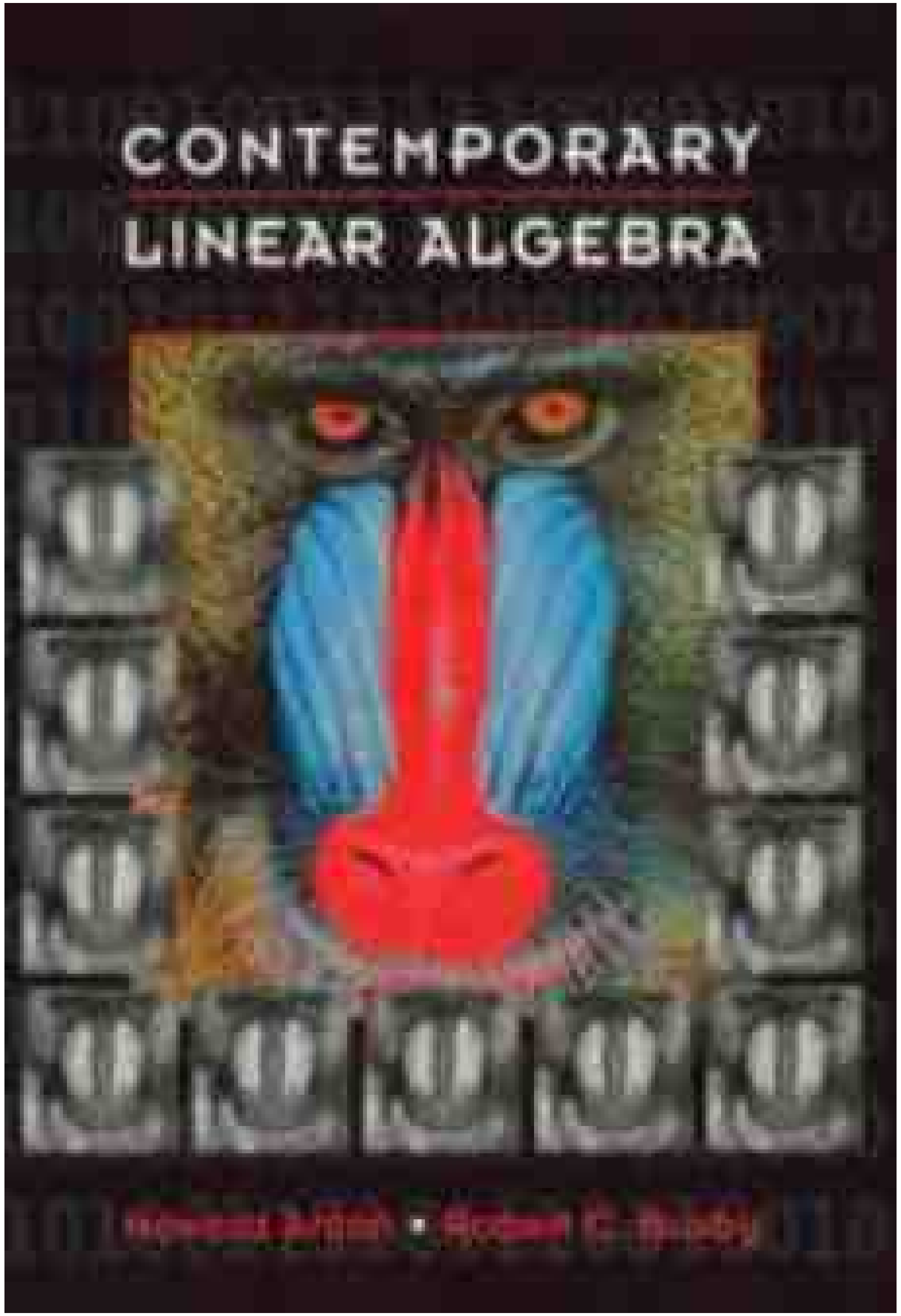

CHAPTER 6
Linear Transformations

6.1 Matrices as Transformations
6.2 Geometry of Linear Operators
6.3 Kernel and Range
6.4 Composition and Invertibility of Linear Transformations
6.5 Computer Graphics

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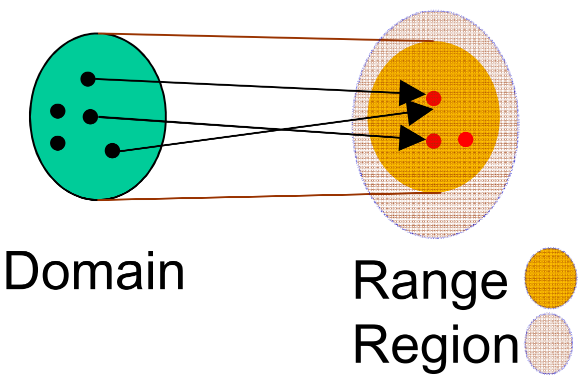
6.1 Matrices as Transformations

A REVIEW OF FUNCTIONS

Definition 6.1.1 Given a set D of allowable inputs, a ***function*** f is a rule that associates a unique output with each input from D; the set D is called the ***domain*** of f. If the input is denoted by x, then the corresponding output is denoted by $f(x)$. The output is also called the ***value*** of f at x or the ***image*** of x under f, and we say that f ***maps*** x into $f(x)$. It is common to denote the output by the single letter y and write $y = f(x)$. The set of all outputs y that results as x varies over the domain is called the ***range*** of f.



$$\mathbf{x} \xrightarrow{T} \mathbf{w}$$

T maps **x** into **w**.

Example 1 A Scaling Transformation

Let T be the transformation that maps $\mathbf{x}=(x_1, x_2)$ into $2\mathbf{x}=(2x_1, 2x_2)$.

May be expressed as

$$T(\mathbf{x}) = 2\mathbf{x}$$

$$T(x_1, x_2) = (2x_1, 2x_2)$$

$$\mathbf{x} \xrightarrow{T} 2\mathbf{x}$$

$$(x_1, x_2) \xrightarrow{T} (2x_1, 2x_2)$$

$$T(-1,\ 3) = (-2,\ 6) \qquad (-1,\ 3) \xrightarrow{T} (-2,\ 6)$$

Example 2 A Component: Squaring Transformation

Let T be the transformation that maps **x**=(x_1, x_2, x_3) into **x**=(x_1^2, x_2^2 , x_3^2).

May be expressed as

$$T(x_1, x_2, x_3) = \left(x_1^2, x_2^2, x_3^2\right)$$

$$(x_1, x_2, x_3) \xrightarrow{T} \left(x_1^2, x_2^2, x_3^2\right)$$

Example 3 A Matrix Multiplication Transformation

Let T_A be the transformation that maps $\mathbf{x}=(x_1, x_2)$ into $A\mathbf{x}$.

$$\mathbf{A}=\begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}$$

May be expressed as

$$T_{\mathbf{A}}(\mathbf{x})=\mathbf{Ax} \qquad \mathbf{x}\xrightarrow{T_{\mathbf{A}}}\mathbf{Ax}$$

$$\mathbf{Ax}=\begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} x_1-x_2 \\ 2x_1+5x_2 \\ 3x_1+4x_2 \end{bmatrix} \qquad T_{\mathbf{A}}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)=\begin{bmatrix} x_1-x_2 \\ 2x_1+5x_2 \\ 3x_1+4x_2 \end{bmatrix}$$

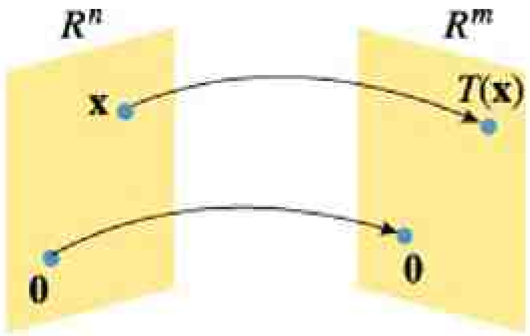
Comma-delimited form: $T_{\mathbf{A}}(x_1, x_2)=(x_1-x_2,\ 2x_1+5x_2,\ 3x_1+4x_2)$

$$T_{\mathbf{A}}\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right)=\begin{bmatrix} -4 \\ 13 \\ 9 \end{bmatrix} \qquad T_A(-1,\ 3)=(-4,\ 13,\ 9)$$

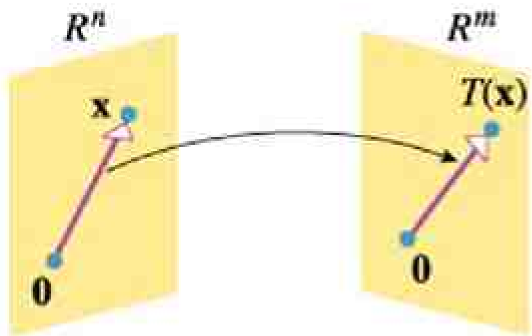
Matrix Transformations

If T is a transformation with domain R^n and image R^m, then

$T : R^n \to R^m$ (T maps R^n into R^m.)



T maps points to points.



T maps vectors to vectors.

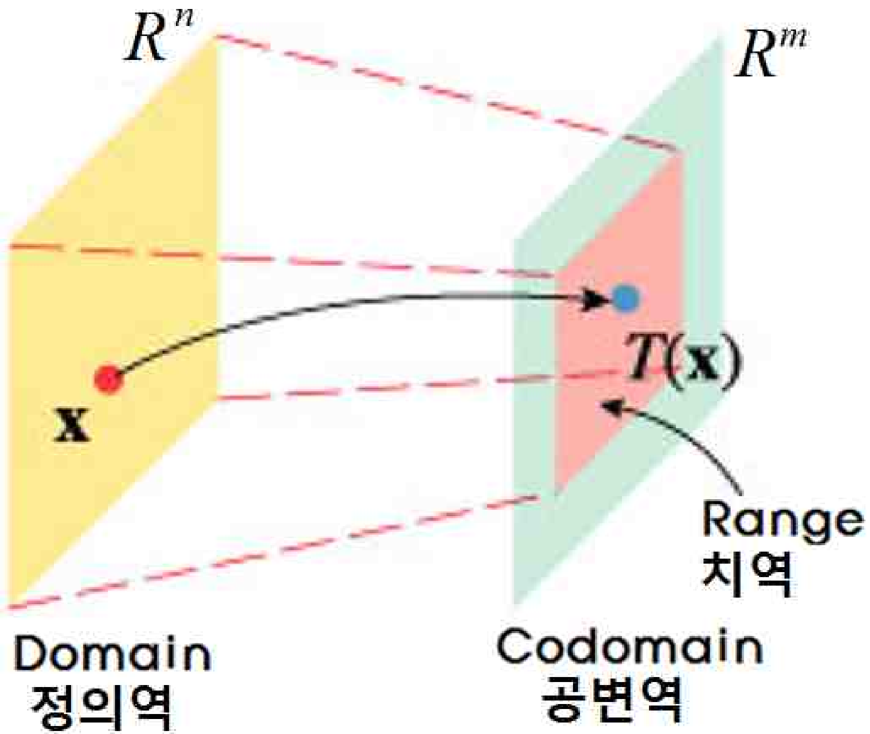
Example 1 $T : R^2 \to R^2$ (5)

Example 2 $T_{\mathbf{A}} : R^2 \to R^3$ (6)

Matrix Transformations-conti

If T is a transformation with domain R^n and image R^m, then

$T : R^n \rightarrow R^m$ (T maps R^n into R^m.)



Examples 4 and 5

Example 4 Zero Transformation

If **0** is the mxn zero matrix, then

$$T_{\mathbf{0}}(\mathbf{x}) = \mathbf{0}\mathbf{x} = \mathbf{0}$$

$T_{\mathbf{0}}$: the *zero transformation* from R^n to R^m.

Example 5 Identity Operators

$$T_{\mathbf{I}}(\mathbf{x}) = \mathbf{I}\mathbf{x} = \mathbf{x}$$

$T_{\mathbf{I}}$: the *identity operator* on R^n.

The solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$:

The vector **x** that mapped to **b** by the transformation **Ax**.

Example 6 A Matrix Transformation

Let $T_A: R^2 \rightarrow R^3$ be the transformation in Example 3.

$$\mathbf{x} \xrightarrow{T_\mathbf{A}} \mathbf{Ax} \qquad \mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}$$

(a) Find a vector x in R^2 whose image under T_A is b=[7 0 7]^T.
(b) Find a vector x in R^2 whose image under T_A is b=[9 -3 -1]^T.

Sol.

(a) Find a vector **x** in R^2 whose image under T_A is **b**=[7 0 7]^T.

$$\begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 7 \\ 2 & 5 & 0 \\ 3 & 4 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Example 6 A Matrix Transformation-conti

(b) Find a vector **x** in R^2 whose image under T_A is b=[9 -3 -1]^T.

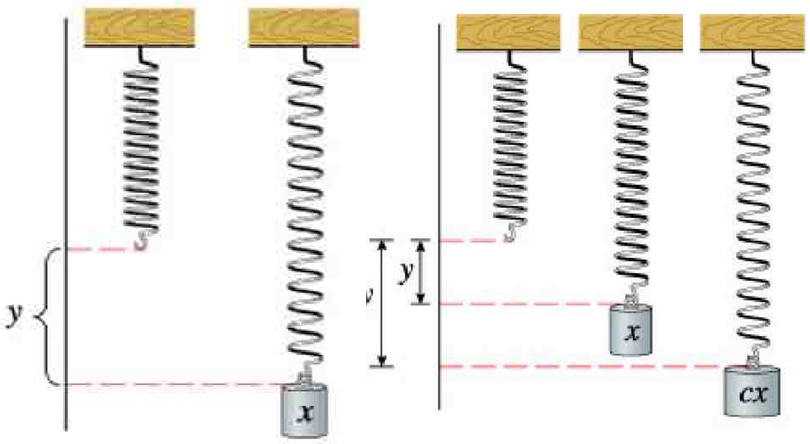
$$\begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 9 \\ 2 & 5 & -3 \\ 3 & 4 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

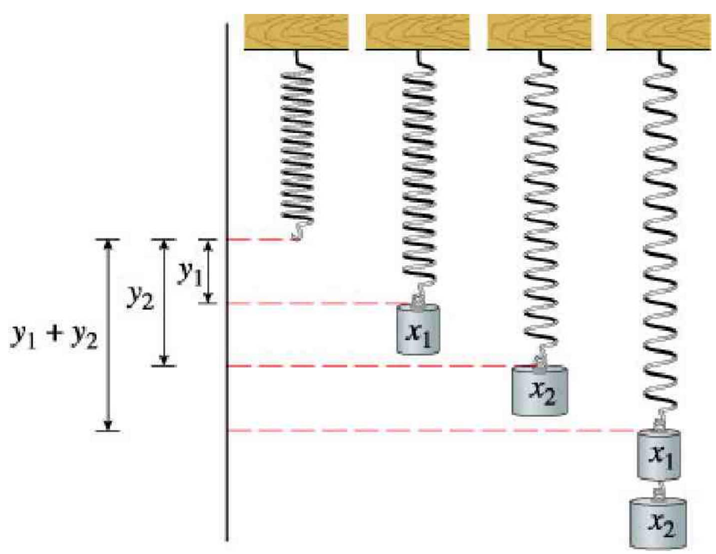
There is no vector **x** in R^2 whose image under T_A is **b**=[9 -3 -1]^T.

Linear Transformations

Hooke's Law: $y = kx$ **or** $y = f(x), \quad f(x) = kx$



(a)



(b)

The stretched length, y, is directly proportional to the weight x.

Definition 6.1.2 Linear Transformation from R^n to R^m

Definition 6.1.2 A function $T: R^n \to R^m$ is called a ***linear transformation*** from R^n to R^m if the following two properties hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n and for all scalars c:

(i) $T(c\mathbf{u}) = cT(\mathbf{u})$ **[Homogeneity property]**

(ii) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ **[Additivity property]**

In the special case where $m = n$, the linear transformation T is called a ***linear operator*** on R^n.

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$$

More generally, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$: vectors in R^n

$c_1, c_2, \cdots, c_k$: any scalars, then

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_kT(\mathbf{v}_k) \qquad (11)$$

Engineers and physicists sometimes call this the superposition principle.

Example 7 Superposition Principle

Show that the transformation $\mathbf{x} \xrightarrow{T_{\mathbf{A}}} \mathbf{Ax}$ is linear.

Sol.

(1) Homogeneity property:

$$T_{\mathbf{A}}(c\mathbf{u}) = \mathbf{A}(c\mathbf{u}) = c(\mathbf{Au}) = cT_{\mathbf{A}}(\mathbf{u})$$

(2) Additivity property:

$$T_{\mathbf{A}}(\mathbf{u}+\mathbf{v}) = \mathbf{A}(\mathbf{u}+\mathbf{v}) = \mathbf{Au}+\mathbf{Av} = T_{\mathbf{A}}(\mathbf{u}) + T_{\mathbf{A}}(\mathbf{v})$$

Example 8 An Example of Nonlinear Transformations

Show that the transformation $T(x_1, x_2, x_3)=(x_1^2, x_2^2, x_3^2)$ is *not* linear.

Sol.

(1) Homogeneity property:

$$T(c\mathbf{u}) = T(cu_1, cu_2, cu_3) = \left(c^2u_1^2, c^2u_2^2, c^2u_3^2\right) = c^2\left(u_1^2, u_2^2, u_3^2\right) = c^2T(\mathbf{u})$$

$$\therefore\ T(c\mathbf{u}) \neq cT(\mathbf{u})$$

(2) Additivity property:

$$T(\mathbf{u}+\mathbf{v}) = T(u_1+v_1, u_2+v_2, u_3+v_3) = \left((u_1+v_1)^2, (u_2+v_2)^2, (u_3+v_3)^2\right)$$

$$T(\mathbf{u})+T(\mathbf{v}) = \left(u_1^2,\ u_2^2,\ u_3^2\right) + \left(v_1^2,\ v_2^2,\ v_3^2\right) = \left(u_1^2+v_1^2,\ u_2^2+v_2^2,\ u_3^2+v_3^2\right)$$

$$\therefore\ T(\mathbf{u}+\mathbf{v}) \neq T(\mathbf{u})+T(\mathbf{v})$$

Some Properties of Linear Transformations

Theorem 6.1.3 *If $T : R^n \to R^m$ is a linear transformation, then:*

(a) $T(\mathbf{0}) = \mathbf{0}$

(b) $T(-\mathbf{u}) = -T(\mathbf{u})$

(c) $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$

Proof

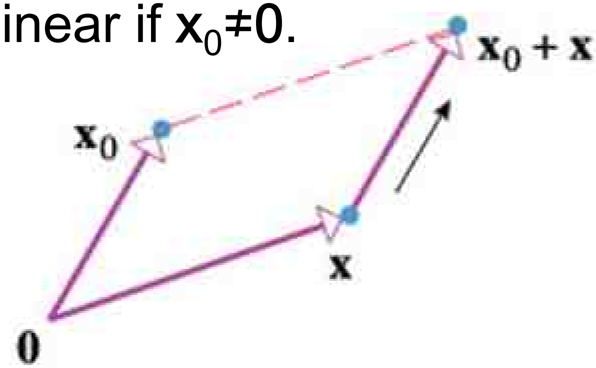
(a) $T(\mathbf{0})=T(0\times\mathbf{0})=0\times T(\mathbf{0})=\mathbf{0}$:

(b) $T(\mathbf{0})=T(\mathbf{u}-\mathbf{u})=T(\mathbf{u})+T(-\mathbf{u})=\mathbf{0}$, $T(-\mathbf{u})=\mathbf{0}-T(\mathbf{u})=-T(\mathbf{u})$

(c) $T(\mathbf{u}-\mathbf{v})=T(\mathbf{u})+(-1)T(\mathbf{v})=T(\mathbf{u})-T(\mathbf{v})$

Example 9 Translations Are Not Linear

Show that the transformation $T(\mathbf{x})=\mathbf{x}_0+\mathbf{x}$ is *not* linear if $\mathbf{x}_0 \neq 0$.



