers. Powers and Roots

EXAMPLE 1 Polar Form of Complex Numbers. Principal Value Arg z

Let $z_1 = 1+i$ and $z_2 = 3+3\sqrt{3}i$.
(1) Denote z_1 and z_2 by the polar form.
(2) Determine absolute values, arguments, and Arguments of z_1 and z_2.

Sol

(1)
$$z_1 = 1 + i = \sqrt{2}\,(\cos \tfrac{1}{4}\pi + i \sin \tfrac{1}{4}\pi)$$
$$z_2 = 3 + 3\sqrt{3}i = 6\,(\cos \tfrac{1}{3}\pi + i \sin \tfrac{1}{3}\pi)$$



(2)
$$|z_1| = \sqrt{2}$$
$$|z_2| = 6$$

$$\arg z_1 = \tfrac{1}{4}\pi \pm 2n\pi \quad (n = 0, 1, \cdots)$$
$$\arg z_2 = \tfrac{1}{3}\pi \pm 2n\pi \quad (n = 0, 1, \cdots)$$

$$\text{Arg } z_1 = \tfrac{1}{4}\pi \qquad \text{Arg } z_2 = \tfrac{1}{3}\pi$$

Triangle Inequality

the **triangle inequality**

$$|z_1 + z_2| \leqq |z_1| + |z_2| \tag{6}$$

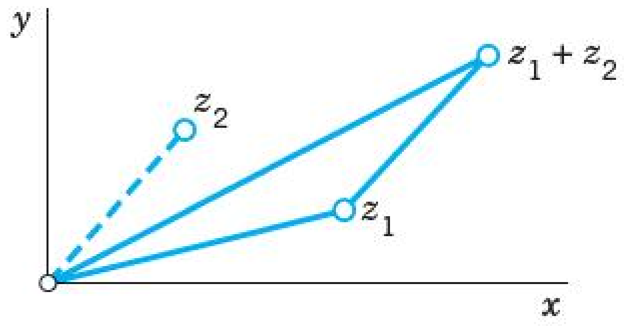


Fig. 326. Triangle inequality

By induction we obtain from (6) the **generalized triangle inequality**

$$|z_1 + z_2 + \cdots + z_n| \leqq |z_1| + |z_2| + \cdots + |z_n|; \tag{6*}$$

that is, *the absolute value of a sum cannot exceed the sum of the absolute values of the terms.*

Example 2 Triangle Inequality

Let $z_1 = 1+i$ and $z_2 = -2+3i$.
Show that the triangle inequality holds.

Sol

$$z_1 = 1+i, \ \ z_2 = -2+3i$$

$$|z_1 + z_2| = |-1+4i| = \sqrt{17}$$
$$|z_1| + |z_2| = \sqrt{2} + \sqrt{13}$$
$$\therefore |z_1 + z_2| = \sqrt{17} = 4.12 < \sqrt{2} + \sqrt{13} = 5.02$$

13.2 Polar Form of Complex Numbers. Powers and Roots

Multiplication and Division in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Multiplication.

By (3) in Sec. 13.1 the product is at first

$$z_1 z_2 = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)].$$

The addition rules for the sine and cosine [(6) in App. A3.1] now yield

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \tag{7}$$

$$|z_1 z_2| = |z_1||z_2|. \tag{8}$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \quad \text{(up to multiples of } 2\pi\text{)}. \tag{9}$$

13.2 Polar Form of Complex Numbers. Powers and Roots

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Division.

We have $z_1 = (z_1/z_2)z_2$.

Hence $|z_1| = |(z_1/z_2)z_2| = |z_1/z_2||z_2|$ and by division by $|z_2|$

(10) $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (z_2 \neq 0).$$

Similarly, $\arg z_1 = \arg[(z_1/z_2)z_2] = \arg(z_1/z_2) + \arg z_2$.

Thus,

(11) $$\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2 \qquad \text{(up to multiples of } 2\pi\text{)}.$$

Combining (10) and (11) we also have the analog of (7),

(12) $$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

13.2 Polar Form of Complex Numbers. Powers and Roots

Example 3 Illustration of Formulas (8)–(11)

Let z_1 = -2+2i and z_2 = 3i. Determine $|z_1 z_2|$, $|z_1/z_2|$, Arg($z_1 z_2$), and Arg(z_1/z_2)

Sol

$$(a)\, z_1z_2 = (-2+2i)(3i) = -6i + 6i^2 = -6-6i$$

$$|z_1z_2| = |-6-6i| = \sqrt{(-6)^2+(-6)^2} = 6\sqrt{2}$$

$$|z_1|\,|z_2| = |(-2+2i)|\,|(3i)| = (2\sqrt{2})\cdot 3 = 6\sqrt{2}$$

$$(b)\, \frac{z_1}{z_2} = \frac{-2+2i}{3i} = \frac{2}{3}+\frac{2}{3}i$$

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{2}{3}+\frac{2}{3}i\right| = \frac{2}{3}\sqrt{2}$$

$$\frac{|z_1|}{|z_2|} = \frac{|(-2+2i)|}{|(3i)|} = \frac{2\sqrt{2}}{3}$$

$$(c)\, z_1z_2 = (-2+2i)(3i) = -6i + 6i^2 = -6-6i$$

$$Arg(z_1z_2) = Arg(-6-6i) = -135^o$$

$$Arg\, z_1 + Arg\, z_2 = 135^0 + 90^o = 225^0 = -135^o$$

$$(d)\, \frac{z_1}{z_2} = \frac{-2+2i}{3i} = \frac{2}{3}+\frac{2}{3}i$$

$$Arg\left(\frac{z_1}{z_2}\right) = Arg\left(\frac{2}{3}+\frac{2}{3}i\right) = 45^o \qquad Arg\left(\frac{z_1}{z_2}\right) = Arg z_1 - Arg z_2 = 135^o - 90^o = 45^o$$

13.2 Polar Form of Complex Numbers. Powers and Roots

Example 4 Integer Powers of z. De Moivre's Formula

(8) $$|z_1 z_2| = |z_1||z_2|.$$

(9) $$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$ (up to multiples of 2π).

by induction for n=0, 1, 2, ⋯, n

(13) $$z^n = r^n(\cos n\theta + i \sin n\theta).$$

(13*) $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta.$$: **De Moivre's formula**

13.2 Polar Form of Complex Numbers. Powers and Roots

(13*) $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$\Big\downarrow\ n = 2$$

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + \mathrm{i}\sin 2\theta$$

$$(\cos^2\theta - \sin^2\theta) + 2i\cos\theta\sin\theta = \cos 2\theta + \mathrm{i}\sin 2\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \qquad \sin 2\theta = 2\cos\theta\sin\theta.$$

Try n=3:

Roots

If $z = w^n$ $(n = 1, 2, \cdots)$,

then to each value of w there corresponds *one* value of z.

