11.4 Approximation by Trigonometric Polynomials

Approximation theory

An area concerned with approximating functions by other functions-usually simpler functions

Let $f(x)$ be a function on [-π π] that can be represented by a Fourier series. Then the Nth partial sum of the Fourier series is an approximation of $f(x)$.

$$(1) \quad f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

Question: Is (1) the best approximation?

Consider an approximation function, $F(x)$, defined by

$$(2) \quad F(x) \approx A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx)$$

Define an error, E, by

$$(3) \quad E = \int_{-\pi}^{\pi} (f - F)^2 dx$$

E: square error of F relative to the function f on the interval [-π π]

Determine F to minimize the error E.
Determine A_0, A_n, and B_n.

$$(3) \quad E = \int_{-\pi}^{\pi} (f - F)^2 dx = \int_{-\pi}^{\pi} (f^2 - 2fF + F^2) dx$$

$$(4) \quad E = \int_{-\pi}^{\pi} f^2 dx - 2\int_{-\pi}^{\pi} fF dx + \int_{-\pi}^{\pi} F^2 dx$$

$$\begin{aligned}\int_{-\pi}^{\pi} fF dx &= \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)\right] \cdot \\ &\quad \left[A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx)\right] dx \\ &= 2\pi a_0 A_0 + \pi(a_1 A_1 + \cdots + a_n A_n) \\ &\quad + \pi(b_1 B_1 + b_2 B_2 + \cdots + b_n B_n)\end{aligned}$$

$$\int_{-\pi}^{\pi} F^2 dx = \int_{-\pi}^{\pi} \left[A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \right]^2 dx$$
$$= 2\pi A_0^2 + \pi (A_1^2 + A_2^2 \cdots + A_n^2 + B_1^2 + B_2^2 + \cdots + B_n^2)$$

$$(5) \quad \therefore E = \int_{-\pi}^{\pi} f^2 dx - 2\pi \left[2A_o a_0 + \sum_{n=1}^{N} (A_n a_n + B_n b_n) \right]$$
$$+ \pi \left[2A_0^2 + \sum_{n=1}^{N} (A_n^2 + B_n^2) \right]$$

Let E^* be the value of E when $A_n = a_n$ and $B_n = b_n$.

Let E^* be the value of E when $A_n = a_n$ and $B_n = b_n$. Then,

$$
\begin{aligned}
(6) \quad E^* &= \int_{-\pi}^{\pi} f^2 dx - 2\pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^{2)}\right] \\
&\quad + \pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right] \\
&= \int_{-\pi}^{\pi} f^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right]
\end{aligned}
$$

$$(5)\quad E=\int_{-\pi}^{\pi} f^2 dx-2\pi\left[2A_o a_0+\sum_{n=1}^{N}(A_n a_n+B_n b_n)\right]$$
$$+\pi\left[2A_0^2+\sum_{n=1}^{N}(A_n^2+B_n^2)\right]$$

$$(6)\quad E^*=\int_{-\pi}^{\pi} f^2 dx-\pi\left[2a_0^2+\sum_{n=1}^{N}(a_n^2+b_n^2)\right]$$

$$E-E^*=2\pi[-2A_0a_0+A_0^2+a_0^2]$$
$$+\pi\sum_{n=1}^{N}[-2(A_na_n+B_nb_n)+(A_n^2+B_n^2)+(a_n^2+b_n^2)]$$
$$=2\pi(A_0-a_0)^2+\sum_{n=1}^{N}[(A_n-a_n)^2+(B_n-b_n)^2]\geq 0$$

$$\therefore E\geq E^*$$

THEOREM 1 Minimum Square Error

The square error of F in (2) (with fixed N) relative to f on the interval -π≤x≤π is minimum if and only if the coefficients of F in (2) are the Fourier coefficients of f. This minimum value E is given by (6).*

$$(2) \quad F(x) \approx A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx)$$

$$(6) \quad E^* = \int_{-\pi}^{\pi} f^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right]$$

$$(6) \quad E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right] \geqq 0$$

$$(7) \quad 2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2) \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 dx$$

Bessell's identity

Ref. Appendix [C12]

$$(8) \quad 2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 dx$$

Parseval's identity

11.5 Sturm-Liouville Problems. Orthogonal Functions

Is a function can be approximated by sets of other orthogonal functions than a trigonometric system?

Yes! For example, by Legendre polunomials, Bessel functions, etc

Generalized Fourier series

Sturm-Liouville Problem

$$(1) \quad [p(x)y']' + [q(x) + \lambda r(x)]y = 0$$

$$(2) \quad \begin{array}{l} (a)\ \ k_1 y(a) + k_2 y'(a) = 0 \\ (b)\ \ l_1 y(b) + l_2 y_2{}'(b) = 0 \end{array}$$

Sturm-Liouville Equation: (1)
Sturm-Liouville Problem: (1)(2)

Eigenfunction: a solution of (1) satisfying (2)
Eigenvalue: a number which an eigenfunction exists

EX. 1 Trigonometric Functions as Eigenfunctions. Vibrating String

Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$(3) \quad y'' + \lambda y = 0, \;\; y(0) = y(\pi) = 0$$

Sol.

$$(1) \quad [p(x)y']' + [q(x) + \lambda r(x)]y = 0$$

(1) and (3): $p = 1, \;\; q = 0, \;\; r = 1$
(2) and (3): $a = 0, \;\; b = \pi, \;\; k_1 = l_1 = 1, \;\; k_2 = l_2 = 0$

Characteristic eq: $\nu^2 + \lambda = 0$

$$(1)\ \ Negative\ \ \lambda:\ \ \nu^2 = -\lambda$$
$$y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x}$$
$$y(0) = c_1 + c_2 = 0,\ \ y(\pi) = c_1 e^{\pi} + c_2 e^{-\pi} = 0$$
$$y(x) = 0e^{\nu x} + 02e^{-\nu x} = 0(trivial\ \ sol)$$

$$(2)\ \ \lambda = 0:$$
$$y(x) = c_1 x + c_2 = 0(trivial\ \ sol)$$

$$(3)\ \ Positive\ \ \lambda:\ \ \nu^2 = -\lambda:$$
$$y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x} = A\cos\nu x + B\sin\nu x$$
$$y(0) = A = 0$$
$$y(\pi) = B\sin\nu\pi = 0,\ \ \nu = 0,\ \ 1,\ \ 2,\cdots$$

(3) *Positive* λ: $\nu^2 = -\lambda$:

$$y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x} = A\cos\nu x + B\sin\nu x$$
$$y(0) = A = 0$$
$$y(\pi) = B\sin\nu\pi = 0,\ \ \nu = 0,\ 1,\ 2,\cdots$$

$$y(x) = B\sin\nu x,\ \ \nu = 0,\ 1,\ 2,\cdots$$

Therefore,

Eigenvalue: $\lambda = \nu^2 (\nu = 1,2,\cdots)$

Eigenfunction: $y = \sin\nu x (\nu = 1,2,\cdots)$

Orthogonal Functions

Functions $y_1(x)$, $y_2(x)$, … defined on some interval $a \leqq x \leqq b$ are called **orthogonal** on this interval with respect to the **weight function** $r(x)>0$ if for all m and n different from m,

$$(4) \quad (y_m,\ y_n) = \int_a^b r(x) y_m(x) y_n(x) dx = 0 (m \neq n)$$

The Norm $\|y_m\|$ of y_m is defined by

$$(5) \quad \|y_m\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x) y_m^2(x) dx}$$

Normal/Orthonormal Functions

y_m is a normal function if and only if $\|y_m\| = 1$.

