Introduction to Discrete Math

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Course Outline

- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion & Induction, Logic, Invariants
- Probability & Combinatronics
  - Basic Counting, Binomial Coeff, Advanced Counting, Probability, Random Variables

Probability & Combinatronics – Random Variables

MARKOV INEQUALITY

- From Expectation to Probability
- Markov Inequality
- Application to Algorithms

Expectation vs Probability

- We have introduced random variables and their expected values
- We will now see how these notions can help in studies of probabilities of events

Lottery Budget

Problem

A lottery ticket costs 10,000 Won. 40% of the lottery budget goes to prizes. Show that the chances to win 500,000 won or more is < 1%.

- We will use proof by contradiction

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- We will use proof by contradiction
- Assume the contrary: the probability to win 50 man wons or more is at least 0.01 = ≥ 0.01 (1%) < 0.01 (1%)
- Denote number of tickets sold as ***n***
- Then the budget of the lottery is ***10n*** won — 10n won
- Therefore, ***10n x 0.4 = 4n*** won are spent on prizes

Lottery Budget

Problem

A lottery ticket costs 10,000 Won. 40% of the lottery budget goes to prizes. Show that the chances to win 500,000 won or more is < 1%.

– By our assumption, at least $\frac{n}{100}$ tickets win at least 500,000 won

– In total, these tickets win $\frac{n}{100} \times 500 = 5n$ won

– This exceeds the total prize budget of ***4n***!

– We arrived into contradiction and the problem is solved

- From Expectation to Probability
- **Markov Inequality**
- Application to Algorithms

Markov's Inequality

Suppose that f is a non-negative random variable. Then for any number $a > 0$, we have

$$\Pr\left[f \geq \frac{\mathrm{E}f}{a}\right]$$

- The inequality allows to use expected value to bound probability of certain events
- For the proof, it is convenient to rewrite the inequality:

$$a \times \Pr\left[f \geq a\right] \leq \mathrm{E}f$$

Markov's Inequality

- We need to prove the inequality $a \times \Pr[f \geq a] \leq \mathrm{E}f$
- Consider the following random variable g on the same probability space: for an outcome such that $f = a_i$ we let $g = a$ on this outcome if $a \leq a_i$ and $g = 0$ otherwise; that is

$$g = \begin{cases} a & \text{if } f = a_i \geq a \\ 0 & \text{if } f = a_i < a \end{cases}$$

- g is less than or equal to f on each outcome
- So, the average value $\mathrm{E}g$ of g is less than or equal to the average value $\mathrm{E}f$ of f

$$\mathrm{E}g \leq \mathrm{E}f$$

Markov's Inequality

$$g = \begin{cases} a & \text{if } f = a_i \geq a \\ 0 & \text{if } f = a_i < a \end{cases}$$

- What is the expectation of **g**?
  - **g** has only one nonzero value
- So **Eg** is this value multiplied by the sum of probabilities of all outcomes for this value
  - But these outcomes are exactly the outcomes that form event "$f \geq a$"
- Thus the sum of their probabilities is equal to $\Pr[f \geq a]$
- Thus $Eg = a \times \Pr[f \geq a]$

$\{f \geq a\}$

Markov's Inequality

$$g = \begin{cases} a & \text{if } f = a_i \geq a \\ 0 & \text{if } f = a_i < a \end{cases}$$

- Finally we have $\mathbf{E}g \leq \mathbf{E}f$
- And $\mathrm{E}g = a \times \Pr[f \geq a]$
- So

