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# Estimation error in mean returns and the mean-variance efficient frontier☆

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ABSTRACT

In this paper, we build estimation error in mean returns into the mean-variance (MV) portfolio theory under the assumption that returns on individual assets follow a joint normal distribution. We derive the conditional sampling distribution of the MV portfolio along with its mean and risk return when the sample covariance matrix is equal to the population covariance matrix. We use the mean squared error (MSE) to characterize the effects of estimation error in mean returns on the joint sampling distributions and examine how such error affects the risk-return tradeoff of the MV portfolios. We show that the negative effects of error in mean returns on the joint sampling distributions increase with the decision maker's risk tolerance and the number of assets in a portfolio, but decrease with the sample size.

## 1. Introduction

In Markowitz's (1952) paradigm, known as the mean-variance (MV henceforth) model, the objective of a portfolio decision maker (DM henceforth) is to choose a portfolio on the efficient set under the assumption that she has the perfect information on the model parameters – the expected returns on individual assets and the corresponding covariance matrix. In the real world, however, the DM has to estimate the parameters using historical data. Numerous studies, (Best & Grauer, 1991; Broadie, 1993; Chopra & Ziemba, 1993; Litterman, 2004; Michaud, 1989), have shown that bad estimates based on the historical approach that arise from estimation errors in the moments of return distributions can render inferior performance ex-post.

Researchers have striven to derive robust estimates for the MV model to mitigate the negative impacts of estimation errors on portfolio performance. Since Merton (1980), many researchers have shifted their efforts to the global minimum variance (GMV henceforth) portfolio, whose weights depend solely on the more stable covariance matrix. For example, Jagannathan and Ma (2003) and DeMiguel, Garlappi, and Uppal (2009) show that the GMV portfolio outperforms portfolios that require estimating mean returns. Motivated by these findings, researchers have proposed alternative approaches to robust covariance matrix estimates. For instance, Ledoit and Wolf (2003) develop a shrinkage approach that generates a lower GMV portfolio variance than the conventional sample covariance matrix; Candelon, Hurlin, and Tokpavi (2012) provide a double shrinkage approach and show based on Monte Carlo simulation that the double shrinkage approach can be more beneficial when the estimation window is small.

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While many past studies have focused on the GMV portfolio and developed robust estimates of the covariance matrix, it is still important and fruitful to solve for a mean-variance optimal portfolio should the DM have a reliable approach to tackling estimation errors in mean returns.[1] For example, Black and Litterman (1992) demonstrate that a small change in the vector of mean returns can cause large variation in portfolio positions. DeMiguel and Nogales (2009) illustrate how fluctuations in mean returns affect both the mean-variance portfolio and the GMV portfolio, and show that the effect on the former is much more pronounced than on the latter.

In this paper, we provide a theoretical framework that builds estimation error in mean returns into the MV paradigm while assuming that asset returns follow a joint normal distribution. Our main results are derived for the conditional case when the sample covariance matrix is equal to the population covariance matrix. We consider the conditional case for two main reasons. First, we are motivated by previous findings that the effect of estimation error in the covariance matrix on portfolio performance is much less severe than the mean returns (see e.g., Chopra and Ziemba (1993)). Second, we trade-off between accuracy and perspicacity. The results for the conditional case are less complicated than the unconditional one. This, therefore, allows us to inspect the implications of estimation errors for portfolio performance in a more tractable manner.

We derive the conditional joint sampling distributions of the weights of the MV portfolio, along with its mean and risk return when the sample covariance matrix is equal to the population covariance matrix. Using the mean squared error (MSE henceforth) to characterize estimation error, we show analytically and numerically that estimation errors in the weights of the MV portfolio and its corresponding mean and variance increase with the level of risk tolerance and the number of assets under consideration, while it decreases with the sample size. These results suggest that when there is a small number of assets in the portfolio, the benefit of diversification outweighs the cost arising from estimation errors. However, as we keep adding more assets to the portfolio, the cost will eventually dominate the benefit.

We are further interested in the effects of estimation error in mean returns on the risk-return tradeoff of efficient portfolios. In the conventional MV model, the tradeoff is deterministic and given by a parabola that defines a set of optimal portfolios, known as the mean-variance efficient frontier (MVE henceforth). This relation, however, does not hold in practice, due to the presence of estimation error, (Cochrane, 2014). In the paper, we show that for every efficient portfolio on the classical MVE frontier there is a joint distribution between the mean and variance of the MV portfolio return.

Our paper is related to the work by Bodnar and Schmid (2009a). The authors derive the marginal sampling distribution for the mean and variance of each portfolio when the expected returns and the covariance matrix of individual assets are both unknown. Different from Bodnar and Schmid (2009a), we explore the implications of estimation errors for the MVE frontier. We do so graphically with emphasis on the estimation error related factors, i.e. sample size, number of assets, and risk tolerance. Okhrin and Schmid (2006), another related paper, also examine what affects estimation errors in the MV portfolio weights when both the expected returns and the covariance matrix of individual assets are both unknown. However, they do not cover the estimation error effect on the mean and risk return of the MV portfolio.

Our paper is also related to Bodnar and Schmid (2011). The authors derive the distribution of a linear combination of the estimated portfolio weights that allows the decision maker to characterize the distribution of specific positions in the portfolio, which we refer to as the unconditional case. We focus on a conditional case in which the sample covariance matrix is equal to the population covariance matrix. In the paper, we explore the differential implications of Bodnar and Schmid's (2011) and our model. Our numerical exercises suggest that the discrepancy between the two models decreases as the sample size increases. Consistent with past studies, this result indicates that estimation error induced by the mean vector plays a more important role than that induced by the covariance matrix.