Functions $y_1(x)$, $y_2(x)$, …are orthonormal(정규직교) if and only if

$$(y_m,\ y_n) = \int_a^b r(x) y_m(x) y_n(x) dx$$

$$= \delta_{mn} = \begin{cases} 0 & (m \neq n) \\ 1 & (m = n) \end{cases}$$

Kronecker symbol, Kronecker delta function

Orthogonal Functions when r(x)=1

If the **weight function** r(x)=1, the term orthogonal is more briefly used than orthogonal with respect to r(x)=1.

$$(y_m,\ y_n) = \int_a^b y_m(x)\, y_n(x)\, dx = 0 \quad (m \neq n)$$

$$\|y_m\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b y_m^2(x)\, dx}$$

EX. 2 Orthogonal Functions, Orthonormal Functions, Notation

Show that the functions $y_m(x)$ = sin mx, m=1, 2, … form an orthogonal set on the interval -π≦x ≦π.

Sol.

$$(y_m,\ y_n) = \int_{-\pi}^{\pi} \sin mx \sin nx\, dx$$

$$= -\frac{1}{2}\int_{-\pi}^{\pi} [\sin(m+n)x - \sin(m-n)x]dx$$

$$= 0 \quad (m \neq n)$$

$$\|y_m\|^2 = (y_m,\ y_m) = \int_{-\pi}^{\pi} \sin^2 mx\,dx$$

$$= \frac{1}{2}\int_{\pi}^{\pi} (1 - \cos 2x)\,dx = \pi$$

$$\|y_m\| = \sqrt{\pi} \quad \Longrightarrow \quad \left\| \frac{y_m}{\sqrt{\pi}} \right\| = 1$$

The corresponding orthonormal set:

$$\frac{\sin x}{\sqrt{\pi}},\ \frac{\sin 2x}{\sqrt{\pi}},\ \frac{\sin 3x}{\sqrt{\pi}},\ \cdots$$

Sturm-Liouville Problem in Sec. 11.5-Revisited

$$(1) \quad [p(x)y']' + [q(x) + \lambda r(x)]y = 0$$

$$(2) \quad \begin{array}{l} (a)\ k_1 y(a) + k_2 y'(a) = 0 \\ (b)\ l_1 y(b) + l_2 {y_2}'(b) = 0 \end{array}$$

Sturm-Liouville Equation: (1)
Sturm-Liouville Problem: (1)(2)

Eigenfunction: a solution of (1) satisfying (2)
Eigenvalue λ: a number which an eigenfunction exists

Theorem1 Orthogonality of Eigenfunctions of Sturm-Liouville Problem

Suppose that the functions p, q, r, and p' in the *Sturm-Liouville equation* (1) are real valued and continuous and $r(x)>0$ on the interval $a \leqq x \leqq b$. Let $y_m(x)$ and $y_n(x)$ be eigenfunctions of the *Sturm-Liouville problem* (1), (2) that correspond to different eigenvalues λ_m and λ_n, respectively. Then y_m, y_n are orthogonal on that interval with respect to the weight function r, that is,

$$(6) \quad (y_m,\ y_n) = \int_a^b r(x) y_m(x) y_n(x) dx = 0 (m \neq n)$$

Theorem1 -continued

If $p(a)=0$, then (2a) can be dropped from the problem.
If $p(b)=0$, then (2b) can be dropped from the problem.

It is then required that y and y' remain bounded at such a point, and the problem is called **singular**, as opposed to a **regular problem** in which (2) is used.

If $p(a)=p(b)$, then (2) can be replaced by the "periodic boundary conditions"

$$y(a)=y(b), \quad y'(a)=y'(b) \tag{7}$$

Proof

Omitted!

Example 3 Application of Theorem 1. Vibrating String

Show that the solutions of the following equation are orthonormal on $0 \leqq x \leqq \pi$.

$$(3) \quad y'' + \lambda y = 0, \; y(0) = y(\pi) = 0$$

Sol.

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0$$

Comparison shows that p=1, q=0, and r=1.
By Theorem 1, the eigenfunctions y_m=sin mx, m=0, 1, 2, … are orthogonal on $0 \leqq x \leqq \pi$.

Example 4 Application of Theorem 1. Orthogonality of the Legendre Polynomials

Show that the following Legendre equation is a Liouville's problem.

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

Sol.

$$[(1-x^2)y']' + \lambda y = 0 \quad where\ \lambda = n(n+1)$$

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0$$

$$p = 1 - x^2, \;\; q = 0, \;\; r = 1$$

(10) $$\int_{-1}^{1} P_m(x) P_n(x) = 0 \quad m \neq n$$

11.6 Ortthogonal Series, Generalized Fourier Series (직교급수, 일반화된 Fourier 급수)

Let y_0, y_1, y_2,... be orthogonal with respect to a weight function r(x) on an interval a≦x≦b, and let f(x) be a function that can be represented by a convergent series

$$(1) \quad f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots$$

(1) is called an **orthogonal series**(직교급수), **orthogonal expansion**(직교전개), or **generalized Fourier series**(일반화된 Fourier 급수)라 부른다.

If y_m are eigenfunctions of a Sturm-Liouville problem, (1) is called an **eigenfunction expansion**(고유함수 전개).

Examples of Generalized Fourier Series

- Fourier-Legendre Series
- Fourier-Bessel Series

(1) $$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots$$

$$(f,\ y_n) = \int_a^b r f y_n dx = \int_a^b r\left(\sum_{m=0}^{\infty} a_m y_m\right) y_n dx$$

$$= \sum_{m=0}^{\infty} a_m \int_a^b r y_m y_n dx = \sum_{m=0}^{\infty} a_m (y_m,\ y_n) dx$$

$$= a_n \|y_n\|^2$$

(2) $$a_m = \frac{(f,\ y_m)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2} \int_a^b r(x) f(x) y_m(x) dx$$

$$(m = 0,\ 1,\ \cdots)$$

Ex 1 Fourier-Legendre Series

Represent a function f(x) in terms of Legendre functions.

