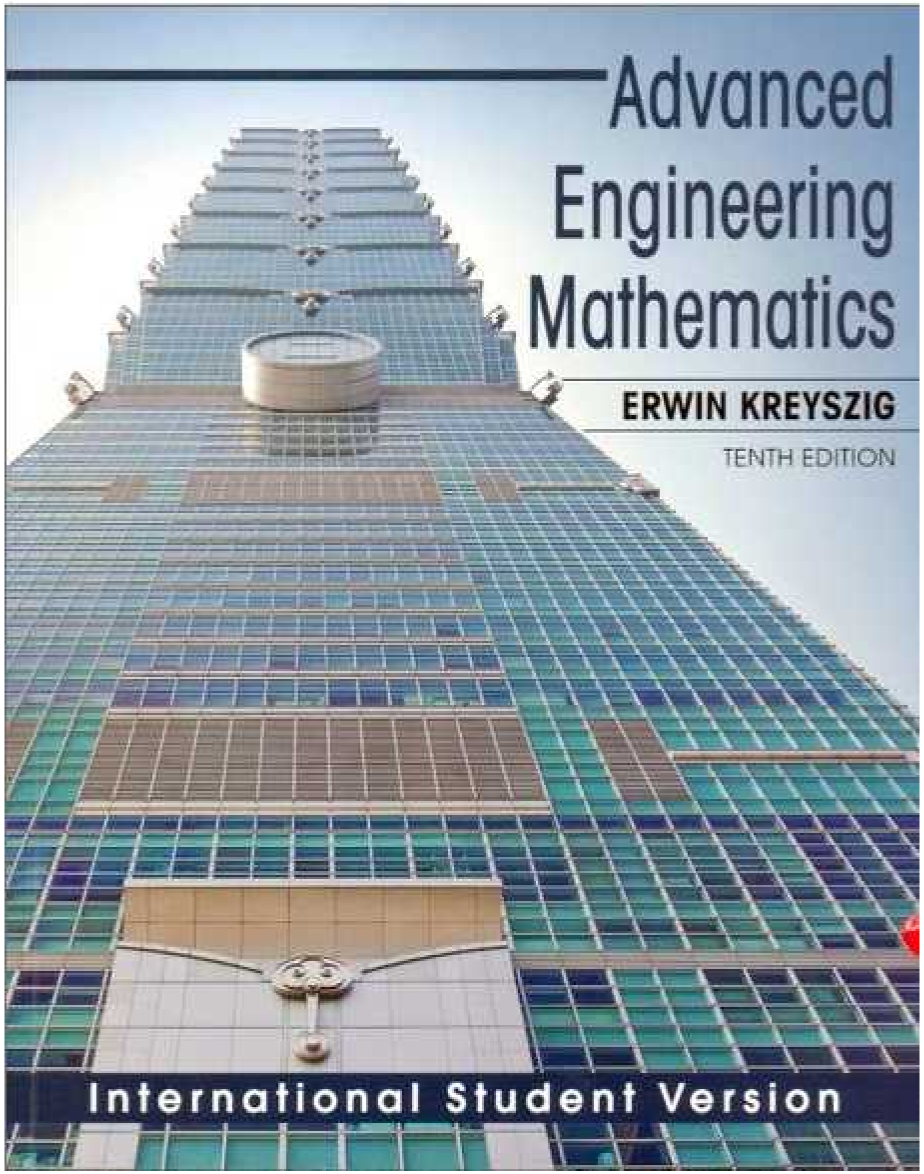


CHAPTER 14

Complex Integration

Ch. 14 Complex Integration (복소적분)

■ Importance of Complex Integration

- Practical reason: Complex integration can evaluate certain real integrals that appear in applications that are not accessible by real integral calculus
- Theoretical reason: Some basic properties of analytic functions are easily proved otherwise difficult to prove.

14.1 Line Integral in the Complex Plane (복소평면에서의 선적분)

Complex definite integrals are called (complex) line integrals.

$$\int_C f(z)dz$$

Path of Integration: C

$$(1) \quad z(t) = x(t) + iy(t) \qquad (a \le t \le b)$$

Positive sense: the sense of increasing t
Negative sense: the sense of decreasing t

Assume C is a smooth curve.

14.1 Line Integral in the Complex Plane

Smooth curve:

C has a continuous and non-zero derivative at each point.

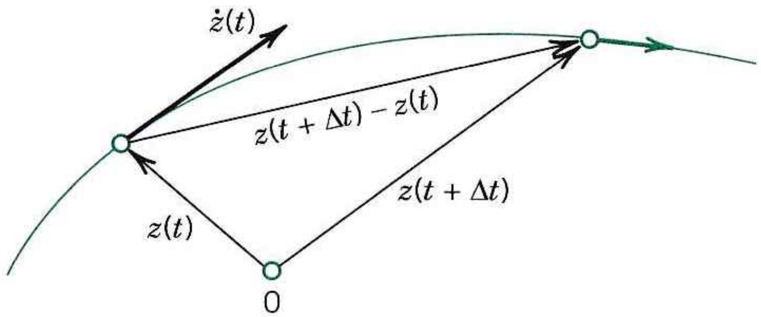
$$\dot{z}(t) = \frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{z(t+\Delta t) - z(t)}{\Delta t}$$

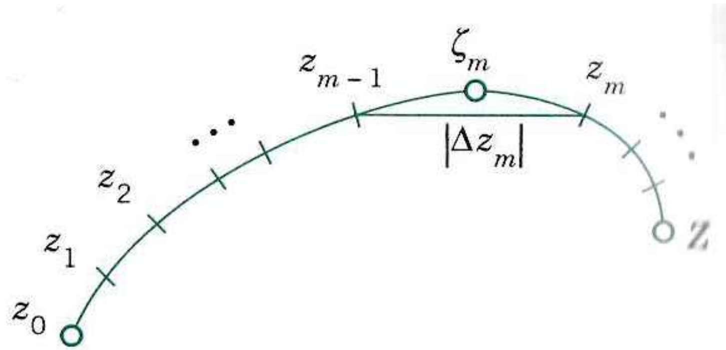
$$= \dot{x}(t) + i\dot{y}(t)$$

$\cdot\ means\ \ d/dt,\ '\ means\ \ d/dz.$

14.1 Line Integral in the Complex Plane

Definition of a Complex Line Integral





$$(2) \qquad S_n = \sum_{m=1}^{n} f(\zeta_m)\Delta z_m \quad where \;\; \Delta z_m = z_m - z_{m-1}$$

$$(3) \qquad \int_C f(z)dz \quad or \quad by \quad \oint_C f(z)dz$$

$f(z)$ *is assumed to be piecewise smooth.*

14.1 Line Integral in the Complex Plane

Basic Properties

1. Linearity

$$(4) \qquad \int_C [k_1 f_1(z) + k_2 f_2(z)]\,dz = k_1 \int_C f_1(z)\,dz + k_2 \int_C f_2(z)\,dz$$

2. Sense reversal

$$(5) \qquad \int_{z_0}^{Z} f(z)\,dz = -\int_{Z}^{z_0} f(z)\,dz$$

3. Partitioning of path

$$(6) \qquad \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

14.1 Line Integral in the Complex Plane

Existence of the Complex Line Integral

$$\int_C f(z)\,dz$$

exists if

1. C is smooth
2. f(z) is piecewise continuous.

14.1 Line Integral in the Complex Plane

First Evaluation Method

THEOREM 1 Indefinite Integration of Analytic Functions

$$\int_{z_0}^{z_1} f(z)\,dz = F(z)\Big|_{z_0}^{z_1} = F(z_1) - F(z_0) \tag{9}$$

$$\text{where } F'(z) = f(z)$$

14.1 Line Integral in the Complex Plane

EX 1 :

$$\int_0^{1+i} z^2 dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{(1+i)^3}{3} = -\frac{2}{3} + \frac{2}{3}i$$

EX 2 :

$$\int_{-\pi i}^{\pi i} \cos z dz = \sin z\Big|_{-\pi i}^{\pi i} = 2\sin \pi i = 2i \sinh \pi = 23.097i$$

EX 3 :

$$\int_{8+\pi i}^{8-3\pi i} e^{z/2} dz = 2e^{z/2}\Big|_{8+\pi i}^{8-3\pi i} = 2(e^{4-3\pi i/2} - e^{4+\pi i/2}) = 0$$

Since e^z *is periodic with period* 2π.

