14.3 Cauchy's Integral Formula

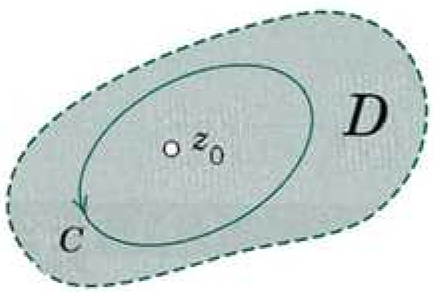
14.3 Cauchy's Integral Formula (Cauchy의 적분공식)

THEOREM 1 Cauchy's Integral Formula

Let f(z) be analytic in a simply connected domain *D*. Then for any point z_0 in *D* and any simple closed path *C* in *D* that encloses z_0,

$$(1) \quad \oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0)$$

$$(1^*) \quad f(z_o) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$



14.3 Cauchy's Integral Formula

PROOF

$$f(z) = f(z_0) + [f(z) - f(z_0)]$$

$$(2) \quad \oint_{C_1} \frac{f(z)}{z - z_0} dz = \oint_{C_1} \frac{f(z_0) + [f(z) - f(z_0)]}{z - z_0} dz$$

$$= f(z_0) \oint_{C_1} \frac{1}{z - z_0} dz + \oint_{C} \frac{f(z) - f(z_0)}{z - z_0} dz$$

$$= f(z_0) \oint_{C_1} 2\pi i + \oint_{C} \frac{f(z) - f(z_0)}{z - z_0} dz$$

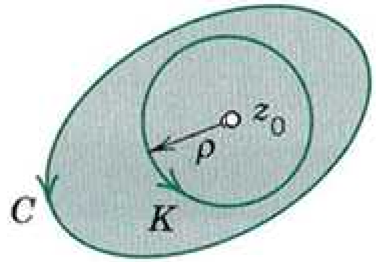
The second term is zero.(See next page.)
Thus,

$$(1) \quad \oint_{C} \frac{f(z)}{z - z_0} dz = 2\pi i\, f(z_0)$$

14.3 Cauchy's Integral Formula

Show that

$$\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0$$



$$\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz = \oint_K \frac{f(z)-f(z_0)}{z-z_0}\,dz < ML$$

For any $\epsilon > 0$, there exists δ such that

if $|z-z_0| < \delta$, then $|f(z)-f(z_0)| < \epsilon$

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \frac{\epsilon}{\rho}$$

$$\left|\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \le ML < \frac{\epsilon}{\rho}\cdot 2\pi\rho = 2\pi\epsilon$$

14.3 Cauchy's Integral Formula

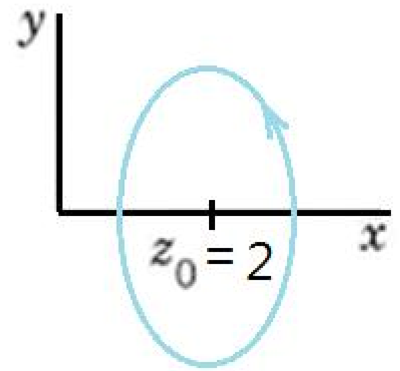
Since ϵ can be chosen arbitrarily small, the integral is zero.

$$\oint_C \frac{f(z) - f(z_0)}{z - z_0} dz = 0$$

14.3 Cauchy's Integral Formula

EX 1. Cauchy's Integral Formula

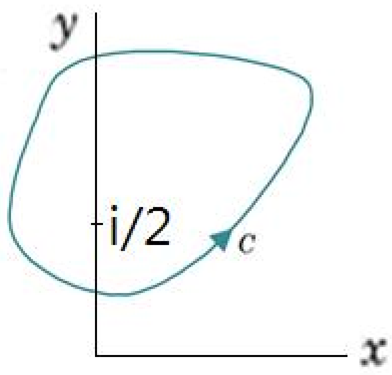
$$\oint_C \frac{e^z}{z-2}dz = 2\pi i e^z|_{z=2} = 2\pi i e^2 = 46.4268i$$



EX 2. Cauchy's Integral Formula

$$\oint_C \frac{z^3-6}{2z-i}dz = \frac{1}{2}\int_C \frac{z^3-6}{z-i/2}dz$$

$$= \frac{1}{2}2\pi i[z^3-6]_{z=i/2} = \frac{\pi}{8} - 6\pi i$$

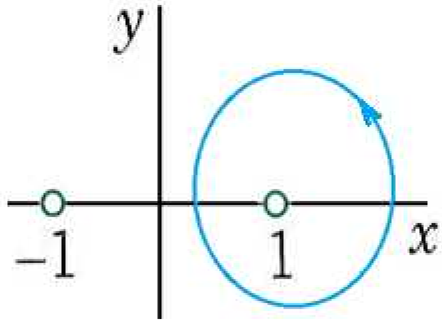


14.3 Cauchy's Integral Formula

EX 3. Integration Around Different Contours

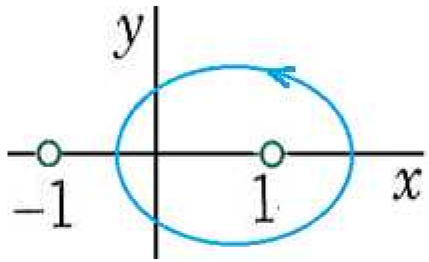
$$g(z)=\frac{z^2+1}{z^2-1}=\frac{z^2+1}{(z-1)(z+1)}$$

(a) Around z_0=1



$$\oint_C \frac{z^2+1}{z^2-1}dz=2\pi i\left[\frac{z^2+1}{z+1}\right]_{z=1}$$
$$=2\pi i$$

