Information Sciences 462 (2018) 278–289

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

# Portfolio selection under different attitudes in fuzzy environment

Xiaoyang Zhou[a,b], Jue Wang[b,*], Xiangping Yang[a], Benjamin Lev[c], Yan Tu[d,*], Shouyang Wang[b]

[a] School of Economics & Management, Xidian University, Xi'an 710126, China
[b] Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
[c] LeBow College of Business, Drexel University, Philadelphia, PA 19104, USA
[d] School of Management, Wuhan University of Technology, Wuhan 430070, China

ARTICLE INFO

*Article history:*
Received 30 December 2016
Revised 28 December 2017
Accepted 4 June 2018
Available online 7 June 2018

*Keywords:*
Portfolio selection
Conservative-neutral-aggressive attitude
Fuzzy variable
Value at risk, ε-constraint method

ABSTRACT

This paper studies stock portfolio selection problem based on varying conservative-neutral-aggressive attitudes. The return rates of stocks are characterized by fuzzy variables. The Pareto-optimal solutions are obtained by maximizing the return and minimizing the risk subject to constraints of transaction cost and value at risk. Since investors with different attitudes may have different understanding of the likelihoods of occurrence, measure Me with the ability of reflecting varying conservative-neutral-aggressive attitudes is adopted. Based on Me, the expected value of fuzzy return and the lower absolute deviation are used to quantify the return and risk levels of a portfolio respectively. Then the ε-constraint method is employed to obtain the efficient frontier. Finally, an empirical study is carried out using the data of 10 stocks in Chinese stock market. Sensitivity comparisons are conducted to demonstrate the effectiveness of the proposed model. The results show that different frontiers can be obtained under different attitudes, confidence levels and values at risk.



## 1. Introduction

Portfolio selection involves allocation of capital among a large number of securities such that the investment yields the most profitable return, while minimizing risk. Investors in stock market always make decisions of portfolio selection for future, the realized value of any stock is difficult to predict precisely due to the uncertainty of future markets. Therefore, investors usually have no choice but rely on the historical data. Most of literatures considered the return rate of a stock was a random variable and the distribution parameters could be estimated from the historical data. Under this assumption, a great deal of portfolio selection models have been built based on the probability theory.

In 1952, Markowitz [7,8] published his pioneering work, which served as the basis for the development of the modern portfolio theory over the past several decades. Markowitz's model used variance to describe the risk by the biased degree between effective rate of return and the expected rate of return. However, variance calculated by the total deviation from the expected return describes both the downside risk and the upside risk. In reality, investors don't like the downside risk, but they are actually willing to accept the upside risk. Using variance thus may limit the potential profits as well. In 1952, Roy

* Corresponding authors.
*E-mail addresses:* wjue@amss.ac.cn (J. Wang), tuyan_belle@163.com (Y. Tu).

https://doi.org/10.1016/j.ins.2018.06.013

[13] proposed the safety-first portfolio model, which only minimized the probability of the downside risk. Another standard benchmark for measuring firm-wide risk is Value at Risk (VaR). For a given time horizon and the confidence level $\beta$, the VaR of a portfolio is the loss in the portfolio's market value over the time horizon that is exceeded with probability $1-\beta$. For example, a 99% VaR for a 10-day holding period implies that the maximum loss incurred over the next 10 days should only exceed the limit once in every100 cases. Hence, the VaR shows the potential downside risk that investors need to face on nominal losses [19].

However, many factors such as social, economic, political and investor's psychology exist in addition to the probabilistic factor. More and more researchers have realized that fuzzy set theory introduced by Zadeh [18] is an appropriate tool to handle the vagueness or ambiguity in the stock markets. In 1988, Dubois and Prade [2] introduced the possibility space which was similar to the pattern space proposed by Nahmias [11] in 1978, and a fuzzy variable was first defined as a mapping from a possibility space to a real number. After that, Dubois and Prade carried out research work on possibility theory and gave definitions for the two basic measures of a fuzzy variable: the possibility and the necessity. For the same fuzzy event, different decision makers may have different understanding of the likelihoods of occurrence. If the decision maker is more aggressive, he/she will have a larger value when measuring the likelihood; if the decision maker is more conservative, he/she will have a smaller value when measuring the likelihood. Because possibility of a fuzzy event is always larger than its necessity, possibility can be considered as a measure with an aggressive attitude, and necessity, a measure with a conservative attitude. In addition, Liu and Liu [5,6] defined a self-dual measure (i.e., credibility measure) which can be used to quantify the likelihood under neutral attitude. Xu et al. [16] developed optimistic and pessimistic portfolio selection models under fuzzy random environment. There are a number of studies of employing the fuzzy set theory to solve portfolio selection problem: Kocadağlı and Keskin [4] introduced a novel fuzzy portfolio selection model that took into accounts the risk preferences in accordance with the market moving trends as well as the risk-return tradeoff, and allowed the decision makers to define a certain importance and priority among their objectives. Zhou et al. [20] demonstrated that the mean-semi-entropy models can significantly improve the dispersion of investment by proposing fuzzy semi-entropy to quantify the downside uncertainty.

In the fuzzy environment, the investors will process the existing information subjectively and then make decisions based on the processed data. Even facing with the same historical data of a stock, investors may hold different conservative-neutral-aggressive attitudes and thus expect different future values for the stock. Someone may be optimistic about it, someone maybe not, so it is reasonable for them to have different expected values of the uncertain return rate of the stock. Recently, Berman et al. [12] studied the effect of a decision maker's risk attitude on the median and center problems. Therefore, in a fuzzy portfolio selection problem, the investor with different attitudes may have various efficient frontiers.

However, to the best of our knowledge, very few of the previous studies only provides one kind of decision making result for different investors. The study about the portfolio selection with different attitude is still in exploration phase. The portfolio selection with considering the behavioral of the investor is attracting more and more attention. Tsaur [15] examined investor risk attitudes—risk-averse, risk-neutral, or risk-seeking behaviors—to discover an efficient method for fuzzy portfolio selection. Momen et al. [10] developed a behavioral portfolio selection model that used a robust estimator for expected returns in order to produce portfolios with less need to change over consecutive periods. And they also considered investor attitudes toward risk through spectral risk measure as well as investor expectations on future returns by means of the Black–Litterman model. Sun [14] proposed a novel fuzzy random portfolio selection model with investor psychology and behavior to provide different investment strategies for different investors with different rational-emotional, optimistic-pessimistic and risk preference attitudes.

