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A class of on-line portfolio selection algorithms based on linear learning

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ABSTRACT

On-line portfolio is a sequential investment algorithm during a long period and makes portfolio decision without any statistical assumption about the behavior of the market. A constant rebalanced portfolio (CRP) is an investment strategy which adopts the same portfolio vector on each trading period. Design of on-line portfolio algorithms which are competitive with the best constant rebalanced portfolio (BCRP) is a hot topic recently. In this paper, we present a new on-line portfolio selection strategy, which computes the new portfolio vector based completely on on-line learning of linear functions. The proposed algorithm is useful since it gives the investor a whole range of choices for the on-line portfolios. Using the technique of taking relative entropy as a distance function, we prove that the new algorithm is a universal portfolio, which exhibits the same asymptotic growth rate in normalized natural logarithmic wealth as the BCRP for any sequence of price relatives. Experiments on several New York Stock Exchange dates also show the good performance of the new strategy.

1. Introduction

The portfolio selection problem, which originates from the seminal paper of Markowitz [1], has been studied for over fifty years. Most of existing portfolio selection models are based on probability theory [2–5] and possibility theory (i.e. fuzzy decision theory) [6–8]. These models are proved to be theoretically useful. However, in some cases the feasibility of these models encounters difficulties since it is difficult to predict the trend of stock prices or to find a suitable distribution to describe the trend. These difficulties become even more obvious in the case that the process that governs stock price behavior changes with time. Cover has proposed a different approach to overcome the problems related to the necessity of making statistical assumptions about the stock prices behavior [9]. Indeed, portfolio selection in Cover's investment framework is based totally on the sequence of the past stock prices. To emphasize the on-line manner of choosing portfolio and the independence from statistical assumptions, such portfolios are called on-line portfolios. Previously, Cover first proposed the notion of universal portfolio and the UP algorithm [9] that performs almost as well as the best constant rebalanced portfolio (BCRP). Later, Cover et al. generalized their algorithm and produced several relevant contributions [9–14]. Simultaneously, the competitive theory of on-line portfolio is studied intensively and systematically [10,15,16]. However, Cover's UP approach yields portfolios which are hard to compute for large number of stocks. To solve the computational issue of UP algorithm, Helmbold et al. [17] proposed the EG(η) (exponentiated gradient) update, which employs a multiplicative update rule derived using a framework introduced by Kivinen and Warmuth [18]. They proved that EG(η) is very simple to implement and achieves almost the same wealth as the BCRP determined in hindsight.

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In this paper, we will generalize the on-line portfolio selection algorithm EG(η) [17]. Mainly, we present a new on-line portfolio selection algorithm LF(δ) (based on on-line learning of linear function), which computes the new portfolio vector as a linear function of the previous portfolio vector, i.e. $w_i^{t+1} = c_i^t w_i^t,\ i = 1, 2, \ldots, N$, where N is the number of stocks and w_i^t is the fraction of the wealth invested in the ith stock on day t. The idea behind our algorithm LF(δ) is that the better the performance of stock i on day t is, the larger the weight w_i^{t+1} is. The performance of stock i on day t is measured by its price relative x_i^t. Therefore, the linear coefficient c_i^t is related to x_i^t. The proposed algorithm is interesting since it gives the investor a whole range of choices for the on-line portfolios by using an interval in which linear coefficient c_i^t must lie. Simultaneously, the interval is determined by the parameter δ, which is used to control the interval length and also can be seen as learning rate. Using the technique provided by Helmbold et al. [17], i.e., taking relative entropy as a distance function, we prove that LF(δ) achieves almost the same wealth as BCRP. The theoretical bound we prove on the performance of our algorithm relative to BCRP is as strong as the bound proved by Helmbold et al.. However, our algorithm simply makes a linear learning and provides a class universal portfolios. LF(δ) is identical to EG(η) when LF(δ) takes the left point of the interval to update and $\delta = \eta$. Experiments on several New York Stock Exchange dates clearly show that LF(δ) performs as well as EG(η) algorithm and outperforms Cover's UP algorithm. Our algorithm LF(δ) yields the same asymptotic growth rate of wealth as BCRP without any information about the future inputs while BCRP is constructed with the perfect knowledge of the whole inputs.

The rest of this paper is organized as follows. In the next section, we provide formal notations and a literature review about the on-line portfolio selection algorithm. In Section 3, we give the methodology and procedure of our algorithm LF(δ). In Section 4, we focus on the asymptotic properties of LF(δ). We show that LF(δ) is a class of universal portfolios through the theoretical bound we prove on its performance relative to BCRP. In Section 5, we present numerical experiments. We compare our approach LF(δ) with BCRP and UP, and analyze when and how our approach has achieved superior performance. In the last section, we conclude the paper and give a discussion on the future research.

2. Formal notations and literature review

We assume that the investment in a market without short selling or borrowing, and stock market evolves in discrete time, each day of discrete time will be referred to as trading day. The stock market is composed of N stocks which prices vary from one trading day to another. Denote the performance of the stocks by a vector of price relatives $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, where x_i is the price relative of the ith stock, i.e., the ratio of closing to opening price for stock i. Denote the set of all possible portfolio vectors by

$$\mathbf{B} = \left\{ \mathbf{w} = (w_1, w_2, \ldots, w_N);\ w_i \geqslant 0,\ \sum_{i=1}^{N} w_i = 1 \right\}.$$

Given a portfolio $\mathbf{w}$ and the price relatives $\mathbf{x}$, investor using this portfolio can decrease or increase his wealth from one day to the next day by a factor of $\mathbf{w} \cdot \mathbf{x} = \sum_{i=1}^{N} w_i x_i$.

