Introduction to Discrete Math

Felipe P. Vista IV

Course Outline

- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion, Logic, Invariants
- Probability & Combinatronics
  - Counting, Probability, Random Variables
- Graph Theory
  - Graphs (cycles, classes, parameters)
- Number Theory & Cryptography
  - Arithmetic in modular form
  - Intro to Cryptography

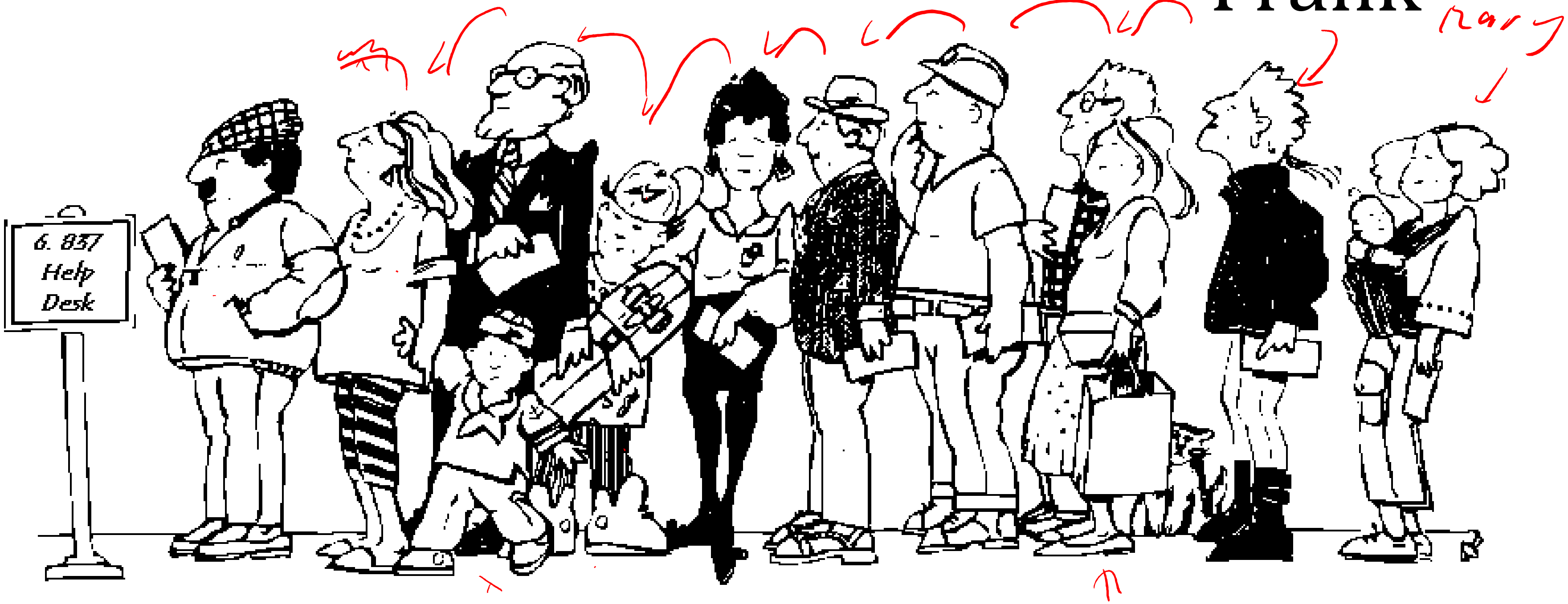
Mathematical Thinking – Recursion & Induction

RECURSION

- Recursion
- The Coin Problem
- Hanoi Towers

Line to process a paper

Frank



http://groups.csail.mit.edu/graphics/classes/6.837/F98/Lecture5/longline.gif

Compute Queue Length

1) Mary gets in line
2) She wonders how many people before her?
3) Mary asks Frank (in front of her)
    - *Could you please tell how many people ahead of you?*
4) Now, Frank has the same problem
    - But Frank was able to find out there are 8
5) Now Mary knows there are 9 people in front

Recursive Program

Algorithm:

numberOfPeopleInFront(F):

if there is no one before A:

return 0

F ← number of people before A

return *numberOfPeopleInFront*(F) + 1

Factorial Function

Definition:

- For a positive integer *n*, its factorial is the product of integers from 1 to n.
  - $5! = \underbrace{\overbrace{1 \text{ x } 2}^{2!} \text{ x } 3 \text{ x } 4}_{4!} \text{ x } 5 = 120 = 4! \text{ X } 5$

Recursive definition:

$$n! = \begin{cases} 1 & \text{if } n = 1 \\ n \times (n-1)! & \text{if } n > 1 \end{cases}$$

$n! = 4! = 4\,(3!)$

Iterative Program

"iterate"

```
def factorial(n):
    assert(n > 0)
    result = 1
    for i in range(1, n+1)
        result *= i
    return result
```

→ result = result * i

res = res + 1 => res += 1

~s 1 * 2 * ...

res = res * 2

res = res * 3

res = res * n

Recursive Program

```
def factorial(n):
    assert(n > 0)
    if n == 1
        return 1
    else:
        return n * factorial(n - 1)
```

Termination

- Must make sure that recursive program (or definition) terminates after finite number of steps
- Achieved by decreasing some parameter until it reaches the base case
  - line length: line of the length decreases by 1 with each recursive call, until it becomes 1
  - Factorial: n decreases by 1

Example of Infinite Recursion

n = 2

```
def infinite(n):
    if n == 1
        return 0
    return n * infinite(n+1)
```

if n == 50, stop

- In theory:
  - will never stop, parameters increase to infinity
- In practice:
  - will cause error message
    - “Stack overload” or “Recursion depth exceeded”

More Examples of Infinite Recursion

- No base case:

```
def factorial(n):
    return n * factorial(n - 1)
```

- Parameter do not change:

