13.5 Exponential Function

13.5 Exponential Function

- 복소 지수함수(Exponential Function) : $e^z = \exp z = e^{x+iy}$

Definition:

$$(1) \qquad e^z = \exp z = e^{x+iy} = e^x(\cos y + i \sin y)$$

Basic Properties

(A) $e^z = e^x$ for real $z = x$ because cos$y$=1 and sin$y$=0 when y=0.

(B) e^z is analytic for all z: entire function

(C) The derivative of e^z is e^z, that is,

$$(2) \qquad (e^z)' = e^z$$: next slide

13.5 Exponential Function

Proof of $(e^z)' = e^z$

$$f'(z) = u_x + iv_x$$: Eq. (4) in Sec. 13.4

$$= -iu_y + v_y$$: Eq. (5) in Sec. 13.4

$$(e^z)' = u_x + iv_x$$: Eq. (4) in Sec. 13.4

$$= (e^x \cos y)_x + i(e^x \sin y)_x$$

$$= e^x \cos y + ie^x \sin y = e^z$$

Further Properties

$$(3) \qquad e^{z_1+z_2} = e^{z_1} e^{z_2}$$

$$\begin{aligned} e^{z_1} e^{z_2} &= e^{x_1}(\cos y_1 + i \sin y_1) \cdot e^{x_2}(\cos y_2 + i \sin y_2) \\ &= e^{x_1+x_2}[(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) \\ &\qquad + i(\cos y_1 \sin y_2 + \sin y_1 \cos y_2)] \\ &= e^{x_1+x_2}[\cos(y_1+y_2) + i \sin(y_1+y_2)] \\ &= e^{x_1+x_2} e^{i(y_1+y_2)} \\ &= e^{(x_1+iy_1)+(x_2+iy_2)} = e^{z_1+z_2} \end{aligned}$$

$$(4) \qquad e^{z} = e^{x+iy} = e^{x} e^{iy}$$

13.5 Exponential Function

Euler Formula

$$(1) \qquad e^z = \exp z = e^{x+iy} = e^x(\cos y + i\sin y)$$

$$(5) \qquad e^{iy} = \cos y + i\sin y$$: Euler Formula

Thus, the Polar Form of a Complex Number,

$z = r(\cos\theta + i\sin\theta)$ is,

$$(6) \qquad z = re^{i\theta}$$

13.5 Exponential Function

Properties

(7) $e^{2\pi i} = 1$

(8) $e^{i\pi/2} = i, \qquad e^{\pi i} = -1$

$e^{-\pi i/2} = -i, \quad e^{-\pi i} = -1$

(9) $|e^{iy}| = |\cos y + i\sin y|$

$= \sqrt{\cos^2 y + \sin^2 y} = 1$

(10) $|e^{x}| = e^{x}$

$\arg e^{z} = y \pm 2n\pi \quad (n = 0, 1, 2 \cdots)$

(11) $|e^{z}| = e^{x} \neq 0 \quad \mathit{for\ all}\ z$

13.5 Exponential Function

Periodicity of e^z with period $2\pi i$

$$e^{z+2\pi i} = e^z \quad \text{for all } z \tag{12}$$

Fundamental Region of e^z:

$$-\pi < y \leq \pi \tag{13}$$

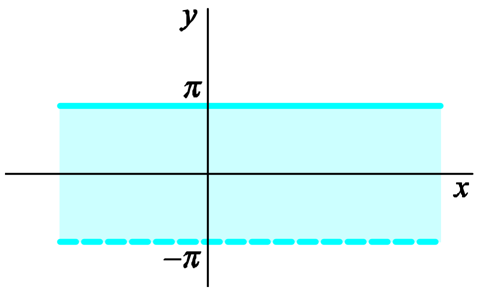


Fig. 336. Fundamental region of the exponential function e^z in the z-plane

13.5 Exponential Function

EX.1 Function Values. Solution of Equations

(a) Compute $z = e^{1.4-0.6i}$

(b) Solve $e^z = 3+4i$

Sol.