EX 4 :

$$\int_{-i}^{i} \frac{dz}{z} = \operatorname{Ln} z\Big|_{-i}^{i} = \operatorname{Ln}(i) - \operatorname{Ln}(-i) = \frac{i\pi}{2} - \frac{-i\pi}{2} = i\pi$$

14.1 Line Integral in the Complex Plane

Second Evaluation Method

THEOREM 2 Integration by the Use of the Path

$$(10) \quad \int_C f(z)\,dz = \int_a^b f[z(t)]\dot{z}(t)\,dt \qquad \left(\dot{z} = \frac{dz}{dt}\right)$$

$$C: \quad z(t), \quad a \le t \le b$$

Steps in Applying Theorem 2

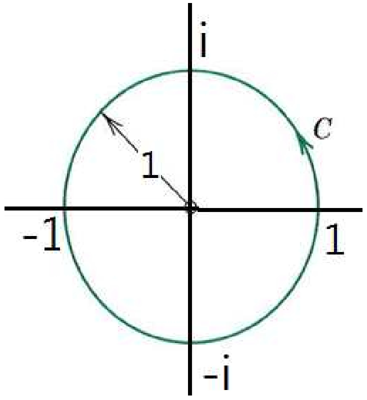
(A) Represent the path C in the form z(t) (a≤t≤b)
(B) Calculate the derivative dz/dt.
(C) Substitute z(t) for every z.
(D) Integrate: $\int_a^b f[z(t)]\dot{z}(t)\,dt$

14.1 Line Integral in the Complex Plane

PROOF

14.1 Line Integral in the Complex Plane

EX 5 A Basic Result: Integral of 1/z Around the Unit Circle



Show the following:

$$(11) \quad \oint_C \frac{dz}{z} = 2\pi i \quad (Unit\ \ Circle,\ \ CCW)$$

Sol.

(A) Represent the path C in the form z(t) (a≤t≤b)

$$z(t) = \cos t + i \sin t = e^{it} \quad (0 \le t \le 2\pi)$$

(B) Calculate the derivative dz/dt.

$$\dot{z}(t) = ie^{it}$$

14.1 Line Integral in the Complex Plane

(C) Substitute z(t) for every z.

$$f(z) = 1/z = 1/e^{it} = e^{-it}$$

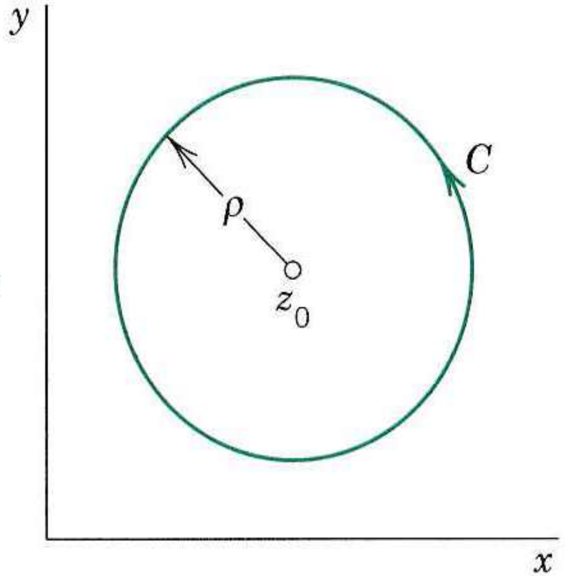
(D) Integrate:

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} e^{-it} \cdot ie^{it} dt = 2\pi i$$

14.1 Line Integral in the Complex Plane

EX 6 Integral of 1/z ^{m} with Integer Power m

$$\oint_C (z-z_0)^m dz$$



Sol.

$$z = z_0 + \rho(\cos t + i \sin t) = z_0 + \rho e^{it}$$

$$(z-z_0)^m = (\rho e^{it})^m = \rho^m e^{imt}$$

$$dz = i\rho e^{it} dt, \quad (0 \le t \le 2\pi)$$

$$\oint_C (z-z_0)^m dz = \int_0^{2\pi} \rho^m e^{imt} \cdot i\rho e^{it} dt$$

$$= i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t} dt$$

14.1 Line Integral in the Complex Plane

$$\oint_C (z-z_0)^m dz = i\rho^{m+1}\int_0^{2\pi} e^{i(m+1)t}dt$$

$m=-1:$

$$\oint_C (z-z_0)^m dz = i\rho^{m+1}\int_0^{2\pi} e^{i(m+1)t}dt = 2\pi i$$

$m \neq -1 \ \ and \ \ integer:$

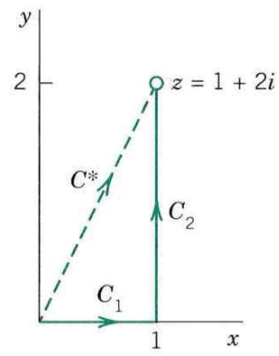
$$\oint_C (z-z_0)^m dz = i\rho^{m+1}\int_0^{2\pi} e^{i(m+1)t}dt$$

$$= i\rho^{m+1}\left[\frac{e^{i(m+1)t}}{i(m+1)}\right]_0^{2\pi} = 0$$

EX 7 Integral of a Nonanalytic Function. Dependence on Path

$$(a)\ \int_{C^*} Re z\, dz$$

$$(b)\ \int_{C_1 - C_2} Re z\, dz$$



Sol.

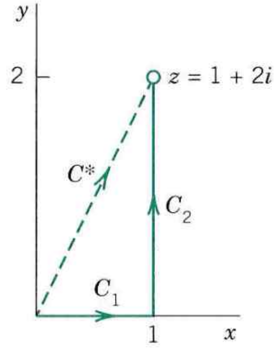
$$(a)\ \int_{C^*} Re z\, dz$$

$$\dot{z}(t) = \frac{d}{dt}(t + i2t) = 1 + i2 \qquad f[z(t)] = x = t$$

$$\int_{C^*} Re z\, dz = \int_0^1 Re z \frac{dz}{dt} dt = \int_0^1 t(1 + i2) dt$$

$$= (1 + 2i)\frac{t^2}{2}\Big|_0^1 = \frac{1}{2} + i$$

14.1 Line Integral in the Complex Plane



$$(b)\ \int_{C_1 - C_2} Re z\, dz$$

$$C_1:\ \ z(t) = t,\ \ \dot{z}(t) = 1,$$
$$f(z(t)) = x(t) = t\ \ (0 \le t \le 1)$$

$$C_2:\ \ z(t) = 1 + it,\ \ \dot{z}(t) = i,$$
$$f(z(t)) = x(t) = 1\ \ (0 \le t \le 2)$$

$$\int_{C_1 - C_2} Re z\, dz = \int_{C_1} Re z \frac{dz}{dt} dt + \int_{C_2} Re z \frac{dz}{dt} dt$$
$$= \int_0^1 t(1) dt + \int_0^2 1(i) dt = \frac{1}{2} + 2i$$