In this paper we focus on the on-line portfolio selection strategy, which is a sequential investment procedure during a long period and becomes a useful complement to traditional methods. At the beginning of each investing day t, the on-line portfolio immediately selects its portfolio $\mathbf{w}^t$ for the day only based on the information of the previous price relatives of the stock market $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^{t-1}$. At the beginning of the day $t+1$, the wealth increases or decreases by a factor of $\mathbf{w}^t \cdot \mathbf{x}^t$ since the price relatives for day t are obtained. After T trading days, a sequence of daily prices relatives $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^T$ is observed and a sequence of portfolios $\mathbf{w}^1, \mathbf{w}^2, \ldots, \mathbf{w}^T$ is selected correspondingly. Therefore, from the beginning of the first day to the beginning of day $T+1$, the wealth will have increased by a factor of

$$S_T(\{\mathbf{w}^t\}, \{\mathbf{x}^t\}) = \prod_{t=1}^{T} \mathbf{w}^t \cdot \mathbf{x}^t,$$

where the initial wealth $S_0 = 1$ is normalized to 1. The independence from statistical assumption is the typical feature of on-line portfolio. Therefore, what can we use to evaluate the on-line portfolios? The answer is the BCRP. A constant rebalanced portfolio (CRP) strategy adopts the same investment vector $\mathbf{w}$ on each trading day and the resulting wealth after T days is

$$S_T(\mathbf{w}) = S_T(\mathbf{w}, \{\mathbf{x}^t\}) = \prod_{t=1}^{T} \mathbf{w} \cdot \mathbf{x}^t.$$

We notice that a CRP strategy might require vast amounts of trading since at the beginning of each trading day the investment portfolio vectors are rebalanced back to $\mathbf{w}$. In this paper we ignore the commission costs that will be focused on in the future. In retrospect, with information of the sequence of daily price relatives vector $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^T$, there exists a BCRP $\mathbf{w}^*$ such that

$$\mathbf{w}^* = \arg\max_{\mathbf{w} \in \mathbf{B}} S_T(\mathbf{w}),$$

where the maximum is taken over all possible portfolio vectors.

Cover [9] proved that the wealth of BCRP exceeds the wealth of the best stock, the wealth of the arithmetic mean of the N stocks and so on that are good properties. Hence, BCRP has been used as an off-line benchmark for the evaluation of on-line portfolio. An on-line portfolio selection algorithm A is said to be universal if it has the same exponential rate of growth as the apparently unachievable BCRP, i.e., the portfolio sequence $\{\mathbf{w}^t\}$ that generated by the on-line portfolio selection algorithm A should satisfy

$$\lim_{T\to\infty}\max_{\mathbf{x}^t}\left[LS^*(\{\mathbf{x}^t\}) - LS(\{\mathbf{w}^t\},\{\mathbf{x}^t\})\right] \leqslant 0, \tag{1}$$

where

$$LS^*(\{\mathbf{x}^t\}) = \frac{1}{T}\ln S_T(\mathbf{w}^*), \quad LS(\{\mathbf{w}^t\},\{\mathbf{x}^t\}) = \frac{1}{T}\ln S_T(\{\mathbf{w}^t\},\{\mathbf{x}^t\}).$$

Cover first proposed UP algorithm [9] and then extended it to the case with side information [13]. Based on it, Blum and Kalai further extended it by introducing transaction cost [19]. Cover's portfolio selection strategy can be understood that the current portfolio is the weighted average over all CRPs and the weight is decided by the performance of the stocks in the previous days, i.e.

$$\mathbf{w}^{t+1} = \frac{\int_{\mathbf{B}}\mathbf{w}S_t(\mathbf{w})d\mathbf{w}}{\int_{\mathbf{B}}S_t(\mathbf{w})d\mathbf{w}}. \tag{2}$$

Cover's approach can be called analytical as it provides an elegant formula. However, it relies on multidimensional integration, which is difficult to compute for relative large number of stocks.

Computational issue of Cover's universal portfolio was addressed in several papers [20–23]. Gaivoronski et al. using a different set of ideas drawn from nonstationary stochastic optimization [24–26], constructed universal portfolio and presented the adaptive portfolio selection policies. When stock price varies greatly, UP strategy cannot get good performance. Hence, switching method is proposed to design on-line portfolio [27]. Particularly, Helmbold et al. [17] proposed an on-line portfolio algorithm EG(η) that used relative entropy as a distance function for motivating updates. And EG(η) derived portfolio vector $\mathbf{w}^{t+1}$ on step $t+1$ from maximization of the objective function

$$F(\mathbf{w}^{t+1},\eta) = \max\left[\eta\ln(\mathbf{w}^{t+1}\cdot\mathbf{x}^t) - d(\mathbf{w}^{t+1},\mathbf{w}^t)\right], \tag{3}$$

with respect to $\mathbf{w}^t,\ \mathbf{w}^{t+1}\in\mathbf{B}$, where $d(\mathbf{w}^{t+1},\mathbf{w}^t) = \sum_{i=1}^N w_i^{t+1}\ln w_i^{t+1}/w_i^t$ and η represents the learning rate. Combining (3) with the additional constraint $\sum_{i=1}^N w_i^{t+1} = 1$, EG($\eta$) gives the update:

$$w_i^{t+1} = \frac{w_i^t\exp\left(\eta x_i^t/\mathbf{w}^t\cdot\mathbf{x}^t\right)}{\sum_{j=1}^N w_j^t\exp\left(\eta x_j^t/\mathbf{w}^t\cdot\mathbf{x}^t\right)}.$$

Helmbold et al. proved that EG(η) is easy to compute and is universal portfolio. Based on on-line learning of linear function, this paper will generalize EG(η) and provide a class of on-line portfolio selection algorithm LF(δ). Using relative entropy as a distance function, we can easily prove that the generalized algorithm LF(δ) is a class of universal portfolio.