Adding $\mathbf{x}_0$ to $\mathbf{x}$ translates the terminal point of $\mathbf{x}$ by $\mathbf{x}_0$.

Sol.

$T(0)= \mathbf{x}_0+0= \mathbf{x}_0 \neq 0$

(1) Homogeneity property:

$$T(c\mathbf{x}) = c\mathbf{x} + \mathbf{x}_0 \qquad cT(\mathbf{x}) = c(\mathbf{x} + \mathbf{x}_0)$$

$$\therefore \; T(c\mathbf{x}) \neq cT(\mathbf{x})$$

(2) Addiviity property:

$$T(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_0$$

$$T(\mathbf{x}_1) + T(\mathbf{x}_2) = (\mathbf{x}_1 + \mathbf{x}_0) + (\mathbf{x}_2 + \mathbf{x}_0)$$

$$\therefore \; T(\mathbf{x}_1 + \mathbf{x}_2) \neq T(\mathbf{x}_1) + T(\mathbf{x}_2)$$

All Linear Transformations from R^n to R^m:Matrix Transformations

Example 7 shows that every matrix transformation from R^n to R^m, T(**x**)=A**x**, is linear.

Now, let's show that every linear transformation from R^n to R^m can be expressed as a matrix transformation.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

$$T(\mathbf{x}) = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) + \cdots + x_n T(\mathbf{e}_n) \tag{12}$$

$$T(\mathbf{x}) = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{A}\mathbf{x}$$

Theorem 6.1.4

Theorem 6.1.4 *Let $T : R^n \to R^m$ be a linear transformation, and suppose that vectors are expressed in column form. If $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard unit vectors in R^n, and if $\mathbf{x}$ is any vector in R^n, then $T(\mathbf{x})$ can be expressed as*

$$T(\mathbf{x}) = A\mathbf{x} \tag{13}$$

where

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)]$$

A: *Standard matrix for T*

T: *the transformation corresponding to* ***A***
the transformation represented by ***A,***
the transformation ***A***

$$[T] = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)] \qquad T(\mathbf{x}) = [T]\mathbf{x} \tag{14}$$

Example 10 Standard Matrix for a Scaling Operator

$$T(\mathbf{x}) = 2\mathbf{x}$$

(1) Show that the transformation is linear.
(2) Find the standard matrix.

Sol.

(1) Show that the transformation is linear.

$$T(c\mathbf{u}) = 2(c\mathbf{u}) = c(2\mathbf{u}) = cT(\mathbf{u})$$

$$T(\mathbf{u}+\mathbf{v}) = 2(\mathbf{u}+\mathbf{v}) = 2\mathbf{u}+2\mathbf{v} = T(\mathbf{u})+T(\mathbf{v})$$

(2) Find the standard matrix.

$$[T] = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)] = [2\mathbf{e}_1 \quad 2\mathbf{e}_2] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$[T]\mathbf{x} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\mathbf{x} = T(\mathbf{x})$$

Necessary and Sufficient Condition for Linear Transformation

A transformation $T : R^n \rightarrow R^m$:

Let $\mathbf{w} = (w_1, w_2, \cdots, w_m)$ be the image for $\mathbf{x} = (x_1, x_2, \cdots, x_n)$.

$$\mathbf{w} = \mathbf{A}\mathbf{x}$$

$$\begin{array}{ccccccccc}
w_1 & = & a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n \\
w_2 & = & a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
w_m & = & a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n
\end{array}$$

T(x)= $(w_1, w_2, \cdots, w_m)$ *is **a linear transformation** if and only if the equations relating the components of **x** and **w** are linear equations by Theorem 6.1.4.*

Example 11 Standard Matrix for a Linear Transformation

$$T(x_1,\ x_2,\ x_3) = (x_1 + x_2,\ x_2 - x_3) \qquad (15)$$

(1) Show that the transformation is linear.
(2) Find the standard matrix.

Sol.

(1) Show that the transformation is linear.

Let $\mathbf{x}=(x_1, x_2, x_3)$, $\mathbf{y}=(y_1, y_2 , y_3)$.

Then,

$T(c\mathbf{x})=(cx_1+cx_2, cx_2-cx_3)=c(x_1+x_2, x_2-x_3)=cT(\mathbf{x})$.
$T(\mathbf{x}+\mathbf{y})=T(\mathbf{x})+T(\mathbf{y})$.

(2) Find the standard matrix.

Example 11 Standard Matrix - cont

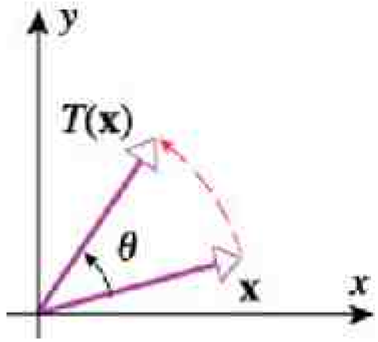
(2) Find the standard matrix for $T(x_1,\ x_2,\ x_3) = (x_1 + x_2,\ x_2 - x_3)$.

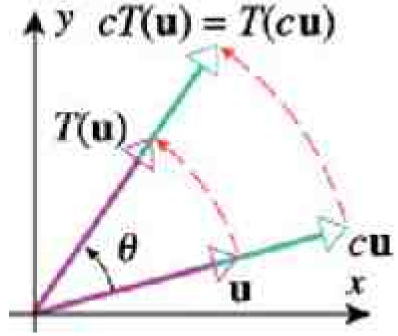
$$T(\mathbf{e}_1) = T(1,\ 0,\ 0) = (1,\ 0)$$
$$T(\mathbf{e}_2) = T(0,\ 1,\ 0) = (1,\ 1)$$
$$T(\mathbf{e}_3) = T(0,\ 0,\ 1) = (0,\ -1)$$

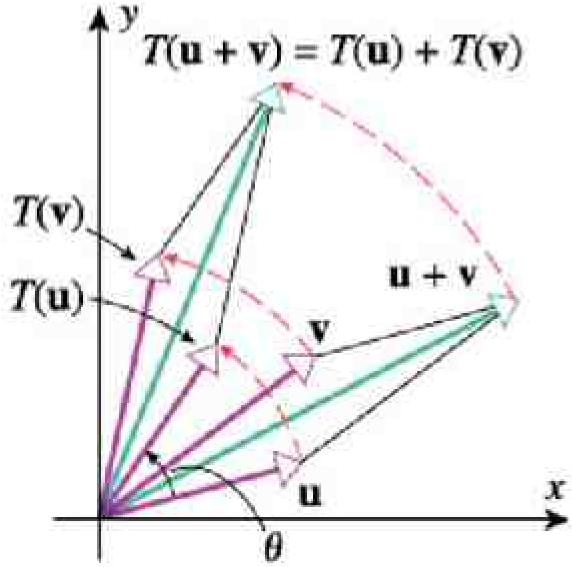
$$\left[T\right] = \left[T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3)\right] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

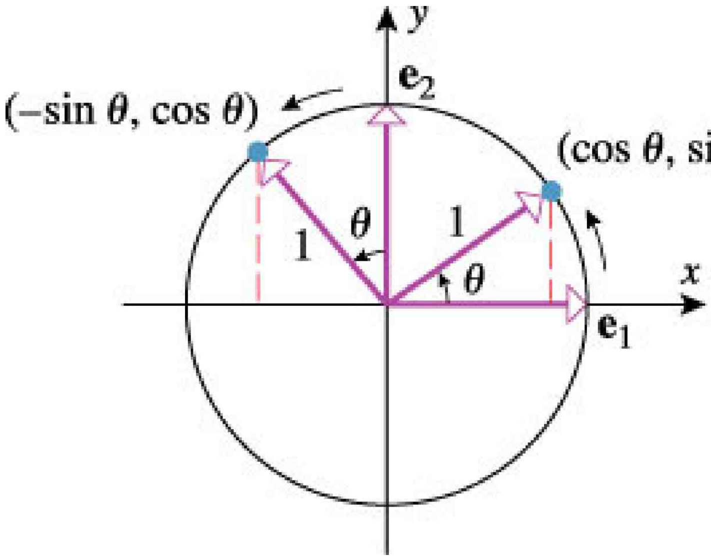
$$\left[T\right]\mathbf{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \end{bmatrix}$$

Rotations About the Origin









$$R_\theta = [T(\mathbf{e}_1)\ \ T(\mathbf{e}_2)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad (16)$$

$$R_\theta \mathbf{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (17)$$

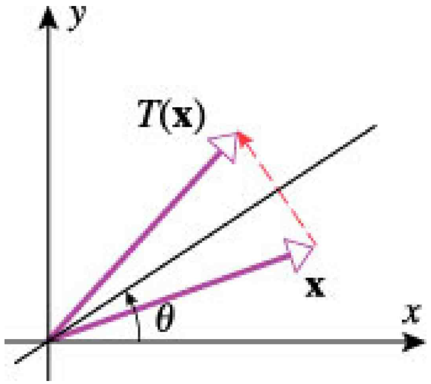
Example 12 A Rotation Operator

Find the image of **x**=(1,1) under a rotation of π/6 radians about the origin.

Sol.

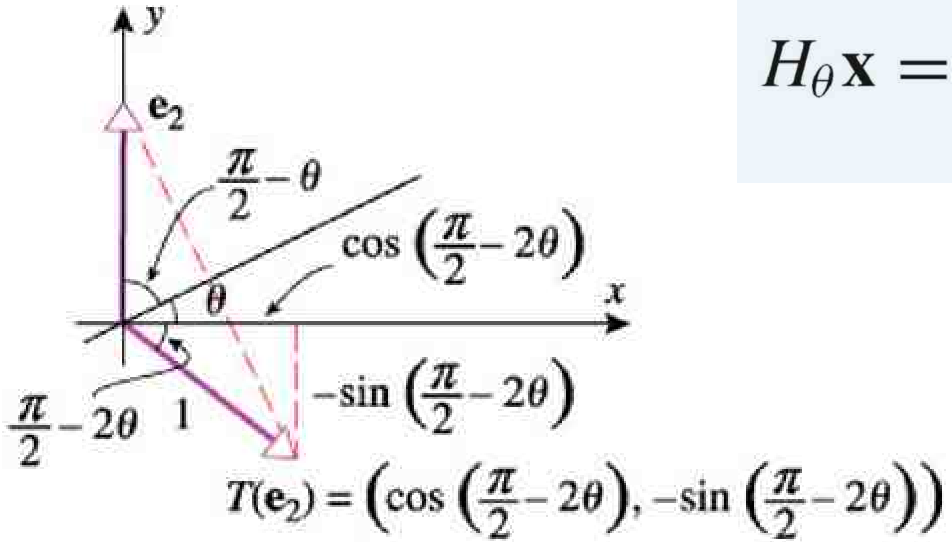
$$\mathbf{R}_{\pi/6}\mathbf{x} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{\sqrt{3}-1}{2} \\ \dfrac{1+\sqrt{3}}{2} \end{bmatrix} \approx \begin{bmatrix} 0.37 \\ 1.37 \end{bmatrix}$$

Reflections About Lines Through the Origin



$$H_\theta = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)] = \begin{bmatrix} \cos 2\theta & \cos\left(\frac{\pi}{2} - 2\theta\right) \\ \sin 2\theta & -\sin\left(\frac{\pi}{2} - 2\theta\right) \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \tag{18}$$

$$H_\theta \mathbf{x} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{19}$$



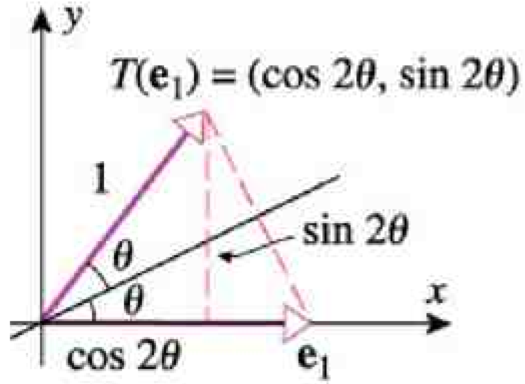
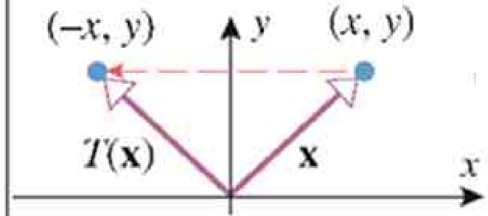
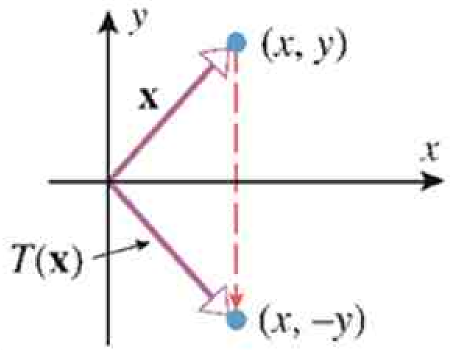
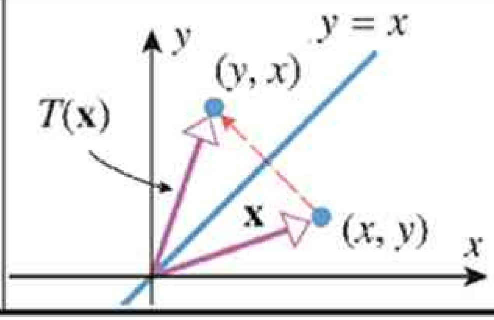


Table 6.1.1 The Most Basic Reflections

Operator	Illustration	Images of $\mathbf{e}_1$, $\mathbf{e}_2$	Standard Matrix
Reflection about the *y*-axis $T(x,y)=(-x,y)$		$T(\mathbf{e}_1)=(-1,0)$ $T(\mathbf{e}_2)=(0,1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the *x*-axis $T(x,y)=(x,-y)$		$T(\mathbf{e}_1)=(1,0)$ $T(\mathbf{e}_2)=(0,-1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the line *y=x* $T(x,y)=(y,x)$		$T(\mathbf{e}_1)=(0,1)$ $T(\mathbf{e}_2)=(1,0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Example 13 A Reflection Operator

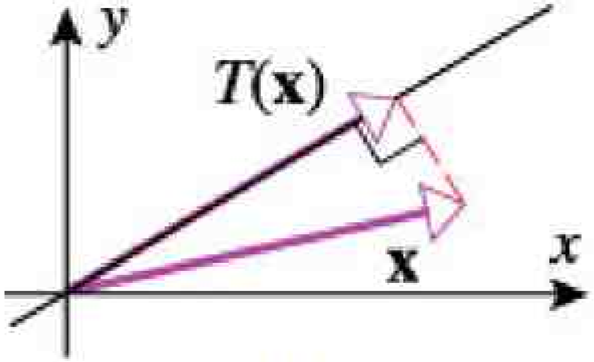
Find the image of **x**=(1,1) under a reflection about the line through the origin that makes an angle of π/6 radians with the positive x-axis.

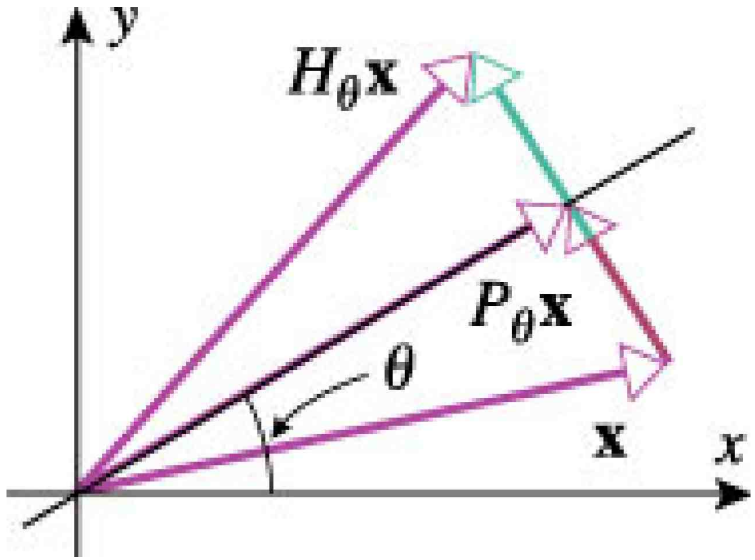
Sol.

$$H_\theta = [T(\mathbf{e}_1)\ T(\mathbf{e}_2)]$$
$$= \begin{bmatrix} \cos 2\theta & \cos\left(\frac{\pi}{2} - 2\theta\right) \\ \sin 2\theta & -\sin\left(\frac{\pi}{2} - 2\theta\right) \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \quad (18)$$

$$\mathbf{H}_{\pi/6}\mathbf{x} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (1+\sqrt{3})/2 \\ (\sqrt{3}-1)/2 \end{bmatrix} \approx \begin{bmatrix} 1.37 \\ 0.37 \end{bmatrix}$$

Orthogonal Projections onto the Lines Through the Origin





$$\mathbf{P}_\theta \mathbf{x} - \mathbf{x} = \frac{1}{2}(\mathbf{H}_\theta \mathbf{x} - \mathbf{x})$$

$$\mathbf{P}_\theta \mathbf{x} = \frac{1}{2}\mathbf{H}_\theta \mathbf{x} + \frac{1}{2}\mathbf{x} = \frac{1}{2}\mathbf{H}_\theta \mathbf{x} + \frac{1}{2}\mathbf{I}\mathbf{x} = \frac{1}{2}(\mathbf{H}_\theta + \mathbf{I})\mathbf{x}$$

$$\mathbf{P}_\theta = \frac{1}{2}(\mathbf{H}_\theta + \mathbf{I})$$

Orthogonal Projections onto the Lines Through the Origin

$$\mathbf{P}_\theta = \frac{1}{2}(\mathbf{H}_\theta + \mathbf{I}) \tag{20}$$

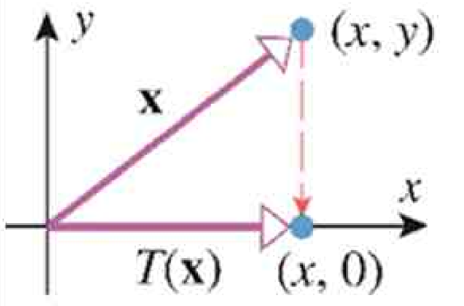
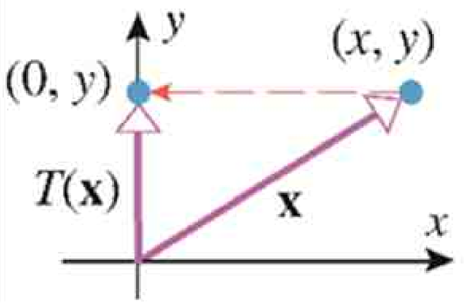
$$\mathbf{P}_\theta = \begin{bmatrix} \frac{1}{2}(1+\cos 2\theta) & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \frac{1}{2}(1-\cos 2\theta) \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \tag{21}$$

$$\cos 2\theta = 2\cos^2\theta - 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\mathbf{P}_\theta\mathbf{x} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \tag{22}$$

Orthogonal Projections in R2 Onto the Coordinate Axes

Table 6.1.2

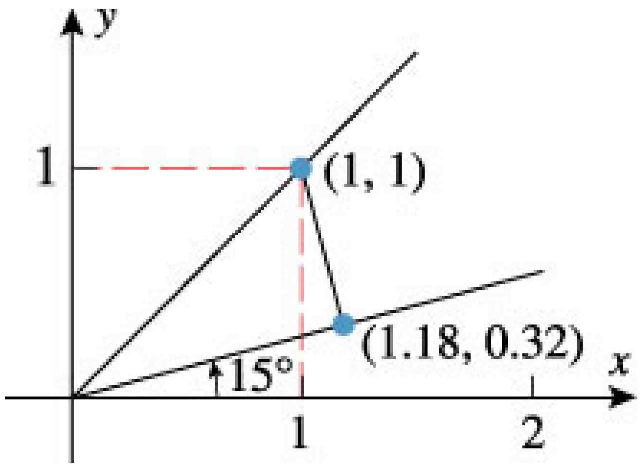
Operator	Illustration	Images of $\mathbf{e}_1, \mathbf{e}_2$	Standard Matrix
Orthogonal Projection on the *x*-axis $T(x, y) = (x, 0)$		$T(\mathbf{e}_1) = (1, 0)$ $T(\mathbf{e}_2) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal Projection on the *y*-axis $T(x, y) = (0, y)$		$T(\mathbf{e}_1) = (0, 0)$ $T(\mathbf{e}_2) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Concept Problem Use (22) to derive the Table 6.1.2.