14.1 Line Integral in the Complex Plane

Bounds for Integrals. ML-Inequality

$$(13) \qquad \left| \int_C f(z)\, dz \right| \le ML \qquad (ML-inequality)$$

PROOF

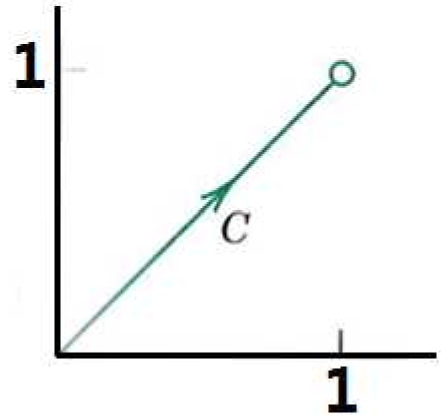
$$|S_n| = \left| \sum_{m=1}^{n} f(\zeta_m) \Delta z_m \right| \le \sum_{m=1}^{n} |f(\zeta_m)|\, |\Delta z_m| \le M \sum_{m=1}^{n} f(\zeta_m) |\Delta z_m|$$

$$\lim_{n\to\infty} |S_m| \le \lim_{n\to\infty} \left[M \sum_{m=1}^{n} f(\zeta_m) |\Delta z_m| \right] = ML$$

$$\left| \int_C f(z)\, dz \right| \le \int_C |f(z)| dz \le M \int_C dz = ML$$

EX 8 Estimation of an Integral

Find an upperbound of $\int_C z^2\, dz$



Sol.

$$M = \max|f(z)| = \max|z^2| = (\max|z|)^2 = (\sqrt{2})^2 = 2$$

$$L = \sqrt{2}$$

$$\left|\int_C z^2\, dz\right| \le ML = 2\sqrt{2}$$

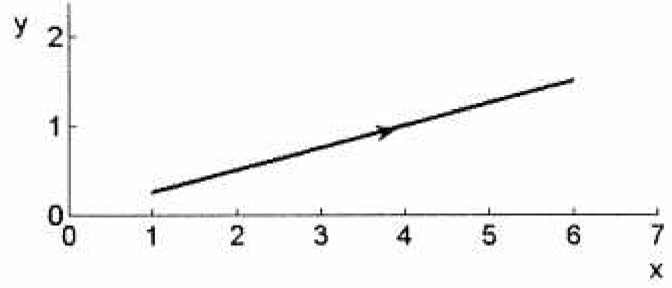
Problems for Sec. 14.1

1. FIND THE PATH and sketch it.

$$z(t) = (1 + \tfrac{1}{4}i)t \quad (1 \leqq t \leqq 6)$$

$= x + iy$

$x = t, \quad y = (1/4)\,t = (1/4)x$



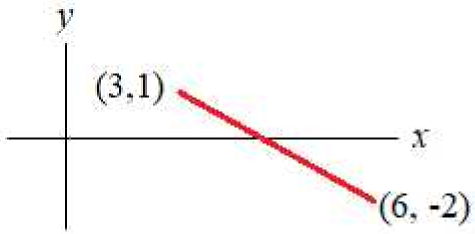
2. FIND THE PATH and sketch it.

$$z(t) = 3 + i + (1 - i)t \quad (0 \leqq t \leqq 3)$$

$= x + iy$

$x = 3 + t,$

$y = 1 - t = 1 - (x - 3) = -x + 4$

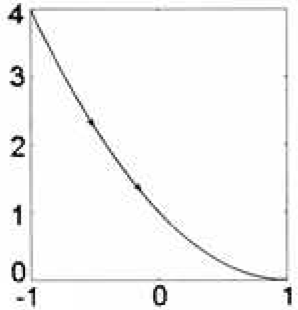


Problems for Sec. 14.1

4. FIND THE PATH and sketch it.

$$z(t) = t + (1-t)^2 i \quad (-1 \leqq t \leqq 1)$$
$$= x + iy$$

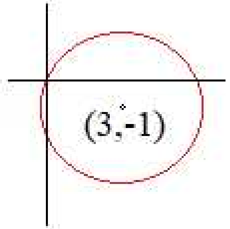
$$x = t,$$
$$y = (1-t)^2 = (1-x)^2$$



5. FIND THE PATH and sketch it.

$$z(t) = 3 - i + \sqrt{10}e^{-it} \quad (0 \leqq t \leqq 2\pi)$$
$$= x + iy$$

$$x = 3 + \sqrt{10}\cos t$$
$$y = -1 - \sqrt{10}\sin t$$
$$\cos^2 t + \sin^2 t = \left(\frac{x-3}{\sqrt{10}}\right)^2 + \left(\frac{y+1}{\sqrt{10}}\right)^2 = 1$$
$$(x-3)^2 + (y+1)^2 = 10$$



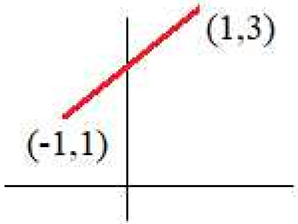
Problems for Sec. 14.1

11. **FIND A PARAMETRIC REPRESENTATION** and sketch the path.

Segment from $(-1, 1)$ to $(1, 3)$

$$\begin{aligned} z(t) &= x + iy \\ &= (-1,1)+(1-(-1),\, 3-1)t \\ &= (-1+2t,\ 1+2t) \quad 0 \le t \le 1 \end{aligned}$$

$x = -1+2t,\ \ y = 1+2t = 1+(1+x)=x+2$



16. **FIND A PARAMETRIC REPRESENTATION** and sketch the path.

Ellipse $4x^2 + 9y^2 = 36$, counterclockwise

$$(x/3)^2 + (y/2)^2 = 1$$

$$z(t) = x + iy = 3\cos t + i\, 2\sin t, \qquad 0 \le t \le \pi$$

Problems for Sec. 14.1

17. FIND A PARAMETRIC REPRESENTATION and sketch the path.

$|z + a + ib| = r$, clockwise

$|(x+iy)+a+ib|=r$

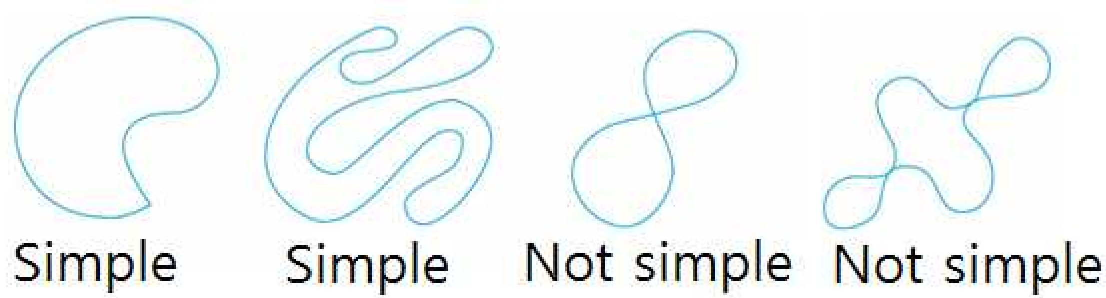
$(x+a)^2 + (y+b)^2 = r^2$

$x+a = r\cos(-t)$, $x = -a + r\cos t$

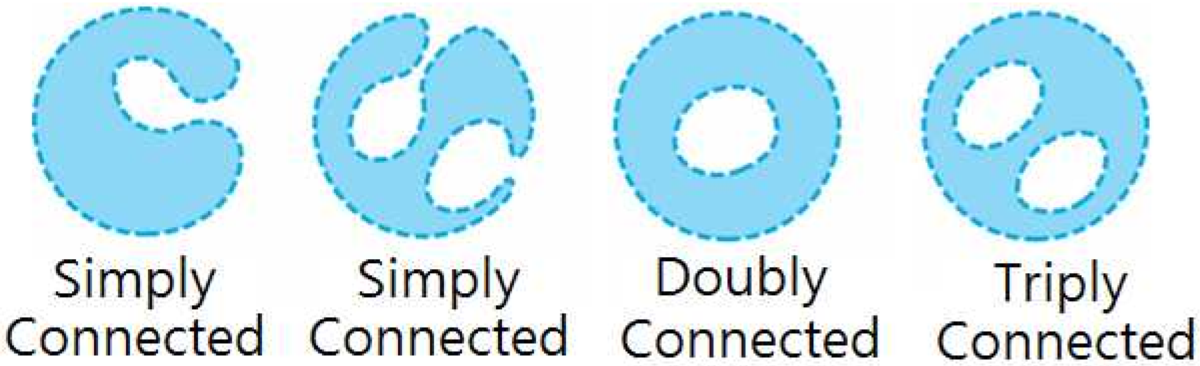
$y+b = r\sin(-t) = -r\sin t$, $y = -b - r\sin t$, $\quad 0 \le t \le 2\pi$

14.2 Cauchy's Integral Theorem (Cauchy의 적분정리)

1. A simple closed path



2. A simply connected domain D



14.2 Cauchy's Integral Theorem

p-fold connected domain if its boundary consists of p closed connected sets without common points.

