Linear Algebra: Course Description

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Classes: 14:00-15:50 Mon, 14:00-14:50 Wed

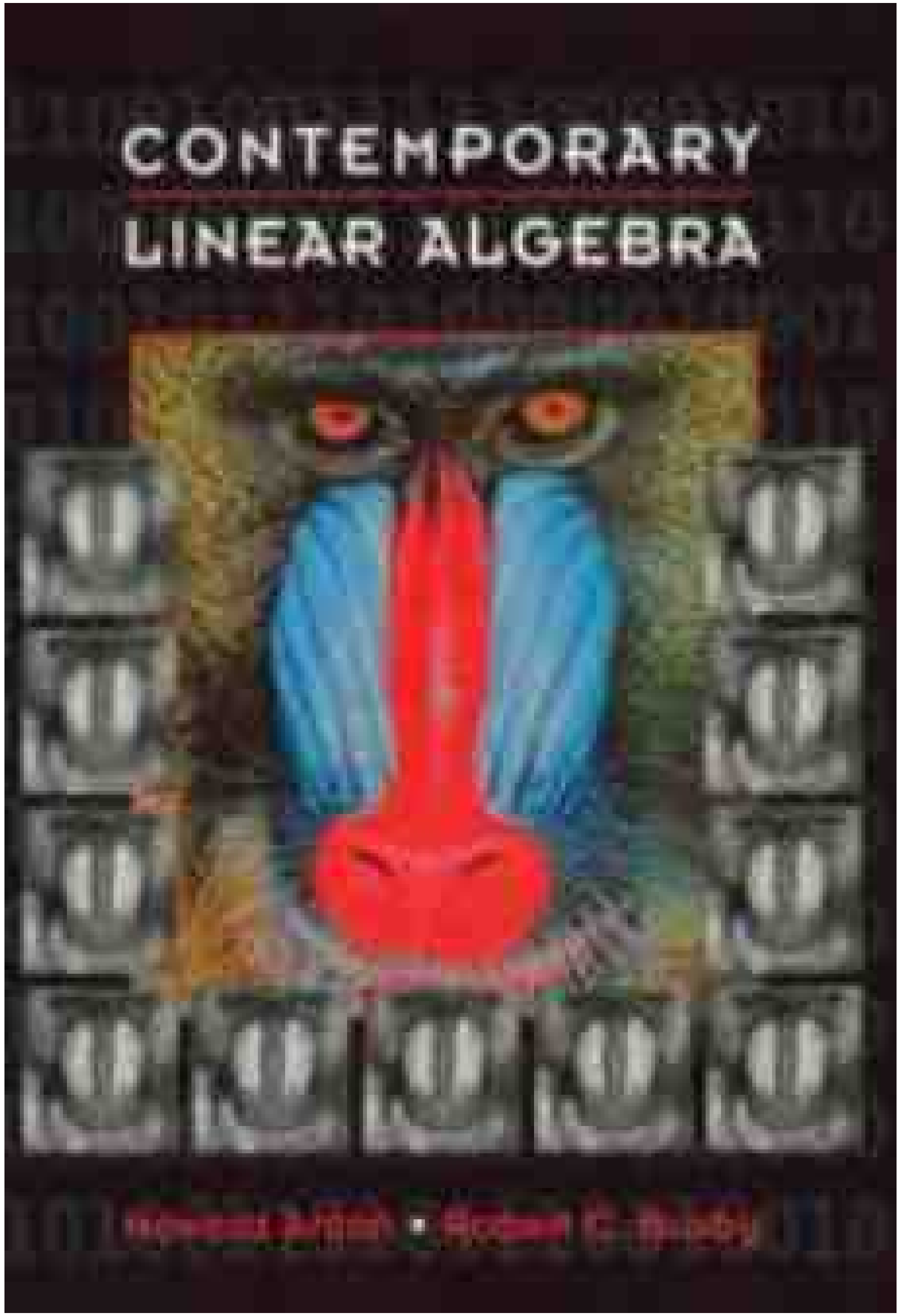
Classroom: NA

On-line class will be provided until notified because of COVID-19 virus.

Evaluation: Midterm(35%), Final(45%),
Homework(10%), Attendance(10%)

Attendance will be checked only by ieilms_old.jbnu.ac.kr.
Therefore every student is strongly required to download class material at the ieilms_old.jbnu.ac.kr.
If not, your class attendance may be regarded as absence.

Tests: Midterm(10.19), Final(12.07)



Contemporary
Linear Algebra,
Anton, Busby
Wiley

Covered Topics

Textbook: Contemporary Linear Algebra, Anton, Busby, Wiley,

Ch 1. Vectors

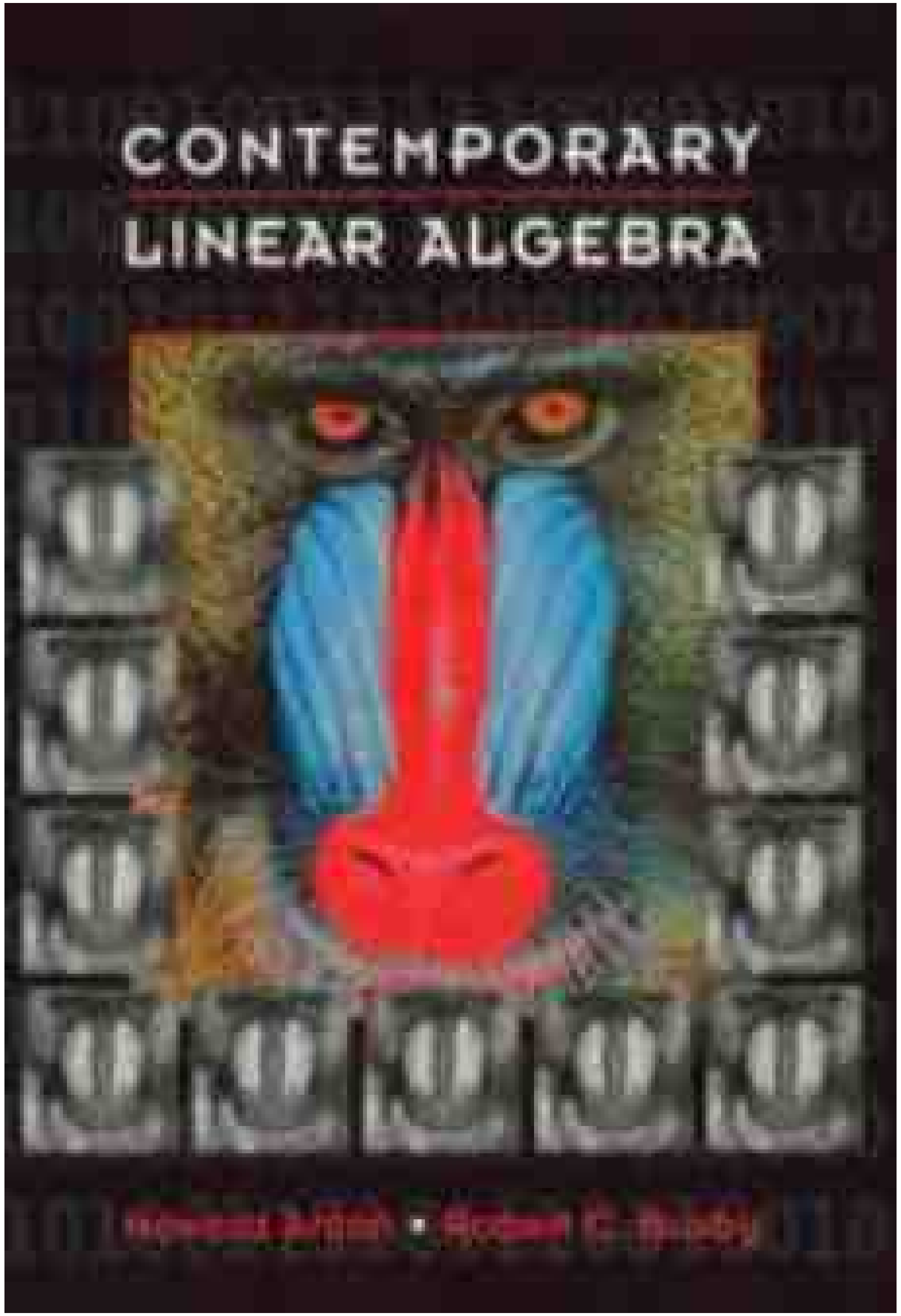
Ch 2. Systems of Linear Equations

Ch 3. Matrices and Matrix Algebra

Ch 4. Determinants

Ch 6. Linear Transformations

Ch 7. Dimension and Structure(7.1-7.9)



CHAPTER 1

Vectors

1.1 Vectors and Matrices in Engineering and Mathematics: n-Space

1.2 Dot Product and Orthogonality

1.3 Vector Equations of Lines and Planes

1.1 Vectors and Matrices in Engineering and Mathematics: n-Space

1.1 공학과 수학에서의 벡터와 행렬 및 n-공간

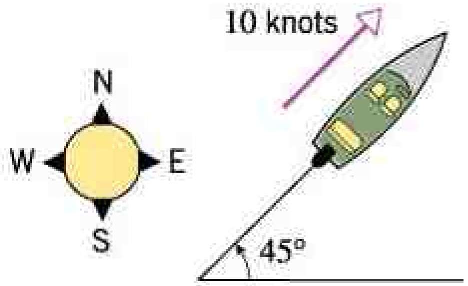
Scalars and Vectors

Scalar: a numerical value alone

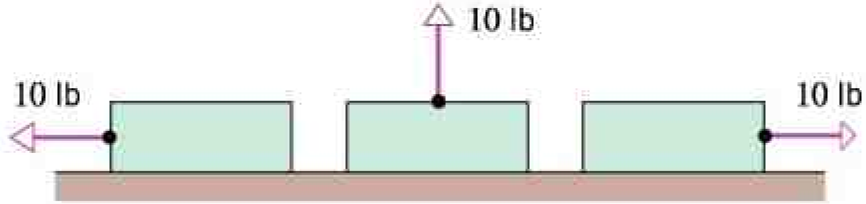
Examples: temperature, length, speed

Vector: both a numerical value and a direction

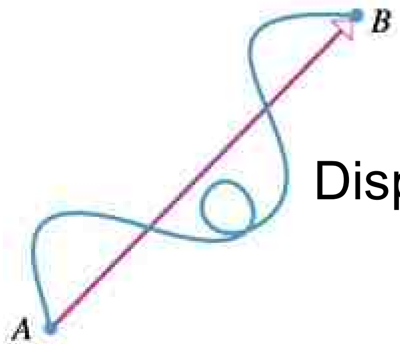
Examples: velocity, force, displacement



Velocity



Force

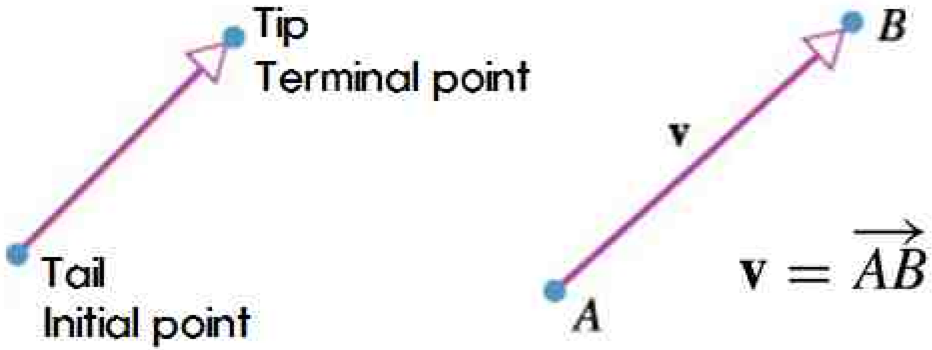


Displacement

Fig. 1.1.1 Vectors

1.1 Vectors and Matrices in Engineering and Mathematics: n-Space

Geometrical representation of vectors: by arrows



In textbook,

Vector: boldface types are used such as **a**, **k**, **v**, **w**, and **x**

Scalar: lower case italic types such as a, k, v, w, and x

Bound/Free Vector

Two types of vectors in applications: Bound vector and Free vector

Bound vector(제한벡터): Physical effect depends on the location of the initial point as well as the magnitude and direction

Free vector(자유벡터): depends on the magnitude and direction alone

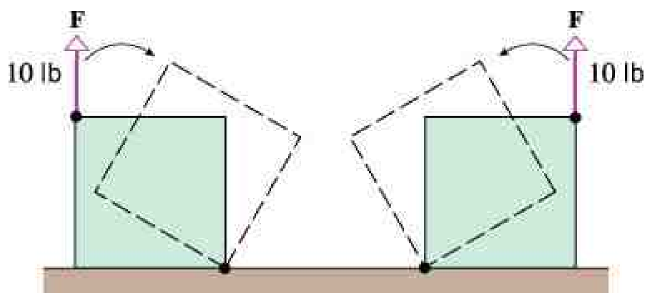


Fig. 1.1.4 Bound vector

In this text we will focus only on free vectors.

