Introduction to Discrete Math

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Course Outline

- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion & Induction, Logic, Invariants
- Probability & Combinatronics
  - Basic Counting, Binomial Coeff, Advanced Counting, Probability, Random Variables

Probability & Combinatronics – Random Variables

THE DICE GAME

- Dice Game
- Playing the Game

Dice Game

Consider the following situation

- You are in a bad neighborhood
- There is a shady person on the corner of the street who offers bypassers to play a game with him

*shady – doubtful, suspicious, questionable character

Rules of the Game

- There are several dices with various numbers on their sides
- You and the shady person pick one dice each
- Both of you throw your dices
- Whoever has the larger number wins
- The winner gets man won from the loser

The Catch

- To give you an advantage, the shady person lets you pick your dice first
- So you can pick the one you find the best
- And he will have to pick from the remaining options
  - So why not, to win all the shady person's money?
- What is the catch?

The Catch

- There are no catches external to our model
  - *in the end, this is a math course*
  - Dices are fair: each outcome has probability exactly 1/6
  - No one will hit you on the head during the game
  - No one will pick your pocket
- Disclaimer: beware, all of these are not guaranteed to you in real life games with scammers!
- In our problem the shady person is not cheating
  - the game will be played exactly as described

The Catch

- The game seems favorable to us
  - *Yet, the shady person is eager to play*

interested

The Catch

- The game seems favorable to us
  - *Yet, the shady person is eager to play*

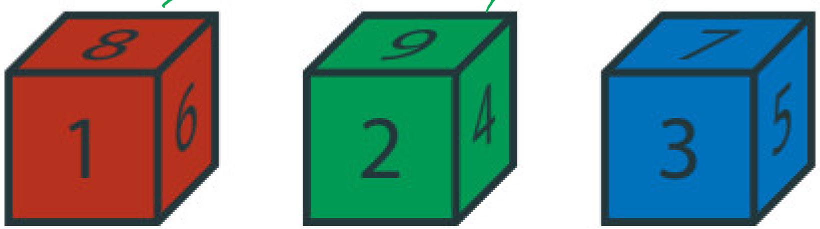
- So, what's wrong with this situation?
  - *It turns out that there is purely a mathematical answer to this puzzle*

- Dice Game

- **Playing the Game**

Playing the Game

- We observe closer & see shady person has just 3 dices
- And here they are:

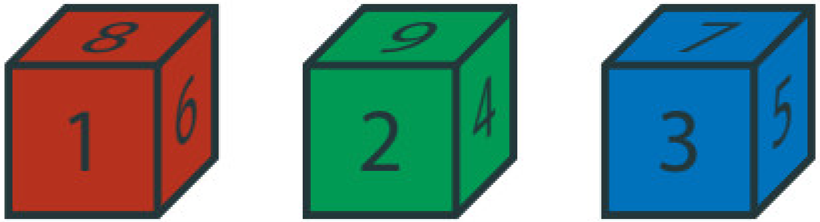


- Dice 1 has numbers: *1, 1, 6, 6, 8, 8*

  Dice 2 has numbers: *2, 2, 4, 4, 9, 9*

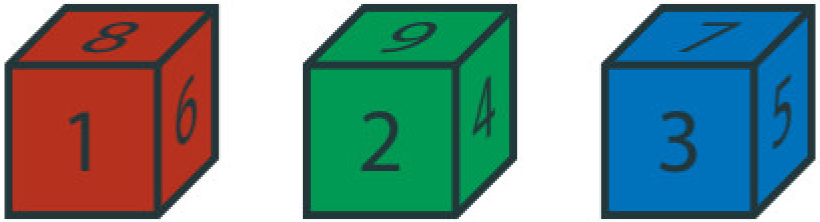
  Dice 3 has numbers: *3, 3, 5, 5, 7, 7*
- Which dice should we pick?

Playing the Game



- We feel we are educated enough

Playing the Game



- We feel we are educated enough

- We should compare dices and compute the probabilities
- Let’s start with Dice 1 and Dice 2

Dice 1 vs Dice 2

- Consider all possible outcomes and count winning outcomes for each of the dices

12	12	14	14	19	19
12	12	14	14	19	19
62	62	64	64	69	69
62	62	64	64	69	69
82	82	84	84	89	89
82	82	84	84	89	89

20/36

16 X

Dice 1 vs Dice 2

- Consider all possible outcomes and count winning outcomes for each of the dices

12	12	14	14	19	19
12	12	14	14	19	19
62	62	64	64	69	69
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- **Dice 1** wins in 16 outcomes
- **Dice 2** wins in 20 outcomes