3-fold connected domain

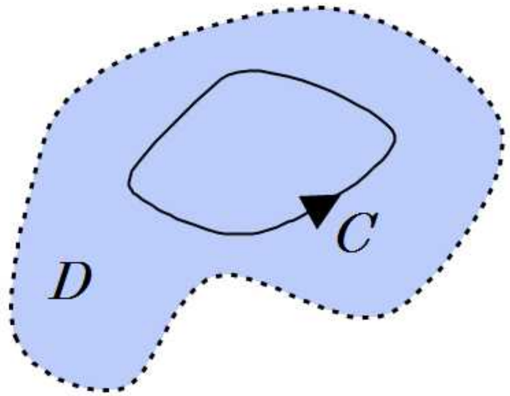
The boundary has 3 closed connected sets without common points.

14.2 Cauchy's Integral Theorem

THEOREM 1 Cauchy's Integral Theorem

If f(z) is analytic in a simply connected domain D.
then for every closed path C in D,

$$\oint_C f(z)\,dz = 0 \qquad (1)$$



A simple closed path is sometimes called a contour.

An integral over a simple closed path is a contour integral.

14.2 Cauchy's Integral Theorem

PROOF

14.2 Cauchy's Integral Theorem

EX 1. Entire Functions

An entire function is a function that is analytic for all z.

$$\oint_C e^z \, dz = 0$$

$$\oint_C \cos z \, dz = 0$$

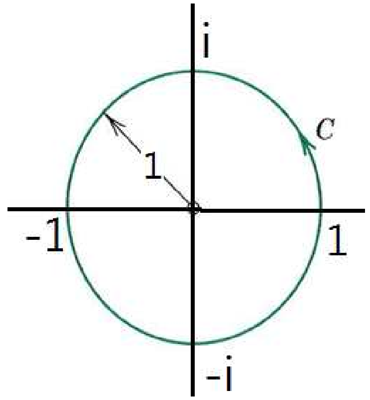
$$\oint_C z^n \, dz = 0, \quad (n = 0, 1, \cdots)$$

14.2 Cauchy's Integral Theorem

EX 2. Points outside the Contour Where f(x) is Not Analytic

$$\oint_C \sec z \, dz = 0,$$

$$\oint_C \frac{dz}{z^2+4} \, dz = 0$$



Sol.

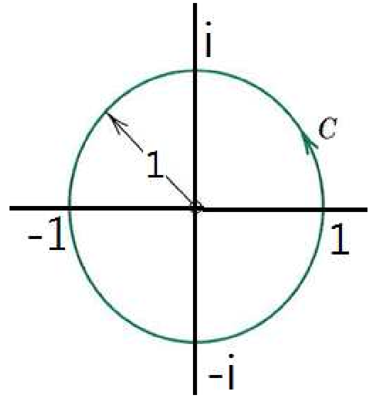
$$\cos z = (1/2)(e^{iz} + e^{-iz}) = 0$$
$$e^{2iz} + 1 = 0, \quad t^2 + 1 = 0$$
$$e^{iz} = e^{i(x+iy)} = e^{-y} e^{ix} = \pm i$$
$$e^{-y} = 1 \quad \therefore y = 0$$
$$e^{ix} = \cos x + i \sin x = \pm i, \quad x = \pm (n+0.5)\pi$$
$$z = \pm (n+0.5)\pi \quad (n = 0, 1, 2, \cdots)$$

D: a circle centered at the origin with 1<radius<0.5π.

14.2 Cauchy's Integral Theorem

EX 3. Nonanalytic Function

$$\oint_C \bar{z}\, dz = 2\pi i$$



Sol.

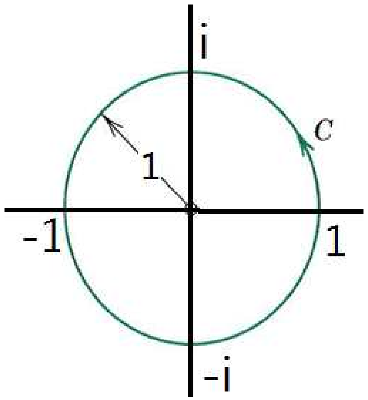
$$\oint_C \bar{z}\, dz = \int \bar{z}\, \frac{dz}{dt}\, dt$$

$$= \int_0^{2\pi} e^{-it} \cdot ie^{it} dt = 2\pi i$$

14.2 Cauchy's Integral Theorem

EX 4. Analyticity Sufficient, Not Necessary

$$\oint_C \frac{1}{z^2}\,dz = 0$$



Sol.

$$\oint_C \frac{1}{z^2}\,dz = \int \frac{1}{z^2}\frac{dz}{dt}\,dt = \int_0^{2\pi} e^{-2it}\cdot ie^{it}dt$$

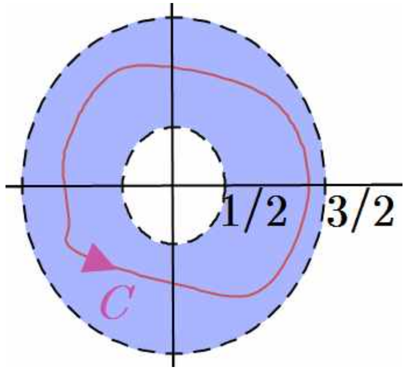
$$= \int_0^{2\pi} ie^{-it}dt = \left.\frac{ie^{-it}}{-i}\right|_0^{2\pi} = 0$$

Not analytic in D, but the closed integral is zero
Thus, the analyticity is sufficient rather than necessary for integral of zero.

14.2 Cauchy's Integral Theorem

EX 5. Simple Connectedness Essential

$$\oint_C \frac{dz}{z} = 2\pi i$$



Sol.

$$\oint_C \frac{dz}{z} = \int \frac{1}{z}\frac{dz}{dt}dt$$
$$= \int_0^{2\pi} e^{-it} \cdot ie^{it} dt = 2\pi i$$