Example 14 An Orthogonal Projection Operator

Find the orthogonal projection of the vector **x**=(1,1) on the line through the origin that makes an angle of π/12 with the x-axis.

Sol.



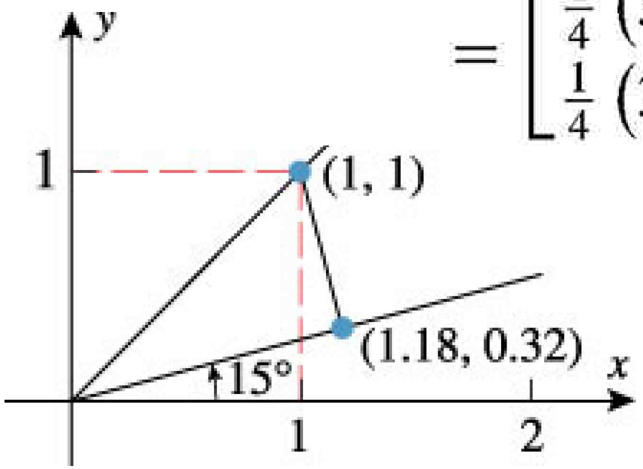
$$P_{\pi/12} = \begin{bmatrix} \frac{1}{2}\left(1+\cos\frac{\pi}{6}\right) & \frac{1}{2}\sin\frac{\pi}{6} \\ \frac{1}{2}\sin\frac{\pi}{6} & \frac{1}{2}\left(1-\cos\frac{\pi}{6}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right) & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right) \end{bmatrix}$$

Example 14 An Orthogonal Projection Operator-conti

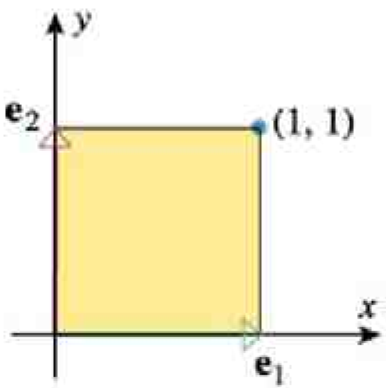
$$P_{\pi/12} = \begin{bmatrix} \frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right) & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right) \end{bmatrix}$$

$$P_{\pi/12}\mathbf{x} = \begin{bmatrix} \frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right) & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

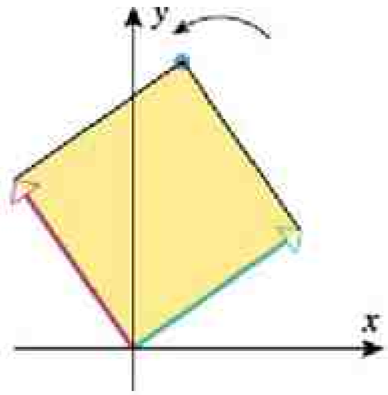
$$= \begin{bmatrix} \frac{1}{4}\left(3+\sqrt{3}\right) \\ \frac{1}{4}\left(3-\sqrt{3}\right) \end{bmatrix} \approx \begin{bmatrix} 1.18 \\ 0.32 \end{bmatrix}$$



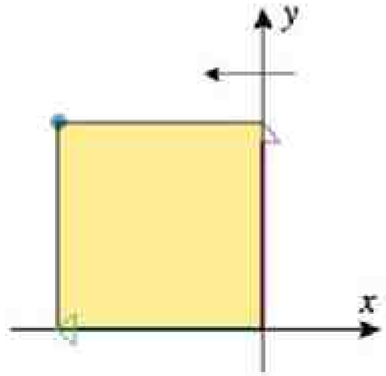
Transformations of the Unit Square



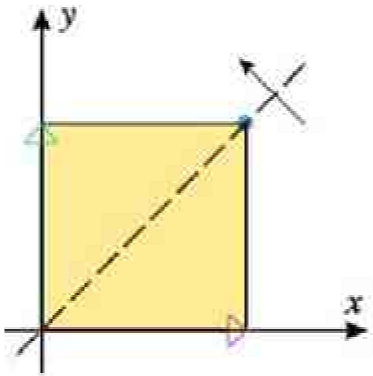
Unit square



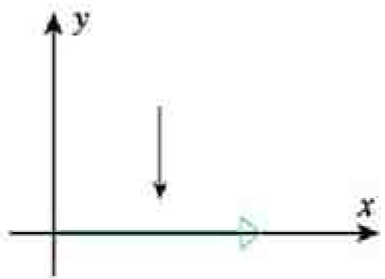
Unit square rotated



Unit square reflected about the y-axis



Unit square reflected about the line y=x



Unit square projected onto the x-axis

Power Sequences

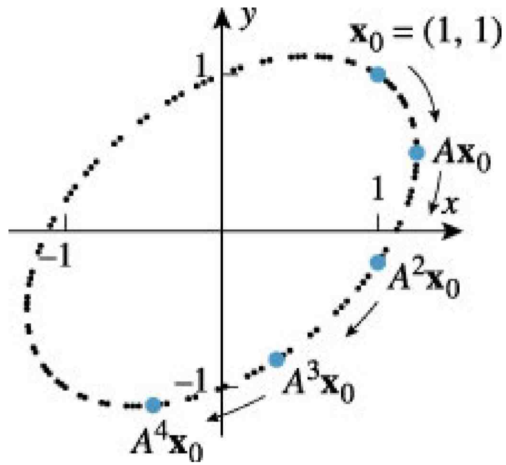
$$\mathbf{x}_0,\ A\mathbf{x}_0,\ A^2\mathbf{x}_0,\ \ldots,\ A^k\mathbf{x}_0,\ \ldots$$

For example,

$$A = \begin{bmatrix} 1/2 & 3/4 \\ -3/5 & 11/10 \end{bmatrix} \qquad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{A}\mathbf{x}_0 = \begin{bmatrix} 1.25 \\ 0.5 \end{bmatrix},\ \mathbf{A}^2\mathbf{x} = \begin{bmatrix} 1.0 \\ -0.2 \end{bmatrix}$$

$$\mathbf{A}^3\mathbf{x} = \begin{bmatrix} 0.35 \\ -0.82 \end{bmatrix},\ \mathbf{A}^4\mathbf{x} = \begin{bmatrix} -0.44 \\ -1.112 \end{bmatrix}$$



NORM PRESERVING OPERATORS

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotation about the origin through an angle θ

$$H_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

Reflection about the line through the origin making an angle θ with the positive x-axis

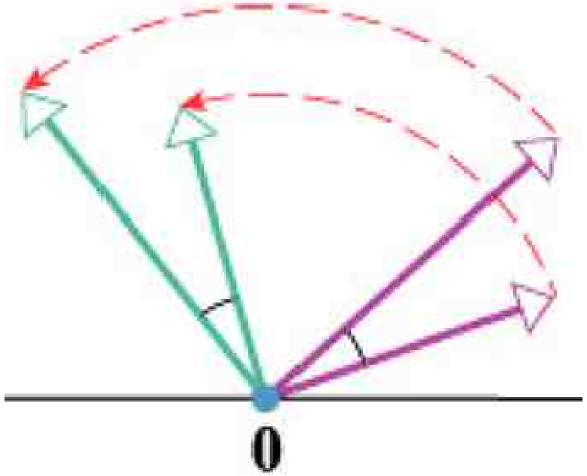
$$P_\theta = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

Orthogonal projection onto the line through the origin making an angle θ with the positive x-axis

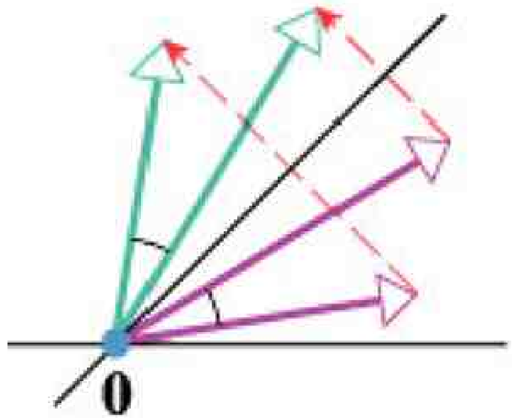
In general, a linear operator T : $R^n \rightarrow R^m$ with the length-preserving property $||T(\mathbf{x})||=||\mathbf{x}||$ is called an *orthogonal operator* or a *linear isometry*.

6.2 Geometry of Linear Operators

Rotations about the origin and reflections about lines through the origin of R^2 are examples of orthogonal operators



A rotation about the origin does not change lengths of vectors or angles between vectors.



A reflection about a line through the origin does not change lengths of vectors or angles between vectors.

Rotations and reflections preserve angles as well as lengths.
Length-preserving linear operators are *dot product preserving*, and conversely.(Theorem 6.2.1)

Theorem 6.2.1 Equivalent Statements

Theorem 6.2.1 *If $T : R^n \to R^n$ is a linear operator on R^n, then the following statements are equivalent.*

(a) $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$ *in* R^n. **[T orthogonal (i.e., length preserving)]**

(b) $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* R^n. **[T is dot product preserving.]**

(a)→(b): $\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\,\mathbf{x}\cdot\mathbf{y}$

$$\|\mathbf{x}-\mathbf{y}\|^2 = (\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\mathbf{x}\cdot\mathbf{y}$$

$$\Rightarrow \quad \mathbf{x}\cdot\mathbf{y} = \frac{1}{4}\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2\right) \tag{4}$$

$$\begin{aligned} T(\mathbf{x})\cdot T(\mathbf{y}) &= \frac{1}{4}\left(\|T(\mathbf{x})+T(\mathbf{y})\|^2 - \|T(\mathbf{x})-T(\mathbf{y})\|^2\right) \\ &= \frac{1}{4}\left(\|T(\mathbf{x}+\mathbf{y})\|^2 - \|T(\mathbf{x}-\mathbf{y})\|^2\right) \\ &= \frac{1}{4}\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2\right) = \mathbf{x}\cdot\mathbf{y} \end{aligned} \tag{5}$$

(b)→(a): $\|\mathbf{x}\| = \sqrt{\mathbf{x}\cdot\mathbf{x}}$

$$\|T(\mathbf{x})\| = \sqrt{T(\mathbf{x})\cdot T(\mathbf{x})} = \sqrt{\mathbf{x}\cdot\mathbf{x}} = \|\mathbf{x}\|$$

Orthogonal Operators Preserve Angles and Orthogonality

The angle between **x** and **y**:

$$\theta = \cos^{-1}\left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\right) \tag{6}$$

The angle between T(**x**) and T(**y**):

$$\cos^{-1}\left(\frac{T(\mathbf{x})\cdot T(\mathbf{y})}{\|T(\mathbf{x})\|\,\|T(\mathbf{y})\|}\right) = \cos^{-1}\left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\right) \tag{7}$$

Thus,

$$\theta = \cos^{-1}\left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}\right) = \cos^{-1}\left(\frac{T(\mathbf{x})\cdot T(\mathbf{y})}{\|T(\mathbf{x})\|\,\|T(\mathbf{y})\|}\right)$$

Orthogonal Matrices

Let A be the standard matrix for a linear operator T: $R^n \rightarrow R^m$.

For all **x** in R^n,

$$T(\mathbf{x}) = \mathbf{Ax}$$
$$\|T(\mathbf{x})\| = \|\mathbf{x}\|$$

$\Rightarrow$

$$\|\mathbf{Ax}\| = \|\mathbf{x}\| \qquad (8)$$

The equation (8) is used to determine the orthogonality of a linear operator.

Definition: Orthogonal Matrix

Definition 6.2.2 A square matrix A is said to be ***orthogonal*** if $A^{-1} = A^T$.

Example 1 Orthogonal Matrix

Determine whether A is orthogonal.

$$\mathbf{A} = \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}$$

Sol.

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 3/7 & -6/7 & 2/7 \\ 2/7 & 3/7 & 6/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix} \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{A}^{-1} = \mathbf{A}^T = \begin{bmatrix} 3/7 & -6/7 & 2/7 \\ 2/7 & 3/7 & 6/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix}$$

Thus, A is orthogonal by definition 6.2.2.

Theorem 6.2.3

Theorem 6.2.3

(*a*) *The transpose of an orthogonal matrix is orthogonal.*

(*b*) *The inverse of an orthogonal matrix is orthogonal.*

(*c*) *A product of orthogonal matrices is orthogonal.*

(*d*) *If A is orthogonal, then* $\det(A) = 1$ *or* $\det(A) = -1$.

(a):

If A is orthogonal, $\mathbf{A}^T\mathbf{A} = \mathbf{I}$

$$(\mathbf{A}^T)^T(\mathbf{A}^T) = \mathbf{A}\mathbf{A}^T = \mathbf{I}$$

(b): $(\mathbf{A}^{-1})^T(\mathbf{A}^{-1}) = (\mathbf{A}^T)^{-1}(\mathbf{A}^{-1}) = (\mathbf{A}\mathbf{A}^T)^{-1} = (\mathbf{I})^{-1} = \mathbf{I}$

(c): $(\mathbf{AB})^T(\mathbf{AB}) = \mathbf{B}^T\mathbf{A}^T\mathbf{AB} = \mathbf{B}^T\mathbf{B} = \mathbf{I}$

(d): $\det(\mathbf{A}^T\mathbf{A}) = \det(\mathbf{A}^T)\det(\mathbf{A}) = \left(\det(\mathbf{A})\right)^2 = 1$ $\quad\quad$ $\det(\mathbf{A}) = \pm 1$

Theorem 6.2.4 Equivalent Statements

Theorem 6.2.4 *If A is an $m \times n$ matrix, then the following statements are equivalent.*

(*a*) $A^T A = I$.

(*b*) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$ *in* R^n.

(*c*) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* R^n.

(*d*) *The column vectors of A are orthonormal.*

(a)→(b): $\|\mathbf{Ax}\|^2 = (\mathbf{Ax}) \cdot (\mathbf{Ax}) = (\mathbf{Ax})^T (\mathbf{Ax}) = \mathbf{x}^T \mathbf{A}^T \mathbf{Ax} = \mathbf{x}^T \mathbf{x} = \mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$

(b)→(c): $(\mathbf{Ax}) \cdot (\mathbf{Ay}) = (\mathbf{Ax})^T (\mathbf{Ay}) = \mathbf{x}^T \mathbf{A}^T \mathbf{Ay} = \mathbf{x}^T \mathbf{y} = \mathbf{x} \cdot \mathbf{y}$

(c)→(d): $T(\mathbf{e}_1) = \mathbf{Ae}_1,\ T(\mathbf{e}_2) = \mathbf{Ae}_2,\ \cdots,\ T(\mathbf{e}_n) = \mathbf{Ae}_n$

$\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$: Orthonormal

Thus, $T(\mathbf{e}_1), T(\mathbf{e}_2), \cdots, T(\mathbf{e}_n)$: Orhtonormal

Theorem 6.2.4 Equivalent Statements-cont

Theorem 6.2.4 *If A is an $m \times n$ matrix, then the following statements are equivalent.*

(a) $A^T A = I$.

(b) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$ *in* R^n.

(c) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* R^n.

(d) *The column vectors of A are orthonormal.*

(d)→(a): Let $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ be orthonormal column vectors of A.

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{a}_2 & \cdots & \mathbf{a}_1^T\mathbf{a}_n \\ \mathbf{a}_2^T\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{a}_2 & \cdots & \mathbf{a}_2^T\mathbf{a}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n^T\mathbf{a}_1 & \mathbf{a}_n^T\mathbf{a}_2 & \cdots & \mathbf{a}_n^T\mathbf{a}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \mathbf{I}$$

Theorem 6.2.5 Equivalent Statements

Theorem 6.2.5 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(a) *A is orthogonal.*

(b) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$ *in* R^n.

(c) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* R^n.

(d) *The column vectors of A are orthonormal.*

(e) *The row vectors of A are orthonormal.*

Proof

In the case of a square matrix, Theorems 6.2.3 and 6.2.4 yield 6.2.5.

Theorem 6.2.6 The Condition for Orthogonal Linear Operator

Theorem 6.2.6 *A linear operator $T : R^n \to R^n$ is orthogonal if and only if its standard matrix is orthogonal.*

Proof

A linear operator T: $R^n \to R^n$ is defined to be orthogonal if and only if $\|T(\mathbf{x})\|=\|\mathbf{x}\|$ for all $\mathbf{x}$ in R^n.

Thus, T is orthogonal if and only if its standard matrix has the property $\|A\mathbf{x}\|=\|\mathbf{x}\|$ for all $\mathbf{x}$ in R^n.

This fact and Theorem 6.2.5 (a) and (b) yield 6.2.6.

Theorem 6.2.5 *If A: $n\times n$ matrix, equivalent statements*
(a) A is orthogonal. *(b) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in R^n.*

Example 2 Orthogonal Standard Matrices

Example 2 Orthogonal Matrix

Show that R_θ and H_θ are orthogonal.

Sol.

$$\mathbf{R}_\theta^T\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{H}_\theta^T\mathbf{H}_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 2\theta + \sin^2 2\theta & 0 \\ 0 & \cos^2 2\theta + \sin^2 2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Example 3 Orthogonal Standard Matrices

Show that A in Example 1 is orthogonal.

$$\mathbf{A} = \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}$$

Sol.

$$\mathbf{r}_1 \cdot \mathbf{r}_1 = 1,\ \mathbf{r}_2 \cdot \mathbf{r}_2 = 1,\ \mathbf{r}_3 \cdot \mathbf{r}_3 = 1,$$
$$\mathbf{r}_1 \cdot \mathbf{r}_2 = 0,\ \mathbf{r}_1 \cdot \mathbf{r}_3 = 0,\ \mathbf{r}_2 \cdot \mathbf{r}_3 = 0$$

Thus, A is orthogonal by Theorem 6.2.5(e).

$$\mathbf{c}_1 \cdot \mathbf{c}_1 = 1,\ \mathbf{c}_2 \cdot \mathbf{c}_2 = 1,\ \mathbf{c}_3 \cdot \mathbf{c}_3 = 1,$$
$$\mathbf{c}_1 \cdot \mathbf{c}_2 = 0,\ \mathbf{c}_1 \cdot \mathbf{c}_3 = 0,\ \mathbf{c}_2 \cdot \mathbf{c}_3 = 0$$

Thus, A is orthogonal by Theorem 6.2.5(d).

All Orthogonal Linear Operators

Theorem 6.2.7 *If $T : R^2 \to R^2$ is an orthogonal linear operator, then the standard matrix for T is expressible in the form*

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad or \quad H_{\theta/2} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \tag{10}$$

That is, T is either a rotation about the origin or a reflection about a line through the origin.

Proof

Assume that T is an orthogonal linear operator on R^2 with its standard matrix A.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ ($\because$ orthogonal)

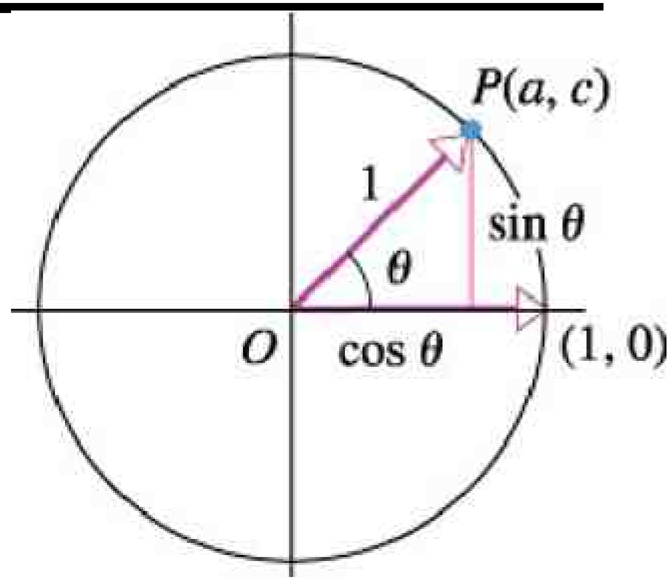
Column vectors of A are orthonormal because A is orthogonal.