Conversely, to a given $z \neq 0$ there correspond precisely n distinct values of w.

Each of these values is called an ***n*th root** of z, and we write

$$w = \sqrt[n]{z}. \tag{14}$$

Hence this symbol is ***multivalued***, namely, *n-valued*.

13.2 Polar Form of Complex Numbers. Powers and Roots

The n values of $\omega = \sqrt[n]{z}$ can be obtained as follows.

$$\omega = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right) (k = 0,1,2\ ,\ldots,\ n-1)$$

$$z = r(\cos\theta + i\sin\theta)$$
$$\omega = R(\cos\phi + i\sin\phi)$$
$$\omega^n = z:\ \ R^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$$
$$R^n = r,\ \ \therefore R = \sqrt[n]{r}$$
$$n\phi = \theta + 2k\pi,\ \ \phi = \frac{\theta}{n} + \frac{2k\pi}{n}$$

$$(15)\quad \therefore \omega = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$$

13.2 Polar Form of Complex Numbers. Powers and Roots

$$(15) \qquad \omega = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)$$

Principle value of $\omega = \sqrt[n]{z}$

$$k = 0: \quad \omega = \sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

13.2 Polar Form of Complex Numbers. Powers and Roots

the **nth roots of unity** : $\sqrt[n]{1}$

$$r=1,\ \theta=0$$

$$\omega=\sqrt[n]{1}=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n},\quad k=0,1,\cdots,n-1$$

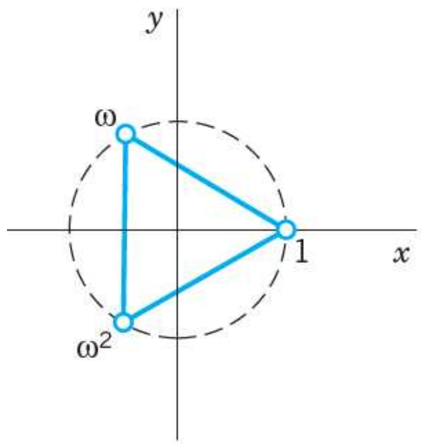


Fig. 327. $\sqrt[3]{1}$

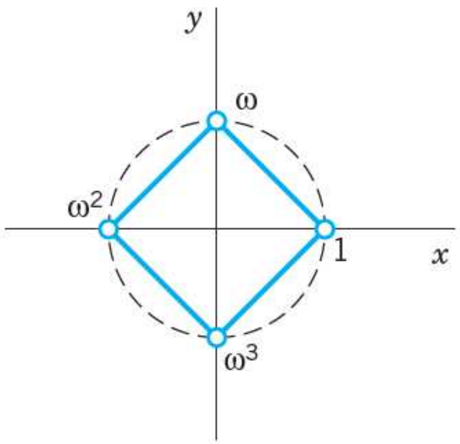


Fig. 328. $\sqrt[4]{1}$

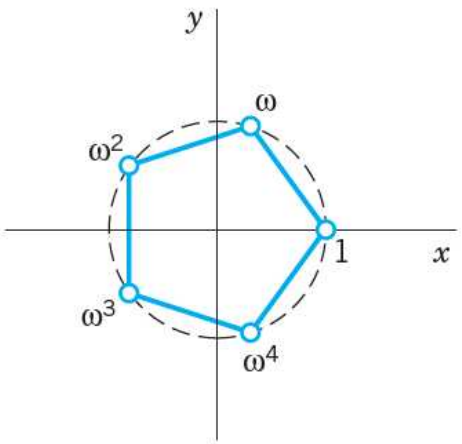


Fig. 329. $\sqrt[5]{1}$

13.2 Polar Form of Complex Numbers. Powers and Roots

$$\omega = \sqrt[n]{1} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0, 1, \cdots, n-1$$

Let $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$,

then, roots are: $1, \omega, \omega^2, \cdots, \omega^{n-1}$.

More generally, if w_1 is any nth root of an arbitrary complex number z ($\neq 0$), then the n values of $\sqrt[n]{z}$ in (15) are

$$(17) \quad w_1, \; w_1\omega, \; w_1\omega^2, \; \cdots, \; w_1\omega^{n-1}$$

because multiplying w_1 by ω^k corresponds to increasing the argument of w_1 by $2k\pi/n$.