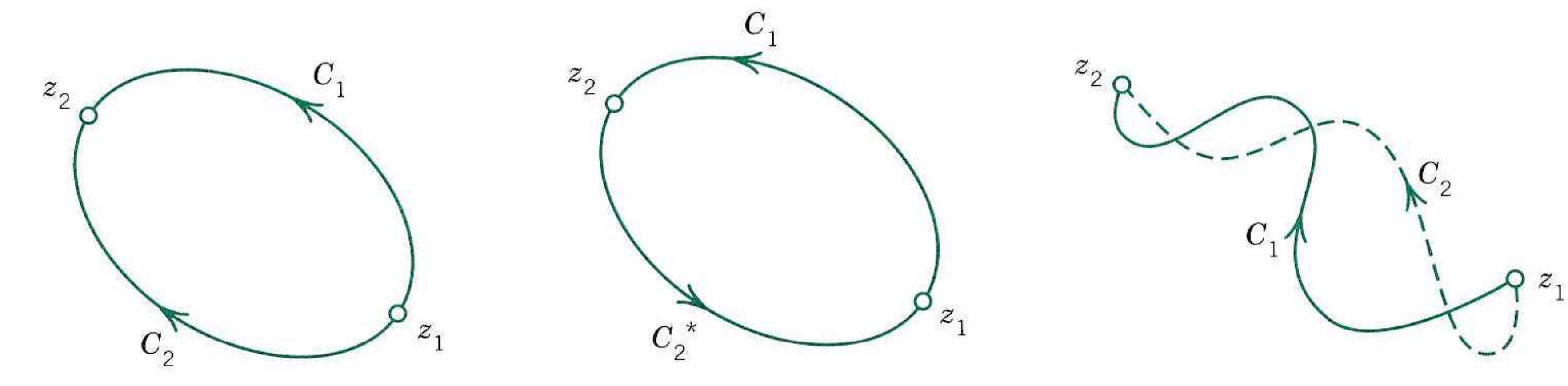
Analytic in D, but D is not simply connected.
Thus, the simple connectedness is necessary for integral of zero.

14.2 Cauchy's Integral Theorem

Independence of Path

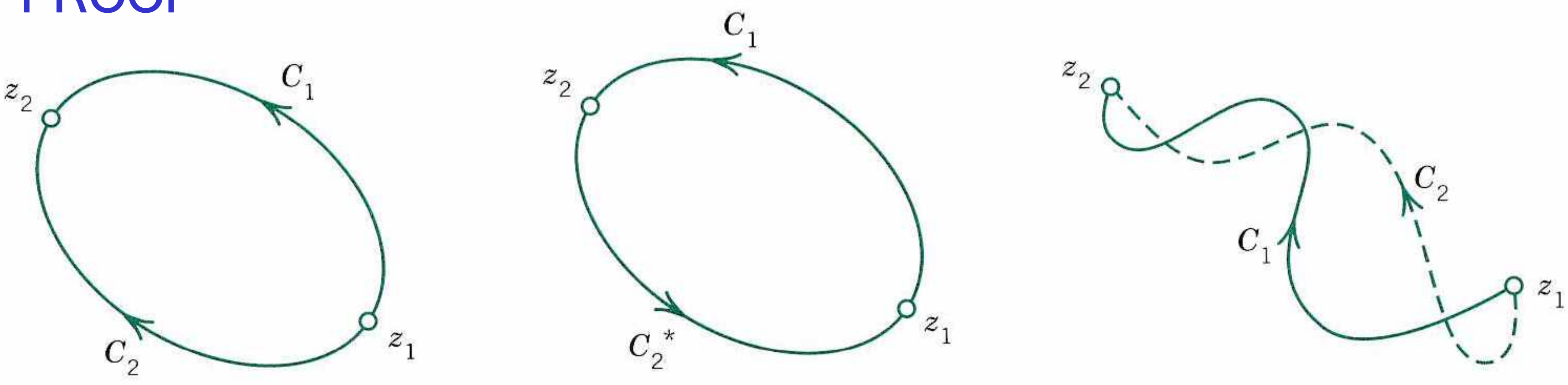
THEOREM 2 Independence of Path

If f(z) is analytic in a simply connected domain D, then the integral of f(x) is independent of path in D.



14.2 Cauchy's Integral Theorem

PROOF



$$\int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz = 0$$

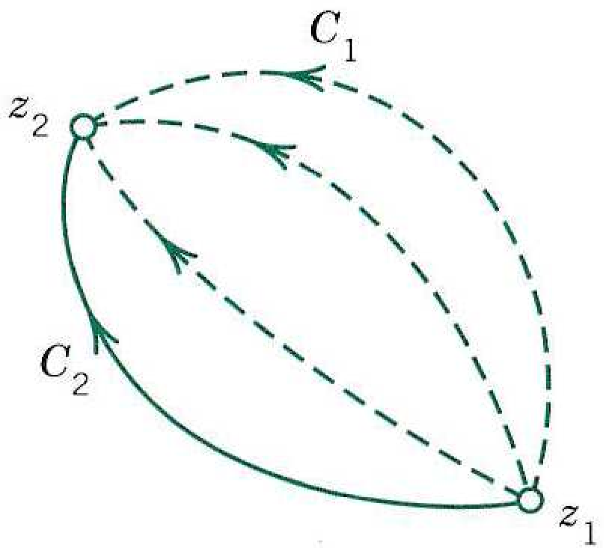
$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\therefore \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

14.2 Cauchy's Integral Theorem

Principle of Deformation of Path(경로변형의 원리)

Consider a deformation of a path of an integral, keeping the ends fixed. As long as the deforming path always contains only points at which f(z) is analytic, the integral retains the same value.

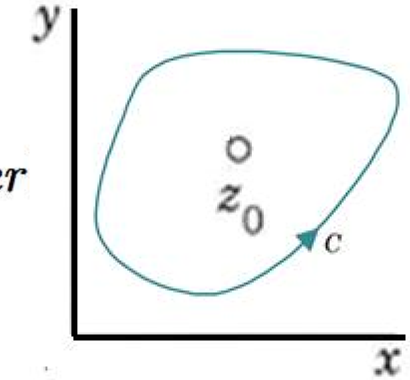


$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\therefore \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

14.2 Cauchy's Integral Theorem

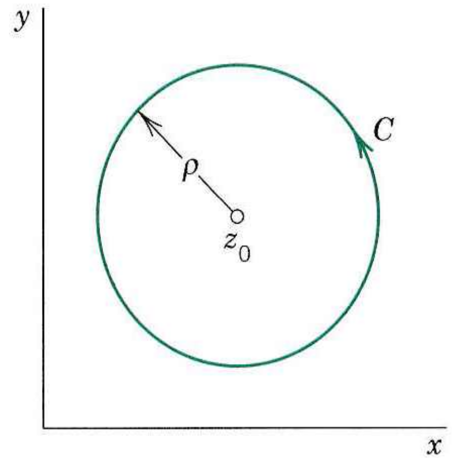
EX 6. A Basic Result: Integral of Integer Powers



$$(3) \qquad \oint_C (z-z_0)^m dz = \begin{cases} 2\pi i, & (m=-1) \\ 0, & (m \neq -1 \text{ and integer} \end{cases}$$

C: *any CCW simple closed path containing z_0 in its interior*

Sol.



$$z = z_0 + \rho(\cos t + i\sin t) = z_0 + \rho e^{it}$$

$$(z-z_0)^m = (\rho e^{it})^m = \rho^m e^{imt}$$

$$dz = i\rho e^{it} dt, \quad (0 \le t \le 2\pi)$$

$$\oint_C (z-z_0)^m dz = \int_0^{2\pi} \rho^m e^{imt} \cdot i\rho e^{it} dt$$

$$= i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t} dt$$

14.2 Cauchy's Integral Theorem

$$\oint_C (z-z_0)^m dz = i\rho^{m+1}\int_0^{2\pi} e^{i(m+1)t} dt$$

$m=-1:$

$$\oint_C (z-z_0)^m dz = i\rho^{m+1}\int_0^{2\pi} e^{i(m+1)t} dt = 2\pi i$$

$m \neq -1 \;\; and \;\; integer:$

$$\oint_C (z-z_0)^m dz = i\rho^{m+1}\int_0^{2\pi} e^{i(m+1)t} dt$$

$$= i\rho^{m+1}\left[\frac{e^{i(m+1)t}}{i(m+1)}\right]_0^{2\pi} = 0$$

14.2 Cauchy's Integral Theorem

Existence of Indefinite Integral

THEOREM 3 Existence of Indefinite Integral

If f(z) is analytic in a simply connected domain D, then

(1) there exists an indefinite integral of f(z)