The remainder of this paper is organized as follows. Section 2 provides an algebraic review of the traditional mean-variance portfolio theory, in which we do not consider estimation errors in the portfolio analysis. In Section 3, we incorporate estimation error into the portfolio problem and summarize our main findings. Section 4 provides robustness check. Section 5 concludes. We present our main findings in four propositions and report proofs in the appendix.

## 2. Review of asset allocation

In this section, we provide a short algebraic review of asset allocation theory and a number of important properties of the MV efficient set. While this review assumes full information and omits estimation error, the analytical framework serves as the building block of our analysis in Section 3.

Let $\mathbf{r} = (r_1, \ldots, r_d)'$ be the vector of returns on $d$ assets. We assume that returns are independent over time and follow a joint normal distribution, such that $\mathbf{r} \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are, respectively, the mean vector and the covariance matrix of the asset returns. The vector of asset weights is denoted by $\boldsymbol{\xi} = (\xi_1, \ldots \xi_d)$, where $\xi_i$ represents the proportion of wealth allocated in asset $i$, $\forall i = 1, \ldots, d$. The investor is assumed to hold an initial wealth of one dollar that she allocates among the $d$ risky assets, such that $\boldsymbol{\xi}'\mathbf{e} = 1$, where $\mathbf{e}$ is a $d \times 1$ vector of ones. The portfolio return is $\mathbf{r}_p = \boldsymbol{\xi}'\mathbf{r}$ and follows a univariate normal distribution with a mean and variance equal to $\boldsymbol{\eta}_p = \boldsymbol{\xi}'\boldsymbol{\mu}$ and $\boldsymbol{\sigma}_p^2 = \boldsymbol{\xi}'\boldsymbol{\Sigma}\boldsymbol{\xi}$, respectively.

The DM with a risk aversion of $\kappa$ chooses her optimal portfolio by solving the following optimization problem:

$$\max_{\boldsymbol{\xi}} \boldsymbol{\xi}'\boldsymbol{\mu} - 0.5\kappa\boldsymbol{\xi}'\boldsymbol{\Sigma}\boldsymbol{\xi} \, \mathrm{s.t.} \, \boldsymbol{\xi}'\mathbf{e} = 1 \tag{2.1}$$

The solution of (2.1) yields

[1] For instance, Zhu (2013) and DeMiguel, Nogales, and Uppal (2014) investigate assets returns predictability and its implications for optimal portfolio selection.

$$\boldsymbol{\xi} = \boldsymbol{\alpha}_0 + \frac{1}{\kappa}\boldsymbol{\alpha}_1 \tag{2.2}$$

whereas

$$\boldsymbol{\alpha}_0 = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{e}}{\mathbf{e}'\boldsymbol{\Sigma}^{-1}\mathbf{e}}, \tag{2.3}$$

$$\boldsymbol{\alpha}_1 = \mathbf{B}\boldsymbol{\mu}, \tag{2.4}$$

and

$$\mathbf{B} = \boldsymbol{\Sigma}^{-1}\left[\mathbf{I} - \frac{\mathbf{e}'\boldsymbol{\Sigma}^{-1}}{\mathbf{e}'\boldsymbol{\Sigma}^{-1}\mathbf{e}}\right] = \boldsymbol{\Sigma}^{-1}\left[\mathbf{I} - \mathbf{e}\alpha_0'\right]. \tag{2.5}$$

A number of comments are in order. First, the optimization problem in (2.1) is consistent with the expected utility framework, where the trade-off between the portfolio mean and variance is represented by an expected utility function that increases with portfolio mean return and decreases with the undertaken risk.[2] Second, the closed form of the MV portfolio from Equation (2.2) is common in the portfolio literature and follows suit with recent work, e.g., Okhrin and Schmid (2006); Bodnar and Schmid (2009a). This formulation simplifies the mathematical representation of the MV portfolio so that it consists of two sets of allocations, $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_1$, and that the relative weight allocated to each set is determined by the DM's level of risk aversion, i.e., $\kappa$. Third, the portfolio $\boldsymbol{\xi}$ in Equation (2.2) represents the efficient set of allocations for which the DM achieves maximum reward for a given level of risk, or minimum risk for a given reward. We summarize the properties of the efficient set in Lemma 1.

**Lemma 1** Properties of the efficient set:

(a) $\boldsymbol{\alpha}_0 = \boldsymbol{\Sigma}^{-1}\mathbf{e}/\mathbf{e}'\boldsymbol{\Sigma}^{-1}\mathbf{e}$ is the global minimum variance (GMV) portfolio, with mean and variance equal to $\boldsymbol{\eta}_0 = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\mathbf{e}/\mathbf{e}'\boldsymbol{\Sigma}^{-1}\mathbf{e}$ and $\boldsymbol{\sigma}_0^2 = 1/\mathbf{e}'\boldsymbol{\Sigma}^{-1}\mathbf{e}$, respectively.
(b) $\boldsymbol{\alpha}_1 = \mathbf{B}\boldsymbol{\mu}$ is an arbitrage portfolio such that $\boldsymbol{\alpha}_1'\boldsymbol{e} = 0$
(c) The GMV portfolio, $\boldsymbol{\alpha}_0$, is orthogonal to $\boldsymbol{\alpha}_1$, where it holds true that $\alpha_{\mathbf{0}}'\boldsymbol{\Sigma}\boldsymbol{\alpha}_1 = 0$.
(d) It follows that $\mathbf{B}\boldsymbol{\Sigma}\mathbf{B} = \mathbf{B}$, so that $\mathbf{B}\boldsymbol{\Sigma}$ is an idempotent matrix.
(e) $\boldsymbol{\eta}_1 = \boldsymbol{\sigma}_1^2$, where $\boldsymbol{\eta}_1 = \boldsymbol{\mu}'\boldsymbol{\alpha}_1$ and $\boldsymbol{\sigma}_1^2 = \boldsymbol{\alpha}_1\boldsymbol{\Sigma}\boldsymbol{\alpha}_1$
(f) The variance of the MV portfolio return is equal to $\boldsymbol{\sigma}_p^2 = \boldsymbol{\sigma}_0^2 + \kappa^{-2}\boldsymbol{\sigma}_1^2$.
(g) For each level of risk $\sigma_p^2$, the MVE frontier is given by