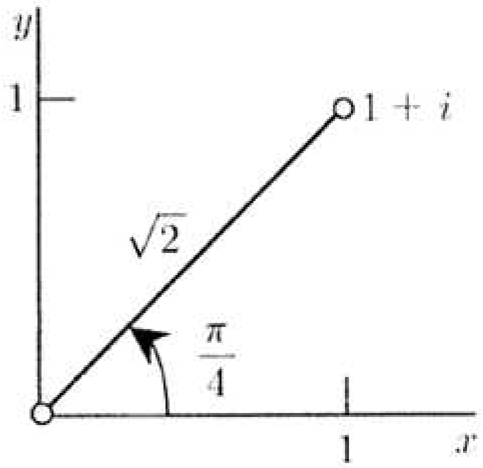
Problems in Sec 13.2

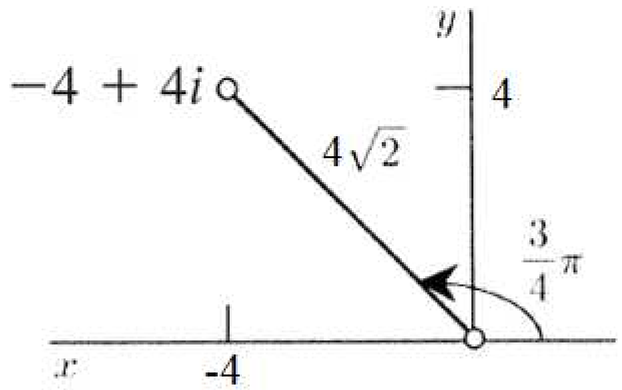
1–8 POLAR FORM

Represent in polar form and graph in the complex plane as in Fig. 325.

1. $1 + i$

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$





2. $-4 + 4i$

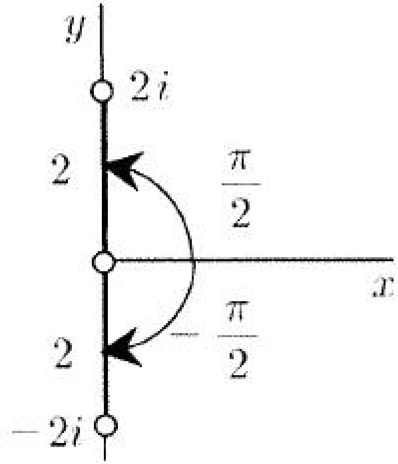
$$-4 + 4i = 4\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Problems in Sec 13.2

3. $2i, \quad -2i$

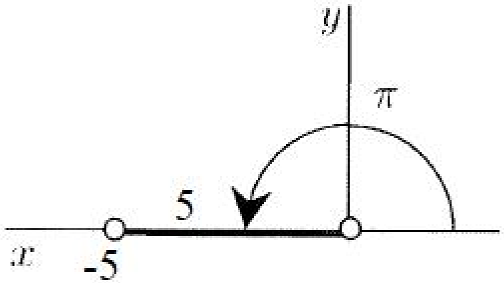
$$2i = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$-2i = 2\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$$



4. -5

$$-5 = 5(\cos\pi + i\sin\pi)$$

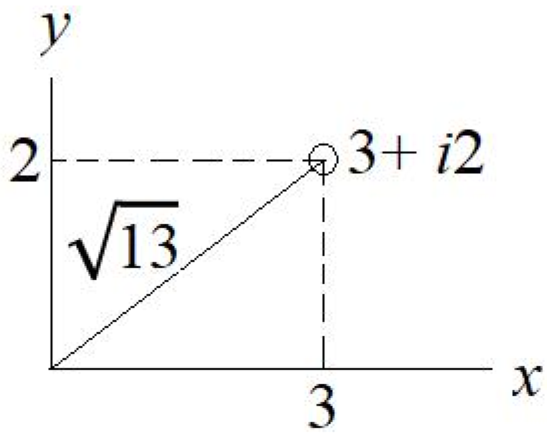


Problems in Sec 13.2

8. $\dfrac{-4 + 19i}{2 + 5i}$

$$\frac{-4+19i}{2+5i} = \frac{(-4+19i)(2-5i)}{(2+5i)(2-5i)} = \frac{(-4\cdot 2+19\cdot 5)+i(19\cdot 2+4\cdot 5)}{2^2+5^2}$$

$$= \frac{87+i58}{29} = 3+i2 = \sqrt{13}\ [\cos(\tan^{-1}2/3) + \mathrm{i}\sin(\tan^{-1}2/3)]$$



Problems in Sec 13.2

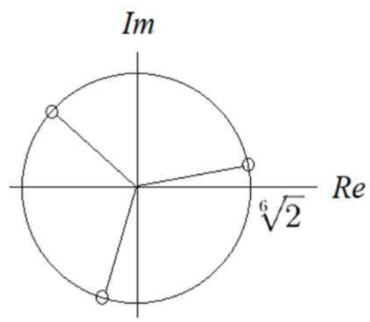
21. Find and graph all roots of $\sqrt[3]{1+i}$ in the complex plane.

$$z = r\left[\cos\theta + i\sin\theta\right]$$

$$z^3 = r^3\left[\cos(3\theta) + i\sin(3\theta)\right] = 1 + i = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

$$r^3 = \sqrt{2}\ ,\ \ 3\theta = \frac{\pi}{4} + 2k\pi\ ,\ k = 0,\ 1,\ 2$$

$$\begin{cases} \sqrt[6]{2}\left[\cos\left(\dfrac{\pi}{12}\right) + i\sin\left(\dfrac{\pi}{12}\right)\right] \\ \sqrt[6]{2}\left[\cos\left(\dfrac{7\pi}{12}\right) + i\sin\left(\dfrac{7\pi}{12}\right)\right] \\ \sqrt[6]{2}\left[\cos\left(\dfrac{5\pi}{4}\right) + i\sin\left(\dfrac{5\pi}{4}\right)\right] \end{cases}$$



Problems in Sec 13.2

28. Solve and graph the solutions. Show details.

$$z^2 - (6 - 2i)z + 17 - 6i = 0$$

The quadratic formula

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b' \pm \sqrt{b'^2 - ac}}{a}, \quad b' = \frac{b}{2}$$

$$\begin{aligned} x &= \frac{-b' \pm \sqrt{b'^2 - ac}}{a} \\ &= (3-i) \pm \sqrt{(3-i)^2 - (17-6i)} \\ &= (3-i) \pm 9i = \begin{cases} 3+8i \\ 3-10i \end{cases} \end{aligned}$$

