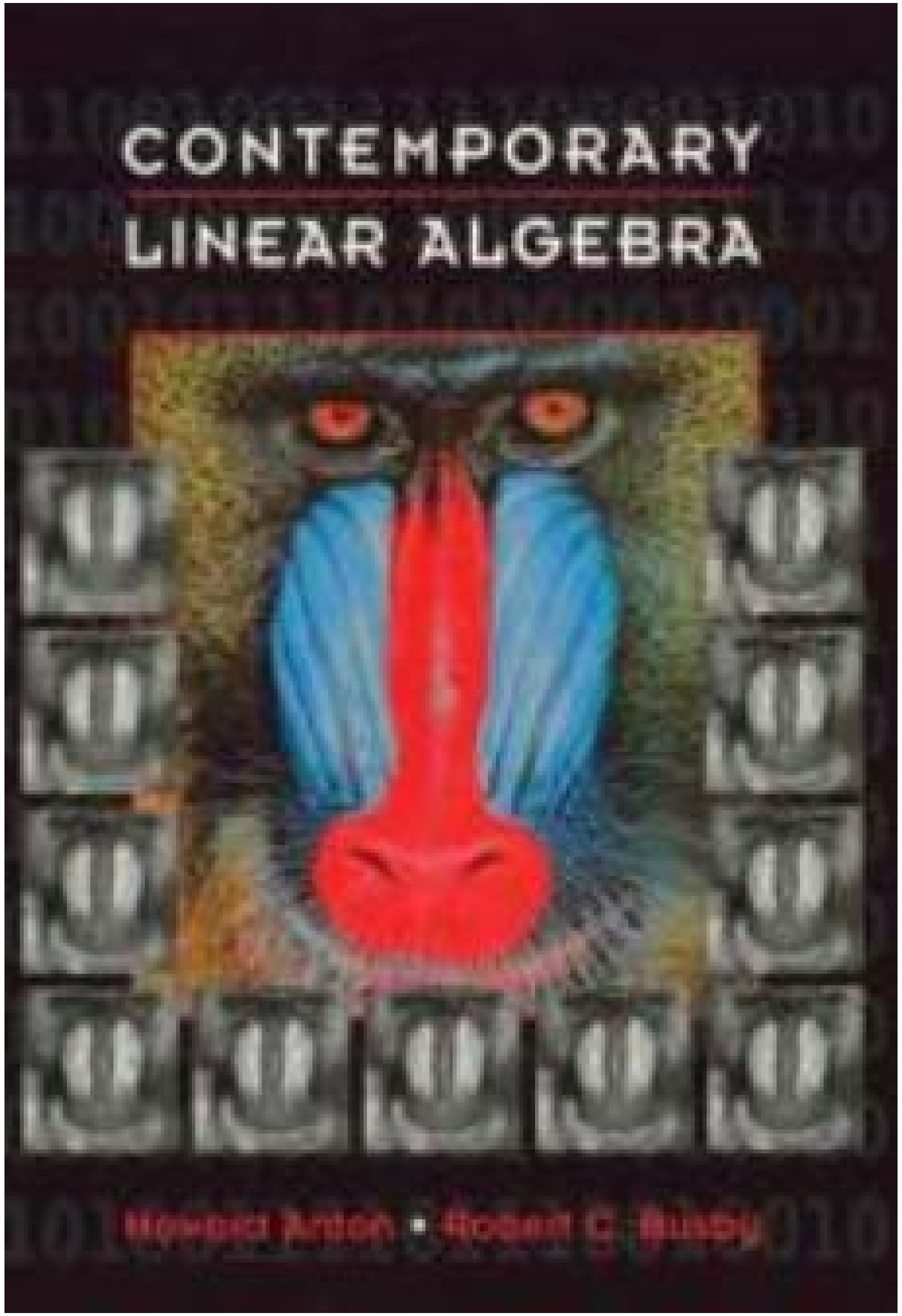

CHAPTER 4
Determinants

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4.1 Determinants; Cofactor Expansion

DETERMINANTS OF 2x2 AND 3x3 MATRCES

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(\mathbf{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right.$$

$$\det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Example 1 Evaluating Determinants

Find the determinant of $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$

Sol.

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \left|\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ -4 & 5 & 6 & -4 & 5 \\ 7 & -8 & 9 & 7 & -8 \end{array}\right.$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240$$

ELEMENTARY PRODUCTS

Elementary product: product of elements containing exactly one entry from each row and one entry from each column

Signed elementary product: the elementary product with its associated + or – sign.

Associated sign:

+ sign: even number of interchanges for natural order

– sign: even number of interchanges for natural order

Permutation of Column Undices	Minimum Number of Interchanges to Put Permutation in Natural Order	Signed Elementary Product
$\{1, 2, 3\}$	0	$+a_{11}a_{22}a_{33}$
$\{1, 3, 2\}$	1	$-a_{11}a_{23}a_{32}$
$\{2, 1, 3\}$	1	$-a_{12}a_{21}a_{33}$
$\{2, 3, 1\}$	2	$+a_{12}a_{23}a_{31}$
$\{3, 1, 2\}$	2	$+a_{13}a_{21}a_{32}$
$\{3, 2, 1\}$	1	$-a_{13}a_{22}a_{31}$

GENERAL DETERMINANTS

Definition 4.1.1 The determinants of a square matrix A is denoted ny det(A) and is defined to be the sum of all signed elementary products from A.

$$\det(\mathbf{A}) = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots\,\vdots\,\vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\det(\mathbf{A}) = \sum \pm a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

where the sign is + if the permutation {j1, j2, · · · , jn} is even and – if it is odd.

Evaluation Difficulties for Higher Order Determinants

The amount of computation to find the determinant of an nxn matrix:

$n!=n(n-1)(n-1)\cdot\cdot\cdot 2\cdot 1$ which increases dramatically as n increases.

Determinants of Matrices with Rows or Columns with All Zeros

Theorem 4.1.2 *If A is a square matrix with a row or a column of zeros, then* det(A)=0.

Theorem 4.1.3 *If A is a triangular matrix , then det(A) is the product of the entries on the main diagonal.*

Proof

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Example 2 Determinant of a Triangular Matrix

Find the determinants of the matrices: $\begin{bmatrix} -2 & 5 & 7 \\ 0 & 3 & 8 \\ 0 & 0 & 5 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 9 & 0 & 0 \\ -7 & 6 & -1 & 0 \\ 3 & 8 & -5 & -2 \end{bmatrix}$

Sol.

$$\begin{vmatrix} -2 & 5 & 7 \\ 0 & 3 & 8 \\ 0 & 0 & 5 \end{vmatrix} = (-2)(3)(5) = -30$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 4 & 9 & 0 & 0 \\ -7 & 6 & -1 & 0 \\ 3 & 8 & -5 & -2 \end{vmatrix} = (1)(9)(-1)(-2) = 18$$

Minors and Cofactors(소행렬식과 여인수)

Definition 4.1.4 If A is a square matrix, then the **minor** of entry a_{ij} (also called the ij th minor of A) is denoted by M_{ij} is defined to be the determinant of the matrix that remains when the i th row and j th column of A are deleted. The number $C_{ij}=(-1)^{i+j} M_{ij}$ is called the cofactor of entry a_{ij} (or ij th cofactor of A).