(a) Compute $z = e^{1.4-0.6i} = e^{1.4}(\cos 0.6 - i \sin 0.6)$

$= 4.055(0.8253 - i0.5646) = 3.347 - 2.289i$

(b) Solve $e^z = 3+4i = e^x(\cos y + i \sin y)$

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y) = 3 + 4i$$
$$e^x = \sqrt{3^2 + 4^2} = 5, \quad x = \ln 5 = 1.609$$
$$e^x \cos y = 5\cos y = 3, \quad \cos y = 0.6$$
$$e^x \sin y = 5\sin y = 4, \quad \sin y = 0.8, \quad y = 0.927 + 2n\pi$$
$$z = x + iy = 1.609 + (0.927 \pm 2n\pi)i \ (n = 0, 1, 2, \cdots)$$

Problems in Sec 13.5

9. Write $4+3i$ in polar form.

$$4+3i = 5\left[\cos\left(\tan^{-1}\frac{3}{4}\right) - i\sin\left(\tan^{-1}\frac{3}{4}\right)\right]$$

$$= 5\exp\left(-i\tan^{-1}\frac{3}{4}\right)$$

20. Find all solutions for $e^z = 4+3i$.

$$e^z = e^{x+iy} = e^x e^{iy} = 4+3i = 5e^{i\theta}, \quad \theta = \tan^{-1}\frac{3}{4}$$

$$e^x = 5, \quad x = \ln 5,$$

$y = \tan^{-1}\frac{3}{4} \pm 2n\pi \quad (n = 0, 1, \cdots)$이다.

$$z = x + iy = \ln 5 + i\left(\tan^{-1}\frac{3}{4} \pm 2n\pi\right) \qquad (n = 0, 1, \cdots)$$

13.6 Trigonometric and Hyperbolic Functions (삼각함수와 쌍곡선 함수)

$$e^{ix} = \cos x + i \sin x, \quad e^{-ix} = \cos x - i \sin x$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\Downarrow$$

(1) $$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

(2) $$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}$$

13.6 Trigonometric and Hyperbolic Functions

(3) $\sec z = \dfrac{1}{\cos z}, \quad \csc z = \dfrac{1}{\sin z}$

(4) $(\cos z)' = -\sin z, \quad (\sin z)' = \cos z$

$(\tan z)' = \sec^2 z$

(5) $e^{iz} = \cos z + i \sin z \qquad \textit{for all } z$

- $\cos z$ and $\sin z$ are entire functions.
- $\tan z$ and $\sec z$ are analytic except where $\cos z = 0$.
- $\cot z$ and $\csc z$ are analytic except where $\sin z = 0$.

13.6 Trigonometric and Hyperbolic Functions

EXAMPLE 1 **Real and Imaginary Parts. Absolute Value. Periodicity**

Show that

$$(6) \quad \begin{aligned} &(a) \quad \cos z = \cos x \cosh y - i \sin x \sinh y \\ &(b) \quad \sin z = \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

$$(7) \quad \begin{aligned} &(a) \quad |\cos z|^2 = \cos^2 x + \sinh^2 y \\ &(b) \quad |\sin z|^2 = \sin^2 x + \sinh^2 x \end{aligned}$$

Sol.

$$
\begin{aligned}
(1) \qquad \cos z &= (e^{iz} + e^{-iz})/2 \\
&= (1/2)[e^{i(x+iy)} + e^{-i(x+iy)}] \\
&= (1/2)[e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)] \\
&= (1/2)(e^{y} + e^{-y})\cos x + (1/2)(e^{y} - e^{-y})(-i \sin x) \\
&= \cos x \cosh y - i \sin x \sinh y
\end{aligned}
$$

13.6 Trigonometric and Hyperbolic Functions

(8) $\cosh y = (1/2)(e^{y} + e^{-y}), \quad \sinh y = (1/2)(e^{y} - e^{-y})$

$$\begin{aligned}\cosh^2 y &= (1/4)(e^{y}+e^{-y})^2 = (1/4)[(e^{y}-e^{-y})^2 + 4e^{y}\cdot e^{-y}] \\ &= [(1/2)(e^{y}-e^{-y})]^2 + 1 = 1 + \sinh^2 y\end{aligned}$$

$$\begin{aligned}|\cos z|^2 &= (\cos x \cosh y)^2 + (\sin x \sinh y)^2 \\ &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\ &= \cos^2 x (1 + \sinh^2 y) + \sin^2 x \sinh^2 y \\ &= (\cos^2 x + \sin^2 x)\sinh^2 y + \cos^2 x \\ &= \cos^2 x + \sinh^2 y\end{aligned}$$