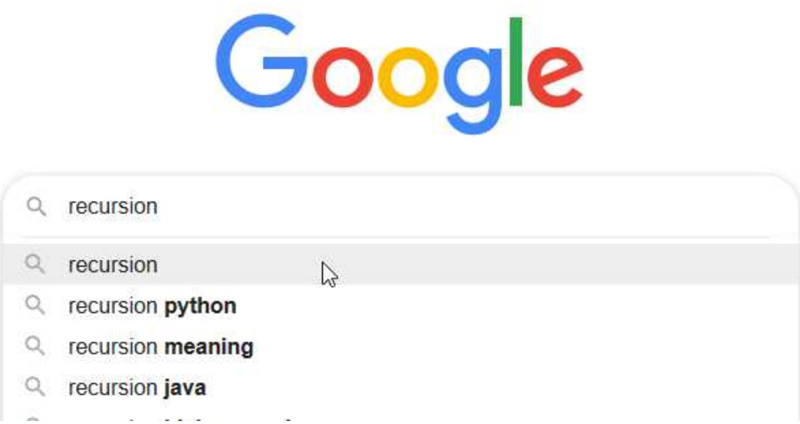
n = 1 , n = 2

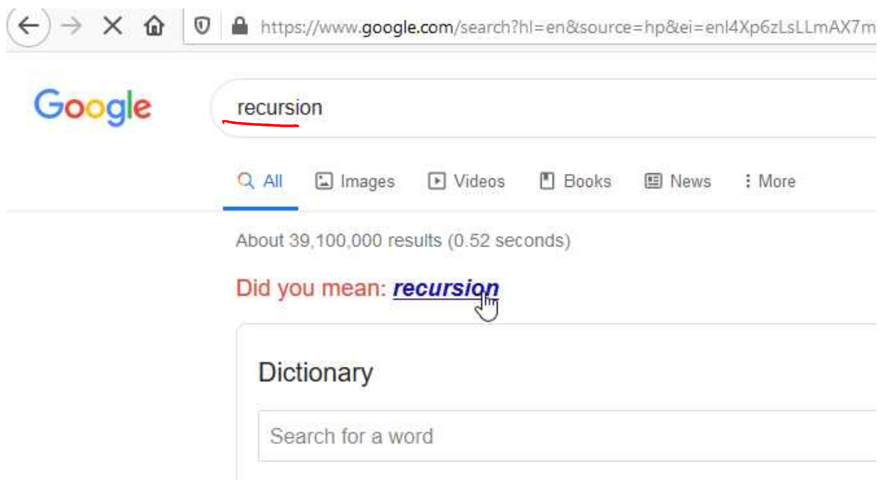
```
def infinite(n):
    if n == 1
        return 0
    return 1 + infinite(n)
```

More Examples (Not so serious :D)

- To understand recursion, one must first understand recursion

- Google is even in with the fun





- Recursion
- The Coin Problem
- Hanoi Towers

The Coin Problem

Problem

Prove that any amount starting from 8 **ewans** can be paid for using only 3 **ewans** and 5 **ewans.**

- $8 = 3+5$, $9=3+3+3$, $10=5+5$, $11=3+3+5$
  - looks promising, seems possible
- How to be sure it will always be possible?

Speculation

- *Speculation* – forming a theory without firm, solid concrete proof or evidence

- 8 can be definitely ~~be~~ done
- 11 is also possible by adding one 3 ewan coin
  - 11 = 8 + 3
  - same principle for 14, 17, 20, etc…
- Same for 9, keep on adding a 3 ewan coin
  - will give 12, 15, 18, 21, etc
- Checking 10 ewans, still keep adding 3 ewan coins
  - we'll get 13, 16, 19, 22, etc

Recursive Program

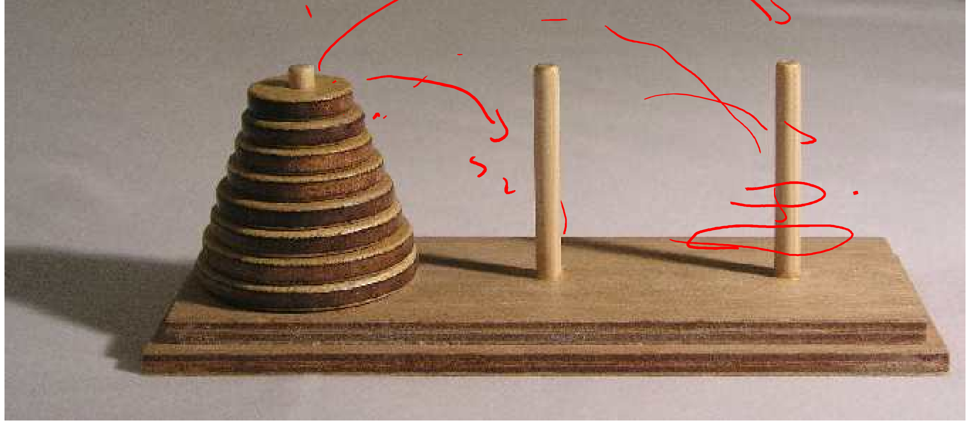
```
def change(amount):
    assert(amount >= 8)
    if amount == 8:
        return [3, 5]
    if amount == 9:
        return [3, 3, 3]
    if amount == 10:
        return [5, 5]

    coins = change(amount - 3)
    coins.append(3)
    return coins
```

- Recursion
- The Coin Problem
- Hanoi Towers

Hanoi Towers

Problem

There are 3 sticks with *n* discs sorted by size on one of the sticks. The goal is to move all *n* disks to another stick subject to 2 constraints: (1) move one disk at a time, and (2) don't put a larger disk over a smaller one.