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\mathbf{M}_{23} = \begin{vmatrix} a_{11} & a_{12} & \boxed{a_{13}} & a_{14} \\ \boxed{a_{21}} & \boxed{a_{22}} & \boxed{a_{23}} & \boxed{a_{24}} \\ a_{31} & a_{32} & \boxed{a_{33}} & a_{34} \\ a_{41} & a_{42} & \boxed{a_{43}} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

$$\mathbf{C}_{23} = (-1)^{2+3}\ \mathbf{M}_{23} = (-1)\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

Example 3 Minors and Cofactors

Find M_{11}, C_{11}, M_{32}, and C_{32} of the matrix $\mathbf{A} = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$$

(9)

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -26$$

Sign of the Minors

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Cofactor Expansions

$$\det(\mathbf{A}) = \sum \pm a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

For a 3x3 matrix,

$$\begin{aligned}
\det(\mathbf{A}) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{21}(a_{13}a_{32} - a_{12}a_{33}) + a_{31}(a_{12}a_{23} - a_{13}a_{22}) \\
&= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}
\end{aligned}$$

$$\begin{aligned}
\det(\mathbf{A}) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \leftarrow \text{1st row} \\
&= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} && \leftarrow \text{1st column} \\
&= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} && \leftarrow \text{2nd row} \\
&= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} && \leftarrow \text{2nd column} \\
&= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} && \leftarrow \text{3rd row} \\
&= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} && \leftarrow \text{3rd column}
\end{aligned}$$

Example 4 Cofactor Expansion

Find the determinant of $\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = (1)(+1)\begin{vmatrix} 5 & 6 \\ -8 & 9 \end{vmatrix} + (-4)(-1)\begin{vmatrix} 2 & 3 \\ -8 & 9 \end{vmatrix} + (7)(+1)\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

$$= (1)(93) + (4)(42) + (7)(-3) = 240$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = (2)(-1)\begin{vmatrix} -4 & 6 \\ 7 & 9 \end{vmatrix} + (5)(+1)\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + (-8)(-1)\begin{vmatrix} 1 & 3 \\ -4 & 6 \end{vmatrix}$$

$$= (-2)(-78) + (5)(-12) + (8)(18) = 240$$

Theorem 4.1.5 The Determinant of an nxn Matrix A

Theorem 4.1.5 *The determinant of an $n \times n$ matrix A can be computed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products; that is, for each $1 \leq i \leq n$ and $1 \leq j \leq n$,*

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

(*cofactor expansion along the j*th *column*)

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(*cofactor expansion along the i*th *row*)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \boxed{a_{13}} & a_{14} \\ a_{21} & a_{22} & \boxed{a_{23}} & a_{24} \\ a_{31} & a_{32} & \boxed{a_{33}} & a_{34} \\ a_{41} & a_{42} & \boxed{a_{43}} & a_{44} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \boxed{a_{21}} & \boxed{a_{22}} & \boxed{a_{23}} & \boxed{a_{24}} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Example 5 Cofactor Expansion of a 4x4 Determinant

Use a cofactor expansion to find the determinant of A.

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 5 \\ -1 & 2 & 4 & 1 \\ 3 & 0 & 0 & 3 \\ 8 & 6 & 0 & 0 \end{bmatrix}$$

Sol.

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 0 & 0 & 5 \\ -1 & 2 & 4 & 1 \\ 3 & 0 & 0 & 3 \\ 8 & 6 & 0 & 0 \end{vmatrix} = (-4)\begin{vmatrix} 2 & 0 & 5 \\ 3 & 0 & 3 \\ 8 & 6 & 0 \end{vmatrix} = (-4)(-6)\begin{vmatrix} 2 & 5 \\ 3 & 3 \end{vmatrix} = -216$$

DETERMINANT OF A^T

$$\det(\mathbf{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = ad - cb = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \det(\mathbf{A}^T)$$

Theorem 4.2.1 *If A is a square matrix, then* det (A) = det (A^T).

Effect of Row Operations on a Determinant

Theorem 4.2.2 *Let A be an nxn matrix.*

(a) *If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then* $\det(B) = k\det(A)$.

(b) *If B is the matrix that results when two rows or two columns of A are interchanged, then* $\det(B) = -\det(A)$.

(c) *If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then* $\det(B) = \det(A)$.

(a)
$$\underset{\mathbf{B}}{\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ & \boxed{\qquad\qquad} & \end{vmatrix}} = k\underset{\mathbf{A}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ & \boxed{\qquad\qquad} & \end{vmatrix}} \qquad \underset{\mathbf{B}}{\begin{vmatrix} ka_{11} & \\ ka_{21} & \boxed{} \\ ka_{31} & \end{vmatrix}} = k\underset{\mathbf{A}}{\begin{vmatrix} a_{11} & \\ a_{21} & \boxed{} \\ a_{31} & \end{vmatrix}}$$

(b)
$$\underset{\mathbf{B}}{\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} = -\underset{\mathbf{A}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad \underset{\mathbf{B}}{\begin{vmatrix} a_{11} & a_{13} & a_{12} \\ a_{21} & a_{23} & a_{22} \\ a_{31} & a_{33} & a_{32} \end{vmatrix}} = -\underset{\mathbf{A}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

(c)
$$\underset{\mathbf{B}}{\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} = \underset{\mathbf{A}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

$$\begin{vmatrix} a_{11} & a_{12}+ka_{11} & a_{13} \\ a_{21} & a_{22}+ka_{21} & a_{23} \\ a_{31} & a_{32}+ka_{31} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Example 1 Effect of Elementary Row Operations

$$\overset{\mathbf{B}}{\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} = k\overset{\mathbf{A}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \Leftarrow \quad \det(\mathbf{B}) = k\det(\mathbf{A})$$

$$\overset{\mathbf{B}}{\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} = -\overset{\mathbf{A}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \Leftarrow \quad \det(\mathbf{B}) = -\det(\mathbf{A})$$

$$\overset{\mathbf{B}}{\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} = \overset{\mathbf{A}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \Leftarrow \quad \det(\mathbf{B}) = \det(\mathbf{A})$$

Theorem 4.2.3

Theorem 4.2.3 *Let A be an nxn matrix.*

(a) *If A has two identical tows or columns, then* det(A)=0.

(b) *If A has two proportional rows or columns, then* det(A)=0.

(c) $\det(kA)=k^n \det(A)$.

Example 2 Some Determinants by Inspection

$$\begin{vmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 15 \end{vmatrix} = 0$$

$$R_4 = -3R_1$$

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 1$$

$$R_1 \leftrightarrow R_4, R_2 \leftrightarrow R_3 \quad \Rightarrow \quad \mathbf{I}$$

$$\begin{vmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{vmatrix} = 0$$

$$C_2 = -2C_1$$

Example 3 Determinant of the Negative of a Matrix

What is the relationship between det(A) and det(-A)?