(b) Around Around z_0=1



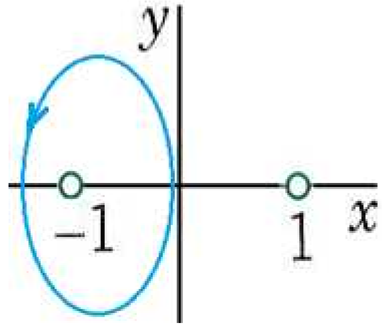
$$\oint_C \frac{z^2+1}{z^2-1}dz=2\pi i\left[\frac{z^2+1}{z+1}\right]_{z=1}$$
$$=2\pi i$$

14.3 Cauchy's Integral Formula

(c) Around z_0=-1

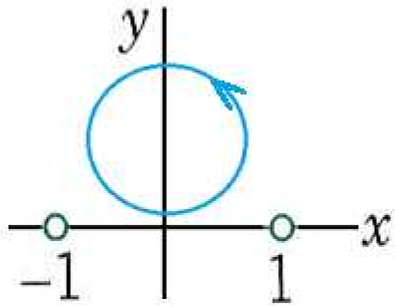
$$\oint_C \frac{z^2+1}{z^2-1}dz = \oint_C \frac{z^2+1}{z-1}\cdot\frac{1}{z+1}dz$$

$$= 2\pi i\left[\frac{z^2+1}{z-1}\right]_{z=-1} = -2\pi i$$



(d) Around no pole

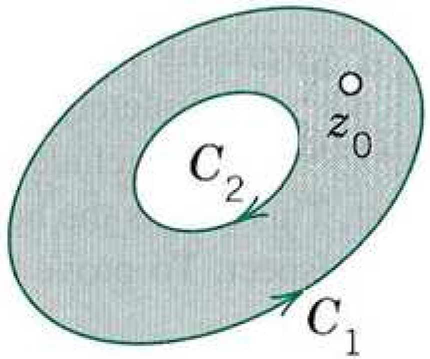
$$\oint_C \frac{z^2+1}{z^2-1}dz = 0$$



14.3 Cauchy's Integral Formula

Multiply Connected Domains

f(z): analytic in multiply connected domain D



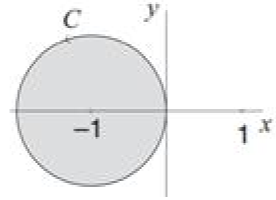
$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z - z_0}dz + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z - z_0}dz$$

$$\begin{aligned}&\textit{where}\\ &\quad \textit{outer contour } C_1:\ \textit{Counterclockwise}(CCW)\\ &\quad \textit{inner contour } C_2:\ \textit{Clockwise}(CW)\end{aligned}$$

Problems at for Sec. 14.3

1. Integrate $z^2/(z^2 - 1)$ by Cauchy's formula counterclockwise around the circle $|z + 1| = 1$.

$$(1) \quad \oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0)$$



$$\oint_C \frac{z^2}{z^2 - 1}\, dz = \oint_C \frac{f(z)}{z - z_0}\, dz = \oint_C \frac{z^2/(z-1)}{z - (-1)}\, dz$$

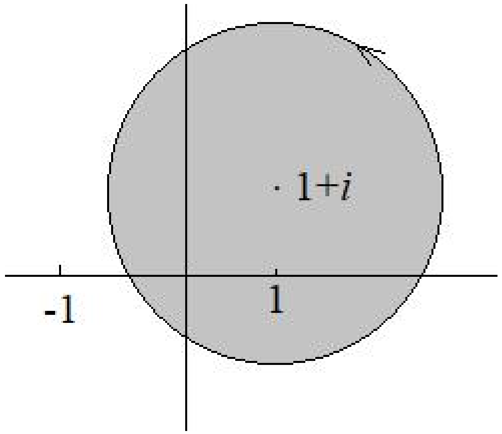
$$= 2\pi i\, f(z_0) = 2\pi i\, f(-1)$$

$$= 2\pi i \cdot \frac{1}{-2} = -\pi i.$$

Problems at for Sec. 14.3

2. Integrate $z^2/(z^2 - 1)$ by Cauchy's formula counterclockwise around the circle $|z - 1 - i| = \pi/2$.

$$(1) \quad \oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0)$$

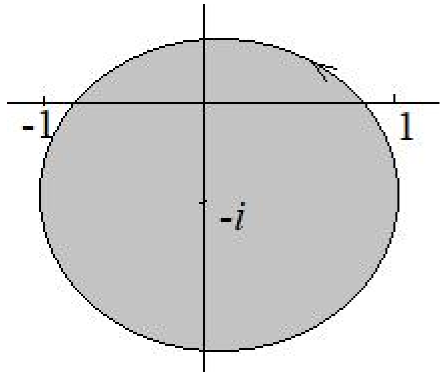


$$\oint_C \frac{z^2}{z^2 - 1}\, dz = \oint_C \frac{z^2/(z+1)}{z-1}\, dz$$

$$= 2\pi i \left.\frac{z^2}{z+1}\right|_{z=1} = \pi i$$

Problems at for Sec. 14.3

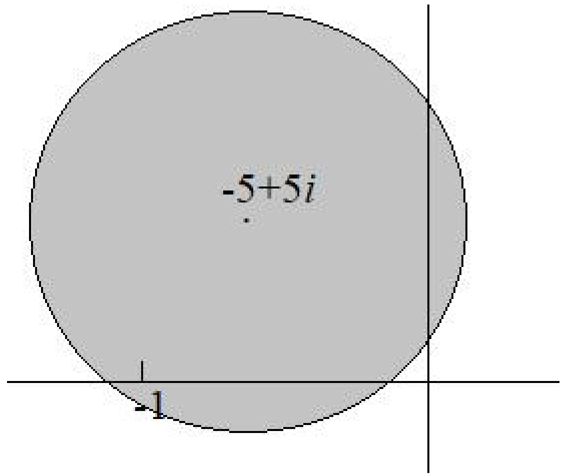
3. Integrate $z^2/(z^2 - 1)$ by Cauchy's formula counterclockwise around the circle $|z + i| = 1.4$.



$$\oint_C \frac{z^2}{z^2-1} dz = 0 \qquad \left[\text{by setting } f(z) = \frac{z^2}{z^2-1} \text{ in (1)}\right]$$