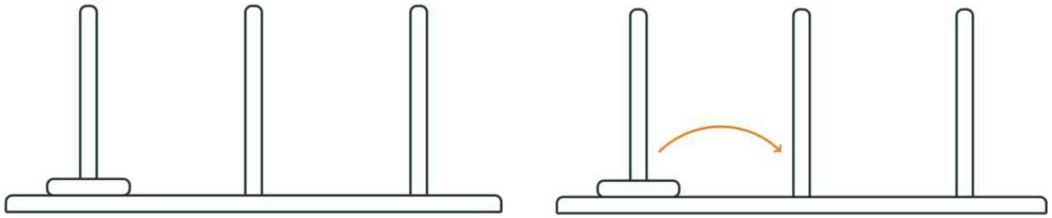


https://en.wikipedia.org/wiki/Tower_of_Hanoi#/media/File:Tower_of_Hanoi.jpeg

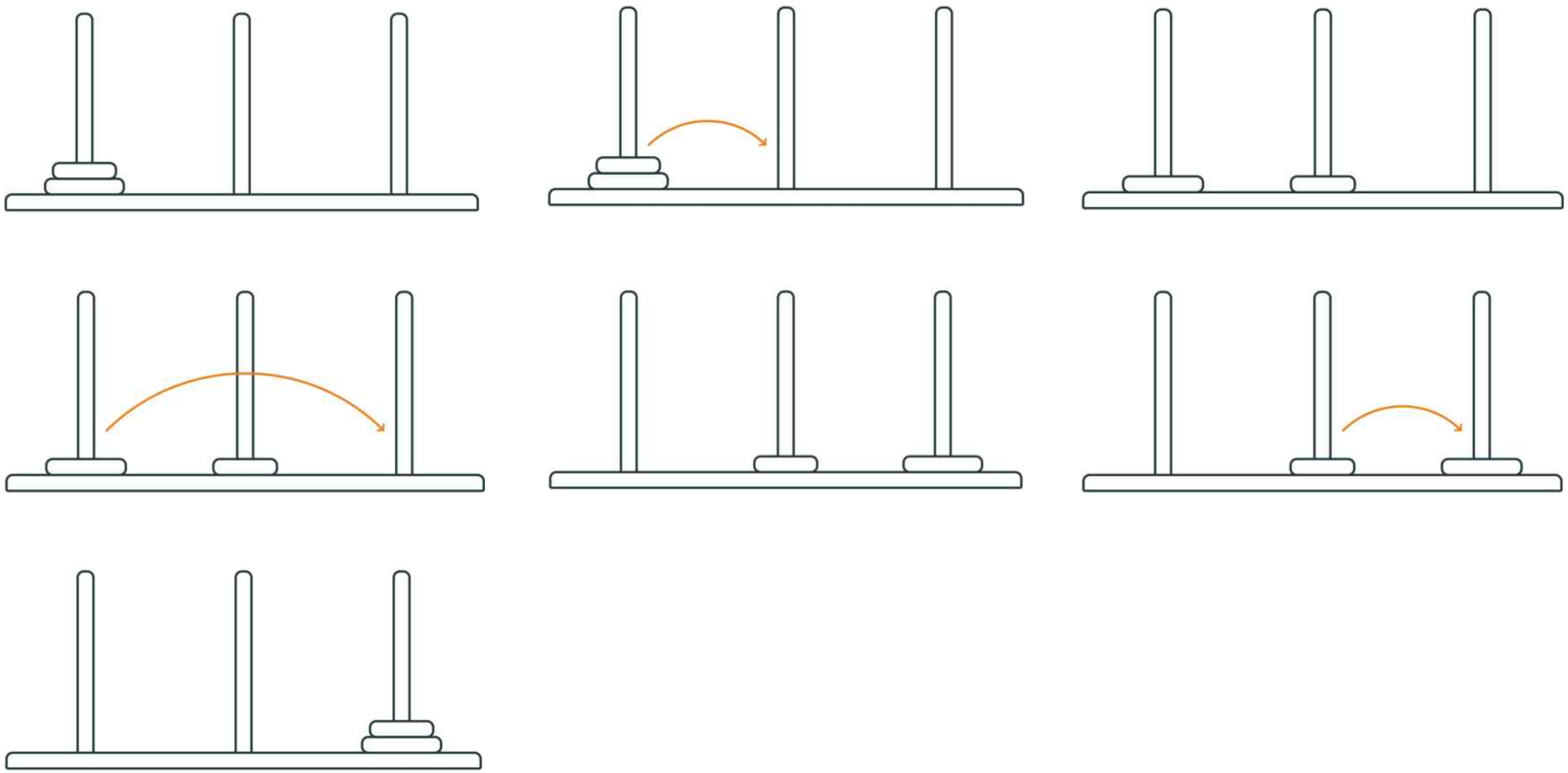
Can it be done?

- For what value of n is this possible?
  - For all!
- How can we be sure?
  - Design a recursive program that will solve the puzzle for every value of n

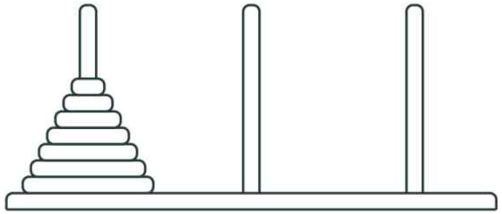
Simplest scenario, n=1 disk

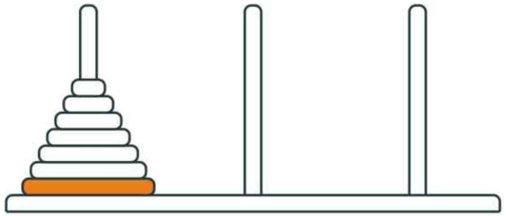


n=2 Disks Scenario



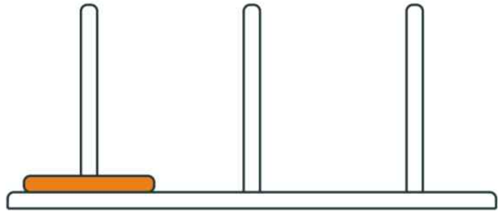
How about for *n* disks? Let's speculate





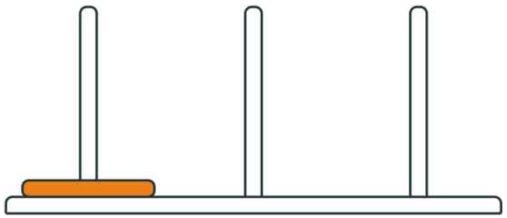
1

Consider largest disk



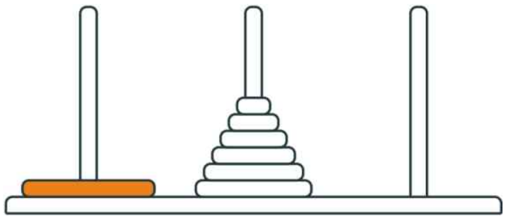
When can we move it?

C_1 → When there are no other disks on the main stick



Where can we move it?

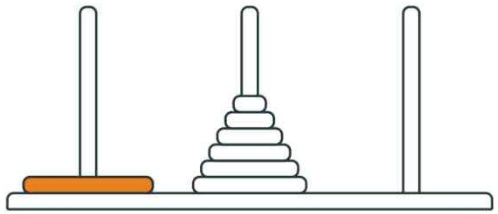
C_2 → Only to an empty stick



2

All other disks are on another disk

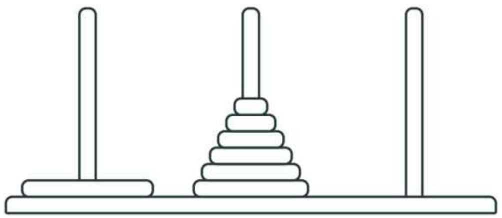
C_1 && C_2



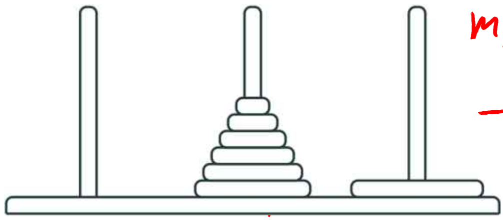
n-1

This is same problem, just n-1 disks!

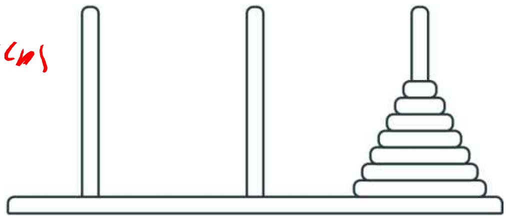
n-1 disks? Let's do it recursively



Move n-1 disks recursively



Move largest disk to free stick



Move n-1 disks recursively until done

Summary

- A solution has been proposed for all values of ***n***:
  - Base scenario: possible for *disks = 1*
    - Therefore, it is possible for *disks = 2*
    - Therefore, it is possible for *disks = 3*, … until *disks = n*
  - Or putting it other way, It is possible to solve *n*, solve first for *n-1*
    - Therefore, it is possible for *n – 1*
    - Therefore, it is possible for *n – 2*; … until n = 1
- recursion & induction
  - *recursion*: method of defining/implementing something
  - *induction*: mathematical method of proving something

Reminder

- Everybody, make sure that your name in ZOOM is in the following format:
  - Ex: 202054321 Juan Dela Cruz

Not changing your name to this format

* you **will** be marked Absent * → absent?