Equivalent Vectors

Free vectors **v** and **w** are ***equal*** (or ***equivalent***), denoted by **v=w**, if they are represented by parallel arrows with

- the same length(magnitude) and
- the same direction.

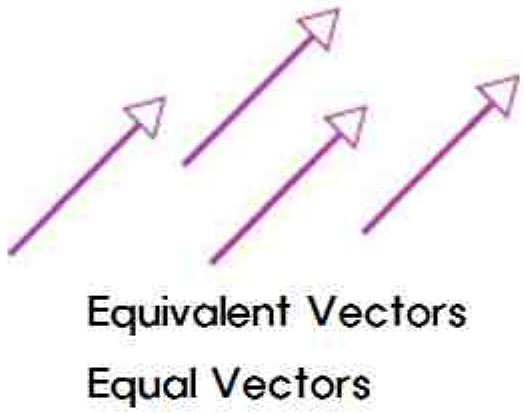


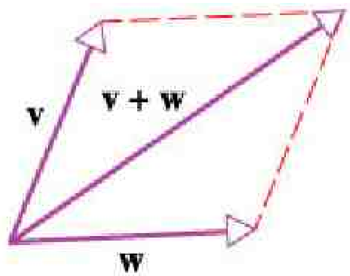
Fig. 1.1.5 Equivalent vectors

A vector with length zero is

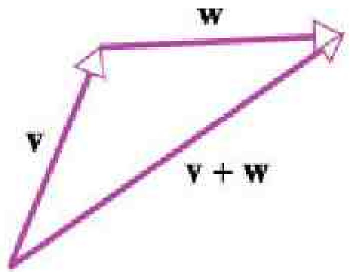
- Initial point=terminal point
- Any direction
- Called the zero vector
- Denoted by **0**

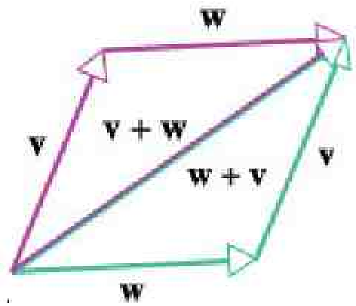
Vector Addition

Parallelogram Rule:



Triangle Rule:





v + w = w + v

Fig. 1.1.6 Vector addition

Vector Addition by Translation

v + **w**

- Translation of **v** by **w**, or
- Translation of **w** by **v**

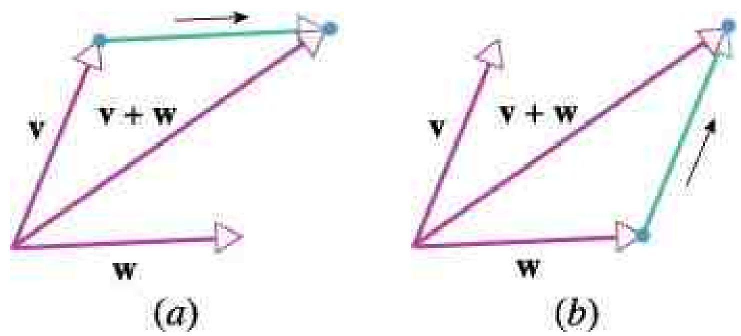
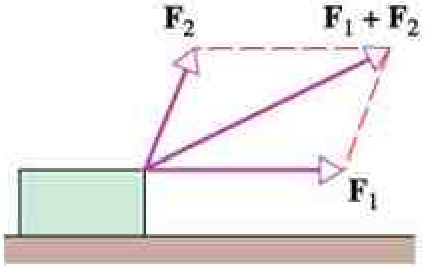


Fig. 1.1.7

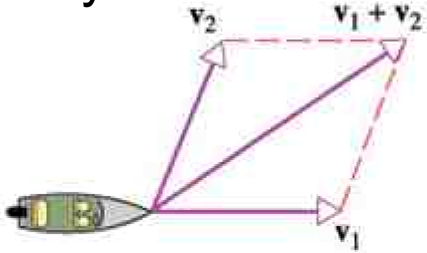
Example 1

(a) Two forces F1 and F2 are applied to a block.



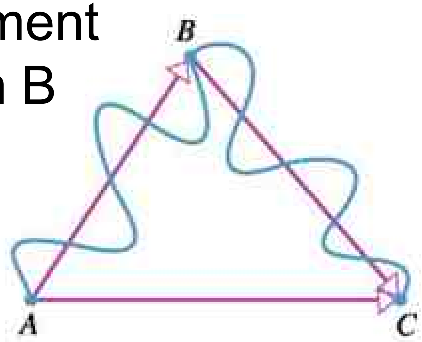
(a)

(b) A boat is influenced by v1 and v2 due to the boat engine and the wind, respectively.



(b)

(c) A particle undergoes a displacement from A to B followed by that from B to C.



(c)

Fig. 1.1.8

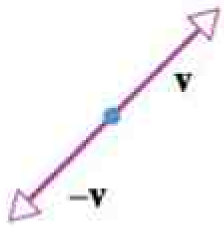
Vector Subtraction

In ordinary arithmetic, $a - b = a + (-b)$

analogy

Negative of a vector:

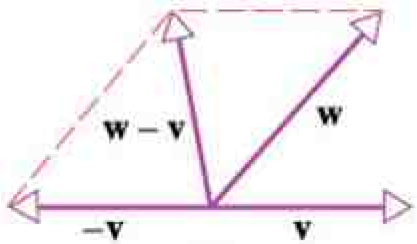
-**v** is a vector with the same length as **v** but oppositely directed



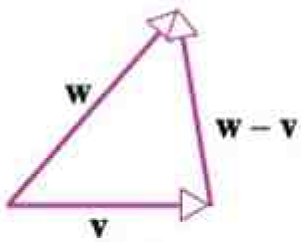
(a)

The difference of **v** from **w**,

- denoted by **w** - **v**,
- **w** - **v** = **w** + (-**v**)



(b)



(c)

Scalar Multiplication

$k\mathbf{v}$:

Length: $|k|$ times
Direction: the same as **v** for $k>0$
and opposite for $k<0$

(-1)**v** = -**v**

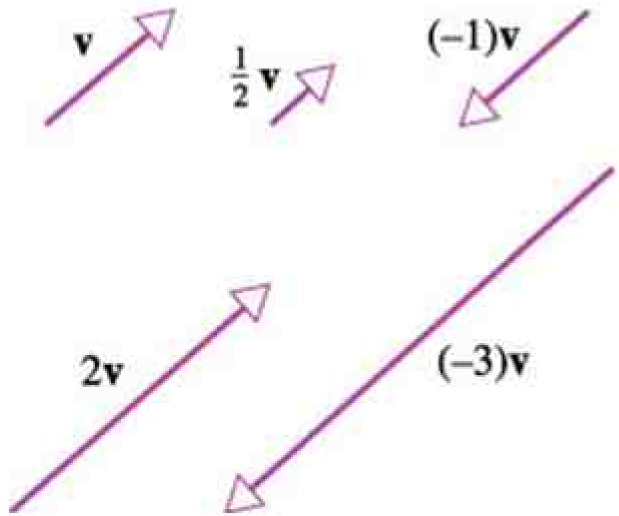
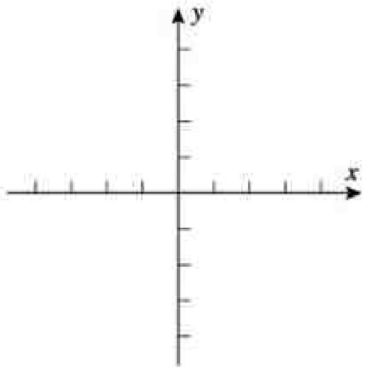
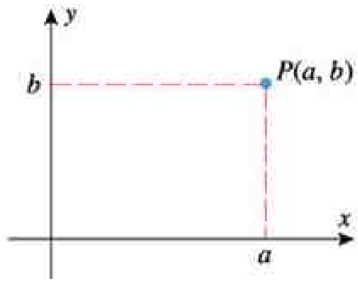


Fig. 1.1.10 Examples of Scalar Multiplication

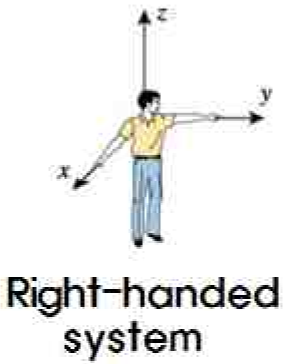
Vectors in Coordinate Systems

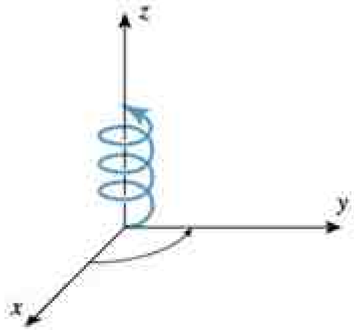
Rectangular coordinate system in 2-space

