Introduction to Discrete Math

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Course Outline

- Mathematical Thinking
  - Convincing Arguments, Find Example, Recursion & Induction, Logic, Invariants
- Probability & Combinatronics
  - Basic Counting, Binomial Coeff, Advanced Counting, Probability, Random Variables

Probability & Combinatronics – Random Variables

LINEARITY OF EXPECTATION

- **Linearity of Expectation**

- Birthday Problem

Linearity of Expectation

- Suppose there are two random variables f and g over the same probability space
- The outcomes for f are $a_1, \dots, a_k$; the outcomes for g are $b_1, \dots, b_k$; the probabilities are $p_1, \dots, p_k$
- Consider $f + g$
  - It is also a random variable over the same probability distribution
- Values of $f + g$ are $a_1 + b_1, \dots, a_k + b_k$
- Can we say anything about the expectation of $f + g$? Yes!

Linearity of Expectation

Linearity of Expectation

Suppose there are random variables f and g on the same probability space, then:

$$E(f+g) = Ef + Eg$$

- Indeed, we have:

$$\begin{aligned} E(f+g) &= (f_1+g_1)p_1 + \ldots + (f_k+g_k)p_k \\ &= (f_1p_1 + \ldots + f_kp_k) + (g_1p_1 + \ldots + g_kp_k) \\ &= E(f+g) \end{aligned}$$

Linearity of Expectation

Linearity of Expectation

Suppose there are random variables f and g on the same probability space, then:

$$\mathrm{E}\left(f+g\right)=\mathrm{E}f+\mathrm{E}g$$

- Linearity is a very useful property
- Greatly simplifies computation of expressions

Linearity of Expectation

Problem

We throw two dices. What is the expected value of the sum of two numbers on them?

- Instead we can consider two random variables on our probability distribution
- f_1 is an outcome of the 1st dice; f_2 is outcome of the 2nd dice
- We are interested in $E(f_1 + f_2)$
  - We already computed the expected value of one dice throw:

Average Outcome of Dice Throw

- Suppose we throw a dice n times for a very large n
- Then among outcomes there are approximately $n/6$ ones, $n/6$ twos, and so on… $n/6$ 3's, $n/6$ 4's, $n/6$ 5's, $n/6$ 6's
- The sum of results is then approximately

$$\frac{n}{6}\times 1+\frac{n}{6}\times 2+\frac{n}{6}\times 3+\frac{n}{6}\times 4+\frac{n}{6}\times 5+\frac{n}{6}\times 6$$

$$=\frac{n(1+2+3+4+5+6)}{6}=\frac{21n}{6}=\frac{3.5\cancel{n}}{\cancel{n}}=3.5$$

- The average can be obtained by dividing with number of throws n
- Thus the average is approximately: $\frac{3.5n}{n}=3.5$
- This is an expected value or expectation of a dice throw

Linearity of Expectation

Problem

We throw two dices. What is the expected value of the sum of two numbers on them?

- Instead we can consider two random variables on our probability distribution
- f_1 is an outcome of the 1st dice; f_2 is outcome of the 2nd dice
- We are interested in $E(f1 + f2)$
  - We already computed the expected value of one dice throw: $Ef_1 = Ef_2 = 3.5$
- Thus, $E(f_1 + f_2) = Ef_1 + Ef_2 = 3.5 + 3.5 = \mathbf{7}$

Linearity of Expectation

Problem

We toss a coin 5 times in a row. What is expected number of tails?

- Again, we can compute directly
- But would require computing probabilities of all possible numbers of tails
- We need to recall Combinatorics, etc…
- On the other hand, Linearity can give the answer almost immediately

Linearity of Expectation

Problem

We toss a coin 5 times in a row. What is expected number of tails?

- Let f_i be an outcome of the i-th coin: it is 1 if the outcome is "tails" and it is 0 if it is "heads"
- We are interested in $E(f_1 + f_2 + f_3 + f_4 + f_5)$!
- It is easy to compute the expectation for a single toss

$$\mathrm{E}f_i = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$$

$$E f_{i_{hnd}} = \cancel{0 \times \frac{1}{2}} + 1 \times \frac{1}{2} = \frac{1}{2}$$

- Thus:

$$\mathrm{E}(f_1 + f_2 + f_3 + f_4 + f_5) =$$
$$= \mathrm{E}f_1 + \mathrm{E}f_2 + \mathrm{E}f_3 + \mathrm{E}f_4 + \mathrm{E}f_5 = 2.5$$

$$E(f_1 + \dots + f_{10})$$
$$= Ef_1 + Ef_2 + \dots + Ef_{10}$$
$$= 10\,(0.5) = 5$$

- Linearity of Expectation

- Birthday Problem

Birthday Problem

Birthday Problem

Consider 28 randomly chosen people. Consider the number of pairs (i, j) such that the i-th person has a birthday on the same day as j-th person. Show that the expectation of this number is greater than 1.