Mathematical Thinking – Recursion & Induction

MATHEMATICAL INDUCTION

Mathematical Induction

- Very powerful method of proving
- Take the case of falling dominos
  - Push 1st one, it'll fall & push 2nd one
  - 2nd one will fall & push 3rd one, etc...
  - So on until all dominos fall
- How to prove for *n* dominos
  - push $k^{th} \rightarrow k^{th}$ falls $\rightarrow$ *push* $k+1 \rightarrow k+1$ *falls* $\rightarrow \ldots \rightarrow n^{th}$ falls
- Lots of Comp Sci algo proved using induction

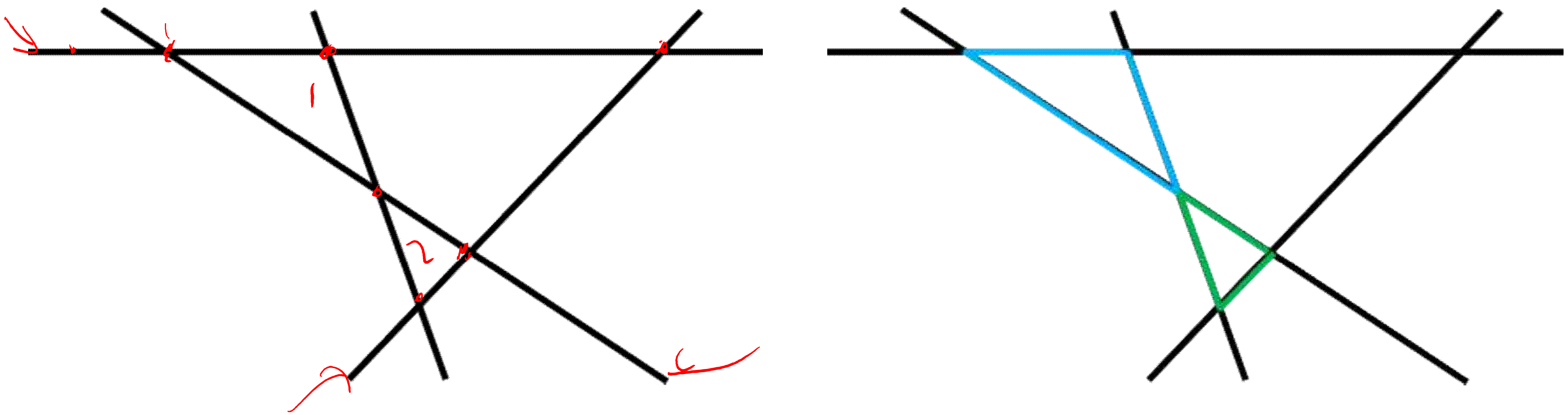
https://spiritualguidancefortheso ul.com/wp-content/uploads/2015/01/dominoes-04.jpg

- Lines and Triangles
- Connecting Points
- Sums of Numbers
- Bernoulli's Inequality
- Coins
- Cutting a Triangle
- Flawed Induction Proofs
- Alternating Sum

Lines & Triangles

Problem

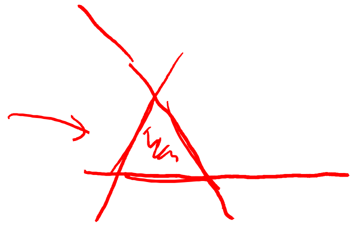
Several straight lines (at least three) cut a plane into pieces. Each line intersects with every other line, and all intersection points are different. **Prove that there is at least one triangular piece.**



Lines & Triangles

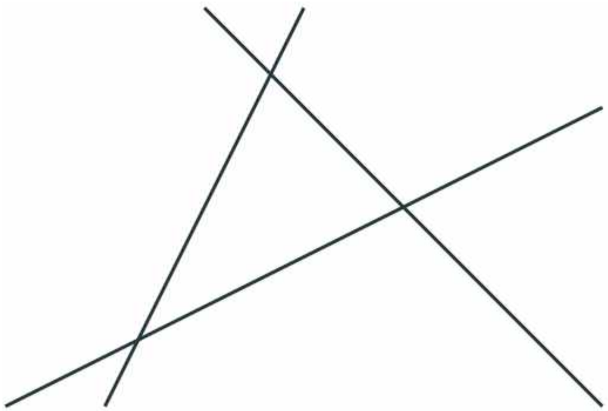
Proof Idea

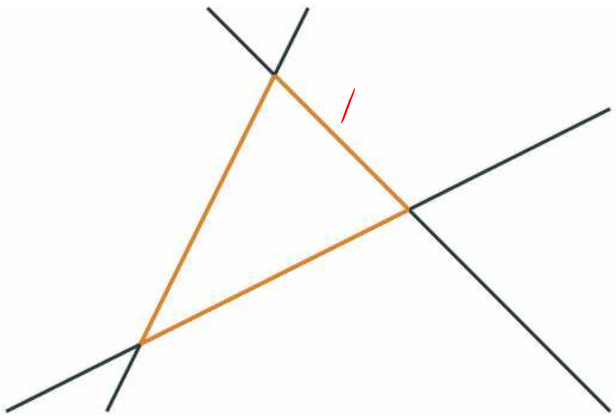
- A triangle appears as soon as there are 3 lines
- Each time we add more lines one at a time,
  - Either the same triangle remains
  - Or a new one is formed