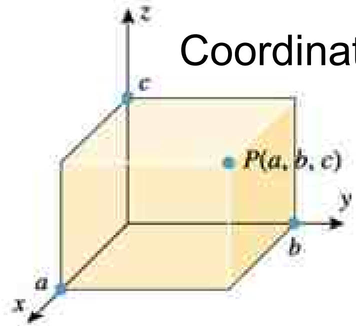




Rectangular coordinate system in 3-space





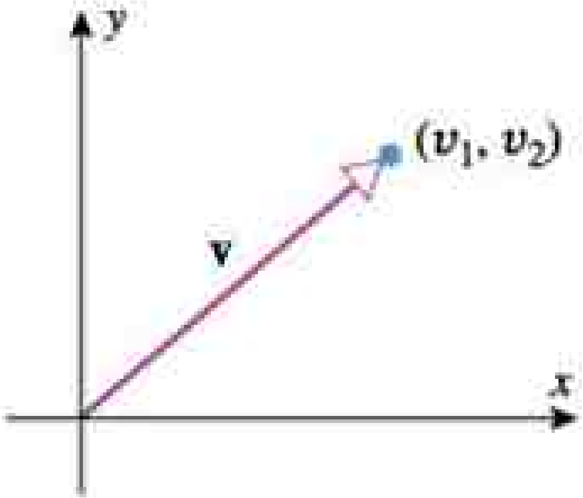


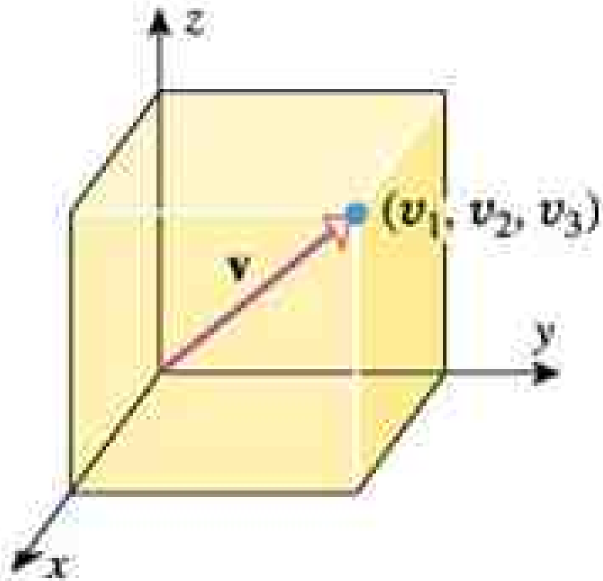
Coordinates of *P*

Interpretation of an Ordered Pair

The ordered pair, $(v_1,\ v_2)$ or $(v_1,\ v_2,\ v_3)$, may be interpreted as

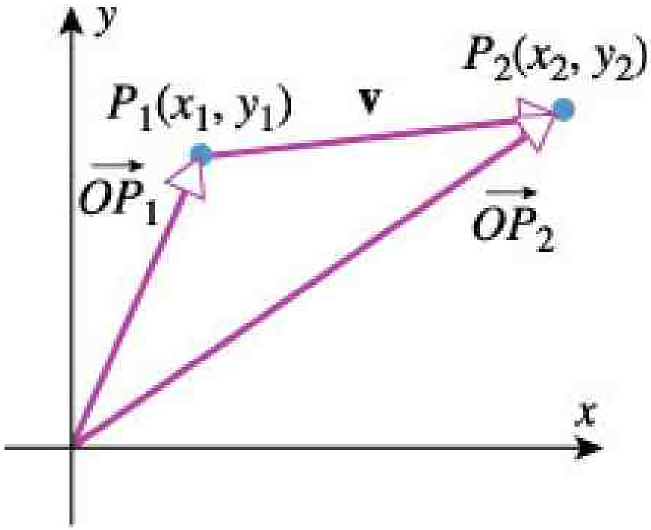
- a point, or
- a vector





Components of a Vector

Components of a vector whose initial point is not at the origin.



$$\mathbf{v} = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

$$\overrightarrow{OP_1} + \overrightarrow{P_1P_2} = \overrightarrow{OP_2}$$

Thus,

$$\mathbf{v} = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

Fig. 1.1.15

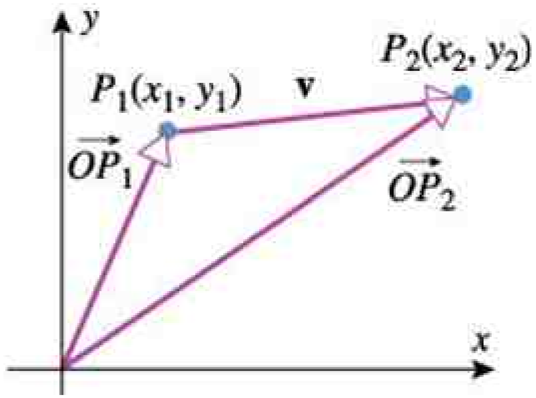
Theorem 1.1.1

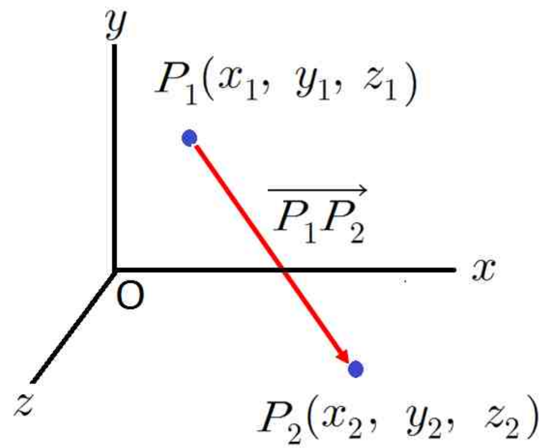
(a) The vector in 2-space that has initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1,\ y_2 - y_1)$$

(b) The vector in 3-space that has initial point $P_1(x_1, y_1, z_1)$ and terminal point $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$$





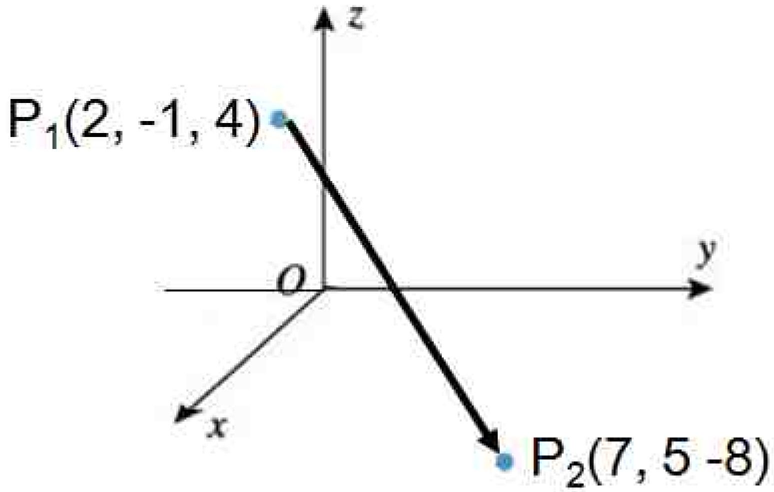
Example 2

Components of a vector whose initial point is not at the origin.

Find components of a vector whose initial point is $P_1(2, -1, 4)$ and its terminal point is $P_2(7, 5\ -8)$.

Sol.

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$
$$= (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$$
$$= (7-2,\ 5+1,\ -8-4)$$
$$= (5,\ 6,\ -12)$$



Vectors in R^n

Definition:

An ordered n-tuple: $(v_1, \; v_2, \; \cdots, \; v_n)$

n-space, R^n: The set of all ordered n-tuples

zero vector, origin of R^n: $\mathbf{0} = (0, \; 0, \cdots, \; 0)$

Visible space: R^1, R^2, R^3

Higher-dimensional spaces: R^4, R^5, …

Example 3

Some examples of vectors in higher-dimensional spaces

- Experimental data: (measured value1, mv_2, …, mv_n)
- Storage and warehousing:
  (# trucks in storage1, # in storage2, …, # in storage n)
- Electrical Circuits: $\mathbf{v}=(v_1, v_2, v_3, v_4)$
- Graphical Images:
  (x-coordinate, y-coordinate, hue, saturation, brightness)
- Economics: $(s_1, s_2, \ldots, s_n)$, s_n: the value for the sector n
- Mechanical systems: Assume six particles move at time t.
  $(x_1, x_2, \ldots, x_6, v_1, v_2, \ldots, v_6, t)$

Definitions for Vectors in R^n

Definition 1.1.3 Equivalent(Equal):

Suppose $\mathbf{v}=(v_1, v_2, \cdots, v_n)$ and $\mathbf{w}=(w_1, w_2, \cdots, w_n)$.
Then **v** and **w** are equivalent(or equal), indicated by **v=w**, if and only if

$$v_1 = w_1, \; v_2 = w_2, \; \cdots, \; v_n = w_n$$

Definition 1.1.4 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in R^n, and if k is any scalar, then we define

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n) \tag{10}$$

$$k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n) \tag{11}$$

$$-\mathbf{v} = (-v_1, -v_2, \ldots, -v_n) \tag{12}$$

$$\mathbf{w} - \mathbf{v} = \mathbf{w} + (-\mathbf{v}) = (w_1 - v_1, w_2 - v_2, \ldots, w_n - v_n) \tag{13}$$

Theorems

Theorem 1.1.5 *If* **u**, **v** *and* **w** *are vectors in R^n, and if k and l are scalars, then*:

(a) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(c) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(d) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

(e) $(k+l)\mathbf{u}=k\mathbf{u}+l\mathbf{u}$

(f) $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$

(g) $k(l\mathbf{u})=(kl)\mathbf{u}$

(h) $1\mathbf{u}=\mathbf{u}$

Theorem 1.1.6 *If* **v** *is a vector in R^n and k is a scalar, then*:

(a) $0\mathbf{v}=\mathbf{0}$

(b) $k\mathbf{0}=\mathbf{0}$

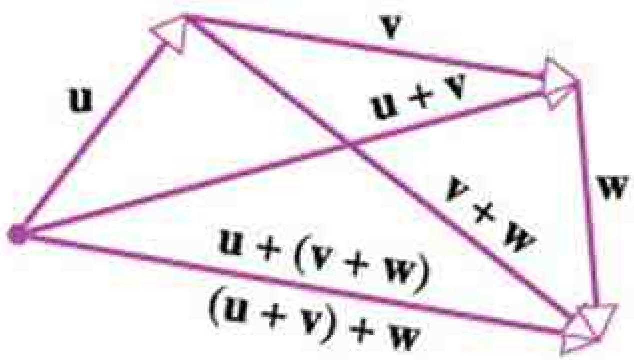
(c) $(-1)\mathbf{v}=-\mathbf{v}$

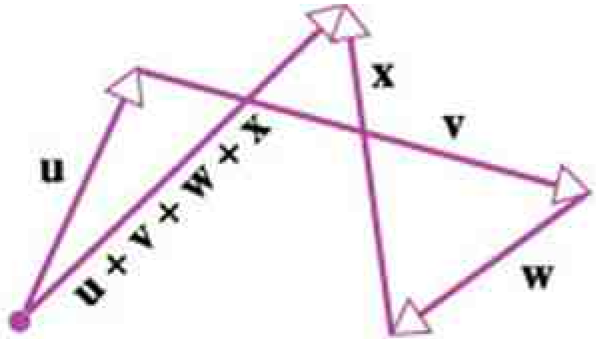
Sums of Three or More Vectors

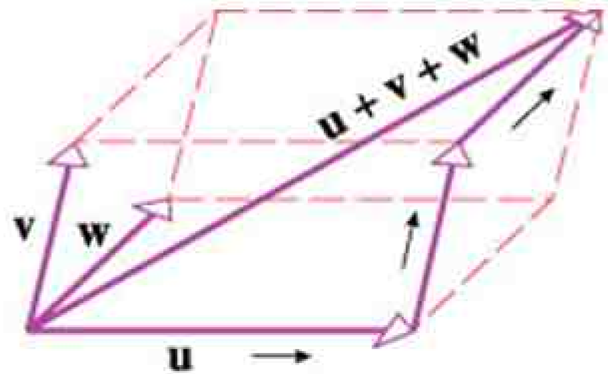
Associtive law for Vector addition

Theorem 1.1.5 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^n,

$$(b)\ \ (\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})=\mathbf{u}+\mathbf{v}+\mathbf{w}$$

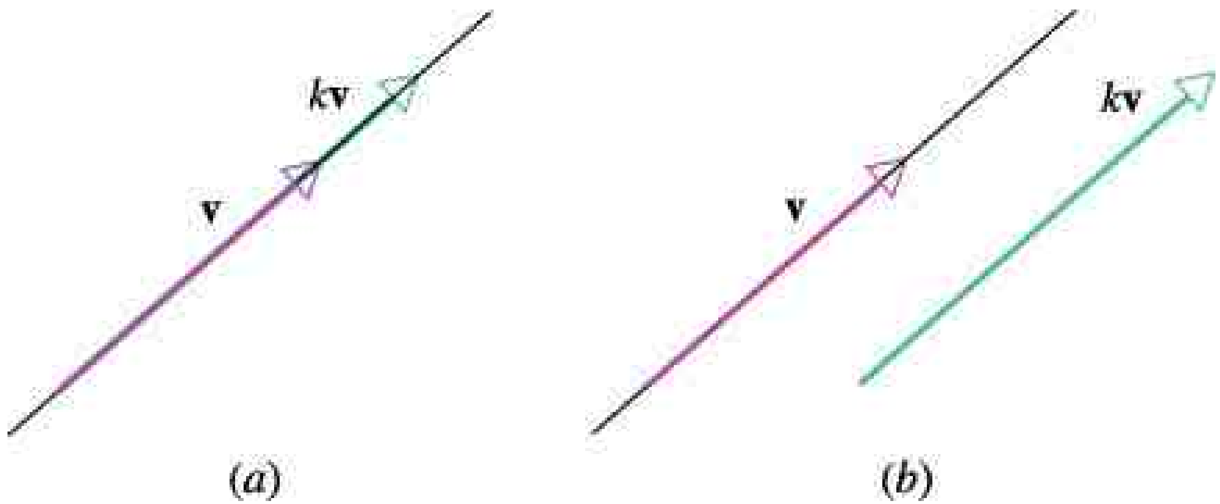






Parallel and Collinear Vectors(평행, 동일 직선상)

Definition 1.1.7 Two vectors in R^n are said to be ***parallel*** or, alternatively, ***collinear*** if at least one of the vectors is a scalar multiple of the other. If one of the vectors is a positive scalar multiple of the other, then the vectors are said to have the ***same direction***, and if one of them is a negative scalar multiple of the other, then the vectors are said to have ***opposite directions***.