Problems at for Sec. 14.3

4. Integrate $z^2/(z^2-1)$ by Cauchy's formula counterclockwise around the circle $|z+5-5i|=7$.



$$(1)\quad \oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$\oint_C \frac{z^2}{z^2-1}\,dz = \oint_C \frac{z^2/(z-1)}{z+1}\,dz$$

$$= 2\pi i \left.\frac{z^2}{z-1}\right|_{z=-1} = -\pi i$$

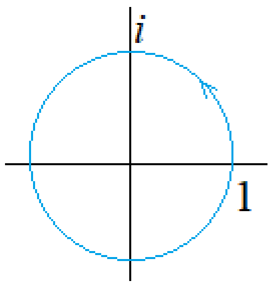
Problems at for Sec. 14.3

5. Integrate the given function $(\cos 3z)/(6z)$ around the unit circle.

$$(1)\quad \oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$\oint_C \frac{\cos 2z}{4z}\,dz = \oint_C \frac{(\cos 2z)/4}{z}\,dz$$

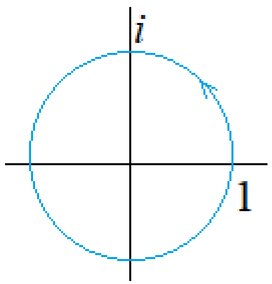
$$= 2\pi i \left.\frac{\cos 2z}{4}\right|_{z=0} = \frac{\pi i}{2}$$



Problems at for Sec. 14.3

6. Integrate the given function $e^{2z}/(\pi z - i)$ around the unit circle.

(1) $$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$



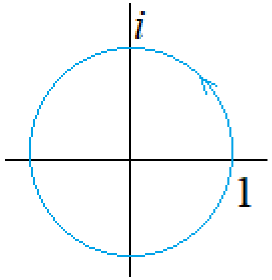
$$\oint_C \frac{e^{2z}}{\pi z - i}\,dz = \frac{1}{\pi}\oint_C \frac{e^{2z}}{z - i/\pi}\,dz = 2ie^{2z}\Big|_{z=i/\pi}$$

$$= 2ie^{2i/\pi} = 2\left(-\sin\frac{2}{\pi} + i\cos\frac{2}{\pi}\right)$$

Problems at for Sec. 14.3

7. Integrate the given function $z^3/(2z - i)$ around the unit circle.

$$(1) \quad \oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0)$$



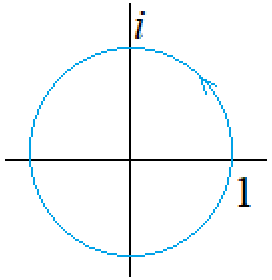
$$\oint_C \frac{z^2}{2z - i}\, dz = \frac{1}{2} \oint_C \frac{z^2}{z - i/2}\, dz$$

$$= \frac{\pi i}{2} z^2 \Big|_{z = i/2} = -\frac{\pi i}{8}$$

Problems at for Sec. 14.3

8. Integrate the given function $(z^2 \sin z)/(4z - 1)$ around the unit circle.

$$(1) \quad \oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0)$$



$$\oint_C \frac{z \sin z}{4z-1}\, dz = \frac{1}{4} \oint_C \frac{z \sin z}{z - 1/4}\, dz$$

$$= \frac{1}{4}\, 2\pi i\, (z \sin z)_{z = 1/4}$$

$$= \frac{\pi i}{8} \sin \frac{1}{4}$$

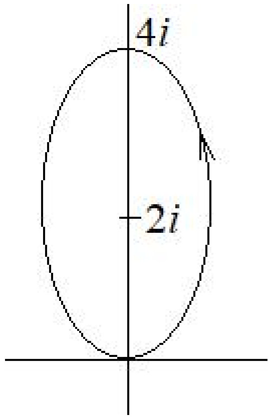
Problems at for Sec. 14.3

11. $\displaystyle\oint_C \frac{dz}{z^2+4}$, $\quad C: 4x^2 + (y-2)^2 = 4 \quad$ counterclockwise

(1) $$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

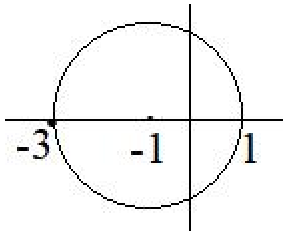
$$\oint_C \frac{dz}{z^2+4} = \oint_C \frac{1/(z+2i)}{z-2i}\,dz$$

$$= 2\pi i \left.\frac{1}{z+2i}\right|_{z=2i} = \frac{\pi}{2}$$



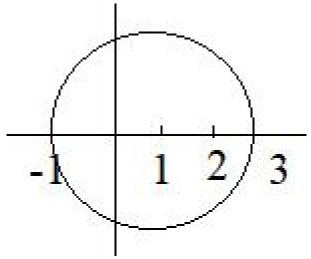
Problems at for Sec. 14.3

12. $\oint_C \frac{z}{z^2+4z+3}\,dz$, C : CCW, the circle with center -1 and radius 2



Problems at for Sec. 14.3

13. $\oint_C \frac{z+2}{z-2}\,dz, \quad C: |z-1| = 2$, CCW



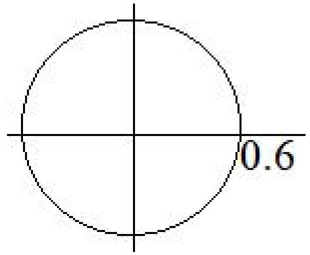
(1) $$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$\oint_C \frac{z+2}{z-2}\,dz = 2\pi i(z+2)\big|_{z=2} = 8\pi i$$

Problems at for Sec. 14.3

14. $\oint_C \frac{e^z}{ze^z - 2iz} dz, \quad C: |z| = 0.6, \text{ CCW}$

(1) $$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$



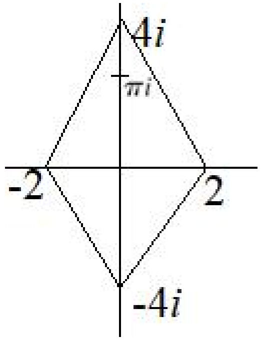
$e^z - 2i = 0, \quad z = \ln 2 + i\left(\frac{\pi}{2} + n\pi\right)$: exterior of the circle

$$\oint_C \frac{e^z}{ze^z - 2iz} dz = \oint_C \frac{e^z/(e^z - 2i)}{z} dz$$

$$= 2\pi i \left. \frac{e^z}{e^z - 2i} \right|_{z=0} = \frac{(2i - 4)\pi}{5}$$

Problems at for Sec. 14.3

15. $\oint_C \frac{\cosh(z^2 - \pi i)}{z - \pi i}\,dz$, C: CCW, the boundary of the square with vertices ± 2, $\pm 4i$.