Therefore, this paper aims to deal with the portfolio selection problem with different conservative-neutral-aggressive attitudes and discuss how the results change when the attitude of investor varies. We use fuzzy variable to characterize the return rate of a stock. A new expected value operator is developed to compute different expected values of a portfolio's return for investors with different attitudes. A lower absolute deviation is introduced to measure the downside risk under the expected return. Meanwhile, chance VaR constraint based on different attitude is also constructed to limit the risk, and the transaction cost is incorporated in the proposed portfolio optimization model.

The remainder of this paper is organized as follows. In Section 2, we briefly review some basic fundamental knowledge. In Section 3, we propose our models with VaR constraints and give the equivalent crisp models. ε-constraint method is presented in Section 4 to solve the proposed model. In Section 5, a numerical example is given to illustrate our proposed effective approaches. Finally, Section 6 presents the conclusions.

## 2. Preliminaries

In this section, let us first briefly review the basic theory of fuzzy variables, which will be used in the following sections.

**Definition 1.** [11] Let $\Theta = \{\omega_1, \omega_2, \cdots\}$, ($\Theta$is a nonempty set) and let $P(\Theta)$ be the power set of $\Theta$. For each $A \in P(\Theta)$, the corresponding $P(A)$ is a nonnegative number, called its possibility, such that

(1) $Pos(\phi) = 0$ and $Pos(\Theta) = 1$;
(2) $Pos(\bigcup_k A_k) = \sup_k Pos(A_k)$ for any arbitrary collection $\{A_k\}$in $P(\Theta)$.

The triple ($\Theta$, $P(\Theta)$, $Pos$) is referred as a possibility space, and the function is called a possibility measure.

**Definition 2.** [2] The necessity measure of $A$ is

$$Nec\{A\} = 1 - Pos\{A^c\}$$

Thus the necessity measure is the dual of possibility measure, that is, $Pos\{A\} + Nec\{A^c\} = 1$, for any $A \in P(\Theta)$.

**Definition 3.** [17] The $Me$ of $A$ is defined as

$$Me\{A\} = Nec\{A\} + \lambda(Pos\{A\} - Nec\{A\})$$

where $\lambda$ $(0 \le \lambda \le 1)$ is the conservative-aggressive attitude to determine the combined attitude of decision maker.

**Theorem 1.** For any $\lambda_1 \ge \lambda_2$, we have $Me^{\lambda_1}\{A\} \ge Me^{\lambda_2}\{A\}$.

**Proof.** According to Definition 1, we have that

$$\begin{aligned} &Me^{\lambda_1}\{A\} - Me^{\lambda_2}\{A\} \\ &= Nec\{A\} + \lambda_1(Pos\{A\} - Nec\{A\}) - Nec\{A\} - \lambda_2(Pos\{A\} - Nec\{A\}) \\ &= (\lambda_1 - \lambda_2)(Pos\{A\} - Nec\{A\}) \end{aligned}$$

Because $\lambda_1 - \lambda_2 \ge 0$, in order to prove $Me^{\lambda_1}\{A\} - Me^{\lambda_2}\{A\} \ge 0$, we just need to prove $Pos\{A\} \ge Nec\{A\}$.

If $Pos\{A\} = 1$, then it is obvious that $Pos\{A\} \ge Nec\{A\}$. Otherwise, we must have $Pos\{A^c\} = 1$, which implies that $Nec\{A\} = 1 - Pos\{A^c\} = 0$. Thus $Pos\{A\} \ge Nec\{A\}$ always holds, so we have

$$(\lambda_1 - \lambda_2)(Pos\{A\} - Nec\{A\}) \ge 0 \Rightarrow Me^{\lambda_1}\{A\} \ge Me^{\lambda_2}\{A\}.$$

The proof is complete. □

For the same event, different decision makers may have different understanding of the likelihoods of occurrence. If the decision maker is more aggressive, he/she will have a larger value when measuring the likelihood; if the decision maker is more conservative, he/she will have a smaller value when measuring the likelihood. Based on the above discussion, we find that $Me$ has the ability to reflect the different conservative-aggressive attitudes by adjusting the value of $\lambda$ and finally provide a certain measurement of the likelihood under a given $\lambda$. Therefore, this paper use $Me$ to evaluate the likelihood degree of a fuzzy event under different attitudes. When $\lambda = 1$, we have $Me = Pos$, which means the decision maker isextremely aggressive, so it shows the measure of the best case of the fuzzy event, and it is the maximal chance that $A$ holds; When $\lambda = 0$, we have $Me = Nec$, which means the decision maker is extremely conservative, so it shows the measure of the worst case of the fuzzy event, and it is the minimal chance that $A$ holds; When $\lambda = 0.5$, we have $Me = Cr$ where $Cr$ is the credibility measure introduced by Liu [8], which is a special case of $Me$, and means that the decision maker has the neutral attitude. $Cr$ is self-dual, that is, $Cr\{A\} + Cr\{A^c\} = 1$ for any $A \in P(\Theta)$.

Triangular fuzzy variable $\xi$ is fully defined by the triplet $(r, \alpha, \beta)$ of crisp number with$r - \alpha < r < r + \beta$, and the membership function is given by

$$\mu(x) = \begin{cases} \frac{x-(r-\alpha)}{\alpha}, \text{if } r-\alpha < x \le r \\ \frac{r+\beta-x}{\beta}, \text{if } r \le x < r+\beta \\ 0, \text{otherwise} \end{cases} \tag{1}$$

where $r$ is the middle value, $\alpha$ and $\beta$ are left and right spreads of the triangular fuzzy variable.

For a triangular fuzzy variable $\xi$, the possibility, necessity, and the general measure of $\xi \le x$ are

$$\begin{aligned} &Pos\{\xi \le x\} = \begin{cases} 0, \text{if } x \le r-\alpha \\ \frac{x-(r-\alpha)}{\alpha}, \text{if } r-\alpha \le x \le r \\ 1, \text{if } x \ge r \end{cases} Nec\{\xi \le x\} = \begin{cases} 0, \text{if } x \le r \\ \frac{x-r}{\beta}, \text{if } r \le x \le r+\beta \\ 1, \text{if } x \ge r+\beta \end{cases} \\ &Me\{\xi \le x\} = \begin{cases} 0, \text{if } x \le r-\alpha \\ \lambda\frac{x-(r-\alpha)}{\alpha}, \text{if } r-\alpha \le x \le r \\ \lambda + (1-\lambda)\frac{x-r}{\beta}, \text{if } r \le x \le r+\beta \\ 1, \text{if } x \ge r+\beta \end{cases} \end{aligned} \tag{2}$$

**Lemma 1.** [3] Let $\xi = (\mu, \alpha, \beta)$ and $\xi' = (\mu', \alpha', \beta')$ be triangular fuzzy numbers, then $\xi + \xi' = (\mu + \mu', \alpha + \alpha', \beta + \beta')$ is also a triangular fuzzy number. Let $k$ be a real number, then $k\xi = (k\mu, k\alpha, k\beta)$ is also a triangular fuzzy number.