Sol.

$$-\mathbf{A} = (-1)\mathbf{A}$$

$$\det\left(-\mathbf{A}\right) = (-1)^n \det\left(\mathbf{A}\right)$$

Simplifying Cofactor Expansions

Example 4 Using Row Operations to Simplify a Cofactor Expansion

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

$$\det(\mathbf{A}) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} = -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$ Cofacor expansion with C_1

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix}$$ $R_1 + R_3 \rightarrow R_3$ $$= -(-1)\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix}$$ Cofacor expansion with C_1

$$= -18$$

Determinants by Gaussian Elimination

Example 5 Evaluating a Determinant by Gaussian Elimination

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

$$\det(\mathbf{A}) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \quad R_1 \leftrightarrow R_2 \quad = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \quad (-2)R_1 + R_3 \to R_3 \qquad = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} \quad (-10)R_2 + R_3 \to R_3$$

$$= (-3)(-55)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix} = (-3)(-55)(1) = 165$$

A Determinant Test for Invertibility

Theorem 4.2.4 *A square matrix is invertible if and only if* $\det(A) \neq 0$*.*

Theorem 4.2.5 *If A and B are square matrices of the same size, then*

$$\det(AB) = \det(A)\det(B) \tag{2}$$

$$\det(A^n) = [\det(A)]^n$$

[By Theorem 4.2.5]

Example 6 An Illustration of Theorem 4.2.5

Show that det(AB)=det(A) det(B)

where

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -1 & 3 \\ 5 & -4 \end{bmatrix}$$

Sol.

$$\mathbf{AB} = \begin{bmatrix} 2 & 5 \\ 13 & -6 \end{bmatrix}$$

$$\det(\mathbf{A}) = 7,\ \det(\mathbf{B}) = -11,\ \det(\mathbf{AB}) = -77$$

$$\therefore \det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

Determinant Evaluation by LU-Decomposition

LU-decomposition of a nxn square matrix requires approximately $(2/3)n^3$ flops for large n.

Thus, computation of det(A)=det(LU) requires $(2/3)n^3$ flops.

Direct computation of det(A) requires n! flops.

Today's typical PC can calculate 30x30 determinant in less than 1ms by using LU-decomposition and roughly 10^{10} years by direct computation.

Determinant Evaluation by LU Decomposition_Review

If A=LU,
then Det(A) = Det(LU) = Det(L) Det(U)

Theorem 4.1.3 *If A is a triangular matrix , then det(A) is the product of the entries on the main diagonal.*

Example 2 in Section 3.7

$$A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

Det(A) = 6(-5-7)-9(-10-0)+3(-2+0)=-72+90-6=12
Det(A) = (6x2x1)(1x1x1)= 12

Determinant of the Inverse of a Matrix

Theorem 4.2.6 *If A is invertible, then*

$$\det(A^{-1})=1/\det(A) \tag{3}$$

$$\mathbf{A}\mathbf{A}^{-1}=\mathbf{I} \quad \Rightarrow \quad \det\left(\mathbf{A}\mathbf{A}^{-1}\right)=\det\left(\mathbf{I}\right)=1$$

$$\det\left(\mathbf{A}\mathbf{A}^{-1}\right)=\det\left(\mathbf{A}\right)\det\left(\mathbf{A}^{-1}\right)=1$$

$$\det\left(\mathbf{A}^{-1}\right)=\frac{1}{\det\left(\mathbf{A}\right)}$$

Example 7 Determinant of A^{-1}

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Sol.

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\det\left(\mathbf{A}^{-1}\right) = \frac{1}{(ad-bc)^2}\det\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{(ad-bc)^2}(ad-bc) = \frac{1}{ad-bc} = \frac{1}{\det(\mathbf{A})}$$

Example 8 Determinant of A+B

$$\det(\mathbf{A}+\mathbf{B}) \neq \det(\mathbf{A}) + \det(\mathbf{B})$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{A}+\mathbf{B} = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

Sol.

$$\det(\mathbf{A}) = 1, \det(\mathbf{B}) = 8, \det(\mathbf{A}+\mathbf{B}) = 23$$

$$\det(\mathbf{A}+\mathbf{B}) \neq \det(\mathbf{A}) + \det(\mathbf{B})$$

A Unifying Theorem

Theorem 4.2.7 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(*a*) *The reduced row echelon form of A is I_n.*

(*b*) *A is expressible as a product of elementary matrices.*

(*c*) *A is invertible.*

(*d*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every vector* $\mathbf{b}$ *in* R^n.

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every vector* $\mathbf{b}$ *in* R^n.

(*g*) *The column vectors of A are linearly independent.*

(*h*) *The row vectors of A are linearly independent.*

(*i*) $\det(A) \neq 0$.

Lemma 4.2.8 and 4.2.9

Lemma 4.2.8 *Let E be an $n \times n$ elementary matrix and I_n the $n \times n$ identity matrix.*

(a) *If E results by multiplying a row of I_n by k, then* $\det(E) = k$.

(b) *If E results by interchanging two rows of I_n, then* $\det(E) = -1$.

(c) *If E results by adding a multiple of one row of I_n to another, then* $\det(E) = 1$.

Lemma 4.2.9 *If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then*

$$\det(EB) = \det(E)\det(B)$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

$$\det\left(\mathbf{A}'\right) = a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$$

$$\Downarrow$$

Theorem 4.3.1 *If the entries in any row (column) of a square matrix are multiplied by the cofactors of the corresponding entries in a different row (column), then the sum of the products is zero.*

Definition 4.3.2 Adjoint Matrix

Adjoint of a Matrix

Definition 4.3.2 If A is an $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the ***matrix of cofactors*** from A.

The transpose of this matrix is called the ***adjoint***
(or sometimes the ***adjugate***) of A and is denoted by $\mathrm{adj}(A)$.

Example 1 Adjoint matrix

Find the adjoint of A, where A is given by $\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$

$$
\begin{array}{lll}
C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\
C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\
C_{31} = 12 & C_{32} = -10 & C_{33} = 16
\end{array}
$$

$$
\mathbf{C} = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}, \quad adj\left(\mathbf{A}\right) = \mathbf{C}^T = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}
$$

Theorem 4.3.3 Inverse Matrix

If A is invertible, then $\mathbf{A}^{-1} = \dfrac{1}{\det(\mathbf{A})} adj(\mathbf{A})$

Proof

$$\mathbf{A} \cdot adj(\mathbf{A}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & \cdots & C_{j2} & \cdots & C_{jn} \\ \vdots & & \vdots & & \vdots \\ C_{1n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix}$$

$$[\mathbf{A} \cdot adj(\mathbf{A})]_{ij} = a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = \begin{cases} \det(\mathbf{A}), & i = j \\ 0, & i \neq j \end{cases}$$

$$\mathbf{A} \cdot adj(\mathbf{A}) = \begin{bmatrix} \det(\mathbf{A}) & 0 & \cdots & 0 \\ 0 & \det(\mathbf{A}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \det(\mathbf{A}) \end{bmatrix} = \det(\mathbf{A})\mathbf{I}$$

Theorem 4.3.3 Inverse Matrix-conti

$$\mathbf{A}\cdot adj(\mathbf{A})=\begin{bmatrix}\det(\mathbf{A}) & 0 & \cdots & 0\\ 0 & \det(\mathbf{A}) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \det(\mathbf{A})\end{bmatrix}=\det(\mathbf{A})\mathbf{I}$$

$$\mathbf{A}\cdot\frac{adj(\mathbf{A})}{\det(\mathbf{A})}=\mathbf{I}\quad\Rightarrow\quad\mathbf{A}^{-1}=\frac{1}{\det(\mathbf{A})}adj(\mathbf{A})$$

Example 2 Inverse by the Adjoint

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{A}^{-1} = ?$$

$$\det(\mathbf{A}) = 64$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} adj(\mathbf{A}) = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Cramer's Rule

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} a_{11} & a_{1`2} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \quad \Rightarrow \quad \text{Cramer's rule} \qquad (4)$$

$$\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} : \begin{array}{l} \text{determinant of matrix} \\ \text{changed coefficients of } x \text{ into } \mathbf{b} \end{array}$$

$$\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} : \begin{array}{l} \text{determinant of matrix} \\ \text{changed coefficients of } y \text{ into } \mathbf{b} \end{array}$$

Example 4 Solution by Cramer's Rule

Use Cramer's rule to solve the system

$$2x - 6y = 1$$
$$3x - 4y = 5$$

Sol.

$$x = \frac{\begin{vmatrix} 1 & -6 \\ 5 & -4 \end{vmatrix}}{\begin{vmatrix} 2 & -6 \\ 3 & -4 \end{vmatrix}} = \frac{26}{10}, \quad y = \frac{\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & -6 \\ 3 & -4 \end{vmatrix}} = \frac{7}{10}$$

Example 5 Solution by Cramer's Rule

Use Cramer's rule to solve the system for x and y in terms of x' and y'.