(1) $$\oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0)$$

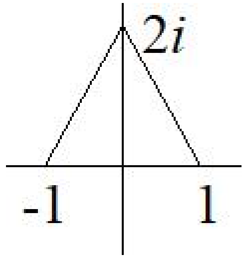
$$\oint_C \frac{\cosh(z^2 - \pi i)}{z - \pi i}\,dz = 2\pi i \cosh(z^2 - \pi i)\big|_{z = \pi i}$$

$$= -2\pi i \cosh \pi^2$$

Problems at for Sec. 14.3

16. $\displaystyle\oint_C \frac{\tan z}{z-i}\,dz,\quad C$: CCW, the boundary of the triangle with vertices 0 and $\pm 1 + 2i$.

$$\frac{\tan z}{z-i} = \frac{\sin z}{(z-i)\cos z}$$



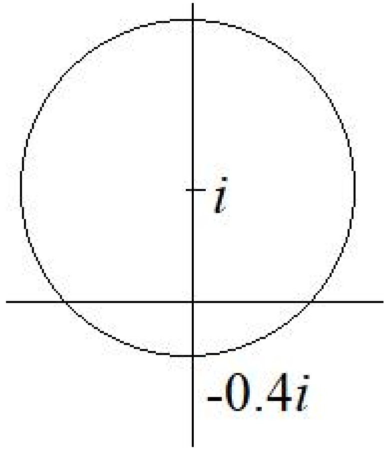
$\cos z = 0,\quad z = \dfrac{\pi}{2} + n\pi$: exterior of the triangle

$$(1)\quad \oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$\oint_C \frac{\tan z}{z-i}\,dz = 2\pi i \tan z\big|_{z=i} = -2\pi \tanh 1$$

Problems at for Sec. 14.3

17. $\oint_C \frac{\mathrm{Ln}\,(z+1)}{z^2+1}\,dz,\ C\!:\mathrm{CCW},\ |z-i| = 1.4$



(1) $$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

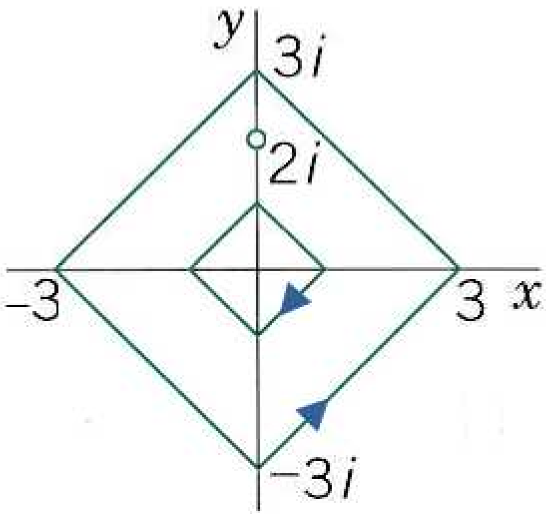
$$\oint_C \frac{\mathrm{Ln}(z+1)}{z^2+1}\,dz = \oint_C \frac{[\mathrm{Ln}(z+1)]/(z+i)}{z-i}\,dz$$

$$= 2\pi i \left.\frac{\mathrm{Ln}(z+1)}{z+i}\right|_{z=i} = \pi\left(\ln\sqrt{2} + \frac{\pi}{4} i\right)$$

Problems at for Sec. 14.3

Problem Set 14.3-18

$$\frac{\sin z}{4z^2 - 8iz} = \frac{(1/4)(\sin z)/z}{z - 2i}$$



(1) $$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0)$$

$$\oint_C \frac{\sin z}{4z^2 - 8iz}\, dz = \oint_C \frac{(1/4)(\sin z)/z}{z - 2i}\, dz$$

$$= 2\pi i \left[\frac{1}{4}\frac{\sin z}{z}\right]_{z=2i}$$

$$= \frac{\pi}{4}\sin 2i = \frac{\pi}{4} i \sinh 2 = 2.8485i$$

Problems at for Sec. 14.3

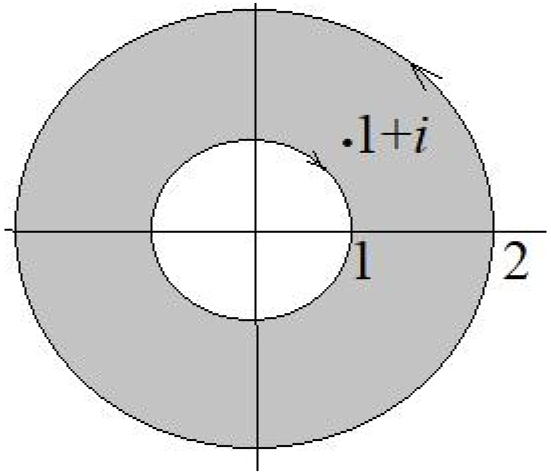
19. $\oint_C \frac{\exp z^2}{z^2(z-1-i)}\,dz$, C consists of $|z|=2$ counterclockwise and $|z|=1$ clockwise.