Dice 1 vs Dice 2

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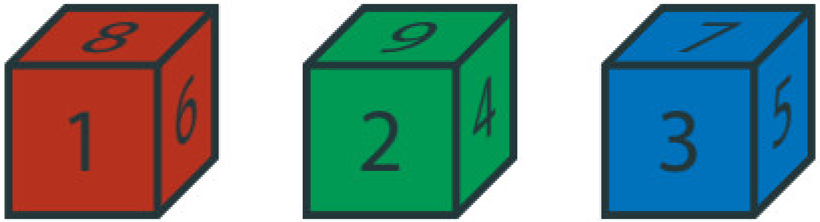
82	82	84	84	89	89
82	82	84	84	89	89

- **Dice 1** wins in **16** outcomes $\frac{16}{36}$
- **Dice 2** wins in **20** outcomes $\frac{20}{36}$
- **Dice 2** wins with probability of:

$$\left(\frac{16}{36} \to \frac{4}{9}\right) < \left(\frac{20}{36} \to \frac{5}{9} > \frac{1}{2}\right)$$

Playing the Game



- Dice 2 is better than Dice 1
- Now let's compare Dice 2 with Dice 3

Dice 2 vs Dice 3

- Consider all possible outcomes and count winning outcomes for each of the dices

23	23	25	25	27	27
23	23	25	25	27	27
43	43	45	45	47	47
43	43	45	45	47	47
93	93	95	95	97	97
93	93	95	95	97	97

20

14

Dice 2 vs Dice 3

- Consider all possible outcomes and count winning outcomes for each of the dices

23	23	25	25	27	27
23	23	25	25	27	27
43	43	45	45	47	47
43	43	45	45	47	47
93	93	95	95	97	97
93	93	95	95	97	97

- **Dice 2** wins in 16 outcomes $\frac{16}{36} =$
- **Dice 3** wins in 20 outcomes

Dice 2 vs Dice 3

- Consider all possible outcomes and count winning outcomes for each of the dices

23	23	25	25	27	27
23	23	25	25	27	27
43	43	45	45	47	47
43	43	45	45	47	47
93	93	95	95	97	97
93	93	95	95	97	97

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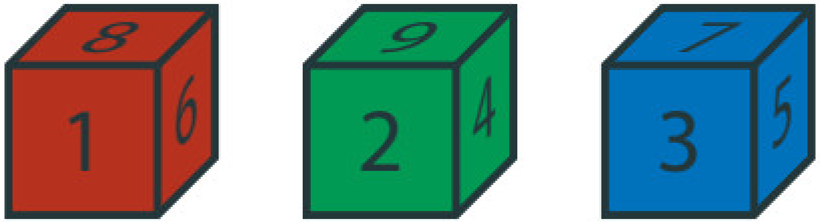
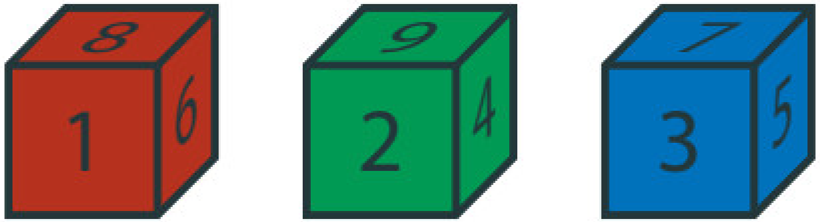
Playing the Game



- Dice 2 is better than Dice 1
- Dice 3 is better than Dice 2
- Clearly, Dice 3 is better than Dice 1, hence we are done!

Playing the Game



- Dice 2 is better than Dice 1
- Dice 3 is better than Dice 2
- Clearly, Dice 3 is better than Dice 1, hence we are done!
-  Or…. Are we? Let's check!

Dice 3 vs Dice 1

- Consider all possible outcomes and count winning outcomes for each of the dices

31	31	36	36	38	38	
31	31	36	36	38	38	20
51	51	56	56	58	58	
51	51	56	56	58	58	
71	71	76	76	78	78	
71	71	76	76	78	78	

16

Dice 3 vs Dice 1

- Consider all possible outcomes and count winning outcomes for each of the dices

31	31	36	36	38	38
31	31	36	36	38	38
51	51	56	56	58	58
51	51	56	56	58	58
71	71	76	76	78	78
71	71	76	76	78	78

- **Dice 3** wins in 16 outcomes
- **Dice 1** wins in 20 outcomes

Dice 3 vs Dice 1

- Consider all possible outcomes and count winning outcomes for each of the dices

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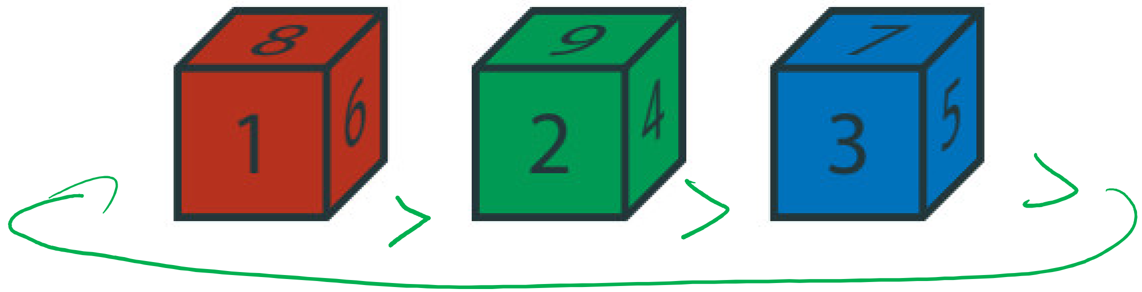
31	31	36	36	38	38
51	51	56	56	58	58
51	51	56	56	58	58
71	71	76	76	78	78
71	71	76	76	78	78

