Introduction to Discrete Math

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Course Outline

- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion, Logic, Invariants
- Probability & Combinatronics
  - Counting, Probability, Random Variables
- Graph Theory
  - Graphs (cycles, classes, parameters)
- Number Theory & Cryptography
  - Arithmetic in modular form
  - Intro to Cryptography

Mathematical Thinking – Logic

EXAMPLES, COUNTEREXAMPLES, LOGIC

- Examples
- Counterexamples
- Logic

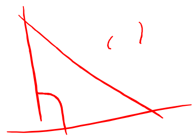
Examples

- Sometimes, one example is already enough
- If we want to prove that white lions exists
  - It is enough to show just one white lion
  - such examples are not always easy to come up with
    - 13 in the wild and approximately 300 in captivity
    - https://whitelions.org/white-lion/faqs/

Examples

Problem I

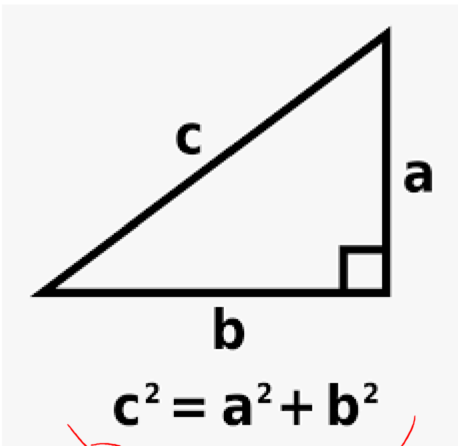
- Is it possible that for three positive integer numbers a, b, and c so that $a^2+b^2 = c^2$?



Examples

Solution

- To come up with this example, recall
  - Pythagorean Theorem with right angle, and sides 3, 4, and 5



$$h = \sqrt{(a^2 + b^2)};\ h = c$$

$$c^2 = a^2 + b^2$$

$$25 = 16 + 9$$

$$5^2 = 4^2 + 3^2$$

Examples

Problem II

- Is it possible that for three positive integer numbers a, b, and c so that $a^3+b^3 = c^3$?

Examples

Solution

* conjecture – conclusion assume to be true due to initial supporting evidence but no proof or disproof has yet been found

- This is in fact impossible, so there is no example
- Fermat's Last Theorem (1637), a famous mathematical conjecture
  - for any integer $n > 2$, there are no such integers a, b, & c such that
    $a^n + b^n = c^n$.
  - Mathematicians failed tried to prove it for hundreds of years
  - Andrew Wiles was able to prove it in 1995

Examples

Problem III

- Is it possible that for positive integer numbers a, b, c, and d so that $a^4+b^4 + c^4 = d^4$?

Examples

Solution

- We could initially assume that it is impossible due to Fermat's Last Theorem
- But the theorem only focus on equations with the form

$$a^n + b^n = c^n$$

- Hence, it is not applicable for this problem a, b c d

Examples

Solution

- It is actually possible, the smallest number though is very big

$$95800^4 + 217519^4 + 414560^4 = 422481^4$$

- Computers used to derive the examples
- But the possible number of values are so huge that it is hard to find examples that satisfy the equation, even with the help of computers

Examples

Problem IV

- Is there a power of 2 that starts with 65?

$x^2 = 65yyyy\ldots$

Examples

Problem IV

- Is there a power of 2 that starts with 65?

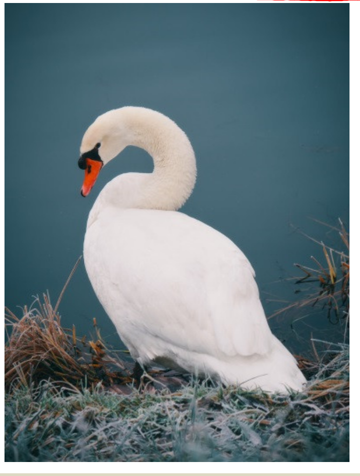
Solution

- The answer is $2^{16} = 65536$, 1 n) - 2 4 - 8 - 16 - 32 - 64
- This is the complete solution
- In fact, there is a power of 2 that starts w/ any integer n, $n > 0$
  - But much more difficult to prove

- Examples
- Counterexamples
- Logic

Counterexamples

- Just one counterexample is enough to disprove a statement
- If we want to prove that all swans are white
  - Just one instance of black swan is enough to disprove it
  - However, it is often difficult to find such counterexamples



Counterexamples

Theorem I

- *All rectangles are squares*

Counterexamples

Theorem I

- *All rectangles are squares*

Solution

- A rectangle with sides of sizes 1 and 2, respectively, is not a square.
- This is a counterexample for the theorem, hence the theorem is wrong

Counterexamples

Theorem II

- *All square are rectangles*

Solution

- There is no counterexample for this case, hence the theorem is true
- Since square is a rectangle with equal sides.

Counterexamples

Theorem II

- *All square are rectangles*

Counterexamples

Theorem III

- Euler came up w/ a generalization of Fermat's Last Theorem
- *For any* $n > 2$, *it is impossible for an* n-*th power of a positive integer to be represented as a sum of* $n - 1$ *numbers w/c are the* n-*th powers of positive integers*
- For $n = 3$, it is the same as Fermat's Last Theorem: It is impossible that $a^3 + b^3 = c^3$.

Counterexamples

Solution

- Lander came up with a counterexample in 1966 for $n=5$:

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

- Elkies found another counterexample in 1986 for $n=4$:

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$$

- Frye found the smallest counterexample for $n=4$ in1988:
  - Which is an example for one statement & counterexample for another

$$95800^4 + 217519^4 + 414560^4 = 422481^4$$

- Examples
- Counterexamples
- Logic

Logic

Logical Operators

- Negation
- Logical AND
- Logical OR
- If-then

Negation

Statement

- All swans are white.

Negation

Statement

- All swans are white.

Negation

- **Not** all swans are white. Or, there are swans that are **not** white.

Negation

Statement

- There exists three positive integers a, b, & c, such that

  $$a^3 + b^3 = c^3.$$

Negation

Statement

- There exists three positive integers a, b, & c, such that

  $$a^3 + b^3 = c^3.$$

Negation

- There are no such positive integer numbers a, b, & c, such that $a^3 + b^3 = c^3$.
- Or for any positive integers a, b, & c, such that $a^3 + b^3 \neq c^3$.

Negation

Statement

- $4 = 2 + 2$
- $5 = 2 + 2$

Negation

Statement

- $4 = 2 + 2$
- $5 = 2 + 2$

Negation

- $4 \neq 2 + 2$
- $5 \neq 2 + 2$

Negation

Negation

iff

- Is true, if and only if the initial statement is wrong
- Is false, if and only if the initial statement is correct

$4 = 2+2$ ✓ $\xrightarrow{N}$ $4 \neq 2+2$ ✗

$5 = 2+2$ ✗ $\quad$ $5 \neq 2+2$ ✓

Logical AND

Statement

- 4 = 2 + 2 **AND** 4 = 2 x 2
- The logical AND of two statements is true if and only if both statements are true.

- 4 = 2 + 2 AND 4 = 2 x 2 → TRUE
- 4 = 2 + 2 AND 5 = 2 x 2 → FALSE
- 5 = 2 + 2 AND 4 = 2 x 2 → FALSE
- 5 = 2 + 2 AND 5 = 2 x 2 → FALSE

Logical OR

Statement

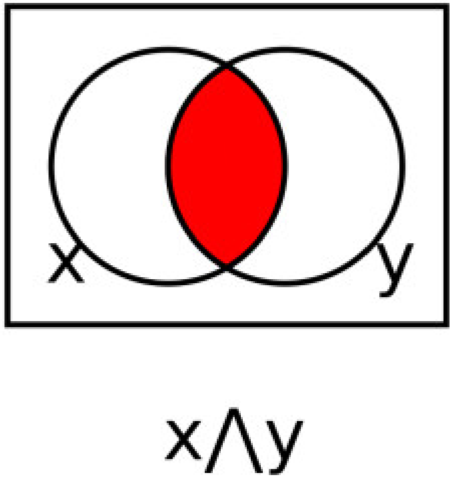
- $4 = 2 + 2$ **OR** $4 = 2 \times 2$
- The logical OR of two statements is true if and only if at least one of the statements is true.