3. On-line portfolio selection algorithm LF(δ)

Our basic on-line portfolio selection algorithm LF(δ) is designed to perform well based on the on-line learning of linear functions set $\tilde{w}_i^{t+1} = w_i^t\mathrm{coe}(\mathbf{w}^t\cdot\mathbf{x}^t, x_i^t,\delta),\ i = 1,2,\ldots,N$. The linear coefficient $\mathrm{coe}(\mathbf{w}^t\cdot\mathbf{x}^t, x_i^t,\delta)$ is chosen arbitrarily from the interval $\left[e^{\frac{\delta x_i^t}{\mathbf{w}^t\cdot\mathbf{x}^t}}, 1+(e^{\frac{\delta}{\mathbf{w}^t\cdot\mathbf{x}^t}}-1)x_i^t\right]$. In the next section, we will prove that this interval has positive length. Therefore, $\tilde{w}_i^{t+1}$ has a range of choices that $\mathrm{coe}(\mathbf{w}^t\cdot\mathbf{x}^t, x_i^t,\delta)$ takes any value in the interval $\left[e^{\frac{\delta x_i^t}{\mathbf{w}^t\cdot\mathbf{x}^t}}, 1+(e^{\frac{\delta}{\mathbf{w}^t\cdot\mathbf{x}^t}}-1)x_i^t\right]$ can update to acquire the new weight for stock i in the $(t+1)$th trading day. Then $\tilde{w}_i^{t+1}$ needs to be normalized to satisfy that the sum of the total weight of the N stocks is 1, i.e., $\sum_{i=1}^N\tilde{w}_i^{t+1} = 1$. Simply, we initialize the portfolio weight vector to uniform distribution, i.e., $w_i = 1/N$. The parameter δ can be seen as learning rate ($\delta > 0$) and cannot be too large as it controls the length of the interval $\left[e^{\frac{\delta x_i^t}{\mathbf{w}^t\cdot\mathbf{x}^t}}, 1+(e^{\frac{\delta}{\mathbf{w}^t\cdot\mathbf{x}^t}}-1)x_i^t\right]$. In each trading day, the linear coefficient $\mathrm{coe}(\mathbf{w}^t\cdot\mathbf{x}^t, x_i^t,\delta)$ is related to $\mathbf{w}^t\cdot\mathbf{x}^t, x_i^t$, and δ. The idea behind our algorithm is that the larger the price relative x_i^t is, the greater the new weight w_i^{t+1} is. This methodology is similar to that in [9,17]. However, LF(δ) gives a whole range of on-line portfolio sequences, each of which is a universal portfolio sequence. The steps of the algorithm LF(δ) is given as follows.

The steps of on-line portfolio selection algorithm LF(δ):

- *Initialization*. At time $t = 0$, set initial portfolio weight vector $\mathbf{w}^0 = (1/N,\ldots,1/N)$.
- *Iteration weight vector*. At the end of tth day, set $\tilde{w}_i^{t+1} = w_i^t\mathrm{coe}(\mathbf{w}^t\cdot\mathbf{x}^t, x_i^t,\delta)$, where $\mathrm{coe}(\mathbf{w}^t\cdot\mathbf{x}^t, x_i^t,\delta)\in\left[e^{\frac{\delta x_i^t}{\mathbf{w}^t\cdot\mathbf{x}^t}}, 1+(e^{\frac{\delta}{\mathbf{w}^t\cdot\mathbf{x}^t}}-1)x_i^t\right]$.
- *Normalization*. Set $w_i^{t+1} = \tilde{w}_i^{t+1}/Z_t$, where $Z_t = \sum_{j=1}^N w_j^t\mathrm{coe}(\mathbf{w}^t\cdot\mathbf{x}^t, x_j^t,\delta)$.
- After T trading days, the sequential portfolio is $\{\mathbf{w}^t\}_{t=0}^{T-1}$.

The natural logarithmic wealth achieved by LF(δ) is $\sum_{t=0}^{T-1}\ln(\mathbf{w}^t\cdot\mathbf{x}^t)$.

In order to analyze LF(δ) algorithm, we use relative entropy as the distance function, i.e., the distance between portfolio $\mathbf{w}^{t+1}$ and $\mathbf{w}^t$ is measured by

$$d(\mathbf{w}^{t+1},\mathbf{w}^t)=\sum_{i=1}^{N}w_i^{t+1}\ln w_i^{t+1}/w_i^t.$$

This measurement is already used by Helmbold et al. [17] to research the on-line portfolio selection algorithm EG(η). Using the method provided in [17], we can easily prove that LF(δ) is a universal portfolio which achieves the same exponential rate of growth as that of BCRP. The main conclusions are listed in the following sections show the reason LF(δ) works.

4. Competitive analysis of on-line portfolio selection algorithm LF(δ)

The goal of on-line portfolio is to provide sequential trading strategies that are competitive in some sense. Previously research initiated by Cover [9,13] who attempts to design portfolio selection strategies that can get good performance with respect to some off-line benchmark. Cover [9] proved that BCRP can be used as an off-line benchmark as it possesses some good properties. And Cover proposed the notion of universal portfolio, i.e., the universal portfolio guarantees a sub-exponential ratio in total trading T days between its return and that of BCRP. Moreover, Cover and Ordentlich designed the UP algorithms [9,13]. Using BCRP as the benchmark of on-line portfolio, Helmbold et al. [17] proposed EG(η) algorithm and proved it is universal portfolio. In this section, we analyze the logarithmic wealth obtained by LF(δ) algorithm. When certain assumptions on the price relative holds, we show that LF(δ) is a class universal portfolios through the theoretical bound we prove on its performance relative to BCRP. And the bound implies that LF(δ) is as good as EG(η). The first assumption is that the lower bound on the price relative x_i^t is r, i.e. $x_i^t\geqslant r$, which can be viewed as a lower bound on the ratio of the worst to best price relatives for the trading day t. The second assumption is that $\max x_i^t=1$, which is reasonable since there exists $c_1\geqslant 1$ that satisfies $x_i^t=c_1y_i^t,\ y_i^t\leqslant 1$ and we can use the new relative price y_i^t to compute without making any difference between the natural logarithmic wealth achieved by the LF(δ) and that achieved by BCRP. In other words, the price relative can be normalized to satisfy $x_i^t\leqslant 1$. Therefore, the natural logarithmic wealth is also called normalized one since the wealth is computed after price relative being normalized. Firstly, under the assumption $r\leqslant x_i^t\leqslant 1$, we can acquire Theorem 4.1, which shows the lower bound on the total natural logarithmic wealth of LF(δ).