## 3. Model development

In this section, we assume the return rate of a stock is described as a triangular fuzzy variable. The expected value return and the lower absolute deviation risk are considered as the two main objectives when optimizing the portfolio selection.

*3.1. Return objective*

To maximize the expected value of the investment return of selected stock portfolio under fuzzy environment, the first objective function is developed as:

$$\max E^{Me}\left(\sum_{j=1}^{n}\xi_j x_j\right) \tag{3}$$

where $\xi_j=(r_j,\alpha_j,\beta_j)$ with $r_j-\alpha_j>0$, i.e., $\xi_j$ is a positive triangular fuzzy variable; $x_j$ is the proportion of investment.

For the expected value of a fuzzy variable, there are several kinds of definitions. However, in realistic investment, different investors may have different expected value towards the same portfolio. Therefore, an adjustable expected value operator should be developed to suit varying conservative-neutral-aggressive attitudes. In order to incorporate the attitude of a decision maker, we define the following expected value operator based on *Me* measure.

**Definition 4.** Let $\xi$ be a fuzzy variable, the *Pos* expected value and *Nec* expected value of $\xi$ can be defined by

$$E^{Pos}[\xi]=\int_0^{+\infty}Pos\{\xi\geq u\}du-\int_{-\infty}^{0}Pos\{\xi\leq u\}du$$

and

$$E^{Nec}[\xi]=\int_0^{+\infty}Nec\{\xi\geq u\}du-\int_{-\infty}^{0}Nec\{\xi\leq u\}du$$

provided that at least one of the two integrals is finite.

Then the *Me* expected value of $\xi$ is defined by

$$\begin{aligned}&E^{Me}[\xi]=\textstyle\int_0^{+\infty}Me\{\xi\geq u\}du-\int_{-\infty}^{0}Me\{\xi\leq u\}du\\&=E^{Nec}[\xi]+\lambda\left(E^{Pos}[\xi]-E^{Nec}[\xi]\right)\\&=\lambda E^{Pos}[\xi]+(1-\lambda)E^{Nec}[\xi]\end{aligned} \tag{4}$$

Since the return rate is assumed as a triangular fuzzy variable, we propose the following theorem to calculate the *Me* expected value of a triangular fuzzy variable.

**Theorem 2.** The *Me* expected value of triangular fuzzy variable $\xi=(r,\alpha,\beta)$ is

$$E^{Me}[\xi]=\frac{(1-\lambda)}{2}(r-\alpha)+\frac{1}{2}r+\frac{\lambda}{2}(r+\beta)$$

where$\lambda(0\leq\lambda\leq1)$is the conservative-neutral-aggressive attitude parameter.

**Proof.** Let's discuss the case based on Eq. (4).

$$\begin{aligned}&E^{Me}[\xi]\\&=\textstyle\int_0^{+\infty}Me\{\xi\geq u\}du-\int_{-\infty}^{0}Me\{\xi\leq u\}du\\&=\textstyle\int_0^{r-\alpha}Me\{\xi\leq u\}du+\int_{r-\alpha}^{r}Me\{\xi\leq u\}du+\int_r^{r+\beta}Me\{\xi\leq u\}du+\int_{r+\beta}^{+\infty}Me\{\xi\leq u\}du\\&=\textstyle\int_0^{r-\alpha}du+\int_{r-\alpha}^{r}\lambda+(1-\lambda)\frac{r-u}{\alpha}du+\int_r^{r+\beta}\lambda\frac{r+\beta-u}{\beta}du\\&=\tfrac{(1-\lambda)}{2}(r-\alpha)+\tfrac{1}{2}r+\tfrac{\lambda}{2}(r+\beta)\end{aligned}$$

The proof is complete.□

For a neutral attitude investor, the attitude parameter $\lambda=0.5$, and the expected value of $\xi$ is $E^{Cr}[\xi]=r+\frac{\beta-\alpha}{4}$. For a extremely conservative investor, the attitude parameter $\lambda=0$, and the corresponding expected value of $\xi$ is $E^{Nec}[\xi]=r-\frac{\alpha}{2}$. For a extremely conservative investor, the attitude parameter $\lambda=1$, and thus the corresponding expected value of $\xi$ is $E^{Pos}[\xi]=r+\frac{\beta}{2}$.

**Theorem 3.** For a triangular fuzzy variable $\xi=(r,\alpha,\beta)$ described in Definition 4, if $\lambda_1\geq\lambda_2$, then we have $E^{Me^{\lambda_1}}[\xi]\geq E^{Me^{\lambda_2}}[\xi]$.

**Proof.** According to Theorem 2, we have

$$\begin{aligned}&E^{Me^{\lambda_1}}[\xi]-E^{Me^{\lambda_2}}[\xi]\\&=\left[\tfrac{(1-\lambda_1)}{2}(r-\alpha)+\tfrac{1}{2}r+\tfrac{\lambda_1}{2}(r+\beta)\right]-\left[\tfrac{(1-\lambda_2)}{2}(r-\alpha)+\tfrac{1}{2}r+\tfrac{\lambda_2}{2}(r+\beta)\right]\\&=\tfrac{(\lambda_1-\lambda_2)[(r+\beta)-(r-\alpha)]}{2}\end{aligned}$$

Since $\lambda_1\geq\lambda_2$ and $r+\beta>r+\alpha$, we have $E^{Me^{\lambda_1}}[\xi]-E^{Me^{\lambda_2}}[\xi]\geq0$.