$$\mathrm{E}f \geq \mathrm{E}g = a \times \Pr[f \geq a]$$

- We have Markov's Inequality

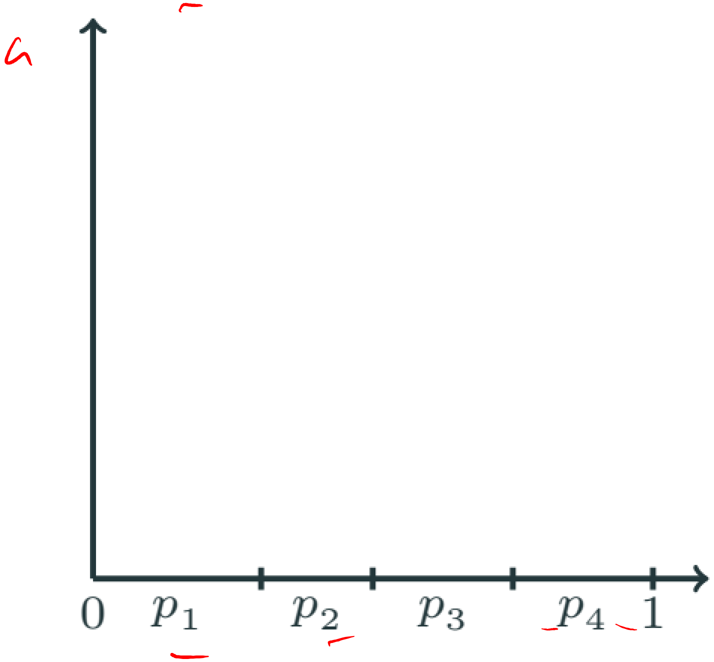
$a \geq b$

$b \leq a$

Geometric Interpretation

$$\mathrm{E}f \geq \mathrm{E}g = a \times \Pr\left[f \geq a\right]$$

- Suppose ***f*** has values a_1, a_2, a_3, a_4 with probabilities p_1, p_2, p_3, p_4



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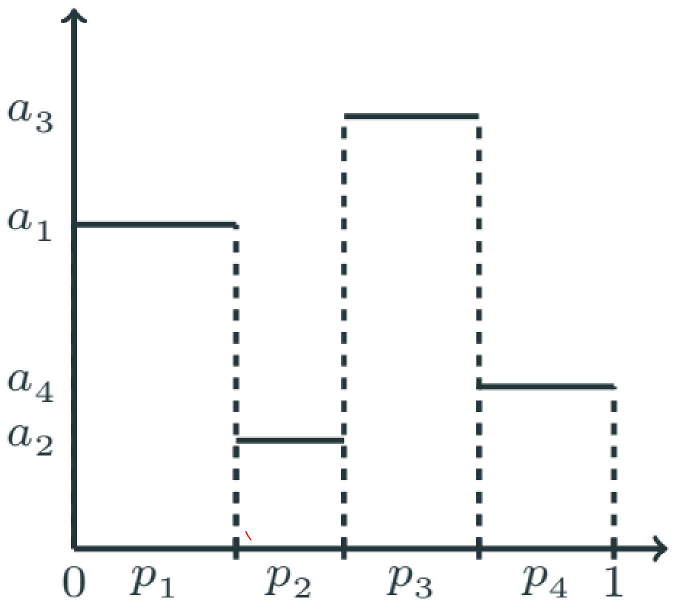

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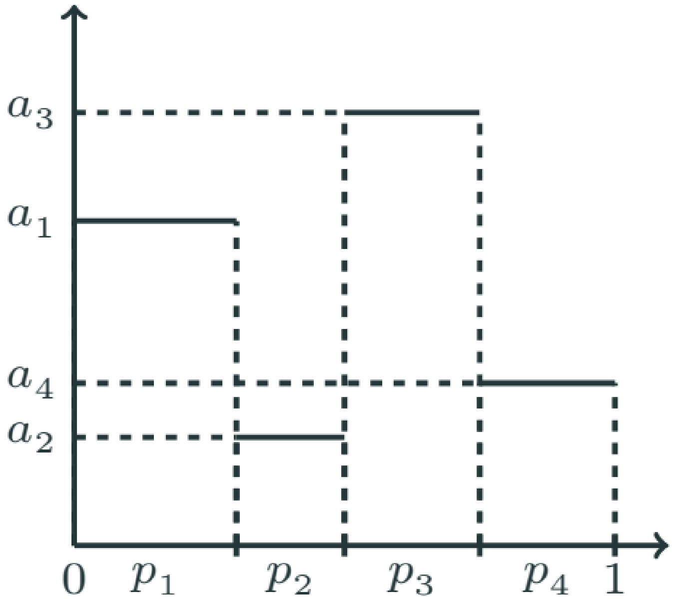