(2) $\int_{z_0}^{z_1} f(z)dz$ can be evaluated by

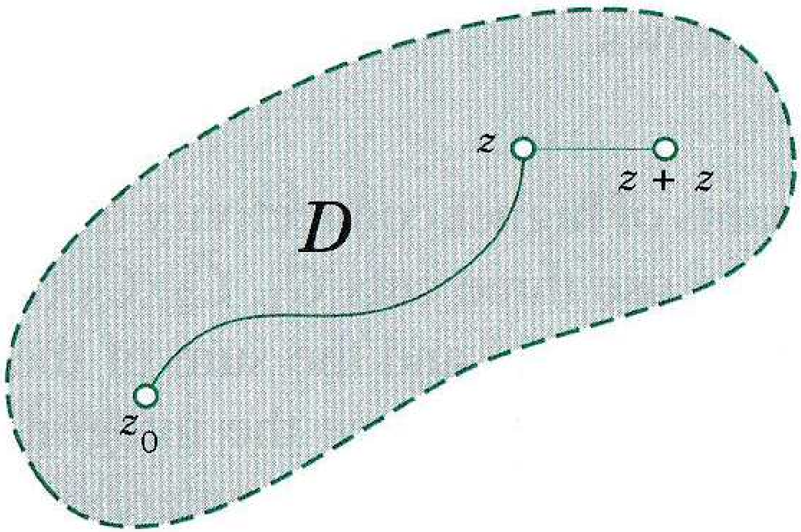
$$\int_{z_0}^{z_1} f(z)dz = F(z)|_{z_0}^{z_1} = F(z_1) - F(z_0)$$

$$where \;\; F'(z) = f(z)$$

Integral paths are any path in D joining any two points in D.

14.2 Cauchy's Integral Theorem

PROOF



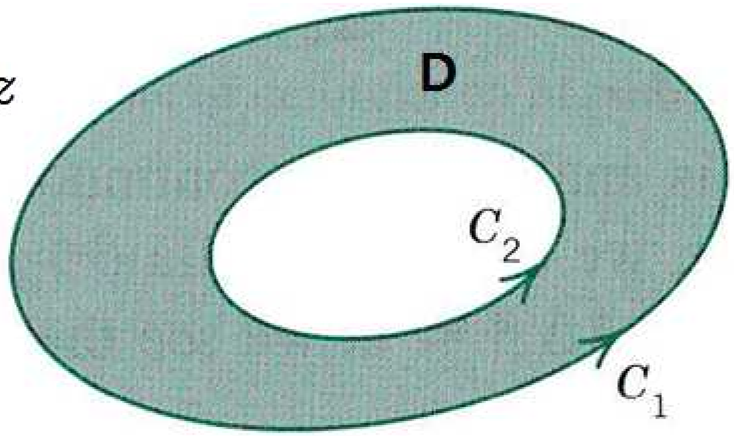
14.2 Cauchy's Integral Theorem

Cauchy's Integral Theorem for Multiply Connected Domains

Let D be a doubly connected domain with outer boundary curve C_1 and inner C_2.

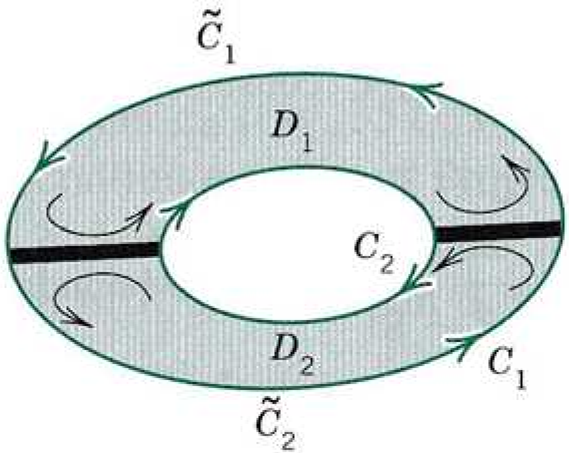
If f(z) is analytic in any domain D* that contains D and its boundary curves, then

$$(6) \qquad \oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$



14.2 Cauchy's Integral Theorem

PROOF



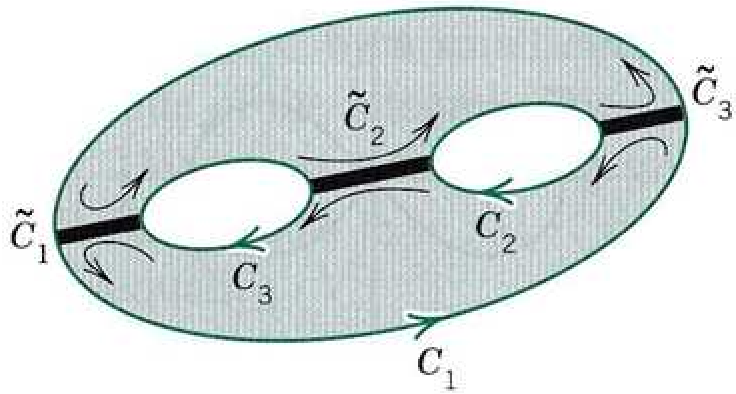
$$\oint_{\tilde{C}_1} f(z)\,dz + \int_{\tilde{C}_2} f(z)\,dz = 0$$

$$\oint_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = 0$$

$$\oint_{C_1} f(z)\,dz = -\int_{C_2} f(z)\,dz$$

14.2 Cauchy's Integral Theorem

Triply connected domain



$$\oint_{C_1} f(z)\,dz = -\int_{C_2} f(z)\,dz - \int_{C_3} f(z)\,dz$$

Problems for Sec 14.2

9. CAUCHY'S THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

$f(z) = \exp(-z^2)$

THEOREM 1 Cauchy's Integral Theorem

If f(z) is analytic in a simply connected domain D, then for every closed path C in D,

$$\oint_C f(z)\,dz = 0 \qquad (1)$$

$$\frac{d}{dz}\left(e^{-z^2}\right) = -2ze^{z^2}$$

Thus, $f(z)$ is analytic in the domain including the unit circle.

By Cauchy's Integral Theorem, $\oint_C e^{-z^2} dz = 0$

Problems for Sec 14.2

10. CAUCHY'S THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

$f(z) = \tan \frac{1}{4}z$

$$\frac{d}{dz}\left(\tan \frac{1}{4}z\right) = \frac{1}{4}\sec^2 \frac{1}{4}z$$

Thus, $f(z)$ is analytic in the domain including the unit circle.

By Cauchy's Integral Theorem, $\oint_C \tan \frac{1}{4} z\,dz = 0$

Problems for Sec 14.2

11. CAUCHY'S THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

$f(z) = 1/(2z - 1)$

$$f(z) = \frac{1}{2z-1} = \frac{1/2}{z-1/2}$$

$f(z)$ is not analytic at $z=1/2$.
Thus, Cauchy's Integral Theorem is not applicable.

$$\int_C f(z)dz = \frac{1}{2}\int_C \frac{1}{z-1/2}dz = \frac{1}{2} \cdot 2\pi i = \pi i$$

Problems for Sec 14.2

12. CAUCHY'S THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

$f(z) = \bar{z}^3$

$f(z)$ is not analytic. Thus, Cauchy's Intergral Theorem is not applicable.

$$f(z) = \bar{z}^3 = (x-iy)^3 = x^3 - 3x^2(iy) + 3x(iy)^2 - (iy)^3$$

$$= (x - 3xy^2) + i(-3x^2y + y^3)$$

$$u = x - 3xy^2, \quad v = -3x^2y + y^3$$

$$u_x = 1 - 3y^2 \neq v_y = -3x^2 + 3y^2 \quad \text{: Not analytic}$$

$$\int_C \overline{z^3}\,dz = \int_C \overline{z^3}\,\frac{dz}{dt}\,dt = \int_0^{2\pi} e^{-3it} ie^{it}\,dt = i\int_0^{2\pi} e^{-2it}\,dt = 0$$

C: $z = e^{it}$

Problems for Sec 14.2

13. CAUCHY'S THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

$f(z) = 1/(z^4 - 1.1)$

$z^4 = r^4 e^{4i\theta} = 1.1 = 1.1e^{i\,2n\pi}$ 에서 $r = (1.1)^{1/4} > 1$

Thus, $f(z)$ is analytic inside and on the unit circle.