Linear Combination

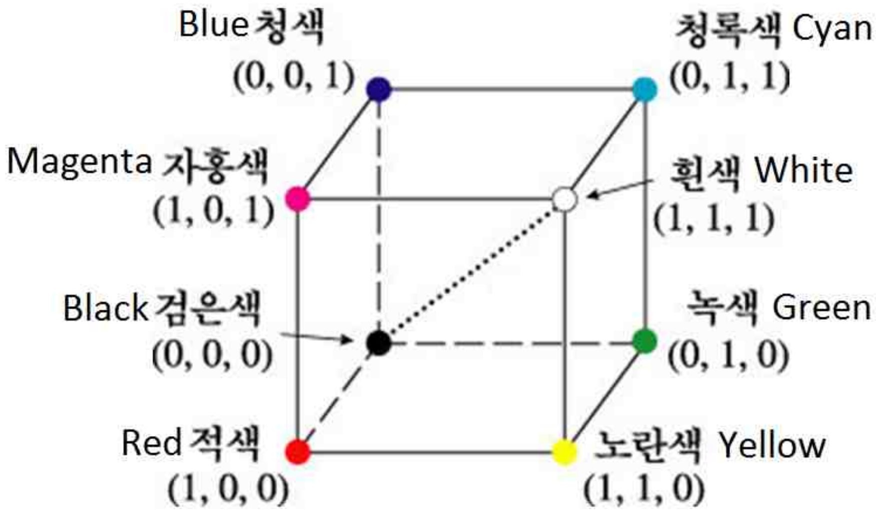
Definition 1.1.8 A vector $\mathbf{w}$ in R^n is said to be a ***linear combination*** of the vectors n $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in R^n if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \tag{14}$$

The scalars $c_1, c_2, \ldots, c_k$ are called the ***coefficients*** in the linear combination. In the case where $k = 1$, Formula (14) becomes $\mathbf{w} = c_1\mathbf{v}_1$, so to say that $\mathbf{w}$ is a linear combination of $\mathbf{v}_1$ is the same as saying that $\mathbf{w}$ is a scalar multiple of $\mathbf{v}_1$.

Application to Computer Color Models

$$\mathbf{r} = (1, 0, 0) \quad \text{(pure red)},$$
$$\mathbf{g} = (0, 1, 0) \quad \text{(pure green)},$$
$$\mathbf{b} = (0, 0, 1) \quad \text{(pure blue)}$$



$$\begin{aligned}\mathbf{c} &= c_1\mathbf{r} + c_2\mathbf{g} + c_3\mathbf{b} \\ &= c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) \\ &= (c_1, c_2, c_3)\end{aligned}$$

Alternate Notations for Vectors

Comma-delimited form:(괄호표기 형식)

$$\mathbf{v} = (v_1, v_2, \ldots, v_n)$$

Row-vector form: $\mathbf{v} = [v_1 \quad v_2 \quad \cdots \quad v_n]$

Column-vector form:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

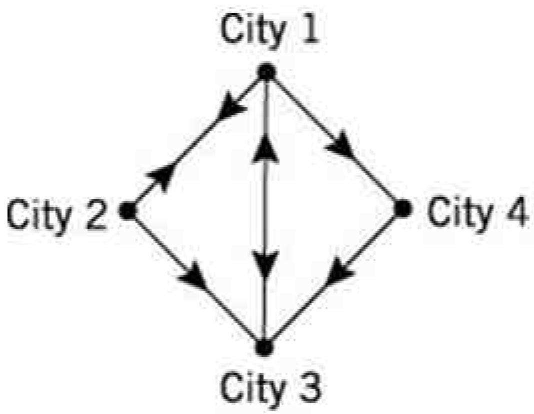
Matrices

	MON	TUE	WED	THU	FRI	SAT	SUN
Math	2	1	2	0	3	0	1
English	2	0	1	3	1	0	1
Chemistry	1	3	0	0	1	0	1
Physics	1	2	4	1	0	0	2

$$\begin{bmatrix} 2 & 1 & 2 & 0 & 3 & 0 & 1 \\ 2 & 0 & 1 & 3 & 1 & 0 & 1 \\ 1 & 3 & 0 & 0 & 1 & 0 & 1 \\ 1 & 2 & 4 & 1 & 0 & 0 & 2 \end{bmatrix}$$

Matrix
Entry
Size mxn,
Row/Column vector

Graph



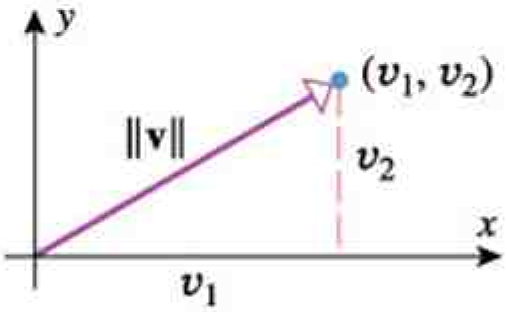
Directed Graph
유향그래프

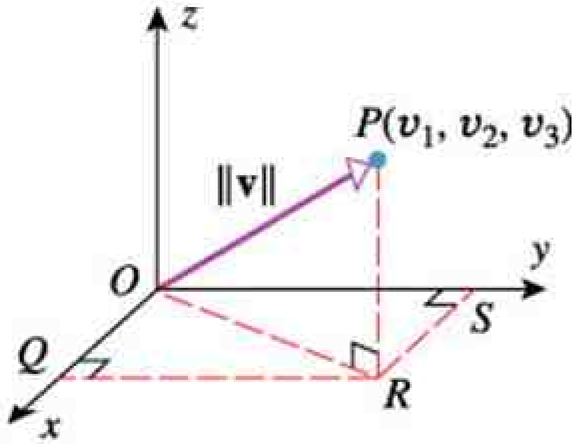
$$\begin{array}{cc} & \text{To} \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \text{From} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Adjacency Matrix
인접성 행렬

Norm of a Vector: By theorem of Pythagoras,

$$\| \mathbf{v} \| = \sqrt{v_1^2 + v_2^2} \qquad \| \mathbf{v} \| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$





1.2 Dot Product and Orthogonality

Norm of a Vector: By theorem of Pythagoras,

$$\| \mathbf{v} \| = \sqrt{v_1^2 + v_2^2} \qquad \| \mathbf{v} \| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Extend to R^n

Definition 1.2.1 If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the ***length*** of $\mathbf{v}$, also called the ***norm*** of $\mathbf{v}$ or the ***magnitude*** of $\mathbf{v}$, is denoted by $\|\mathbf{v}\|$ and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2} \tag{3}$$

Example 1

Find the norms of vectors v_1=(-3, 2, 1) and v_2=(2, -1, 3, -5).

$$\|\mathbf{v}_1\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$$

$$\|\mathbf{v}_2\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2} = \sqrt{2^2 + (-1)^2 + 3^2 + (-5)^2} = \sqrt{39}$$

Theorem 1.2.2 Properties

Theorem 1.2.2 Properties

If $\mathbf{v}$ *is a vector in* R^n*, and if* k *is any scalar, then;*

(a) $||\mathbf{v}|| \geq 0$

(b) $||\mathbf{v}|| = 0$ *if and only if* $\mathbf{v}=\mathbf{0}$

(c) $||k\mathbf{v}|| = |k|\ ||\mathbf{v}||$

Unit Vectors

A vector of length 1 is called a *unit vector*.

A unit vector **u** in the direction of **v** is given by

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$: Normalizing

Example 2 Find the unit vector parallel to **v**=(2, 2, -1).

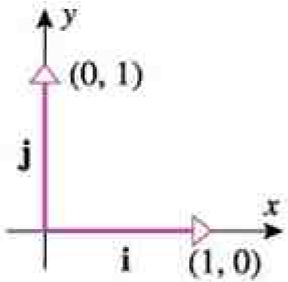
$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

$$\mathbf{u} = \frac{1}{3}(2, 2, -1) = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$$

Standard Unit Vectors

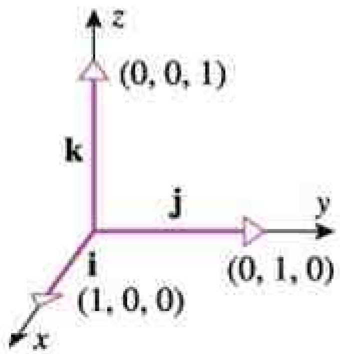
Standard unit vectors:
The unit vectors in the positive direction of the coordinate axes

In R^2, **i**=(1, 0) and **j**=(0, 1)



$$\begin{aligned}\mathbf{v} &= (v_1, v_2)\\ &= v_1(1,0) + v_2(0,1)\\ &= v_1\mathbf{i} + v_2\mathbf{j}\end{aligned}$$

In R^3, **i**=(1, 0, 0), **j**=(0, 1, 0) and **k**=(0, 0, 1).



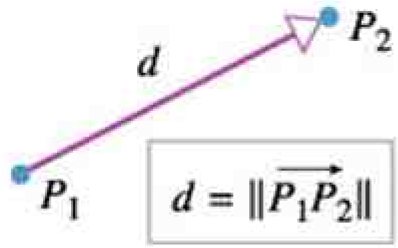
$$\begin{aligned}\mathbf{v} &= (v_1, v_2, v_3)\\ &= v_1(1,0,0) + v_2(0,1,0) + v_3(0,0,1)\\ &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}\end{aligned}$$

In R^n, $\mathbf{v} = (v_1, v_2, \cdots, v_n) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$

Distance between Points in R^n

$$d = \| \overrightarrow{P_1P_2} \| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \| \overrightarrow{P_1P_2} \| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



Higher dimensions

Definition 1.2.3 Distance

If **u**$=(u_1, u_2, \ldots, u_n)$ and **v** $=(v_1, v_2, \ldots, v_n)$ are points in R^n, then the ***distance*** between **u** and **v**, denoted by $d(\mathbf{u}, \mathbf{v})$, *is defined by*

$$d(\mathbf{u}, \mathbf{v}) = \| \mathbf{u} - \mathbf{v} \| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2} \qquad (11)$$

Distance between Points in R^n

Theorem 1.2.4 Properties

If **u** *and* **v** *are points in* R^n*, then;*

(a) $d(\mathbf{u}, \mathbf{v}) \geq 0$

(b) $d(\mathbf{u}, \mathbf{v}) = 0$ *if and only if* $\mathbf{u} = \mathbf{v}$

(c) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

Definition of Dot Products

Definition 1.2.5 Dot product, Euclidean inner product

If **u**$=(u_1, u_2, \ldots, u_n)$ and **v** $=(v_1, v_2, \ldots, v_n)$ *are vectors in* R^n*, then the dot product of* **u** *and* **v***, also called the* ***Euclidean inner product*** *of* **u** *and* **v***, is*

(a) denoted by **u·v***, and*

(b) defined by the formula

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1+u_2v_2+\ldots+u_nv_n \tag{12}$$

Application of a Dot Product

Ex. 3 ISBN(International Standard Book Number)

10 ISBN digits = 3 digits for the country + 3 for the publisher
+ 3 for the title + 1 for the check digit

First nine digits of ISBN: Let a=(1, 2, 3, 4, 5, 6, 7, 8, 9)

The Check digit is computed by the following procedure.