Lines & Triangles

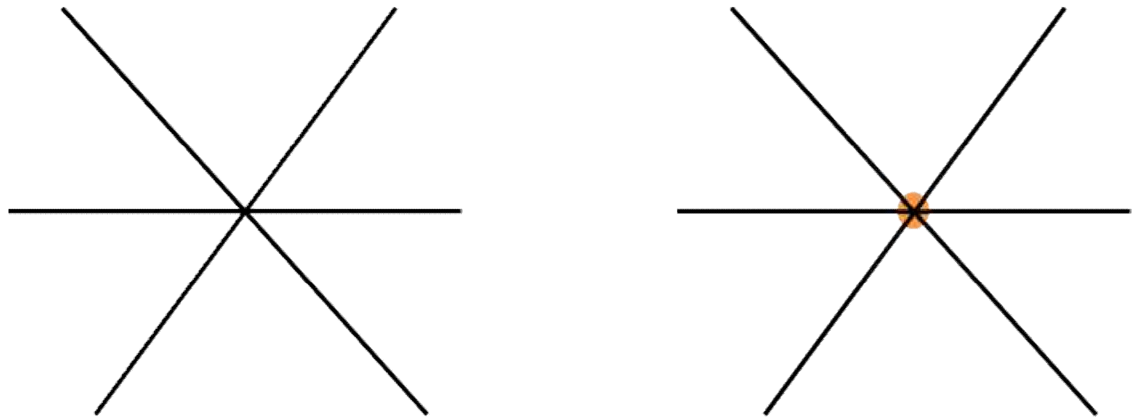
Three lines





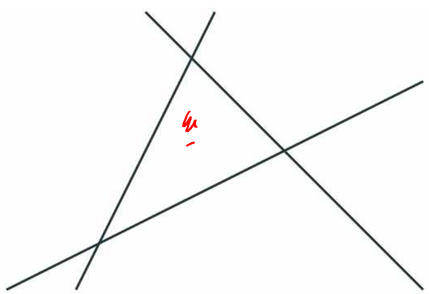
Lines & Triangles

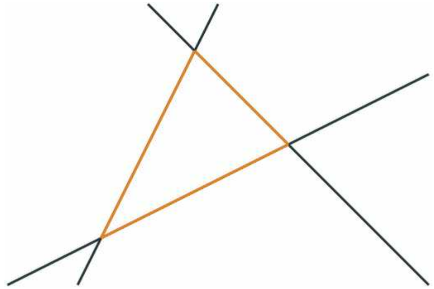
Three lines – Bad Case

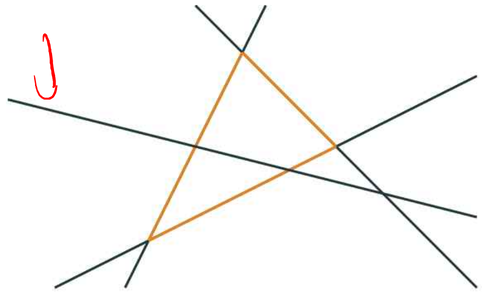


Lines & Triangles

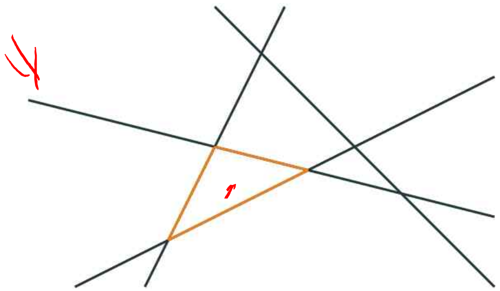
Three lines – Adding one more line



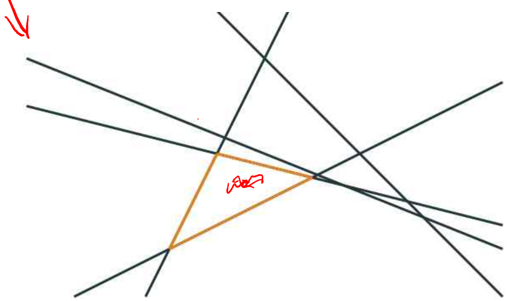




When new line intersects triangle…



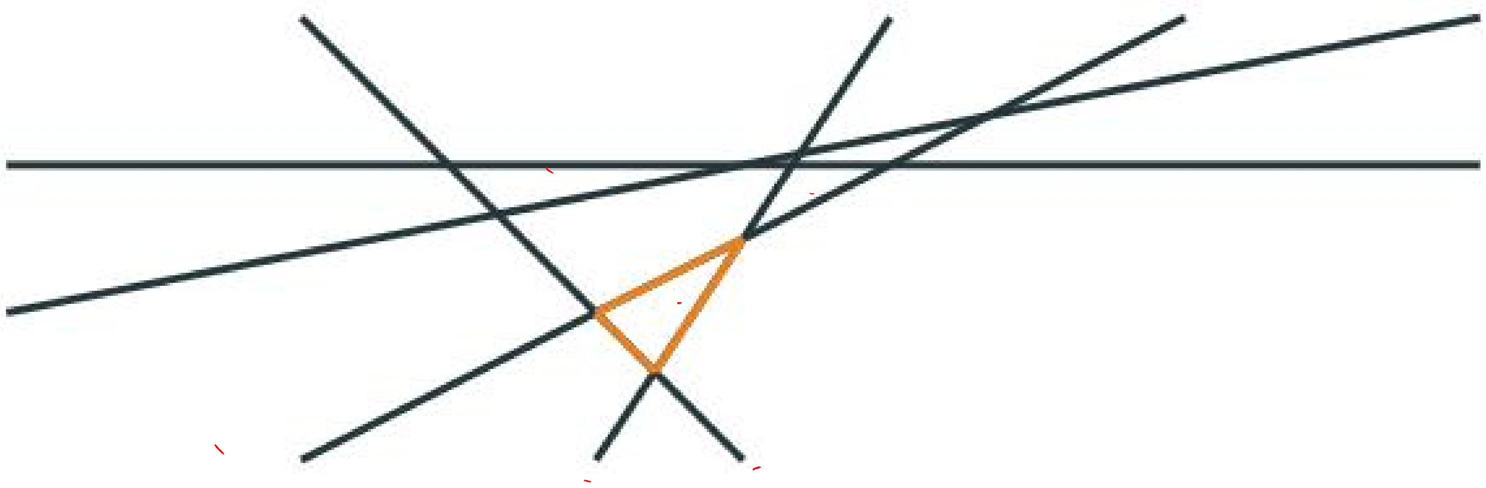
When new line intersects triangle, a new triangle appears



When new line doesn't touch existing triangle, the triangle remains intact

Lines & Triangles

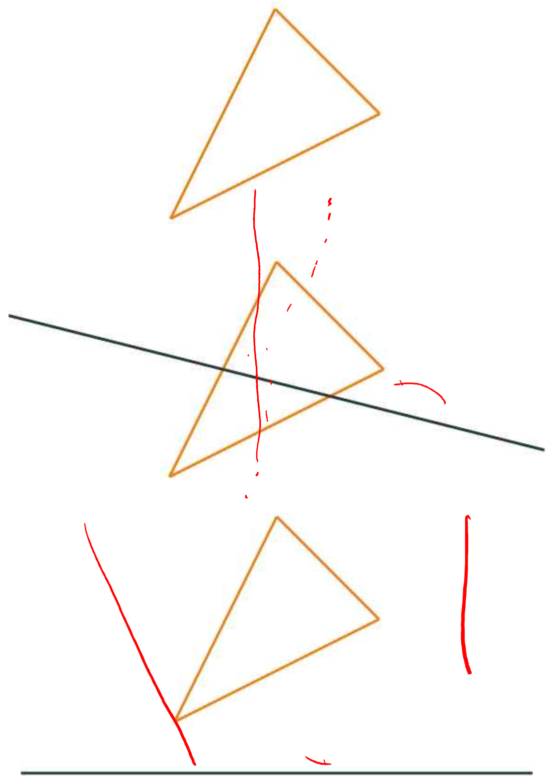
General Case



With a lot of lines, we might spot the triangle.
But with thousands of lines, we might not be able to see it.

Lines & Triangles

General Case – Adding a line



We assume lot of lines have already been added.
A triangle has already been formed

When new line intersects 2 sides of the triangle,
a new triangle appears

When new line doesn't touch existing triangle,
the triangle remains intact

Lines & Triangles

Theorem

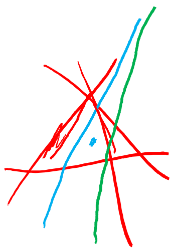
- For any $n \geq 3$ and n number of straight lines on a plane, if every two lines intersect, and all the intersection points are different, there is a triangular piece among the pieces into which these lines cut the plane.