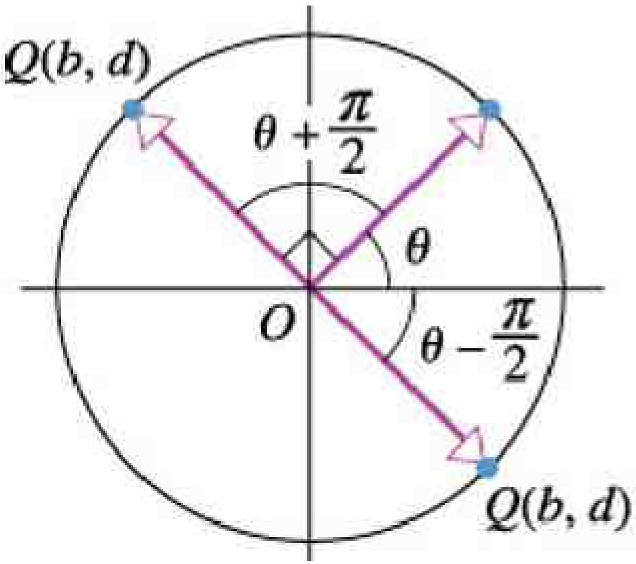
$$a^2+c^2=1 \qquad b^2+d^2=1$$

All Orthogonal Linear Operators

$a^2+c^2=1 \Rightarrow P(a,c)=(\cos\theta,\ \sin\theta)$

$\Rightarrow \mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} \cos\theta & b \\ \sin\theta & d \end{bmatrix}$





$b^2+d^2=1$

$$\begin{cases} b=\cos(\theta+\pi/2)=-\sin\theta \\ d=\sin(\theta+\pi/2)=\cos\theta \end{cases}$$

$$\begin{cases} b=\cos(\theta-\pi/2)=\sin\theta \\ d=\sin(\theta-\pi/2)=-\cos\theta \end{cases}$$

$\Rightarrow \mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} \cos\theta & b \\ \sin\theta & d \end{bmatrix}=\begin{bmatrix} \cos\theta & \mp\sin\theta \\ \sin\theta & \pm\cos\theta \end{bmatrix}$

All Orthogonal Linear Operators

How to distinguish between a rotation and a reflection?

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad or \quad H_{\theta/2} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \tag{10}$$

(a) A rotation about the origin

$$\det(\mathbf{R}_\theta) = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

(b) A reflection about a line through the origin

$$\det(\mathbf{H}_{\theta/2}) = \begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{vmatrix} = -(\cos^2\theta + \sin^2\theta) = -1$$

Example 4

In each part, describe the linear operator on R^2 corresponding to the standard matrices A in (a) and (b.

(a) $\mathbf{A} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ (b) $\mathbf{A} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

Sol.

(a) Column vectors of A are orthonormal, so A is orthogonal.

$$\cos\theta = \sin\theta = 1/\sqrt{2} \Longrightarrow \theta = \pi/4$$

Thus, a rotation of θ=π/4 about the origin.

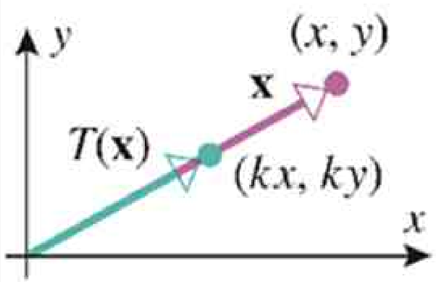
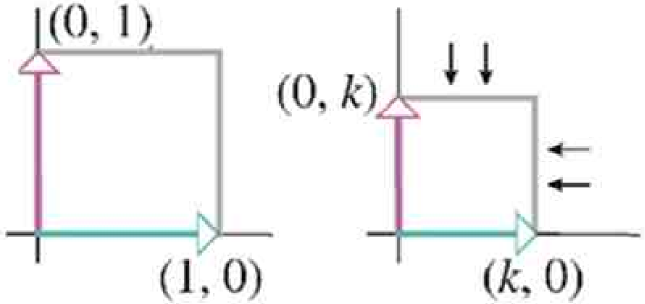
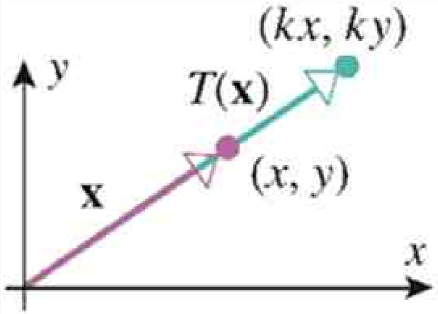
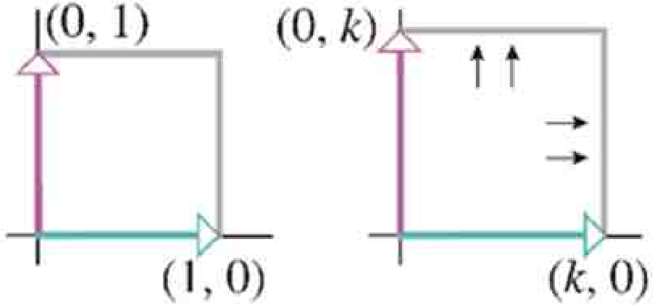
(b) Column vectors of A are orthonormal, so A is orthogonal.

$$\cos 2\theta = \sin 2\theta = 1/\sqrt{2} \Longrightarrow \theta = \pi/8$$

Thus, a reflection about a line of θ=π/8 through the origin.

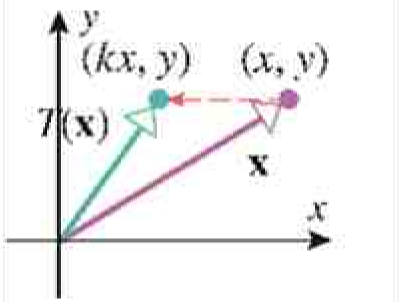
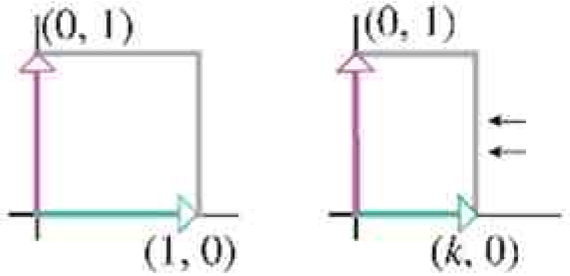
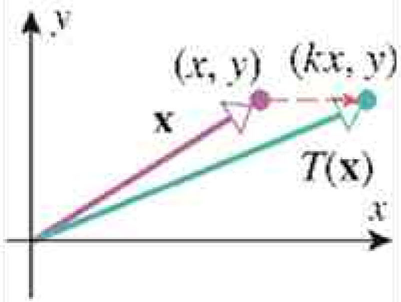
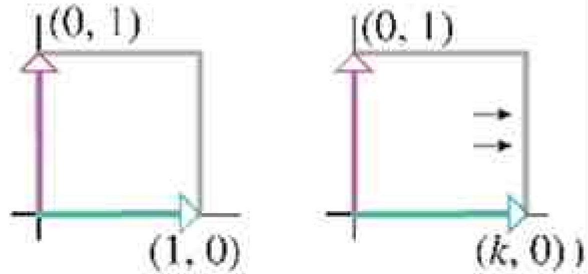
Contractions and Dilations of R^2(축약, 확대변환)

Table 6.2.1

Operator	**Illustration** $T(x, y) = (kx, ky)$	**Effect on the Unit Square**	**Standard Matrix**
Contraction with factor k on R^2 $(0 \le k < 1)$			$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Dilation with factor k on R^2 $(k > 1)$			

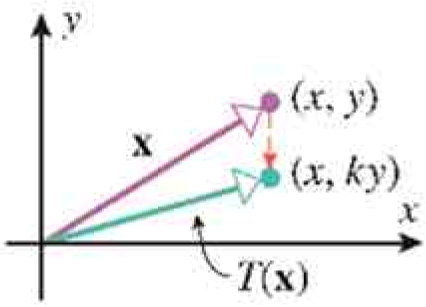
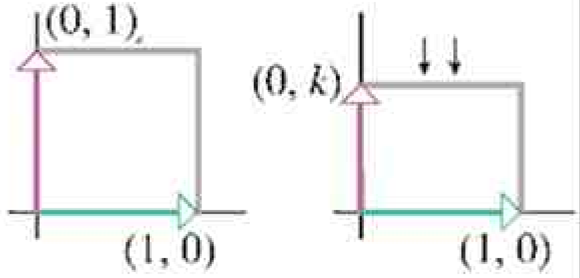
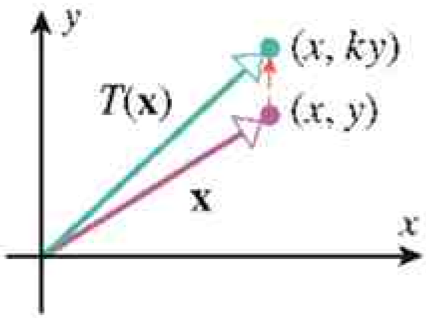
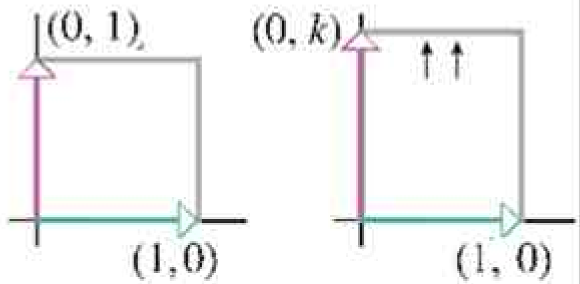
Vertical and Horizontal Compressions and Expansions of R^2

Table 6.2.2 part(a)

Operator	Illustration $T(x,y)=(kx,y)$	Effect on the Unit Square	Standard Matrix
Compression of R^2 in the x-direction with factor k $(0 \le k < 1)$ 압축			$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Expansion of R^2 in the x-direction with factor k $(k > 1)$ 확대			

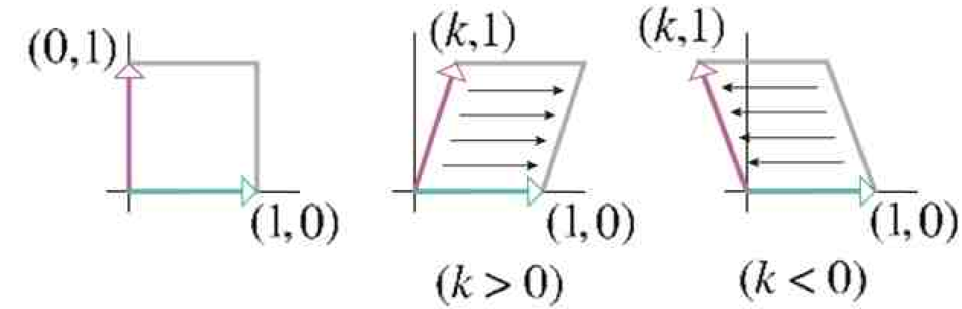
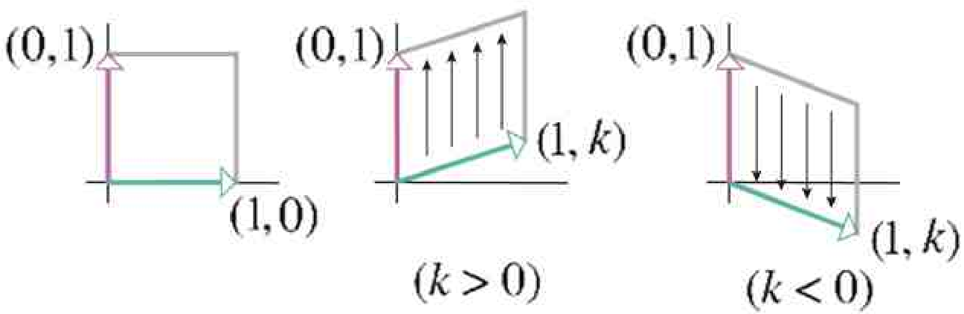
Vertical and Horizontal Compressions and Expansions of R^2

Table 6.2.2 part(b)

Operator	Illustration $T(x, y) = (x, ky)$	Effect on the Unit Square	Standard Matrix
Compression of R^2 in the y-direction with factor k $(0 \leq k < 1)$ 압축			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Expansion of R^2 in the y-direction with factor k $(k > 1)$ 확대			

Shears(층밀림 변환)

Table 6.2.3

Operator	Effect on the Unit Square	Standard Matrix
Shear of R^2 in the x-direction with factor k $T(x,y)=(x+ky,y)$		$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear of R^2 in the y-direction with factor k $T(x,y)=(x,y+kx)$		$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Example 5 Some Basic Linear Operators on R^2

In each part, describe the linear operator corresponding to A and show its effect on the unit square.

(a) $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) $A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (c) $A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

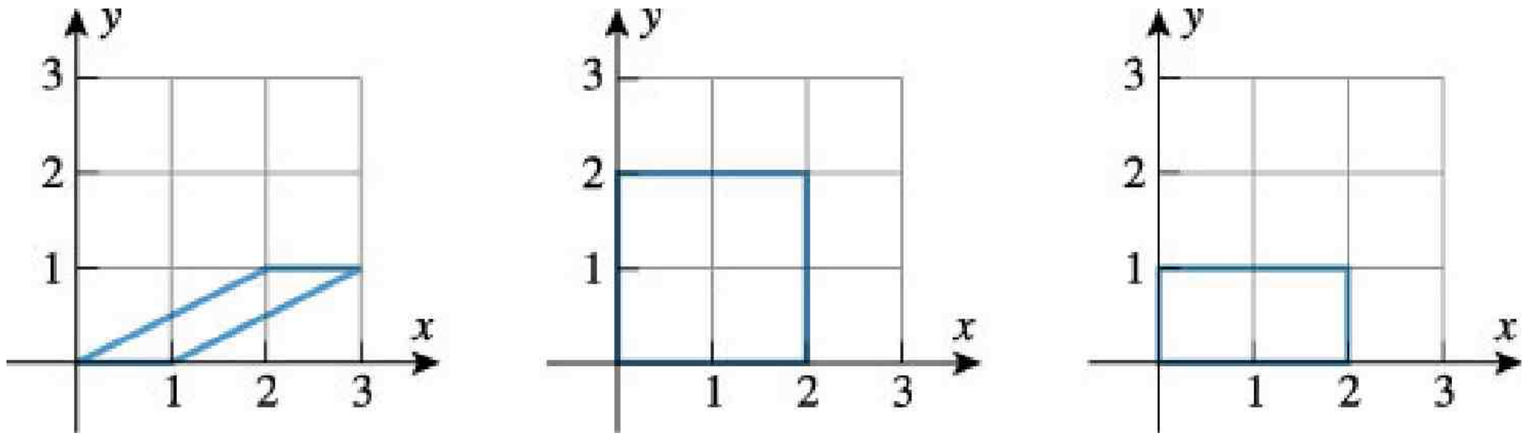
Sol.

By comparing to Tables 6.2.1-6.2.3:

A_1: A shear in the x-direction with factor 2

A_2: A dilation with factor 2

A_3: An expansion in the x-direction with factor 2



Example 6

Examples of linear transformations:

- Rotation
- Shear
- Compression

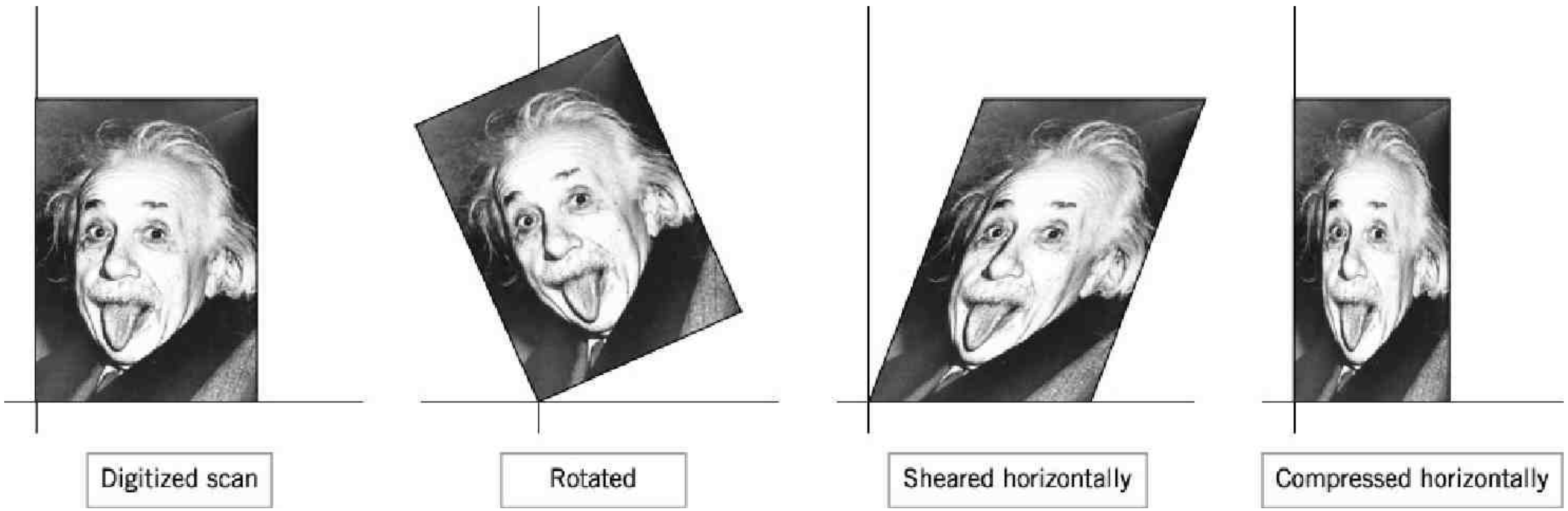
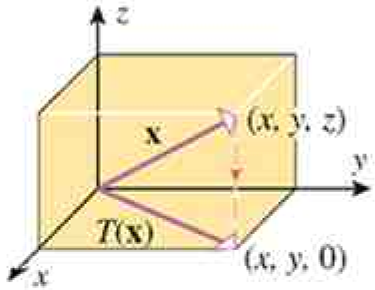
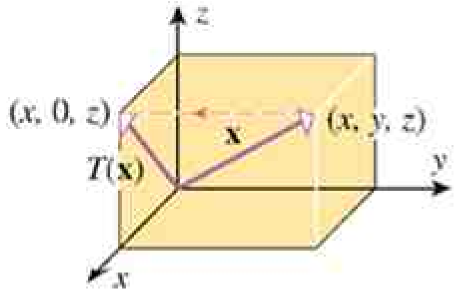
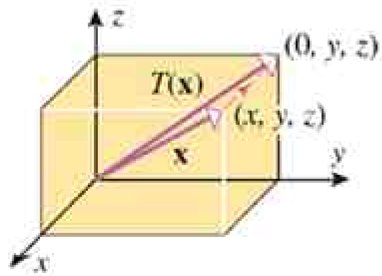


Table 6.2.4 Linear Operators on R^3

Operator	Illustration	Standard Matrix
Orthogonal Projection on the xy-plane $T(x,y,z)=(x,y,0)$ xy 평면으로의 정사영		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal Projection on the xz-plane $T(x,y,z)=(x,0,z)$ xz 평면으로의 정사영		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal Projection on the yz-plane $T(x,y,z)=(0,y,z)$ yz 평면으로의 정사영		$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3x3 Orthogonal Matrices Correspond to Linear Operators on R^3

2x2 orthogonal matrices correspond to rotations about the origin or reflections about lines through the origin in R^2.