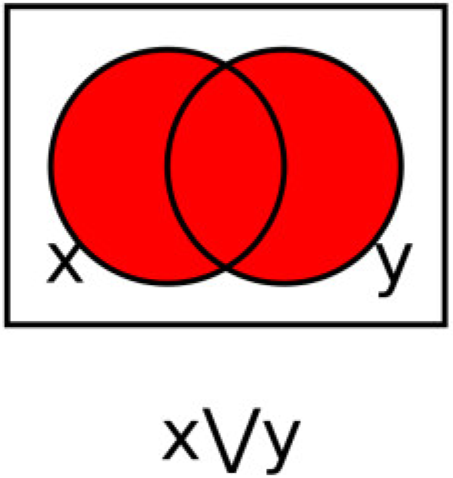
- $4 = 2 + 2$ OR $4 = 2 \times 2 \rightarrow$ TRUE
- $4 = 2 + 2$ OR $5 = 2 \times 2 \rightarrow$ TRUE
- $5 = 2 + 2$ OR $4 = 2 \times 2 \rightarrow$ TRUE
- $5 = 2 + 2$ OR $5 = 2 \times 2 \rightarrow$ FALSE

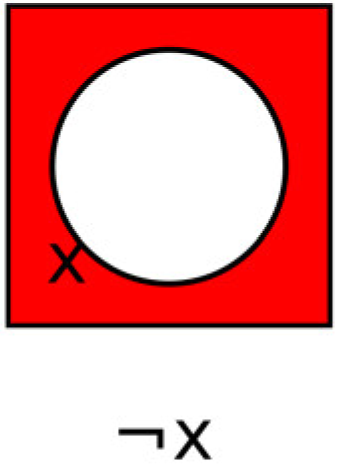
Venn Diagram

Symbols:

- Logical NOT (¬), logical AND (∧), and logical OR (∨)







https://en.wikipedia.org/wiki/Logical_connective

Negation of AND

Statement

Negation of AND is OR of negations:

- Negation of “A AND B” is “Not A OR Not B”

$\neg A \quad \vee \quad \neg B$

Negation of AND

Statement

Negation of AND is OR of negations:

- Negation of “A AND B” is “Not A OR Not B”

A AND B

T T

Negation of $(4 = 2 + 2)$ AND $(4 = 2 \times 2)$: T

- $4 \neq 2 + 2$ OR $4 \neq 2 \times 2$. Negated AND

F F → F.

Negation of OR

Statement

Negation of OR is AND of negations:

- Negation of “A OR B” is “Not A AND Not B”

$\neg A \wedge \neg B$

Negation of OR

Statement

Negation of OR is AND of negations:

- Negation of “A OR B” is “Not A AND Not B”

Negation of $(4 = 2 + 2)$ OR $(5 = 2 \text{ x } 2)$:

- $(4 \neq 2 + 2)$ AND $(5 \neq 2 \text{ x } 2.)$

Reminder

- Everybody, make sure that your name in ZOOM is in the following format:
  - Ex: 202054321 Juan Dela Cruz

Not changing your name to this format

* you **will** be marked Absent * → absent?

If-Then

Consider a promise:

- If there is an elephant in the refrigerator, then I’ll give you 100,000 won.

If-Then

Consider a promise:

- If there is an elephant in the refrigerator, then I'll give you 100,000 won.

The interesting case:

- There is no elephant in the refrigerator, but I gave you 100,000 won anyways, did I keep my promise?
- In terms of technical sciences, it is considered a kept promise, hence the statement is true

If-Then

Statement

- In general, the phrase “If P, Then Q” is true whenever either P is false or Q is true.

If-Then

Statement

- In general, the phrase "If P, Then Q" is TRUE whenever either P is FALSE or Q is TRUE.

T = (P = F) && (Q = T)

Negation:

- If $n = 6$, then n is even → TRUE
- If $n = 5$, then n is even → FALSE
- If 1 $= 2$, then $2 = 3$ is even → TRUE
- If $1 = 2$, then I am an elephant → TRUE

(1≠2) && NOT (F) → F

IF-THEN $\xrightarrow{neg}$ P AND NOT Q

If-Then

True = P=F; & Q=T

If-Then Generalization

- "If n is divisible by 4, then n is divisible by 2"
  - means "For all n, if n is divisible by 4, then n is divisible by 2"
  - statement is TRUE
- For any number n that is divisible by 4, and so is divisible by 2
  - both "if" and "then" parts are true, statement is TRUE
- For any number n that is not divisible by 4 P=F ; Q=?
  - "if" part is false, statement is TRUE, no matter if n divisible by 2 or not

If-Then

n= 4 = n/4
4 = 2/2 ✓ n/4 ✗

P=T, Q=?

If-Then Generalization

- "If n is divisible by 2, then n is divisible by 4"
  - means "For all n, if n is divisible by 2, then n is divisible by 4"
  - statement is FALSE

- Because when $n = 2$, n is divisible by 2 but not by 4, hence
  - "if" part is true but "then" statement is false
  - statement is FALSE

If-Then

Direct & Converse

- "If n is divisible by 4, then n is divisible by 2"
  - direct statement
- "If n is divisible by 2, then n is divisible by 4"
  - converse statement

- When both direct ("If P, then Q") and converse ("If Q, then P") statements are true, they are equivalent to
  - "P if and only if Q"

If-Then

Negation of If-Then

- The negation of the phrase "If P, then Q" is "P and not Q"

  不 AND

- Check if it is true whenever the corresponding If-Then is false
- Check if it is false whenever the corresponding If-Then is true

If-Then

Universal Quantification

- Statements that are examples of universal quantification:
  - All swans are white
  - All integers ending with the digit "2" are even
  - For all n, $2 \times n = n + n$
- Fermat's Last Theorem states that for all $n > 2$, equation $a^n + b^n = c^n$ does not have solutions with positive integers a, b, & c.
  - This is another example universal negation

$n = 1$ ✓ ; n $c^2 = a^2 + b^2$

$a^n + b^n = c^n$ $n > 2$

If-Then

Existential Quantification

- Statements that are examples of existential quantification:
  - There are black swans
  - There is a way to get a change of 12 ewans with 4 ewan & 5 ewan coins
  - There exists such integers a, b, c, and d that $a^4 + b^4 + c^4 = d^4$
  - There exists a power of 2 starting with 65

If-Then

Combination of Quantifiers

- Mathematical statements are usually combinations of universal and existential qualifications.
- Example is a corollary from Fermat's Last Theorem:

- Theorem:
  - There exists such integer m that for any integer $n > m$, equation $a^n + b^n = c^n$ has no solutions with positive integers a, b, and c.

If-Then

Combination of Quantifiers

- Theorem: EQ UQ
  - There exists such integer m that for any integer $n > m$, equation $a^n + b^n = c^n$ has no solutions with positive integers a, b, and c.

- If we take $m = 2$, then it follows from Fermat's last theorem. It is a combination of
  - existential quantifier: "there exists such integer m" and
  - universal quantifier: "for any integer $n > m$…"

If-Then

Combination of Quantifiers

- Theorem:
  - There exists such integer m that for any integer $n > m$, equation $a^n + b^n = c^n$ has no solutions with positive integers a, b, and c.

- If we take $m = 2$, then it follows from Fermat's last theorem. It is a combination of
  - existential quantifier: "there exists such integer m" and
  - universal quantifier: "for any integer $n > m$..."

If-Then

Negation of Quantifications

- Negation of universal quantification is a corresponding existential quantification
- Negation of existential quantification is a corresponding universal quantification

If-Then

Negation of Quantifications

- Negation of universal quantification is a corresponding existential quantification
- Negation of existential quantification is a corresponding universal quantification

- Example: uQ
  - UQ: “For **all** n, statement A is **true**”
  F
  - Negation of UQ: “There exists **such** n that statement A is **false**”

EQ

If-Then

Negation of Quantifications

- Euler's hypothesis is a combination of two universal quantifications:
  - For **any** $n > 3$, for **any** positive integer a, it is **impossible to represent** a^n as a sum of $n - 1$ numbers which are the $n\text{-}th$ powers of positive integers.