(1) $$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

$$\oint_C \frac{\exp z^2}{z^2(z-1-i)}\,dz = \oint_C \frac{(\exp z^2)/z^2}{z-1-i}\,dz$$

$$= 2\pi i \left.\frac{\exp z^2}{z^2}\right|_{z=1+i}$$

$$= \pi(\cos 2 + i\sin 2)$$



14.4 Derivatives of Analytic Functions (해석함수의 도함수)

THEOREM 1. Derivatives of an Analytic Function

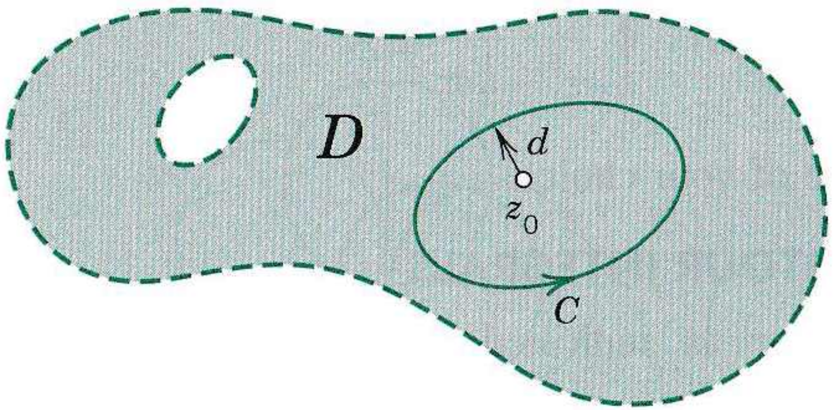
If f(z) is analytic in a domain D, then it has derivatives of all orders, which are then also analytic functions in D.

The values of derivatives at a point z_0 are given by the formulas

$$(1') \quad f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} dz$$

$$(1'') \quad f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^3} dz$$

$$(1) \quad f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$



14.4 Derivatives of Analytic Functions

PROOF

Applications of Theorem 1

EX 1. Evaluation of Line Integrals: $\oint_C \frac{\cos z}{(z-\pi i)^2} dz$

THEOREM 1. Derivatives of an Analytic Function

$$(1') \quad f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} dz$$

$$\oint_C \frac{f(z)}{(z-z_0)^2} dz = 2\pi i f'(z_0)$$

$$\oint_C \frac{\cos z}{(z-\pi i)^2} dz = 2\pi i (\cos z)' \Big|_{z=\pi i}$$

$$= 2\pi i(-\sin \pi i)$$

$$= -2\pi i(i \sinh \pi) = 2\pi \sinh \pi$$

14.4 Derivatives of Analytic Functions

EX 2. Evaluation of Line Integrals: $\oint_C \frac{z^4-3z^2+6}{(z+i)^3}dz$

THEOREM 1. Derivatives of an Analytic Function

$$(1'') \quad f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}dz$$

$$\oint_C \frac{f(z)}{(z-z_0)^3}dz = \pi i\, f''(z_0)$$

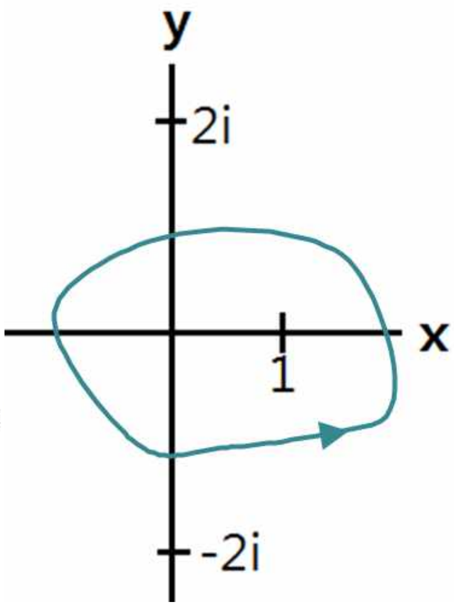
$$\oint_C \frac{z^4-3z^2+6}{(z+i)^3}dz = \pi i(z^4-3z^2+6)''\Big|_{z=-i}$$

$$= \pi i(12z^2-6)\Big|_{z=-i} = -18\pi i$$

EX 3. Evaluation of Line Integrals

Let C be a CCW closed contour with 1 inside and ∓2i outside. Evaluate the line integral given by

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)}dz$$

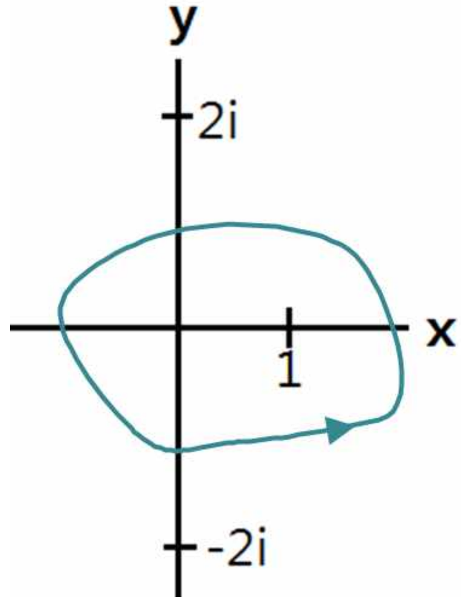


Sol.

$$(1') \quad f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$

14.4 Derivatives of Analytic Functions

$$(1') \quad f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} dz$$



$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)} dz = \oint_C \frac{e^z/(z^2+4)}{(z-1)^2} dz$$

$$= 2\pi i \left[\frac{e^z}{z^2+4} \right]' \Bigg|_{z=1}$$

$$= 2\pi i \left[\frac{e^z(z^2+4) - e^z 2z}{(z^2+4)^2} \right]_{z=1}$$

$$= \frac{6e\pi}{25} i$$

14.4 Derivatives of Analytic Functions

Cauchy's Inequality. Liouville's and Morera's Theorems

$$(1) \quad f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz \qquad (n = 1, 2, 3, \cdots)$$

$$\begin{aligned} |f^{(n)}(z_0)| &= \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \\ &\le \frac{n!}{2\pi}\oint_C \left|\frac{f(z)}{(z-z_0)^{n+1}}\right| dz \\ &\le \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r = \frac{n!\,M}{r^n} \end{aligned}$$

$where$
$|f(z)| \le M \ on \ C$

Cauchy's Inequality:

$$(2) \qquad |f^{(n)}(z_0)| \le \frac{n!\,M}{r^n}$$

14.4 Derivatives of Analytic Functions

THEOREM 2 Liouville's Theorem

If an entire function is bounded in absolute value in the whole complex plane, then this function must be a constant.