1. Form the dot product **a•b** where **b** is the first 9 ISBN digits.
2. Divide **a•b** by 11 to find the remainder c. The check digit is c when $c \neq 10$, and 0 when c=10.

0-471-15307-9 ; Calculus by Howard Anton

1. **a•b** =(1,2,3,4,5,6,7,8,9) •(0,4,7,1,1,5,3,0,7)=152
2. 152=13x11+9, Thus, c=9.

From Jan.1, 2007, ISBN digits are increased to 13 digits.

Application of a Dot Product

Ex. Korean National Id Number(주민등록번호)

13 digits = 6 digits for DOB + 1 for M/F+4 for area
+ 1 for serial order + 1 for the check digit

a=(2,3,4,5,6,7,8,9,2,3,4,5)

b=the row vector formed by the first 12 digits of an Id

C=the remainder of (**a**•**b**)/11.

The check digit c is

c=C when C<10

c=0 when C=10

Algebraic Properties of the Dot Product

Theorem 1.2.6 Properties of the Dot Product in R^n

If **u**, **v**, *and* **w** *are vectors in* R^n*, and if k is a scalar, then;*

(a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$ **[Symmetry property]**

(b) $\mathbf{u}\cdot(\mathbf{v} + \mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$ **[Distributive property]**

(c) $k(\mathbf{u}\cdot\mathbf{v}) = (k\mathbf{u})\cdot\mathbf{v}$ **[Homogeneity property]**

(d) $\mathbf{v}\cdot\mathbf{v} \geq 0$ *and* $\mathbf{v}\cdot\mathbf{v}=0$ *if and only if* $\mathbf{v}=\mathbf{0}$ **[Positivity property]**

Theorem 1.2.7 Properties of the Dot Product in R^n

If **u**, **v**, *and* **w** *are vectors in* R^n*, and if k is a scalar, then;*

(a) $\mathbf{0}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{0} = 0$

(b) $(\mathbf{u} + \mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

(c) $\mathbf{u}\cdot(\mathbf{v} - \mathbf{w}) = \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{w}$

(d) $(\mathbf{u} - \mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} - \mathbf{v}\cdot\mathbf{w}$

(e) $k(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(k\mathbf{v})$

Example 4 Calculating with Dot Products

Ex. 4 Calculating with Dot Products

$$
\begin{aligned}
(\mathbf{u}-2\mathbf{v})\cdot(3\mathbf{u}+4\mathbf{v}) &= \mathbf{u}\cdot(3\mathbf{u}+4\mathbf{v})-2\mathbf{v}\cdot(3\mathbf{u}+4\mathbf{v}) \\
&= 3(\mathbf{u}\cdot\mathbf{u})+4(\mathbf{u}\cdot\mathbf{v})-6(\mathbf{v}\cdot\mathbf{u})-8(\mathbf{v}\cdot\mathbf{v}) \\
&= 3\,\|\,\mathbf{u}\,\|^2 - 2(\mathbf{u}\cdot\mathbf{v})-8\,\|\,\mathbf{v}\,\|^2
\end{aligned}
$$

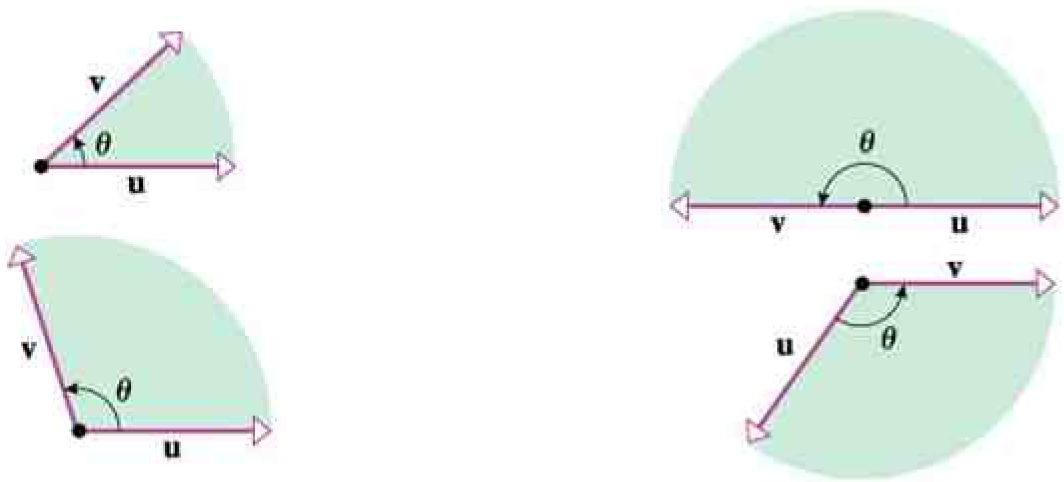
Angle between Vectors in R^2 and R^3

Definition: Angle between **u** and **v**

Smallest nonnegative angle θ between **u** and **v**

Algebraically,

- $0 \leqq \theta \leqq \pi$
- Counterclockwise rotation



Theorem 1.2.8 Angle between Two Vectors

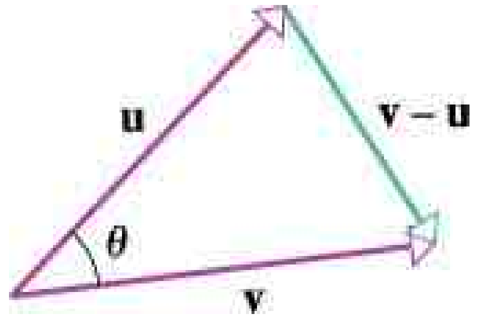
Theorem 1.2.8 *If* **u** *and* **v** *are nonzero vectors in* R^2 *or* R^3*, and if* θ *is the angle between these vectors, then*

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \quad \textit{or equivalently,} \quad \theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right) \qquad (15\text{–}16)$$

Proof

The law of cosines shows

$$\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \quad \textbf{(17)}$$



$$\|\mathbf{v}-\mathbf{u}\|^2 = (\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{u})$$

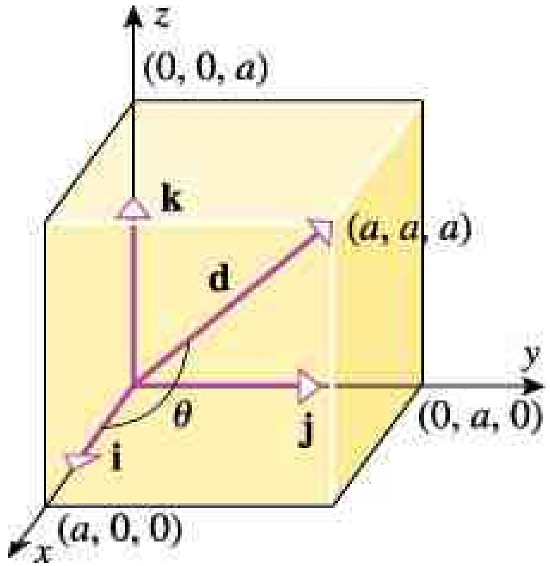
$$= (\mathbf{v}-\mathbf{u})\cdot\mathbf{v} - (\mathbf{v}-\mathbf{u})\cdot\mathbf{u} = \mathbf{v}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{u}$$

$$= \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2$$

$$\therefore\ \mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \quad \Longrightarrow \quad \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

Example 5 Find the Angle

Ex. 5 Find the angle between a diagonal of a cube and one of its edges.



Sol.

$$\mathbf{d} = (a, a, a) \qquad \|\mathbf{d}\| = \sqrt{3a^2} = \sqrt{3}\,a$$

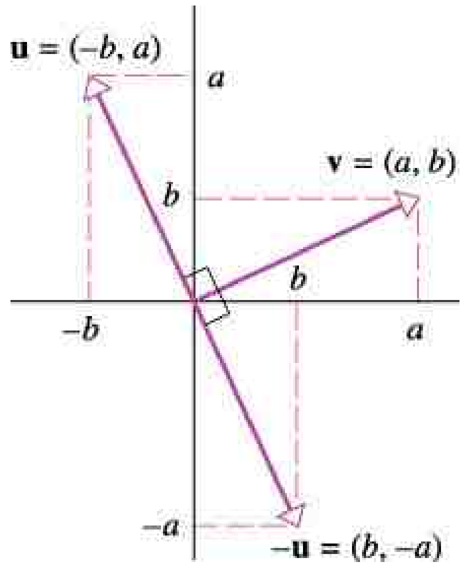
$$\mathbf{v}_1 = (a, 0, 0) \qquad \|\mathbf{v}_1\| = a$$

$$\cos\theta = \frac{\mathbf{d}\cdot\mathbf{v}_1}{\|\mathbf{d}\|\|\mathbf{v}_1\|} = \frac{a^2}{a\sqrt{3a^2}} = \frac{1}{\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.7^\circ$$

Example 6 Find the Nonzero Vector

Ex. 6 Find the nonzero vector in R^2 that is perpendicular to the nonzero vector $\mathbf{v}=(a, b)$.



Sol.

Let $\mathbf{u}=(u_1, u_2)$ be the vector with $\mathbf{u}\cdot\mathbf{v} = 0$.

$\mathbf{u}\cdot\mathbf{v} = (u_1, u_2) \cdot (a, b) = u_1a+u_2b=0.$

If $a\neq0$,

$u_1a+u_2b=0 \Rightarrow u_1=-u_2b/a$

Thus, $\mathbf{u}=(u_1, u_2)=(-u_2b/a, u_2)=(u_2/a)(-b, a)=k(-b, a)$

If $a=0$,

$u_1a+u_2b=0 \Rightarrow u_2b=0, \quad u_2=0$

u_1=any nonzero number

Thus, $\mathbf{u}=(u_1, u_2)=(u_1, 0)$

Orthogonality(직교성)

Definition 1.2.9 *Orthonormal and Orthogonal set*

(a) Two vectors **u** and **v** in R^n are said to be ***orthogonal*** if **u·v**=0.

(b) A nonempty set of vectors in R^n is said to be an ***orthogonal set*** if each pair of distinct vectors in the set is orthogonal.

Ex. 7 Show that the following vectors form an orthogonal set in R^4.

$\mathbf{v}_1=(1, 2, 2, 4)$, $\mathbf{v}_2=(-2, 1, -4, 2)$, $\mathbf{v}_3=(-4, 2, 2, -1)$

Sol.

$\mathbf{v}_1 \cdot \mathbf{v}_2 = (1, 2, 2, 4)\cdot(-2, 1, -4, 2) = -2+2-8+8=0$

$\mathbf{v}_1 \cdot \mathbf{v}_3 = (1, 2, 2, 4)\cdot(-4, 2, 2, -1) = -4+4+4-4=0$

$\mathbf{v}_2 \cdot \mathbf{v}_3 = (-2, 1, -4, 2)\cdot(-4, 2, 2, -1) = 8+2-8-2=0$

Vector Orthogonal to R^n

Definition **Vector Orthogonal to R^n**

A vector $\mathbf{v}$ is said to be orthogonal to the set $\mathbf{S}$ if $\mathbf{v}$ is orthogonal to every vector in $\mathbf{S}$.

Ex. 8 Show that the zero vector is orthogonal to R^n.

Sol.

$$\mathbf{0} \cdot \mathbf{v} = (0, 0, \ldots, 0) \cdot (v_1, v_2, \ldots, v_n) = 0$$

Thus, $\mathbf{0}$ is orthogonal to R^n.