If-Then

Negation of Quantifications

- Euler's hypothesis is a combination of two universal quantifications:
  - For **any** $n > 3$, for **any** positive integer a, it is **impossible to represent** a^n as a sum of $n - 1$ numbers which are the $n\text{-}th$ powers of positive integers.
- Negation:
  - There exists **such** $n > 3$ and **such** positive integer a that a^n **can be represented** as a sum of $n - 1$ numbers which are the $n\text{-}th$ powers of positive integers.

A AND B $\overset{neg}{\Longrightarrow}$ (NOT A) OR (NOT B)

If-Then

A OR B $\overset{neg}{\Longrightarrow}$ (NOT A) AND (NOT B)

Negation of Quantifications

- UQ: "**All** positive integers n are either even **OR** odd"
- Negation: "There exists **such** positive integer *n* that is not even **AND** not odd"

- To negate:
  - We switch universal quantification (UQ) to existential qualification (EQ) and switch OR to AND
  - We switch existential quantification (EQ) to universal qualification (UQ) and switch AND to OR

If-Then

Negation of Quantifications

- UQ: "All positive integers are either even OR odd"
- Negation: "There exists such positive integer n that is not even AND not odd"

- To negate:
  - We switch universal quantification (UQ) to existential qualification (EQ) and switch OR to AND
  - We switch existential quantification (EQ) to universal qualification (UQ) and switch AND to OR

If-Then

Examples & Counterexamples

- Counterexample for Euler's hypothesis that was previously given is an example of negation
- By proving negation of Euler's hypothesis, we prove that the Euler's hypothesis itself is false

If-Then

Reductio ad Absurdum

- A popular way of proving mathematical statements is through Reductio ad Absurdum
  - basically proving that the negation of the initial statement is false, hence the statement itself is true.

- We will study about this next

Mathematical Thinking – Reductio ad absurdum

REDUCTIO AD ABSURDUM

- Reduction ad absurdum
- Balls in Boxes
- Numbers in Boxes
- Pigeonhole Principle
- An(-1,0,1) Antimagic Square
- Handshakes

Reductio ad absurdum

Reductio ad absurdum

- How to prove that something is true?
- Show that the opposite is impossible
- Reductio ad absurdum
  - Proof by contradiction
- One of the base methods of reasoning
  - Used everywhere
- Often combined with other methods
- We will use constantly throughout the course

Reductio ad absurdum

Socratic Method

- Reductio ad absurdum is classic
  - Used in Socratic method (Plato, ~400BC)
- Socrates revealed contradictions in what his student believed
  - By asking them questions step-by-step

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Socratic Method

- ***Nontrivial*** – Not obvious or easy to prove

Problem

- There are boys and girls in the class. They are divided into two groups for the foreign language: there are students studying French, and there are students studying German. Each student picks one of the two languages. Show that there is a boy and a girl who study different languages.

Seems impossible at first: we know basically nothing and we claim something nontrivial.

Socratic Method

Solution

- There are boys and girls in the class. They are divided into two groups for the foreign language: there are students studying French, and there are students studying German. Each student picks one of the two languages. Show that there is a boy and a girl who study different languages.

Our statement: there are no boy & no girl studying different languages. (Assume original statement is wrong)

Socratic Method

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G → some girls learn German

Socratic Method

Solution

- There are boys and girls in the class. They are divided into two groups for the foreign language: there are students studying French, and there are students studying German. Each student picks one of the two languages. Show that there is a boy and a girl who study different languages.

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G → some girls learn German

B B … B → then all boys learn German

Socratic Method

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- There are boys and girls in the class. They are divided into two groups for the foreign language: there are students studying French, and there are students studying German. Each student picks one of the two languages. Show that there is a boy and a girl who study different languages.

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G → some girls learn German

B B … B → then all boys learn German

G G … G → then all girls learn German

Socratic Method

Solution

- There are boys and girls in the class. They are divided into two groups for the foreign language: there are students studying French, and there are students studying German. Each student picks one of the two languages. Show that there is a boy and a girl who study different languages.

Our statement: there are no boy & no girl studying different languages. (Assume original statement is wrong)

G → some girls learn German

B B … B → then all boys learn German

G G … G → then all girls learn German

Everyone learns German!
It is a contradiction!
Therefore our assumption is wrong
& original statement is true.

- Reduction ad absurdum
- Balls in Boxes
- Numbers in Boxes
- Pigeonhole Principle
- An(-1,0,1) Antimagic Square
- Handshakes

Balls in Boxes

Problem

- We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

1 2 3 4 5 6 7 8 9 10

Balls in Boxes

Problem

- We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

Suppose we can do it. Let's see what will happen. Let us enumerate all boxes with increasing number of black balls.

1 2 3 4 5 6 7 8 9 10

Balls in Boxes

Problem

- We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

Suppose we can do it. Let's see what will happen. Let us enumerate all boxes with increasing number of black balls.

1	2	3	4	5	6	7	8	9	10
≥ 0									

Balls in Boxes

Problem

- We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

Suppose we can do it. Let's see what will happen. Let us enumerate all boxes with increasing number of black balls.

1	2	3	4	5	6	7	8	9	10
≥ 0	≥ 1								

Balls in Boxes

Problem

- We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

Suppose we can do it. Let's see what will happen. Let us enumerate all boxes with increasing number of black balls.

1	2	3	4	5	6	7	8	9	10
≥ 0	≥ 1	≥ 2							

Balls in Boxes

Problem

- We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

Suppose we can do it. Let's see what will happen. Let us enumerate all boxes with increasing number of black balls.

1	2	3	4	5	6	7	8	9	10
≥ 0	≥ 1	≥ 2	≥ 3						

Balls in Boxes

Problem

- We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

Suppose we can do it. Let's see what will happen. Let us enumerate all boxes with increasing number of black balls.

1	2	3	4	5	6	7	8	9	10
≥ 0	≥ 1	≥ 2	≥ 3	≥ 4					

Balls in Boxes

Problem

- We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

Suppose we can do it. Let's see what will happen. Let us enumerate all boxes with increasing number of black balls.

1	2	3	4	5	6	7	8	9	10
≥ 0	≥ 1	≥ 2	≥ 3	≥ 4	≥ 5				

Balls in Boxes

Problem

- We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

Suppose we can do it. Let's see what will happen. Let us enumerate all boxes with increasing number of black balls.

1	2	3	4	5	6	7	8	9	10
≥ 0	≥ 1	≥ 2	≥ 3	≥ 4	≥ 5	≥ 6			

Balls in Boxes

Problem

- We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

Suppose we can do it. Let's see what will happen. Let us enumerate all boxes with increasing number of black balls.

1	2	3	4	5	6	7	8	9	10
≥ 0	≥ 1	≥ 2	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7		

Balls in Boxes

Problem

- We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

Suppose we can do it. Let's see what will happen. Let us enumerate all boxes with increasing number of black balls.

1	2	3	4	5	6	7	8	9	10
≥0	≥1	≥2	≥3	≥4	≥5	≥6	≥7	≥8	

Balls in Boxes

Problem

- We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

Suppose we can do it. Let's see what will happen. Let us enumerate all boxes with increasing number of black balls.

1	2	3	4	5	6	7	8	9	10
≥ 0	≥ 1	≥ 2	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7	≥ 8	≥ 9

Balls in Boxes

Problem

- We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

Suppose we can do it. Let's see what will happen. Let us enumerate all boxes with increasing number of black balls.

$$\overset{1}{\boxed{\geq 0}} + \overset{2}{\boxed{\geq 1}} + \overset{3}{\boxed{\geq 2}} + \overset{4}{\boxed{\geq 3}} + \overset{5}{\boxed{\geq 4}} + \overset{6}{\boxed{\geq 5}} + \overset{7}{\boxed{\geq 6}} + \overset{8}{\boxed{\geq 7}} + \overset{9}{\boxed{\geq 8}} + \overset{10}{\boxed{\geq 9}} \geq 45$$

Balls in Boxes

Problem

- We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

Suppose we can do it. Let's see what will happen. Let us enumerate all boxes with increasing number of black balls.