PROOF

14.4 Derivatives of Analytic Functions

THEOREM 3 Morera's Theorem(Converse of Cauchy's Integral Theorem)

If f(z) is continuous in a simply connected domain D and if

$$(3) \qquad \oint_C f(z)\,dz = 0$$

for every closed path in D, then f(z) is analytic.

PROOF

Problems for Sec 14.4

1. $\oint_C \frac{\sin z}{z^4} dz \qquad C$: CCW, $|z|=1$

Theorem 1: (1) $f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \qquad (n=1,2,3,\cdots)$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

$n=3$

$f(z)=\sin z, \qquad f^{(3)}(z)=-\cos z$

$$\oint_C \frac{\sin z}{z^4} dz = \frac{2\pi i}{3!} (-\cos z)\Big|_{z=0} = -\frac{\pi i}{3}$$

Problems for Sec 14.4

2. $\oint_C \frac{z^6}{(2z-1)^6}\,dz \qquad C\text{: CCW, } |z|=1$

Theorem 1: (1) $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (n=1,2,3,\cdots)$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!}\,f^{(n)}(z_0)$$

$n = 5$

$f(z) = z^6, \quad f^{(5)}(z) = 6!z$

$$\oint_C \frac{z^6}{(2z-1)^6}\,dz = \frac{1}{2^6}\oint_C \frac{z^6}{(z-1/2)^6}\,dz$$

$$= \frac{1}{2^6}\,\frac{2\pi i}{5!}\,(6!z)\Big|_{z=1/2} = \frac{3\pi i}{32}$$

Problems for Sec 14.4

3. $\oint_C \frac{e^z}{z^n}\,dz, \quad n = 1, 2, \cdots \qquad C\text{: CCW}, |z|=1$

Theorem 1: (1) $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz \qquad (n = 1,2,3,\cdots)$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(z_0)$$

$$f(z) = e^{-z}, \qquad f^{(n)}(z) = (-1)^n e^{-z}$$

$$\oint_C \frac{e^{-z}}{z^n}dz = \frac{2\pi i}{(n-1)!}(-1)^{n-1}e^{-z}\Big|_{z=0} = \frac{2\pi i(-1)^{n-1}}{(n-1)!}$$

$$= \frac{1}{2^6}\frac{2\pi i}{5!}(6!z)\Big|_{z=1/2} = \frac{3\pi i}{32}$$

Problems for Sec 14.4

4. $\oint_C \frac{e^z \cos z}{(z-\pi/4)^3}\,dz \qquad C\text{: CCW, } |z|=1$

Theorem 1: (1) $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (n=1,2,3,\cdots)$

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!}\,f^{(n)}(z_0)$$

$f(z)=e^z\cos z\,, \quad f''(z)=-2e^z\sin z$

$$\oint_C \frac{e^z\cos z}{(z-\pi/4)^3}\,dz = \frac{2\pi i}{2!}\left(-2e^z\sin z\right)\Big|_{z=\pi/4}$$

$$=-\sqrt{2}\,\pi i e^{\pi/4}$$

Problems for Sec 14.4

5. $\displaystyle\oint_C \frac{\cosh 2z}{(z-\frac{1}{2})^4}\,dz \quad C\text{: CCW, } |z|=1$

Theorem 1: $\displaystyle\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

$f(z) = \sinh 2z, \quad f^{(3)}(z) = 8\cosh 2z$

$$\oint_C \frac{\sinh 2z}{(z-1/2)^4}\,dz = \frac{2\pi i}{3!}(8\cosh 2z)\Big|_{z=1/2}$$

$$= \frac{8\pi i}{3}\cosh 1$$

Problems for Sec 14.4

6. $\displaystyle\oint_C \frac{dz}{(z-2i)^2(z-i/2)^2}$ C: CCW, $|z|=1$

Theorem 1: $\displaystyle\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

$$f(z)=\frac{1}{(z-2i)^2},\quad f\,'(z)=\frac{-2}{(z-2i)^3}$$

$$\oint_C \frac{dz}{(z-2i)^2(z-i/2)^2} = \oint_C \frac{1/(z-2i)^2}{(z-i/2)^2}dz$$

$$= \frac{2\pi i}{1!}\left.\frac{-2}{(z-2i)^3}\right|_{z=i/2} = -\frac{32\pi}{27}$$

Problems for Sec 14.4

7. $\oint_C \frac{\cos z}{z^{2n+1}} dz, \quad n = 0, 1, \cdots \quad C$: CCW, $|z|=1$

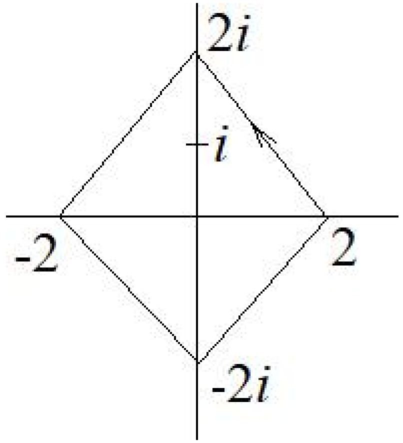
Theorem 1: $\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

$$f(z) = \cos z, \quad f^{(2n)}(z) = \cos(z + n\pi)$$

$$\oint_C \frac{\cos z}{z^{2n+1}} dz = \frac{2\pi i}{(2n)!} \cos(z + n\pi)\Big|_{z=0} = \frac{2\pi i (-1)^n}{(2n)!}$$

Problems for Sec 14.4

8. $\displaystyle\oint_C \frac{z^3 + \sin z}{(z - i)^3}\, dz,$ C the boundary of the square with vertices ± 2, $\pm 2i$ counterclockwise.