Extend to 3x3 matrices

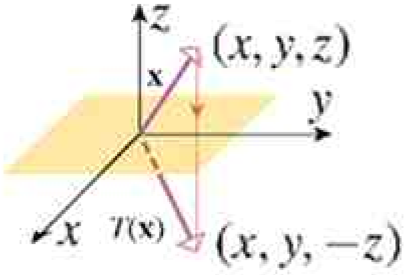
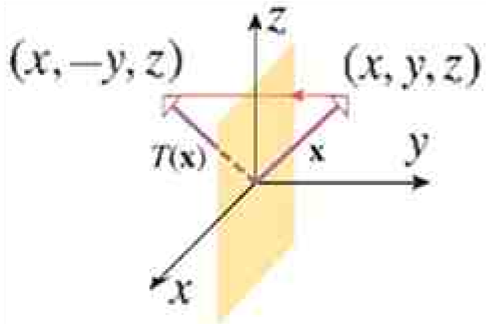
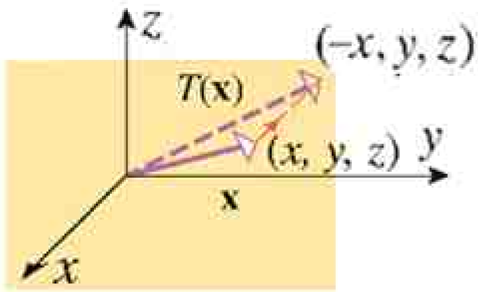
3x3 orthogonal matrices correspond to linear operators on R^3 of the following types:

Type 1: Rotations about lines through the origin.
Type 2: Rotations about planes through the origin.
Type 3: A rotation about a line through the origin followed by a reflection about the plane through the origin that is perpendicular to the line.

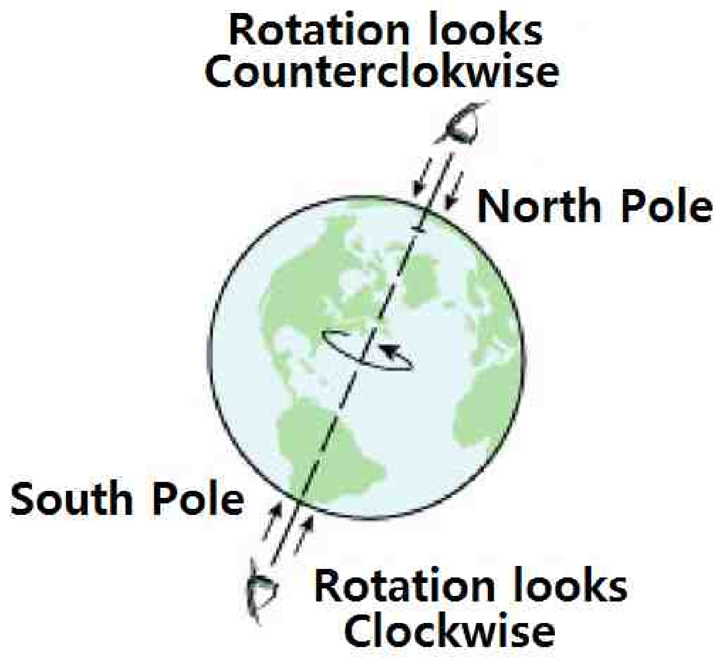
Let A be a 3x3 orthogonal matrix represents a rotation or a reflection, then

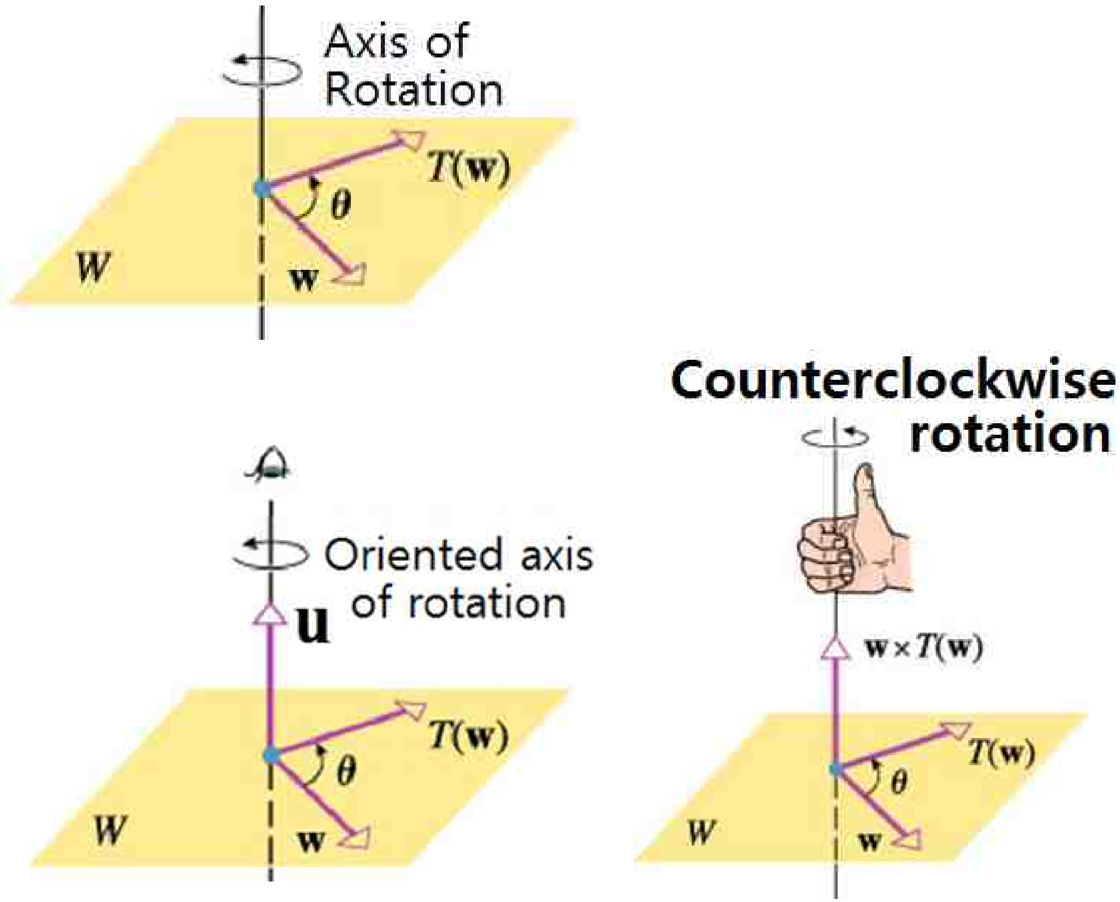
det(A)=1: Type 1, Rotation matrix
det(A)=-1: Type 2 or 3

Table 6.2.5 Reflections about Coordinate Planes

Operator	Illustration	Standard Matrix
Reflection about the xy-plane $T(x,y,z)=(x,y,-z)$ xy 평면에 대한 반사		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane $T(x,y,z)=(x,-y,z)$ xz 평면에 대한 반사		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane $T(x,y,z)=(-x,y,z)$ yz 평면에 대한 반사		$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rotations in R^3





$$\mathbf{u} = \mathbf{w} \times T(\mathbf{w}) \qquad (12)$$

Rotations in R^3-cont

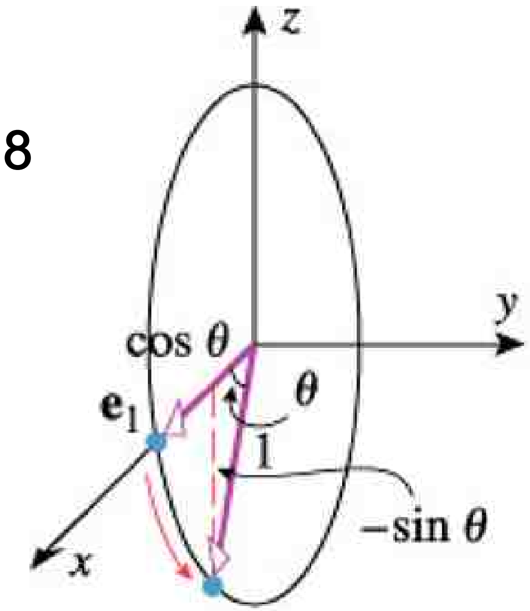
$R_{y,\theta}$: the standard matrix for a rotation about the positive y-axis through an angle θ

$e_1=(1,0,0) \xrightarrow{R_{y,\theta}} (\cos\theta, 0, -\sin\theta)$

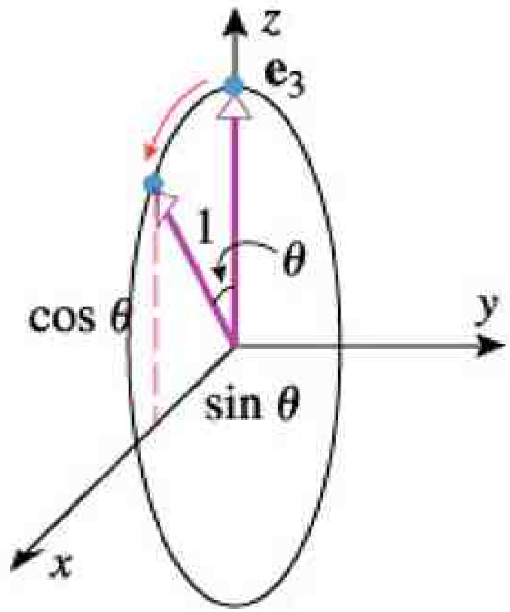
$e_2=(0,1,0) \xrightarrow{R_{y,\theta}} (0, 1, 0)$

$e_3=(0,0,1) \xrightarrow{R_{y,\theta}} (\sin\theta, 0, \cos\theta)$

Figure 6.2.8



$T(\mathbf{e}_1) = (\cos\theta, 0, -\sin\theta)$



$T(\mathbf{e}_3) = (\sin\theta, 0, \cos\theta)$

General Rotations

A complete analysis of general rotation in R^3 involves too much detail to present here.

So, only the following two basic problems are discussed.

1. Find the standard matrix for a rotation whose axis of rotation and angle of rotation are known. ⇒ Theorem 6.2.8
2. Given the standard matrix, find the axis and angle of rotation

Find the standard matrix for the rotation through the angle θ

Theorem 6.2.8 *If* $\mathbf{u} = (a, b, c)$ *is a unit vector, then the standard matrix* $R_{\mathbf{u},\theta}$ *for the rotation through the angle* θ *about an axis through the origin with orientation* $\mathbf{u}$ *is*

$$R_{\mathbf{u},\theta} = \begin{bmatrix} a^2(1-\cos\theta)+\cos\theta & ab(1-\cos\theta)-c\sin\theta & ac(1-\cos\theta)+b\sin\theta \\ ab(1-\cos\theta)+c\sin\theta & b^2(1-\cos\theta)+\cos\theta & bc(1-\cos\theta)-a\sin\theta \\ ac(1-\cos\theta)-b\sin\theta & bc(1-\cos\theta)+a\sin\theta & c^2(1-\cos\theta)+\cos\theta \end{bmatrix} \quad (13)$$

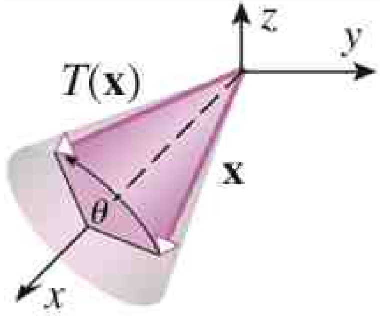
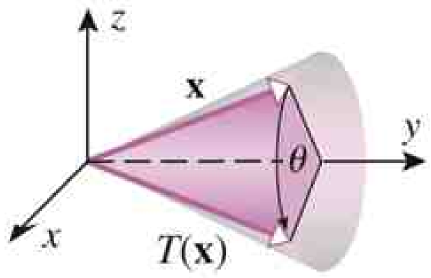
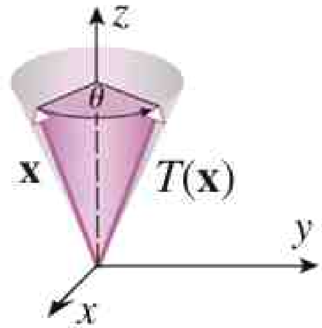
Proof: *Principles of Interactive Computer Graphics*, by W.M. Newman and Sproull, McGraw-Hill, New York, 1979

Several special cases are summarized in Table 6.2.6.

Rotations about standard axis.

- The rotation about x-axis: **u**=(1, 0, 0)
- The rotation about y-axis: **u**=(0, 1, 0)
- The rotation about z-axis: **u**=(0, 0, 1)

Table 6.2.6 Rotations about Standard Axis

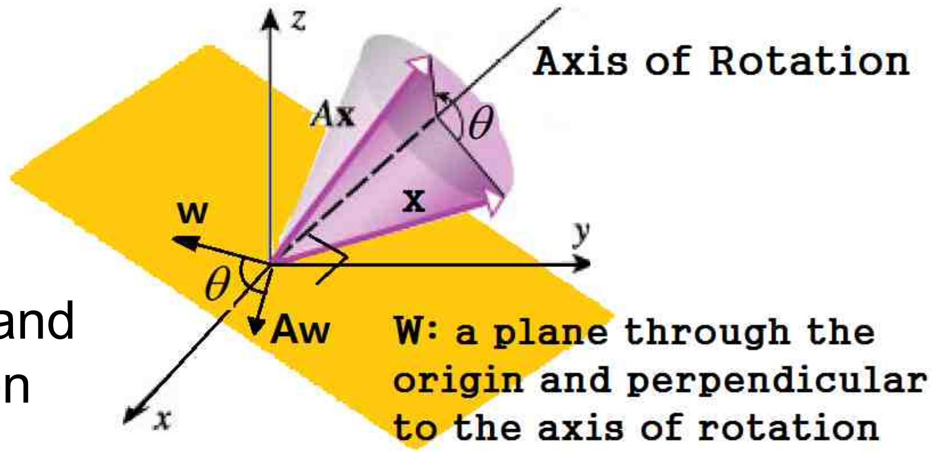
Operator	Illustration	Standard Matrix
Rotation about the positive x-axis through an angle θ		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Rotation about the positive y-axis through an angle θ		$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Rotation about the positive z-axis through an angle θ		$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Find the axis and angle of rotation for a given standard matrix

The axis of rotation consists of the fixed points of A.

$$\mathbf{Ax} = \mathbf{x}$$

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$$



W : a plane through the origin and perpendicular to the rotation of axis

$\mathbf{w}$: any nonzero vector in W

Angle of rotation: $\cos\theta = \dfrac{\mathbf{w} \cdot \mathbf{Aw}}{\|\mathbf{w}\|\|\mathbf{Aw}\|}$ (14)

Axis of rotation: $\mathbf{u} = \mathbf{w} \times \mathbf{Aw}$

Example 7 Rotation

(a) Show that the matrix represents a rotation about a line through the origin of R^3.

(b) Find the axis and the angle of rotation.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Sol.

(a) a rotation about a line through the origin

Rotation because of orthogonality and det(A)=1.

(b) the axis and the angle of rotation.

$$\mathbf{A}\mathbf{x} = \mathbf{x}$$

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$$

Example 7-cont

$(\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

the axis of rotation: the line through the origin that passes through the point (1,1,1)

the plane through the origin that passes through the point (1,1,1)

$$x+y+z=0$$

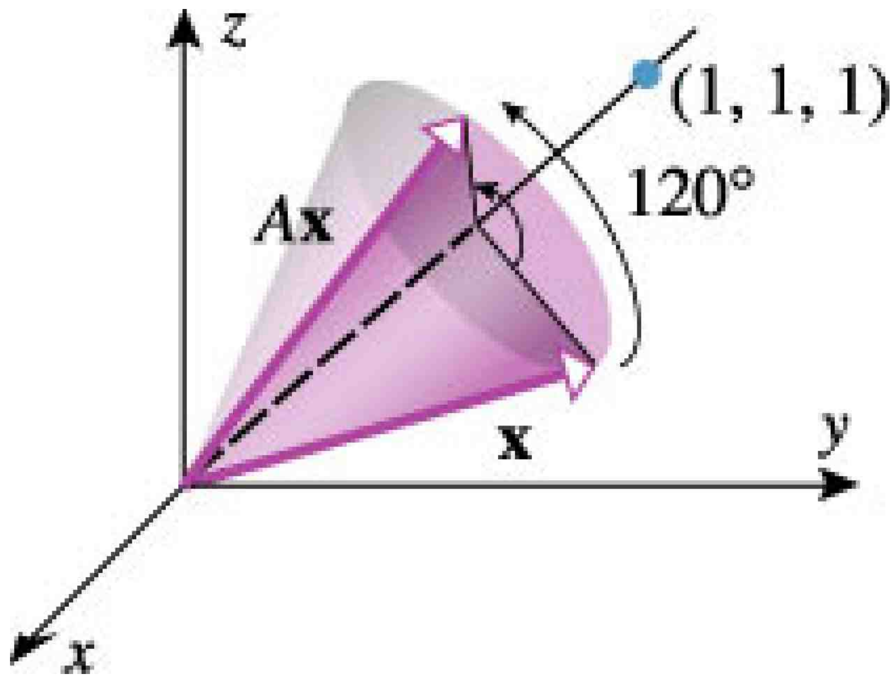
$$x=1,\ y=-1 \quad \Rightarrow \quad z=0 \quad \Rightarrow \quad \mathbf{w}=(1,-1,0)$$

$$\mathbf{w}=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ \mathbf{A}\mathbf{w}=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},\ \mathbf{w}\times\mathbf{A}\mathbf{w}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Example 7-cont

Rotation angle θ

$$\cos\theta = \frac{\mathbf{w}\cdot\mathbf{Aw}}{\|\mathbf{w}\|\|\mathbf{Aw}\|} = \frac{-1}{\sqrt{2}\sqrt{2}} = -\frac{1}{2} \tag{15}$$



A Formula for the Cosine of the Rotation Angle

$$R_{\mathbf{u},\theta} = \begin{bmatrix} a^2(1-\cos\theta)+\ \ \cos\theta & ab(1-\cos\theta)-c\sin\theta & ac(1-\cos\theta)+b\sin\theta \\ ab(1-\cos\theta)+c\sin\theta & b^2(1-\cos\theta)+\ \ \cos\theta & bc(1-\cos\theta)-a\sin\theta \\ ac(1-\cos\theta)-b\sin\theta & bc(1-\cos\theta)+a\sin\theta & c^2(1-\cos\theta)+\ \ \cos\theta \end{bmatrix} \quad (13)$$

$$\begin{aligned} \mathrm{tr}(A) &= (a^2+b^2+c^2)(1-\cos\theta)+3\cos\theta \\ &= 1-\cos\theta+3\cos\theta = 1+2\cos\theta \end{aligned}$$

$$\cos\theta = \frac{\mathrm{tr}(A)-1}{2} \quad (16)$$

If A is rotation matrix, then for any **x≠0**, the vector

$$\mathbf{v} = A\mathbf{x} + A^T\mathbf{x} + [1-\mathrm{tr}(A)]\mathbf{x} \quad (17)$$

is along the axis of rotation when **x** has its initial point at the origin.

Example 8 Example 7 Revisited

Use formulas (16) and (17) to solve Example 7(b) which is to find the axis and the angle of rotation.

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Sol.