There is a contradiction.
Total number of balls is at least 45, $45 > 30$.
Therefore our assumption is wrong. It cannot be done.

$$\overset{1}{\boxed{\geq 0}} + \overset{2}{\boxed{\geq 1}} + \overset{3}{\boxed{\geq 2}} + \overset{4}{\boxed{\geq 3}} + \overset{5}{\boxed{\geq 4}} + \overset{6}{\boxed{\geq 5}} + \overset{7}{\boxed{\geq 6}} + \overset{8}{\boxed{\geq 7}} + \overset{9}{\boxed{\geq 8}} + \overset{10}{\boxed{\geq 9}} \geq 45$$

- Reduction ad absurdum
- Balls in Boxes
- **Numbers in Tables**
- Pigeonhole Principle
- An(-1,0,1) Antimagic Square
- Handshakes

Numbers in Boxes

Puzzle

- There is a sequence of 10 cells. The leftmost contains “1” while the rightmost has “30”. Is it possible to fill other cells with consecutive numbers in such a way that they differ by 3 at the most?

1									30
	2								
	3								
	4								

Numbers in Boxes

Puzzle

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1									30

Suppose we can do it. Let's see what will happen

Numbers in Boxes

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1	4								30

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Numbers in Boxes

Puzzle

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1	4	7							30

Suppose we can do it. Let’s see what will happen

Numbers in Boxes

Puzzle

- There is a sequence of 10 cells. The leftmost contains “1” while the rightmost has “30”. Is it possible to fill other cells with consecutive numbers in such a way that they differ by 3 at the most?

1	4	7	10						30

Suppose we can do it. Let’s see what will happen

Numbers in Boxes

Puzzle

- There is a sequence of 10 cells. The leftmost contains "1" while the rightmost has "30". Is it possible to fill other cells with consecutive numbers in such a way that they differ by 3 at the most?

1	4	7	10	13					30

Suppose we can do it. Let's see what will happen

Numbers in Boxes

Puzzle

- There is a sequence of 10 cells. The leftmost contains “1” while the rightmost has “30”. Is it possible to fill other cells with consecutive numbers in such a way that they differ by 3 at the most?

1	4	7	10	13	16				30

Suppose we can do it. Let’s see what will happen

Numbers in Boxes

Puzzle

- There is a sequence of 10 cells. The leftmost contains "1" while the rightmost has "30". Is it possible to fill other cells with consecutive numbers in such a way that they differ by 3 at the most?

1	4	7	10	13	16	19			30

Suppose we can do it. Let's see what will happen

Numbers in Boxes

Puzzle

- There is a sequence of 10 cells. The leftmost contains “1” while the rightmost has “30”. Is it possible to fill other cells with consecutive numbers in such a way that they differ by 3 at the most?

1	4	7	10	13	16	19	22		30

Suppose we can do it. Let’s see what will happen

Numbers in Boxes

Puzzle

- There is a sequence of 10 cells. The leftmost contains "1" while the rightmost has "30". Is it possible to fill other cells with consecutive numbers in such a way that they differ by 3 at the most?

+3 +3 +3 +3 +3 +3 +3 +3 (5)

1	4	7	10	13	16	19	22	25	30

Suppose we can do it. Let's see what will happen

Numbers in Boxes

Puzzle

- There is a sequence of 10 cells. The leftmost contains “1” while the rightmost has “30”. Is it possible to fill other cells with consecutive numbers in such a way that they differ by 3 at the most?

1	4	7	10	13	16	19	22	25	30

Suppose we can do it. Let’s see what will happen

There is a contradiction.
The numbers in the cells grow too slow ☹.
Therefore our assumption is wrong. **It cannot be done**.
This is a very common way to estimate running time of some algorithm.

Reminder

- Everybody, make sure that your name in ZOOM is in the following format:
  – Ex: 202054321 Juan Dela Cruz

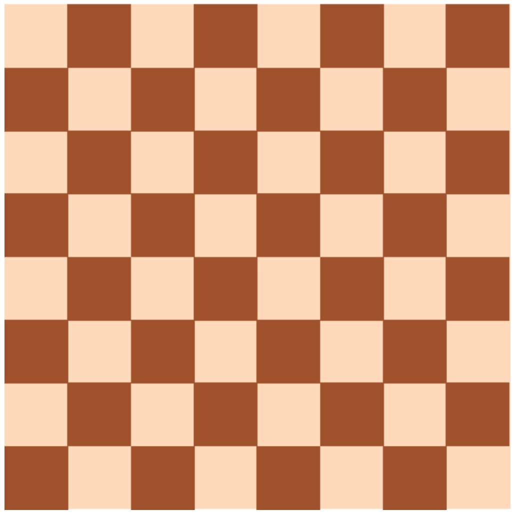
Not changing your name to this format

* you **will** be marked Absent * → absent?

Numbers on a Chessboard

Puzzle

- Is is possible to put the numbers 1, 2, 3, … , 63, 64 on a chessboard in such a way that neighbors (cells which share the same side) will differ by 4 at the most?



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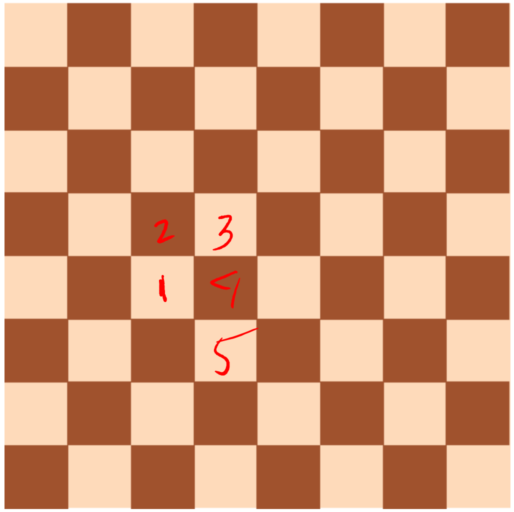
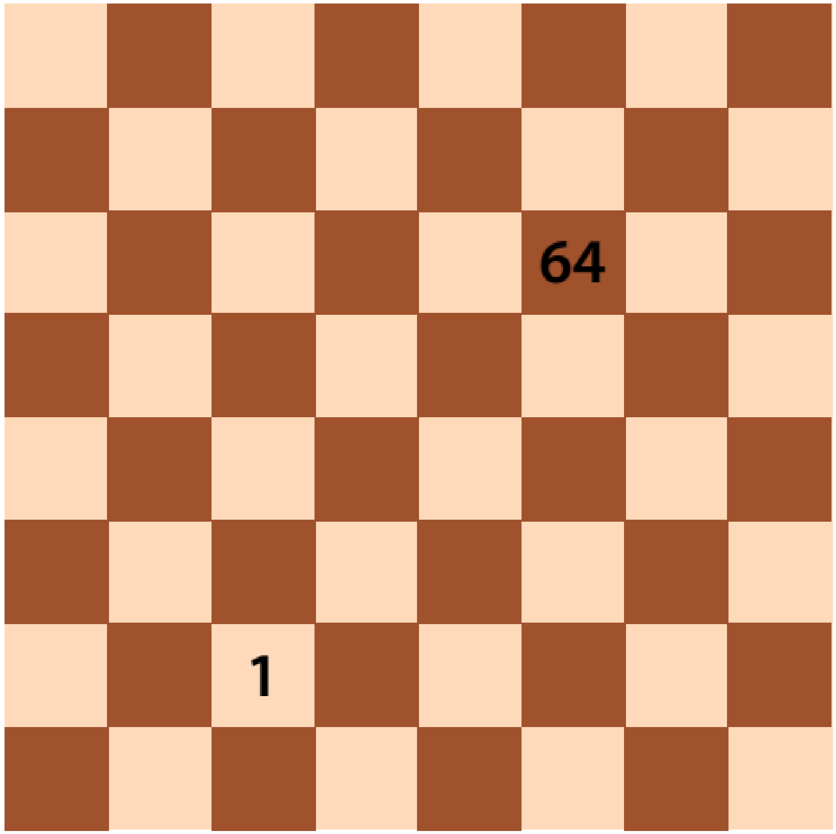


Let us again suppose we can do it. Let's see what will happen.