Theorem 1: $$\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

$f(z) = z^3 + \sin z$, $f''(z) = 6z - \sin z$

$$\oint_C \frac{z^3 + \sin z}{(z - i)^3}\, dz = \frac{2\pi i}{2!} (6z - \sin z)\Big|_{z = i} = -\pi \sin 1$$

Problems for Sec 14.4

9. $\oint_C \frac{\tan \pi z}{z^2}\, dz$, C the ellipse $16x^2 + y^2 = 1$ clockwise.

$\cos \pi z = 0$, $\pi z = \frac{\pi}{2} + n\pi$, $z = \frac{1}{2} + n$: exterior of the ellipse

Theorem 1: $\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

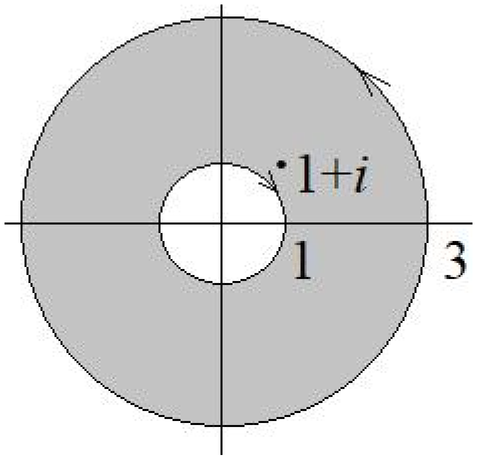
$f(z) = \tan \pi z$, $f'(z) = \pi \sec^2 \pi z$

$$\oint_C \frac{\tan \pi z}{z^2} dz = -\frac{2\pi i}{1!} \pi \sec^2 \pi z \bigg|_{z=0} = -2\pi^2 i$$

Problems for Sec 14.4

10. $\oint_C \frac{4z^3 - 6}{z(z-1-i)^2}\, dz$, C consists of $|z| = 3$ counterclockwise and $|z| = 1$ clockwise.

Theorem 1: $\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$



$$f(z) = \frac{4z^3 - 6}{z} = 4z^2 - \frac{6}{z}, \quad f'(z) = 8z + \frac{6}{z^2}$$

$$\int_C \frac{4z^3 - 6}{z(z-1-i)^2} dz = \int_C \frac{4z^2 - 6/z}{(z-1-i)^2} dz$$

$$= \frac{2\pi i}{1!}\left(8z + \frac{6}{z^2}\right)\Bigg|_{z=1+i} = 2\pi(-5+8i)$$

Problems for Sec 14.4

11. $\oint_C \frac{(1+z)\sin z}{(2z-1)^2}\,dz, \quad C: |z-i| = 2$ counterclockwise.

Theorem 1: $\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

$f(z) = (1+z)\sin z$, $f'(z) = \sin z + (1+z)\cos z$

$$\begin{aligned}
\oint_C \frac{(1+z)\sin z}{(2z-1)^2}\,dz &= \frac{1}{2^2}\oint_C \frac{(1+z)\sin z}{(z-1/2)^2}\,dz \\
&= \frac{1}{2^2}\frac{2\pi i}{1!}\left[\sin z + (1+z)\cos z\right]\Big|_{z=1/2} \\
&= \frac{\pi i}{2}\left(\sin\frac{1}{2} + \frac{3}{2}\cos\frac{1}{2}\right)
\end{aligned}$$

Problems for Sec 14.4

12. $\oint_C \frac{\exp(z^2)}{z(z-2i)^2}\,dz, \quad C: |z-3i| = 2$ clockwise.

Theorem 1: $\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

$$f(z) = \frac{\exp(z^2)}{z}, \quad f'(z) = \frac{(2z^2-1)\exp(z^2)}{z^2}$$

$$\oint_C \frac{\exp(z^2)}{z(z-2i)^2}dz = \oint_C \frac{[\exp(z^2)]/z}{(z-2i)^2}dz$$

$$= \frac{2\pi i}{1!} \left.\frac{(2z^2-1)\exp(z^2)}{z^2}\right|_{z=2i} = \frac{9\pi i}{2e^4}$$

Problems for Sec 14.4

13. $\oint_C \frac{\operatorname{Ln} z}{(z-2)^2}\, dz$, C: $|z-3| = 2$ counterclockwise.

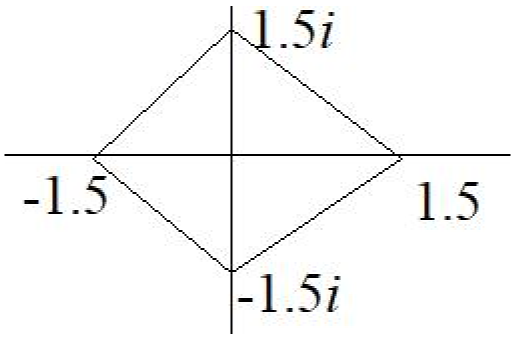
Theorem 1: $\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

$f(z)=$ $f(z)=\operatorname{Ln} z$, $f'(z)=\frac{1}{z}$

$$\oint_C \frac{\operatorname{Ln} z}{(z-4)^2} dz = \frac{2\pi i}{1!} \frac{1}{z}\bigg|_{z=4} = \frac{\pi i}{2}$$

Problems for Sec 14.4

14. $\oint_C \frac{\operatorname{Ln}(z+3)}{(z-2)(z+1)^2}\,dz$, C the boundary of the square with vertices ± 1.5, $\pm 1.5i$, counterclockwise.