13.3 Derivative. Analytic Function

Circles and Disks. Half-Planes

unit circle

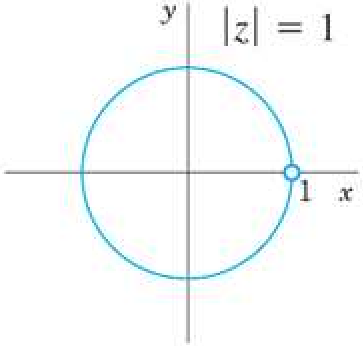


Fig. 330. Unit circle

general circle of radius ρ

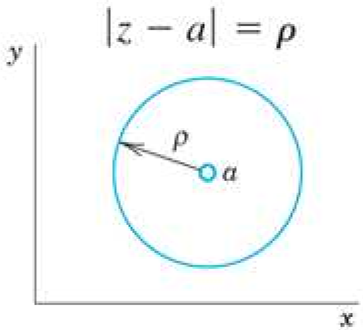


Fig. 331. Circle in the complex plane

an **open annulus** (circular ring)

$$\rho_1 < |z - a| < \rho_2$$

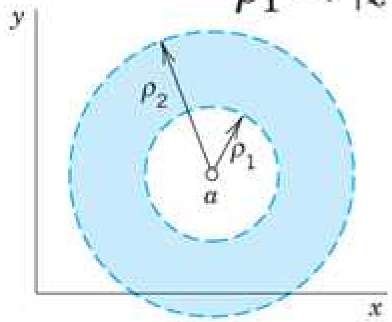
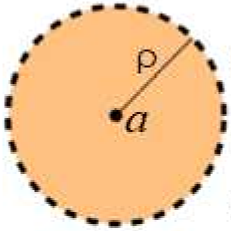


Fig. 332. Annulus in the complex plane

open circular disk

$|z - a| < \rho$



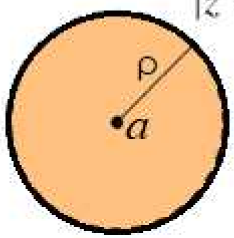
a **neighborhood** of a

or, more precisely,

a ρ-*neighborhood* of a

closed circular disk

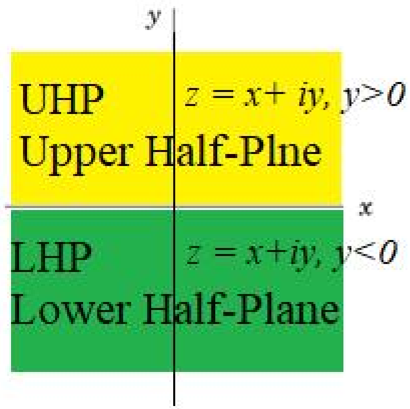
$|z - a| \leqq \rho$

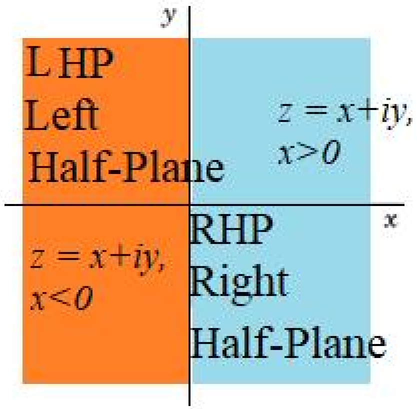


the **closed annulus**

$$\rho_1 \leqq |z - a| \leqq \rho_2$$

Half-Planes.


y
UHP
Upper Half-Plne
z = x+ iy, y>0
x
LHP
Lower Half-Plane
z = x+iy, y<0


y
L HP
Left
Half-Plane
z = x+iy,
x>0
x
z = x+iy,
x<0
RHP
Right
Half-Plane

For Reference: Concepts on Sets in the Complex Plane

a **point set**: any sort of collection of finitely many or infinitely many points.

A set S is called **open** if every point of S has a neighborhood consisting entirely of points that belong to S. For example, the right half-plane $\text{Re } z = x > 0$.

A set S is called **closed** if its complement is open.
For example, the points on and inside the unit circle form a closed set ("closed unit disk") since its complement $|z| > 1$ is open.

A set S is called **connected** if any two of its points can be joined by a chain of finitely many straight-line segments all of whose points belong to S.

An open and connected set is called a **domain.**
Thus an open disk and an open annulus are domains.

13.3 Derivative. Analytic Function

The **complement** of a set S in the complex plane is the set of all points of the complex plane that ***do not belong*** to S.

A **boundary point** of a set S is a point every neighborhood of which contains both points that belong to S and points that do not belong to S.

For example, the boundary points of an annulus are the points on the two bounding circles.

Clearly, if a set S is open, then no boundary point belongs to S; if S is closed, then every boundary point belongs to S.

The set of all boundary points of a set S is called the **boundary** of S.

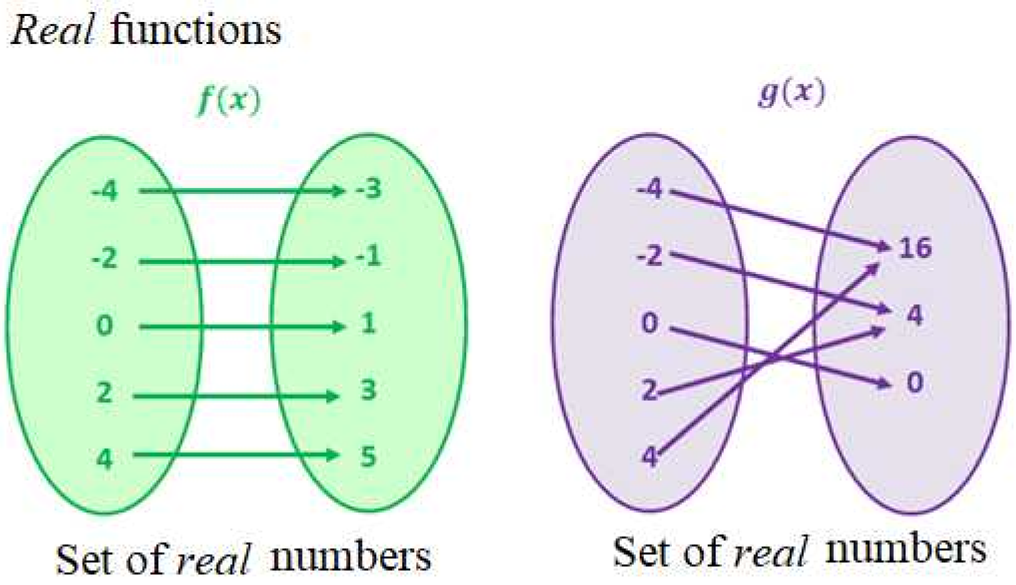
A **region** is a set consisting of a domain plus, perhaps, some or all of its boundary points.