$$\boldsymbol{\eta}_p = \boldsymbol{\eta}_0 + \sqrt{\boldsymbol{\sigma}_1^2\left(\boldsymbol{\sigma}_p^2 - \boldsymbol{\sigma}_0^2\right)}, \tag{2.6}$$

where $\boldsymbol{\eta}_p$ is the expected return of the MV portfolio, i.e., $\boldsymbol{\eta}_p = \boldsymbol{\xi}'\boldsymbol{\mu}$.

In finding her optimal portfolio, the DM takes an initial position in the GMV portfolio, $\boldsymbol{\alpha}_0$. She then shifts her allocation to the arbitrage portfolio, $\boldsymbol{\alpha}_1$, depending on her degree of risk aversion, characterized by $\kappa$. For instance, if the DM is extremely risk averse, i.e., $\kappa \to \infty$, then she will allocate all her wealth to the GMV portfolio. As the DM's risk tolerance increases, i.e., smaller $\kappa$, she allocates more wealth to the arbitrage portfolio. Equation (2.6) formulates the trade-off between the mean and variance of the MV portfolio. That is, for each level of risk, $\boldsymbol{\sigma}_p^2$, there is a maximum level of award, $\boldsymbol{\eta}_p$, with the weights of the optimal portfolio set to $\boldsymbol{\xi}$.

Equation (2.6) gives the MVE frontier under full information. In the real world, however, it is not clear how to derive the MVE frontier when the DM does not possess information on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (see e.g., (Cochrane, 2014)). We address this issue in the following section.

## 3. Estimation error in portfolio selection

In this section, we extend the asset allocation problem described in Section 2 by incorporating estimation error into the MV paradigm. In particular, we require that the mean returns on individual assets be estimated while assuming that returns on individual assets have a joint normal distribution. Our analysis considers the conditional case in which the sample covariance matrix is equal to the population covariance matrix. Hence, we only focus on the case of "perfect" knowledge about the covariance matrix in this paper.

Under full information, the MV portfolio, $\boldsymbol{\xi}$, from Equation (2.2) is a function of the mean vector, $\boldsymbol{\mu}$, and the covariance matrix, $\boldsymbol{\Sigma}$. In practice, however, neither the mean vector nor the covariance matrix is known. The estimated portfolio is, hence, a function of each, such that

$$\boldsymbol{\xi}(\mathbf{m}, \mathbf{S}) = \widehat{\boldsymbol{\alpha}}_0 + \frac{1}{\kappa}\widehat{\mathbf{B}}\mathbf{m} \tag{3.1}$$

[2] See Markowitz (1991, 2014); Levy and Markowitz (1979); Kroll, Levy, and Markowitz (1984); Simaan (1993, 2014) for further discussion on the consistency between MV and expected utility portfolios.

where $\mathbf{m}$ and $\mathbf{S}$ correspond, respectively, to the sample mean and the sample covariance matrix based on a sample of $n$ periods. Moreover, $\widehat{\boldsymbol{\alpha}}_0$ and $\widehat{\mathbf{B}}$ denote the estimated GMV portfolio and the $\mathbf{B}$ matrix from Equations (2.3) and (2.5) using the sample covariance matrix $\mathbf{S}$, respectively.

In this paper, we focus on the case when the DM possess some information about the covariance matrix while she estimates the mean vector using the sample mean. Put formally, we concentrate on the portfolio from Equation (3.1) when the sample covariance matrix is equal to the population covariance matrix, i.e. $\boldsymbol{\xi}(\mathbf{m}, \mathbf{S} = \boldsymbol{\Sigma})$. In the rest of this paper, we refer to this portfolio as the estimated MV portfolio and denote it by $\mathbf{x}$, where

$$\mathbf{x} = \boldsymbol{\xi}(\mathbf{m}, \boldsymbol{\Sigma}) = \boldsymbol{\alpha}_0 + \frac{1}{\kappa}\mathbf{Bm}. \tag{3.2}$$

The DM realizes that the mean returns are not the true parameters but are subject to estimation errors. Therefore, each portfolio on the efficient set is unknown and estimated using a random vector $\mathbf{x}$. Moreover, the mean and variance of the MV portfolio, $\boldsymbol{\eta}_p$ and $\boldsymbol{\sigma}_p^2$, are also unknown and estimated using $\tilde{E}_p$ and $\tilde{v}_p$ respectively, such that $\tilde{E}_p = \mathbf{x}'\mathbf{m}$ and $\tilde{v}_p = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}$.[3] From Equation (3.2), it follows that

$$\tilde{E}_p = \mathbf{m}'\boldsymbol{\alpha}_0 + \kappa^{-1}\mathbf{m}'\mathbf{Bm} \tag{3.3}$$

and

$$\tilde{v}_p = \boldsymbol{\alpha}_0\boldsymbol{\Sigma}\boldsymbol{\alpha}_0 + 2\kappa^{-1}\alpha_0'\boldsymbol{\Sigma}\mathbf{Bm} + \kappa^{-2}\mathbf{m}'\mathbf{B}\boldsymbol{\Sigma}\mathbf{Bm}. \tag{3.4}$$