The proof is complete.□

Based on Theorem 1, we can equivalently transform Eq. (3) into the following Eq. (5):

$$E^{Me}\left(\sum_{j=1}^{n}\xi_j x_j\right) = \sum_{j=1}^{n}\left(\frac{(1-\lambda)}{2}\left(r_j-\alpha_j\right)+\frac{r_j}{2}+\frac{\lambda}{2}\left(r_j+\beta_j\right)\right)x_j \tag{5}$$

### 3.2. Risk objective

**Definition 6.** Let $\xi$ be a fuzzy variable with finite expected value $E^{Me}(\xi)$. Then, its Lower Absolute Deviation (LAD) is defined as

$$LAD(\xi) = E^{Cr}\left[\left|\left(\xi - E^{Me}(\xi)\right)^{-}\right|\right] = \int_0^{+\infty}\left\{Cr\left(\xi - E^{Me}(\xi)\right)^{-} \ge u\right\}du - \int_{-\infty}^{0}\left\{Cr\left(\xi - E^{Me}(\xi)\right)^{-} \le u\right\}du$$

where$(\xi - E^{Me}(\xi))^{-} = \{\begin{matrix}\xi - E^{Me}(\xi), \text{if}\xi \le E^{Me}(\xi) \\ 0, \text{if}\xi \ge E^{Me}(\xi)\end{matrix}$.

Note that LAD only measures the volatility of $\xi$ below its expected value $E^{Me}(\xi)$. Thus, it can be used to characterize the risk of a portfolio in a fuzzy environment, and the risk objective is

$$\min LAD\left(\xi_j x_j\right) \tag{4}$$

**Theorem 4.** The $LAD(\xi)$ of triangular fuzzy variable $\xi = (r, \alpha, \beta)$ can be computed by

$$LAD(\xi) = \begin{cases}\frac{[(r-\alpha)(1+\lambda)-r-\lambda(r+\beta)]^2}{16\alpha} & if\ r > E^{Me}(\xi) \\ \frac{[(\lambda-1)(r-\alpha)-r+(2-\lambda)(r+\beta)]^2}{16\beta} & if\ r \le E^{Me}(\xi)\end{cases} \tag{5}$$

Based on Theorem 2, the objective function (4) is equivalent to

$$LAD\left(\xi_j x_j\right) = E\left[\left|\left(\xi_j - E\left(\xi_j\right)\right)^{-}\right|\right] \\ = \frac{\sum_{j=1}^{n}\left[(\lambda-1)\left(r_j-\alpha_j\right)x_j - r_j x_j + (2-\lambda)\left(r_j+\beta_j\right)x_j\right]^2}{16\sum_{j=1}^{n}\beta_j x_j} \tag{6}$$

### 3.3. VaR constraint formulation

VaR is a return which is a negative number, and shows the biggest potential loss of portfolio with the confidence level $\gamma$. It is shown in the following Eq. (7).

$$Me\left(\sum_{j=1}^{n}\xi_j x_j \le (-VaR)\right) \le 1-\gamma \tag{7}$$

It is obvious that Eq. (7) cannot be handled directly, so we propose the following theorem to convert it into a workable constraint.

**Theorem 5.** For specific values of determined attitude parameter $\lambda$ and confidence level $\gamma$, the VaR constraint $Me(\sum_{j=1}^{n}\xi_j x_j \le (-VaR)) \le 1-\gamma$ is equivalent to

$$\begin{cases}(-VaR) \le \frac{\sum_{j=1}^{n}\alpha_j x_j(1-\gamma)}{\lambda} + \sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n}\alpha_j x_j, \text{if}\lambda+\gamma \ge 1 \\ (-VaR) \le \frac{(1-\lambda-\gamma)\sum_{j=1}^{n}\beta_j x_j}{1-\lambda} + \sum_{j=1}^{n} r_j x_j, \text{if } \lambda+\gamma < 1\end{cases} \tag{8}$$

### 3.4. Portfolio selection model with VaR constraint under different attitudes

People's attitude to investment varies from conservative to aggressive. In the following, we formulate model (9) to obtain the frontier of the pareto-optimal portfolios based on different attitudes in fuzzy environments as a multi-objective programming,

$$\begin{aligned}
&\max E^{Me}\left(\sum_{j=1}^{n}\xi_j x_j\right)\\
&\min LAD\left(\xi_j x_j\right)\\
&s.t.\begin{cases}
Me\left\{\sum_{j=1}^{n}\xi_j x_j \le (-VaR)\right\} \le 1-\gamma\\
\sum_{j=1}^{n} x_j = 1\\
E^{Me}\left(\sum_{j=1}^{n}\xi_j x_j\right) - \sum_{j=1}^{n} C_j x_j \ge g
\end{cases}
\end{aligned}\tag{9}$$

where $C_j$ is the transaction cost rate of the $j$th security. People who are not willing to take any risk will invest their money in a treasury bond which yields $g$, so the return on the risky investment has to be greater than $g$.

Based on the Theorems 2, 4 and 5, the following crisp equivalent multi-objective portfolio selection model can be obtained.

$$\begin{aligned}
&\max E^{Me}\left(\sum_{j=1}^{n}\xi_j x_j\right) = \sum_{j=1}^{n}\left(\tfrac{(1-\lambda)}{2}\left(r_j-\alpha_j\right)+\tfrac{r_j}{2}+\tfrac{\lambda}{2}\left(r_j+\beta_j\right)\right)x_j\\
&\min LAD\left(\sum_{j=1}^{n}\xi_j x_j\right) = \frac{\sum_{j=1}^{n}\left[(\lambda-1)\left(r_j-\alpha_j\right)x_j - r_j x_j + (2-\lambda)\left(r_j+\beta_j\right)x_j\right]^2}{16\sum_{j=1}^{n}\beta_j x_j}\\
&s.t.\begin{cases}
\begin{cases}
-VaR \le \frac{\sum_{j=1}^{n}\alpha_j x_j(1-\gamma)}{\lambda} + \sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n}\alpha_j x_j, \text{ if } \sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n}\alpha_j x_j \le -VaR \le \sum_{j=1}^{n} r_j x_j\\
-VaR \le \frac{(1-\lambda-\gamma)\sum_{j=1}^{n}\beta_j x_j}{1-\lambda} + \sum_{j=1}^{n} r_j x_j, \text{ if } \sum_{j=1}^{n} r_j x_j \le -VaR \le \sum_{j=1}^{n} r_j x_j + \sum_{j=1}^{n}\beta_j x_j
\end{cases}\\
\sum_{j=1}^{n} x_j = 1\\
E^{Me}\left(\sum_{j=1}^{n}\xi_j x_j\right) - \sum_{j=1}^{n} C_j x_j \ge g
\end{cases}
\end{aligned}\tag{10}$$