WARNING! "Domain" is the *modern* term for an open connected set. Nevertheless, some authors still call a domain a "region" and others make no distinction between the two terms.

13.3 Derivative. Analytic Function

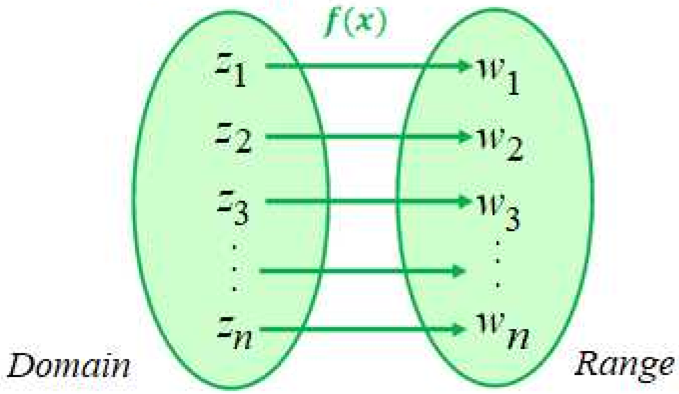
Complex Function

Recall from calculus that a *real* function f defined on a set S of real numbers (usually an interval) is a rule that assigns to every x in S a real number $f(x)$, called the *value* of f at x.

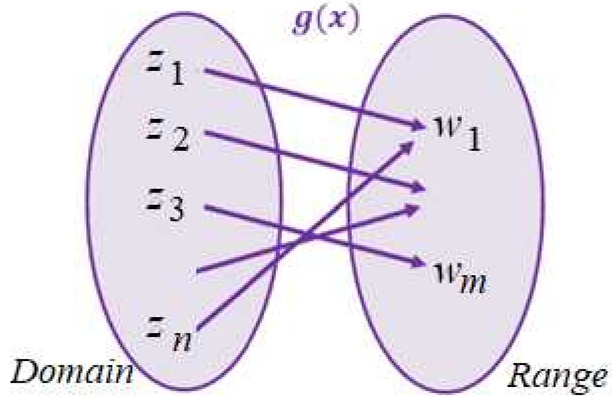


13.3 Derivative. Analytic Function

Complex functions



Set of *complex* numbers



Set of *complex* numbers

A **function** f defined on a complex S is a rule that assigns to every z in S a complex number w, called the *value* of f at z. We write

$$w = f(z).$$

Here z varies in S and is called a **complex variable**. The set S is called the *domain of definition* of f or, briefly, the **domain** of f. (In most cases S will be open and connected, thus a domain as defined just before.)

13.3 Derivative. Analytic Function

- Example: a complex function

  $w = f(z) = z^2 + 3z$ is a complex function defined for all z; that is, its domain S is the whole complex plane.

w depends on $z = x + iy$. Hence u becomes a real function of x and y, and so does v. We may thus write

$$w = f(z) = u(x, y) + iv(x, y).$$

EXAMPLE 1 **Function of a Complex Variable**

Let $w = f(z) = z^2 + 3z$. Find u and v and calculate the value of f at $z = 1 + 3i$.

Sol.

$$f(z) = z^2 + 3z = (x+iy)^2 + 3(x+iy) = (x^2 - y^2 + 3x) + i(2xy + 3y)$$

$$u = \operatorname{Re} f(z) = x^2 - y^2 + 3x, \quad v = 2xy + 3y$$

$$f(1+3i) = (1+3i)^2 + 3(1+3i) = 1 - 9 + 6i + 3 + 9i = -5 + 15i$$

13.3 Derivative. Analytic Function

EXAMPLE 2 Function of a Complex Variable

Let $w = f(z) = 2iz + 6\bar{z}$. Find u and v and the value of f at $z = \frac{1}{2} + 4i$.

Sol.

$$w = f(z) = 2iz + 6\bar{z} = 2i(x + iy) + 6(x - iy) = (6x - 2y) + i\,(2x - 6y)$$

$$\Rightarrow \quad \begin{aligned} u(x, y) &= 6x - 2y \\ v(x, y) &= 2x - 6y \end{aligned}$$

$$\begin{aligned} f(\tfrac{1}{2} + 4i) &= 2i(\tfrac{1}{2} + 4i) + 6(\tfrac{1}{2} - 4i) \\ &= i - 8 + 3 - 24i = -5 - 23i. \end{aligned}$$

Remarks on Notation and Terminology

1. Strictly speaking, $f(z)$ denotes the value of f at z, but it is a convenient abuse of language to talk about *the function* $f(z)$ (instead of *the function* f), thereby exhibiting the notation for the independent variable.

2. We assume all functions to be ***single-valued relations***, as usual: to each z in S there corresponds but *one* value $w = f(z)$ (but, of course, several z may give the same value $w = f(z)$ just as in calculus). Accordingly, we shall *not use* the term "multivalued function" (used in some books on complex analysis) for a multivalued relation, in which to a z there corresponds more than one w.

13.3 Derivative. Analytic Function

Limit, Continuity

A function $f(z)$ is said to have the **limit** l as z approaches a point z_0, written

$$\lim_{z \to z_0} f(z) = l, \tag{1}$$

if f is defined in a neighborhood of z_0(except perhaps at z_0 itself) and if the values of f are "close" to l for all z "close" to z_0;

in precise terms, if for every positive real ϵ we can find a positive real δ such that for all $z \neq z_0$ in the disk (Fig. 333) we have

$$|f(z) - l| < \epsilon; \tag{2}$$

geometrically, if for every $z \neq z_0$ in that δ-disk the value of f lies in the disk (2).