Before moving on to the main findings of the paper, a number of comments are in order. First, the MV efficient frontier under estimation error is determined by Equation (3.2) through (3.4). Second, when the sample covariance matrix is equal to the population covariance matrix, the $\mathbf{B}$ matrix and the GMV portfolio, $\boldsymbol{\alpha}_0$, are known. Third, the presence of estimation error in the mean vector not only affects the portfolio's mean return but also its variance; this is evident in Equations (3.3) and (3.4), where each of $\tilde{E}_p$ and $\tilde{v}_p$ is a function of the sample mean, $\mathbf{m}$. Last, under estimation error, the weights of the MV portfolio along with its mean and variance are unknown (since $\mathbf{m}$ is random). Therefore, to understand the impacts of estimation error in mean returns on the MV paradigm as well as on trade-off between mean and variance (MVE frontier), we need to address the statistical properties of $\mathbf{x}$, $\tilde{E}_p$, and $\tilde{v}_p$.

In the rest of this section, we discuss the main findings of the paper. Specifically, we derive (i) the conditional sampling distribution of the MV portfolio weights ($\mathbf{x}$); (ii) the conditional sampling distribution of the portfolio mean return ($\tilde{E}_p$); (iii) the conditional sampling distribution of the portfolio risk return ($\tilde{v}_p$); and (iv) the conditional joint sampling distribution of $\tilde{E}_p$ and $\tilde{v}_p$. Using the MSE as a proxy for estimation error, we find the corresponding closed form expression to characterize the effects of error in mean returns on each of the three components.[4] We summarize the key findings in four main propositions.

### 3.1. Weights in individual assets

First, we derive the statistical properties of the weights of the MV portfolio, i.e., $\mathbf{x}$ from Equation (3.2).

**Proposition 1** Under the condition that the sample covariance matrix, $\mathbf{S}$, is equal to the population covariance matrix, $\boldsymbol{\Sigma}$, the estimated MV portfolio is both unbiased and consistent. Moreover, it follows a multivariate joint normal distribution, such that

$$\mathbf{x} \sim N\left(\boldsymbol{\xi}, n^{-1}\kappa^{-2}\mathbf{B}\right) \tag{3.5}$$

Proposition 1 provides a number of direct insights. Since $\mathbf{x}$ is an unbiased estimate of $\boldsymbol{\xi}$, $MSE(\mathbf{x}) = Var(\mathbf{x})$, the estimation error of $\mathbf{x}$, is given by the $n^{-1}\kappa^{-2}\mathbf{B}$ matrix. This implies that the estimation error in $\mathbf{x}$ decreases with the sample size ($n$), or the amount of information that the DM possesses. Additionally, as the degree of risk aversion ($\kappa$) goes to infinity, the estimation error of $\mathbf{x}$ becomes trivial as the DM invests all her wealth in the GMV portfolio, which is associated with lower estimation error.[5]

To understand the impact of the $\mathbf{B}$ matrix on the estimation error in $\mathbf{x}$, we take a closer look at the element of row $i$ and column $j$ of $Var(\mathbf{x})$; this corresponds to the covariance between the weights allocated to asset $i$ and $j$ in the MV portfolio, for every $i, j = 1, \ldots, d$.

$$Cov\left(\mathbf{x}_i, \mathbf{x}_j\right) = \frac{1}{n\kappa^2}\left[v_{ij} - \frac{\boldsymbol{\alpha}_{0i}\boldsymbol{\alpha}_{0j}}{\boldsymbol{\sigma}_0^2}\right] \tag{3.6}$$

where $v_{ij}$ is the element of the $i$-th row and $j$-th column of $\boldsymbol{\Sigma}^{-1}$ and $\boldsymbol{\alpha}_{0i}$ is the weight allocated to asset $i$ in the GMV portfolio, for every $i$, $j = 1, \ldots, d$.

A couple of comments are in order. First, Equation (3.6) indicates that for all $i = j$ and holding everything else constant, the estimation error of the weight in asset $i$ increases as it shifts away from the corresponding weight of the GMV portfolio ($\boldsymbol{\alpha}_{0i}^2$), it increases with the variance of the GMV portfolio ($\boldsymbol{\sigma}_0^2$), and it is negatively associated with the ratio of $\boldsymbol{\alpha}_{0i}^2$ to $\boldsymbol{\sigma}_0^2$. Second, for $i \neq j$, there is a cross-sectional effect on the estimation error between two positions. For instance, if all asset returns are uncorrelated, where $v_{ij} = 0$, $\forall i \neq j$, and $\boldsymbol{\Sigma}$ is

[3] Note that in reality it should be $\tilde{v}_p = \mathbf{x}'\mathbf{S}\mathbf{x}$. However, since we focus on the conditional case in which $\mathbf{S} = \boldsymbol{\Sigma}$ , then it follows that $\tilde{v}_p = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}$.

[4] Recall that if $\widehat{\theta}$ is the estimate of the parameter θ, then the MSE of $\widehat{\theta}$ is given by $MSE(\widehat{\theta}) = Var(\widehat{\theta}) + (E[\widehat{\theta}] - \theta)^2$.

[5] We note that in the unconditional case, the estimation error does not necessarily converge to zero as κ goes to infinity. However, since estimation error in the mean vector is much more severe than in the covariance matrix, the GMV portfolio should result in a lower estimation error.

a diagonal matrix, then all weights of the GMV portfolio are positive, i.e. $\boldsymbol{\alpha}_{0i} > 0$, $\forall i$, and, thus, such cross-sectional effect is negative.[6]

The result from Proposition 1 denotes the conditional distribution when the sample covariance matrix is equal to the population covariance matrix. This result is a specific case for the unconditional distribution of the portfolio weights proposed by Bodnar and Schmid (2011). Since the unconditional case serves as the more accurate one, we compare the proposed distribution with theirs. We devote Section 4 to this discussion.