Numbers on a Chessboard

Puzzle

Let us again suppose we can do it. Let's see what will happen.

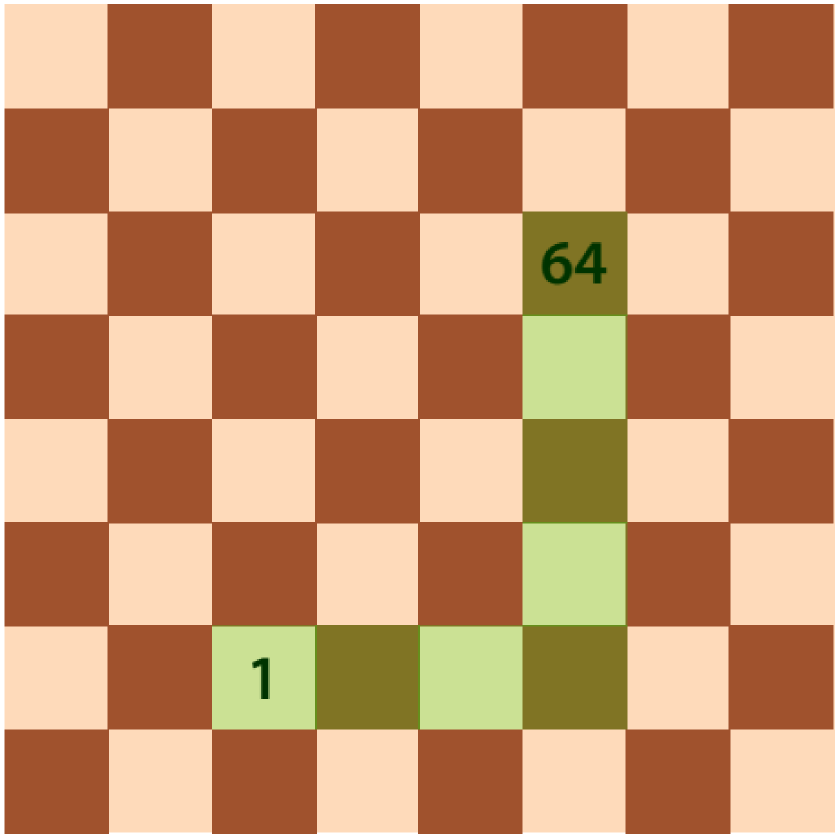


Numbers on a Chessboard

Puzzle

Let us again suppose we can do it. Let's see what will happen.

For this:
We need 7 steps to get from 1 to 64



Numbers on a Chessboard

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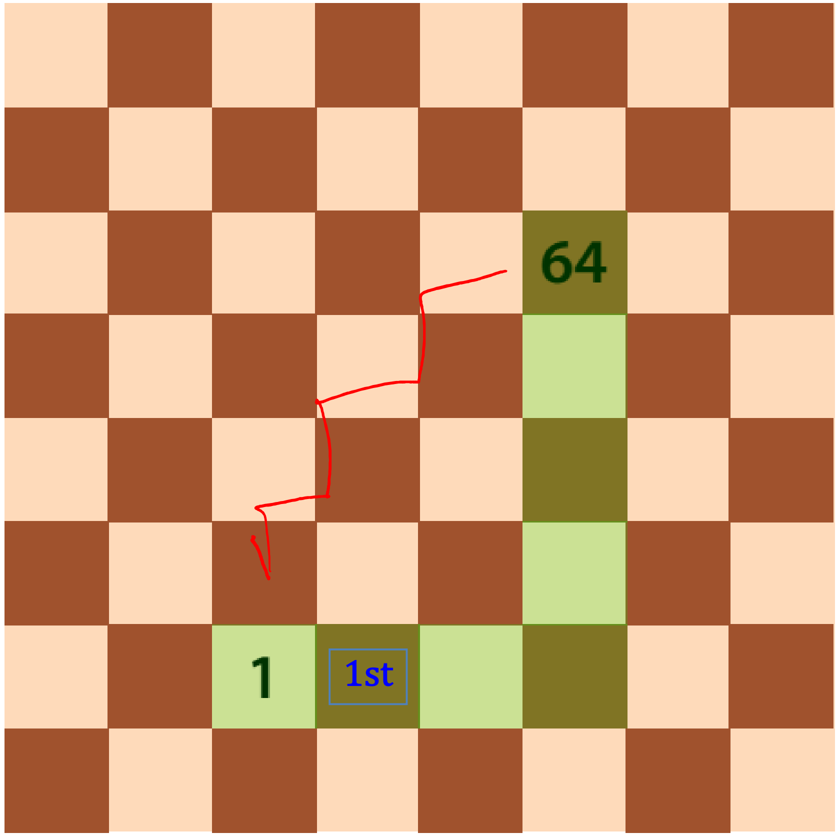

Numbers on a Chessboard

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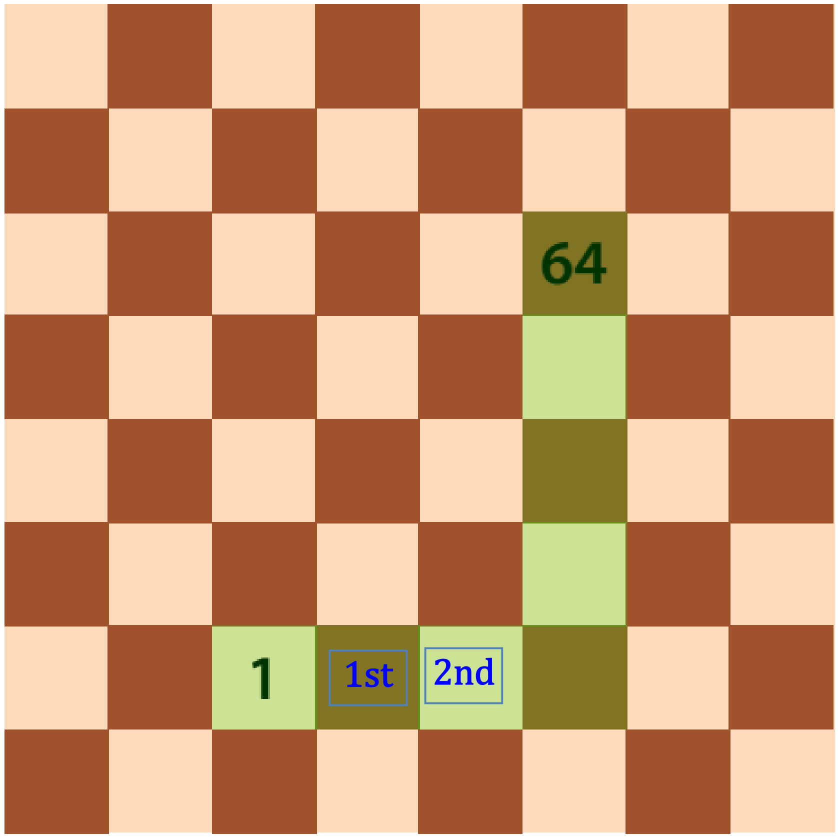