Theorem 4.1. *Let $\mathbf{w}\in\mathbf{B}$ be a portfolio vector, and $\mathbf{x}^0,\ldots,\mathbf{x}^{T-1}$ be a sequence of price relatives. Under the assumption that $r\leqslant x_i^t\leqslant 1$, the portfolio vectors $\{\mathbf{w}^t\}_{t=0}^{T-1}$ generated by LF(δ) has the following properties:*

(i) For $0<\delta\leqslant 1$, the lower bound of the normalized natural logarithmic wealth of the portfolio vectors $\{\mathbf{w}^t\}_{t=0}^{T-1}$ is as follows:

$$\sum_{t=0}^{T-1}\ln(\mathbf{w}^t\cdot\mathbf{x}^t)\geqslant\sum_{t=0}^{T-1}\ln(\mathbf{w}\cdot\mathbf{x}^t)-\frac{\ln N}{\delta}-\frac{\delta T}{8r^2}.$$

(ii) If $\delta\in[2r\sqrt{2\ln N/T},r\sqrt{2\ln N/T}]$ (T is suitable large), then LF(δ) is a universal portfolio.

Proof

(i) Let $\Pi_t=d(\mathbf{w},\mathbf{w}^{t+1})-d(\mathbf{w},\mathbf{w}^t)$, then

$$\Pi_t=-\sum_{i=1}^{N}w_i\ln\frac{w_i^{t+1}}{w_i^t}=-\sum_{i=1}^{N}w_i\ln\mathrm{coe}(\mathbf{w}^t\cdot\mathbf{x}^t,x_i^t,\delta)+\ln Z_t,$$

where

$$Z_t=\sum_{j=1}^{N}w_j^t\mathrm{coe}(\mathbf{w}^t\cdot\mathbf{x}^t,x_j^t,\delta)$$

and

$$\mathrm{coe}(\mathbf{w}^t\cdot\mathbf{x}^t,x_i^t,\delta)\in\left[e^{\frac{\delta x_i^t}{\mathbf{w}^t\cdot\mathbf{x}^t}},1+\left(e^{\frac{\delta}{\mathbf{w}^t\cdot\mathbf{x}^t}}-1\right)x_i^t\right].$$

We claim that $1+(a-1)x>a^x$ holds for $a>0$ and $x\in[0,1]$. In fact, setting $f(x)=a^x+(1-a)x$, it is equivalent to prove that $f(x)\leqslant 1$. Obviously, $f''(x)=a^x(\ln a)^2\geqslant 0$ and $f(0)=f(1)=1$. Thus $f(x)$ is convex and has maximum 1 on the interval [0,1]. Hence, $f(x)\leqslant 1$ holds, which assures us that the interval in which $\mathrm{coe}(\mathbf{w}^t\cdot\mathbf{x}^t,x_i^t,\delta)$ must lie has positive length. Therefore,

$$\Pi_t\leqslant-\sum_{i=1}^{N}w_i\ln\exp\left(\delta x_i^t/\mathbf{w}^t\cdot\mathbf{x}^t\right)+\ln Z_t=-\delta\frac{\mathbf{w}\cdot\mathbf{x}^t}{\mathbf{w}^t\cdot\mathbf{x}^t}+\ln Z_t\leqslant-\delta\frac{\mathbf{w}\cdot\mathbf{x}^t}{\mathbf{w}^t\cdot\mathbf{x}^t}+\ln\left(1+\left(e^{\delta/\mathbf{w}^t\cdot\mathbf{x}^t}-1\right)\mathbf{w}^t\cdot\mathbf{x}^t\right).$$

On the other hand, the inequality $\ln(1-\alpha(1-e^x)) \leqslant \alpha x + x^2/8$ holds for $\alpha \in [0,1]$ and $x \in [0,+\infty)$. In fact, we consider function

$$g(x) = \alpha x + x^2/8 - \ln(1-\alpha(1-e^x)).$$

The first two derivatives of $g(x)$ are

$$g'(x) = \alpha + \frac{x}{4} - \frac{\alpha e^x}{1-\alpha+\alpha e^x}$$

and

$$g''(x) = \frac{1}{4} - \frac{\alpha(1-\alpha)e^x}{(1-\alpha+\alpha e^x)^2} = \frac{(1-\alpha-\alpha e^x)^2}{4(1-\alpha+\alpha e^x)^2} > 0.$$

Hence, $g'(x) > g'(0) = 0$ and $g(x) > g(0) = 0$.
Therefore,

$$\Pi_t \leqslant -\delta \frac{\mathbf{w}\cdot\mathbf{x}^t}{\mathbf{w}^t\cdot\mathbf{x}^t} + \delta + \frac{\delta^2}{8(\mathbf{w}^t\cdot\mathbf{x}^t)^2} = \delta\left(1 - \frac{\mathbf{w}\cdot\mathbf{x}^t}{\mathbf{w}^t\cdot\mathbf{x}^t}\right) + \frac{\delta^2}{8(\mathbf{w}^t\cdot\mathbf{x}^t)^2}.$$

By inequalities $e^x \geqslant 1+x$ and $x_i^t \geqslant r$, we have

$$\Pi_t \leqslant \delta \ln \frac{\mathbf{w}^t\cdot\mathbf{x}^t}{\mathbf{w}\cdot\mathbf{x}^t} + \frac{\delta^2}{8(\mathbf{w}^t\cdot\mathbf{x}^t)^2} \leqslant \delta \ln \frac{\mathbf{w}^t\cdot\mathbf{x}^t}{\mathbf{w}\cdot\mathbf{x}^t} + \frac{\delta^2}{8r^2}.$$

Furthermore, we obtain

$$-d(\mathbf{w},\mathbf{w}^0) \leqslant d(\mathbf{w},\mathbf{w}^T) - d(\mathbf{w},\mathbf{w}^0) = \sum_{t=0}^{T-1} \Pi_t \leqslant \sum_{t=0}^{T-1} \delta \ln \frac{\mathbf{w}^t\cdot\mathbf{x}^t}{\mathbf{w}\cdot\mathbf{x}^t} + \frac{\delta^2 T}{8r^2}.$$