- **Dice 3** wins in **16** outcomes $\frac{16}{36}$
- **Dice 1** wins in **20** outcomes $\frac{20}{36}$
- **Dice 1** wins with probability of:

$$\left(\frac{16}{36} \to \frac{4}{9}\right) < \left(\frac{20}{36} \to \frac{5}{9} > \frac{1}{2}\right)$$

Playing the Game



- Dice 2 is better than Dice 1
- Dice 3 is better than Dice 2….
- But Clearly, Dice 1 is better than Dice 3!
- Now this is interesting…. How is this even possible?!

Numbers vs Random Variables

- We are used to comparing numbers
- And we are used that certain properties hold
- One of them : if $a > b$ and $b > c$, then $a > c$
- This is called transitivity
- This translates to real life experience:
  faster, higher, stronger

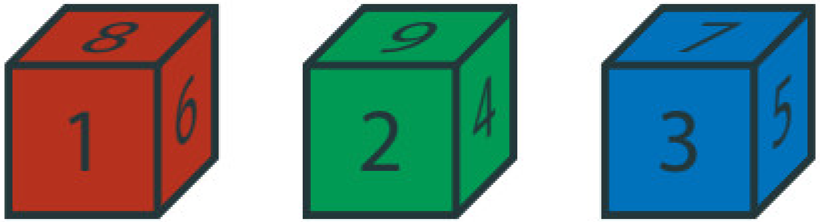
Numbers vs Random Variables

- But random variables are not numbers!
- It is way harder to compare them
- If we find some way of comparison, still usual properties are not guaranteed!
- For instance: no transitivity in our game!

Xlun: if $a > b$, $b > c$, $c > a$ ✓

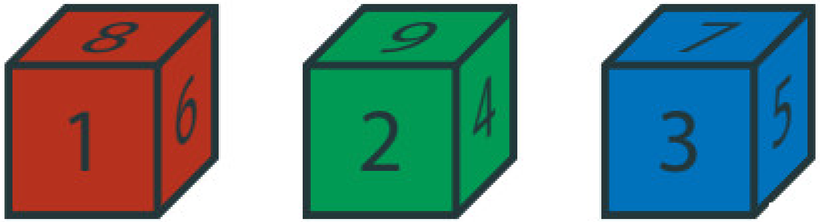
Random Vars: if $a > b$, $b > c$, $c > a$? ✗

Who Wins the Game?



- But what does this mean to the game?
- Dice 2 is better than Dice 1
- Dice 3 is better than Dice 2
- Dice 1 is better than Dice 3

Who Wins the Game?



- But what does this mean to the game?
- Dice 2 is better than Dice 1
- Dice 3 is better than Dice 2
- Dice 1 is better than Dice 3
- The shady person, who allows us to choose a dice first
  - Is actually having an advantage because of that!

Who Wins the Game?

Remember:

- Dice 2 is better than Dice 1
- Dice 3 is better than Dice 2
- Dice 1 is better than Dice 3

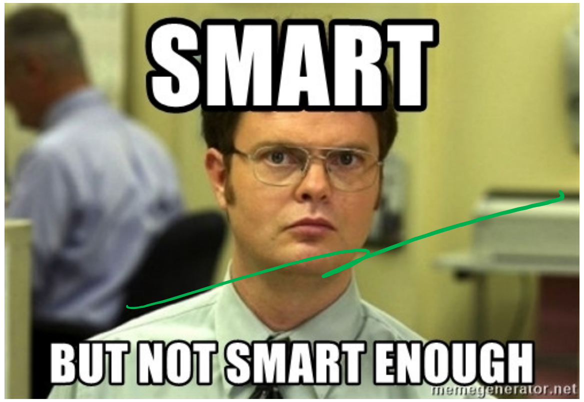
So, how does the shady person play the game to win?

Who Wins the Game?

Remember:

5/9 4/9

- Dice 2 is better than Dice 1
- Dice 3 is better than Dice 2
- Dice 1 is better than Dice 3



So, how does the shady person play the game to win?

- If we pick Dice 1, the shady person picks Dice 2
- If we pick Dice 2, the shady person picks Dice 3
- If we pick Dice 3, the shady person picks Dice 1
- Smart!

Main Lessons

- Probability is tricky!
- We should be very careful when applying usual intuition to probability
- In the end, we should just avoid scam games ☺…

Thank you.