13.6 Trigonometric and Hyperbolic Functions

$|\cos z|$ and $|\sin z|$ are no longer bounded but approach infinity in absolute value as $y \to \infty$, since then $\sinh y \to \infty$ in (7).

$$
(7) \qquad \begin{aligned} &(a) \quad |\cos z|^2 = \cos^2 x + \sinh^2 y \\ &(b) \quad |\sin z|^2 = \sin^2 x + \sinh^2 x \end{aligned}
$$

13.6 Trigonometric and Hyperbolic Functions

EXAMPLE 2 **Solutions of Equations. Zeros of cos z and sin z**

Solve $(a)\ \cos z = 5$

$(b)\ \cos z = 0$

$(c)\ \sin z = 0$

Sol.

$(a)\ \cos z = 5$

$$\cos z = (1/2)(e^{iz} + e^{-iz}) = 5$$

$$e^{2iz} - 10e^{iz} + 1 = 0, \quad t^2 - 10t + 1 = 0$$

$$e^{iz} = e^{i(x+iy)} = e^{-y}e^{ix} = 5 \pm \sqrt{5^2 - 1} = 9.899,\ 0.101$$

$$e^{-y} = 9.899,\ 0.101 \quad \therefore y = \pm 2.292$$

$$e^{ix} = \cos x + i\sin x = 1, \quad x = \pm 2n\pi$$

$$z = \pm 2n\pi \pm 2.292i \quad (n = 0, 1, 2, \cdots)$$

13.6 Trigonometric and Hyperbolic Functions

$(b)\ \cos z = 0$

$$\cos z = (1/2)(e^{iz} + e^{-iz}) = 0$$
$$e^{2iz} + 1 = 0, \quad t^2 + 1 = 0$$
$$e^{iz} = e^{i(x+iy)} = e^{-y} e^{ix} = \pm i$$
$$e^{-y} = 1 \quad \therefore y = 0$$
$$e^{ix} = \cos x + i \sin x = \pm i, \quad x = \pm (n + 0.5)\pi$$
$$z = \pm (n + 0.5)\pi \quad (n = 0, 1, 2, \cdots)$$

$$(c)\ \sin z = 0$$

$$\sin z = (e^{iz} - e^{-iz})/(2i) = 0$$
$$e^{2iz} - 1 = 0, \quad t^2 - 1 = 0$$
$$e^{iz} = e^{i(x+iy)} = e^{-y}e^{ix} = \pm 1$$
$$e^{-y} = 1 \quad \therefore y = 0$$
$$e^{ix} = \cos x + i \sin x = \pm 1, \quad x = \pm n\pi$$
$$z = \pm n\pi \quad (n = 0, 1, 2, \cdots)$$

13.6 Trigonometric and Hyperbolic Functions

General formulas

$$(9)\qquad \begin{aligned} \cos(z_1 \pm z_2) &= \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2 \\ \sin(z_1 \pm z_2) &= \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2 \end{aligned}$$

$$(10)\qquad \cos^2 z + \sin^2 z = 1$$

Hyperbolic Functions

Definition

$$(11) \quad \cosh z = (e^{z} + e^{-z})/2, \quad \sinh z = (e^{z} - e^{-z})/2$$

Ref. $\cos x = (e^{ix} + e^{-ix})/2, \quad \sin x = (e^{ix} - e^{-ix})/(2i)$

$$(13) \quad \begin{aligned} \tanh z &= \frac{\sinh z}{\cosh z}, \quad & \coth z &= \frac{\cosh z}{\sinh z} \\ \operatorname{sech} z &= \frac{1}{\cosh z}, \quad & \operatorname{csch} z &= \frac{1}{\sinh z} \end{aligned}$$