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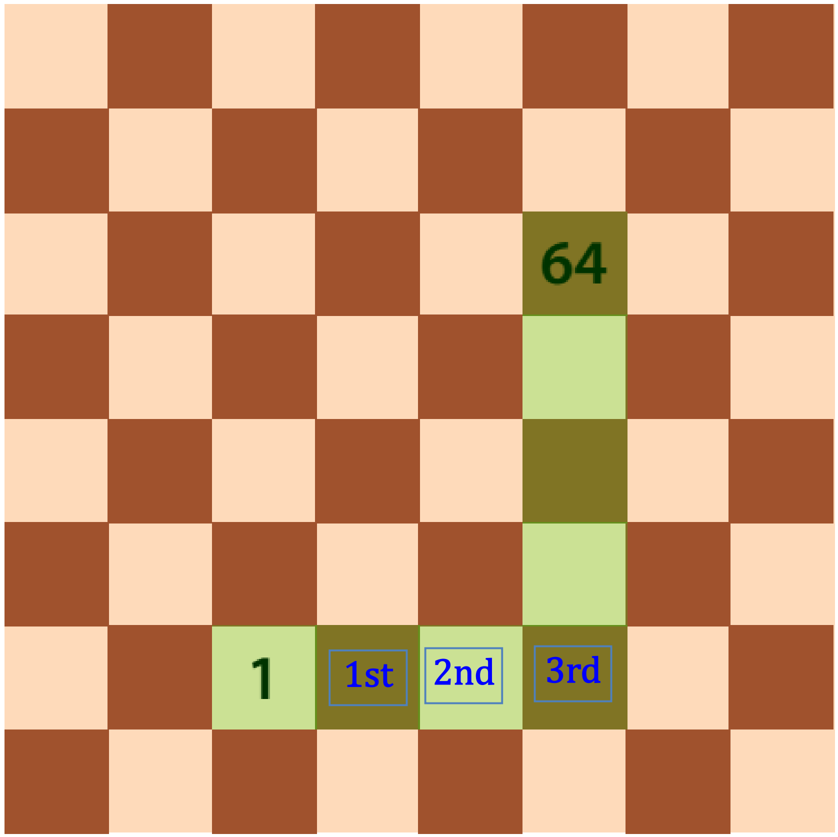

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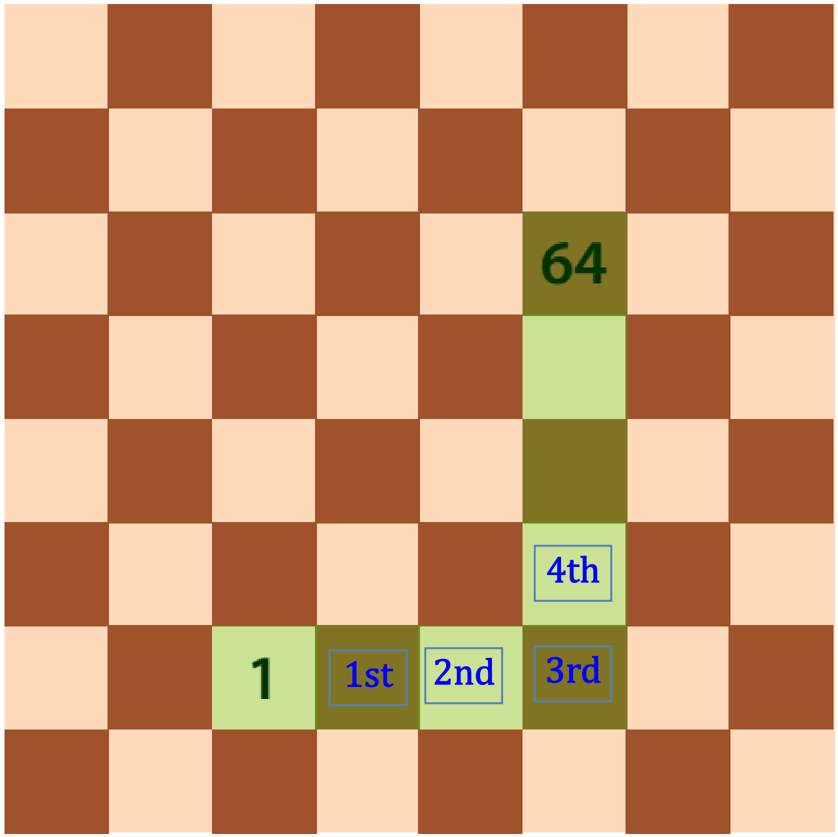

Numbers on a Chessboard

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For this:
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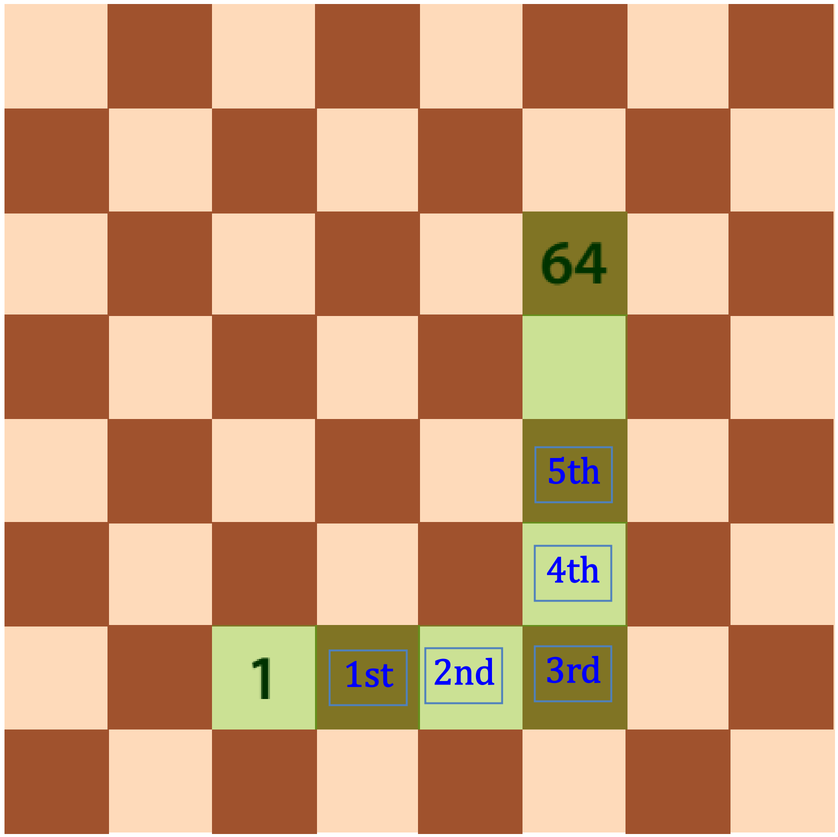