### 3.2. The portfolio mean and variance

In this section, we investigate the impact of estimation error on the portfolio's mean and variance. The corresponding results are summarized in Proposition 2 and Proposition 3, both of which depend on the following two lemmas:

**Lemma 2.** The quadratic form $n\mathbf{m}'\mathbf{B}\mathbf{m}$ follows a Non-central Chi-squared distribution with $d-1$ degrees of freedom and a non-central parameter $\delta = n\boldsymbol{\mu}'\mathbf{B}\boldsymbol{\mu}$, where $\mathbf{B}$ is a singular matrix with $rank(\mathbf{B}) = d-1$.

**Lemma 3.** *Let* $\tilde{E}_1 = n\mathbf{m}'\mathbf{B}\mathbf{m}$ *and* $\tilde{E}_0 = \mathbf{m}'\boldsymbol{\alpha}_0$, *then* $\tilde{E}_1$ *and* $\tilde{E}_0$ *are mutually independent, i.e.,* $\tilde{E}_1 \perp \tilde{E}_0$.

**Proposition 2.** Under the condition that the sample covariance matrix, $\mathbf{S}$, is equal to the population covariance matrix, $\boldsymbol{\Sigma}$, the mean return of the MV portfolio, i.e., $\widetilde{E_p} = \mathbf{m}'\mathbf{x}$, shares the following properties:

(a) $\widetilde{E_p}$ is a biased estimate of the actual MV portfolio mean return, $\boldsymbol{\eta}_p$. The bias is positive and increases with risk tolerance and number of assets under consideration, however it shrinks with the sample size:

$$E\left[\tilde{E}_p\right] = \boldsymbol{\eta}_p + \frac{d-1}{n\kappa} \tag{3.7}$$

(b) $\widetilde{E_p}$ is a consistent estimate of $\boldsymbol{\eta}_p$. The estimation error increases with the level of risk tolerance and the number of assets within the portfolio, while it decreases with the sample size:

$$MSE\left[\tilde{E}_p\right] = \frac{1}{n}\left[\boldsymbol{\sigma}_p^2 + 3\kappa^{-2}\boldsymbol{\sigma}_1^2\right] + \frac{d^2-1}{n^2\kappa^2} \tag{3.8}$$

(c) The moment generating function of $\widetilde{E_p}$ is

$$M_{\tilde{E}_p}(t) = \left(1 - \frac{t}{n\kappa}\right)^{-\frac{d-1}{2}} exp\left(t\left(\boldsymbol{\eta}_0 + \frac{n}{n\kappa - 2t}\boldsymbol{\mu}'\mathbf{B}\boldsymbol{\mu}\right) + \frac{t^2}{2n}\boldsymbol{\sigma}_0^2\right) \tag{3.9}$$

**Proposition 3** Under the condition that the sample covariance matrix, $\mathbf{S}$, is equal to the population covariance matrix, $\boldsymbol{\Sigma}$, the variance of the MV portfolio, i.e., $\widetilde{v_p} = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}$, shares the following properties:

(a) $\tilde{v}_p$ overstates the actual MV portfolio variance, $\boldsymbol{\sigma}_p^2$. The bias increases with the level of risk tolerance and the number of assets under consideration; nevertheless,it decreases with the sample size, such that

$$E\left[\tilde{v}_p\right] = \boldsymbol{\sigma}_p^2 + \frac{d-1}{n\kappa^2} \tag{3.10}$$

2 $\tilde{v}_p$ is a consistent estimate of $\boldsymbol{\sigma}_p^2$. The estimation error of $\tilde{v}_p$ increases with the level of risk tolerance and the number of assets, but it decreases with the sample size, such that:

$$MSE\left[\tilde{v}_p\right] = \frac{4}{n\kappa^2}\left[\boldsymbol{\sigma}_p^2 - \boldsymbol{\sigma}_0^2\right] + \frac{d^2-1}{n^2\kappa^4} \tag{3.11}$$

3 $\tilde{v}_p$ follows a shifted, scaled non-central Chi-squared distribution with the following moment generating function:

$$M_{\tilde{v}_p}(t) = exp\left(t\boldsymbol{\sigma}_0^2 + \frac{t\boldsymbol{\mu}'\mathbf{B}\boldsymbol{\mu}}{\kappa^2 - 2tn^{-1}}\right)\cdot\left(1 - \frac{2t}{n\kappa^2}\right)^{-\frac{d-1}{2}} \tag{3.12}$$

The main take from Proposition 2 and Proposition 3 is the estimation errors in the MV portfolio mean and variance increase with the level of risk tolerance and the number of assets included the portfolio, while it decreases with the sample size. With respect to this, a number of comments are in order. First, the sample size $n$ represents the set of the information that the DM possesses, so that estimation

[6] Note that as long as $\boldsymbol{\Sigma}$ is positive definite, it is an invertible matrix. When asset returns are uncorrelated, $\boldsymbol{\Sigma}$ is a diagonal matrix, and so is its inverse, i.e. $\boldsymbol{\Sigma}^{-1}$. Hence, the off-diagonal elements of $\boldsymbol{\Sigma}^{-1}$ are equal to zero as well, i.e. $v_{ij} = 0$ for all $i \neq j$.

error decreases as the sample size increases.