Together with the condition that $-d(\mathbf{w},\mathbf{w}^0) \geqslant -\ln N$, it holds that

$$\sum_{t=0}^{T-1} \ln(\mathbf{w}^t\cdot\mathbf{x}^t) \geqslant \sum_{t=0}^{T-1} \ln(\mathbf{w}\cdot\mathbf{x}^t) - \frac{\ln N}{\delta} - \frac{\delta T}{8r^2}.$$

(ii) When $\delta \in [2r\sqrt{2\ln N/T}, r\sqrt{2\ln N/T}]$, we have

$$\frac{\ln N}{\delta T} + \frac{\delta}{8r^2} \leqslant \sqrt{\frac{\ln N}{8r^2 T}} + \sqrt{\frac{\ln N}{32r^2 T}}.$$

Hence,

$$\lim_{T\to\infty} \frac{\sum_{t=0}^{T-1}\ln(\mathbf{w}^t\cdot\mathbf{x}^t) - \sum_{t=0}^{T-1}\ln(\mathbf{w}\cdot\mathbf{x}^t)}{T} \geqslant -\lim_{T\to\infty}\left(\frac{\ln N}{\delta T} + \frac{\delta}{8r^2}\right) \geqslant -\lim_{T\to\infty}\left(\sqrt{\frac{\ln N}{8r^2T}} + \sqrt{\frac{\ln N}{32r^2T}}\right) = 0.$$

Therefore, the portfolio vectors $\{\mathbf{w}^t\}_{t=0}^{T-1}$ satisfies (1) and thus $\mathrm{LF}(\delta)$ is a class of universal portfolio algorithms. □

However, under the assumption of Theorem 4.1, choosing the proper δ requires knowledge of both the number of trading days T and the price relatives' lower bound r in advance. The two difficulties can be dealt with by special techniques presented in [17].

When the lower bound of x_i^t is unknown, let $\tilde{\mathbf{x}}^t = (1-\alpha/N)\mathbf{x}^t + (\alpha/N)\mathbf{1},\ \alpha \in [0,1]$. We obtain the updated portfolio vector $\mathbf{w}^{t+1}$ by using $\tilde{\mathbf{x}}^t$ rather than $\mathbf{x}^t$. Then, the new weight is expressed by

$$w_i^{t+1} = \frac{w_i^t \mathrm{coe}(\mathbf{w}^t\cdot\tilde{\mathbf{x}}^t, \tilde{x}_i^t, \delta)}{\sum_{j=1}^N w_j^t \mathrm{coe}(\mathbf{w}^t\cdot\tilde{\mathbf{x}}^t, \tilde{x}_j^t, \delta)},$$

where $\mathrm{coe}(\mathbf{w}^t\cdot\tilde{\mathbf{x}}^t, \tilde{x}_i^t, \delta) \in \left[e^{\frac{\delta\tilde{x}_i^t}{\mathbf{w}^t\cdot\tilde{\mathbf{x}}^t}}, 1 + \left(e^{\frac{\delta}{\mathbf{w}^t\cdot\tilde{\mathbf{x}}^t}} - 1\right)\tilde{x}_i^t\right]$.

Meanwhile, the invested portfolio vector is also changed a little. The modified algorithm denoted by $\widetilde{\mathrm{LF}}(\alpha,\delta)$ uses the portfolio vector

$$\tilde{\mathbf{w}}^t = (1-\alpha)\mathbf{w}^t + \left(\frac{\alpha}{N}\right)\mathbf{1}.$$

Therefore, the normalized natural logarithmic wealth achieved is $\sum_{t=0}^{T-1}\ln(\tilde{\mathbf{w}}^t\cdot\mathbf{x}^t)$.

The following conclusions are obtained exactly based on [17].

Theorem 4.2. *Let $\mathbf{w} \in \mathbf{B}$ be a portfolio vector, and $\mathbf{x}^0, \ldots, \mathbf{x}^{T-1}$ be a sequence of price relatives with $0 < x_i^t \leqslant 1$. For $\alpha \in (0, 1/2]$ and $\delta > 0$, the normalized natural logarithmic wealth due to the portfolio vectors produced by the strategy $\widetilde{LF}(\alpha, \delta)$ is bounded from below as follows:*

$$\sum_{t=0}^{T-1} \ln(\tilde{\mathbf{w}}^t \cdot \mathbf{x}^t) \geqslant \sum_{t=0}^{T-1} \ln(\mathbf{w} \cdot \mathbf{x}^t) - \frac{\ln N}{\delta} - 2\alpha T - \frac{\delta T}{8(\alpha/N)^2}.$$

Furthermore, if $\mathbf{w}^0 = (1/N, 1/N, \ldots, 1/N)$, $T \geqslant 2N^2 \ln N$, and setting $\alpha = (N^2 \ln N/(8T))^{1/4}$, $\delta = \sqrt{8\alpha^2 \ln N/(N^2 T)}$, we have

$$\sum_{t=0}^{T-1} \ln(\tilde{\mathbf{w}}^t \cdot \mathbf{x}^t) \geqslant \sum_{t=0}^{T-1} \ln(\mathbf{w} \cdot \mathbf{x}^t) - 2(2N^2 \ln N)^{1/4} T^{3/4}.$$

Theorem 4.2 shows that the growth rate achieved by $\widetilde{\mathrm{LF}}(\alpha, \delta)$ converges to that of BCRP. However, the learning rate δ is dependent on the total trading days T. Hence, Theorem 4.2 is still not strong enough to show that $\widetilde{\mathrm{LF}}(\alpha, \delta)$ is a universal portfolio selection algorithm. Helmbold et al. proposed a technique of dividing the total on-line trading days into stages to deal with the problem. For large enough T ($T > 2N^2 \ln N$), the staged technique works as follows. The first and second $2N^2 \ln N$ trading days is numbered by stage 0 and 1, respectively. And the following $2^i N^2 \ln N$ ($i > 1$) trading days is numbered by stage i. At the start of each stage the portfolio vector is reinitialized to the uniform portfolio vector and α and δ are set as in Theorem 4.2 using $2^i N^2 \ln N$ in stage $i (i \geqslant 1)$ as the value for T (number of trading days in stage 0 is $2N^2 \ln N$). In each stage i, the $\widetilde{\mathrm{LF}}(\alpha, \delta)$ algorithm is carried out to obtain the staged portfolio vectors and we call this algorithm for staged $\widetilde{\mathrm{LF}}(\alpha, \delta)$.