Sol.

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0$$

Comparison shows that

$$p = 1 - x^2,\ q = 0,\ r = 1, \lambda = n(n+1)$$

Example 4 in section 11.5 shows that Pm(x) are orthogonal on -1≦x ≦1.

$$f(x) = \sum_{m=0}^{\infty} a_m p_m(x) = a_0 P_0(x) + a_1 P_1(x) + \cdots$$

$$= a_0 + a_1 x + a_2[(3/2)x^2 - (1/2)] + \cdots$$

$$a_m = \frac{(f, y_m)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2} \int_a^b r(x) f(x) y_m(x)\, dx$$

$$= \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx \quad (m = 0, 1, 2, \cdots)$$

Because, without proof,

(4) $\|P_m\| = \sqrt{\int_{-1}^{1} p_m(x)^2 dx} = \sqrt{\frac{2}{2m+1}}$ **(m=0, 1, 2, ...)**

Legendre Polynomials(Sec 5.2)

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = (1/2)(3x^2 - 1)$$
$$P_3(x) = (1/2)(5x^3 - 3x)$$
$$P_4(x) = (1/8)(35x^4 - 30x^2 + 3)$$
$$P_5(x) = (1/8)(63x^5 - 70x^3 + 15x)$$

An Example of Fourier-Legendre Series

$$f(x) = \sin \pi x$$

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x) dx \quad (m = 0, 1, 2, \cdots)$$

$$= \frac{2m+1}{2} \int_{-1}^{1} \sin \pi x P_m(x) dx \quad (m = 0, 1, 2, \cdots)$$

$$a_1 = \frac{3}{2} \int_{-1}^{1} x \sin x dx = \frac{3}{\pi} = 0.95493, \quad \textbf{etc.}$$

The Fourier-Legendre series of f(x) = sin πx:

$$\sin \pi x = 0.95493 P_1(x) - 1.15824 P_3(x) + \cdots$$

EX. 2 Fourier-Bessel Series

Represent a function f(x) in terms of Bessel functions.

Sol.

Step 1. Bessel's equation as a Sturm-Liouville equation

Bessel's equation:

$$\tilde{x}^2 \ddot{J}_n(\tilde{x}) + \tilde{x}\dot{J}_n(x) + (\tilde{x}^2 - n^2) J_n(\tilde{x}) = 0$$

Let $\tilde{x} = kx$, then

$$x = \tilde{x}/k, \quad \dot{J}_n = dJ_n/d\tilde{x} = (dJ_n/dx)/k$$

$$\ddot{J}_n = d^2J_n/d\tilde{x} = J_n''/k^2$$

$$\tilde{x}^2 \ddot{J}_n(\tilde{x}) + \tilde{x}\dot{J}_n(x) + (\tilde{x}^2 - n^2) J_n(\tilde{x}) = 0$$

$$x = \tilde{x}/k, \quad \dot{J}_n = dJ_n/d\tilde{x} = (dJ_n/dx)/k$$
$$\ddot{J}_n = d^2 J_n/d\tilde{x} = J_n''/k^2$$

$$(kx)^2[J_n''(kx)/k^2)] + (kx)[J_n'(kx)/k]$$
$$+ [(kx)^2 - n^2] J_n(kx) = 0$$

$$x^2 J_n''(kx) + x J_n'(kx) + (k^2x^2 - n^2) J_n(kx) = 0$$
$$x J_n''(kx) + J_n'(kx) + (k^2 x - n^2/x) J_n(kx) = 0$$

$$xJ_n''(kx) + J_n'(kx) + (k^2x - n^2/x)J_n(kx) = 0$$

$(xJ_n'(kx))' = xJ_n''(kx)+J_n'(kx)$

$$[xJ_n'(kx)]' + \left(-\frac{n^2}{x} + k^2x\right)J_n(kx) = 0$$

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0$$: Sturm-Liouville Equation, equation (1) in Sec. 11.5

Comparing two equations:

$p(x)=x$, $q(x)=-n^2/x$, $r(x)=x$, $\lambda=k^2$

Since $p(0)=0$, Theorem 1 in Sec. 11.5 implies orthogonality on an interval $0 \leqq x \leqq R$ (R given, fixed) of those solutions $J_n(kx)$ that are zero at R, that is,

$$J_n(kR) = 0 \qquad (n \text{ fixed}) \tag{6}$$

Step 2. Orthogonality

It can be shown(Ref. [A13]) that $J_n(\tilde{x})$ has many zeros, say, $\tilde{x} = \alpha_{n,1} < \alpha_{n,1} < \alpha_{n,1} < \cdots$ (see Fig. 110 in Sec. 5.4 for n=0 and 1. Hence we must have

$$(7) \qquad kR = \alpha_{(n,m)} \ thus \ k_{n,m} = \alpha_{(n,m)}/R \qquad (m = 1, 2, \cdots)$$

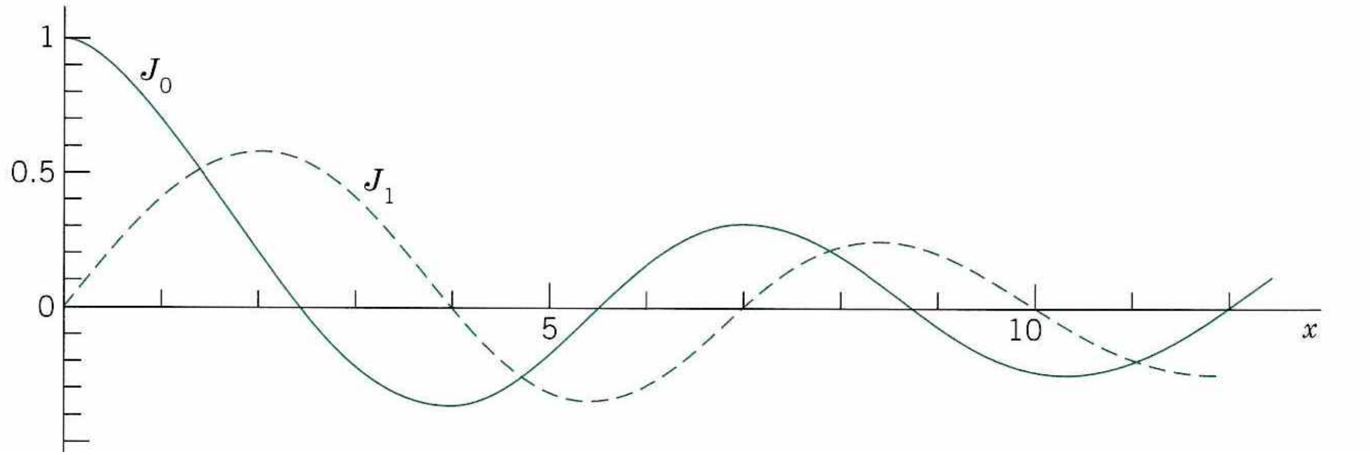


Fig. 110 in Sec. 5.4 Bessel functions of the first kind J_0 and J_1

Step 2. Orthogonality

The results in step 1 and the equation (7) lead the following Theorem for the orthogonality of solutions for Sturm-Liouville equation.

