THEOREM 2 Representation by a Fourier Series

Let f(x) be periodic with period 2π and piecewise continuous in the interval $-\pi \le x \le \pi$. Furthermore, let f(x) have a left- and right-hand derivatives at each point of that interval. Then the Fourier series (5) of f(x) with the coefficients in (6) converges. Its sum is f(x), except at point x_0 where f(x) is discontinuous. There the sum of the series is the average of left- and right-hand limits of f(x) at x_0.

PROOF Omitted!

$$(5) \quad f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

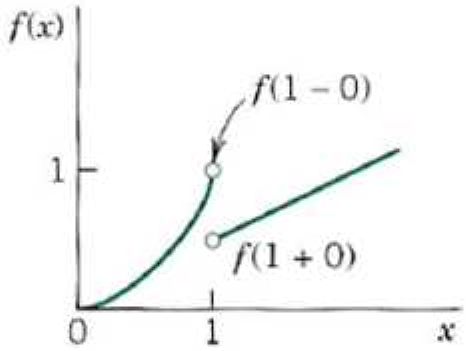
$$(0) \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$(6) \quad (a) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx \qquad n = 1, 2, \cdots$$

$$(b) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx \qquad n = 1, 2, \cdots$$

Left- and Right-hand limits

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x/2 & \text{if } x \geqq 1 \end{cases}$$



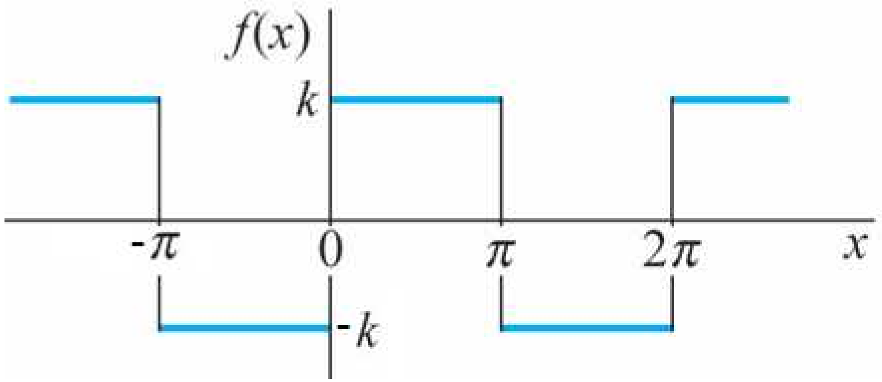
$$f(1-0) = 1,$$
$$f(1+0) = \tfrac{1}{2}$$

EX 2 Convergence at a Jump as Indicated in Theorem 2

Show that values of the Fourier series for the following function agree with Theorem 2.

$$(7) \qquad f(x) = \begin{cases} -k & if \quad -\pi < x < 0 \\ k & if \quad 0 < x < \pi \end{cases} \quad and \quad f(x+2\pi) = f(x)$$

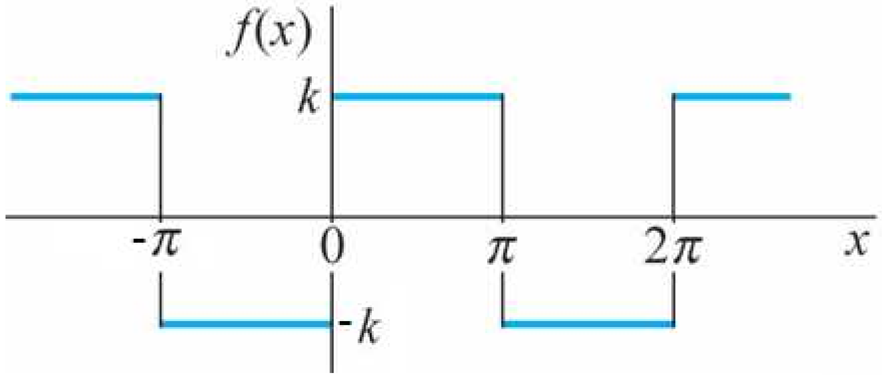
Fig. 260



Sol.

$$(5) \quad f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

Fig. 260



$$(5)\quad f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

$$f(0) = \frac{4k}{\pi}(0+0+0+\cdots) = 0$$

$$\textit{The average of } f(0^-) \textit{ and } f(0^+) = \frac{k-k}{2} = 0$$

Similarly, the relation holds at ∓nπ.