Orthonormal Sets(정규직교 집합)

Definition 1.2.10 *Orthonormal and Orthogonal set*

(a) Two vectors **u** and **v** in R^n are said to be ***orthonormal*** if they are orthogonal and have length 1.

(b) A set of vectors is said to be an ***orthogonal set*** if every vector in the set has length 1 and each pair of distinct vectors is orthogonal.

Ex. 9 Show that the standard unit vectors form an orthonormal set in R^n.

Sol.

For $i \neq j$, $\mathbf{e}_i \cdot \mathbf{e}_j = (0, 0, \ldots, 1, \ldots, 0, 0) \cdot (0, 0, \ldots, 1, \ldots, 0, 0) = 0$

i^{th} element $\qquad$ j^{th} element

$$\|\mathbf{e}_i\|^2 = \|(0, 0, \ldots, 1, \ldots, 0, 0)\|^2 = 0^2 + 0^2 + \ldots + 1^2 + 0^2 + 0^2 = 1$$

Orthonormal Sets(정규직교 집합)

Ex. 10 Show that the following vectors form an orthonormal set in R^4.

$\mathbf{q}_1$=(1/5, 2/5, 2/5, 4/5), q_2=(-2/5, 1/5, -4/5, 2/5),
$\mathbf{q}_3$=(-4/5, 2/5, 2/5, -1/5)

Sol.

$$\|\mathbf{q}_1\|^2 = \|(1/5, 2/5, 2/5, 4/5)\|^2 = (1/5)^2 + (2/5)^2 + (2/5)^2 + (4/5)^2 = 1$$
$$\|\mathbf{q}_2\|^2 = \|(-2/5, 2/5, -4/5, 2/5)\|^2 = (-2/5)^2 + (1/5)^2 + (4/5)^2 + (-2/5)^2 = 1$$
$$\|\mathbf{q}_3\|^2 = \|(-4/5, 2/5, 2/5, -1/5)\|^2 = (-4/5)^2 + (2/5)^2 + (2/5)^2 + (-1/5)^2 = 1$$

$$\mathbf{q}_1 \cdot \mathbf{q}_2 = (1/5, 2/5, 2/5, 4/5)\cdot(-2/5, 1/5, -4/5, 2/5) = 0$$
$$\mathbf{q}_1 \cdot \mathbf{q}_3 = (1/5, 2/5, 2/5, 4/5)\cdot(-4/5, 2/5, 2/5, -1/5) = 0$$
$$\mathbf{q}_2 \cdot \mathbf{q}_3 = (-2/5, 1/5, -4/5, 2/5)\cdot(-4/5, 2/5, 2/5, -1/5) = 0$$

$$\Downarrow$$

Therefore, the set is an orthonormal set in R^4.

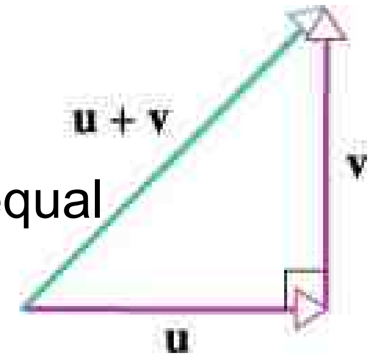
Euclidean Geometry in R^n

Euclidean Norm, Euclidean Distance in R^n **:**

$$\|\mathbf{v}\| = \|(v_1, v_2, \ldots, v_n)\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2} \tag{3}$$

$$d(\mathbf{u}, \mathbf{v}) = \| \mathbf{u} - \mathbf{v} \| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2} \tag{11}$$

For n=2 or 3



1. In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides(Theorem of Pythagoras).
2. The sum of lengths of two sides of a triangle is at least as large as the length of the third side.
3. The shortest distance between two points is along a straight line.

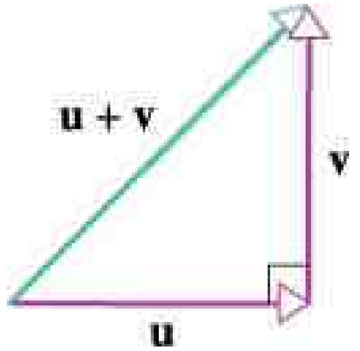
Theorem 1.2.11 *Theorem of Pythagoras*

Extension of the three properties to R^n:

Theorem 1.2.11 *Theorem of Pythagoras*

If **u** *and* **v** *are orthogonal vectors in* R^n, *then*

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 \qquad (18)$$



Proof

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$$
$$= ||\mathbf{u}||^2 + 2(\mathbf{u}\cdot\mathbf{v}) + ||\mathbf{v}||^2$$
$$= ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

Angle between Nonzero Vectors in R^n

In R^2 and R^3, the angle between nonzero vectors is given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right) \qquad (19)$$

Extendable to R^n?

Yes!
Because, the following inequality holds for R^n.

$$-1 \le \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \le 1 \qquad (20)$$

Cauchy-Schwartz Inequality

Theorem 1.2.12 *Cauchy-Schwartz Inequality in R^n*

Theorem 1.2.12 *Cauchy-Schwartz Inequality in R^n*

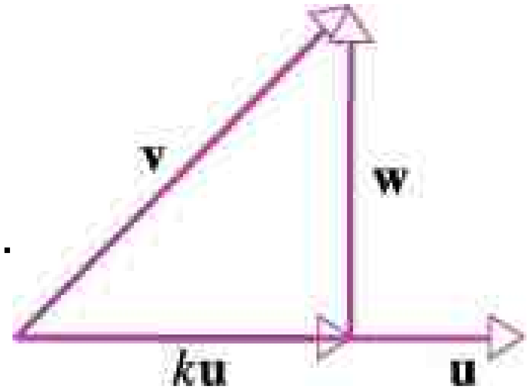
If **u** *and* **v** *are vectors in R^n, then*

$$(\mathbf{u}\cdot\mathbf{v})^2 \le \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 \qquad (21)$$

or equivalently(by taking square roots)

$$|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\| \qquad (22)$$

Proof



If $\|\mathbf{u}\|=0$ or $\|\mathbf{v}\|=0$, then the inequality holds.

If $\|\mathbf{u}\|\neq 0$ and $\|\mathbf{v}\| \neq 0$, let $\mathbf{v} = k\mathbf{u} + \mathbf{w}$.

The appropriate scalar k can be computed by

(1) Setting $\mathbf{w} = \mathbf{v} - k\mathbf{u}$ and

(2) Using the orthogonality condition $\mathbf{u}\cdot\mathbf{w}=0$.

Theorem 1.2.12 *Cauchy-Schwartz Inequality in R^n - cont*

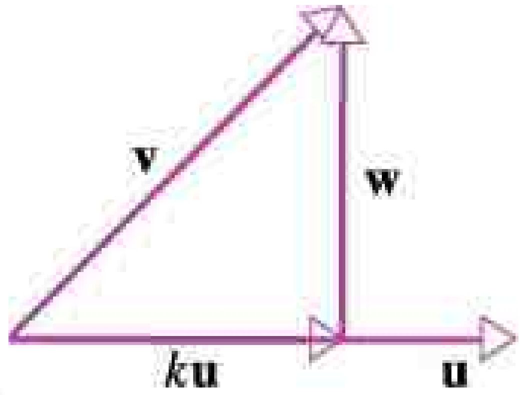
The appropriate scalar k *can be computed by*

(1) Setting $\mathbf{w} = \mathbf{v} - k\mathbf{u}$ *and*

(2) Using the orthogonality condition **u·w=**0.

$$\mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} - k\mathbf{u}) = \mathbf{u} \cdot \mathbf{v} - k(\mathbf{u} \cdot \mathbf{u}) = \mathbf{0} \quad \Rightarrow \quad k = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}$$

By the theorem of Pythagoras,



$$\|\mathbf{v}\|^2 = \|k\mathbf{u}\|^2 + \|\mathbf{w}\|^2 = k^2\|\mathbf{u}\|^2 + \|\mathbf{w}\|^2$$

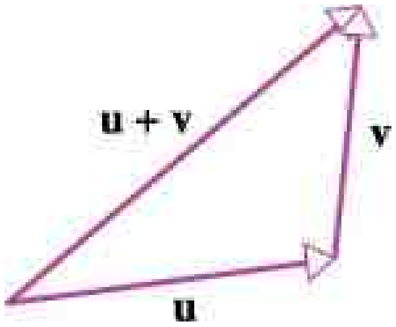
$$= \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^4}\|\mathbf{u}\|^2 + \|\mathbf{w}\|^2 = \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2} + \|\mathbf{w}\|^2$$

$$\|\mathbf{u}\|^2\|\mathbf{v}\|^2 = (\mathbf{u} \cdot \mathbf{v})^2 + \|\mathbf{u}\|^2\|\mathbf{w}\|^2 \geq (\mathbf{u} \cdot \mathbf{v})^2$$

Theorem 1.2.13 *Triangle Inequality for Vectors*

If **u** , **v** and **w** are points in R^n, then

$$||\mathbf{u}+\mathbf{v}|| \le ||\mathbf{u}|| + ||\mathbf{v}|| \tag{26}$$



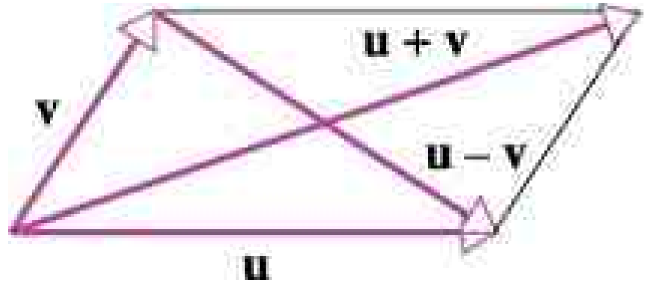
Proof

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\ &= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \\ &\le \|\mathbf{u}\|^2 + 2|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2 \\ &\le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 \\ &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2\end{aligned}$$

Theorem 1.2.14 *Parallelogram Equation for Vectors*

If **u** and **v** are points in R^n, then

$$||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2(||\mathbf{u}||^2 + ||\mathbf{v}||^2) \qquad (27)$$



Proof

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$$
$$= (\mathbf{u}\cdot\mathbf{u} + 2\,\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{u} - 2\,\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v})$$
$$= 2(\mathbf{u}\cdot\mathbf{u}) + 2(\mathbf{v}\cdot\mathbf{v})$$
$$= 2\left(||\,\mathbf{u}\,||^2 + ||\,\mathbf{v}\,||^2\right)$$

Theorem 1.2.15 *Triangle Inequality for Distances*

Theorem 1.2.15 *Triangle Inequality for Distances*

If **u** , **v** and **w** are points in R^n, then

$$d(\mathbf{u}, \mathbf{v}) \le d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v}) \tag{28}$$

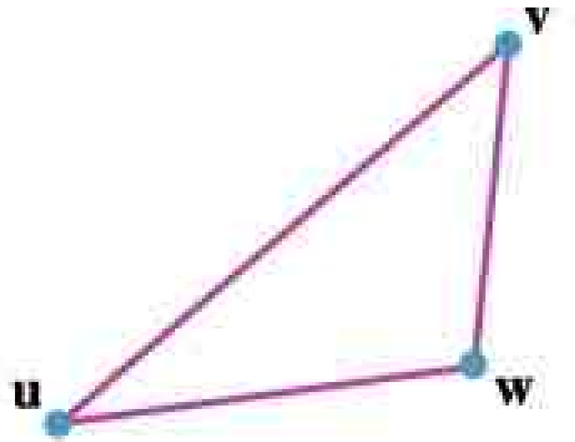
Proof

$$
\begin{aligned}
d(\mathbf{u},\mathbf{v}) &= \| \mathbf{u} - \mathbf{v} \| \\
&= \| (\mathbf{u} - \mathbf{w}) + (\mathbf{w} - \mathbf{v}) \|
\end{aligned}
$$