Derivatives

$$(12) \quad (\cosh z)' = \sinh z, \quad (\sinh z)' = \cosh z$$

13.6 Trigonometric and Hyperbolic Functions

$$(11)\quad \cosh z = (e^{z} + e^{-z})/2, \quad \sinh z = (e^{z} - e^{-z})/2$$

$$\cos x = (e^{ix} + e^{-ix})/2, \quad \sin x = (e^{ix} - e^{-ix})/(2i)$$

Relationships

$$(14)\quad \cosh iz = \cos z, \quad \sinh iz = i \sin z$$

$$(15)\quad \cos iz = \cosh z, \quad \sin iz = i \sinh z$$

Problems in Sec 13.6

6. Find $\sin 2\pi i$ in the form $u + iv$.

$$\sin 2\pi i = i \sinh 2\pi$$

10. Find $\sinh(3 + 4i)$ in the form $u + iv$.

$$\sinh(3+4i) = \sinh 3 \cos 4 + i \cosh 3 \sin 4$$

Problems in Sec 13.6

16. Find all solutions of $\sin z = 100$.

$$\sin z = \sin x \cosh y + i \cos x \sinh y = 100$$

$$\cos x \sinh y = 0,\ \ x = \frac{1}{2}\pi \pm 2n\pi$$

$$\cosh y = 100 \approx \frac{1}{2}e^{y},\ \ (\text{ sufficient large } y)$$

$$e^{y} \approx 200,\ \ y \approx 5.29832$$

$$z = x + iy = \frac{1}{2}\pi \pm 2n\pi \pm 5.29832i$$

13.7 Logarithm. General Power. Principal Value

The natural logarithm of $z = x+iy$, denoted by ln z, and is defined by the inverse function of e^z.

$w = \ln z$ is defined for $z \neq 0$.

$e^w = z$

$z = 0$ is impossible since $z = e^w \neq 0$ for all w.

13.7 Logarithm. General Power

$$e^{w} = z$$

Let $w = u + iv$ and $z = re^{i\theta}$.

$$e^{w} = e^{u+iv} = re^{i\theta}$$

$$\begin{cases} e^{u} = r, & \therefore u = \ln r \\ v = \theta \end{cases}$$

$$(1) \quad w = u + iv = \ln r + i\theta \quad (r = |z| > 0, \ \theta = \arg z)$$

$\ln z = \ln(re^{i\theta}) = \ln r + i\theta$:Infinitely many-valued

13.7 Logarithm. General Power

Principal Value of the Logarithm

$$\ln z = \ln\left(re^{i\theta}\right) = \ln r + i\theta$$

(2) $\quad \operatorname{Ln} z = \ln r + i \operatorname{Arg} z$: Principal Value of the Logarithm

(3) $\quad \ln z = \operatorname{Ln} z \pm 2n\pi i \quad (n = 1, 2, 3, \cdots)$

If z is positive real: $\operatorname{Ln} z = \ln z$

If z is negative real: $\operatorname{Ln} z = \ln|z| + \pi i$

13.7 Logarithm. General Power

$$(1) \quad \ln z = \ln r + i\theta \qquad \theta = \arg z$$

$$(4a) \quad e^{\ln z} = e^{\ln r + i\theta} = re^{i\theta} = z$$

$$(4b) \quad \ln(e^{z}) = z \pm 2n\pi i \qquad n = 0, 1, 2, \cdots$$

EXAMPLE 1 **Natural Logarithm. Principal Value**

$$\ln 1 = \ln r + i\operatorname{Arg} z \pm 2n\pi i = \ln 1 + 0 \pm 2n\pi i = \pm 2n\pi i,$$
$$\operatorname{Ln} 1 = \ln r + i\operatorname{Arg} z = \ln 1 + i \cdot 0 = 0 \qquad n = 0, 1, 2, \cdots$$

$$\ln 4 = \ln r + i\operatorname{Arg} z \pm 2n\pi i = \ln 4 + 0 \pm 2n\pi i = 1.386 \pm 2n\pi i,$$
$$\operatorname{Ln} 4 = \ln r + i\operatorname{Arg} z = \ln 4 + i \cdot 0 = 1.386 \qquad n = 0, 1, 2, \cdots$$