By Cauchy's Integral Theorem, $\oint_C \frac{1}{z^4 - 1.1} dz = 0$

Problems for Sec 14.2

14. CAUCHY'S THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

$f(z) = 1/\bar{z}$

$f(z)$ is not analytic at $z=0$ which is inside the unit circle.
Thus, Cauchy's Integral Theorem is not applicable.

$$\int_C \frac{1}{\bar{z}}\,dz = \int_C \frac{1}{\bar{z}}\frac{dz}{dt}\,dt = \int_0^{2\pi} \frac{1}{e^{-it}}\, ie^{it}\,dt$$

$$C\colon z = e^{it}$$

$$= i\int_0^{2\pi} e^{2it}\,dt = i\,\frac{e^{2it}}{2i}\Big|_0^{2\pi} = 0$$

Problems for Sec 14.2

15. CAUCHY'S THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

$f(z) = \operatorname{Im} z$

$f(z) = \operatorname{Im} z = \operatorname{Im}(x+iy) = y$

$u = 0, \quad v = y$

$u_x = 0 \neq v_y = 1$: Not analytic

$$\int_C \operatorname{Im} z \, dz = \int_C \operatorname{Im} z \, \frac{dz}{dt} \, dt = \int_0^{2\pi} \sin t \cdot i(\cos t + i \sin t) \, dt$$

$C: z = e^{it}$

$$= \int_0^{2\pi} (i \sin t \cos t - \sin^2 t) \, dt = \int_0^{2\pi} \left(i \frac{\sin 2t}{2} - \frac{1 - \cos 2t}{2} \right) dt = -\pi$$

Problems for Sec 14.2

16. CAUCHY'S THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

$f(z) = 1/(\pi z - 1)$

$f(z)$ is not analytic at $z=1/\pi$ which is inside the unit circle. Thus, Cauchy's Integral Theorem is not applicable.

$$\int_C f(z)dz = \frac{1}{\pi}\int_C \frac{1}{z-1/\pi}dz = \frac{1}{\pi}\cdot 2\pi i = 2i$$

Problems for Sec 14.2

17. CAUCHY'S THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

$f(z) = 1/|z|^2$

$f(z) = 1/|z|^2$ is not analytic at $|z|=0$ or $z=0$ which is inside the unit circle.

Thus, Cauchy's Integral Theorem is not applicable for this problem.

$$\oint_C \frac{1}{|z|^2}\,dz = \oint_C \frac{1}{|z|^2}\frac{dz}{dt}\,dt = \int_0^{2\pi} \frac{1}{1^2}\,ie^{it}\,dt$$

$$C\colon z = e^{it}$$

$$= i\left.\frac{e^{it}}{i}\right|_0^{2\pi} = e^{2\pi i} - 1 = 0$$

Problems for Sec 14.2

18. CAUCHY'S THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

$f(z) = 1/(4z - 3)$

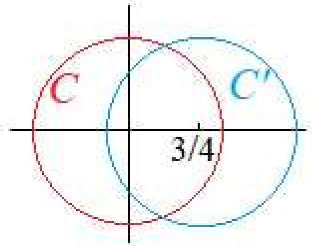
$f(z) = 1/(4z - 3)$ is not analytic at $4z - 3 = 0$ or $z = 3/4$ which is inside the unit circle.

Thus, Cauchy's Integral Theorem is not applicable for this problem.

$$\oint_C \frac{1}{4z-3}\,dz = \oint_C \frac{1}{4}\,\frac{1}{z-3/4}\,\frac{dz}{dt}\,dt \quad C\colon z = e^{it}$$

$$= \oint_{C'} \frac{1}{4}\,\frac{1}{z-3/4}\,\frac{dz}{dt}\,dt \quad C'\colon z - 3/4 = e^{it}$$

$$= \oint_{C'} \frac{1}{4}\,\frac{1}{e^{it}}\,ie^{it}\,dt = \frac{1}{4}\,i \cdot 2\pi = \frac{1}{2}\pi i$$



Problems for Sec 14.2

19. CAUCHY'S THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle. Indicate whether Cauchy's integral theorem applies. Show the details.

$f(z) = z^3 \cot z$

$$\frac{d}{dz}\left(z^3 \cot z\right) = 3z^2 \cot z - z^3 \operatorname{cosec}^2 z$$

Thus, $f(z)$ is analytic inside and on the unit circle.

By Cauchy's Integral Theorem, $\oint_C z^3 \cot z \, dz = 0$

Problems for Sec 14.2

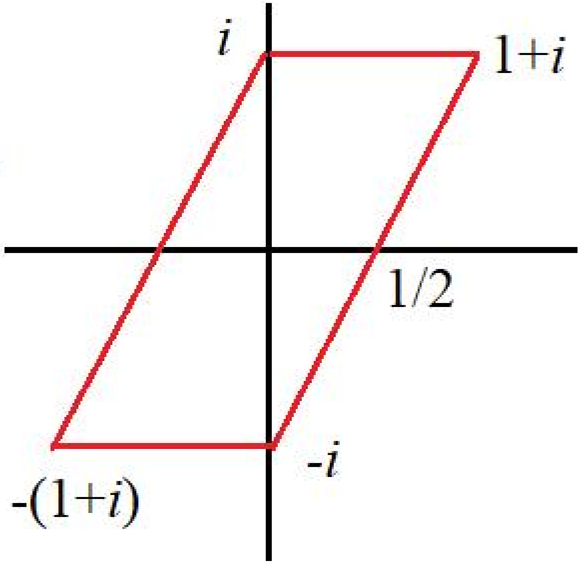
20. FURTHER CONTOUR INTEGRALS

Evaluate the integral. Does Cauchy's theorem apply? Show details.

$\oint_C \operatorname{Ln}(1 - z)\, dz$, C the boundary of the parallelogram with vertices $\pm i$, $\pm(1 + i)$.

$\operatorname{Ln}(1 - z)$ is not analytic if 1-z is negative real number or z is a real number larger than 1.

Thus, the function $\operatorname{Ln}(1 - z)$ is analytic inside and on the unit circle.



By Cauchy's Integral Theorem,

$$\oint_C \operatorname{Ln}(1 - z)\, dz = 0$$

Problems for Sec 14.2

21. FURTHER CONTOUR INTEGRALS

Evaluate the integral. Does Cauchy's theorem apply? Show details.

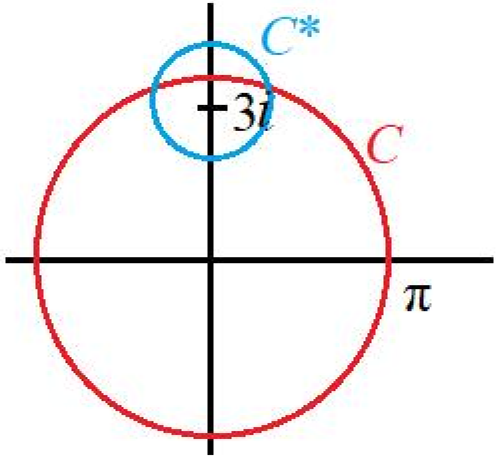
$\oint_C \frac{dz}{z-3i}$, C the circle $|z| = \pi$ counterclockwise.