Theorem 4.3. *The staged $\widetilde{LF}(\alpha, \delta)$ is a universal portfolio selection algorithm.*

Proof. We first compute the upper bound b of the last stage number. From the staged $\widetilde{\mathrm{LF}}(\alpha, \delta)$ algorithm above, we have inequalities

$$2N^2 \ln N + 2N^2 \ln N + 2^2 N^2 \ln N + \cdots + 2^{b-1} N^2 \ln N < T$$

and

$$T \leqslant 2N^2 \ln N + 2N^2 \ln N + 2^2 N^2 \ln N + \cdots + 2^b N^2 \ln N.$$

Hence, we have

$$\log_2(T/2N^2 \ln N) \leqslant b < \log_2(T/2N^2 \ln N) + 1.$$

Therefore, $b = [\log_2(T/2N^2 \ln N)]$. From Theorem 4.2 and [17], we obtain

$$\lim_{T\to\infty} \frac{\sum_{t=0}^{T-1} \ln(\tilde{\mathbf{w}}^t \cdot \mathbf{x}^t) - \sum_{t=0}^{T-1} \ln(\mathbf{w} \cdot \mathbf{x}^t)}{T} \geqslant -\lim_{T\to\infty} \frac{6N^2 \ln N}{T} \left(1 + \left(\frac{T}{2N^2 \ln N}\right)^{3/4}\right) = 0.$$

Hence, the staged $\widetilde{\mathrm{LF}}(\alpha, \delta)$ is a universal portfolio selection algorithm. □

Thus, LF(δ) gives a whole range of on-line portfolio sequences, each portfolio sequence of which is a universal portfolio selection algorithm.

5. Numerical analysis

In this section, we present numerical experiments which illustrate the performance of the portfolio selection algorithm LF(δ). For each experiment, the data is obtained from the New York Stock Exchange (NYSE) accumulated from 1981 to 2007 year including 6460 trading days. The code, the name abbreviation, the full name and the increase factor of each stock we used are listed in Table 1.

In particular we compare the performance of our portfolio, namely the LF(δ) algorithm, with that of UP and EG(η) algorithm. We also compare the wealth achieved by our LF(δ) algorithm with the wealth achieved by BCRP, which is obtained

Table 1
The code, the name abbreviation, the full name and the increase factor of each stock.

Code	Name abbreviation	Full name	Increase factor
9N3570	XOM	Exxon Mobil Corp	47.7778
9N2106	MRK	Merck Co Inc	29.4017
9N1969	MCD	McDonald's Corp	53.625
9N1793	KO	Coca-Cola Co (Coke)	123.9166
9N2531	PG	The Procter & Gamble Co	206.2951

Table 2
Comparison between BCRP and LF(δ) on MCD and XOM.

Stocks	BCRP	LF(δ)	δ
MCD and XOM	69.1955	68.9111	0.005
		68.8063	0.01
		68.0034	0.05

Table 3
Comparison between BCRP and LF(δ) on PG and KO.

Stocks	BCRP	LF(δ)	δ
PG and KO	208.0622	188.4235	0.005
		188.3004	0.01
		187.3523	0.05

Table 4
Comparison between BCRP and LF(δ) on MCD-KO-PG.

Stocks	BCRP	LF(δ)	δ
MCD-KO-PG	208.0622	158.7033	0.005
		158.5062	0.01
		156.9792	0.05

Table 5
Comparison between BCRP and LF(δ) on XOM-MRK-MCD.

Stocks	BCRP	LF(δ)	δ
XOM-MRK-MCD	69.3320	63.7500	0.005
		63.6307	0.01
		62.7088	0.05

using widely available MATLAB environment. The same environment is also used to implement the LF(δ) algorithm and the EG(η) algorithm and the UP algorithm.

The first four examples use the following subsets of stocks: MCD and XOM, PG and KO, MCD-KO-PG, XOM-MRK-MCD. The first two portfolios consist of two stocks while last two portfolios consist of three stocks. Relative wealth achieved by LF(δ) by the end of the 6460 day period for different values of δ is reported in Tables 2–5. From these tables, we conclude that the small values of $\delta(\delta = 0.005)$ are good choices in each example. The intuition explanation is that the total length of the period is very long and smooth effects of δ should be felt through reasonable portion of the whole period. These tables also show that LF(δ) performs almost as well as BCRP, taking into account that LF(δ) knows only the past while BCRP knows everything about the future stock behavior.