Numbers on a Chessboard

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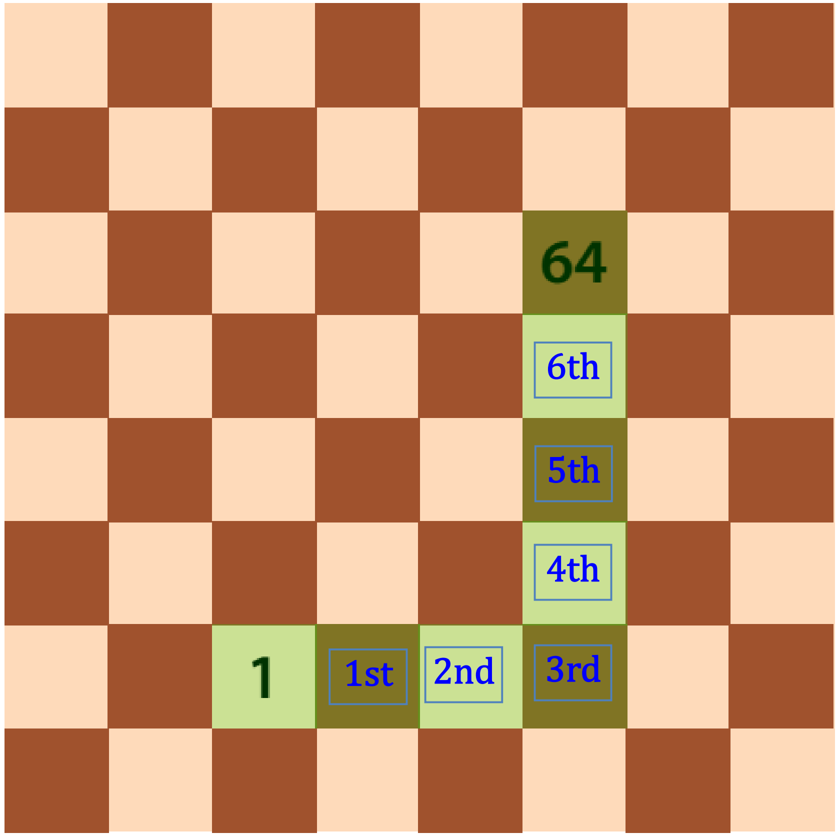
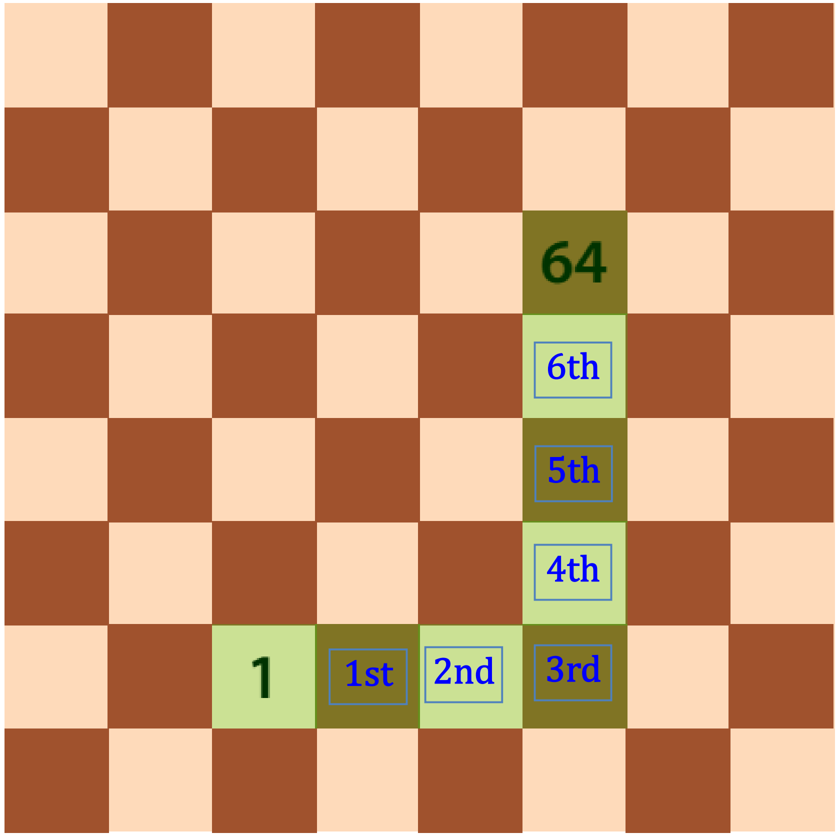


Numbers on a Chessboard

Puzzle

Let us again suppose we can do it. Let's see what will happen.

For this:
We need 7 steps to get from 1 to 64



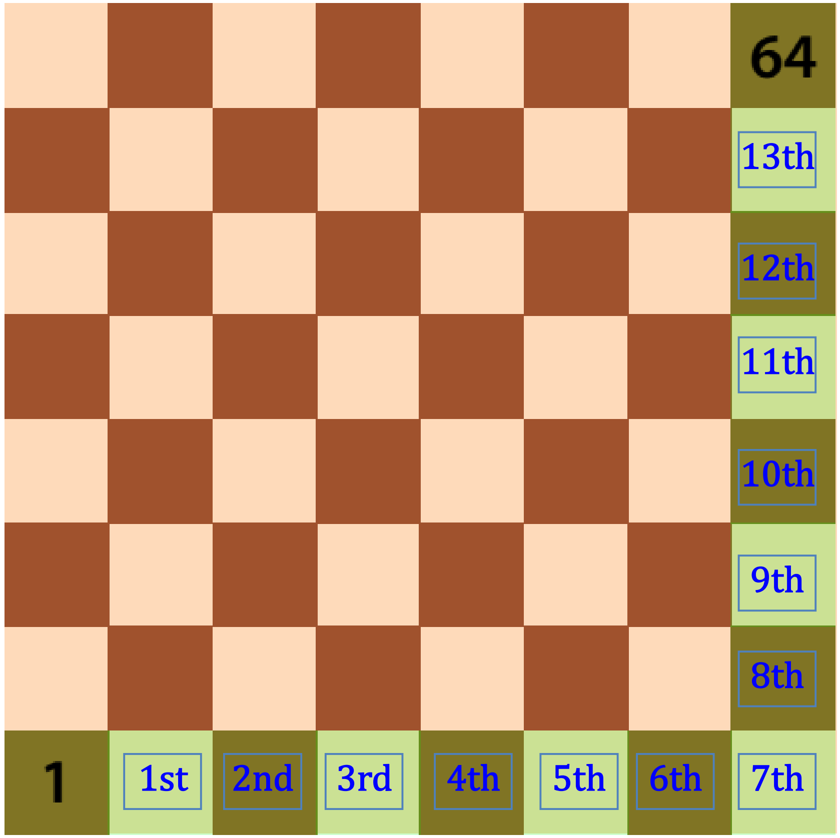
For this:
4 x 7 = 28, too small.
Doesn't work.

Numbers on a Chessboard

Puzzle

Let us again suppose we can do it. Let's see what will happen.

Let us try to put 1 & 64 with the maximum number of steps possible:
14 steps!



For this:
4 x 14= 56, still not enough.
Hence, by showing it is not possible, we proved our statement is wrong.

- Reduction ad absurdum
- Balls in Boxes
- Numbers in Boxes
- Pigeonhole Principle
- An(-1,0,1) Antimagic Square
- Handshakes

Same number of Hairs

Problem

- Show that there are two people in Seoul with the same number of hairs. Suppose we can do it. Let's see what will happen

https://upload.wikimedia.org/wikipedia/commons/a/a6/%EA%B2%BD%EB%B3%B5%EA%B6%81_%EC%A0%84%EA%B2%BD.jpg

https://sookyeong.files.wordpress.com/2009/07/200907160836341130_1.jpg

Same number of Hairs

Problem

- Show that there are two people in Seoul with the same number of hairs. Suppose we can do it. Let's see what will happen

  - How many people in Seoul?
    - 9,963,000 (2020), (macroctrends.net)

Same number of Hairs

Problem

- Show that there are two people in Seoul with the same number of hairs. Suppose we can do it. Let's see what will happen

  - How many people in Seoul?
    - 9,963,000 (2020), (macroctrends.net)
  - How many hairs a person has?
    - 100,000 on the average, (healthline.com)

Same number of Hairs

Problem

- Show that there are two people in Seoul with the same number of hairs. Suppose we can do it. Let's see what will happen

Jeonju ≈ 400K

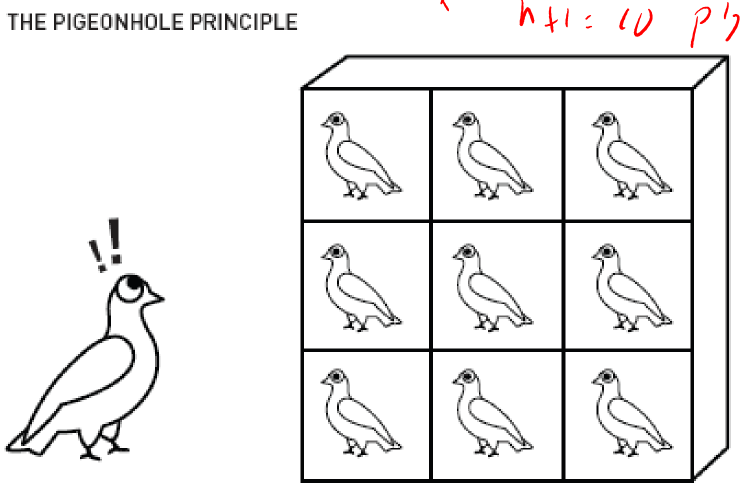
  - How many people in Seoul?
    - 9,963,000 (2020), (macroctrends.net)
  - How many hairs a person has?
    - 100,000 on the average, (healthline.com)
  - Number of people in Seoul >>> no of hairs/ person
    - Hence there should be people with same number of hairs

Pigeonhole Principle

The Pigeonhole Principle

- Suppose that there are $n + 1$ pigeons & n number of pigeon holes. One of the pigeonholes must then be occupied by at least two pigeons.