The *fundamental period* is the smallest positive period. Find it for

1. $\cos x$, $\sin x$,

$\cos 2x$, $\sin 2x$,

$\cos \pi x$, $\sin \pi x$,

$\cos 2\pi x$, $\sin 2\pi x$

2. $\cos nx$, $\sin nx$,

$$\cos \frac{2\pi x}{k}, \qquad \sin \frac{2\pi x}{k},$$

$$\cos \frac{2\pi n x}{k}, \qquad \sin \frac{2\pi n x}{k}$$

3. Linear combinations of periodic functions. Vector space. If $f(x)$ and $g(x)$ have period p, show that $h(x) = af(x) + bg(x)$ has the period p (a, b, constant). Thus all functions of period p form a **vector space**.

Proof. Assume $f(x)$ and $g(x)$ have period p. Then

$$\text{(A)} \qquad f(x+p) = f(x), \qquad g(x+p) = g(x).$$

Now

$$\begin{aligned} h(x+p) &= af(x+p) + bg(x+p) && \text{(by definition of } h\text{)} \\ &= af(x) + bg(x) && \text{[by (A)]} \\ &= h(x) && \text{(by definition of } h\text{).} \end{aligned}$$

$$\text{(B)} \qquad h(x+p) = h(x).$$

Sketch or graph $f(x)$ which for $-\pi < x < \pi$ is given as follows.

6. $f(x) = |x|$

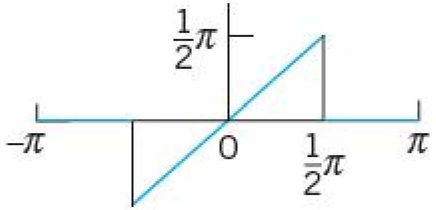
7. $f(x) = |\sin x|, \quad f(x) = \sin |x|$

8. $f(x) = e^{-|x|}, \quad f(x) = |e^{-x}|$

9. $$f(x) = \begin{cases} x & \text{if} \quad -\pi < x < 0 \\ \pi - x & \text{if} \quad 0 < x < \pi \end{cases}$$

10. $$f(x) = \begin{cases} -\cos^2 x & \text{if} \quad -\pi < x < 0 \\ \cos^2 x & \text{if} \quad 0 < x < \pi \end{cases}$$

16 Find the Fourier series of the following function, which is assumed to have the period 2π.



Sol.

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$$(b) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx \qquad n = 1, 2, \cdots$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \; dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad n = 1,2,\cdots$$

$$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,\cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad n = 1,2,\cdots$$

$$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\,\sin nx\,dx \;= \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} x\,\sin nx\,dx$$

formula for integration by parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

formula for integration by parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx \qquad \int u\,dv = uv - \int v\,du$$

$$\int x \sin nx\,dx = x\frac{-\cos nx}{n} - \int 1\cdot\frac{-\cos nx}{n}dx$$

$$= -\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\ \sin nx dx$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\frac{\pi}{2}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]_0^{\frac{\pi}{2}} = \begin{cases} \dfrac{2\sin\frac{n\pi}{2}}{n^2\pi} & (n : \text{odd}\ ) \\ \dfrac{-\cos\frac{n\pi}{2}}{n} & (n : \text{even}\ ) \end{cases}$$

(5) $$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = 0 \qquad a_n = 0$$

$$b_n = \begin{cases} \dfrac{2\sin\frac{n\pi}{2}}{n^2\pi} & (n : \text{odd}\ ) \\ \dfrac{-\cos\frac{n\pi}{2}}{n} & (n : \text{even}\ ) \end{cases}$$

$$= \frac{2}{\pi}\left[\sin x - \frac{1}{9}\sin 3x + \frac{1}{25}\sin 5x - \cdots\right]$$

$$+ \frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x + \frac{1}{6}\sin 6x - \cdots$$