## 4. ε-Constraint method

In this section, we illustrate how to use the ε-constraint method [1,9] to solve the proposed portfolio selection models (9) and (10). The essence of this method is splitting the model into two single objective models (11) and (12):

$$\begin{aligned}
&\max E^{Me}\left(\sum_{j=1}^{n}\xi_j x_j\right)\\
&s.t.\begin{cases}
LAD\left(\sum_{j=1}^{n}\xi_j x_j\right) \le LAD_0\\
Me\left\{\sum_{j=1}^{n}\xi_j x_j \le (-VaR)\right\} \le 1-\gamma\\
\sum_{j=1}^{n} x_j = 1\\
E^{Me}\left(\sum_{j=1}^{n}\xi_j x_j\right) - \sum_{j=1}^{n} C_j x_j \ge g
\end{cases}
\end{aligned}\tag{11}$$

and

$$\begin{aligned} & \min LAD\left(\sum_{j=1}^{n}\xi_j x_j\right) \\ & s.t.\begin{cases} E^{Me}\left(\sum_{j=1}^{n}\xi_j x_j\right) \geq R_0 \\ Me\left\{\sum_{j=1}^{n}\xi_j x_j \leq (-VaR)\right\} \leq 1-\gamma \\ \sum_{j=1}^{n} x_j = 1 \\ E^{Me}\left(\sum_{j=1}^{n}\xi_j x_j\right) - \sum_{j=1}^{n} C_j x_j \geq g \end{cases} \end{aligned} \tag{12}$$

where $LAD_0$ and $R_0$ are predetermined risk level and return level, respectively. We can get a Pareto solution of model (10) through solving models (11) or (12).

Next, we discuss the value ranges of $LAD_0$ and $R_0$. Consider the following two single objective models:

$$\begin{aligned} & \min LAD\left(\sum_{j=1}^{n}\xi_j x_j\right) \\ & s.t.\begin{cases} Me\left\{\sum_{j=1}^{n}\xi_j x_j \leq (-VaR)\right\} \leq 1-\gamma \\ \sum_{j=1}^{n} x_j = 1 \\ E^{Me}\left(\sum_{j=1}^{n}\xi_j x_j\right) - \sum_{j=1}^{n} C_j x_j \geq g \end{cases} \end{aligned} \tag{13}$$

and

$$\begin{aligned} & \max E^{Me}\left(\sum_{j=1}^{n}\xi_j x_j\right) \\ & s.t.\begin{cases} Me\left\{\sum_{j=1}^{n}\xi_j x_j \leq (-VaR)\right\} \leq 1-\gamma \\ \sum_{j=1}^{n} x_j = 1 \\ E^{Me}\left(\sum_{j=1}^{n}\xi_j x_j\right) - \sum_{j=1}^{n} C_j x_j \geq g \end{cases} \end{aligned} \tag{14}$$

The optimal objective value of model (13) is $R_{\max}$, the solution is $x^R$. We use $x^R$ to compute the corresponding $LAD_{\max}$. Similarly, The optimal objective value of model (14) is $LAD_{\min}$, the solution is $x^{LAD}$, and we use $x^{LAD}$ to get the corresponding $R_{\min}$. Then $LAD_0$ and $R_0$ can be determined in the ranges $[LAD_{\min}, LAD_{\max}]$ and $[R_{\min}, R_{\max}]$, respectively. After determining different values of $LAD_0$ or $R_0$, we can solve model (11) or (12) to obtain a portfolio's efficient frontier.

## 5. Case study

In order to illustrate our proposed approaches for the portfolio selection problem, we carry out the following case study. After investigating 10 stocks in Chinese stock market, we list the return rates $\xi_j = (r_j, \alpha_j, \beta_j)$ and transaction costs $C_j$ in Table 1. The return rate $g$ of treasury bond is 0.339%.

### *5.1. Solution process and results*

Assume that $\lambda = 0.5$, $VaR = -0.28\%$ and confidence level $\gamma = 0.7$. According to the ε-constraint method, we solve models (13) and (14) to get ranges of the return level and risk level, respectively:

$$[R_{\min}, R_{\max}] = [0.301\%, 0.392\%], [LAD_{\min}, LAD_{\max}] = [0.033\%, 0.133\%]$$

Based on model (13), we maximize the return without considering the risk, and can obtain the optimal solution as reported in the second column of Table 2. When we consider model (14), we minimize the risk without considering the return, and we can obtain the optimal portfolio given in the third column of Table 2.

Each solution includes a pair of two numbers: the return and the corresponding risk. The pairs are called efficient when under a certain risk, the return is the best that can be achieved; or with the certain return, the risk is the smallest that can be reached. Also, efficient pairs are the points on the efficient frontier curve in Fig. 1.

**Table 1**
Data of 10 stocks.

| Stock no. | $r_j(\%)$ | $\alpha_j(\%)$ | $\beta_j(\%)$ | $C_j(\%)$ |
|---|---|---|---|---|
| 1 | 0.237 | 0.090 | 0.400 | 0.030 |
| 2 | 0.133 | 0.250 | 0.130 | 0.025 |
| 3 | 0.229 | 0.400 | 0.400 | 0.018 |
| 4 | 0.103 | 0.050 | 0.500 | 0.035 |
| 5 | 0.303 | 0.010 | 0.450 | 0.015 |
| 6 | 0.013 | 0.250 | 0.020 | 0.025 |
| 7 | 0.457 | 0.390 | 0.050 | 0.020 |
| 8 | 0.044 | 0.030 | 0.050 | 0.015 |
| 9 | 0.093 | 0.150 | 0.390 | 0.025 |
| 10 | 0.025 | 0.550 | 0.290 | 0.025 |

**Table 2**
Optimal portfolio based on single objective models.

| Proportion of stock no. | Maximizing return | Minimizing LAD |
|---|---|---|
| 1 | 0 | 0.052 |
| 2 | 0 | 0.102 |
| 3 | 0 | 0.032 |
| 4 | 0 | 0.012 |
| 5 | 0.499 | 0.062 |
| 6 | 0 | 0 |
| 7 | 0.501 | 0.621 |
| 8 | | 0.104 |
| 9 | 0 | 0.014 |
| 10 | 0 | 0 |
| Total | 1.00 | 1.00 |
| Return | 0.392 | 0.301 |
| LAD | 0.133 | 0.033 |