$$\ln(-1) = \ln r + i\operatorname{Arg} z \pm 2n\pi i = \ln 1 + i\pi \pm 2n\pi i = (1 \pm 2n)\pi i,$$
$$\operatorname{Ln}(-1) = \ln r + i\operatorname{Arg} z = \ln 1 + i \cdot \pi = \pi i \qquad n = 0, 1, 2, \cdots$$

$$\ln(-4) = \ln r + i\operatorname{Arg} z \pm 2n\pi i = \ln 4 + i\pi \pm 2n\pi i$$
$$= 1.386 + (1 \pm 2n)\pi i, \quad n = 0, 1, 2, \cdots$$

$$\operatorname{Ln}(-4) = \ln r + i\operatorname{Arg} z = \ln 4 + i \cdot \pi = 1.386 + \pi i$$

13.7 Logarithm. General Power

$$\ln i = \ln r + i\,\mathrm{Arg}z \pm 2n\pi i = \ln 1 + i\pi/2 \pm 2n\pi i$$
$$= (0.5 \pm 2n)\pi i,\ n = 0,1,2,\cdots$$
$$\mathrm{Ln}\,i = \ln r + i\,\mathrm{Arg}z = \ln 1 + i \cdot \pi/2 = \pi i/2$$

$$\ln 4i = \ln 4 + i\pi/2 \pm 2n\pi i = 1.386 + (0.5 \pm 2n)\pi i,$$
$$\mathrm{Ln}\,4i = \ln 4 + i \cdot \pi/2 = 1.386 + \pi i/2 \qquad n = 0,1,2,\cdots$$

$$\ln(-4i) = \ln 4 - i\pi/2 \pm 2n\pi i = 1.386 - (0.5 \pm 2n)\pi i,$$
$$\mathrm{Ln}(-4i) = \ln 4 + i \cdot (-\pi/2) = 1.386 - \pi i/2 \quad n = 0,1,2,\cdots$$

$$\ln(3-4i) = \ln 5 + i\,\mathrm{Arg}(3-4i) \pm 2n\pi i$$
$$= 1.609 - 0.927i \pm 2n\pi i,\ n = 0,1,2,\cdots$$
$$\mathrm{Ln}(3-4i) = \ln 4 + \mathrm{Arg}(3-4i) \pm 2n\pi i$$
$$= 1.609 - 0.927i \quad n = 0,1,2,\cdots$$

$$\ln(3-4i) = 1.609 - 0.927i \pm 2n\pi i,\ n = 0, 1, 2, \cdots$$

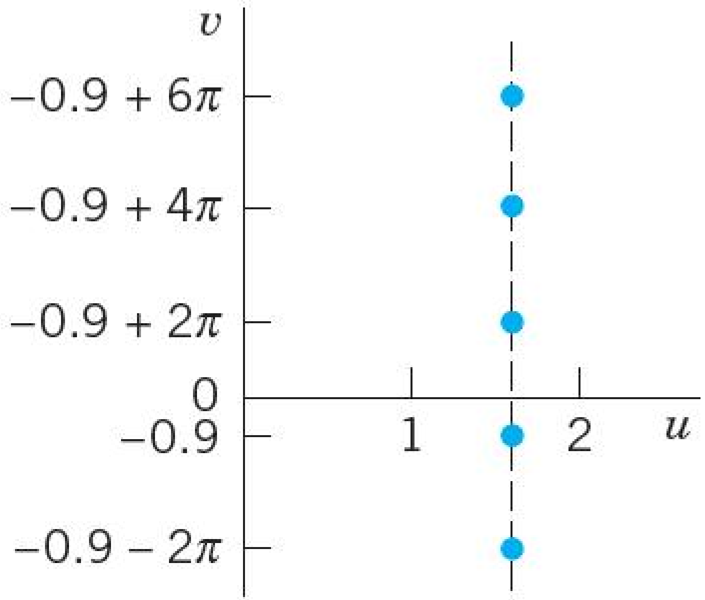


Fig. 337. Some values of ln (3 − 4*i*) in Example 1

13.7 Logarithm. General Power

The familiar relations for the natural logarithm continue to hold for complex values, that is,

$$(5) \qquad \begin{aligned} &(a) \quad \ln(z_1 z_2) = \ln z_1 + \ln z_2 \\ &(b) \quad \ln(z_1/z_2) = \ln z_1 - \ln z_2 \end{aligned}$$

but these relations are to be understood in the sense that each value of one side is also contained among the values of the other side; see the next example.