THEOREM 1 Orthogonality of Bessel Functions

$$\int_0^R x J_n(k_{n,m}x) J_n(k_{n,j}x)\,dx = 0 \quad (j \neq m,\ n \text{ fixed})$$

where

$$(7) \quad kR = \alpha_{n,m}, \ \ thus \ \ k_{n,m} = \alpha_{n,m}/R \quad (m = 1.2.\cdots)$$

Step 3. Fourier-Bessel Series

$$(9) \quad f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{n,m} x) = a_1 J_n(k_{n,1} x) + a_2 J_n(k_{n,2} x) + a_3 J_n(k_{n,3} x) + \cdots \quad (n \ \ fixed)$$

where

$$(7) \quad kR = \alpha_{n,m}, \quad thus \quad k_{n,m} = \alpha_{n,m}/R \quad (m = 1.2. \cdots)$$

The coefficients are:

$$(10) \qquad a_m = \frac{2}{R^2 J_{n+1}^2(\alpha_{n,m})} \int_0^R x f(x) J_n(k_{n,m} x) dx \qquad m = 1, 2, \cdots$$

EXAMPLE 3 Special Fourier-Bessel Series

Represent $f(x)=1-x^2$ in terms of Bessel functions.

Sol.

SKIP!

Mean Square Convergence. Completeness

A sequence of functions f_k is called convergent in the norm, also called mean-square convergent with the limit f if

$$(12^*) \qquad \lim_{k\to\infty} \|f_k - f\| = 0.$$

$$(12) \qquad \lim_{k\to\infty} \int_a^b r(x)[f_k(x) - f(x)]^2 dx = 0$$

$$(13) \qquad \lim_{k\to\infty} \int_a^b r(x)[s_k(x) - f(x)]^2 dx = 0$$

where s_k *is the partial sum of* (1).

$$(1) \quad f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots$$

Completeness

An orthonormal set y_0, y_1, … on $a \leqq x \leqq b$ is *complete* in a set of functions S defined on $a \leqq x \leqq b$ if every f in S can be approximated by a linear combination $a_0y_0+a_1y_1+\ldots+a_k y_k$. That is, for every $\varepsilon>0$, there exist constants a_0, a_1, …, a_k such that

$$(15) \qquad \|f-(a_0y_0+a_1y_1+\cdots+a_ky_k)\|<\epsilon$$

Bessel's Inequality

$$(13)\quad \lim_{k\to\infty}\int_a^b r(x)[s_k(x)-f(x)]^2dx=0$$

$$\int_a^b r(x)[s_k(x)-f(x)]^2dx=\int_a^b rs_k^2dx-2\int_a^b rfs_kdx+\int_a^b rf^2dx$$

The first term:

$$\begin{aligned}\int_a^b rs_k^2dx&=\int_a^b r\left(\sum_{m=0}^{k}a_my_m\right)^2dx\\&=\int_a^b r\sum_{m=0}^{k}a_m^2y_m^2dx+2\int_a^b r\sum_{m\neq n}^{k}y_my_ndx\\&=\sum_{m=0}^{k}a_m^2+0\quad(\because orthonormal\ \ set)\end{aligned}$$

The second term:

$$\int_a^b r f s_k dx = \int_a^b r \sum_{m=0}^{\infty} a_m y_m \sum_{m=0}^{k} a_m y_m dx = \sum_{m=1}^{k} a_m^2$$

$$\int_a^b r(x)[s_k(x) - f(x)]^2 dx = \int_a^b r s_k^2 dx - 2\int_a^b r f s_k dx + \int_a^b r f^2 dx$$

$$\int_a^b r s_k^2 dx = \sum_{m=0}^{k} a_m^2 \qquad \int_a^b r f s_k dx = \sum_{m=1}^{k} a_m^2$$

$$\int_a^b r(x)[s_k(x) - f(x)]^2 dx = -\sum_{m=1}^{k} a_m^2 + \int_a^b r f^2 dx \geq 0$$

$$(\because the\ \ integrand \geq 0)$$

$$\int_a^b r(x)[s_k(x)-f(x)]^2 dx = -\sum_{m=1}^{k} a_m^2 + \int_a^b r f^2 dx \geq 0$$

Bessel's Inequality:

$$(16)\qquad \sum_{m=1}^{k} a_m^2 \leq \int_a^b r f^2 dx = \|f\|^2 \quad (k = 1, 2, \cdots)$$

$As\ k \to \infty$

$$(17)\qquad \sum_{m=1}^{\infty} a_m^2 \leq \|f\|^2$$

Furthermore, if y_0, y_1, … is complete in a set of functions S, then (13) holds for every f in S.

$$(13)\quad \lim_{k\to\infty}\int_a^b r(x)[s_k(x)-f(x)]^2dx=0$$

Therefore, in case of completeness, the following relationship follows.

Parseval's Theorem:

$$(18)\quad \sum_{m=1}^{\infty} a_m^2=\|f\|^2=\int_a^b r(x)f(x)^2dx$$

THEOREM 2 Completeness

Let y_0, y_1, … be a complete orthonormal set on $a \leqq x \leqq b$ in a set of functions S. Then if a function f belongs to S and is orthogonal to every y_0, y_1, … , it must have norm zero. In particular, if f is continuous, then f must be identically zero.

Sol.

$$(18) \qquad \sum_{m=1}^{\infty} a_m^2 = \|f\|^2 = \int_a^b r(x) f(x)^2 dx$$

If f is orthogonal to every y_0, y_1, …, then

$$a_m = \frac{(f,\ y_m)}{\|y_m\|^2}$$

$$= \frac{1}{\|y_m\|^2} \int_a^b r(x) f(x) y_m(x) dx$$

$$= 0$$

Therefore,

$$\sum_{m=1}^{\infty} a_m^2 = \|f\|^2 = \int_a^b r(x) f(x)^2 dx = 0$$

11.7 Fourier Integral(푸리에 적분)

Fourier Series: powerful for problems involving functions that are periodic or are of interest on a finite interval only