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

Sol.

$$(\cos\theta)x + (\sin\theta)\,y = x'$$
$$(-\sin\theta)x + (\cos\theta)\,y = y'$$

By Cramer's Rule:

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{\begin{vmatrix} x' & \sin\theta \\ y' & \cos\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}} = x'\cos\theta - y'\sin\theta$$

$$y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{\begin{vmatrix} \cos\theta & x' \\ -\sin\theta & y' \end{vmatrix}}{\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}} = x'\sin\theta + y'\cos\theta$$

Theorem 4.3.4 Cramer's Rule

Theorem 4.3.4 (***Cramer's Rule***) *If* $A\mathbf{x} = \mathbf{b}$ *is a linear system of* n *equations in* n *unknowns, then the system has a unique solution if and only if* $\det(A) \neq 0$, *in which case the solution is*

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j *is the matrix that results when the* jth *column of* A *is replaced by* $\mathbf{b}$.

Example 6 Solution by Cramer's Rule

Solve the system

$$\begin{aligned} x_1 + \quad\quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Sol.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

$$x_1 = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})} = \frac{-40}{44}$$

$$x_2 = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})} = \frac{72}{44}$$

$$x_3 = \frac{\det(\mathbf{A}_3)}{\det(\mathbf{A})} = \frac{152}{44}$$

Geometric Interpretation of Determinants

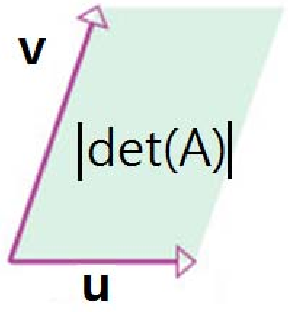
Theorem 4.3.5

(*a*) *If A is a* 2×2 *matrix, then* $|\det(A)|$ *represents the area of the parallelogram determined by the two column vectors of A when they are positioned so their initial points coincide.*

(*b*) *If A is a* 3×3 *matrix, then* $|\det(A)|$ *represents the volume of the parallelepiped determined by the three column vectors of A when they are positioned so their initial points coincide.*

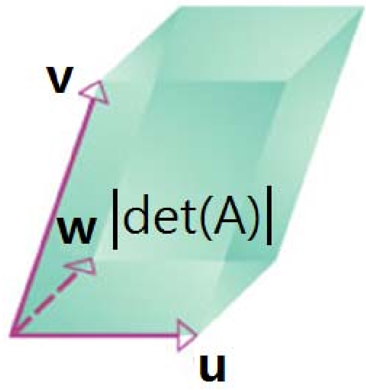
(a)

$A = [\,\mathbf{u}\ \ \mathbf{v}\,]$



(b)

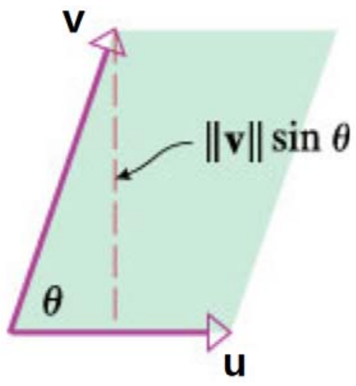
$A = [\,\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}\,]$



Geometric Interpretation of Determinants

Proof

(a)



$$(\text{Area})^2 = (\|\mathbf{u}\|\ \|\mathbf{v}\| \sin\theta)^2 = \|\mathbf{u}\|^2\ \|\mathbf{v}\|^2\ (1-\cos^2\theta)$$

$$= \|\mathbf{u}\|^2\ \|\mathbf{v}\|^2 - (\|\mathbf{u}\|^2\ \|\mathbf{v}\| \cos\theta)^2 \qquad (6)$$

$$= \|\mathbf{u}\|^2\ \|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2$$

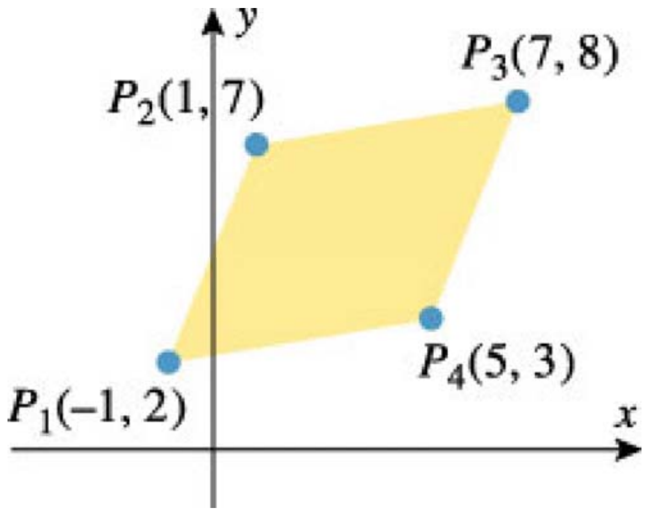
$$= (x_1^2 + y_1^2)(x_2^2 + y_2^2) - (x_1x_2 + y_1y_2)^2$$

$$= (x_1^2y_2^2 + x_2^2y_1^2) - 2(x_1x_2)(y_1y_2)$$

$$= (x_1y_2 - x_2y_1)^2 = [\det(A)]^2$$

Example 7 Area of the Parallelogram

Find the area of the parallelogram with vertices $P_1(-1,2)$, $P_2(1,7)$, $P_3(7,8)$, and $P_4(5,3)$.



Sol.

$$\overrightarrow{P_1P_2} = (2,\ 5)$$

$$\overrightarrow{P_1P_4} = (6,\ 1)$$

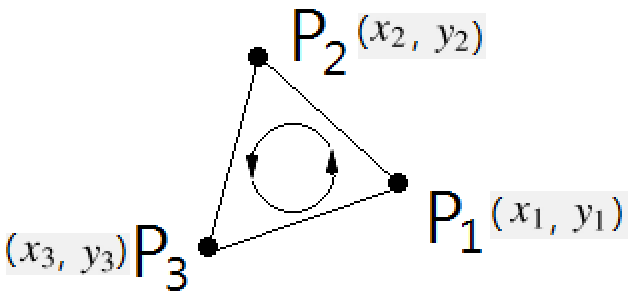
$$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$$

$$\text{Area of the parallelogram} = \pm \begin{vmatrix} 2 & 6 \\ 5 & 1 \end{vmatrix} = \pm(-28) = 28$$

Theorem 4.3.6

Theorem 4.3.6 *Suppose that a triangle in the xy-plane has vertices $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ and that the labeling is such that the triangle is traversed counterclockwise from P_1 to P_2 to P_3. Then the area of the triangle is given by*

$$\text{area } \triangle P_1P_2P_3 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \tag{7}$$