EX 2 **Illustration of the Functional Relation (5) in Complex**

Show that (5a) holds when $z_1 = z_2 = -1$

$$(5a) \quad \ln(z_1 z_2) = \ln z_1 + \ln z_2$$

Sol.

$$\ln(z_1 z_2) = \ln 1 = \ln 1 + 0 \pm 2n\pi i = \pm 2n\pi i \quad n = 0, 1, 2, \cdots$$
$$\ln z_1 = \ln(-1) = \ln 1 + 0 \pm 2m_1\pi i = \pm 2m_1\pi i \quad m_1 = 0, 1, 2, \cdots$$
$$\ln z_2 = \ln(-1) = \ln 1 + 0 \pm 2m_2\pi i = \pm 2m_2\pi i \quad m_2 = 0, 1, 2, \cdots$$
$$\ln z_1 + \ln z_2 = \pm 2(m_1 + m_2)\pi i = \pm 2n\pi i = \ln(z_1 z_2)$$

An example for $\mathrm{Ln}(z_1 z_2) \neq \mathrm{Ln}\, z_1 + \mathrm{Ln}\, z_2$

$$z_1 = z_2 = -1$$

$$\mathrm{Ln}(z_1 z_2) = \mathrm{Ln}\, 1 = \ln 1 + 0 = 0$$

$$\mathrm{Ln}\, z_1 = \mathrm{Ln}(-1) = \ln 1 + \pi i = \pi i$$
$$\mathrm{Ln}\, z_2 = \mathrm{Ln}(-1) = \ln 1 + \pi i = \pi i$$

$$\mathrm{Ln}\, z_1 + \mathrm{Ln}\, z_2 = \pi i + \pi i = 2\pi i$$

$$\therefore \mathrm{Ln}(z_1 z_2) \neq \mathrm{Ln}\, z_1 + \mathrm{Ln}\, z_2$$

13.7 Logarithm. General Power

THEOREM 1 Analyticity of the Logarithm

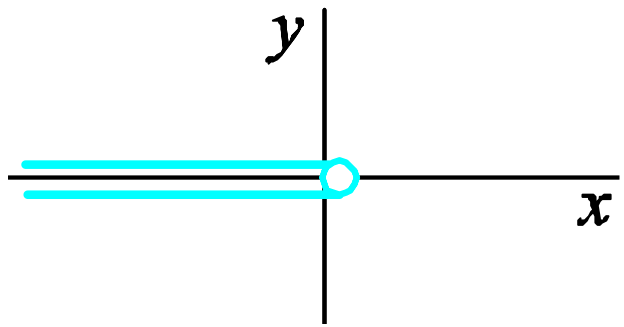
For every n=0, ∓1, ∓2, ⋯, formula (3) defines a function, which is

(1) analytic except at 0 and the negative real axis, and

(2) has the derivative

$$(6) \quad (\ln z)' = \frac{1}{z} \qquad (z \neq 0 \ \ or \ \ neagtive \ \ real)$$

$$(3) \quad \ln z = \mathrm{Ln}\, z \pm 2n\pi i \qquad n = 1, 2, \cdots$$



PROOF

(1) analytic except at 0 and the negative real axis

$$\ln z = \ln r + i\theta + 2n\pi i = \ln\sqrt{x^2+y^2} + i(\tan^{-1} y/x + c)$$
$$= \frac{1}{2}\ln(x^2+y^2) + i\left(\tan^{-1}\frac{y}{x} + c\right)$$

$$\begin{cases} u_x = \dfrac{1}{2}\dfrac{1}{x^2+y^2} \cdot 2x = \dfrac{x}{x^2+y^2} \\ v_y = \dfrac{1}{1+(y/x)^2} \cdot \dfrac{1}{x} = \dfrac{x}{x^2+y^2} \end{cases} \qquad \therefore u_x = v_y$$

$$\begin{cases} u_y = \dfrac{1}{2}\dfrac{}{x^2+y^2} \cdot 2y = \dfrac{y}{x^2+y^2} \\ v_x = \dfrac{1}{1+(y/x)^2} \cdot \dfrac{-y}{x^2} = \dfrac{-y}{x^2+y^2} \end{cases} \qquad \therefore u_y = -v_x$$