Lines & Triangles

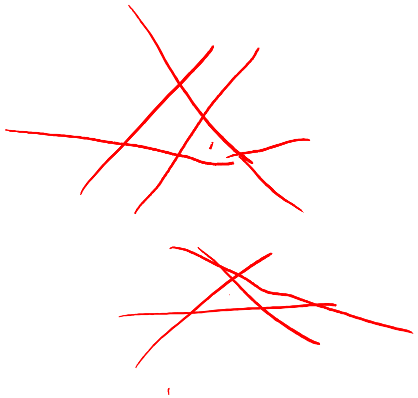
Proof Structure

Number of lines

$$3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow n$$



$$k \rightarrow K+1$$



Mathematical Induction

- Prove induction base: $n = 3$, three lines
- Prove that if theorem is true for $n = 3$, then it is true for $n = 4$
- Prove that if theorem is true for $n = 4$, then it is true for $n = 5$
- …
- Prove induction step: from n to $n + 1$, adding one more line in the general case
- …
- Profit! (Proved the whole statement)

But wait, we can shorten this, can you see how?

Mathematical Induction

Prove only induction base is $n = 3$ and prove induction step from n to $n+1$ in the general case

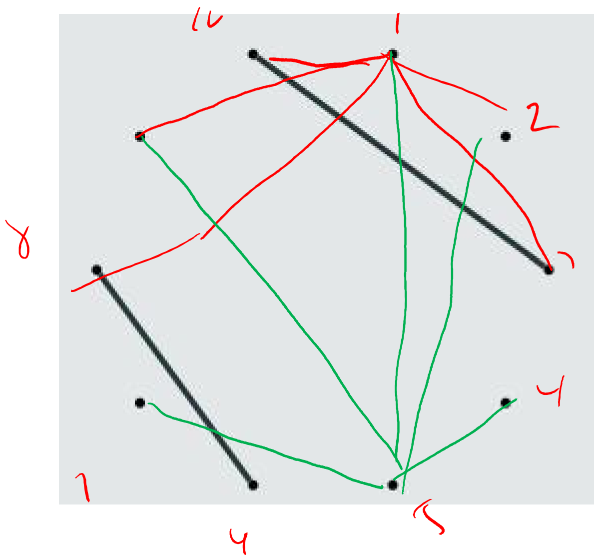
- Prove induction base: $n = 3$, three lines
- Prove induction step: from n to $n + 1$, adding one more line in the general case
- Profit!

- Lines and Triangles
- Connecting Points
- Sums of Numbers
- Bernoulli's Inequality
- Coins
- Cutting a Triangle
- Flawed Induction Proofs
- Alternating Sum

Connecting Points

Problem

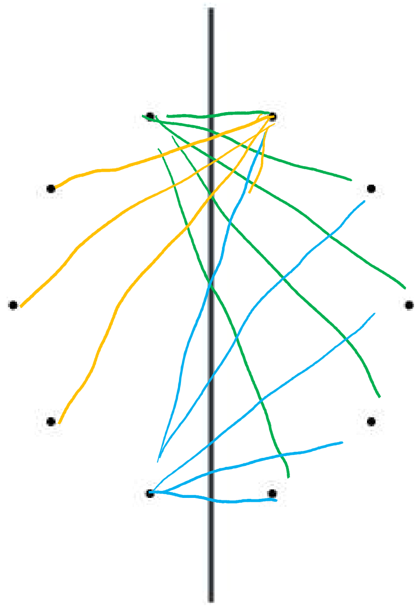
Connect some of these 10 points with segments, such that every point is connected with 5 other points.



Connecting Points

Solution

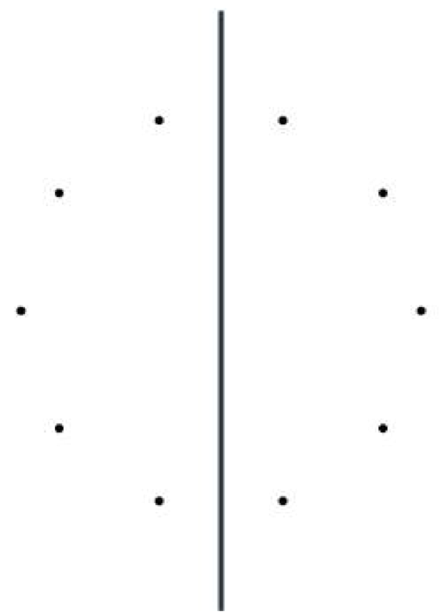
Separate the points into the left half and the right half. Each half has 5 points.



Connecting Points

Solution

Connect each half from the left half to each point of the right half.



Connecting Points

Solution

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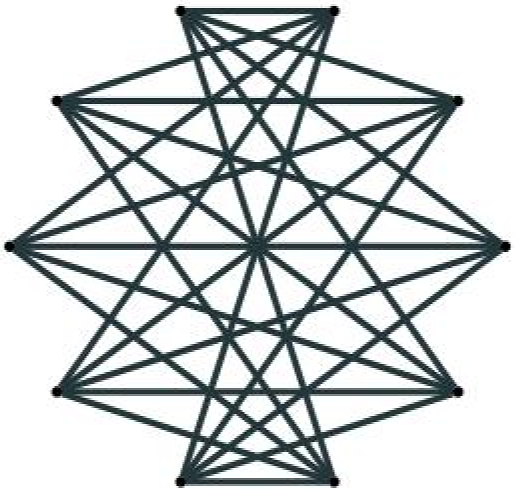


Connecting Points

Problem

Now we have 9 points. Can we connect some of them with segments so that each point is connected with 5 other points?

10 pts → 5

9 pts → 5

Connecting Points

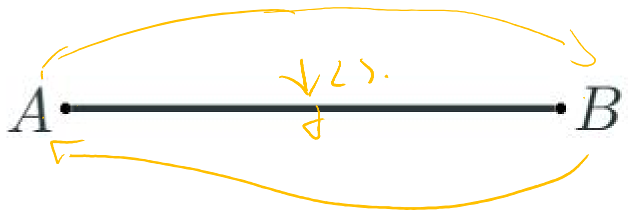
Even and Odd

- The numbers 0, 2, 4, 6, 8, … are called even
  - Even numbers are divisible by 2
- Numbers 1, 3, 5, 7, 9, … are called odd
  - Odd numbers are not divisible by 2

Connecting Points

Neighbors

- Let us call point B the neighbor of point A if points A & B are connected with a segment.
- If point B is a neighbor of A, then A is also a neighbor of B.



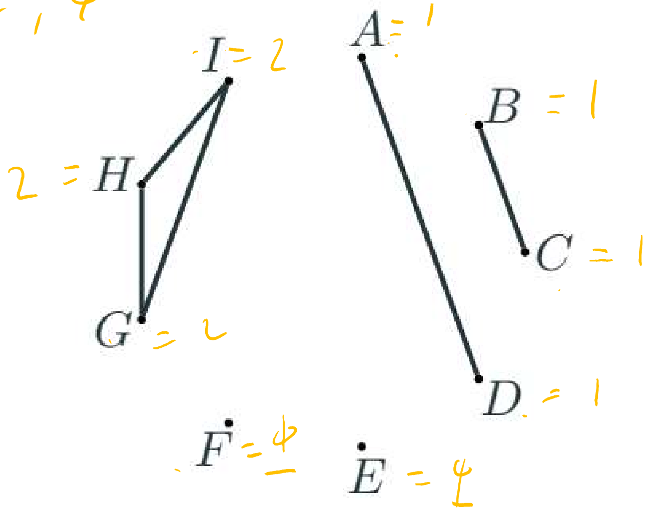
Connecting Points

Even and Odd

- Let us call a point **even** if it has even number of neighbors, otherwise we call it odd.
- For the given example, points A, B, C & D are odd while all the other points are even.

odd = 2

even = 2, 4



e = 5

o = 4

Connecting Points

Theorem

- The number of odd points is always even, regardless of how many points & segments are there and which pairs are connected by segments.