- If there are two people with the same birthday, they will contribute 1 to the number of pairs in the problem
- If three people with the same birthday, they form 3 pairs
  - *So they will contribute 3 to the number of pairs in the problem*
- Looks surprising: not many people
- But we will prove it!

Birthday Problem

Birthday Problem

Consider 28 randomly chosen people. Consider the number of pairs (i, j) such that the i-th person has a birthday on the same day as j-*th* person. Show that the expectation of this number is greater than 1.

- Formalization is needed
- We assume that birthdays are distributed uniformly among 365 days of the year
- We will not discuss it, but a non-uniform distribution on days of the year only increases the expectation!
- People are chosen independently

Birthday Problem

Birthday Problem

- We will use the linearity of expectation; denote the number of pairs of people with the same birthday by f
- Enumerate people from 1 to 28; consider a random variable g_{ij} that is equal to 1 if persons i and j have birthday on the same day, and is equal to 0 otherwise (g_{ij} is same birthday, "1", "0")
- Observation:
  f is equal to the sum of g_{ij} over all (unordered) pairs of i and j!
- Why?

Birthday Problem

Consider an example of 5 people

- Five people: 1, 2, 3, 4, 5
- List of all pairs:

{1,2}	{2,4}
{1,3}	{2,5}
{1,4}	{3,4}
{1,5}	{3,5}
{2,3}	{4,5}

Birthday Problem

Consider an example of 5 people

- Five people: 1, 2, 3, 4, 5
- List of all pairs:

$$\begin{array}{ll} \{1,2\}\ g_{1,2}=0 & \{2,4\}\ g_{2,4}=0 \\ \{1,3\}\ g_{1,3}=0 & \{2,5\}\ g_{2,5}=0 \\ \{1,4\}\ g_{1,4}=0 & \{3,4\}\ g_{3,4}=0 \\ \{1,5\}\ g_{1,5}=0 & \{3,5\}\ g_{3,5}=0 \\ \{2,3\}\ g_{2,3}=0 & \{4,5\}\ g_{4,5}=0 \end{array}$$

- Note that f is the number of pairs $\{i, j\}$ with $g_{ij} = 1$;
- The sum of g_{ij} is the same number!

Birthday Problem

Birthday Problem

Consider 28 randomly chosen people. Consider the number of pairs of people among them having birthday on the same day. Show that the expectation of this number is greater than 1.

- Let's get back to the proof
- We know $\mathrm{E}f$ is equal to the sum of $\mathrm{E}g_{ij}$ over all pairs $\{i, j\}$
- We need to compute $\mathrm{E}g_{ij}$
- We also need to count how many pairs of i and j do we have

Birthday Problem

- Expectation of individual g_{ij} is easy to compute:

$$Eg_{ij} = 1 \times \frac{1}{356} + 0 \times \frac{364}{356} = \frac{1}{356}$$

- Why $\frac{1}{356}$?
- There are 365 × 365 outcomes for birthdays of two people
- And only 365 outcomes with birthdays on the same day

Birthday Problem

- How many pairs of *i* and *j* do we have?
  There are 28 people in total
- There are $\binom{28}{2} = \frac{28 \times 27}{2} = 378$ ways to choose an unordered pair among them
- Short reminder: we have 28 options for the first one in the pair, we have 27 options for the second one, and we counted each pair twice

Birthday Problem

Birthday Problem

Consider 27 randomly chosen people. Consider the number of pairs of people among them having birthday on the same day. Show that the expectation of this number is greater than 1.

- Finally, we have the following
- Ef is the sum of Eg_{ij} over all pairs $\{i, j\}$
- $Eg_{ij} = \dfrac{1}{356}$
- There are 378 pairs of people =
- Overall, we have

$\binom{n}{2}$ $\quad Eg_{ij}$

$$Ef = 378 \times \frac{1}{356} > 1$$

Thank you.