Second, in line with the findings from Proposition 1, as $\kappa$ increases, the DM shifts her portfolio toward the GMV portfolio. This can be seen directly from Equations (3.3) and (3.4), which based on the notation of Lemma 3 can be written, respectively, as

$$\tilde{E}_p = \tilde{E}_0 + n^{-1}\kappa^{-1}\tilde{E}_1 \tag{3.13}$$

and

$$\tilde{v}_p = \boldsymbol{\sigma}_0^2 + n^{-1}\kappa^{-2}\tilde{E}_1. \tag{3.14}$$

The result that estimation error decreases with greater level of risk aversion from Proposition 2 and 3 is implied directly from Equations (3.13) and (3.14). Clearly, the increase in $\kappa$ diverts from the stochastic component $\tilde{E}_1$, which contains greater estimation error. Moreover, as $\kappa$ increases, both $\tilde{E}_p$ and $\tilde{v}_p$ converge to the mean and variance of the GMV portfolio, respectively. Since the GMV portfolio is associated with lower estimation error, so is the case for its mean and variance.

Third, as the DM allocates her wealth across larger set of assets, she faces greater uncertainty about the parameters, and hence, the estimation error of the mean and variance of the MV portfolio should increase. It is well established that estimation error causes the MV portfolio to deviate from the true optimal rule, and hence, result in poor out-of-sample performance, (Jorion, 1986; Klein & Bawa, 1976; Michaud, 1989). The increase of estimation error with respect to increase in the number of assets held in the portfolio is consistent with the familiarity of the DM. If the DM is not familiar with the underlying assets, then she is more likely to diversify among larger number of assets. On the other hand, when she feels confident about certain assets, she is likely to hold a smaller set of assets. Thus, while diversification across a larger number of assets is beneficial, it also comes at the cost of higher estimation error. In this regard, there should be trade-off between diversification and familiarity (see e.g., (Boyle, Garlappi, Uppal, & Wang, 2012)). Moreover, while we show that estimation error decreases with greater risk aversion, investors with larger degree of risk aversion are also likely to invest in smaller set of assets. For instance, Levy and Simaan (2016) show that there is a boundary of risk aversion, where the possession of more assets results in greater punishment from an estimation error perspective. Hence, investors whose risk aversion falls within a given set are better off not diversifying.

Fourth, even though the sample mean and the estimated portfolio are unbiased, the mean and variance of the MV portfolio overshoot their corresponding parameters. This fact can be explained intuitively by Jensen inequality.[7] From (3.3) and (3.4), we know that the portfolio mean and variance are quadratic (convex) functions of the sample mean, $\mathbf{m}$. Thus, if we denote both $\tilde{E}_p$ and $\tilde{v}_p$ by $f_E(\mathbf{m})$ and $f_v(\mathbf{m})$ respectively, then we have $E[f_E(\mathbf{m})] \geq f_E(\boldsymbol{\mu}) = \boldsymbol{\eta}_p$ and $E[f_v(\mathbf{m})] \geq f_v(\boldsymbol{\mu}) = \boldsymbol{\sigma}_p^2$. This implies that the bias in the portfolio mean and variance is integrated by the construction of the MV portfolio. For instance, anticipating this systematic bias while allocating her wealth, the DM can shrink $\mathbf{m}$ with an ad-hoc vector such that the bias of either/both $\tilde{E}_p$ and $\tilde{v}_p$ is omitted/minimized.[8] This is common with shrinkage estimates used in the literature, (see e.g., Jorion (1986)).

### 3.2.1. Empirical Validation

The message from the closed form expressions for MSE in Propositions 2 and 3 is clear: holding everything else constant, estimation error increases with larger $d$, smaller $n$, and smaller $\kappa$. Nonetheless, each expression depends on components related to the asset returns moment. In order to validate this message, we compute the MSE expressions from Equations (3.8) and (3.11) using historical data on asset returns and inspect numerically how these measures are affected with respect to $d$, $n$, and $\kappa$.

As our universe of assets, we consider the Fama-French 48 value-weighted industry portfolios. The sample spans the period of Jan 1970 to Dec 2015. Hence, for each industry portfolio, we have 552 monthly returns. For a fixed sample size of $n = 240$, we let $\kappa = 2, 5, 10, 15$ and $d = 2, 3, 4, 5, 6, 12, 24, 48$. For each possible combination we randomly pick $n$ months and $d$ assets (industry portfolios) with a fixed seed.[9] Hence, given each $\{n, d, \kappa\}$ combination we compute the MSE from Equations (3.8) and (3.11) and report the results in Table 1. Similar to Table 1, Table 2 reports the MSE values with respect to changes in the sample size and level of risk aversion, while keeping the number of assets fixed. Specifically, we set $d = 12$ while letting $n = 120, 240, 360, 480$ and $\kappa = 2, 5, 10, 15$.

The empirical evidence from Tables 1 and 2 is comprehensible and consistent with our theoretical findings. Holding, $n$ and $d$ fixed, we discern from both tables that estimation error (for both $\tilde{E}_p$ and $\tilde{v}_p$) decreases as $\kappa$ increases. From Table 1, we discern that for fixed $n$ and $\kappa$ estimation error increases as the number of assets in the portfolio increases. Nonetheless, it appears that there is a slight drop (or no increase) in estimation error, when the number of assets increases from 3 to 4. However, as we add further assets, we observe that estimation error increases exponentially. A possible explanation for this is the tradeoff between uncertainty and diversification, (Boyle et al., 2012). When the number of assets is very small, i.e. $d = 3$, the DM faces less uncertainty, such that as she adds one more asset she ends up with better diversification. However, as she adds more assets into her portfolio, the welfare gained from diversification is, eventually, outweighed by the increased estimation error.[10] Finally, the take from Table 2 follows suit: for fixed $d$ and $\kappa$ it is clear that estimation error decreases as the sample size increases.