\# $n_{odd\,pts}$ = even ; n_{pts} ; $P_{segments}$

Connecting Points

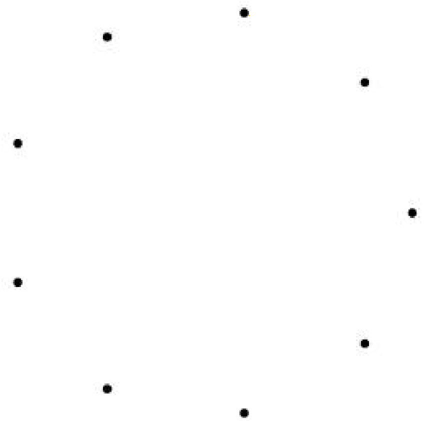
Proof Idea

- When there are no segments, there are no odd points (recall 0 is even), so the number of odd points is indeed even.
- When we add segments one by one, the number of odd points either doesn't change, increases by 2 or decreases by 2. hence the number of odd points stays even.

Connecting Points

Easy Case

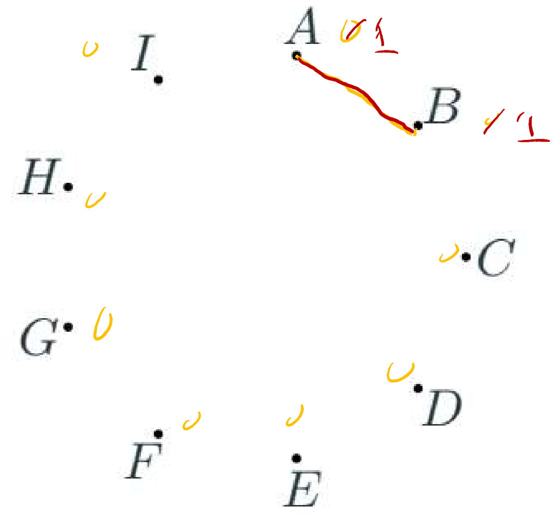
- When there are no segments, each points has 0 neighbors, hence there are no odd points. The number of odd points is 0, which is even, hence there is indeed an even number of points.



Connecting Points

Adding a Segment

- Segment AB adds two odd points A & B.

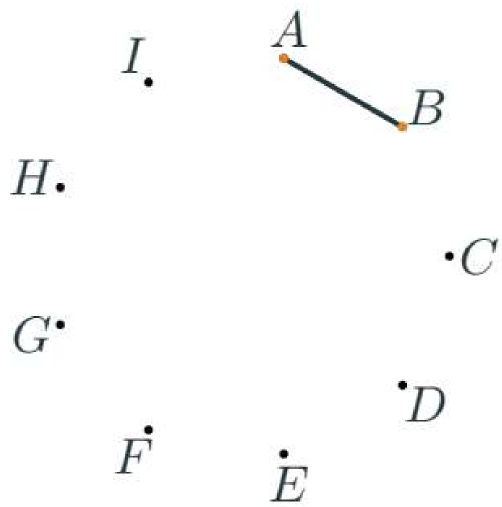


No of odd points: 0

Connecting Points

Adding a Segment

- Segment AB adds two odd points A & B.

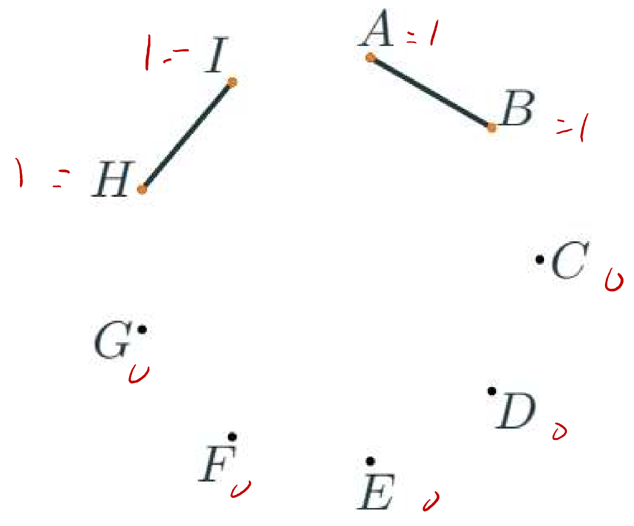


No of odd points: 2

Connecting Points

Adding a Segment

- Segment *HI* adds two odd points *H* & *I*.

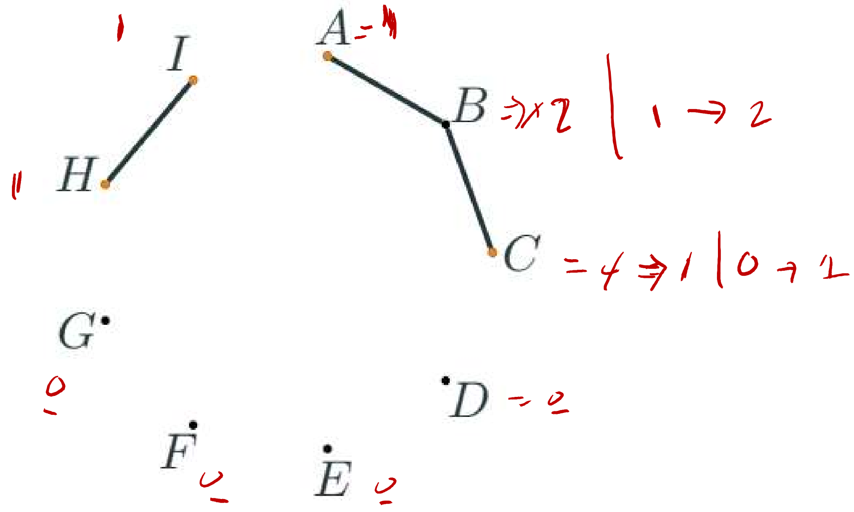


No of odd points: 4

Connecting Points

Adding a Segment

- Segment BC makes B even and C odd.

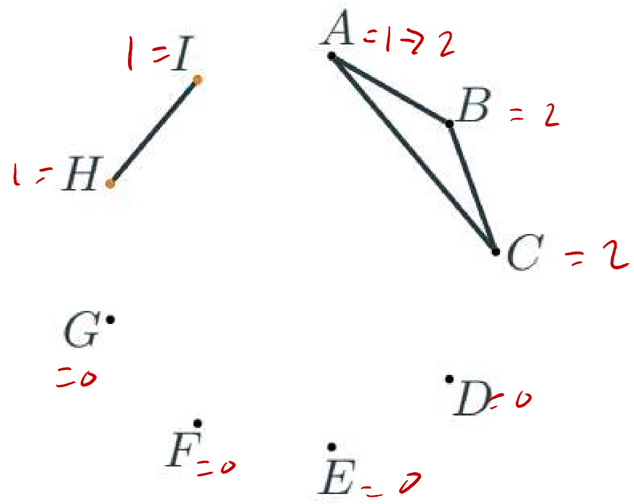


No of odd points: 4

Connecting Points

Adding a Segment

- Segment AC makes A and C even.