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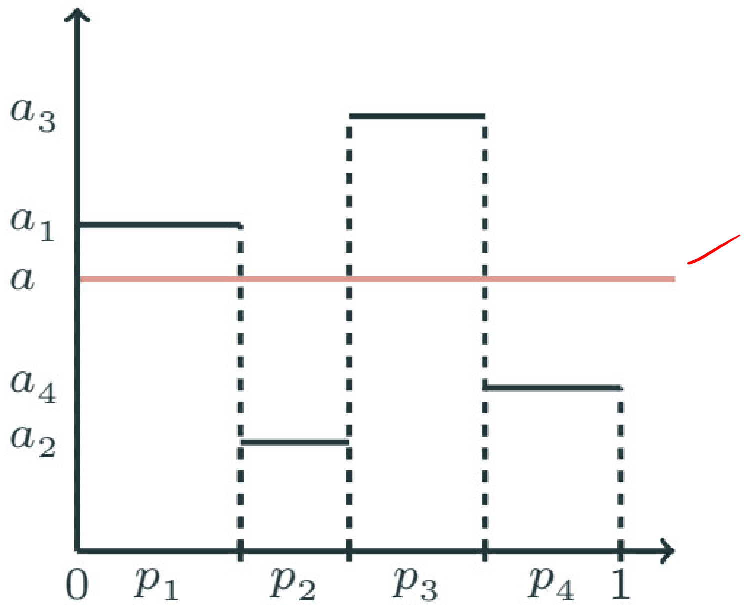

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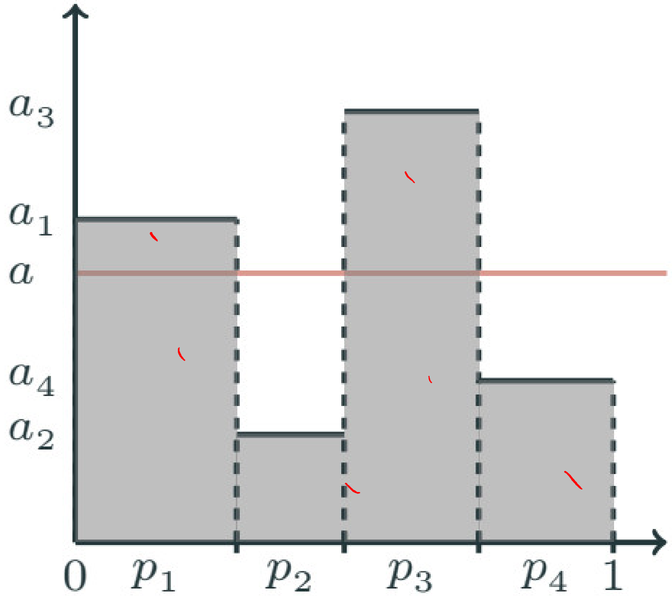