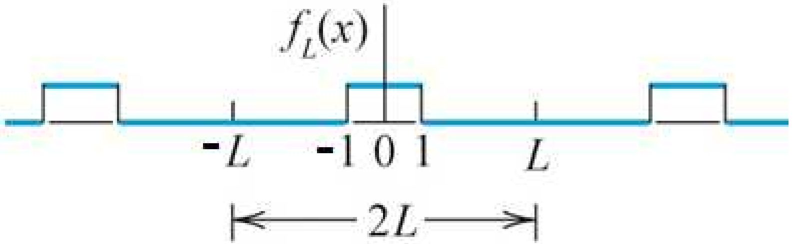
Increase the period $L \rightarrow \infty$

Fourier Integral

EX. 1 Rectangular Wave (직사각형파)

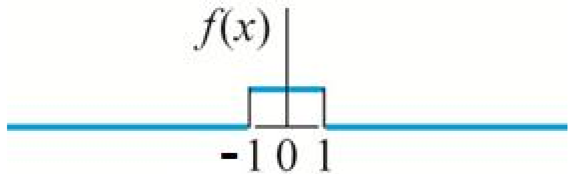
Period=2L>2

$$f_L(x)=\begin{cases}0 & (-L<x<-1)\\ 1 & (-1<x<1)\\ 0 & (\ 1<x<L)\end{cases}$$



$f_L(x)$ becomes nonperiodic as L→∞

$$f(x)=\lim_{L\to\infty}f_L(x)=\begin{cases}1 & (-1<x<1)\\ 0 & elsewhere\end{cases}$$

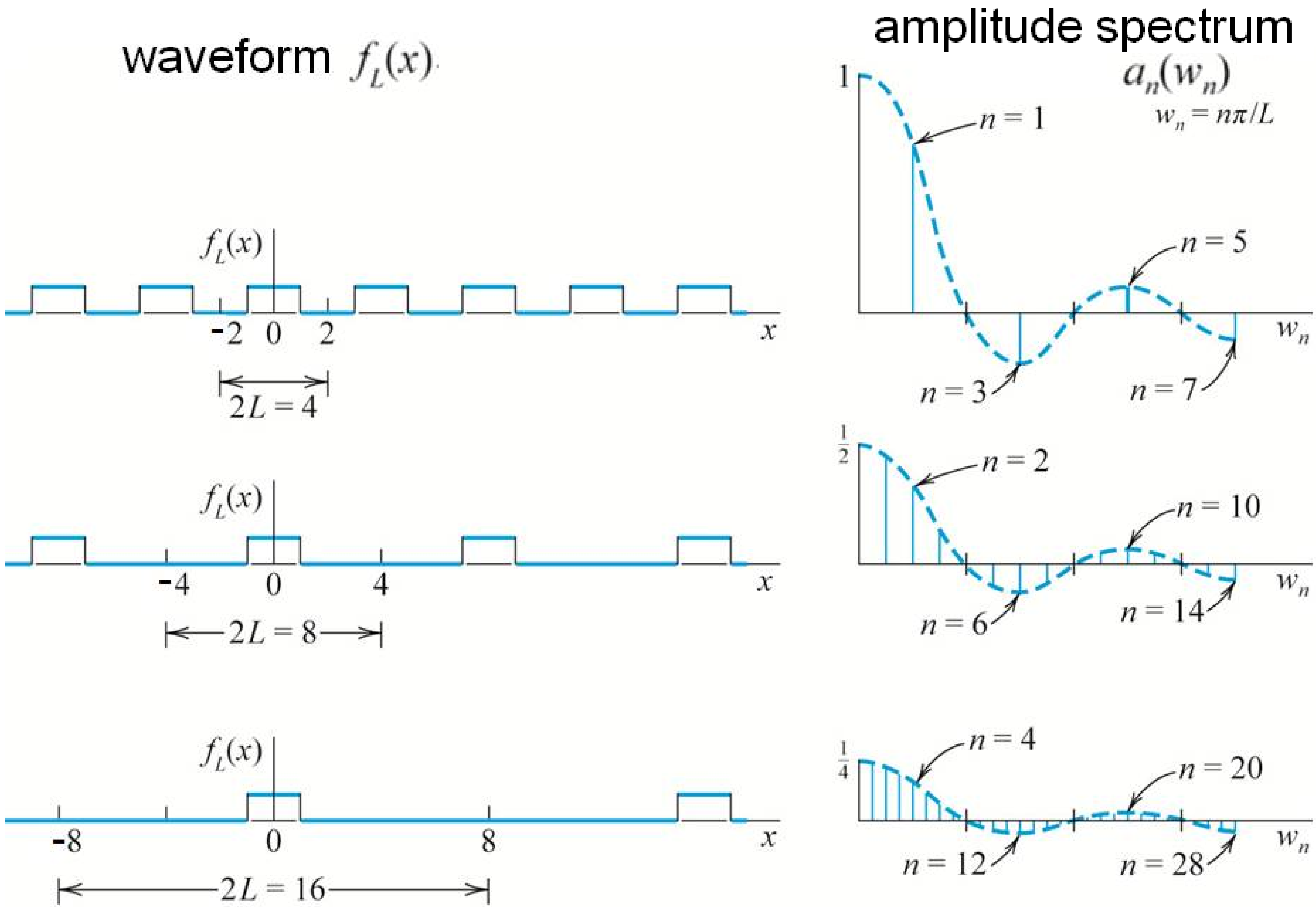


$b_n=0$ since $f_L(x)$ is an even function.

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}$$

$$a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L} dx = \frac{2}{L}\int_{0}^{1} \cos\frac{n\pi x}{L} dx = \frac{2}{L} \cdot \frac{\sin(n\pi/L)}{(n\pi/L)}$$

The sequence a_n is called the **amplitude spectrum** of f_L.



amplitude spectrum
waveform $f_L(x)$
$a_n(w_n)$
$w_n = n\pi/L$
$f_L(x)$
-2 0 2
x
$2L = 4$
1
$n = 1$
$n = 3$
$n = 5$
$n = 7$
w_n
$f_L(x)$
-4 0 4
x
$2L = 8$
$\frac{1}{2}$
$n = 2$
$n = 6$
$n = 10$
$n = 14$
w_n
$f_L(x)$
-8 0 8
x
$2L = 16$
$\frac{1}{4}$
$n = 4$
$n = 12$
$n = 20$
$n = 28$
w_n

From Fourier Series to Fourier Integral： L→∞

$f_L(x)$: function of period 2L

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos\omega_n x + b_n \sin\omega_n x), \quad \omega_n = n\pi/L$$

$$a_n = \frac{1}{L}\int_{-L}^{-L} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{-L} f(x)\sin\frac{n\pi x}{L}dx$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos\omega_n x\int_{-L}^{L} f_L(v)\cos\omega_n v dv + \sin\omega_n x\int_{-L}^{L} f_L(v)\sin\omega_n v dx\right]$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos\omega_n x \int_{-L}^{L} f_L(v)\cos\omega_n v\,dv \right.$$
$$\left. + \sin\omega_n x \int_{-L}^{L} f_L(v)\sin\omega_n v\,dx \right]$$