Theorem 1.2.13

$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$ (26)

$$
\begin{aligned}
&\le \| \mathbf{u} - \mathbf{w} \| + \| \mathbf{w} - \mathbf{v} \| \\
&= d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})
\end{aligned}
$$



$$\boxed{d(\mathbf{u}, \mathbf{v}) \le d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})}$$

1.3 Vector Equations of Lines and Planes

Review : In R^2, the general equation of a line has the form

$$Ax + By = C \quad \text{(A and B not both 0)} \tag{1}$$

The line through the origin has the form

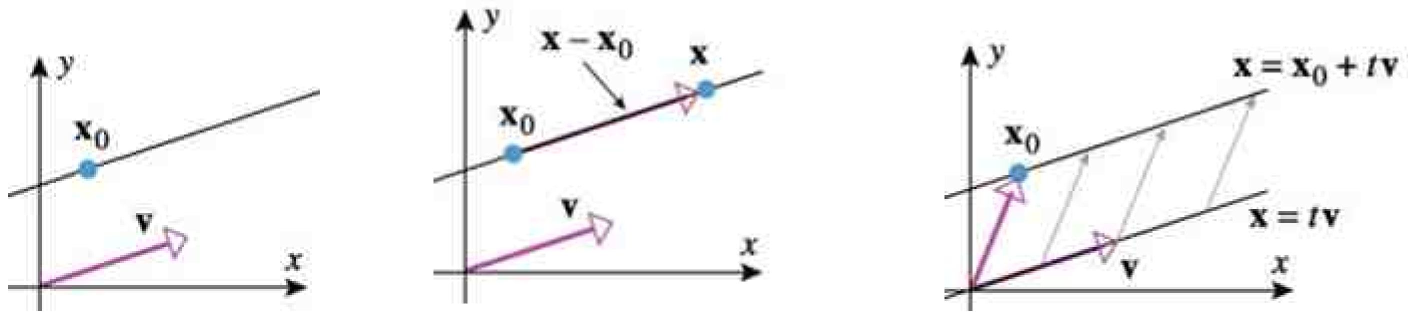
$$Ax + By = 0 \quad \text{(A and B not both 0)} \tag{2}$$

Vector and Parametric Equations of Lines

The line through $\mathbf{x}_0$ and parallel to $\mathbf{v}$ is $\mathbf{x} - \mathbf{x}_0 = t\mathbf{v}$

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \tag{3}$$

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \quad (-\infty < t < +\infty) \quad t\text{ : parameter} \tag{4}$$



1.3 Vector Equations of Lines and Planes-cont

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \quad (-\infty < t < +\infty) \quad t\text{ : parameter} \qquad (4)$$

The line through the origin

$$\mathbf{x} = t\mathbf{v} \quad (-\infty < t < +\infty) \qquad (5)$$

In component form

$$(x, y) = (x_0, y_0) + t(a, b) \quad (-\infty < t < +\infty)$$

Parametric equations

$$x = x_0 + at, \quad y = y_0 + bt \quad (-\infty < t < +\infty) \qquad (6)$$

1.3 Vector Equations of Lines and Planes-cont

Similarly, in R_3,

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \quad (-\infty < t < +\infty) \quad t: \text{parameter} \tag{4}$$

$$\mathbf{x}_0 = (x_0, y_0, z_0)$$
$$\mathbf{v} = (a, b, c)$$

$$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c) \quad (-\infty < t < +\infty)$$

In parametric equations,

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned} \quad (-\infty < t < +\infty) \tag{7}$$

Example 1

(a) Find a vector equation and parametric equation of the line in R^2 that passes through the origin and is parallel to the vector **v**=(-2, 3).
(b) Find a vector equation and parametric equation of the line in R^3 that passes through the point P_0(1, 2, -3) and is parallel to the vector **v**=(4, -5, 1).
(c) Use the vector equation in part (b) to find two points on the line that are different from P_0.

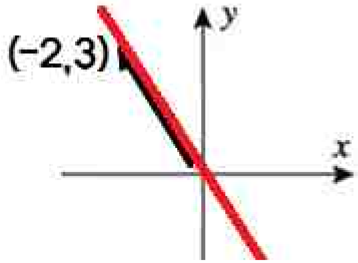
Sol.

(a) the line passing through the origin and parallel to the vector **v**=(-2, 3)

Vector equation: $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ **(4)**

$$= (0, 0) + t(-2, 3)$$

$$= (-2t, 3t)$$



parametric equation: $x = -2t, \quad y = 3t$

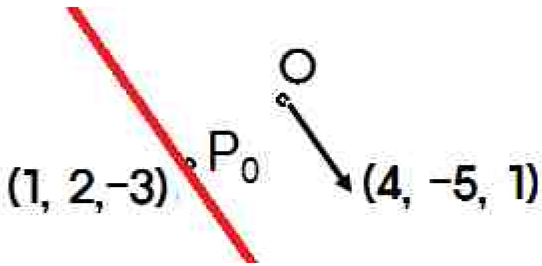
Example 1-cont

(b) the line passing through P_0(1, 2, -3) and parallel to **v**=(4, -5, 1)

Vector equation: $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ **(4)**

$$= (1, 2, -3) + t(4, -5, 1)$$

$$= (1+4t,\ 2-5t,\ -3+t)$$



Parametric eq. : $x = 1 + 4t, \quad y = 2 - 5t, \quad z = -3 + t$

(c) find two points on the line in (b)

Let t=any non-zero value

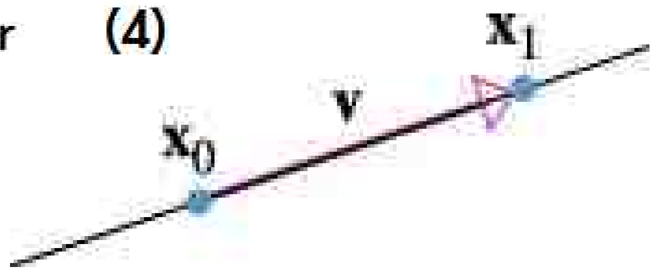
$$t = 1 \quad \rightarrow \quad P(5, -3, -2)$$

$$t = -1 \quad \rightarrow \quad Q(-3, 7, -4)$$

Lines Through Two Points

The line passing through $\mathbf{x}_0$ and $\mathbf{x}_1$:

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \quad (-\infty < t < +\infty) \quad t : \text{parameter} \qquad (4)$$



$$\mathbf{v} = \mathbf{x}_1 - \mathbf{x}_0$$

$$\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0) \quad (-\infty < t < +\infty) \qquad (9)$$

$$\mathbf{x} = (1-t)\mathbf{x}_0 + t\mathbf{x}_1 \quad (-\infty < t < +\infty) \qquad (10)$$

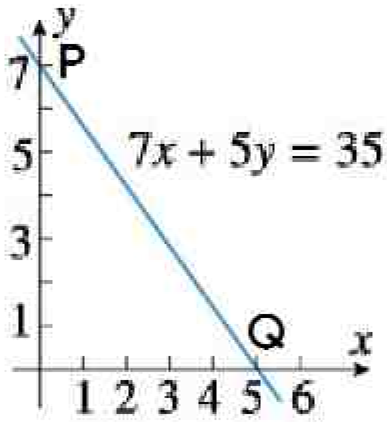
The line segment from $\mathbf{x}_0$ to $\mathbf{x}_1$:

$$\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0) \quad (0 \le t \le 1)$$

$$\mathbf{x} = (1-t)\mathbf{x}_0 + t\mathbf{x}_1 \quad (0 \le t \le 1)$$

Example 2

Find a vector and parametric equations of the line in R^2 that passes through the points P(0, 7) and Q(5, 0).



Sol.

$$\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0) \qquad (9)$$

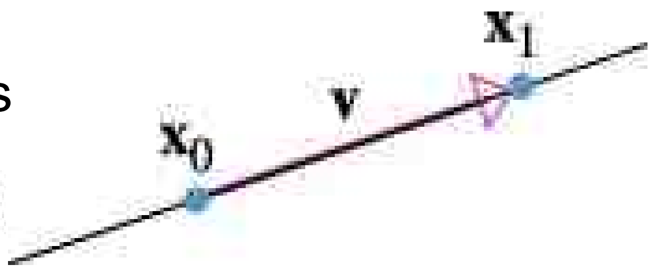
$= P + t(Q-P)$

$= (0, 7) + t\,[(5, 0)-(0, 7)]$

$= (5t, 7-7t)$

Remark

The same line with different representations



$$\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0) \quad (-\infty < t < +\infty)$$

$$\mathbf{x} = \mathbf{x}_1 + t(\mathbf{x}_0 - \mathbf{x}_1) \quad (-\infty < t < +\infty)$$

Point-Normal Equations of Planes

n : normal to the plane

The plane perpendicular to n and passing through $\mathbf{x}_0$:

$$\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)=0 \qquad (15)$$

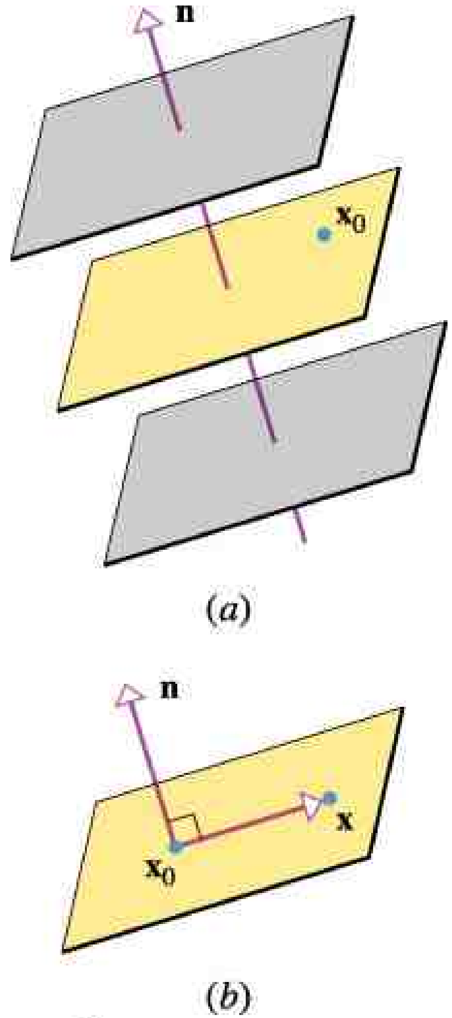
$$\mathbf{x}=(x,y,z),\quad \mathbf{x}_0=(x_0,y_0,z_0)$$
$$\mathbf{n}=(A,B,C)$$

Point-normal equation of the plane through $\mathbf{x}_0$ with normal **n** :

$$(A,B,C)\cdot(x-x_0,y-y_0,z-z_0)=0 \qquad (16)$$

Rearranged to the following ***general equation*** of a plane:

$$Ax+By+Cz=D \quad (A,\ B,\ C \text{ not all zero}) \qquad (17)$$



Point-Normal Equations of Planes - cont

$$\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0)=0 \tag{15}$$

If $\mathbf{x}_0 = (0, 0, 0)$

$$\mathbf{n}\cdot\mathbf{x}=0 \tag{18}$$

$$Ax+By+Cz=0 \;\; (A, B, C \text{ not all zero}) \tag{19}$$

Example 3. Find a point-normal equation and a general equation of the plane that passes through (3, -1, 7) and has normal **n**=(4, 2, -5).

Sol.