7 Recall that if $f: \mathbb{R}\to\mathbb{R}$ is a convex function, then $f(E[X]) \leq E[f(X)]$.

8 To demonstrate this, let $\widehat{\mathbf{m}} = \mathbf{m} + \mathbf{e}\cdot\theta$ for some constant $\theta$, such that the mean return of the MV portfolio becomes $\widehat{E}_p = \alpha_0'(\mathbf{m}+\mathbf{e}\cdot\theta) + \kappa^{-1}(\mathbf{m}+\mathbf{e}\cdot\theta)'\mathbf{B}(\mathbf{m}+\mathbf{e}\cdot\theta)$. Then, the DM can omit the bias by finding the $\theta$ such that satisfies the identity that $E[\widehat{E}_p] = \boldsymbol{\eta}_p$.

9 By setting a fixed seed, we make sure that the assets which show up when $d = 2$ will also be included when $d = 3$, and so on. The same logic holds true for the sample size, by setting a seed we make sure that the same periods are included when the sample size increases.

10 Recall that the MSE measure is the sum of the variance of the estimate and its squared bias. Hence, a decrease in the MSE could be due to decrease in the variance of the estimate. When the number of assets increases, estimation error increase but it could also be the case that the variance of the estimate decreases due to better diversification.

**Table 1**
**Estimation error sensitivity with respect to $d$ and $\kappa$.** This table reports the numerical values of the mean squared error (MSE) of the mean and risk of the MV portfolio return, using the closed form expression derived in Equations (3.8) and (3.11) and historical monthly returns of the Fama-French 48 value-weighted industry portfolios between Jan 1970 and Dec 2015. For a fixed sample size $n = 240$, MSE values are reported in basis points with respect to different number of assets, $d$, and levels of risk aversion, $\kappa$. Panel (a) corresponds to estimation error of the MV portfolio mean return, i.e. Equation (3.8) and Panel (b) corresponds to the estimation error of the MV portfolio risk return, i.e. Equation (3.11).

| $d\backslash\kappa$ | 2 | 5 | 10 | 15 |
|---|---|---|---|---|
| Panel (a) MSE($\tilde{E}_p$) | | | | |
| 2 | 0.2627 | 0.1141 | 0.0651 | 0.1460 |
| 3 | 1.0598 | 0.1603 | 0.1162 | 0.1338 |
| 4 | 0.9942 | 0.2710 | 0.1344 | 0.1137 |
| 5 | 1.3805 | 0.2700 | 0.1636 | 0.1126 |
| 6 | 2.3728 | 0.3359 | 0.1527 | 0.1334 |
| 12 | 7.7499 | 1.2435 | 0.3216 | 0.1754 |
| 24 | 27.8509 | 4.3456 | 1.1186 | 0.5059 |
| 48 | 106.5658 | 16.7277 | 4.1945 | 1.8652 |
| Panel (b) MSE($\tilde{v}_p$) | | | | |
| 2 | 0.0331 | 0.0008 | 0.0001 | 0.0000 |
| 3 | 0.3316 | 0.0048 | 0.0001 | 0.0000 |
| 4 | 0.2556 | 0.0110 | 0.0004 | 0.0001 |
| 5 | 0.3507 | 0.0076 | 0.0008 | 0.0001 |
| 6 | 0.6711 | 0.0140 | 0.0012 | 0.0002 |
| 12 | 2.1453 | 0.0578 | 0.0032 | 0.0007 |
| 24 | 7.3759 | 0.1900 | 0.0123 | 0.0023 |
| 48 | 27.6187 | 0.7101 | 0.0450 | 0.0087 |

**Table 2**
**Estimation error sensitivity with respect to $n$ and $\kappa$.** This table reports the numerical values of the mean squared error (MSE) of the mean and risk of the MV portfolio return, using the closed form expression derived in Equations (3.8) and (3.11) and historical monthly returns of the Fama-French 48 value-weighted industry portfolios between Jan 1970 and Dec 2015. For a fixed number of assets size $d = 12$, MSE values are reported in basis points with respect to different sample sizes, $n$, and levels of risk aversion, $\kappa$. Panel (a) corresponds to estimation error of the MV portfolio mean return, i.e. Equation (3.8) and Panel (b) corresponds to the estimation error of the MV portfolio risk return, i.e. Equation (3.11).

| $n\backslash\kappa$ | 2 | 5 | 10 | 15 |
|---|---|---|---|---|
| Panel (a) MSE($\tilde{E}_p$) | | | | |
| 120 | 28.3374 | 4.4578 | 1.2036 | 0.6106 |
| 240 | 7.2155 | 1.1570 | 0.3354 | 0.1859 |
| 360 | 3.3903 | 0.5464 | 0.1675 | 0.0990 |
| 480 | 1.8950 | 0.3108 | 0.0991 | 0.0608 |
| Panel (b) MSE($\tilde{v}_p$) | | | | |
| 120 | 7.5548 | 0.1934 | 0.0121 | 0.0024 |
| 240 | 1.9282 | 0.0494 | 0.0031 | 0.0006 |
| 360 | 0.9244 | 0.0237 | 0.0015 | 0.0003 |
| 480 | 0.5131 | 0.0131 | 0.0008 | 0.0002 |

### 3.3. The sampling distribution of the MVE frontier

Recall that under full information the MVE efficient is deterministic, such that the relation between the mean and risk is governed by Equation (2.6). However, under estimation error, such relation is determined by the joint distribution between $\tilde{E}_p$ and $\tilde{v}_p$, which we refer to as the sampling distribution of the MVE frontier. The distribution is identified by the moment generating function as well as the probability density function (PDF), conditional on the case that $\mathbf{S} = \mathbf{\Sigma}$. We outline each in Proposition 4.