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} \qquad 1/L = \Delta\omega/\pi$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv$$
$$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos\omega_n x)\Delta\omega \int_{-L}^{L} f_L(v)\cos\omega_n v\,dv \right.$$
$$\left. + (\sin\omega_n x)\Delta\omega \int_{-L}^{L} f_L(v)\sin\omega_n v\,dx \right]$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\Big[(\cos\omega_n x)\,\Delta\omega\int_{-L}^{L} f_L(v)\cos\omega_n v\,dv + (\sin\omega_n x)\,\Delta\omega\int_{-L}^{L} f_L(v)\sin\omega_n v\,dx\Big]$$

$$\big\downarrow\; L\to\infty$$

$$f(x) = \lim_{L\to\infty} f_L(x) = \frac{1}{\pi}\int_{0}^{\infty}\Big[(\cos\omega x)\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv + (\sin\omega x)\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv\Big]\,d\omega$$

(3) $$f(x) = \frac{1}{\pi}\int_0^\infty \left[(\cos\omega x)\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv + (\sin\omega x)\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv\right]d\omega$$

Representation of f(x) by Fourier Integral:

(5) $$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega$$

(4) $$where \quad A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv$$

THEOREM 1 Fourier integral

If $f(x)$

- is piecewise continuous in every finite interval and
- has a right-hand derivative and a left-hand derivative at every point, and
- is absolutely integrable,

then $f(x)$ can be represented by a Fourier integral (5) with A and B given by (4).

At a discontinuous point the value of the Fourier integral is the average of the left- and right-hand limits of $f(x)$ at that point.

f(x) is absolutely integrable if the following integral exists.

$$(2) \quad \int_{-\infty}^{\infty} |f(x)|dx = \lim_{a \to -\infty} \int_{a}^{0} |f(x)|dx + \lim_{b \to \infty} \int_{0}^{b} |f(x)|dx$$

$$\textbf{(5)} \qquad f(x) = \int_{0}^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$$

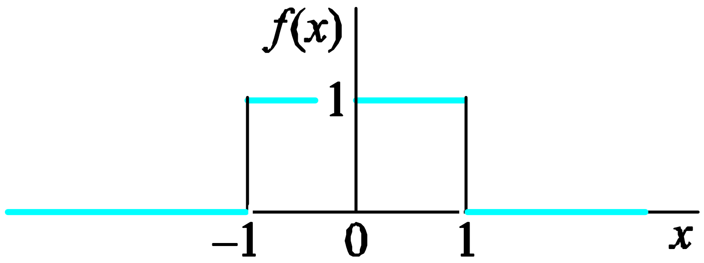
$$\textbf{(4)} \qquad where \quad A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\, dv$$

EX. 2 Single Pulse, Sine Integral. Dirichlet's Discontinuous Factor. Gibbs Phenomenon

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & (|x| < 1) \\ 0 & (|x| > 1) \end{cases}$$



Sol.

(5) $$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$$

(4) where $$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv$$

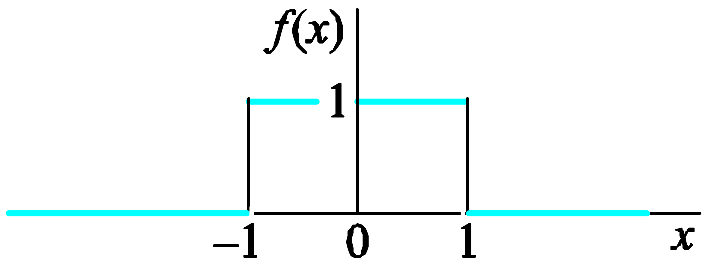
$$A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv=\frac{1}{\pi}\int_{-1}^{1}\cos\omega v\,dv=\frac{2\sin\omega}{\pi\omega}$$

$$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv=\frac{1}{\pi}\int_{-1}^{1}\sin\omega v\,dv=0$$

$$\therefore f(x)=\frac{2}{\pi}\int_{0}^{\infty}\frac{\cos\omega x\sin\omega}{\omega}d\omega$$

Dirichlet discontinuous factor(불연속인자)

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x \sin\omega}{\omega} d\omega$$



Dirichlet discontinuous factor(불연속인자)

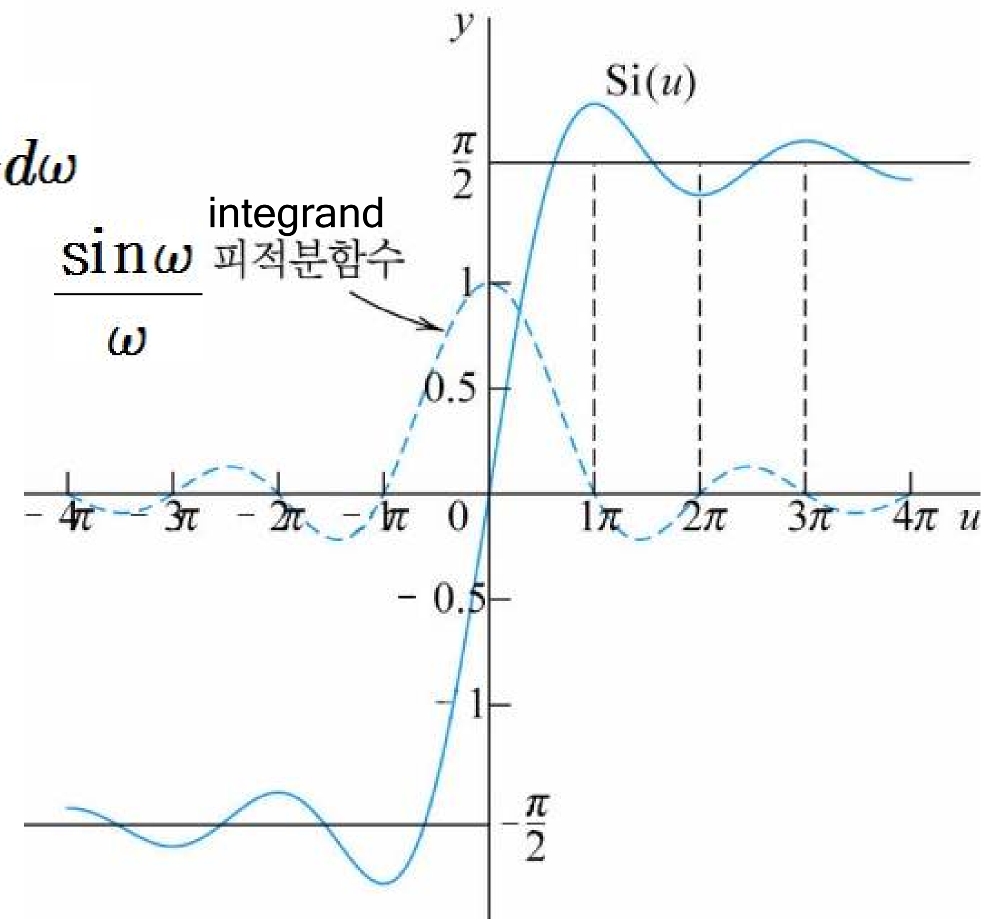
(7) $$\int_0^\infty \frac{\cos\omega x \sin\omega}{\omega} d\omega = \begin{cases} \pi/2, & (-1 < x < 1) \\ \pi/4, & (x = 1) \\ 0 & (x > 1) \end{cases}$$