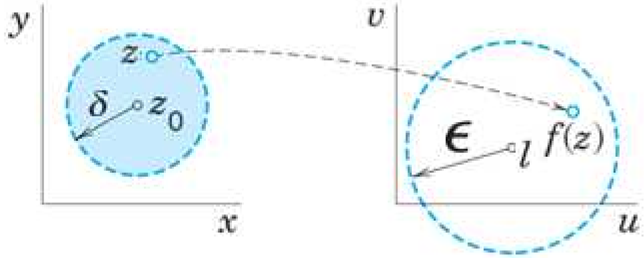


Fig. 333. Limit

13.3 Derivative. Analytic Function

Derivative

The **derivative** of a complex function f at a point z_0 is written $f'(z_0)$ and is defined by

(4) $$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided this limit exists. Then f is said to be **differentiable** at z_0.

If we write $\Delta z = z - z_0$, we have $z = z_0 + \Delta z$ and (4) takes the form

(4′) $$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Differentiability at z_0 means that, along whatever path z approaches z_0, the quotient in (4') always approaches a certain value and all these values are equal. This is important and should be kept in mind.

EXAMPLE 3 Differentiability. Derivative

Show that the function $f(z) = z^2$ is differentiable for all z and determine the derivative.

Sol.

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{2z\Delta z + (\Delta z)^2}{\Delta z} = 2z$$

13.3 Derivative. Analytic Function

Differentiation Rules

- *the same as in real calculus,* since their proofs are literally the same.
- Thus for any differentiable functions f and g and constant c we have

$$(cf)' = cf', \qquad (f+g)' = f' + g'$$

$$(fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

- the chain rule and the power rule

$$(z^n)' = nz^{n-1} \ (n \text{ integer}).$$

- Also, if $f(z)$ is differentiable at z_0, it is continuous at z_0. (See Team Project 24.)

EXAMPLE 4 $\bar{z}$ not Differentiable

Show that $f(z) = \bar{z} = x - iy$ is not differentiable.

Sol.

$$(5)\qquad \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z}$$

$$= \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y} = \begin{cases} 1 & \Delta y = 0 \\ -1 & \Delta x = 0 \end{cases}$$

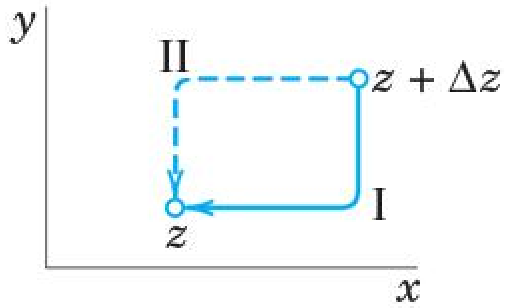


Fig. 334. Paths in (5)

Thus, the function is not differentiable at any point.

13.3 Derivative. Analytic Function

Analytic Functions

DEFINITION

Analyticity

A function $f(z)$ is said to be *analytic in a domain D* if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be *analytic at a point* $z = z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0.

Also, by an **analytic function** we mean a function that is analytic in *some* domain.

A more modern term for *analytic in D* is *holomorphic in D*.

13.3 Derivative. Analytic Function

EXAMPLE 5 Polynomials, Rational Functions

The nonnegative integer powers 1, z, z^2,⋯ are analytic in the entire complex plane, and so are **polynomials**, that is, functions of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$$

where c_0, , ⋯ c_n are complex constants.

The quotient of two polynomials $g(z)$ and $h(z)$

$$f(z) = \frac{g(z)}{h(z)},$$

is called a **rational function**. This f is analytic except at the points where $h(z) = 0$; here we assume that common factors of g and h have been canceled.

Problems in Sec 13.3

18–23 **Differentiation.** Find the value of the derivative of

18. $(z - i)/(z + i)$ at i

$$\left(\frac{z-i}{z+i}\right)' = \frac{(z+i)-(z-i)}{(z+i)^2} = \left.\frac{2i}{(z+i)^2}\right|_{z=i} = -\frac{i}{2}$$

19. $(z - 4i)^8$ at $z = 3 + 4i$

$$[(z - 4i)^8]' = 8(z - 4i)^7\Big|_{z=3+4i} = 8 \cdot 3^7$$

20. $(1.5z + 2i)/(3iz - 4)$ at any z. Explain the result.

$$\left(\frac{1.5z+2i}{3iz-4}\right)' = \left(\frac{1.5z+2i}{2i(1.5z+2i)}\right)' = 0$$

Problems in Sec 13.3

21. $i(1-z)^n$ at 0

$$\left[i(1-z)^n\right]' = -in(1-z)^{n-1}\Big|_{z=0} = -in$$

22. $(iz^3 + 3z^2)^3$ at $2i$

$$\left[\left(iz^3+3z^2\right)^3\right]' = 3\left(iz^3+3z^2\right)^2\left(3iz^2+6z\right)\Big|_{z=2i} = 0$$

23. $z^3/(z+i)^3$ at i

$$\left[\frac{z^3}{(z+i)^3}\right]' = \frac{3z^2(z+i)^3 - z^3 3(z+i)^2}{(z+i)^6} = \frac{3iz^2}{(z+i)^4}\Bigg|_{z=i} = -\frac{3i}{4}$$

13.4 Cauchy-Riemann Equations. Laplace's Equation

13.4 Cauchy-Riemann Equations. Laplace's Equation

The Cauchy–Riemann equations are the most important equations in this chapter and one of the pillars on which complex analysis rests. They provide a criterion (a test) for the analyticity of a complex function

$$w = f(z) = u(x, y) + iv(x, y).$$

Roughly, f is analytic in a domain D if and only if the first partial derivatives of u and satisfy the two **Cauchy–Riemann equations**

$$\textbf{(1)} \qquad u_x = v_y, \qquad u_y = -v_x$$

everywhere in D; here $u_x = \partial u / \partial x$ and $u_y = \partial u / \partial y$ (and similarly for v) are the usual notations for partial derivatives.