1. the axis of rotation:

Let $\mathbf{x} = \mathbf{e}_1 = (1, 0, 0)$

$$\mathbf{v} = \mathbf{A}\mathbf{x} + \mathbf{A}^T\mathbf{x} + [1 - \mathrm{tr}(\mathbf{A})]\mathbf{x} = (\mathbf{A} + \mathbf{A}^T + \mathbf{I})\mathbf{x} \qquad (17)$$

$$= (\mathbf{A} + \mathbf{A}^T + \mathbf{I})\mathbf{e}_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

2. the angle of rotation: cosθ=[tr(A)-1]/2

$$\mathrm{tr}(\mathbf{A}) = 0 \qquad \cos\theta = \frac{\mathrm{tr}(\mathbf{A}) - 1}{2} = -\frac{1}{2} \qquad (16)$$

6.3 Kernel and Range(핵과 치역)

KERNEL OF A LINEAR TRANSFORMATION

If $\mathbf{x}=t\mathbf{v}$ a line through the origin of R^n,
and if T is a linear operator on R^n,

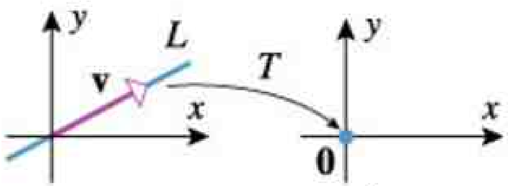
then the image of the line through the transformation T is the set of vectors of the form

$$T(\mathbf{x}) = T(t\mathbf{v}) = tT(\mathbf{v})$$

Geometrically, there are two possibilities for this image:

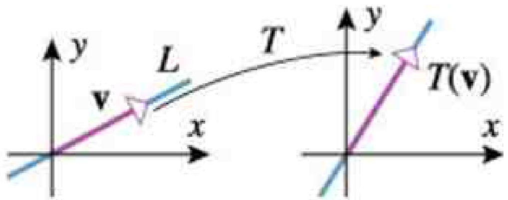
1. If $T(\mathbf{v})=\mathbf{0}$, then $T(\mathbf{x})=\mathbf{0}$ for all $\mathbf{x}$, so the image is the single point **0**.
2. If $T(\mathbf{v})\neq\mathbf{0}$, then the image is the line through the origin determined by $T(\mathbf{v})$.

Case 1



T maps L into the point **0** if $T(\mathbf{v})=\mathbf{0}$.

Case 2



T maps L into the line spanned by $T(\mathbf{v})$ if $T(\mathbf{v})\neq\mathbf{0}$.

6.3 Kernel and Range(핵과 치역)

Similarly, if $\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$ is a plane through the origin of R^n,

then the image of this plane through the transformation T is the set of vectors of the form

$$T(\mathbf{x}) = T(t_1\mathbf{v}_1 + t_2\mathbf{v}_2) = t_1 T(\mathbf{v}_1) + t_2 T(\mathbf{v}_2)$$

Geometrically, there are three possibilities for this image:

1. If $T(\mathbf{v}_1) = T(\mathbf{v}_2) = \mathbf{0}$, then $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$, so the image is the single point **0**.
2. If $T(\mathbf{v}_1) \neq \mathbf{0}$ and $T(\mathbf{v}_2) \neq \mathbf{0}$, and if $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$ are not scalar multiples of one other, then the image is a plane through the origin.
3. The image is a line through the origin in the remaining cases.

Definition 6.3.1 Kernel

Definition 6.3.1 If $T: R^n \rightarrow R^m$ is a linear transformation, then the set of vectors in R^n that T maps into $\mathbf{0}$ is called the ***kernel*** of T and is denoted by $\ker(T)$.

Example 1 Kernels of Some Basic Operators

Find the kernel of the standard linear operator on R^3.

(a) The zero operator $T_0(\mathbf{x})=\mathbf{0}$.
(b) The identity operator $T_I(\mathbf{x})=I\mathbf{x}=\mathbf{x}$.
(c) The orthogonal projection T on the xy-plane.
(d) A rotation T about a line through the origin through an angle θ.

Sol.

(a) The zero operator $T_0(\mathbf{x})=\mathbf{0}$.

$$\ker(T_0) = R^3$$

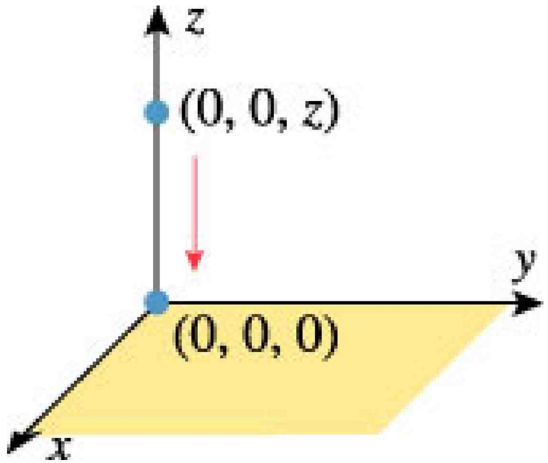
Example 1 Kernels of Some Basic Operators-cont

(b) The identity operator $T_I(\mathbf{x})=I\mathbf{x}=\mathbf{x}$.

$$\ker(T_{\mathbf{I}}) = \{\mathbf{0}\}$$

(c) The orthogonal projection T on the xy-plane.

$$\ker(T) = \{z \text{ axis}\}$$



(d) A rotation T about a line through the origin through an angle θ.

$$\ker(T) = \{\mathbf{0}\}$$

Theorem 6.3.2 Kernel of a Linear Transformation

Theorem 6.3.2 *If $T : R^n \to R^m$ is a linear transformation, then the kernel of T is a subspace of R^n.*

Proof

Definition 3.4.1 A nonempty set of vectors in R^n is called a **subspace** of R^n if it is closed under scalar multiplication and addition.

1. Closed under scalar multiplication:

$$T(c\mathbf{u}) = cT(\mathbf{u}) = c\mathbf{0} = \mathbf{0}$$

2. Closed under addition:

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

Kernel of a Matrix Operator

Theorem 6.3.3 *If A is an $m \times n$ matrix, then the kernel of the corresponding linear transformation is the solution space of $A\mathbf{x} = \mathbf{0}$.*

Proof

Theorem 6.1.4 *Let $T : R^n \to R^m$ be a linear transformation, then $T(\mathbf{x})$ can be expressed as*

$$T(\mathbf{x}) = A\mathbf{x} \tag{13}$$

Definition 6.3.1 If $T : R^n \to R^m$ is a linear transformation, then the set of vectors in R^n that T maps into $\mathbf{0}$ is called the ***kernel*** of T and is denoted by $\ker(T)$.

Example 2 Find the Kernel by the Theorem 6.3.3

Example 2 Find the Kernel by the Theorem 6.3.3

Find the kernel by the Theorem 6.3.3.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol.

Theorem 6.3.3 *If A is an $m \times n$ matrix, then the kernel of the corresponding linear transformation is the solution space of $A\mathbf{x} = \mathbf{0}$.*

$$\mathbf{Ax} = \mathbf{0} \longrightarrow x = 0, \quad y = 0, \quad z = t$$: z-axis

Definition 6.3.4 Null Space of a Matrix

Definition 6.3.4 If A is an $m \times n$ matrix, then the solution space of the linear system $A\mathbf{x} = \mathbf{0}$, or, equivalently, the kernel of the transformation T_A, is called the ***null space*** of the matrix A and is denoted by null(A).

Null(A)=ker(T_A)

Example 3 Null Space of a Matrix

Find the null space of the matrix.

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

Sol.

$\mathbf{Ax} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{x} = r\mathbf{v}_1 + s\mathbf{v}_2 + t\mathbf{v}_3$: by Example 7 in Sec. 2.2

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{null}(\mathbf{A}) = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

Theorem 6.3.5

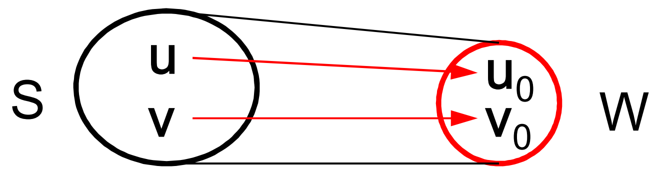
Theorem 6.3.5 *If $T : R^n \rightarrow R^m$ is a linear transformation, then T maps subspaces of R^n into subspaces of R^m.*

Proof

Let S: any subspace of R^n,
and W=T(S) be its image under T.

then → Is W a subspace?

Closed under
1. scalar multiplication
2. vector addition



For any vectors **u** and **v** in W, there exist $\mathbf{u}_0$ and $\mathbf{v}_0$ in S such that

$$\mathbf{u}=T(\mathbf{u}_0) \text{ and } \mathbf{v}=T(\mathbf{v}_0)$$

For any scalar c, $c\mathbf{u}=cT(\mathbf{u}_0)=T(c\mathbf{u}_0)\in W$ since $c\mathbf{u}_0\in S$ which is a subspace.

$\mathbf{u}+\mathbf{v}=T(\mathbf{u}_0)+T(\mathbf{v}_0)=T(\mathbf{u}_0+\mathbf{v}_0)\in W$ since $\mathbf{u}_0+\mathbf{v}_0\in S$ which is a subspace.

Range of a Linear Transformation

Definition 6.3.6 If $T: R^n \rightarrow R^m$ is a linear transformation, then the ***range*** of T, denoted by $\mathrm{ran}(T)$, is the set of all vectors in R^m that are images of at least one vector in R^n. Stated another way, $\mathrm{ran}(T)$ is the image of the domain R^n under the transformation T.

Example 4 Ranges of Some Basic Operators on R^3

Describe the ranges of the following linear operators on R^3.

(a) The zero operator $T_0(\mathbf{x})=\mathbf{0}$.
(b) The identity operator $T_I(\mathbf{x})=I\mathbf{x}=\mathbf{x}$.
(c) The orthogonal projection T on the xy-plane.
(d) A rotation T about a line through the origin through an angle θ.

Sol.

(a) The zero operator $T_0(\mathbf{x})=\mathbf{0}$.

$$\mathrm{ran}(T_0) = \{\mathbf{0}\}$$

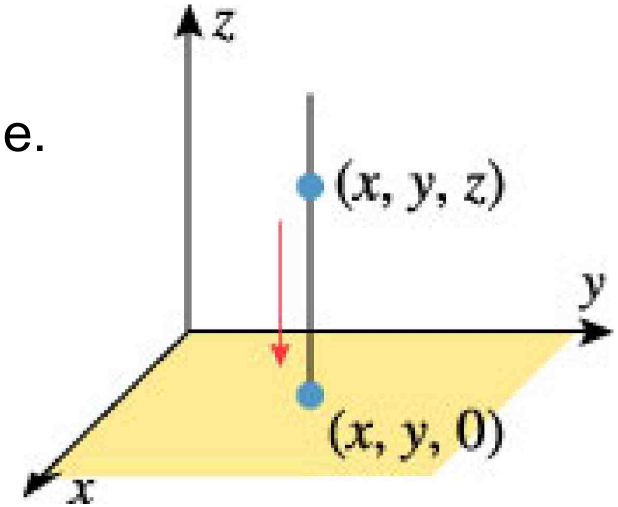
Example 4 Range of Some Basic Operators-cont

(b) The identity operator $T_I(\mathbf{x})=\mathbf{I}\mathbf{x}=\mathbf{x}$.

$$\mathrm{ran}(T_{\mathbf{I}}) = R^3$$

(c) The orthogonal projection T on the xy-plane.

$$\mathrm{ran}(T) = \{xy \text{ plane}\}$$



(d) A rotation T about a line through the origin through an angle θ.

$$\mathrm{ran}(T) = R^3$$

Theorem 6.3.7 and 6.3.8

Theorem 6.3.7 *If $T : R^n \to R^m$ is a linear transformation, then* $\mathrm{ran}(T)$ *is a subspace of R^m.*

Special case of Theorem 6.3.5.

Theorem 6.3.8 *If A is an $m \times n$ matrix, then the range of the corresponding linear transformation is the column space of A.*

$$T_A(\mathbf{x}) = A\mathbf{x} = [\,\mathbf{a}_1\,\mathbf{a}_2 \cdots \mathbf{a}_n] \, [x_1\; x_2 \;\cdots\; x_n]^{\mathsf{T}}$$
$$= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Theorem 3.5.5 *A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if* $\mathbf{b}$ *is in the column space of A.*

Example 5 Range of a Matrix Operator

Solve Example 4(c) by the Theorem 6.3.8 and considering the standard matrix for the projection.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 4 (c) Find the kernel of the orthogonal projection T on the xy-plane.

Sol.

Theorem 6.3.8 *If A is an $m \times n$ matrix, then the range of the corresponding linear transformation is the column space of A.*

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Example 6 Column Space of a Matrix

Determine

(a) whether **b** is in the column space of A,

(b) and, if so, express it as a linear combination of the column vectors of A.

$$A = \begin{bmatrix} 1 & -8 & -7 & -4 \\ 2 & -3 & -1 & 5 \\ 3 & 2 & 5 & 14 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 8 \\ -10 \\ -28 \end{bmatrix}$$

Sol.

Ax=b

$$\begin{bmatrix} 1 & -8 & -7 & -4 & -8 \\ 2 & -3 & -1 & 5 & -10 \\ 3 & 2 & 5 & 14 & -28 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 4 & -8 \\ 0 & 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= -8 - s - 4t \\ x_2 &= -2 - s - t \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

Let $s = 0,\ t = 0$, then $x_1 = -8,\ x_2 = -2,\ x_3 = 0,\ x_4 = 0$

$$\mathbf{b} = \begin{bmatrix} 8 \\ -10 \\ -28 \end{bmatrix} = -8 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} -8 \\ -3 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -7 \\ -1 \\ 5 \end{bmatrix} + 0 \begin{bmatrix} -4 \\ 5 \\ 14 \end{bmatrix}$$

Existence and Uniqueness Issues

Two questions about a linear transformation $T : R^n \rightarrow R^m$:

- **The Existence Question**: Is every vector in R^m the image of at least one vector in R^n? (Fig. 6.3.4)
- **The Uniqueness Question**: Can two different vectors in R^n have the same image in R^m? (Fig. 6.3.5)

The Existence Question

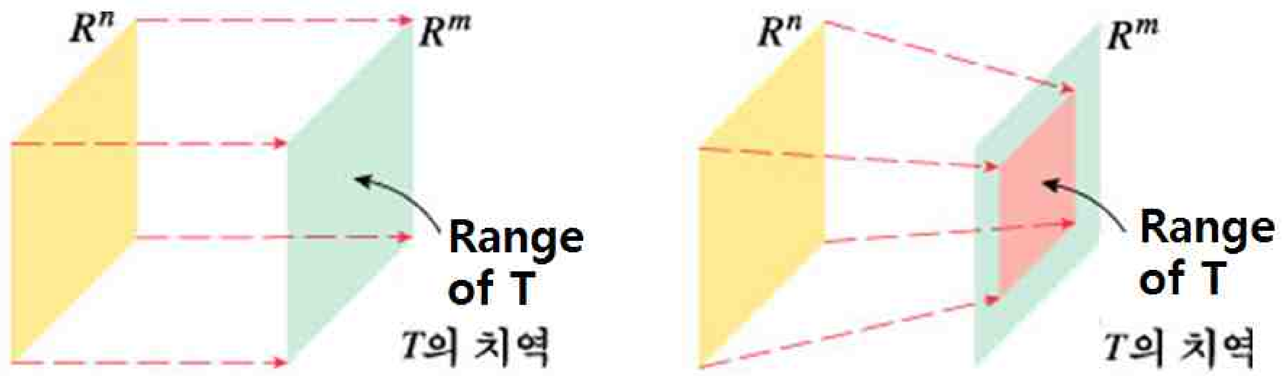


Fig. 6.3.4

The range is R^m, so every vector in R^m is the image of at least one vector in R^n.

There are vectors in R^m that are not images of any vectors in R^n.

Existence and Uniqueness Issues-cont

- **The Uniqueness Question**: Can two different vectors in R^n have the same image in R^m? (Fig. 6.3.5)

The Uniqueness Question

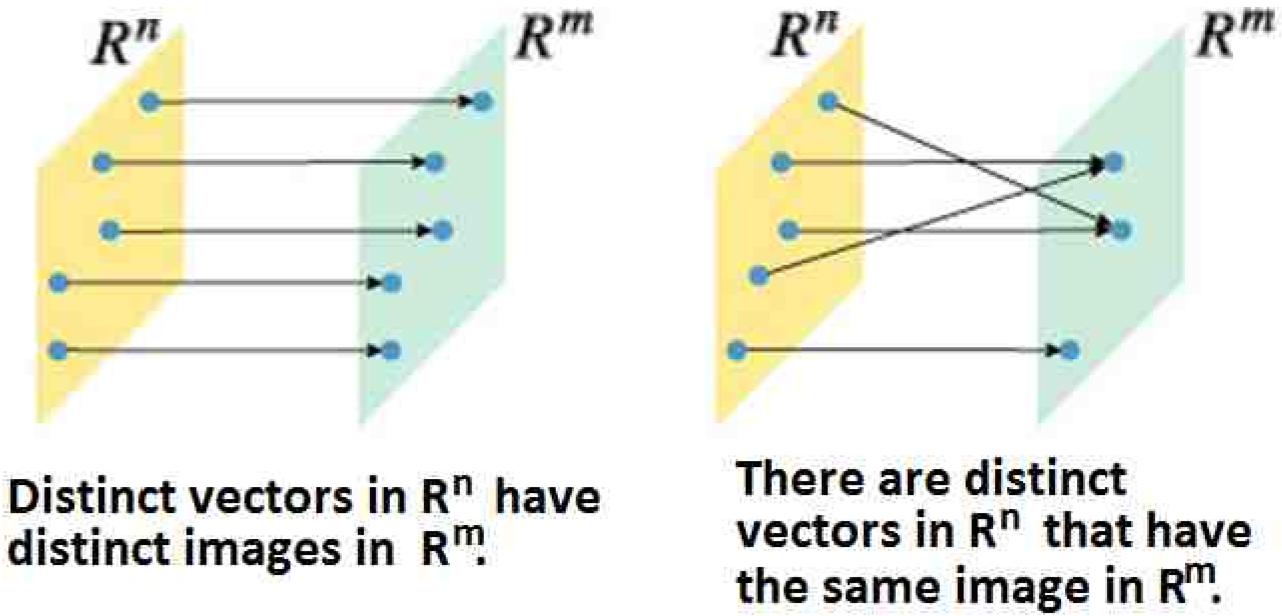
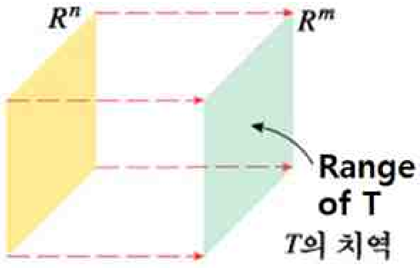


Fig. 6.3.5

Definitions 6.3.9 and 6.3.10

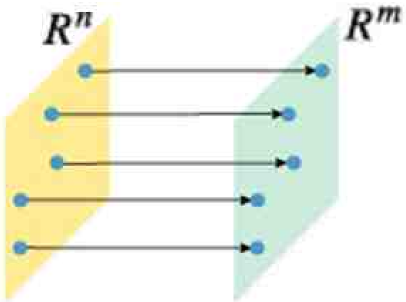
Definition 6.3.9 A transformation $T: R^n \rightarrow R^m$ is said to be ***onto*** if its range is the entire codomain R^m; that is, every vector in R^m is the image of at least one vector in R^n.

onto



The range is R^m, so every vector in R^m is the image of at least one vector in R^n.

one-to-one
1-1



Distinct vectors in R^n have distinct images in R^m.

Definition 6.3.10 A transformation $T: R^n \rightarrow R^m$ is said to be ***one-to-one*** (sometimes written 1–1) if T maps distinct vectors in R^n into distinct vectors in R^m.