- $\mathrm{E}f$ is gray region area

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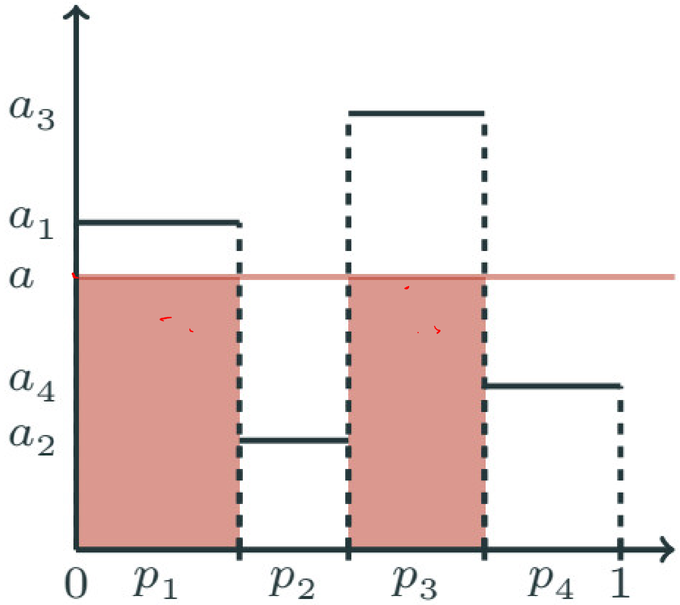

- $\mathrm{E}f$ is gray region area
- $a \times \Pr\left[f \geq a\right]$ is reddish region area

Geometric Interpretation

$\left\{ \left\{ \mathrm{E}f \geq \mathrm{E}g \right\} = a \times \Pr[f \geq a] \right\}$

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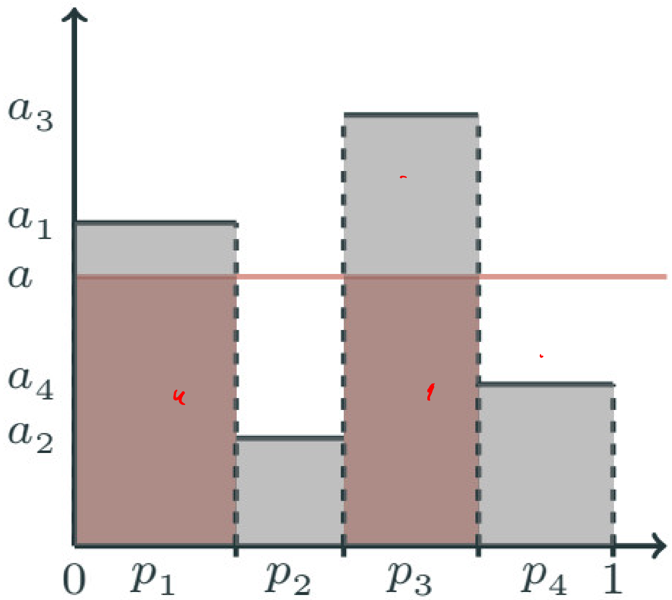

- **E*f*** is gray region area
  Eg
- $a \times \Pr[f \geq a]$ is reddish region area
- Gray region is large and inequality follows

- From Expectation to Probability
- Markov Inequality
- **Application to Algorithms**

Application to Algorithms

Problem

Suppose there is a randomized algorithm that runs on average, assume n^2, where n is the size of input. The algorithm outputs the correct answer. Construct another randomized algorithm that always stops in time cn^2 for some constant c and makes a mistake with probability of at most 10^{-3}

– We will use **Markov's Inequality**

Application to Algorithms

- Running time of algorithm is a random variable, set it as f
- Recall that $Ef = n^2$
- Here is a new algorithm:
  - Run original algorithm for 10^3n^2 steps
  - If it stops, we also stop
  - If not, stop and output, for example "*0*"

Application to Algorithms

Claim

The probability that the original algorithm does not stop after ***10^3n^2*** number of steps is at most ***10^{-3}***

- Indeed, the probability is $\Pr\left[f \geq 10^{3} n^2\right]$
- By Markov's Inequality, it is bounded by

$$\Pr\left[f \geq 10^{3} n^2\right] \leq \frac{\mathrm{E}f}{10^3 n_2} = \frac{\cancel{n^2}}{10^3 \cancel{n^2}} = 10^{-3}$$

Conclusion

- We studied random variables
- Allowed us to study the quantitative aspects of randomness
- Allowed us to apply many analytical tools to study probability
- Expected values is one of the main characteristics of a random variable
- On one side, expectation bears a lot of information of a random variable
- On the other side, expectation has very convenient mathematical properties

Thank you.