13.4 Cauchy-Riemann Equations. Laplace's Equation

THEOREM 1 Cauchy–Riemann Equations

Let $f(z) = u(x, y) + i\,v(x, y)$ be defined and continuous in some neighborhood of a point $z = x+iy$ and differentiable at z itself.

Then, at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy–Riemann equations (1).

Hence, if $f(z)$ is analytic in a domain D, those partial derivatives exist and satisfy (1) *at all points of D.*

(1) $$u_x = v_y, \quad u_y = -v_x$$ **: Cauchy–Riemann Equations**

13.4 Cauchy-Riemann Equations. Laplace's Equation

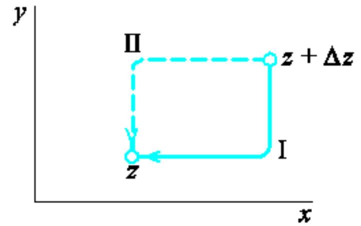
PROOF

$$(2) \quad f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$(3) \quad = \lim_{\Delta z \to 0} \left[\frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)]}{\Delta z} - \frac{[u(x,y) + iv(x,y)]}{\Delta z} \right]$$

$\Delta z \to 0$ *means*

I. $\Delta y \to 0$ *first and* $\Delta x \to 0$, *or*
II. $\Delta x \to 0$ *first and* $\Delta y \to 0$



Case I.

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y) + iv(x+\Delta x, y)] - [u(x,y) + iv(x,y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{[v(x+\Delta x, y) - v(x,y)]}{\Delta x}$$

$$(4) \quad = u_x + iv_x$$

13.4 Cauchy-Riemann Equations. Laplace's Equation

Case II.

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x, y+\Delta y) + iv(x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x}$$

$$= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\frac{[v(x, y+\Delta y) - v(x,y)]}{i\Delta y}$$

$$(5) \quad = -iu_y + v_y$$

$$\begin{cases} (4) \quad f'(z) = u_x + iv_x \\ (5) \quad f'(z) = -iu_y + v_y \end{cases}$$

For differentiability, equations (4) and (5):

$$\boldsymbol{u}_x = \boldsymbol{v}_y, \quad \boldsymbol{u}_y = -\boldsymbol{v}_x$$: Cauchy-Riemann Equations

Ex. 1 Cauchy-Riemann Equations

Show that

(a) $f(z) = z^2$ is analytic for all z.

(b) $f(z) = \overline{z}$ is not analytic for all z.

Sol.

(a) $f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + i2xy$

$$u_x = 2x, \quad v_y = 2x, \qquad \therefore u_x = v_y$$
$$u_y = -2y, \quad v_x = 2y, \qquad \therefore u_y = -v_x$$

(b) $f(z) = \overline{z} = x + i(-y)$

$$u_x = 1, \quad v_y = -1, \qquad \therefore u_x \neq v_y$$
$$u_y = 0, \quad v_x = 0, \qquad \therefore u_y = -v_x$$

13.4 Cauchy-Riemann Equations. Laplace's Equation

THEOREM 2 Cauchy-Riemann Equations

If $u(x,y)$ and $v(x,y)$ satisfy the two Cauchy-Riemann equations in some domain D, then $f(z) = u(x,y) + iv(x,y)$ is analytic in D.

Ex. 2 Cauchy-Riemann Equations, Exponential Function

Show that $f(z) = u(x, y) + i\, v(x, y) = e^x (\cos y + i \sin y)$ **is analytic.**

Sol.

$$u_x = e^x \cos y, \quad v_y = e^x \cos y, \quad \therefore u_x = v_y$$

$$u_y = -e^x \sin y, \quad v_x = e^x \sin y, \quad \therefore u_y = -v_x$$

$f(z)$ is analytic for all z since $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann equations.

EX 3 An Analytic Function of Constant Absolute Value Is Constant

The Cauchy–Riemann equations also help in deriving general properties of analytic functions.

For instance, show that if $f(z)$ is analytic in a domain D and $|f(z)| = k =$ const in D, then $f(z) =$ const in D. (We shall make crucial use of this in Sec. 18.6 in the proof of Theorem 3.)

Sol.

By assumption, $|f|^2 = |u + iv|^2 = u^2 + v^2 = k^2$.

By differentiation,

$$\begin{aligned} uu_x + vv_x &= 0, \\ uu_y + vv_y &= 0. \end{aligned} \xrightarrow[v_y = u_x]{v_x = -u_y} (6) \begin{aligned} &\text{(a)} \quad uu_x - vu_y = 0, \\ &\text{(b)} \quad uu_y + vu_x = 0. \end{aligned}$$

$$(6a)\times u + (6b)\times v: \quad (u^2 + v^2)u_x = 0,$$

$$(6a)\times(-v) + (6b)\times u: \quad (u^2 + v^2)u_y = 0.$$

If $k^2 = u^2 + v^2 = 0$, then $u = v = 0$; hence $f = 0$.
If $k^2 = u^2 + v^2 \neq 0$, then $u_x = u_y = 0$.
Hence, by the Cauchy–Riemann equations, also $v_x = {}_y = 0$. ■

Cauchy-Riemann Equations for the Polar Coordinates

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$
$$f(z) = u(r,\theta) + iv(r,\theta)$$

$$u_r = \frac{1}{r} v_\theta$$
$$(7) \qquad \qquad (r > 0)$$
$$v_r = -\frac{1}{r} u_\theta$$

THEOREM 3

Laplace's Equation

If $f(z) = u(x, y) + iv(x, y)$ *is analytic in a domain* D, *then both* u *and* v *satisfy* **Laplace's equation**

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \tag{8}$$

(∇^2 read "nabla squared") *and*

$$\nabla^2 v = v_{xx} + v_{yy} = 0, \tag{9}$$

in D *and have continuous second partial derivatives in* D.