If x=0:

(8*) $$\int_0^\infty \frac{\sin\omega}{\omega} d\omega = \frac{\pi}{2}$$

Sine integral(사인적분)

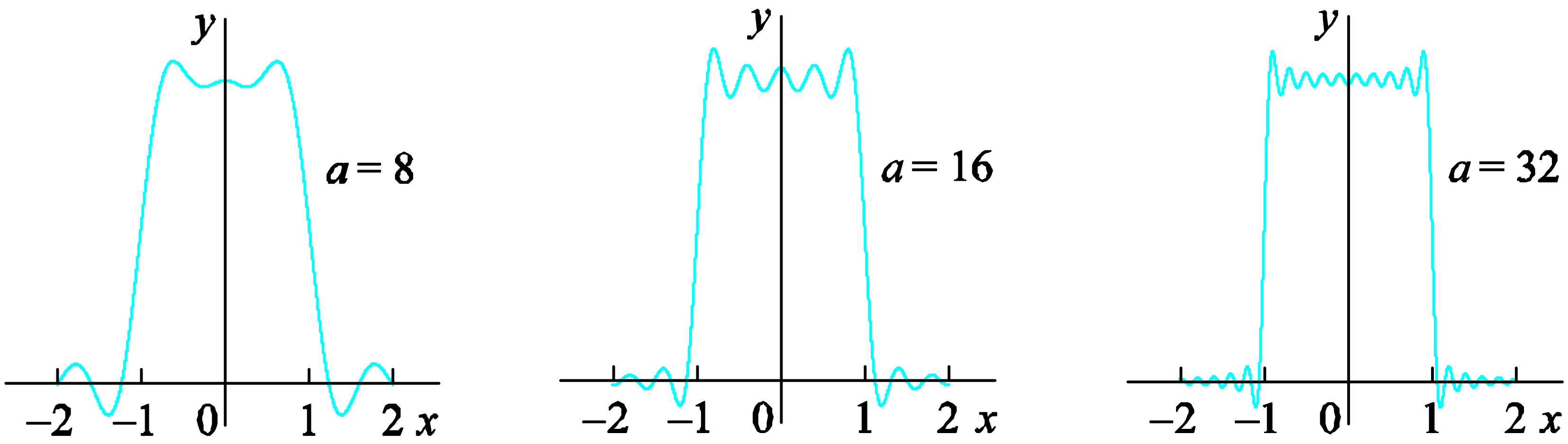
(8) $$\mathrm{Si}(u) = \int_0^u \frac{\sin\omega}{\omega}\,d\omega$$



$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x \sin\omega}{\omega} d\omega \simeq \frac{2}{\pi}\int_0^a \frac{\cos\omega x \sin\omega}{\omega} d\omega$$

Gibbs Phenomena(Gibbs 현상)

The oscillations near the points of discontinuity do not disappear, but shift closer to the points as the increase of a.



$$\frac{2}{\pi}\int_0^a \frac{\cos\omega x \sin\omega}{\omega} d\omega$$

$$= \frac{1}{\pi}\int_0^a \frac{\sin(\omega x+\omega)-\sin(\omega x-\omega)}{\omega} d\omega$$

$$= \frac{1}{\pi}\int_0^a \frac{\sin(\omega x+\omega)}{\omega} d\omega - \frac{1}{\pi}\int_0^a \frac{\sin(\omega x-\omega)}{\omega} d\omega$$

The first term:

$$\frac{1}{\pi}\int_0^a \frac{\sin(\omega x+\omega)}{\omega} d\omega = \frac{1}{\pi}\int_0^{(x+1)a} \frac{\sin t}{t} dt \quad = \mathrm{Si}[a(x+1)]$$

$$\omega(x+1)=t,\ \ (x+1)d\omega = dt,\ \ \therefore \frac{d\omega}{\omega}=\frac{dt}{t}$$

The second term:

$$\frac{1}{\pi}\int_0^a \frac{\sin(\omega x-\omega)}{\omega}d\omega=\frac{1}{\pi}\int_0^{(x-1)a}\frac{\sin t}{t}dt \quad = \mathbf{Si[a(x-1)]}$$

$$\omega(x-1)=t,\ \ (x-1)d\omega=dt,\ \ \therefore \frac{d\omega}{\omega}=\frac{dt}{t}$$

$$\therefore\ f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\cos\omega x\sin\omega}{\omega}d\omega \simeq \frac{2}{\pi}\int_0^{a}\frac{\cos\omega x\sin\omega}{\omega}d\omega$$

$$=\frac{1}{\pi}\int_0^{(x+1)a}\frac{\sin t}{t}dt-\frac{1}{\pi}\int_0^{(x-1)a}\frac{\sin t}{t}dt$$

$$=\frac{1}{\pi}\,\mathbf{Si[a(x+1)]}-\frac{1}{\pi}\,\mathbf{Si[a(x-1)]}$$

Fourier Cosine Integral

Fourier Integral:

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega$$

$$where \quad A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv$$

If f(x) is even, B(ω)=0,

Fourier Cosine Integral:

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\,d\omega \qquad A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos\omega v\,dv$$

Fourier Sine Integral

Fourier Integral:

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega$$

$$where \quad A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv$$

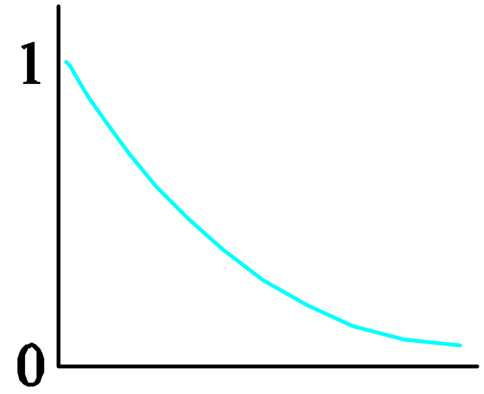
$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv$$

If f(x) is odd, A(ω)=0,

Fourier Sine Integral:

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\,d\omega \qquad B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin\omega v\,dv$$

EX. 3 Laplace Integrals



Derive the Fourier cosine and Fourier sine Integrals for $f(x)=e^{-kx}$, where x>0 and k>0.