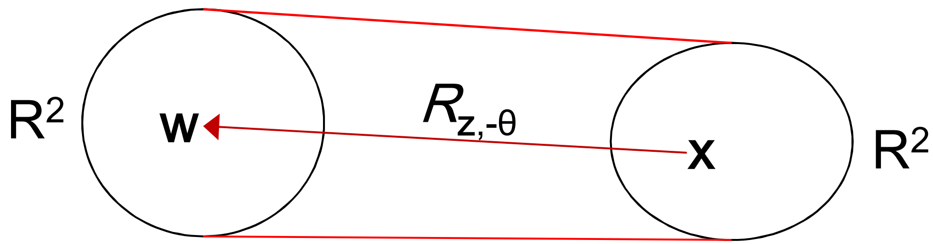
Example 7 One-to-One and Onto

Let T: $R^2 \rightarrow R^2$ be the operator that rotates each vector in the xy-plane about the origin through an angle θ.
Show that the operator is
(a) One-to-one
(b) Onto.

Sol.

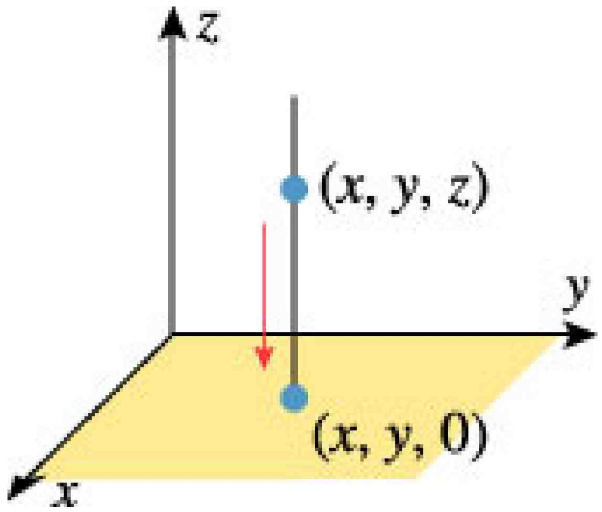
(a) One-to-one because rotating distinct vectors through the same angle produces distinct vectors.

(b) Onto because any vector **x** in R^2 is the image of some vector **w** under the rotation -θ.



Example 8 Neither One-to-One Nor Onto

Let T: $R^3 \rightarrow R^3$ be the orthogonal projection on the *xy*-plane.
Show that the operator is
(a) Neither one-to-one
(b) Nor onto.



Sol.

(a) Neither one-to-one

T: $(x,y,z\neq 0) \rightarrow (x,y,0)$
Thus, not one-to-one mapping

(b) Nor onto.

$(x,y,z\neq 0) \notin R^3$De
Thus, not onto mapping

Example 9 One-to-One but Not Onto

Let $T: R^2 \rightarrow R^3$ be the linear transformation defined by the formula $T(x, y)=(x, y, 0)$. Show that the operator is

(a) Linear

(b) One-to-one

(c) But not Onto.

Sol. Let $\mathbf{x}_1=(x_1,y_1)$, $\mathbf{x}_2=(x_2,y_2)$

(a) Linear: $T(c\mathbf{x}_1)=(cx_1,cy_1,c\cdot 0)= c(x_1,y_1,0)= cT(\mathbf{x}_1)$,

$T(\mathbf{x}_1+\mathbf{x}_2)=(x_1+y_1, x_1+y_1,0+0)=(x_1, y_1,0)+(x_2, y_2,0)=T(\mathbf{x}_1)+T(\mathbf{x}_2)$

(b) one-to-one

If $\mathbf{x}_1\neq\mathbf{x}_2$, then $(x_1,y_1) \neq(x_2,y_2)$. $\therefore T(x_1,y_1)=(x_1,y_1,0) \neq T(x_2,y_2)=(x_2,y_2,0)$

If $T(\mathbf{x}_1)\neq T(\mathbf{x}_2)$, then $(x_1,y_1,0) \neq(x_2,y_2,0)$. $\therefore (x_1,y_1)=\mathbf{x}_1 \neq\mathbf{x}_2=(x_2,y_2)$,

(b) Not onto.

$(x,y,z\neq 0)\notin R^3$. Thus, not onto mapping

Example 10 Onto but Not One-to-One

Let T: $R^3 \rightarrow R^2$ be the linear transformation defined by the formula $T(x, y, z)=(x, y)$.

Show that the operator is

(a) Onto

(b) But not one-to-one.

Sol.

(a) Onto

Any vector $\mathbf{w}=(x,y)$ in R^2 is the image of vectors (x,y,z) in R^3.

(b) But not one-to-one.

T maps vectors (x,y,z) in R^3 to the same point $\mathbf{w}=(x,y)$ in R^2.

Theorem 6.3.11

Theorem 6.3.11 *If $T : R^n \to R^m$ is a linear transformation, then the following statements are equivalent.*

(a) *T is one-to-one.*

(b) $\ker(T) = \{\mathbf{0}\}$.

Proof

(a)⇒(b):

T is linear. Thus, $T(\mathbf{0})=T(0\cdot\mathbf{0})=0T(\mathbf{0})=\mathbf{0}$(Theorem 6.1.3)
Since T is one-to-one, $\ker(T)=\mathbf{0}$.

(b)⇒(a): Let $\mathbf{x}_1, \mathbf{x}_2 \in R^n$

If $\mathbf{x}_1 \neq \mathbf{x}_2$, then $\mathbf{x}_1-\mathbf{x}_2 \neq \mathbf{0}$. $T(\mathbf{x}_1-\mathbf{x}_2)=T(\mathbf{x}_1)-T(\mathbf{x}_2)\neq\mathbf{0}$.($\because \ker(T)=\mathbf{0}$)
Thus, $T(\mathbf{x}_1)\neq T(\mathbf{x}_2)$

If $T(\mathbf{x}_1)\neq T(\mathbf{x}_2)$, then $T(\mathbf{x}_1)-T(\mathbf{x}_2)\neq\mathbf{0}$. $T(\mathbf{x}_1)-T(\mathbf{x}_2)=T(\mathbf{x}_1-\mathbf{x}_2)\neq\mathbf{0}$.
Thus, $\mathbf{x}_1\neq\mathbf{x}_2$($\because \ker(T)=\mathbf{0}$)

One-to-One and Onto from the Viewpoint of Linear Systems

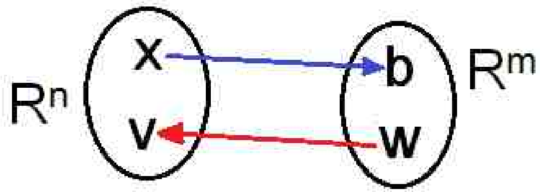
Theorem 6.3.12 *If A is an $m \times n$ matrix, then the corresponding linear transformation $T_A : R^n \rightarrow R^m$ is one-to-one if and only if the the linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

Theorem 6.3.11 says that ker(T)=**0** if and only if T is one-to-one.
ker(T)=**0** means that A**x**=**0** has only one solution **x**=**0**, which is trivial.

Theorem 6.3.13 *If A is an $m \times n$ matrix, then the corresponding linear transformation $T_A : R^n \rightarrow R^m$ is onto if and only if the linear system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in R^n.*

→: Self-evident.

←: Consistency of A**x**=**b** for every **b** in R^m means that, for any vector **w** in R^m, there exists at least a vector **v** in R^n. Thus, T_A is onto.



Example 11 Mapping "Bigger" Spaces into "Smaller" Spaces

Example 11 Mapping

T: $R^n \rightarrow R^m$, n>m, A: standard matrix for A.
Then T is not one-to-one without computation.

Sol.

Since n>m, A**x**=**0** has nontrivial solutions without computation.

The linear system A**x**=**0** has more variables than equations to satisfy.

Thus, by Theorem 6.3.12, T is not one-to-one.

Theorem 6.3.14

Theorem 6.3.14 *If $T : R^n \to R^n$ is a linear operator on R^n, then T is one-to-one if and only if it is onto.*

Proof

Let A be the standard matrix for T.

By Theorem 4.4.7(d)(e), the system A**x**=**0** has only the trivial solution if and only if A**x**=**b** is consistent for every **b** in R^n.

Combining this with Theorem 6.3.13 completes the proof.

Theorem 4.4.7 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(*d*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every vector* $\mathbf{b}$ *in* R^n.

Example 12 Example 7 and 8 Revisited

Example 12 Example 7 and 8 Revisited

Ex. 7: Rotation is both one-to-one and onto.
Ex. 8: Orthogonal projection is neither one-to-one nor onto.

The rotation and the orthogonal projection are both linear operators.

Thus, the “both” and “neither” are consistent with Theorem 6.3.14

Theorem 6.3.15 A Unifying Theorem

Theorem 6.3.15 *If T_A is the linear operator on R^n with $n \times n$ standard matrix A, then the following statements are equivalent.*

(a) *The reduced row echelon form of A is I_n.*

(b) *A is expressible as a product of elementary matrices.*

(c) *A is invertible.*

(d) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every vector* $\mathbf{b}$ *in* R^n.

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every vector* $\mathbf{b}$ *in* R^n.

(g) *The column vectors of A are linearly independent.*

(h) *The row vectors of A are linearly independent.*

(i) $\det(A) \neq 0$.

(j) $\lambda = 0$ *is not an eigenvalue of* A.

(k) T_A *is one-to-one.*

(l) T_A *is onto.*

Example 13 Examples 7 and 8 Revisited Using Determinants

Show that

(1) The rotation about the origin is one-to-one-and onto.

(2) The orthogonal projection of R^3 on the xy-plane is neither one-to-one nor onto.

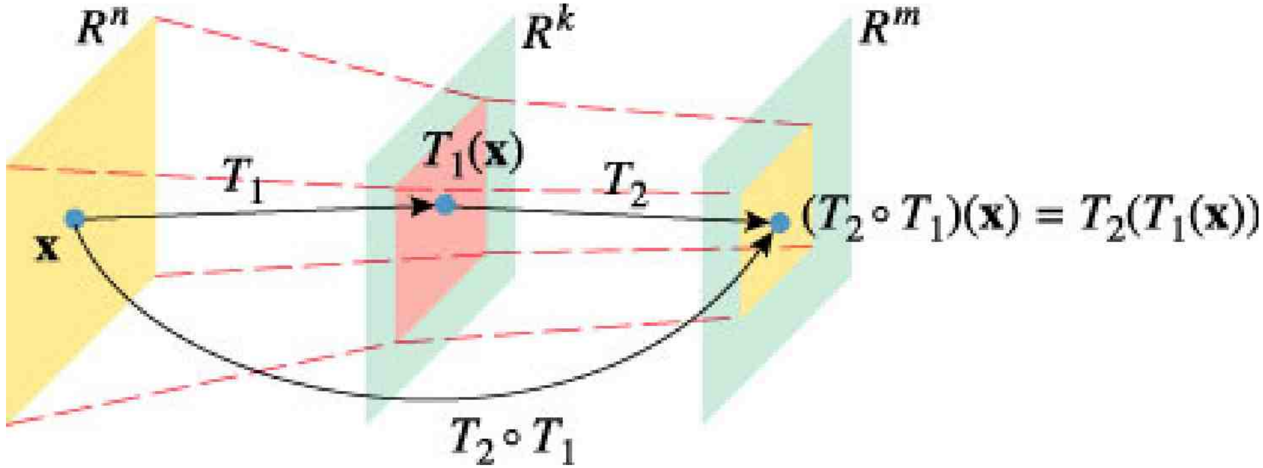
Sol.

$$\det(R_\theta) = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0$$

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

COMPOSITIONS OF LINEAR TRANSFORMATIONS

$$(T_2 \circ T_1)(\mathbf{x}) = T_2(T_1(\mathbf{x})) \tag{1}$$



Theorem 6.4.1

Theorem 6.4.1 *If $T_1 : R^n \to R^k$ and $T_2 : R^k \to R^m$ are both linear transformations, then $(T_2 \circ T_1) : R^n \to R^m$ is also a linear transformation.*

Proof

$T_2 \circ T_1$ (read, "T_2 circle T_1")

Additivity:

$$\begin{aligned}(T_2 \circ T_1)(\mathbf{u}+\mathbf{v}) = T_2(T_1(\mathbf{u}+\mathbf{v})) &= T_2(T_1(\mathbf{u}) + T_1(\mathbf{v})) \\ &= T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) \\ &= (T_2 \circ T_1)(\mathbf{u}) + (T_2 \circ T_1)(\mathbf{v})\end{aligned}$$

Homogeneity:

$$(T_2 \circ T_1)(c\mathbf{u}) = T_2(T_1(c\mathbf{u})) = T_2(cT_1(\mathbf{u})) = cT_2(T_1(\mathbf{u})) = c(T_2 \circ T_1)(\mathbf{u})$$

Theorem 6.4.2

Theorem 6.4.2 *If A is a $k \times n$ matrix and B is an $m \times k$ matrix, then the $m \times n$ matrix BA is the standard matrix for the composition of the linear transformation corresponding to B with the linear linear transformation corresponding to A.*

Proof

Suppose that $T_1 : R^n \to R^k$ has standard matrix $[T_1]$
and that $T_2 : R^k \to R^m$ has standard matrix $[T_2]$.

$$(T_2 \circ T_1)(\mathbf{e}_i) = T_2(T_1(\mathbf{e}_i)) = T_2([T_1]\mathbf{e}_i) = [T_2]([T_1]\mathbf{e}_i) = ([T_2][T_1])\mathbf{e}_i$$

which implies that $[T_2][T_1]$ is the standard matrix for $T_2 \circ T_1$.

Thus,

$$[T_2 \circ T_1] = [T_2][T_1] \tag{2}$$

$$T_B \circ T_A = T_{BA} \tag{3}$$

Example 1 Composing Rotations in R^2

Let T_1 and T_2 : $R^2 \to R^2$ be the rotations about the origin of R^2 through an angle θ_1 and θ_2, respectively.

1. Find the standard matrices for T_1 and T_2 .
2. Find the standard matrix for $T_2 \circ T_1$.

Sol.

1. Find the standard matrices for T_1 and T_2 .

$$R_{\theta_1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \qquad R_{\theta_2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

2. Find the standard matrix for $T_2 \circ T_1$.

$$(T_2 \circ T_1)(\mathbf{x}) = T_2(T_1(\mathbf{x}))$$

Example 1 Composing Rotations in $\mathbf{R}^2$-continued

$$R_{\theta_1+\theta_2} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$\begin{aligned}
R_{\theta_2} R_{\theta_1} &= \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & -\cos\theta_2\sin\theta_1 - \sin\theta_2\cos\theta_1 \\ \sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1 & -\sin\theta_2\sin\theta_1 + \cos\theta_2\cos\theta_1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}
\end{aligned}$$

Example 2 Composing Reflections

Find the standard matrices for rotations :

1. about the line through the origin making angle of θ with the x-axis.
2. about the line through the origin with angle θ_1 and about the line with angle θ_2, consecutively

Sol.

By Formula (18) in section 6.1: $H_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

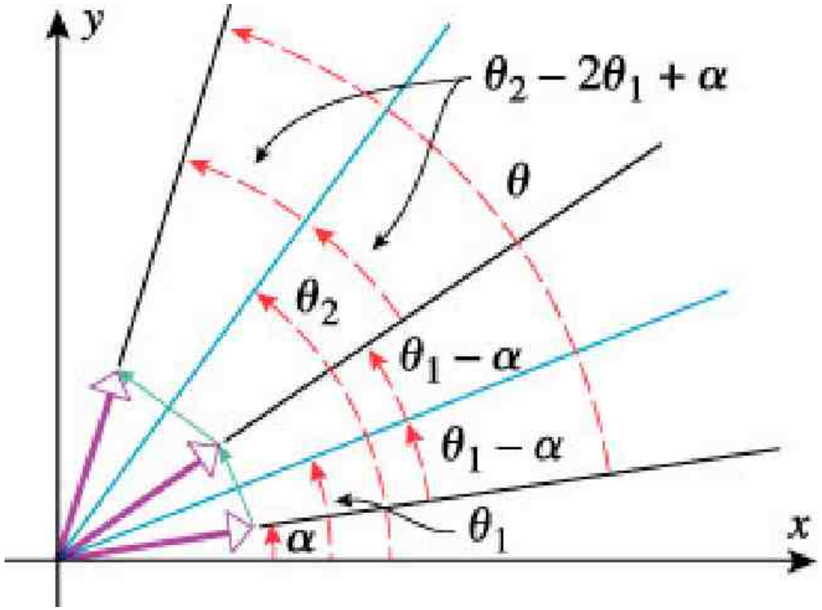
$$H_{\theta_2} H_{\theta_1} = \begin{bmatrix} \cos 2\theta_2 & \sin 2\theta_2 \\ \sin 2\theta_2 & -\cos 2\theta_2 \end{bmatrix} \begin{bmatrix} \cos 2\theta_1 & \sin 2\theta_1 \\ \sin 2\theta_1 & -\cos 2\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta_2 \cos 2\theta_1 + \sin 2\theta_2 \sin 2\theta_1 & \cos 2\theta_2 \sin 2\theta_1 - \sin 2\theta_2 \cos 2\theta_1 \\ \sin 2\theta_2 \cos 2\theta_1 - \cos 2\theta_2 \sin 2\theta_1 & \sin 2\theta_2 \sin 2\theta_1 + \cos 2\theta_2 \cos 2\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta_2 - 2\theta_1) & -\sin(2\theta_2 - 2\theta_1) \\ \sin(2\theta_2 - 2\theta_1) & \cos(2\theta_2 - 2\theta_1) \end{bmatrix}$$

Example 2

$$H_{\theta_2} H_{\theta_1} = R_{2(\theta_2 - \theta_1)}$$



$$\theta = 2(\theta_2 - 2\theta_1 + \alpha) + 2(\theta_1 - \alpha) = 2\theta_2 - 2\theta_1$$

Example 3 Composition Is Not a Commutative Operation

(a) Find the standard matrix for the linear operator on R^2 that first shears by a factor of 2 in the x-direction and then reflects about the line $y = x$.

(b) Find the standard matrix for the linear operator on R^2 that first reflects about the line $y = x$ and then shears by a factor of 2 in the x-direction.

Sol.