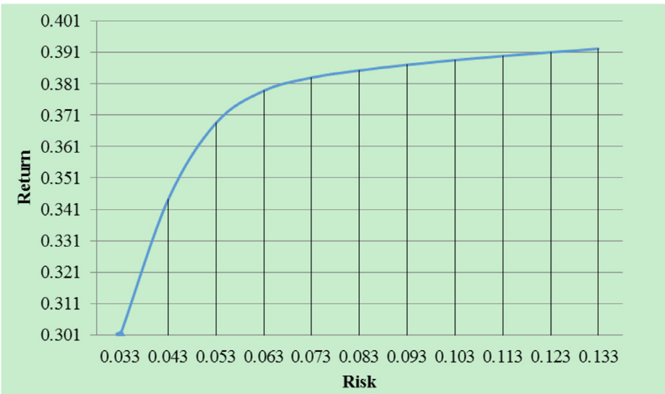


**Fig. 1.** EF under $\lambda = 0.5, \gamma = 0.7, VaR = -0.28\%$.

When $R_0$ includes all of the values in the range $[R_{\min}, R_{\max}]$, we can get all of the efficient pairs of the model (12); Similarly, when $LAD_0$ includes all of the values in the range $[LAD_{\min}, LAD_{\max}]$, we can obtain all of the efficient pairs of model (11).

In what follows, we first choose 11 values of $LAD_0$ from 0.033 to 0.133 with step size 0.01, then solve model (11) for 11 times by using the ε-constraint method, and finally obtain the optimal returns: 0.301, 0.344, 0.368, 0.379, 0.383, 0.385, 0.387, 0.389, 0.390, 0.391, 0.392. Thus we can obtain the efficient frontier (EF) under the under $\lambda = 0.5, \gamma = 0.7, VaR = -0.28\%$ as shown in Fig. 1.

### 5.2. Sensitivity analysis

There are three kinds of parameters needed to be predetermined before producing the solutions. In order to show how does each parameter influence the EF, we conduct the following three analysis.

Firstly, we fix the confidence level$\gamma = 0.7$, and set the values of VaR to be $-0.28\%, -0.29\%, -0.3\%$ respectively. The EFs shown in Fig. 2 are obtained.

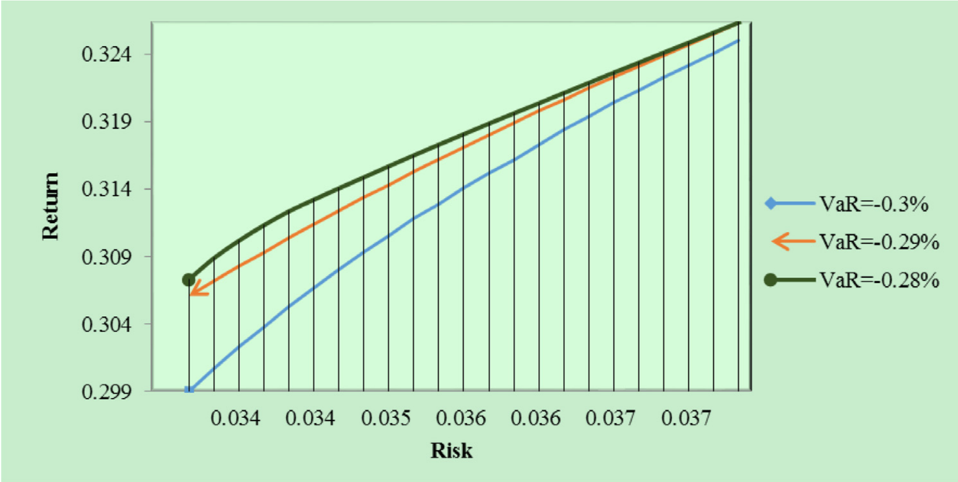


**Fig. 2.** EFs under different values of VaR ($\lambda = 0.5, \gamma = 0.7$).

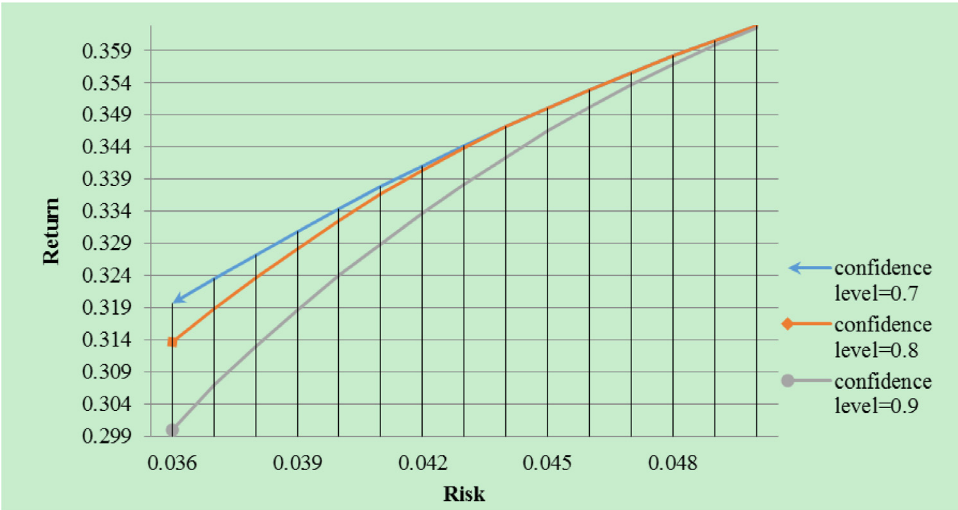


**Fig. 3.** EFs under different values of confidence level ($\lambda = 0.5, VaR = -0.28\%$).

From Fig. 2 we can get that the value of the expected return arises under a fixed risk level or the value of LAD risk decreases under a fixed return level when $(-VaR)$ decreases. After a certain LAD risk level, the change in VaR barely has effect on the return level.

Secondly, we set the same $VaR = -0.28\%$, and change the values of confidence level $\gamma$. We suppose $\gamma = 0.7, 0.8, 0.9$ respectively and obtain the EFs under different confidence levels, as shown in Fig. 3.

We can draw from Fig. 3 that a larger value of expected return under a fixed risk level or a smaller value of LAD risk under a fixed return level can be obtained when the confidence level decreases. After a certain LAD risk level, the change in confidence level barely has effect on the return level.

Finally, we set $VaR = -0.28\%$ and the confidence level $\gamma = 0.7$. When conservative-aggressive attitude $\lambda = 0.2, 0.5, 0.8$, the EFs in Fig. 4 can be obtained.

From Fig. 4 we can draw that when the investor is more aggressive, the return of the portfolio becomes larger under the same LAD risk and VaR. The EFs for investors with different attitudes are quite different. More options can be provided for more aggressive investors.