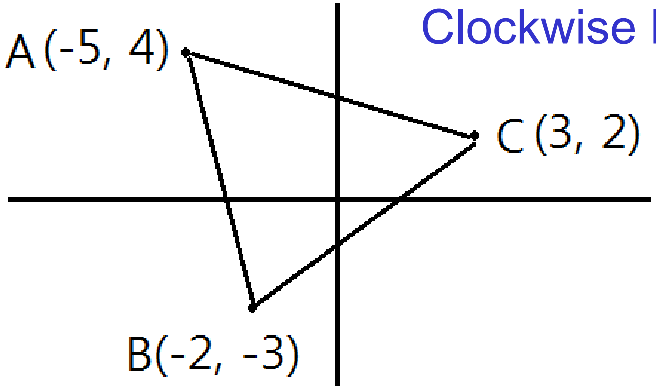


$$\text{area } \triangle P_1P_2P_3 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Example 8 Area of a Triangle Using Determinants

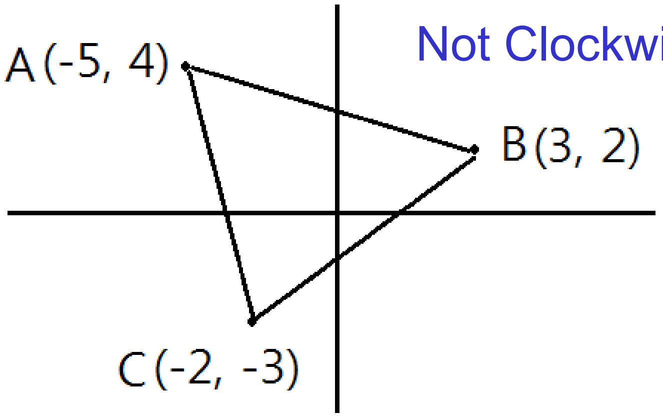
Example 8 Area of the triangle with vertices A(-5,4), B(3,2), and C(-2,-3).

Sol.



Clockwise Numbering

$$\text{Area}\,\triangle ABC = \frac{1}{2}\begin{vmatrix} -5 & 4 & 1 \\ -2 & -3 & 1 \\ 3 & 2 & 1 \end{vmatrix} = \frac{1}{2}(50) = 25$$



Not Clockwise Numbering

$$\text{Area}\,\triangle ABC = \pm\frac{1}{2}\begin{vmatrix} -5 & 4 & 1 \\ 3 & 2 & 1 \\ -2 & -3 & 1 \end{vmatrix} = \pm\frac{1}{2}(-50) = 25$$

Polynomial Interpolation and the Vandermonde Determinant

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$\begin{array}{ccccccccc} a_0 & + & a_1 x_1 & + & a_2 x_1^2 & + \cdots + & a_{n-1} x_1^{n-1} & = & y_1 \\ a_0 & + & a_1 x_2 & + & a_2 x_2^2 & + \cdots + & a_{n-1} x_2^{n-1} & = & y_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_0 & + & a_1 x_n & + & a_2 x_n^2 & + \cdots + & a_{n-1} x_n^{n-1} & = & y_n \end{array}$$

The linear system has a solution if and only if the Vandermonde determinant is not zero.

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} \neq 0$$

Vandermonde Determinant

The Vandermonde determinant for n=3:

$$
\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 \end{vmatrix}
$$

$$
= (x_2 - x_1)(x_3 - x_1) \begin{vmatrix} 1 & x_1 & x_1^2 \\ 0 & 1 & x_2 + x_1 \\ 0 & 1 & x_3 + x_1 \end{vmatrix}
$$

$$
= (x_2 - x_1)(x_3 - x_1) \begin{vmatrix} 1 & x_2 + x_1 \\ 1 & x_3 + x_1 \end{vmatrix}
$$

$$
= (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)
$$

Vandermonde Determinant-conti

The Vandermonde determinant for n:

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i) \tag{9}$$

Cross Products

Definition 4.3.7 If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in R^3, then the ***cross product of*** $\mathbf{u}$ ***with*** $\mathbf{v}$, denoted by $\mathbf{u} \times \mathbf{v}$, is the vector in R^3 defined by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \tag{10}$$

or equivalently,

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \tag{11}$$

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \end{aligned} \tag{12}$$

Example 9 Calculating a Cross Product

Find **uxv, vxu,** and **uxu** where **u**=(1,2,-2), **v**=(3,0,1)

Sol.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \mathbf{k}$$

$$= 2\mathbf{i} - 7\mathbf{j} - 6\mathbf{k} = (2, -7, -6)$$

$$\mathbf{v} \times \mathbf{u} = -\,\mathbf{u} \times \mathbf{v} = (-2, 7, 6)$$

$$\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 1 & 2 & -2 \end{vmatrix} = (0, 0, 0)$$

Theorem 4.3.8

Theorem 4.3.8 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^3 *and* k *is a scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

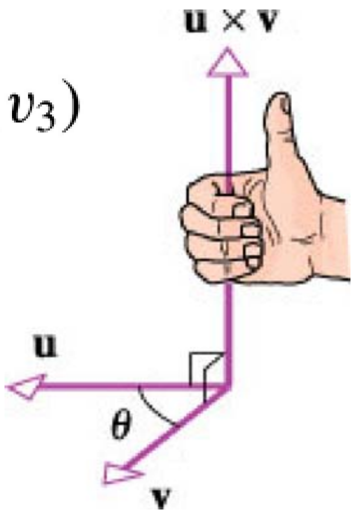
Theorem 4.3.9

Theorem 4.3.9 *If* **u** *and* **v** *are vectors in* R^3*, then:*

(*a*) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ **[u x v is orthogonal to u]**

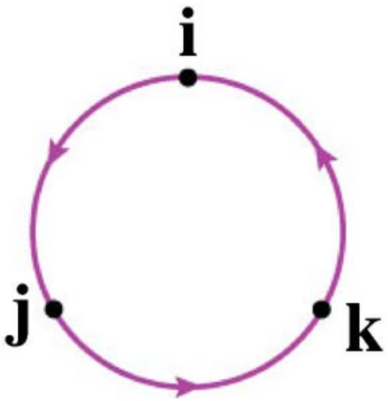
(*b*) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ **[u x v is orthogonal to v]**

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0$$



Cross Product of the standard Unit Vectors

$$\mathbf{i} \times \mathbf{j} = (1, 0, 0) \times (0, 1, 0)$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} = \mathbf{k}$$



$$\begin{array}{ll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j} & \mathbf{j} \times \mathbf{i} = -\mathbf{k} \\ \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array} \tag{13}$$

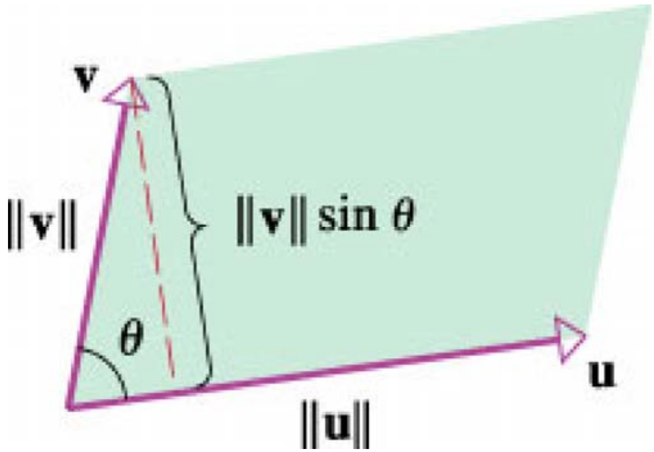
Theorem 4.3.10

Theorem 4.3.10 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be nonzero vectors in* R^3, *and let* θ *be the angle between these vectors.*