$1/(z-3i)$ is not analytic at $z = 3i$ which lies inside of the unit circle.
Thus, Cauchy's Integral Theorem is not applicable for this problem.

$C^*: z-3i = e^{it}\ (0 \le t \le 2\pi)$

$$\oint_C \frac{dz}{z-3i} = \oint_{C^*} \frac{1}{z-3i} \frac{dz}{dt}\, dt$$

$$= \int_0^{2\pi} \frac{1}{e^{it}}\, i e^{it} dt = 2\pi i$$

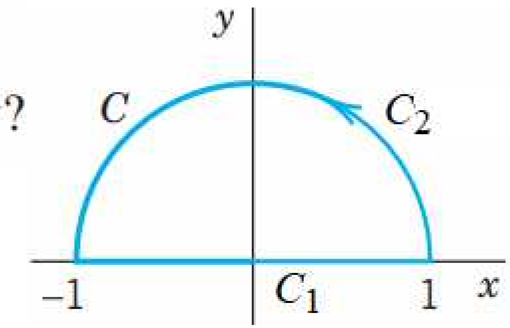


Problems for Sec 14.2

22. FURTHER CONTOUR INTEGRALS

Evaluate the integral. Does Cauchy's theorem apply? Show details.

$$\oint_C \operatorname{Re} z \, dz,$$



$$\oint_C \operatorname{Re} z \, dz = \int_{C_1} \operatorname{Re} z \, dz + \int_{C_2} \operatorname{Re} z \, dz = \int_{C_1} \operatorname{Re} z \, dz + \int_{C_2} \operatorname{Re} z \, \frac{dz}{dt} \, dt$$

$$= \int_{-\pi}^{\pi} x \, dx + \int_0^{\pi} \cos t \; i e^{it} dt = x^2 \Big|_{-\pi}^{\pi} + i \int_0^{\pi} \cos t \, (\cos t + i \sin t) \, dt$$

$$= i \int_0^{\pi} \cos^2 t + i \sin t \cos t \, dt = i \int_0^{\pi} \left[\frac{1+\cos 2t}{2} + \frac{i}{2} \sin 2t \right] dt$$

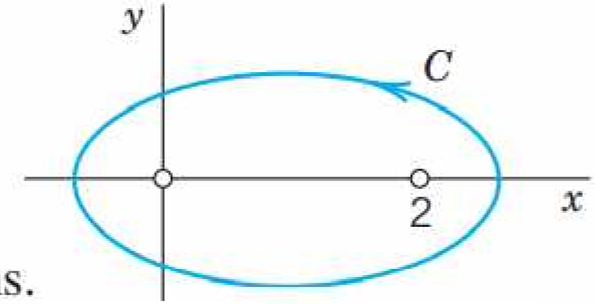
$$= i \left[\frac{t + (\sin 2t)/2}{2} + \frac{i}{2} \frac{\cos 2t}{2} \right]_0^{\pi} = \frac{\pi i}{2}$$

Problems for Sec 14.2

23. FURTHER CONTOUR INTEGRALS

Evaluate the integral. Does Cauchy's theorem apply? Show details.

$$\oint_C \frac{2z-1}{z^2-z}\,dz, \quad C:$$



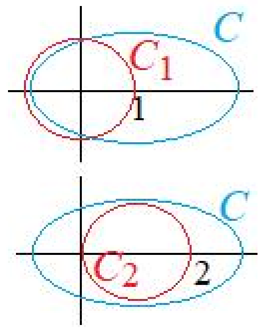
Use partial fractions.

$$\oint_C \frac{2z-1}{z^2-z}dz = \oint_C \left(\frac{1}{z}+\frac{1}{z-1}\right)dz$$

$$= \oint_C \frac{1}{z}\,dz + \oint_C \frac{1}{z-1}\,dz$$

$$= \oint_{C_1} \frac{1}{z}\,dz + \oint_{C_2} \frac{1}{z-1}\,dz$$

$$= 2\pi i + 2\pi i = 4\pi i$$



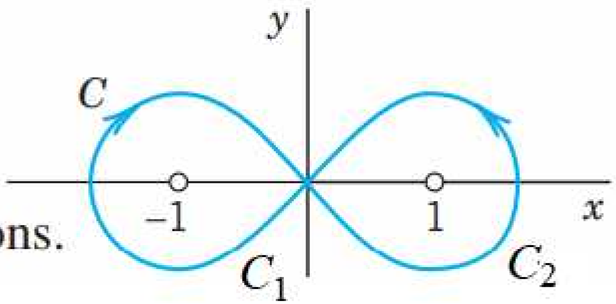
Problems for Sec 14.2

24. FURTHER CONTOUR INTEGRALS

Evaluate the integral. Does Cauchy's theorem apply? Show details.

$$\oint_C \frac{dz}{z^2 - 1}, \quad C:$$

Use partial fractions.



$$\oint_C \frac{dz}{z^2-1} = \oint_C \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)dz$$

$$= \oint_{C_1} \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)dz + \oint_{C_2} \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)dz$$

$$= \; 0 - \frac{1}{2}\oint_{C_1} \frac{1}{z+1}dz + \frac{1}{2}\oint_{C_2} \frac{1}{z-1}\, dz = -\,\frac{1}{2}\; 2\pi i + \frac{1}{2}\; 2\pi i = \; 0$$

Problems for Sec 14.2

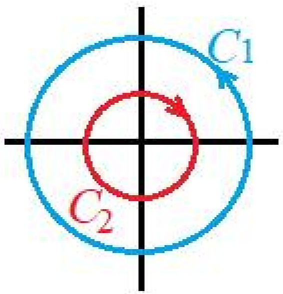
25. FURTHER CONTOUR INTEGRALS

Evaluate the integral. Does Cauchy's theorem apply? Show details.

$\oint_C \frac{e^z}{z}\,dz$, C consists of $|z| = 2$ counterclockwise and $|z| = 1$ clockwise.

$$\oint_C \frac{e^z}{z}\,dz = \oint_{C_1} \frac{e^z}{z}\,dz + \oint_{C_2} \frac{e^z}{z}\,dz$$

$$= 2\pi i - 2\pi i = 0$$



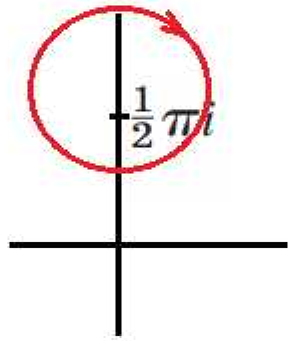
Problems for Sec 14.2

26. FURTHER CONTOUR INTEGRALS

Evaluate the integral. Does Cauchy's theorem apply? Show details.

$$\oint_C \coth \tfrac{1}{2}z \, dz, \ C \text{ the circle } \left|z - \tfrac{1}{2}\pi i\right| = 1 \text{ clockwise.}$$

$$\oint_C \coth \tfrac{1}{2}z \, dz = \oint_C \frac{1}{\tanh \frac{1}{2}z} dz$$



$\tanh \frac{1}{2}z = 0 \Rightarrow 0, \ \pm 2\pi i, \ \pm 4\pi i, \ \cdots$

Outside of the circle

$$\oint_C \coth \tfrac{1}{2}z \, dz = \oint_C \frac{1}{\tanh \frac{1}{2}z} dz = 0$$

Problems for Sec 14.2

27. FURTHER CONTOUR INTEGRALS

Evaluate the integral. Does Cauchy's theorem apply? Show details.

$$\oint_C \frac{\cos z}{z}\, dz, \quad C \text{ consists of } |z| = 1 \text{ counterclockwise and } |z| = 3 \text{ clockwise.}$$

$\dfrac{\cos z}{z}$ is analytic in the given domain.

Thus, Cauchy's Integral Theorem is applicable.

$$\oint_C \frac{\cos z}{z}\, dz = 0$$