To summarize, by using the proposed approach, the aggressive-neutral-conservative attitude, the confidence level and VaR can be incorporated into the decision making of stock portfolio selection. If an investor holds an aggressive (conservative) attitude towards the objectives, they can choose a larger (smaller) value for attitude$\lambda$. If they are very cautious on the constraints, they can set a larger value for confidence level $\gamma$ and VaR. After determining the parameters, the investor will be given a solution that fits their behavior, which is the advantage of using the proposed models and approaches.

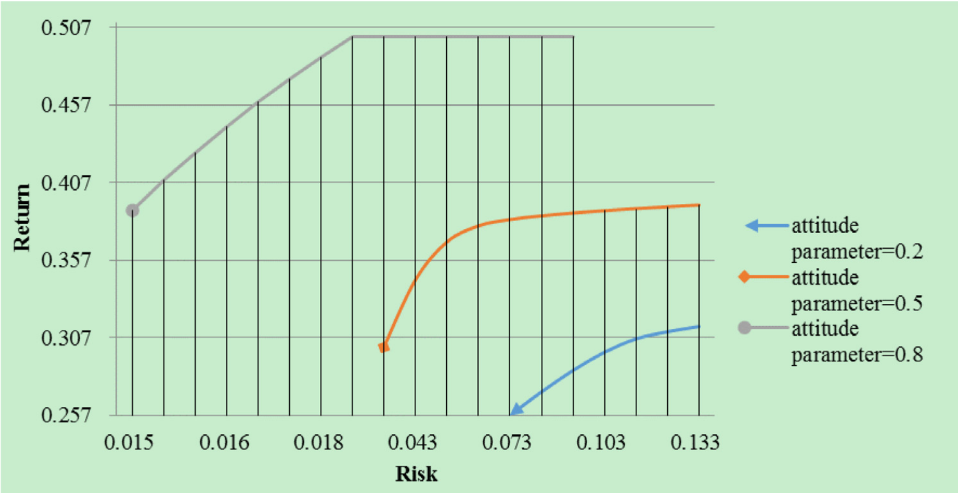


**Fig. 4.** EFs under different attitude parameters ($\gamma = 0.7, VaR = -0.28\%$).

## 6. Conclusion

In this paper, we proposed a stock portfolio selection model with incorporating the aggressive-neutral-conservative attitude and VaR constraint. The general expected value and lower absolute deviation was employed to deal with the fuzzy return and risk. In the proposed model, we assumed that the investor has two decisions objective, viz., maximizing the return and minimizing the risk over the whole investment horizon. We used the ε-constraint method to solve the proposed model. Finally, we applied the proposed approaches to 10 stocks in Chinese market to illustrate the effectiveness. The experimental results revealed that the decision of portfolio selection problem is dependent on the risk attitude of the investor, the value of VaR determined, and the confidence level of the investor. It has been demonstrated that the investors who have different aggressive-neutral-conservative attitudes have diverse efficient frontiers when they select portfolio. For aggressive investors, there are more options of portfolio. Our research findings indicated a way to find different efficient frontiers for varying investors.

For future research, we intend to extract more characteristic of investors and conduct more behavior analysis, and then incorporate them into the portfolio selection problem.

## Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript.

## Acknowledgments

The authors thank those that have given constructive comments and feedback to help improve this paper. Support was provided by the National Natural Science Foundation of China (grant nos. 71401093, 71771208), China Scholarship Council (grant no. 201606875006), Soft Science Research Project of Shaanxi Province (grant no. 2016KRM089), Research Center for Systems Science & Enterprise Development (grant no. Xq16B01), and the Fundamental Research Funds for the Central Universities (WUT: 2016VI002).

## Appendix

Proof of Theorem 2**.** Since the triangular fuzzy variable $\xi$ is fully determined by the triplet $(r, \alpha, \beta)$, then according to Lemma 1, we can define the fuzzy variable $\eta = \xi - E^{Me}(\xi) = (r - E^{Me}(\xi), \alpha, \beta)$. Then based on Eq. (1), we can obtain

$$Cr\{\eta \geq y\} = \begin{cases} 1, \text{ if } y \leq r - \alpha - E^{Me}(\xi) \\ \frac{1}{2} + \frac{r-(y+E^{Me}(\xi))}{2\alpha}, \text{ if } r - \alpha - E^{Me}(\xi) \leq y \leq r - E^{Me}(\xi) \\ \frac{r+\beta-(y+E^{Me}(\xi))}{2\beta}, \text{ ifr} - E^{Me}(\xi) \leq y \leq r + \beta - E^{Me}(\xi) \\ 0, \text{ ify} \geq r + \beta - E^{Me}(\xi) \end{cases}$$

When $r \leq E^{Me}(\xi)$

$$LAD = \int_0^{r+\beta-E^{Me}(\xi)} Cr\{\eta \geq y\}dy = \frac{[r+\beta-E(\xi)]^2}{4\beta} = \frac{[(\lambda-1)(r-\alpha)-r+(2-\lambda)(r+\beta)]^2}{16\beta}$$

Similarly,

$$Cr\{\eta \le y\} = Cr\{\xi \le y + E^{Me}(\xi)\} = \begin{cases} 0, \text{if} y \le r - \alpha - E^{Me}(\xi) \\ \frac{y+E^{Me}(\xi)-(r-\alpha)}{2\alpha}, \text{if} r - \alpha - E^{Me}(\xi) \le y \le r - E^{Me}(\xi) \\ \frac{1}{2} + \frac{y+E^{Me}(\xi)-r}{2\beta}, \text{if} r - E^{Me}(\xi) \le y \le r + \beta - E^{Me}(\xi) \\ 1, \text{if} y \ge r + \beta - E^{Me}(\xi) \end{cases}$$

When $r > E^{Me}(\xi)$

$$LAD(\xi) = \int_{r-\alpha-E^{Me}(\xi)}^{0} Cr\{\eta \le y\}dy = \frac{[r-\alpha-E(\xi)]^2}{4\alpha} = \frac{[(r-\alpha)(1+\lambda)-r-\lambda(r+\beta)]^2}{16\alpha}$$