(*a*) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$

(*b*) *The area* A *of the parallelogram that has* $\mathbf{u}$ *and* $\mathbf{v}$ *as adjacent sides is*

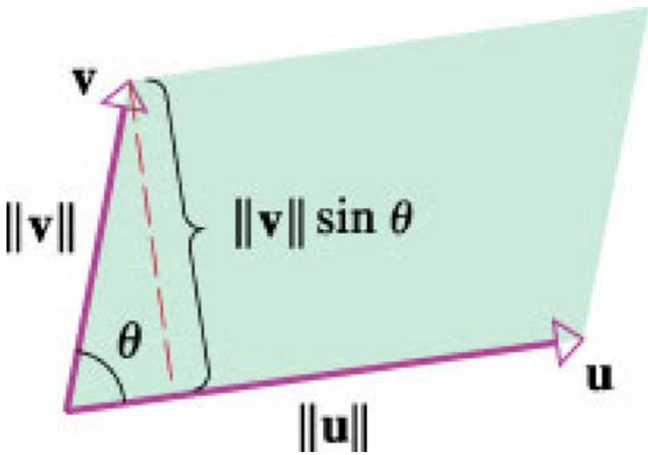
$$A = \|\mathbf{u} \times \mathbf{v}\| \tag{14}$$



Theorem 4.3.10-proof

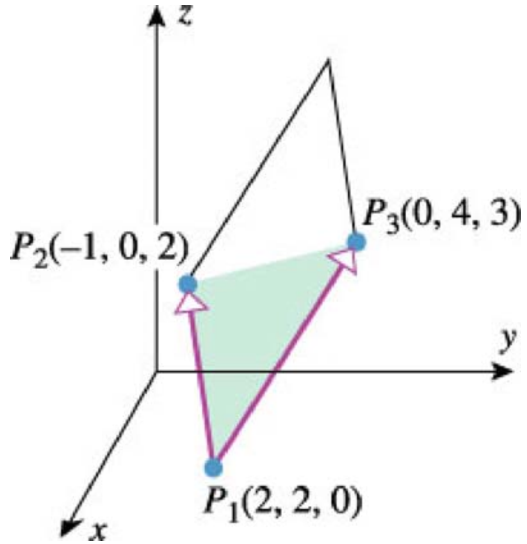
(a)
$$\begin{aligned}
\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta &= \|\mathbf{u}\|\,\|\mathbf{v}\|\sqrt{1-\cos^2\theta} \\
&= \|\mathbf{u}\|\,\|\mathbf{v}\|\sqrt{1-\frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}} \qquad \text{[by Theorem 1.2.8]} \\
&= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2} \\
&= \sqrt{\left(u_1^2+u_2^2+u_3^2\right)\left(v_1^2+v_2^2+v_3^2\right)-(u_1v_1+u_2v_2+u_3v_3)^2} \\
&= \sqrt{(u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2} \\
&= \|\mathbf{u}\times\mathbf{v}\| \qquad \text{[by (10)]}
\end{aligned}$$

(b)



Example 10 Area of a Triangle in 3-Space

Find the area of the triangle with vertices $P_1(2,2,0)$, $P_2(-1,0,2)$, $P_3(0,4,3)$



Sol.

$$A = \frac{1}{2}\left\| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \right\|$$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & 2 \\ -2 & 2 & 3 \end{vmatrix}$$

$$= -10\mathbf{i} + 5\mathbf{j} - 10\mathbf{k}$$

$$= (-10, 5, -10)$$

$$A = \frac{1}{2}\left\| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \right\| = \frac{1}{2}\sqrt{225} = \frac{15}{2}$$

4.4 Eigenvalues and Eigenvectors

Fixed point: $\mathbf{Ax} = \mathbf{x}$ $\quad (\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

Trivial Fixed Point: $\mathbf{x} = \mathbf{0}$

Nontrivial Fixed Point: Points $\mathbf{x} \neq \mathbf{0}$ satisfying $\mathbf{Ax} = \mathbf{x}$

Theorem 4.4.1 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(*a*) *A has nontrivial fixed points.*

(*b*) *$I - A$ is singular.*

(*c*) $\det(I - A) = 0$.

Example 1. Fixed Points

Determine whether the matrix has nontrivial fixed points; and if so. graph the subspace of fixed points in an xy-coordinate system.

$$\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$$

Sol.

(a) $\det(\mathbf{I}-\mathbf{A}) = \begin{vmatrix} -2 & -6 \\ -1 & -1 \end{vmatrix} = -4 \neq 0$ Thus, A has only a trivial fixed point.

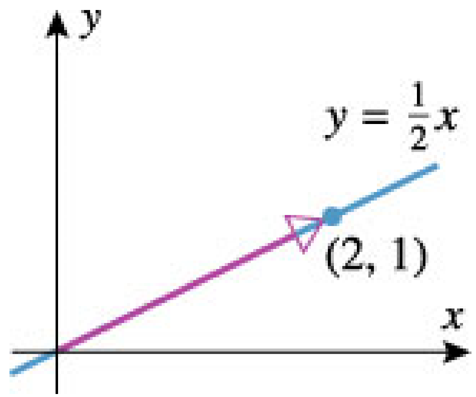
$\det(\mathbf{I}-\mathbf{B}) = \begin{vmatrix} 1 & -2 \\ 0 & 0 \end{vmatrix} = 0$ Thus, A has only nontrivial fixed points.

(b) Assume fixed points: $\mathbf{x} = (x, y)$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $x = 2t, \quad y = t$

$$y = \frac{1}{2}x$$



Example 1. Fixed Points

$$x = 2t, \quad y = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

As a check,

$$\mathbf{Ax} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = \mathbf{x}$$

Eigenvalues and Eigenvectors

λ: Eigenvalue

$\mathbf{x}$: Eigenvector for the eigenvalue λ

$$\mathbf{Ax} = \lambda\mathbf{x}$$

Problem 4.4.2 If A is an $n \times n$ matrix, for what values of the scalar λ, if any, are there nonzero vectors in R^n such that $A\mathbf{x} = \lambda\mathbf{x}$?

nonzero vectors in R^n such that $A\mathbf{x} = \lambda\mathbf{x}$?

Definition: Eigenvalues and Eigenvectors

Definition 4.4.3 If A is an $n \times n$ matrix, then a scalar λ is called an ***eigenvalue*** of A if there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. If λ is an eigenvalue of A, then every nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ is called an ***eigenvector*** of A corresponding to λ.

$$\mathbf{Ax} = \lambda\mathbf{x} \qquad\qquad (\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$$

For nontrivial solution,

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0$$: Characteristic equation

A nonzero solution space of $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ is called the ***eigenspace*** of A corresponding to λ.

Theorem 4.4.4

Theorem 4.4.4 *If A is an $n \times n$ matrix and λ is a scalar, then the following statements are equivalent.*

(*a*) λ *is an eigenvalue of A.*

(*b*) λ *is a solution of the equation* $\det(\lambda I - A) = 0$.

(*c*) *The linear system* $(\lambda I - A)\mathbf{x} = \mathbf{0}$ *has nontrivial solutions.*

Example 2 Eigenvalues

(a) Find the eigenvalues and the corresponding eigenvectors.
(b) Graph the eigenvalues in the xy-plane.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

Sol.