Theorem 1: $\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

$$f(z)=\frac{\operatorname{Ln}(z+3)}{z-2}, \quad f'(z)=\frac{1}{(z+3)(z-2)}-\frac{\operatorname{Ln}(z+3)}{(z-2)^2}$$

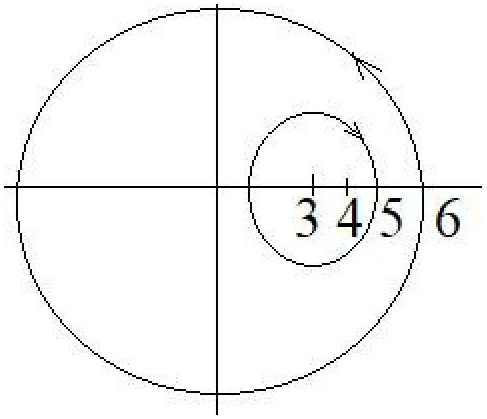
$$\oint_C \frac{\operatorname{Ln}(z+3)}{(z-2)(z+1)^2}dz=\oint_C \frac{\operatorname{Ln}(z+3)/(z-2)}{(z+1)^2}dz$$

$$=\frac{2\pi i}{1!}\left[\frac{1}{(z+3)(z-2)}-\frac{\operatorname{Ln}(z+3)}{(z-2)^2}\right]\Bigg|_{z=-1}$$

$$=\frac{\pi i}{9}(-3-2\operatorname{Ln}2)$$

Problems for Sec 14.4

15. $\oint_C \frac{\cosh 4z}{(z-4)^3}\,dz$, C consists of $|z| = 6$ counterclockwise and $|z-3| = 2$ clockwise.



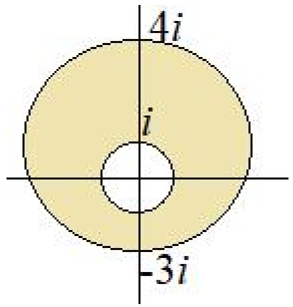
Integrand is analytic on the domain enclosed by $|z|=6$ and $|z-3|=2$.

Therefore the integral is 0.

Problems for Sec 14.4

16. $\oint_C \frac{e^{4z}}{z(z-2i)^2}\,dz$, C consists of $|z-i|=3$ counterclockwise and $|z|=1$ clockwise.

Theorem 1: $\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$



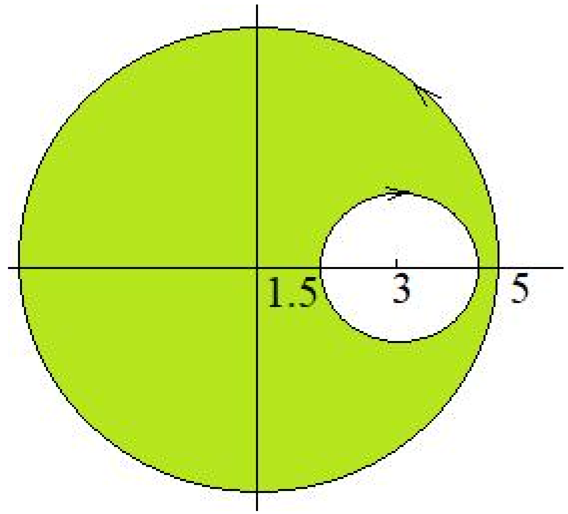
$$f(z)=\frac{e^{4z}}{z}, \qquad f'(z)=\frac{4ze^{4z}-e^{4z}}{z^2}$$

$$\oint_C \frac{e^{4z}}{z(z-2i)^2}dz=\oint_C \frac{e^{4z}/z}{(z-2i)^2}dz$$

$$=\frac{2\pi i}{1!}\left.\frac{4ze^{4z}-e^{4z}}{z^2}\right|_{z=2i}=\frac{(8+i)\pi i}{2}e^{8i}$$

Problems for Sec 14.4

17. $\oint_C \frac{e^{-z}\sin z}{(z-4)^3}\,dz$, C consists of $|z| = 5$ counterclockwise and $|z-3| = \frac{3}{2}$ clockwise.



Integrand is analytic on the domain enclosed by $|z|=5$ and $|z-3|=3/2$.

Therefore the integral is 0.

Problems for Sec 14.4

18. $\oint_C \frac{\sinh z}{z^n}\, dz, \quad C: |z| = 1$ counterclockwise, n integer.

Theorem 1: $$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

$$f(z) = \sinh z\,, \quad f^{(n)}(z) = \begin{cases} \cosh z & (n:\ \text{odd}) \\ \sinh z & (n:\ \text{even}) \end{cases}$$

for odd n:

$$\oint_C \frac{\sinh z}{z^n} dz = \frac{2\pi i}{(n-1)!} \cosh z \bigg|_{z=0} = \frac{2\pi i}{(n-1)!}$$

for even n:

$$\oint_C \frac{\sinh z}{z^n} dz = \frac{2\pi i}{(n-1)!} \sinh z \bigg|_{z=0} = 0$$

Problems for Sec 14.4

19. $\oint_C \frac{e^{3z}}{(4z - \pi i)^3}\,dz, \quad C\colon |z| = 1$, counterclockwise.

Theorem 1: $\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

$$f(z) = e^{3z}, \quad f''(z) = 9e^{3z}$$

$$\oint_C \frac{e^{3z}}{(4z - \pi i)^3}\,dz = \frac{1}{4^3} \oint_C \frac{e^{3z}}{(z - \pi i/4)^3}\,dz$$

$$= \frac{1}{4^3}\,\frac{2\pi i}{2!}\,9e^{3z}\Big|_{z = \pi i/4} = \frac{-9\pi(1+i)}{64\sqrt{2}}$$

Homework for Chapter 14

1. Solve the following problems.

REVIEW QUESTIONS AND PROBLEMS at the end of Chapter 14

21, 24, 26, 28, 30

2. Send your solution to twjeong@jbnu.ac.kr by Nov. 14.

3. File name of your solution: AEM2_Your-Name_Ch14.