The proof is complete.□

Proof of Theorem 3**.** Since $\xi_j$ is a triangular fuzzy variable, based on Lemma 1, we can obtain that $\sum_{j=1}^{n}\xi_j x_j = (\sum_{j=1}^{n} r_j x_j, \sum_{j=1}^{n}\alpha_j x_j, \sum_{j=1}^{n}\beta_j x_j)$ is still a triangular variable, then by following Eq. (1), we can obtain

$$Me\left(\sum_{j=1}^{n}\xi_j x_j \le -VaR\right) = \begin{cases} 0, \text{if}(-VaR) \le \sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n}\alpha_j x_j \\ \lambda\frac{(-VaR)-\left(\sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n}\alpha_j x_j\right)}{\sum_{j=1}^{n}\alpha_j x_j}, \text{if} \sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n}\alpha_j x_j \le (-VaR) \le \sum_{j=1}^{n} r_j x_j \\ \lambda + (1-\lambda)\frac{(-VaR)-\sum_{j=1}^{n} r_j x_j}{\sum_{j=1}^{n}\beta_j x_j}, \text{if} \sum_{j=1}^{n} r_j x_j \le (-VaR) \le \sum_{j=1}^{n} r_j x_j + \sum_{j=1}^{n}\beta_j x_j \\ 1, \text{if}(-VaR) \ge \sum_{j=1}^{n} r_j x_j + \sum_{j=1}^{n}\beta_j x_j \end{cases}$$

Hence, if $\sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n}\alpha_j x_j \le (-VaR) \le \sum_{j=1}^{n} r_j x_j$, we have

$$Me\left(\sum_{j=1}^{n}\xi_j x_j \le -VaR\right) \le 1-\gamma \Leftrightarrow \lambda\frac{(-VaR)-\left(\sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n}\alpha_j x_j\right)}{\sum_{j=1}^{n}\alpha_j x_j} \le 1-\gamma$$

$$\Leftrightarrow (-VaR) \le \frac{\sum_{j=1}^{n}\alpha_j x_j(1-\gamma)}{\lambda} + \sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n}\alpha_j x_j$$

Or if $\sum_{j=1}^{n} r_j x_j \le (-VaR) \le \sum_{j=1}^{n} r_j x_j + \sum_{j=1}^{n}\beta_j x_j$, we have

$$Me\left(\sum_{j=1}^{n}\xi_j x_j \le -VaR\right) \le 1-\gamma \Leftrightarrow (-VaR) \le \frac{(1-\lambda-\gamma)\sum_{j=1}^{n}\beta_j x_j}{1-\lambda} + \sum_{j=1}^{n} r_j x_j$$

It is worth note that if $\lambda+\gamma \ge 1$, then it is obvious that the condition $\sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n}\alpha_j x_j \le (-VaR) \le \sum_{j=1}^{n} r_j x_j$ holds; and if $\lambda+\gamma \le 1$, the condition $\sum_{j=1}^{n} r_j x_j \le (-VaR) \le \sum_{j=1}^{n} r_j x_j + \sum_{j=1}^{n}\beta_j x_j$ holds, so we can also obtain

$$\begin{cases} (-VaR) \le \frac{\sum_{j=1}^{n}\alpha_j x_j(1-\gamma)}{\lambda} + \sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n}\alpha_j x_j, \text{if}\lambda+\gamma \ge 1 \\ (-VaR) \le \frac{(1-\lambda-\gamma)\sum_{j=1}^{n}\beta_j x_j}{1-\lambda} + \sum_{j=1}^{n} r_j x_j, \text{if}\lambda+\gamma < 1 \end{cases}$$

The proof is thus complete.□

## References

[1] P.J. Copado-Méndez, C. Pozo, G. Guillén-Gosálbez, et al., Enhancing the $\varepsilon$-constraint method through the use of objective reduction and random sequences: Application to environmental problems, Comput. Chem. Eng. 87 (2016) 36–48.

[2] D. Dubois, H. Prade, Furzy Sets and Systems: Theory and Applications, Academic Press, New York, 1988.
[3] P. Dubois, H. Prade, Possibility Theory: an Approach to Computerized Processing of Uncertainty, Plenum Press, New York, 1988.
[4] O. Kocadağlı, R. Keskin, A novel portfolio selection model based on fuzzy goal programming with different importance and priorities, Expert Syst. Appl. 42 (20) (2015) 6,898-6,912.
[5] B. Liu, Y. Liu, Expected value of fuzzy variable and fuzzy expected value models, IEEE Trans. Fuzzy Syst. 10 (2002) 445–450.
[6] Y. Liu, B. Liu, A class of fuzzy random optimization: expected value models, Inf. Sci. 155 (1) (2003) 89–102.
[7] H. Markowitz, Portfolio selection, J. Finance 7 (1) (1952) 77–91.
[8] H. Markowitz, Portfolio Selection: Efficient Diversification of Investments, John Wiley & Sons, 1959.
[9] G. Mavrotas, Effective implementation of the $\varepsilon$-constraint method in multi-objective mathematical programming problems, Appl. Math. Comput. 213 (2) (2009) 455–465.
[10] O. Momen, A. Esfahanipour, A. Seifi, A robust behavioral portfolio selection: model with investor attitudes and biases, Oper. Res. (2) (2017) 1–20.
[11] S. Nahmias, Fuzzy variables, Fuzzy Sets Syst. 1 (2) (1978) 97–110.
[12] O. Berman, N. Sanajian, JM. Wang, Location choice and risk attitude of a decision maker, Omega 66 (2017) 170–181.
[13] A.D. Roy, Safety-first and holding of assets, Economics 20 (1952) 431–449.
[14] W. Sun, Portfolio selection strategies with investor psychology and behavior under fuzzy random environment, Int. Symp. Comput. Intell. Des. (2016) 208–211.
[15] R.C. Tsaur, Fuzzy portfolio model with different investor risk attitudes, Eur. J. Oper. Res. 227 (2) (2013) 385–390.
[16] J. Xu, X. Zhou, S. Li, A class of chance constrained multi-objective portfolioselection model under fuzzy random environment, J. Optim. Theory Appl. 150 (3) (2011) 530–552.
[17] J. Xu, X. Zhou, Approximation based fuzzymulti-objective models with expectedobjectives and chance constraint: Application to earth-rock work allocation, Inf. Sci. 238 (7) (2013) 75–95.
[18] L.A. Zadeh, Furzy sets, Inf. Control 8 (1965) 338–353.
[19] P. Zhao, Q. Xiao, Portfolio selection problem with Value-at-Risk constraints under non-extensive statistical mechanics, J. Comput. Appl. Math. 298 (2016) 64–71.
[20] J. Zhou, X. Li, W. Pedrycz, Mean-semi-entropy models of fuzzy portfolio selection, IEEE Trans. Fuzzy Syst. 99 (2016) 1-1.