(a) eigenvalues and eigenvectors.

$$\lambda\mathbf{I} - \mathbf{A} = \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{bmatrix}$$

$$\text{Det}(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{vmatrix} = \lambda^2 - 3\lambda - 10$$
$$= (\lambda + 2)(\lambda - 5) = 0$$

$$\lambda = -2, \quad \lambda = 5$$

Example 2 Eigenvalues-conti

corresponding eigenvectors:

$$\lambda = -2: \quad (\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \begin{bmatrix} -3 & -3 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = -t, \quad y = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

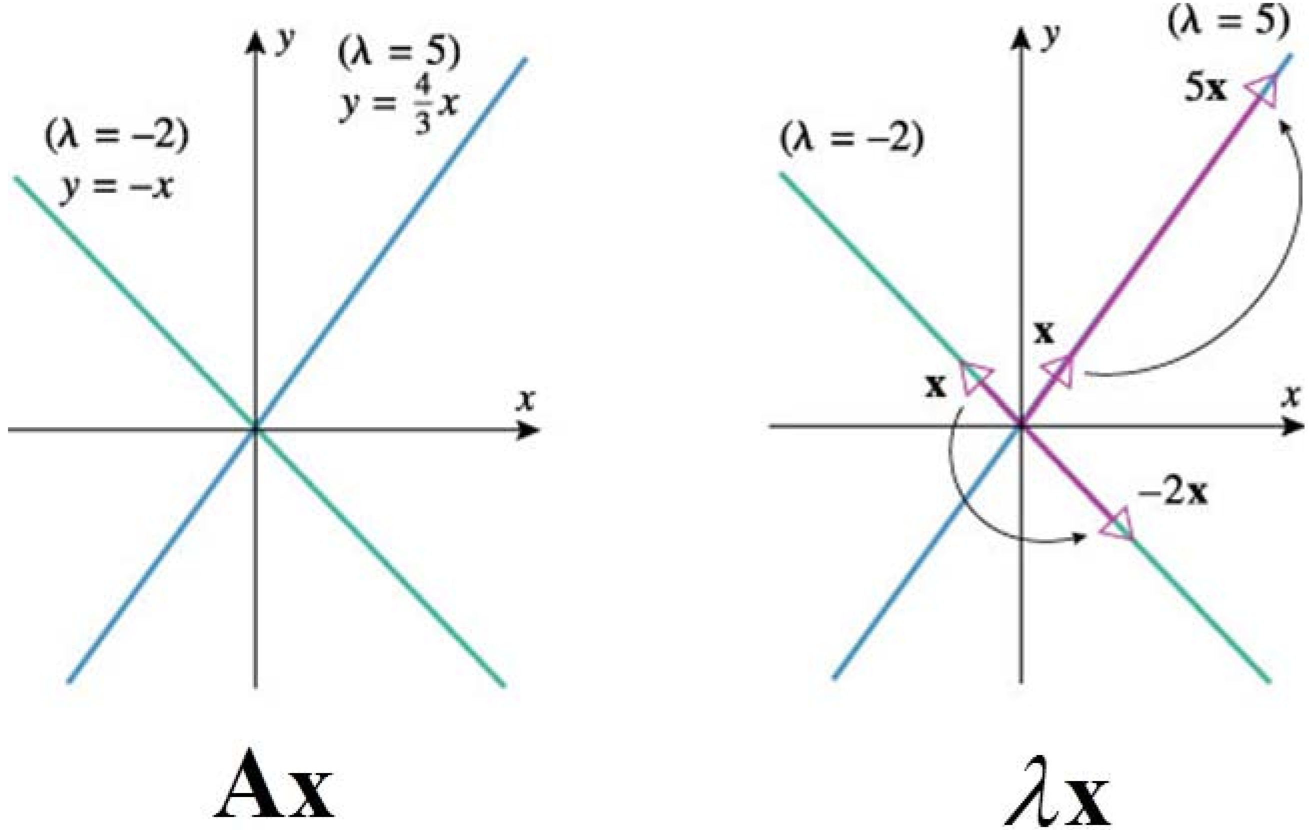
$$\lambda = 5: \quad (\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = 3/4, \quad y = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (3/4)t \\ t \end{bmatrix} = t \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$$

Example 2 Eigenvalues-conti

(b) Graph the eigenvalues in the xy-plane.



Example 3 Eigenvalues

Find the eigenvalues of the matrix. $\mathbf{A} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -4 & -17 & 8 \end{bmatrix}$

Sol.

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda & 1 & 0 \\ 0 & \lambda & -1 \\ 4 & 17 & \lambda - 8 \end{vmatrix} = \lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0 \qquad (13)$$

$$\lambda = 4, \quad \lambda = 2 + \sqrt{3}, \quad \lambda = 2 - \sqrt{3}$$

Eigenvalues of the Triangular Matrices

Let A be a triangular matrix.

Then

$$\det(\lambda \mathbf{I} - \mathbf{A}) = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn})$$

Thus, eigenvalues are

$$\lambda_1 = a_{11},\ \lambda_2 = a_{22}, \cdots,\ \lambda_n = a_{nn}$$

Theorem 4.4.5 *If A is a triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.*

Example 4 Eigenvalues of Triangular Matrices

Find the eigenvalues of the matrix.

$$\mathbf{A} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ -1 & -2/3 & 0 & 0 \\ 7 & 5/8 & 6 & 0 \\ 4/9 & -4 & 3 & 6 \end{bmatrix}$$

Sol.

By inspection the eigenvalues are

$$p(\lambda) = \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{2}{3}\right)(\lambda - 6)^2$$

$$\lambda = \frac{1}{2}, \quad \lambda = -\frac{2}{3}, \quad \lambda = 6$$

Eigenvalues of Powers of a Matrix

Theorem 4.4.6 *If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is a corresponding eigenvector, and if k is any positive integer, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.*

$$\mathbf{Ax} = \lambda\mathbf{x}$$

$$\mathbf{A}^2\mathbf{x} = \mathbf{A}(\mathbf{Ax}) = \mathbf{A}(\lambda\mathbf{x}) = \lambda(\mathbf{Ax}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$$

$$\mathbf{A}^2\mathbf{x} = \lambda^2\mathbf{x}$$

Thus, λ^2 is an eigenvalue for the matrix $\mathbf{A}^2$.

Similarly, λ^k is an eigenvalue for the matrix $\mathbf{A}^k$.

A Unifying Theorem

Theorem 4.4.7 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(a) *The reduced row echelon form of A is I_n.*

(b) *A is expressible as a product of elementary matrices.*

(c) *A is invertible.*

(d) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(e) $A\mathbf{x} = \mathbf{b}$ *is consistent for every vector* $\mathbf{b}$ *in R^n.*

(f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every vector* $\mathbf{b}$ *in R^n.*

(g) *The column vectors of A are linearly independent.*

(h) *The row vectors of A are linearly independent.*

(i) $\det(A) \neq 0$.

(j) $\lambda = 0$ *is not an eigenvalue of A.*

Complex Eigenvalues

$$\mathbf{A} = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$$

$$\begin{vmatrix} \lambda + 2 & 1 \\ -5 & \lambda - 2 \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\lambda = \pm j$$

Algebraic Multiplicity(대수적 중복도)

$$\det(\lambda\mathbf{I}-\mathbf{A}) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n$$
$$= (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k}$$

m_i : algebraic multiplicity of the eigenvalue λ_i

Theorem 4.4.8 *If A is an $n \times n$ matrix, then the characteristic polynomial of A can be expressed as*

$$\det(\lambda I - A) = (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of A and $m_1 + m_2 + \cdots + m_k = n$.