(2) Derivative

$$(\ln z)' = u_x + iv_x = \frac{x}{x^2+y^2} + i\frac{1}{1+(y/x)^2}\left(-\frac{y}{x^2}\right)$$

$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} = \frac{z^*}{zz^*} = \frac{1}{z}$$

In the negative real axis, ln *z* is not continuous, and thus not analytic.

Im(ln z)→π along the path in the second quadrant, and
Im(ln z)→-π along the path in the third quadrant.

Thus ln *z* is not continuous on the negative real axis.

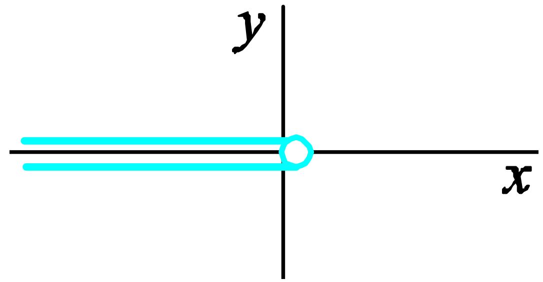
13.7 Logarithm. General Power

Branch of the Logarithm

Each of the infinitely many functions in (3) is called a branch of the logarithm.

$$\text{(3)} \quad \ln z = \operatorname{Ln} z \pm 2n\pi i \qquad n = 1, 2, \cdots$$

The negative real axis is known as a branch cut and is usually graphed as shown in the Figure.



The branch for n=0 is called the principal branch of lnz.

13.7 Logarithm. General Power

General Powers

Definition: General powers of a complex number $z = x + iy$

$$(7) \quad z^c = \exp(\ln z^c) = e^{c \ln z}$$

$$(z = x + iy \neq 0, \; c \; complex)$$

z^c is multi-valued since $\ln z$ is infinitely many-valued.

Principal Value of z^c : $\quad z^c = e^{c \operatorname{Ln} z}$

From (7) we see that for any complex number a,

$$\textbf{(8)} \qquad a^z = e^{z \ln a}.$$

13.7 Logarithm. General Power

Real powers of a complex number: z^c

1. c=integer: single-valued
2. c=1/n: n-distinct values

$$z^c = \sqrt[n]{z} = e^{(1/n)\ln z} \qquad (z \neq 0),$$

3. c=real rational number q/p: finitely-many values
4. c=irrational number: infinitely many-valued

13.7 Logarithm. General Power

EX. 3 General Power

Calculate i^i and $(1+i)^{2-i}$.

Sol.

1. Calculate i^i.

$$i^i = e^{\ln i^i} = e^{i\ln i} = e^{i(\ln 1 + \pi i/2 \pm 2n\pi i)} = e^{-(\pi/2) \mp 2n\pi}$$

$$Principal\ \ value: \ \ e^{-\pi/2}$$

2. Calculate $(1+i)^{2-i}$.

$$
\begin{aligned}
(1+i)^{2-i} &= e^{(2-i)\ln(1+i)} = e^{(2-i)(\ln\sqrt{2}+\pi i/4\pm 2n\pi i)} \\
&= e^{(2\ln\sqrt{2}+\pi/4\mp 2n\pi)+i(-\ln\sqrt{2}+\pi/2\pm 4n\pi)} \\
&= 2e^{\pi/4\mp 2n\pi}e^{i(-\ln\sqrt{2}+\pi/2)} \\
&= 2e^{\pi/4\mp 2n\pi}[\cos(-\ln\sqrt{2}+\pi/2) \\
&\qquad\qquad +i\sin(-\ln\sqrt{2}+\pi/2)] \\
&= 2e^{\pi/4\mp 2n\pi}[\sin(\ln\sqrt{2})+i\cos(\ln\sqrt{2})]
\end{aligned}
$$