13.4 Cauchy-Riemann Equations. Laplace's Equation

PROOF

$$\mathit{Cauchy-Riemann\ \ Equations:}$$

$$\begin{cases} u_x = v_y \xrightarrow{\partial/\partial x} u_{xx} = v_{yx} = v_{xy} = -u_{yy} \\ \quad \therefore u_{xx} + u_{yy} = 0 \\ \\ u_x = v_y \xrightarrow{\partial/\partial y} u_{xy} = -v_{xy} = v_{yy},\ u_{xy} = u_{yx} = -v_{xx} \\ \therefore v_{xx} + v_{yy} = 0 \end{cases}$$

13.4 Cauchy-Riemann Equations. Laplace's Equation

Harmonic Functions and Conjugate Functions

Solutions of a Laplace equation are called as **harmonic functions**.

Let u and v be the real part and the imaginary part of an analytic function.

Then v is called as the **harmonic conjugate function** of u and vice versa.

EXAMPLE 4 How to Find a Harmonic Conjugate Function by the Cauchy–Riemann Equations

Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a harmonic conjugate function v of u.

Sol.

$$\nabla^2 u = u_{xx} + u_{yy} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-2y-1) = 2-2 = 0$$

Hence because of the Cauchy–Riemann equations a conjugate v of u must satisfy

$$\begin{cases} v_y = u_x = 2x, \\ v_x = -u_y = 2y + 1. \end{cases} \quad \Longrightarrow \quad v = \int 2x\, dy = 2xy + h(x), \quad v_x = 2y + \frac{dh}{dx}$$

$$v_x = 2y + \frac{dh}{dx} = 2y + 1 \quad \Longrightarrow \quad h(x) = x + c$$

Hence $v = 2xy + x + c$ (c any real constant)

The corresponding analytic function is

$$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c) = z^2 + iz + ic. \qquad \blacksquare$$

Problems in Sec 13.4

2–11 CAUCHY–RIEMANN EQUATIONS

Are the following functions analytic? Use (1) or (7).

2. $f(z) = iz\bar{z}$

$$f(z) = iz\bar{z} = i(x^2 + y^2)$$

$$u = 0, \quad v = x^2 + y^2$$

$$u_x = 0 \neq 2y = v_y$$ Not analytic

Problems in Sec 13.4

3. $f(z) = e^{-2x}(\cos 2y - i \sin 2y)$

$$u = e^{-2x}\cos 2y, \quad v = -e^{-2x}\sin 2y$$

$$u_x = -2\,e^{-2x}\cos 2y = v_y\,,$$

$$u_y = -2\,e^{-2x}\sin 2y = -v_x$$

Thus, analytic.

4. $f(z) = e^{x}(\cos y - i \sin y)$

$$u = e^{x}\cos y\,, \quad v = -e^{x}\sin y$$

$$u_x = e^{x}\cos y \neq -e^{x}\cos y = v_y$$

Not analytic

Problems in Sec 13.4

5. $f(z) = \text{Re}\,(z^2) - i\,\text{Im}\,(z^2)$

$$f(z) = \text{Re}\,(z^2) - i\,\text{Im}\,(z^2) = \text{Re}\left[\left(x^2 - y^2\right) + 2xyi\right] - i\,\text{Im}\left[\left(x^2 - y^2\right) + 2xyi\right]$$
$$= \left(x^2 - y^2\right) - i\,2xy$$

$$u = \text{Re}\left(z^2\right) = x^2 - y^2\,, \quad v = -\text{Im}\left(z^2\right) = -2xy$$

$$u_x = 2x \neq -2x = v_y$$ Not analytic

6. $f(z) = 1/(z - z^5)$

Analytic except at z-$z^5 = 0$

z-$z^5 = z(1$-$z^4) = z(1$-$z^2)(1+z^2) = z(1$-$z)(1+z)(1$-$iz)(1+iz) = 0.$

$z = 0,\ 1,\ -1,\ i,\ -i$

Problems in Sec 13.4

7. $f(z) = i/z^8$

$$f(z) = i/z^8 = i/[r^8 (\cos 8\theta + i \sin 8\theta)] = i[\, r^{-8} (\cos 8\theta - i \sin 8\theta)]$$

$$u = r^{-8} \sin 8\theta\,, \quad v = r^{-8} \cos 8\theta$$

$$u_r = -8r^{-7} \sin 8\theta = \frac{1}{r} v_\theta, \quad v_r = -8r^{-7} \cos 8\theta = -\frac{1}{r} u_\theta$$

Thus, analytic except at r=0 or z=0.

8. $f(z) = \text{Arg } 2\pi z$

$$f(z) = \text{Arg } 2\pi z = \theta$$

$$u = \theta\,, \quad v = 0$$

$$v_r = 0 \neq -\frac{1}{r} = -\frac{1}{r} u_\theta$$

Not analytic

Problems in Sec 13.4

9. $f(z) = 3\pi^2/(z^3 + 4\pi^2 z)$

Analytic except at $z^3 + 4\pi^2 z = 0$

$z^3 + 4\pi^2 z = z\,(z^2 + 4\pi^2) = z\,(z + 2\pi i)(z - 2\pi i) = 0$

$z = 0, \pm 2\pi i$

10. $f(z) = \ln|z| + i \operatorname{Arg} z$

$f(z) = \ln|z| + i \operatorname{Arg} z = \ln r + i\,\theta$

$u = \ln r\,, \quad v = \theta$

$u_r = \dfrac{1}{r} = \dfrac{1}{r} v_\theta\,, \quad v_r = 0 = -\dfrac{1}{r} u_\theta$

Thus, analytic

Problems in Sec 13.4

11. $f(z) = \cos x \cosh y - i \sin x \sinh y$

$$u = \cos x \cosh y, \qquad v = -\sin x \sinh y$$

$$u_x = -\sin x \cosh y = v_y, \quad u_y = \cos x \sinh y = -v_x$$

Thus, analytic