At the beginning of each trading day $t+1$, the on-line investor has a whole range choices of w_i^{t+1} since $\mathrm{coe}(\mathbf{w}^t \cdot \mathbf{x}^t, x_i^t, \delta)$ can take any value in the interval $\left[e^{\frac{\delta x_i^t}{\mathbf{w}^t \cdot \mathbf{x}^t}}, 1 + (e^{\frac{\delta}{\mathbf{w}^t \cdot \mathbf{x}^t}} - 1)x_i^t\right]$ to produce a portfolio weight for stock *i*. Theorem 4.1 shows that all the portfolio sequences generated by LF(δ) are universal while the difference between the portfolio sequences that generated by different $\mathrm{coe}(\mathbf{w}^t \cdot \mathbf{x}^t, x_i^t, \delta)$ is unknown. For simplicity, we only take three choices of the $\mathrm{coe}(\mathbf{w}^t \cdot \mathbf{x}^t, x_i^t, \delta)$, i.e., the left point, the middle point, and the right point of the interval $\left[e^{\frac{\delta x_i^t}{\mathbf{w}^t \cdot \mathbf{x}^t}}, 1 + (e^{\frac{\delta}{\mathbf{w}^t \cdot \mathbf{x}^t}} - 1)x_i^t\right]$. The algorithms under the three special choices of $\mathrm{coe}(\mathbf{w}^t \cdot \mathbf{x}^t, x_i^t, \delta)$ are denoted by $\mathrm{LF}_{\mathrm{lef}}(\delta)$, $\mathrm{LF}_{\mathrm{mid}}(\delta)$, $\mathrm{LF}_{\mathrm{rig}}(\delta)$, respectively. Obviously, algorithm $\mathrm{LF}_{\mathrm{left}}(\delta)$ is exactly identical to EG(η) when $\delta = \eta$. The comparison among the wealth achieved by LF(δ) under different choices of $\mathrm{coe}(\mathbf{w}^t \cdot \mathbf{x}^t, x_i^t, \delta)$ is given in Table 6. We can see that the effect of $\mathrm{coe}(\mathbf{w}^t \cdot \mathbf{x}^t, x_i^t, \delta)$ on the achieved wealth is very small. And the difference between the portfolio sequences that generated by different $\mathrm{coe}(\mathbf{w}^t \cdot \mathbf{x}^t, x_i^t, \delta)$ can be negligible.

In Table 7, a comparison of wealths achieved by BCRP, UP, EG(η) and our algorithm LF(δ) is reported. Summarizing the results in Table 7, we find that LF(δ) performs as well as EG(η) and outperforms UP. In all the experiments we implemented, the wealth achieved by LF(δ) is a little larger than that achieved by EG(η), and the difference of relative wealth is often around 1. Notice in the last column the ratio of the wealth achieved by LF(δ) to the wealth achieved by BCRP is reported as L/B. In several cases the performance of LF(δ) is almost as good as the performance of BCRP. Three examples comparing

Table 6
Comparison between the wealth achieved by LF(δ) under different choices of coe($\mathbf{w}^t \cdot \mathbf{x}^t, x_i^t, \delta$).

Stocks	MCD-KO	MCD-XOM	PG-KO	MCD-KO-PG	XOM-MRK-MCD
$LF_{lef}(\delta)$	112.8595	68.9109	188.4232	158.7028	63.7497
$LF_{mid}(\delta)$	112.8596	68.9110	188.4234	158.7030	63.7499
$LF_{rig}(\delta)$	112.8598	68.9111	188.4235	158.7033	63.7500

Table 7
Comparison of the wealth achieved by BCRP, UP, EG(η) and LF(δ).

Stocks	BCRP	UP	EG(η)	LF(δ)	L/B
MCD and KO	129.2834	104.4105	111.6458	112.8598	0.8730
MCD and XOM	69.1955	62.4158	67.9839	68.9111	0.9959
PG and KO	208.0622	179.4921	187.3192	188.4235	0.9056
MCD-KO-PG	208.0622	109.7007	156.9251	158.7033	0.7628
XOM-MRK-MCD	69.3320	44.3428	62.6847	63.7500	0.9041

the daily wealth achieved by LF(δ) and BCRP are depicted in Figs. 1–3. Note that when the two stocks show a lock-step performance, as in the case of MCD and XOM, the wealth achieved by LF(δ) and EG(η) as well as BCRP greatly outperforms the individual stocks (see the increase factor in Table 1 and the wealth achieved in Table 7). The case of MCD and XOM also shows that LF(δ) approximates quite well the performance of BCRP (see Fig. 1). The KO-MCD portfolio in Fig. 2 shows that LF (δ) performs almost as well as BCRP in the first 5000 trading days. The discrepancy grows larger between days 5000 and 6000. However, in the last trading day 6460, the discrepancy grows smaller as Table 7 shows the difference is only 16.42. The PG-KO portfolio plotted in Fig. 3 exhibits some interesting behavior that LF(δ) performs almost as well as BCRP at every point.

It is clear from the tables and figures that the wealth achieved by LF(δ) is consistently higher than that of EG(η) and UP. More interesting, LF(δ) has generalize EG(η) that it not only contains EG(η) as a special case, but also can perform a little better than EG(η). And all the examples show that LF(δ) performs almost as well as BCRP. Thus LF(δ) exhibits competitive performance compared with previous approaches.

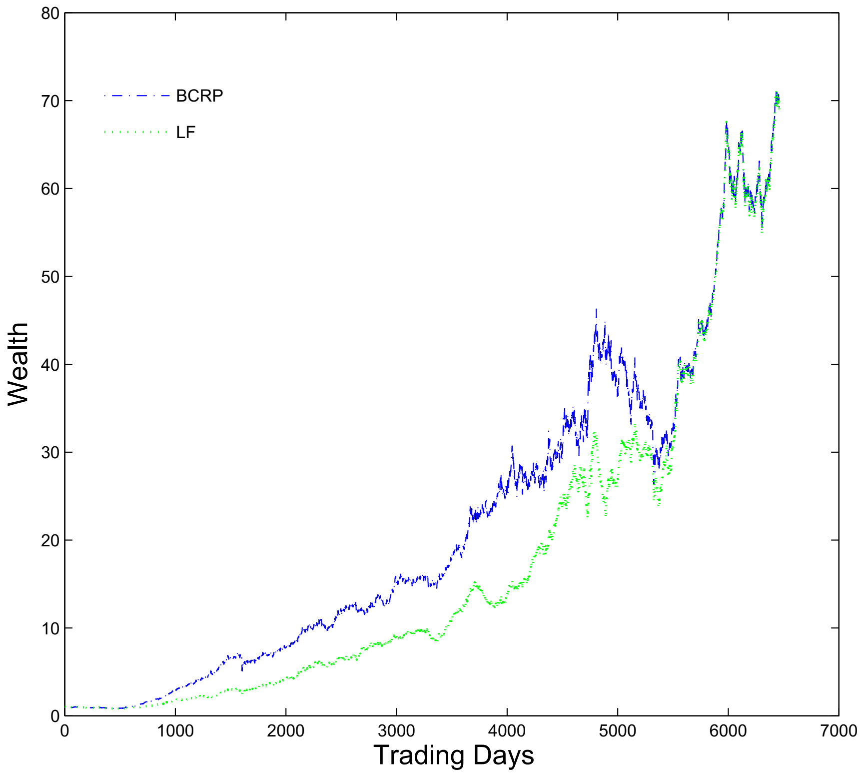

Fig. 1. Comparison of wealths achieved by BCRP and LF(δ) for a portfolio consisting of MCD and XOM.