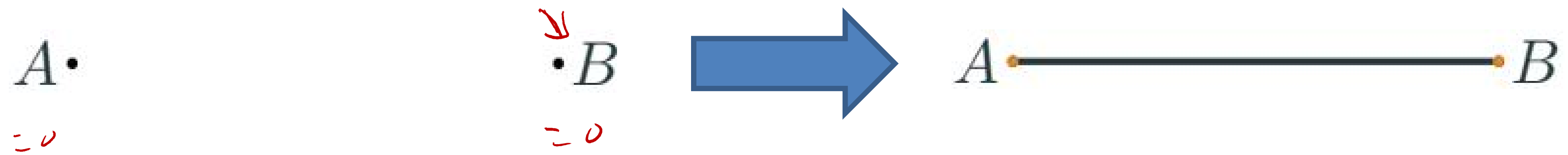


No of odd points: 2

Connecting Points

Adding a Segment in General

- If A & B are even, segment AB makes them both odd and adds 2 odd points.

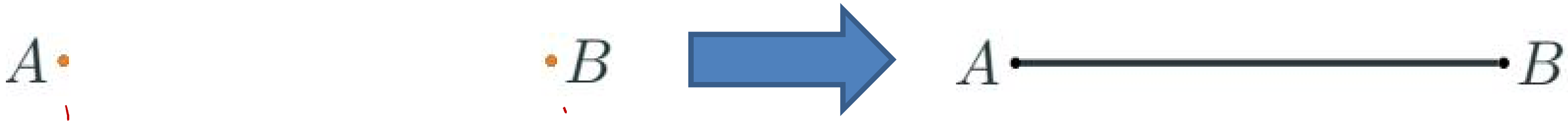


*Note:
- A●: represents even point
- A●: represents odd point

Connecting Points

Adding a Segment in General

- If A & B are odd, segment AB makes them both even and removes 2 odd points.

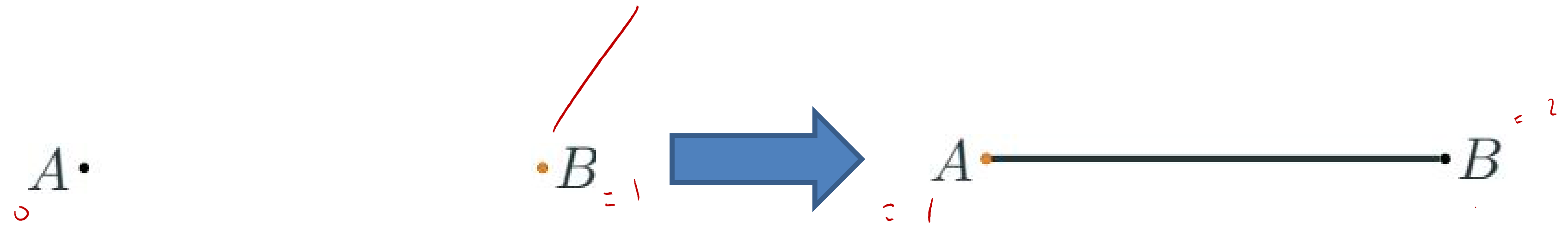


*Note:
- A●: represents even point
- A●: represents odd point

Connecting Points

Adding a Segment in General

- If A is even & B is odd, segment AB swaps their status, keeping number of odd points the same.



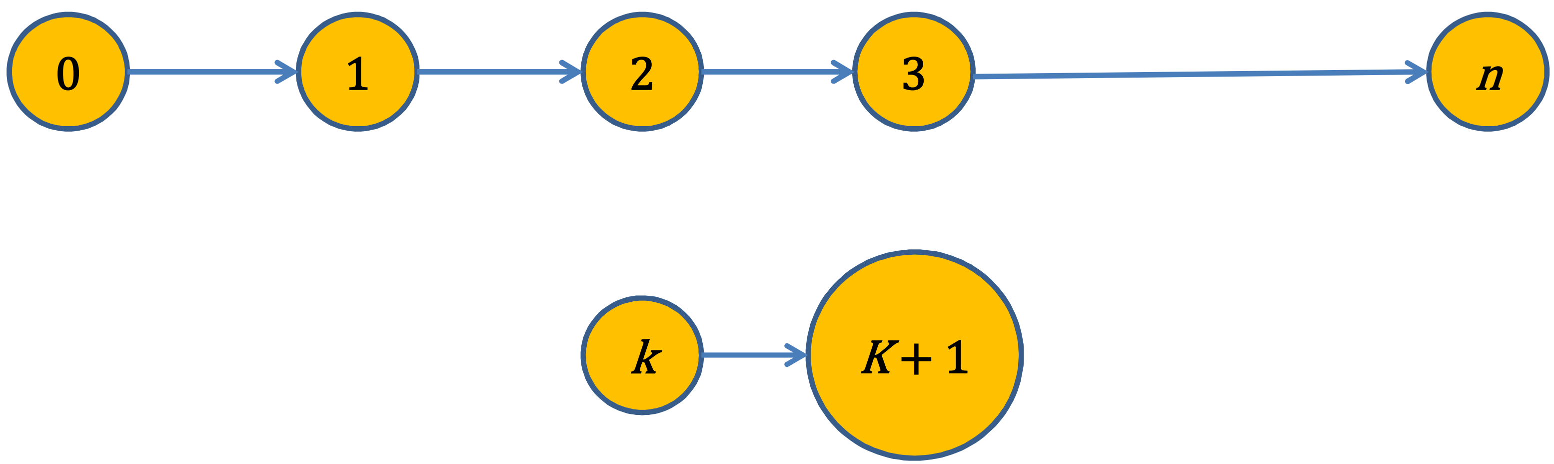
*Note:

- A●: represents even point
- A●: represents odd point

Connecting Points

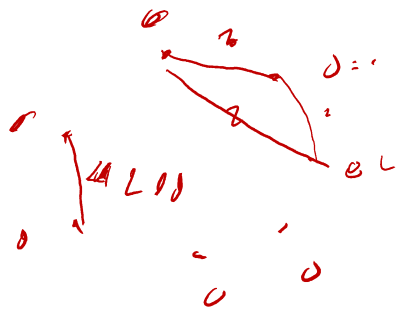
Proof Structure

Number of segments



Mathematical Induction

- Prove induction base: $n = 0$, no segments
- Prove that if theorem is true for $n = 0$, then it is true for $n = 1$
- Prove that if theorem is true for $n = 1$, then it is true for $n = 2$
- ...
- Prove induction step: from n to $n + 1$, adding one more line in the general case
- ...
- Profit!



Mathematical Induction

- Prove induction base: $n = 0$, no segments
- Prove induction step: from n to $n + 1$, adding one more line in the general case
- ...
- Profit!

Thank you.