Problems in Section 13.7

5. Ln(-11) $\qquad \mathrm{Ln}(-7) = \ln 7 + i\pi$

6. Ln(4+4*i*) $\qquad \mathrm{Ln}(4+4i) = \ln\sqrt{32} + \frac{\pi}{4}i$

8. Ln(1± *i*) $\qquad \mathrm{Ln}(1 \pm i) = \ln\sqrt{2} \pm \frac{\pi}{4}i$

9. Ln(0.6 + 0.8*i*) $\qquad \mathrm{Ln}(0.6 + 0.8i) = \ln 1 + i\tan^{-1}\frac{4}{3}$

Problems in Section 13.7

12. $\ln e$ $\quad \ln e = \mathrm{Ln}\, e + 2n\pi i = 1 + 2n\pi i\,,\ \ n \in \mathbb{Z}$ (integer)

14. $\ln(-7)$ $\quad \ln(-7) = \ln 7 + \pi i + 2n\pi i = \ln 7 + \pi i + (2n+1)\pi i$

(n: integer)

16. $\ln(4+3i)$

$$\ln(4+3i) = \mathrm{Ln}(4+3i) + 2n\pi i$$
$$= \left(\ln 5 + i\tan^{-1}\frac{3}{4}\right) + 2n\pi i$$
$$= \ln 5 + i\left(\tan^{-1}\frac{3}{4} + 2n\pi\right),\quad n \in \mathbb{Z}$$

20. Solve for z when $\ln z = e - \pi i$.

$$z = \exp(e - \pi i) = e^{e}\, e^{-\pi i} = e^{e}(\cos\pi - i\sin\pi) = -e^{e}$$

Problems in Section 13.7

23. Find the principal value of $(1 + i)^{1-i}$.

$$(1+i)^{1-i} = e^{(1-i)\mathrm{Ln}(1+i)} = e^{(1-i)(\ln\sqrt{2}+\pi i/4)}$$
$$= \exp\left[\ln\sqrt{2} + \frac{\pi}{4} + i\left(\frac{\pi}{4} - \ln\sqrt{2}\right)\right]$$
$$= \sqrt{2}\, e^{\frac{\pi}{4}}\left[\cos\left(\frac{\pi}{4} - \ln\sqrt{2}\right) + i\sin\left(\frac{\pi}{4} - \ln\sqrt{2}\right)\right]$$

25. Find the principal value of $(-3)^{3-i}$.

$$(-3)^{3-i} = e^{(3-i)\mathrm{Ln}(-3)} = e^{(3-i)(\ln 3+\pi i)}$$
$$= e^{(3\ln 3+\pi)+i(3\pi-\ln 3)}$$
$$= 27e^{\pi}\left[\cos(3\pi - \ln 3) + i\sin(3\pi - \ln 3)\right]$$

SUMMARY OF CHAPTER 13

(1) $z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$where\ r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x)$

$f(z)$ is analytic if it has a derivative

(2) $f'(z) = \lim_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$

$f(z) = u(x,y) + iv(x,y)$ is analytic if

(3) $u_x = v_y, \quad u_y = -v_x$:Cauchy-Riemann Equations

then, satisfy Laplace's equations

(4) $\nabla^2 u = u_{xx} + u_{yy} = 0, \quad \nabla^2 v = v_{xx} + v_{yy} = 0$

(5) $$e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$

(6) $$\cos z = (e^{iz} + e^{-iz})/2 = \cos x\cosh y - i\sin x\sinh y$$
$$\sin z = (e^{iz} - e^{-iz})/(2i) = \sin x\cosh y + i\cos x\sinh y$$

$$\tan z = (\sin z)/(\cos z)$$
$$\cot z = 1/\tan z = (\cos z)/(\sin z)$$

(7) $$\cosh z = (e^z + e^{-z})/2 = \cos iz,$$
$$\sinh z = (e^z - e^{-z})/2 = -i\sin iz$$

(8) $$\ln z = \ln|z| + i\arg z = \ln|z| + i\mathrm{Arg}\, z \pm 2n\pi i$$

(9) $$z^c = e^{c\ln z}$$