Sol.

(a) Fourier Cosine Integral

$$\int f(v)\cos\omega v\,dv = \int e^{-kv}\cos\omega v\,dv$$

$$= -\frac{k}{k^2+\omega^2}e^{-kv}\left(-\frac{\omega}{k}\sin\omega v + \cos\omega v\right)$$

$$\therefore A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos\omega v\,dv = \frac{2k/\pi}{k^2+\omega^2}$$

$$\therefore f(x) = e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos\omega x}{k^2+\omega^2}d\omega \qquad (x>0,\ k>0)$$

(13) $$\int_0^\infty \frac{\cos\omega x}{k^2+\omega^2}d\omega = \frac{\pi}{2k}e^{-kx} \quad (x>0,\ k>0)$$

(b) Fourier Sine Integral

$$\int f(v)\sin\omega v\,dv = \int e^{-kv}\sin\omega v\,dv$$

$$= -\frac{\omega}{k^2+\omega^2}e^{-kv}\left(\frac{k}{\omega}\sin\omega v + \cos\omega v\right)$$

(14) $$B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin\omega v\,dv = \frac{2}{\pi}\frac{\omega}{k^2+\omega^2}$$

$$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{\omega \sin \omega x}{k^2+\omega^2}\,d\omega$$

$$\Rightarrow \quad (15) \int_0^\infty \frac{\omega \sin \omega x}{k^2+\omega^2}\,d\omega = \frac{\pi}{2}e^{-kx} \quad (x>0,\ k>0)$$

Laplace Integral

$$(13) \quad \int_0^\infty \frac{\cos \omega x}{k^2+\omega^2}\,d\omega = \frac{\pi}{2k}e^{-kx} \qquad (x>0,\ k>0)$$

$$(15) \quad \int_0^\infty \frac{\omega \sin \omega x}{k^2+\omega^2}\,d\omega = \frac{\pi}{2}e^{-kx} \quad (x>0,\quad k>0)$$

f is periodic: Fourier series

$$(5)\qquad f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$(6)\qquad (0)\ a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \qquad (a)\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$(b)\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

f is non-periodic: Fourier integral

$$f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega$$

$$where \quad A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv$$

f is non-periodic and even: Fourier cosine integral

$$f(x) = \int_0^{\infty} A(\omega)\cos\omega x\,d\omega \qquad A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v)\cos\omega v\,dv$$

f is non-periodic and odd: Fourier sine integral

$$f(x) = \int_0^{\infty} B(\omega)\sin\omega x\,d\omega \qquad B(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v)\sin\omega v\,dv$$

1. Find the Fourier-Legendre series for $63x^5 - 90x^3 + 35x$.

$$f(x) = \sum_{m=0}^{\infty} a_m p_m(x) = a_0 P_0(x) + a_1 P_1(x) + \cdots$$

(3) $a_m = \dfrac{2m+1}{2}\displaystyle\int_{-1}^{1} f(x) P_m(x)\,dx$

$$a_0 = \frac{1}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)P_0(x)dx = \frac{1}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)dx = 0,$$

$$a_1 = \frac{3}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)P_1(x)dx = \frac{3}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)x\,dx = 8,$$

$$a_2 = \frac{5}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)P_2(x)dx = \frac{5}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)\cdot\frac{1}{2}(3x^2 - 1)dx = 0,$$

$$a_3 = \frac{7}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)P_3(x)dx = \frac{7}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)\cdot\frac{1}{2}(5x^3 - 3x)dx = -8,$$

$$a_4 = \frac{9}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)P_4(x)dx = \frac{9}{16}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)\cdot(35x^4 - 30x^2 + 3)dx = 0,$$

$$a_5 = \frac{11}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)P_5(x)dx = \frac{11}{2}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)(1/8)(63x^5 - 70x^3 + 15x)dx = 8.$$

$$\begin{aligned}63x^5 - 90x^3 + 35x &= a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x) + a_4 P_4(x) + a_5 P_5(x)\\ &= 0\,P_0(x) + 8\,P_1(x) + 0\,P_2(x) - 8\,P_3(x) + 0\,P_4(x) + 8\,P_5(x)\\ &= 8P_1(x) - 8P_3(x) + 8P_5(x)\end{aligned}$$

1. Show that $\displaystyle\int_0^{\infty} \frac{\cos xw + w \sin xw}{1 + w^2}\,dx = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$.

$$f(x) = \begin{cases} 0 & (x<0) \\ \pi e^{-x} & (x>0) \end{cases} \Rightarrow \quad \textbf{(5)} \quad f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega$$

$$\textbf{(4)} \quad where \quad A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv$$

$$A(w) = \int_0^{\infty} e^{-x}\cos wx\,dx = \left[\frac{e^{-x}}{1+w^2}(-\cos wx + w\sin wx)\right]_0^{\infty} = \frac{1}{w^2+1}$$

$$B(w) = \int_0^{\infty} e^{-x}\sin wx\,dx = \left[\frac{e^{-x}}{1+w^2}(-\sin wx - w\cos wx)\right]_0^{\infty} = \frac{w}{w^2+1}$$

$$f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega = \int_0^{\infty} \frac{\cos xw + w\sin xw}{w^2+1}\,dw = \begin{cases} 0 & (x<0) \\ \pi/2 & (x=0) \\ \pi e^{-x} & (x>0) \end{cases}$$

1. Represent $f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$ as a Fourier cosine integral.

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\, d\omega \qquad A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos\omega v\, dv$$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos\omega v\, dv$$

$$= \frac{2}{\pi}\int_0^1 \cos wv\, dv = \frac{2}{\pi}\frac{\sin wv}{w}\bigg|_{v=0}^{v=1} = \frac{2}{\pi}\frac{\sin w}{w}$$

$$\therefore\ f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin w \cos xw}{w}\, dw$$

16 Represent $f(x) = \begin{cases} x & \text{if} \quad 0 < x < a \\ 0 & \text{if} \quad x > a \end{cases}$ as a Fourier sine integral.

$$f(x) = \int_0^\infty B(\omega)\sin\omega x \, d\omega \qquad B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin\omega v \, dv$$

$$B(w) = \frac{2}{\pi}\int_0^a v\sin wv dv = \frac{2}{\pi}\left[-\frac{v}{w}\cos wv + \frac{1}{w^2}\sin wv\right]_0^a$$

$$= \frac{2}{\pi}\left(-\frac{a}{w}\cos aw + \frac{1}{w^2}\sin aw\right)$$

$$\therefore \; f(x) = \frac{2}{\pi}\int_0^\infty \left(-\frac{a}{w}\cos aw + \frac{1}{w^2}\sin aw\right)\sin xw dw$$