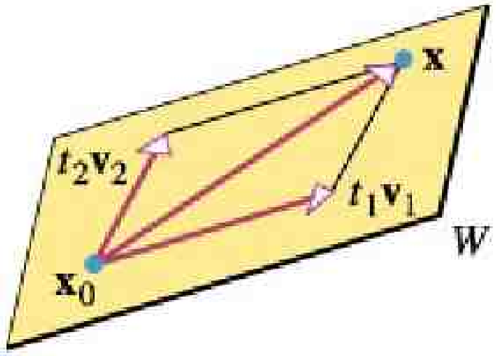
$$(A,B,C)\cdot(x-x_0, y-y_0, z-z_0)=0 \tag{16}$$

$$4(x-3)+2(y+1)-5(z-7)=0$$

$$4x+2y-5z=-25$$

Vector Equations of Planes

The ***vector equation of the plane*** passing through $\mathbf{x}_0$ and parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$



$$\mathbf{x} - \mathbf{x}_0 = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

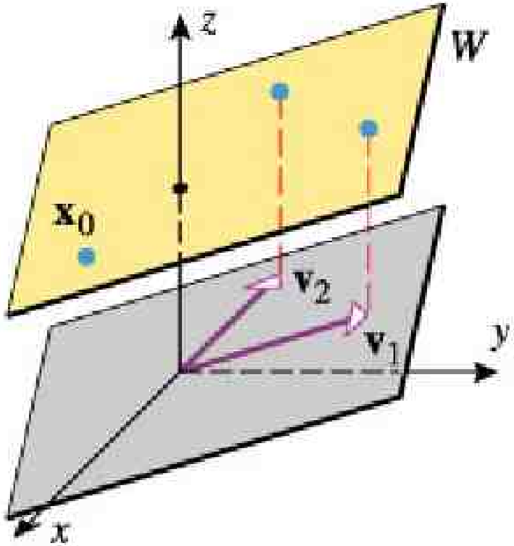
$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \quad (-\infty < t_1, t_2 < +\infty)$$

t_1, t_2 : parameters

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \quad (-\infty < t_1, t_2 < +\infty) \quad (20)$$

If $\mathbf{x}_0 = (0, 0, 0)$

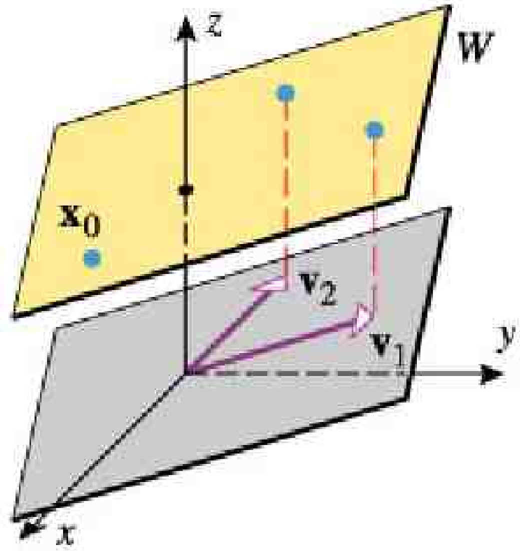
$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \quad (-\infty < t_1, t_2 < +\infty) \quad (21)$$



The plane in (20) is the translation by $\mathbf{x}_0$ of the plane in (21)

Parametric Equations of Planes

The ***parametric equation of the plane*** passing through $\mathbf{x}_0$ and parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$



$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \quad (-\infty < t_1, t_2 < +\infty) \quad (20)$$

$$\mathbf{x} = (x,\ y,\ z),\ \mathbf{x}_0 = (x_0, y_0, z_0)$$

$$\mathbf{v}_1 = (a_1, b_1, c_1),\ \mathbf{v}_2 = (a_2, b_2, c_2)$$

$$\mathbf{x} = (x,\ y,\ z) = (x_0,\ y_0,\ z_0) + t_1(a_1,\ b_1,\ c_1) + t_2(a_2,\ b_2,\ c_2) \quad (21)$$

Parametric equations:

$$x = x_0 + a_1t_1 + a_2t_2,\quad y = y_0 + b_1t_1 + b_2t_2,\quad z = z_0 + c_1t_1 + c_2t_2 \quad (22)$$

$$(-\infty < t_1 < +\infty,\ -\infty < t_2 < +\infty)$$

Example 4 **Vector and Parametric Equations of Planes**

(a) Find vector and parametric equations of the plane that passes through the origin of R^3 and is parallel to the vectors $\mathbf{v}_1=(1, -2, 3)$ and $\mathbf{v}_2=(4, 0, 5)$.

(b) Find three points in the plane obtained in part (a).

Sol.

(a) Vector and parametric equations of the plane through the origin and parallel to the vectors $\mathbf{v}_1=(1, -2, 3)$ and $\mathbf{v}_2=(4, 0, 5)$

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \quad (-\infty < t_1, t_2 < +\infty) \qquad (20)$$

$$= (0, 0, 0) + t_1(1, -2, 3) + t_2(4, 0, 5)$$

$$= (t_1 + 4t_2,\ -2t_1,\ 3t_1 + 5t_2)$$

$$x = t_1 + 4t_2, \quad y = -2t_1, \quad z = 3t_1 + 5t_2$$

(b) Three points in the plane in part (a).

Find three points by assigning suitable values to t_1 and t_2.

Example 5 **A Plane Passing Three Points**

The plane passing three points $\mathbf{x}_0$, $\mathbf{x}_1$, and $\mathbf{x}_2$.

$\mathbf{v}_1 = \mathbf{x}_1 - \mathbf{x}_0$ and $\mathbf{v}_2 = \mathbf{x}_2 - \mathbf{x}_0$.

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \quad (-\infty < t_1, t_2 < +\infty) \tag{20}$$

$\mathbf{v}_1 = \mathbf{x}_1 - \mathbf{x}_0$ and $\mathbf{v}_2 = \mathbf{x}_2 - \mathbf{x}_0$.

$$= \mathbf{x}_0 + t_1(\mathbf{x}_1 - \mathbf{x}_0) + t_2(\mathbf{x}_2 - \mathbf{x}_0) \tag{24}$$

Example 5 A Plane passing three points

Find vector and parametric equations of the plane that passes through the points **P**(2, -4, 5), Q(-1, 4, -3) and R(1, 1-, -7).

$$\mathbf{x} = \mathbf{x}_0 + t_1(\mathbf{x}_1 - \mathbf{x}_0) + t_2(\mathbf{x}_2 - \mathbf{x}_0) \tag{24}$$

$$= (2, -4, 5) + t_1[(-1, 4, -3) - (2, -4, 5)] + t_2[(1, 10, -7) - (2, -4, 5)]$$

$$= (2 - 3t_1 - t_2,\ -4 + 8t_1 + 14t_2,\ 5 - 8t_1 - 12t_2)$$

Example 6 **A Vector Equation from Parametric Equations**

Find a vector equation of the plane whose parametric equations are

$$x = 4 + 5_1 - t_2, \quad y = 2 - t_1 + 8t_2, \quad z = t_1 + t_2$$

Sol.

$$\begin{aligned}(x, y, z) &= (4 + 5t_1 - t_2,\ 2 - t_1 + 8t_2,\ t_1 + t_2) \\ &= (4, 2, 0) + (5t_1,\ -t_1,\ t_1) + (-t_2,\ 8t_2,\ t_2) \\ &= (4, 2, 0) + t_1(5, -1, 1) + t_2(-1, 8, 1)\end{aligned}$$

The plane passing (4, 2, 0) and parallel to (5, -1, 1) and (-1, 8, 1).

Example 7 **Vector and Parametric Equations in R^4**

Find parametric equations of the plane $x - y + 2z = 5$.

Sol.

Let $y = t_1,\ z = t_2,$ then

$$x = 5 + y - 2z = 5 + t_1 - 2t_2$$

$$x = 5 + t_1 - 2t_2$$
$$y = t_1$$
$$z = t_2$$

Let $x = t_1,\ \mathrm{y} = t_2,$ then

$$z = (5 - x + y)/2 = (5 - t_1 + t_2)/2$$

$$x = t_1$$
$$\mathrm{y} = t_2$$
$$z = (5 - t_1 + t_2)/2$$

Different equations of the plane for the same plane .

Lines and Planes in R^n

Definition 1.3.1 *Line through* $\mathbf{x_0}$ *that is parallel to* $\mathbf{v}$

(a) If $\mathbf{x_0}$ is a vector in R^n, and if $\mathbf{v}$ is a nonzero vector in R^n, then we define the ***line through*** $\mathbf{x_0}$ ***that is parallel to*** $\mathbf{v}$ to be the set of all vectors $\mathbf{x}$ in R^n that are expressed in the form

$$\mathbf{x} = \mathbf{x_0} + t\mathbf{v} \quad (-\infty < t < \infty) \tag{27}$$

(b) If $\mathbf{x_0}$ is a vector in R^n, and if $\mathbf{v}_1$ and $\mathbf{v}_2$ are a nonzero vectors in R^n that are not scalar multiples of another, then we define the ***plane through*** $\mathbf{x_0}$ ***that is parallel to*** $\mathbf{v}_1$ **and** $\mathbf{v}_2$ to be the set of all vectors $\mathbf{x}$ in R^n that are expressed in the form

$$\mathbf{x} = \mathbf{x_0} + t_1\mathbf{v}_1 + t_2\mathbf{v}_1 \quad (-\infty < t_1 < \infty,\ -\infty < t_2 < \infty) \tag{28}$$

Example 8 **Parametric Equations from a General Equation**

(a) Find vector and parametric equations of the line through the origin of R^4 and is parallel to the vector **v**=(5, -3, 6, 1).

(b) Find vector and parametric equations of the plane in R^4 that passes through the point $\mathbf{x}_0$=(2, -1, 0, 3) and is parallel to the vectors $\mathbf{v}_1$=(1, 5, 2, -4) and $\mathbf{v}_2$=(0, 7, -8, 6).

Sol.

(a) $$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v} \quad (-\infty < t < +\infty) \quad t: \text{parameter} \tag{27}$$

$$= (0, 0, 0, 0) + t(5, -3, 6, 1)$$

$$= t(5, -3, 6, 1)$$

(b) $$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \quad (-\infty < t_1, t_2 < +\infty) \tag{28}$$

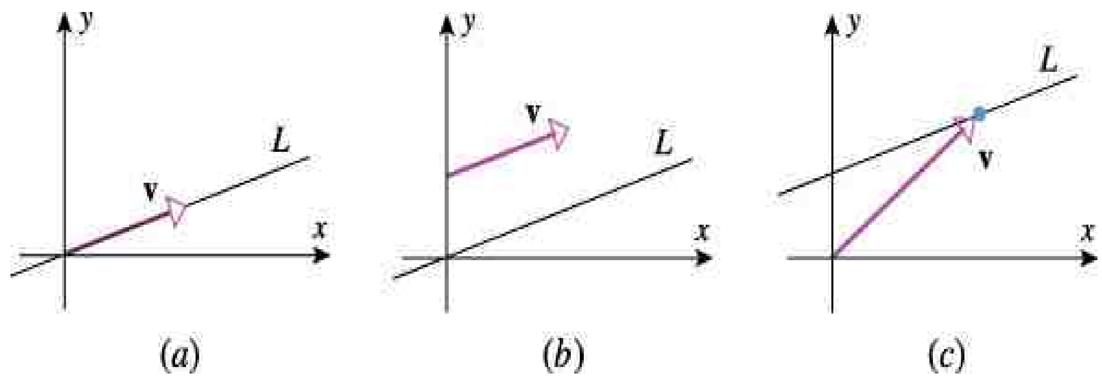
$$= (2, -1, 0, 3) + t_1(1, 5, 2, -4) + t_2(0, 7, -8, 6)$$

$$= (2+t_1,\ -1+5t_1+7t_2,\ 2t_1-8t_2,\ 3-4t_1+6t_2)$$

Comments on Terminology

A vector **v** ***lies on a line L*** in R^2 or R^3 if the terminal point of the vector lies on the line when the vector is positioned with its initial point at the origin.

In all three cases, v lies on the line L.



A vector **v** ***lies in a plane w*** in R^3 if the terminal point of the vector lies in the plane when the initial point of the vector is at the origin.