Eigenvalue Analysis of 2x2 Matrices

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{aligned} \det(\lambda \mathbf{I} - \mathbf{A}) &= \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} \\ &= (\lambda - a)(\lambda - d) - bc \\ &= \lambda^2 - (a+b)\lambda + (ad - bc) \end{aligned}$$

Characteristic equation: $\lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$

Theorem 4.4.9

Theorem 4.4.9 *If A is a 2×2 matrix with real entries, then the characteristic equation of A is*

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$$

and

(*a*) *A has two distinct real eigenvalues if* $\operatorname{tr}(A)^2 - 4\det(A) > 0$;

(*b*) *A has one repeated real eigenvalue if* $\operatorname{tr}(A)^2 - 4\det(A) = 0$;

(*c*) *A has two conjugate imaginary eigenvalues if* $\operatorname{tr}(A)^2 - 4\det(A) < 0$.

$$ax^2 + bx + c = 0$$

$b^2 - 4ac > 0$ **[Two distinct real roots]**

$b^2 - 4ac = 0$ **[One repeated real root]**

$b^2 - 4ac < 0$ **[Two conjugate imaginary roots]**

Example 5 Eigenvalues of a 2x2 Matrix

Find the eigenvalues for the matrices.

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 5 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$$

Sol.

Ch. eq:

$$\lambda^2 - 7\lambda + 12 = 0$$

$$\lambda = 3,\ \lambda = 4$$

Ch. eq:

$$\lambda^2 - 2\lambda + 1 = 0$$

$$\lambda = 1$$

Ch. eq:

$$\lambda^2 - 4\lambda + 13 = 0$$

$$\lambda = 2 \pm 3j$$

Theorem 4.4.10

Theorem 4.4.10 *A symmetric 2×2 matrix with real entries has real eigenvalues. Moreover, if A is of the form*

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \tag{23}$$

then A has one repeated eigenvalue, namely $\lambda = a$; otherwise it has two distinct eigenvalues.

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

Ch. eq.: $\lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$

$$\begin{aligned} D &= \mathrm{tr}(A)^2 - 4\det(A) \\ &= (a+d)^2 - 4(ad-b^2) \\ &= (a-d)^2 + 4b^2 \geq 0 \end{aligned}$$

Thus, real eigenvalues.

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \qquad D = (a-d)^2 + 4b^2 = 0$$

Thus, one repeated eigenvalue.

Theorem 4.4.11

Theorem 4.4.11

(*a*) *If a* 2×2 *symmetric matrix with real entries has one repeated eigenvalue, then the eigenspace corresponding to that eigenvalue is* R^2.

(*b*) *If a* 2×2 *symmetric matrix with real entries has two distinct eigenvalues, then the eigenspaces corresponding to those eigenvalues are perpendicular lines through the origin of* R^2.

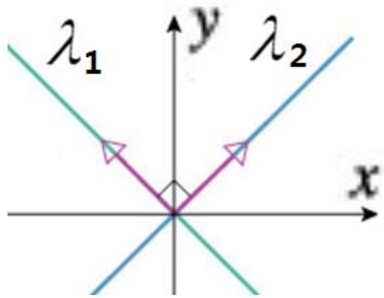
(a) One repeated eigenvalue

Symmetric Matrix

Eigenspace is R^2.

(b) Distinct eigenvalues

Symmetric Matrix



Example 6 Eigenvalues of a Symmetric 2x2 Matrix

Graph the eigenspaces of the symmetric matrix. $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

Sol.

Ch. eq.: $\lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5) = 0 \Rightarrow \lambda = 1, 5$

Case $\lambda = 1$: $(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x = -t, \ y = t \Rightarrow y = -x$

Case $\lambda = 5$: $(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x = t, \ y = t \Rightarrow y = x$

Example 6 Eigenvalues of a Symmetric 2x2 Matrix

In vector form,

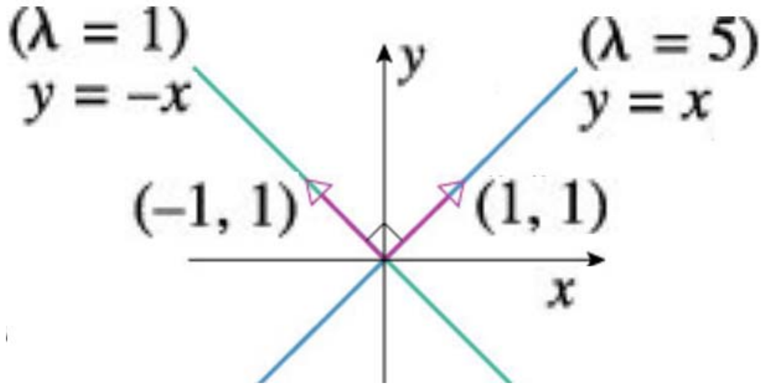
$$\lambda = 5: \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 1: \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Spanning vectors:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Theorem 4.4.12

Theorem 4.4.12 *If A is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (repeated according to multiplicity), then:*

(a) $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

(b) $\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

(a) $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

$$p(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \tag{28}$$

$$\Big\downarrow \ \lambda = 0$$

$$\det(-\mathbf{A}) = (-1)^n (\lambda_1 \lambda_2 \cdots \lambda_n)$$

$$\det(\mathbf{A}) = \lambda_1 \lambda_2 \cdots \lambda_n \tag{29}$$

Theorem 4.4.12 –cont.

$$(b)\quad \operatorname{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

$$\begin{aligned} p(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}) &= (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) \\ &= \lambda^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)\lambda^{n-1} + \cdots \end{aligned} \tag{28}$$

$$p(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots\vdots\vdots & \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix} \tag{30}$$

$$= (\lambda - a_{11}) \begin{vmatrix} \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots\vdots\vdots & \\ -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix} + \underbrace{a_{21} \begin{vmatrix} -a_{12} & \cdots & -a_{1n} \\ -a_{32} & \cdots & -a_{2n} \\ \vdots & \vdots\vdots\vdots & \\ -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix} + \cdots}_{\text{Highest order} \leq n-2}$$

Theorem 4.4.12 –cont.

Thus, the coefficient of λ^{n-1} in $p(\lambda)$is the same as the coefficient of λ^{n-1} in the product

$$(\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn})$$
$$= \lambda^n - (a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + \cdots \tag{31}$$

Comparison of equations (28) and (31) gives:

$$\mathrm{tr}(\mathbf{A}) = a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n \qquad \blacksquare$$

Example 7

Find the determinant and trace of a 3x3 matrix whose characteristic polynomial is $p(\lambda) = \lambda^3 - 3\lambda + 2$

Sol.

$$p(\lambda) = \lambda^3 - 3\lambda + 2 = (\lambda - 1)^2(\lambda + 2)$$

$$\det(\mathbf{A}) = \lambda_1\lambda_2\lambda_3 = (-1)^2(2) = -2$$

$$\mathrm{tr}(\mathbf{A}) = \lambda_1 + \lambda_2 + \lambda_3 = 0$$

Eigenvalues by Numerical Methods

Eigenvalues are rarely obtained by solving the characteristic equation primarily for two reasons:

1. Requires too much time to compute Det(λI-A) in typical applications
2. No algebraic formula or finite algorithm to obtain the exact solutions for a general nxn matrix when $n \geq 5$.