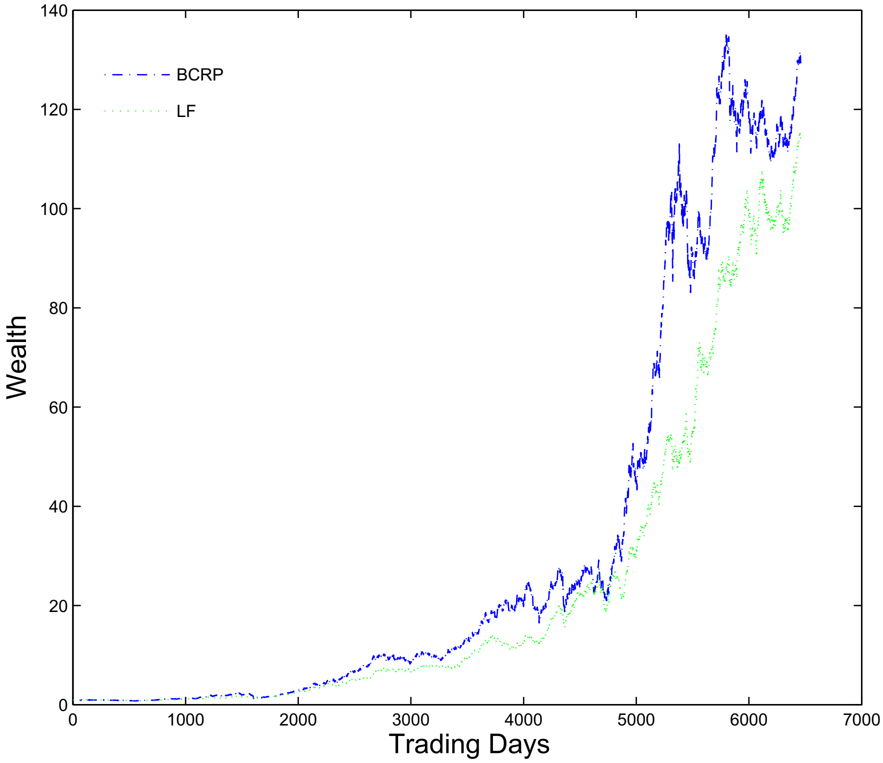

Fig. 2. Comparison of wealths achieved by BCRP and LF(δ) for a portfolio consisting of KO and MCD.

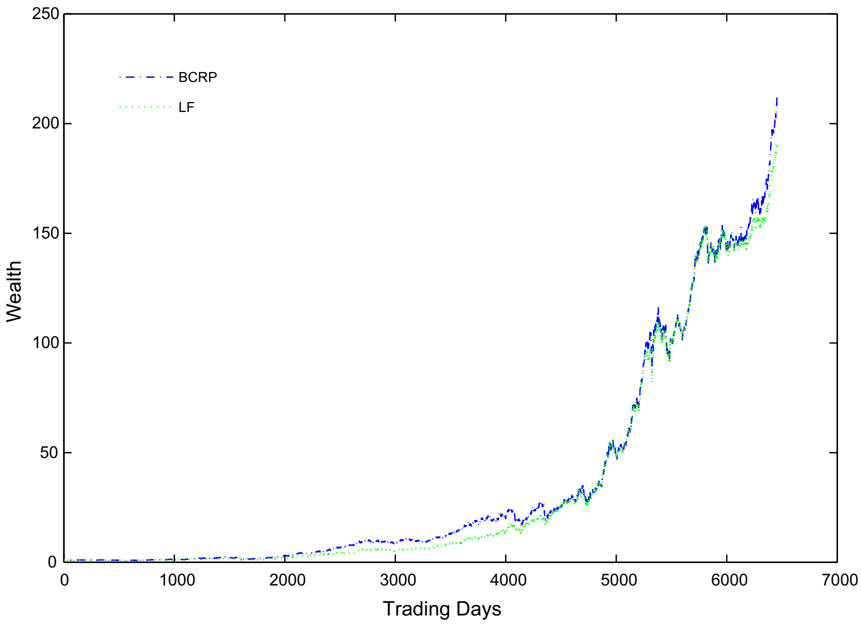

Fig. 3. Comparison of wealths achieved by BCRP and LF(δ) for a portfolio consisting of PG and KO.

6. Conclusions

In this paper, we have considered the on-line portfolio selection problem without any statistical assumption about the behavior of the market. First, we propose the algorithm LF(δ) based on the on-line learning of linear function that the weight update is actually based on linear function of the previous days' performance. Second, we prove that LF(δ) is a universal portfolio selection algorithm. Our proposed algorithm LF(δ) is interesting since it provides the investor a class of universal portfolios. And LF(δ) has generalized the EG(η) algorithm since it contains EG(η) as a special case. However, this method also has its shortcoming that the updated portfolio is obtained only based on current performance. Hence, LF(δ) is similar to EG(η) and UP that can only get good performance when the stocks change slowly and stably. In other words, the investment strategies are proposed under the assumption that the market is stationary. However, this assumption is a little far from the reality. Therefore, how to construct a good on-line portfolio selection strategy in a changing market is of great importance. On the other hand, this paper has ignored the trading costs, which is an important aspect of the real market. Therefore, how to introduce the trading cost into the LF(δ) algorithm is also interesting.

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