Let A_1 and A_2 be the standard matrix for the shear and reflection, respectively.

$$A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(a) Thus, the standard matrix for the shear followed by the reflection is

$$A_2 A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

(b) The standard matrix for the reflection followed by the shear is

$$A_1 A_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Example 3 Composition Is Not a Commutative Operation

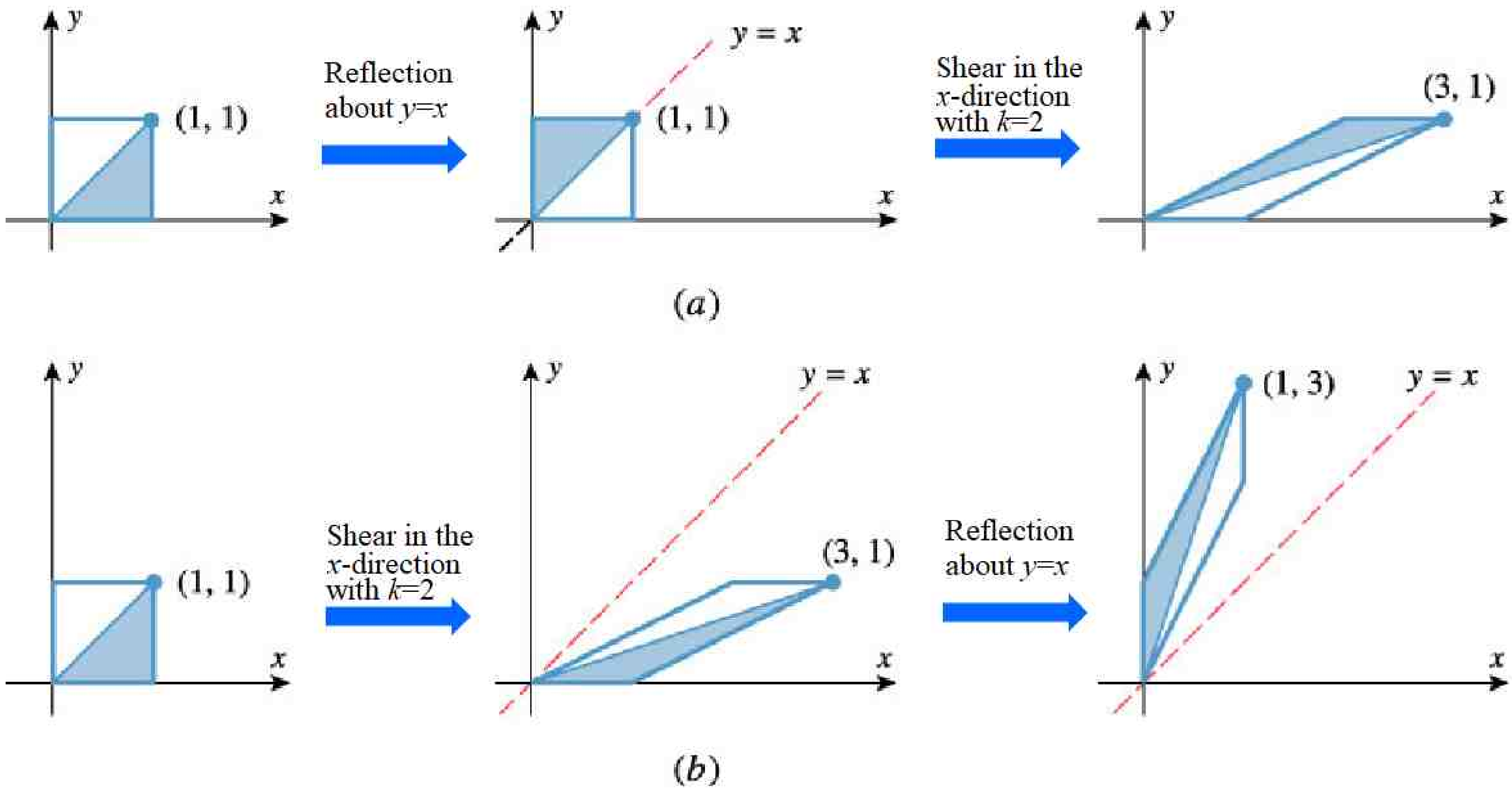


Figure 6.4.3

Composition of Three or more Linear Transformations

If $T_1 : R^n \to R^k, \quad T_2 : R^k \to R^l, \quad T_3 : R^l \to R^m$

then the composition $(T_3 \circ T_2 \circ T_1) : R^n \to R^m$ is defined by

$$(T_3 \circ T_2 \circ T_1)(\mathbf{x}) = T_3(T_2(T_1(\mathbf{x}))) \tag{6}$$

In this case the analog of of formula (2) is

$$[T_3 \circ T_2 \circ T_1] = [T_3][T_2][T_1] \tag{7}$$

Also, if we let A, B, and C denote the standard matrices for the linear transformations T_A, T_B, and T_C, respectively, then the analog of Formula (3) is

$$T_C \circ T_B \circ T_A = T_{CBA} \tag{8}$$

The extensions of (6), (7), and (8) to four or more transformations should be clear.

Example 4 A Composition of Three Matrix Transformations

Find the standard matrix for the linear operator $T: R^3 \to R^3$ that first rotates a vector about the z-axis through an angle θ, then reflects the resulting vector about the yz-plane, and then projects that vector orthogonally onto the xy-plane.

Sol.

The operator can be expressed as the composition

$$T = T_C \circ T_B \circ T_A = T_{CBA}$$

where A:rotation, B:reflection about the yz-plane, and C:projection onto the xy-plane.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The standard matrix for T:

$$CBA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem 6.4.3

Theorem 6.4.3 *If $T_1, T_2, \ldots, T_k$ is a succession of rotations about axes through the origin of R^3, then the rotations can be accomplished by a single rotation about some appropriate axis through the origin of R^3.*

Proof

Let $A_1, A_2, \ldots, A_k$ be the standard matrices for the rotations. Each matrix is orthogonal and has determinant 1, so the same is true for the product

$$A = A_k \cdots A_2 A_1$$

Thus, A represents a rotation about some axis through the origin of R^3. Since A is the standard matrix for the composition $T_k \circ \cdots \circ T_2 \circ T_1$, the result is proved.

Example 5 A Rotation Problem

Suppose that a vector in R^3 is first rotated 45° about the positive x-axis, then the resulting vector is rotated 45° about the positive y-axis, and then that vector is rotated 45° about the positive z-axis.

Find an appropriate axis and angle of rotation that achieves the same result in one rotation.

Sol.

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad R_y = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \quad R_z = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = R_z R_y R_x = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{4} - \frac{1}{2} & \frac{\sqrt{2}}{4} + \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{4} + \frac{1}{2} & \frac{\sqrt{2}}{4} - \frac{1}{2} \\ \frac{-1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \approx \begin{bmatrix} 0.5 & -0.1464 & 0.8536 \\ 0.5 & 0.8536 & -0.1464 \\ -0.7071 & 0.5 & 0.5 \end{bmatrix}$$

Example 5 A Rotation Problem

To find the axis of rotation **v** we will apply the Formula (17) of Section 6.2, taking the arbitrary vector **x** to be $\mathbf{e}_1$.

If A is rotation matrix, then for any $\mathbf{x} \neq \mathbf{0}$, the vector

$$\mathbf{v} = A\mathbf{x} + A^T\mathbf{x} + [1 - \mathrm{tr}(A)]\mathbf{x} \qquad \text{Section 6.2} \qquad (17)$$

$$= A\mathbf{e}_1 + A^T\mathbf{e}_1 + [1\text{-}(\tfrac{3}{2} + \tfrac{\sqrt{2}}{4})]\mathbf{e}_1$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{4} - \frac{1}{2} & \frac{\sqrt{2}}{4} + \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{4} - \frac{1}{2} \\ \frac{\sqrt{2}}{4} + \frac{1}{2} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} + \frac{\sqrt{2}}{4} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} \\ \frac{1}{2} - \frac{\sqrt{2}}{4} \end{bmatrix} \approx \begin{bmatrix} 0.1464 \\ 0.3536 \\ 0.1464 \end{bmatrix}$$

:ation satisfies

$$\cos\theta = \frac{\mathrm{tr}(A) - 1}{2} \qquad \text{Section 6.2} \qquad (16)$$

$$= \frac{2 + \sqrt{2}}{8} \approx 0.4268 \Rightarrow \theta = \cos^{-1} 0.4268 \approx 64.74°$$

Example 6 Transforming with a Diagonal Matrix

A diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

can be factored as

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & 1 \end{bmatrix} = D_2 D_1$$

Multiplication by D_1 produces a compression in the x-direction if $0 \le \lambda_1 \le 1$, an expansion in the x-direction if $\lambda_1 > 1$; multiplication by D_2 produces analogous result in the y-direction.

Thus, for example, multiplication by

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Causes an expansion by a factor of 3 in the x-direction and a compression by a factor of ½ in the y-direction.

A More General Result about Diagonal Matrices

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad \lambda_i \geq 0$$

Multiplication by D maps the standard unit vector $\mathbf{e}_i$ into the vector $\lambda_i \mathbf{e}_i$, so this operator causes compressions or expansions in the directions of the standard unit vectors.

Because of these geometric properties, diagonal matrices with nonnegative entries are called ***scaling matrices***.

Example 7 Transforming with 2x2 Elementary Matrices

The 2x2 elementary matrices have five possible forms.

$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Type 1 shear in the x-direction	Type 2 shear in the y-direction	Type 3 reflection about $y=x$	Type 4 compression in the x-direction	Type 5 compression in the y-direction

If k is negative in types 4 and 5,

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -k_1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ 0 & 1 \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -k_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & k_1 \end{bmatrix} \tag{10}$$

Then type 4 matrix with negative k represents a compression or expansion followed by a reflection about the y-axis; and type 5 matrix with negative k represents a compression or expansion followed by a reflection about the x-axis.

Theorem 6.4.4

Theorem 6.4.4 *If A is an invertible 2×2 matrix, then the corresponding linear operator on R^2 is a composition of shears, compressions, and expansions in the directions of the coordinate axes, and reflections about the coordinate axes and about the line $y = x$.*

Example 8 Describe the geometric effect of multiplication by

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

In terms of shears, compressions, expansions, and reflections.

Sol.

Since det(A)≠0, the matrix is invertible and hence can be reduced to I by a sequence of elementary row operations.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \underset{-3R_1+R_2 \rightarrow R_2}{\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}} \rightarrow \underset{R_2/(-2) \rightarrow R_2}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}} \rightarrow \underset{R_1-2R_2 \rightarrow R_1}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} \xrightarrow{-3R_1+R_2 \to R_2} \begin{bmatrix}1 & 2\\0 & -2\end{bmatrix} \xrightarrow{R_2/(-2) \to R_2} \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} \xrightarrow{R_1-2R_2 \to R_1} \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$E_1 = \begin{bmatrix}1 & 0\\-3 & 1\end{bmatrix}, \quad E_2 = \begin{bmatrix}1 & 0\\0 & -\frac{1}{2}\end{bmatrix}, \quad E_3 = \begin{bmatrix}1 & -2\\0 & 1\end{bmatrix}$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix}1 & 0\\3 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\3 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$$

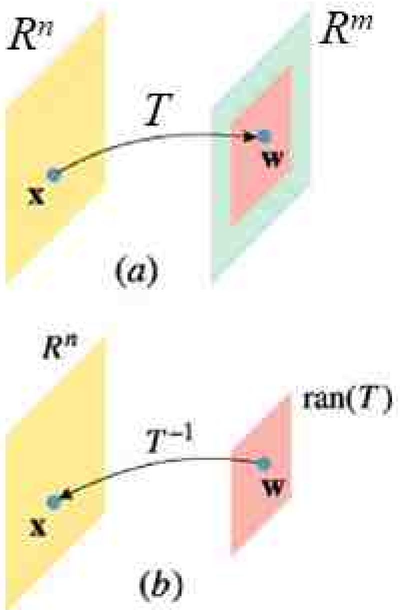
Thus, reading right to left, the geometric effect of A is to successively shear by a factor of 2 in the x-direction, expand by a factor of 2 in the y-direction, reflect about the x-axis, and shear by a factor of 3 in the y-direction.

Inverse of a Linear Transformation

Our next objective is to find a relationship between the linear operators represented by A and A^{-1} when A is invertible.

If T: $R^n \rightarrow R^m$ is a one-to-one linear transformation, then each vector w in the range of T is the image of a unique vector x in the domain of T (Figure *6.4.5a);* we call **x** the *preimage* of **w**.

The uniqueness of the preimage allows us to create a new function that maps w into x; we call this function the *inverse* of T and denote it by T^{-1}.



$$T^{-1}(\mathbf{w}) = \mathbf{x} \quad \text{if and only if} \quad T(\mathbf{x}) = \mathbf{w}$$

$$T^{-1}: \operatorname{ran}(T) \rightarrow R^n \tag{11}$$

Stated informally, T and T^{-1} cancel out" the effect of one another in the sense that if $\mathbf{w} = T(\mathbf{x})$, then

$$T(T^{-1}(\mathbf{w})) = T(\mathbf{x}) = \mathbf{w} \tag{12}$$

$$T^{-1}(T(\mathbf{x})) = T^{-1}(\mathbf{w}) = \mathbf{x} \tag{13}$$

Theorem 6.4.5

Theorem 6.4.5 *If T is a one-to-one linear transformation, then so is T^{-1}.*

Invertible Linear Operators

Theorem 6.4.6 *If T is a one-to-one linear operator on R^n, then the standard matrix for T is invertible and its inverse is the standard matrix for T^{-1}.*

Proof

Let A and B be the standard matrices for T and T^{-1}, respectively, and let $\mathbf{x}$ be any vector in R^n. By (13),

$$T^{-1}(T(\mathbf{x})) = \mathbf{x} \qquad (13)$$

in matrix form,

$$B(A\mathbf{x}) = \mathbf{x} \quad \text{or} \quad B(A\mathbf{x}) = \mathbf{x} = I\,\mathbf{x}$$

Thus, by Theorem 3.4.4, $BA = I$ *or* $A^{-1} = B$ ■

$$[T^{-1}] = [\,T\,]^{-1} \qquad (14)$$

Alternatively, if we use the notation T_A to denote a one-to-one linear operator with standard matrix $A,$ then (14) implies that

$$T_A^{-1} = T_{A^{-1}} \qquad (15)$$

Example 9 Inverse of a Rotation Operator

The linear operator corresponding to the rotation through an angle θ is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Find the inverse of this operator by
(a) The rotation through the angle $-\theta$,
(b) Theorem 6.4.6 which states $[T^{-1}] = [T]^{-1}$

Sol.

(a) The rotation through the angle $-\theta$

It is evident that the inverse of this operator is the rotation through the angle $-\theta$, since rotating $\mathbf{x}$ through the angle θ and then rotating the image through the angle $-\theta$ produces the vector $\mathbf{x}$ back again.

(b) Theorem 6.4.6 which states $[T^{-1}] = [T]^{-1}$

$$A^{-1} = A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

Example 10 Inverse of a Compression Operator

The linear operator on R^2 corresponding to the compression in the y-direction by a factor of ½ is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Find the inverse of this operator by
(a) the expansion,
(b) Theorem 6.4.6 which states $[T^{-1}] = [T]^{-1}$

Sol.

(a) the expansion

It is evident that the inverse of this operator is the expansion in the y-direction by a factor of 2. Thus, the inverse operator is

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

(b) Theorem 6.4.6 which states $[T^{-1}] = [T]^{-1}$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}^{-1} = \frac{1}{\frac{1}{2}} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Example 11 Inverse of a Reflection Operator

The linear operator on R^2, corresponding to the reflection about the line through the origin that makes an angle of $\theta/2$ with the positive x-axis, is

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Find the inverse of this operator by

(a) Reflecting again,

(b) Theorem 6.4.6 which states $[T^{-1}] = [T]^{-1}$

Sol.

(a) Reflecting again

It is evident geometrically that A must be its own inverse, since reflecting $\mathbf{x}$ about this line, and then reflecting the image of $\mathbf{x}$ about the line produces $\mathbf{x}$ back again.

(b) Theorem 6.4.6 which states $[T^{-1}] = [T]^{-1}$

Ex. 12 Inverse of a Linear Operator Defined by a Linear System

Consider the linear operator $T(x_1, x_2, x_3) = (w_1, w_2, w_3)$ that is defined by the linear equations

$$\begin{aligned} w_1 &= x_1 + 2x_2 + 3x_3 \\ w_2 &= 2x_1 + 5x_2 + 3x_3 \\ w_3 &= x_1 \qquad\quad + 8x_3 \end{aligned} \qquad \mathbf{w} = A\,\mathbf{x}$$

(a) Show that the linear operator T is one-to-one,
(b) Find a set of linear equations that define T^{-1}.

Sol.

(a) The standard matrix for the linear operator T is

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

The linear operator T is one-to-one since A is invertible.

(b) Find a set of linear equations that define T^{-1}.

By Theorem 6.4.6, $[T^{-1}] = A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$ $\qquad \mathbf{x} = A^{-1}\,\mathbf{w}$

Geometric Properties of Invertible Linear Operators on R^2

Theorem 6.4.7 *If $T: R^2 \to R^2$ is an invertible linear operator, then:*

(*a*) *The image of a line is a line.*

(*b*) *The image of a line passes through the origin if and only if the original line passes through the origin.*

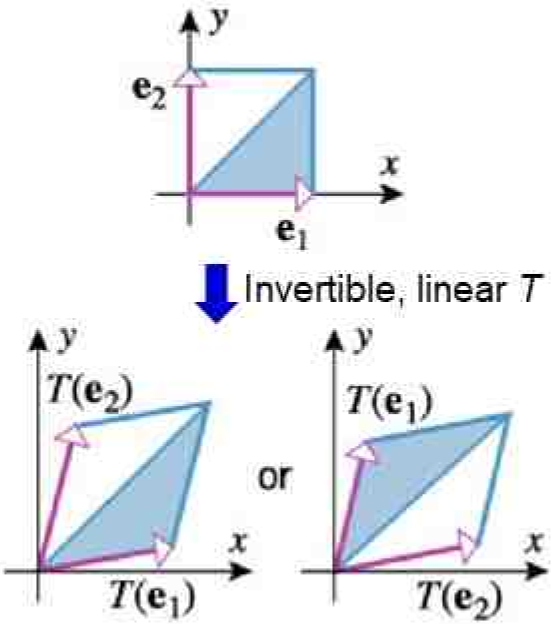
(*c*) *The images of two lines are parallel if and only if the original lines are parallel.*

(*d*) *The images of three points lie on a line if and only if the original points lie on a line.*

(*e*) *The image of the line segment joining two points is the line segment joining the images of those points.*

Image of the Unit Square

Let us see what we can say about the image of the unit square under an invertible linear operator T on R^2.



The vertex at the origin remains fixed under the transformation since all linear operators maps **0** into **0**.

The images of the other three vertices must be distinct, for otherwise they would lie on a line, and this is impossible by part (*d*) of Theorem 6.4.7.

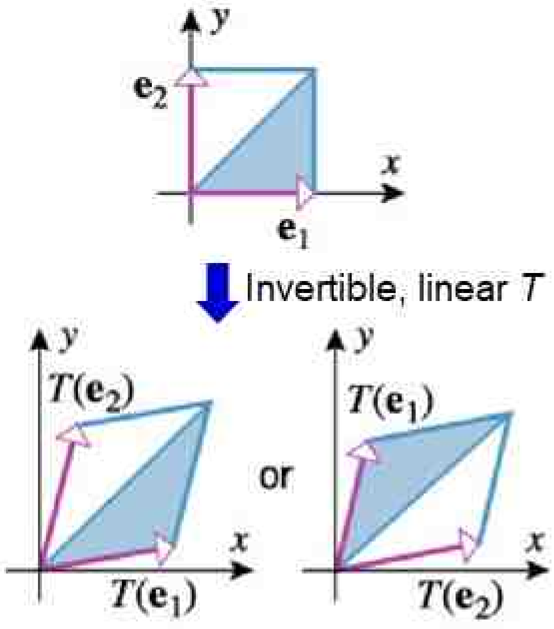
Finally, since the images of the parallel sides remain parallel, we can conclude that the image of the unit square is a nondegenerate parallelogram that has a vertex at the origin and whose adjacent sides are $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$(Figure 6.4.6).

If A, shown below, denotes the standard matrix for T , then it follows from Theorem 4.3.5 that |det(A)| is the area of the parallelogram with adjacent sides $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$.

$$A = \left[T(\mathbf{e}_1)\ \ T(\mathbf{e}_2)\right] = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

Theorem 6.4.8

Theorem 6.4.8 *If $T: R^2 \rightarrow R^2$ is an invertible linear operator, then T maps the unit square into a nondegenerate parallelogram that has a vertex at the origin and has adjacent sides $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$. The area of this parallelogram is $|\det(A)|$, where $A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)]$ is the standard matrix for T.*



Unit square

an invertible linear operator T on R^2

nondegenerate parallelogram

Area of the parallelogram $= |\det(A)|$

where

$$A = \left[T(\mathbf{e}_1) \ T(\mathbf{e}_2)\right] = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

Example 13 Determinants of Rotation and Reflection Operators

Let R_θ be the standard matrix for the rotation about the origin of R^2 through the angle θ, and H_θ be the standard matrix for the reflection about the line making an angle θ with the x-axis of R^2.

Show that $|\det(R_\theta)| = 1$ and $|\det(H_\theta)| = 1$.

Sol.

The rotation and reflection do not change the area of the unit square.
Thus, $|\det(R_\theta)| = 1$ and $|\det(H_\theta)| = 1$ since the area of the unit square is 1.

This is consistent with our observation in Section 6.2 that $\det(R_\theta) = 1$ and $\det(H_\theta) = -1$.

observation in Section 6.2 : $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *or* $H_{\theta/2} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

$$\det(\mathbf{R}_\theta) = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$\det(\mathbf{H}_{\theta/2}) = \begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{vmatrix} = -(\cos^2\theta + \sin^2\theta) = -1$$