$n = 9$ pholes
$n+1 = 10$ pigeons



https://mathimages.swarthmore.edu/imgUpload/Pigeon.gif

Pigeonhole Principle

The Pigeonhole Principle

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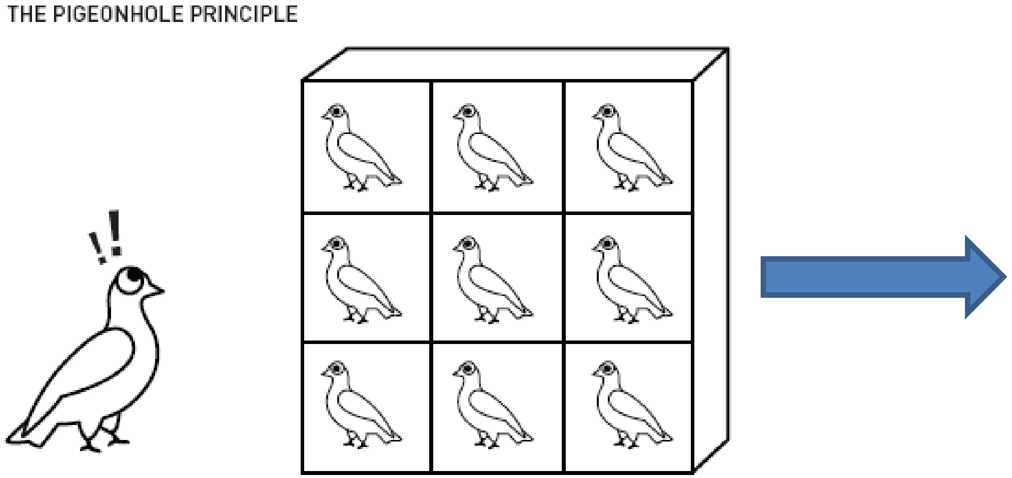

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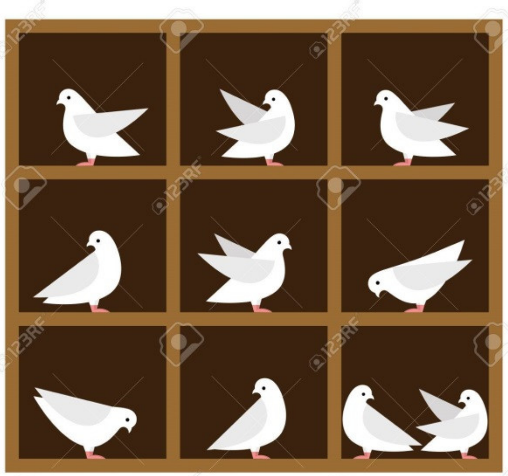


https://previews.123rf.com/images/pimpinello/pimpinello1609/pimpinello160900708/63423979-vector-pigeonhole-principle-illustration-flat-illustration-vector-icons-of-pigeons-.jpg

Pigeonhole Principle

The Pigeonhole Principle

- Suppose that there are $n + 1$ pigeons & n number of pigeon holes. One of the pigeonholes must then be occupied by at least two pigeons.
  - Simple but very useful principle
  - Proof by contradiction:
    - If there is at most one pigeon in each hole, then we would have at most just n pigeons in total for all holes

https://mathimages.swarthmore.edu/imgUpload/Pigeon.gif

Pigeonhole Principle

The Pigeonhole Principle

- Suppose that there are $n + 1$ pigeons & n number of pigeon holes. One of the pigeonholes must then be occupied by at least two pigeons.
  - Simple but very useful principle
  - Proof by contradiction:
    - If there is at most one pigeon in each hole, then we would have at most just n pigeons in total for all holes
  - Recalling the previous example
    - people of Seoul → are the "pigeons"
    - possible number of hairs → are the "pigeon holes"

https://mathimages.swarthmore.edu/imgUpload/Pigeon.gif

- Reduction ad absurdum
- Balls in Boxes
- Numbers in Boxes
- Pigeonhole Principle
- An(-1,0,1) Antimagic Square
- Handshakes

An (-1,0,-1) Antimagic Square

Puzzle

[range]

- Is it possible to fill a *3 x 3* table with the numbers *{-1,0,1}* so that the sum of each row, column and both diagonals produce different numbers?

Suppose we can do it. Let's see what will happen

			w
			u
			v
x	y	z	p

An (-1,0,-1) Antimagic Square

Puzzle

- Is it possible to fill a *3 x 3* table with the numbers *{-1,0,1}* so that the sum of each row, column and both diagonals produce different numbers?

Suppose we can do it. Let's see what will happen

1	0	-1
-1	0	-1
1	1	0

1 1

- The sum for first 2 columns are the same
- It violates the statement
- Seems like the statement is not possible

An (-1,0,-1) Antimagic Square

Puzzle

- Is it possible to fill a *3 x 3* table with the numbers *{-1,0,1}* so that the sum of each row, column and both diagonals produce different numbers?

Suppose we can do it. Let's see what will happen

			h
1	0	-1	g
-1	0	-1	f
1	1	0	e
a	b	c	d

- There are 3 rows, 3 columns, and 2 diagonals; for a total of 8

An (-1,0,-1) Antimagic Square

Puzzle

- Is it possible to fill a 3×3 table with the numbers $\{-1, 0, 1\}$ so that the sum of each row, column and both diagonals produce different numbers?

Suppose we can do it. Let's see what will happen

1	0	-1
-1	0	-1
1	1	0

- There are 3 rows, 3 columns, and 2 diagonals; for a total of 8
- The sum for each line ranges from [-3, 3], total of 7 values:
  - $-1 + -1 + -1 = -3$ (lowest), $1 + 1 + 1 = 3$ (highest)

An (-1,0,~~-~~1) Antimagic Square

Puzzle

- Is it possible to fill a *3 x 3* table with the numbers *{-1,0,1}* so that the sum of each row, column and both diagonals produce different numbers?

Suppose we can do it. Let's see what will happen

D_2

1	0	-1	r_1
-1	0	-1	r_2
1	1	0	r_3
c_1	c_2	c_3	D_1

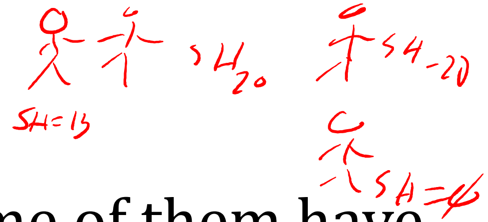
- There are 3 rows, 3 columns, and 2 diagonals; for a total of 8
- The sum for each line ranges from [-3, 3], total of 7 values: (-3,-2,-1,0,1,2,3)
  - *-1+ -1+ -1 = -3* (lowest), *1 + 1 + 1 = 3* (highest)
- Based on Pigeonhole Principle
  - There would be at least 2 lines with same sum
  - Hence the statement is not possible!

PH = 7 ; P = 8

- Reduction ad absurdum
- Balls in Boxes
- Numbers in Boxes
- Pigeonhole Principle
- An(-1,0,1) Antimagic Square
- Handshakes

Handshakes

$n=30$



Problem

- There are 30 persons in the room, and some of them have shaken hands. Prove that there are two persons who shook hands with equal number of people.
  - We're experienced with Pigeonhole, let's count
    - pigeons → 30 people
    - pigeonholes → [0, 29], no of hand shaken, total of 30
  - Pigeonhole principle is not applicable
    - since *#pigeons = #pigeonholes, 30 = 30* ✗
  - But wait
    - A = 0 & B = 29 handshakes, not possible at the same time
      - If A = 0, then max handshakes for B is only 28
    - Now 29 pigeonholes, we can use Pigeonhole Principle!

Proof by Contradiction

Conclusion

- Proof by contradiction
  - basic way of argument (reasoning)
  - used everywhere, usually combined w/ other ideas
  - can sometimes help a lot just on it's own
- Pigeonhole Principle
  - one of the most basic proof ideas
  - used a lot in mathematics
  - very simple principle, basically amounts to counting

Thank you.