**Proposition 4** Under the condition that the sample covariance matrix, $\mathbf{S}$, is equal to the population covariance matrix, $\mathbf{\Sigma}$, the sampling distribution of the MVE frontier can be described by either

(a) the joint moment generating function of $\tilde{E}_p$ and $\tilde{v}_p$

$$M_{\tilde{E}_p,\tilde{v}_p}(t_1,t_2) = exp\left(t_2\boldsymbol{\sigma}_0^2 + \boldsymbol{\eta}_0 t_1 + \frac{\boldsymbol{\sigma}_0^2 t_1^2}{2n} + \boldsymbol{\mu}\prime\mathbf{B}\boldsymbol{\mu}\frac{\frac{\kappa t_1+t_2}{\kappa^2}}{1-2\frac{\kappa t_1+t_2}{n\kappa^2}}\right)\cdot\left(1-2\frac{\kappa t_1+t_2}{n\kappa^2}\right)^{-\frac{d-1}{2}} \tag{3.15}$$

or

(b) the joint PDF of $\tilde{E}_p$ and $\tilde{v}_p$

$$f_{\tilde{E}_p\tilde{v}_p}\left(E_p, v_p\right) = f_N\left(E_p;\ \kappa\left(v_p-\boldsymbol{\sigma}_0^2\right)+\boldsymbol{\eta}_0,\ n^{-1}\boldsymbol{\sigma}_0^2\right)\cdot n\kappa^2 f_{\chi^2}\left(v_p;\ d-1,\ n\boldsymbol{\mu}\prime\mathbf{B}\boldsymbol{\mu}\right) \tag{3.16}$$

where $f_N(x;\ \mu,\ \sigma^2)$ and $f_{\chi^2}(y;\ r,\ \delta)$, respectively, refer to the PDF of (i) a normal random variable at point x with mean μ and variance $\sigma^2$, and (ii) a Non-central Chi-squared random variable at point y with r degrees of freedom and non-centrality parameter δ.

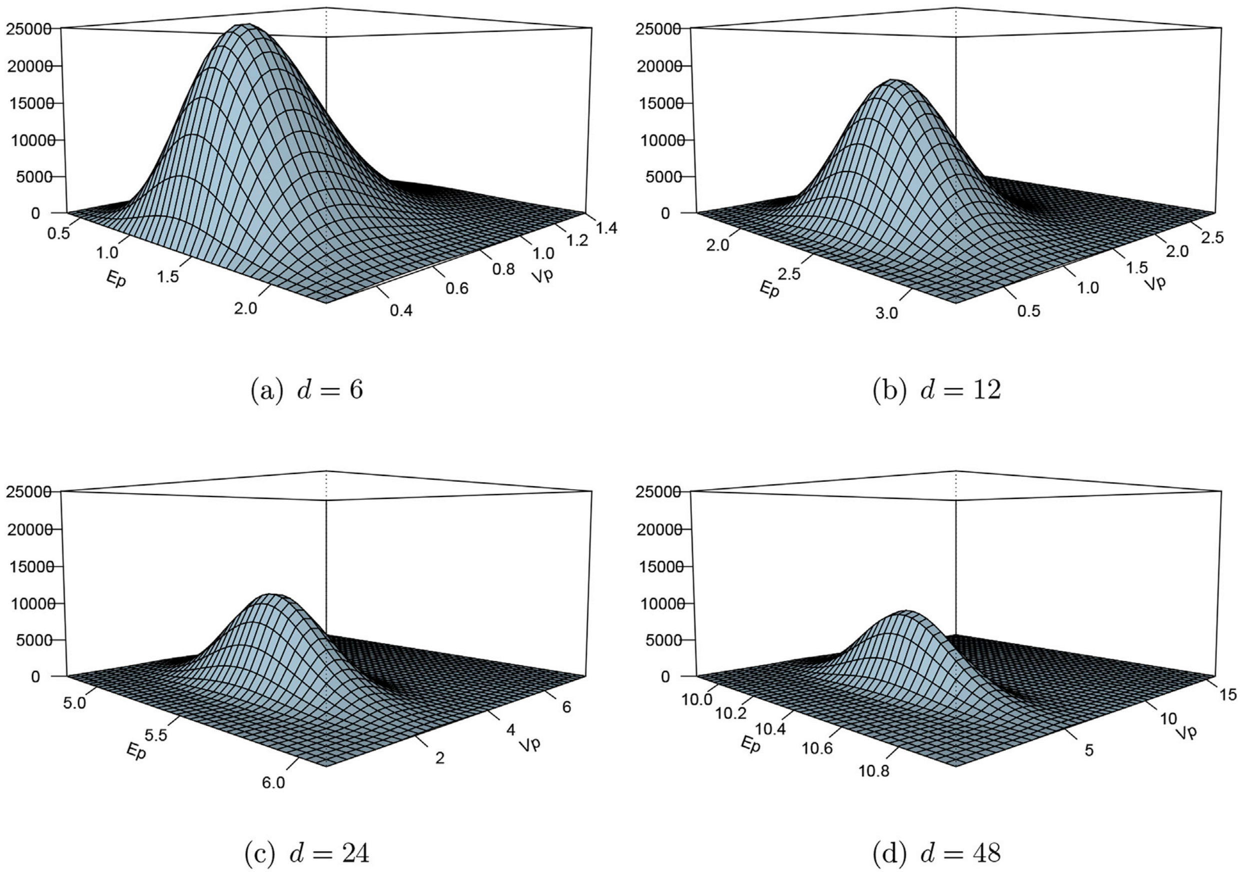
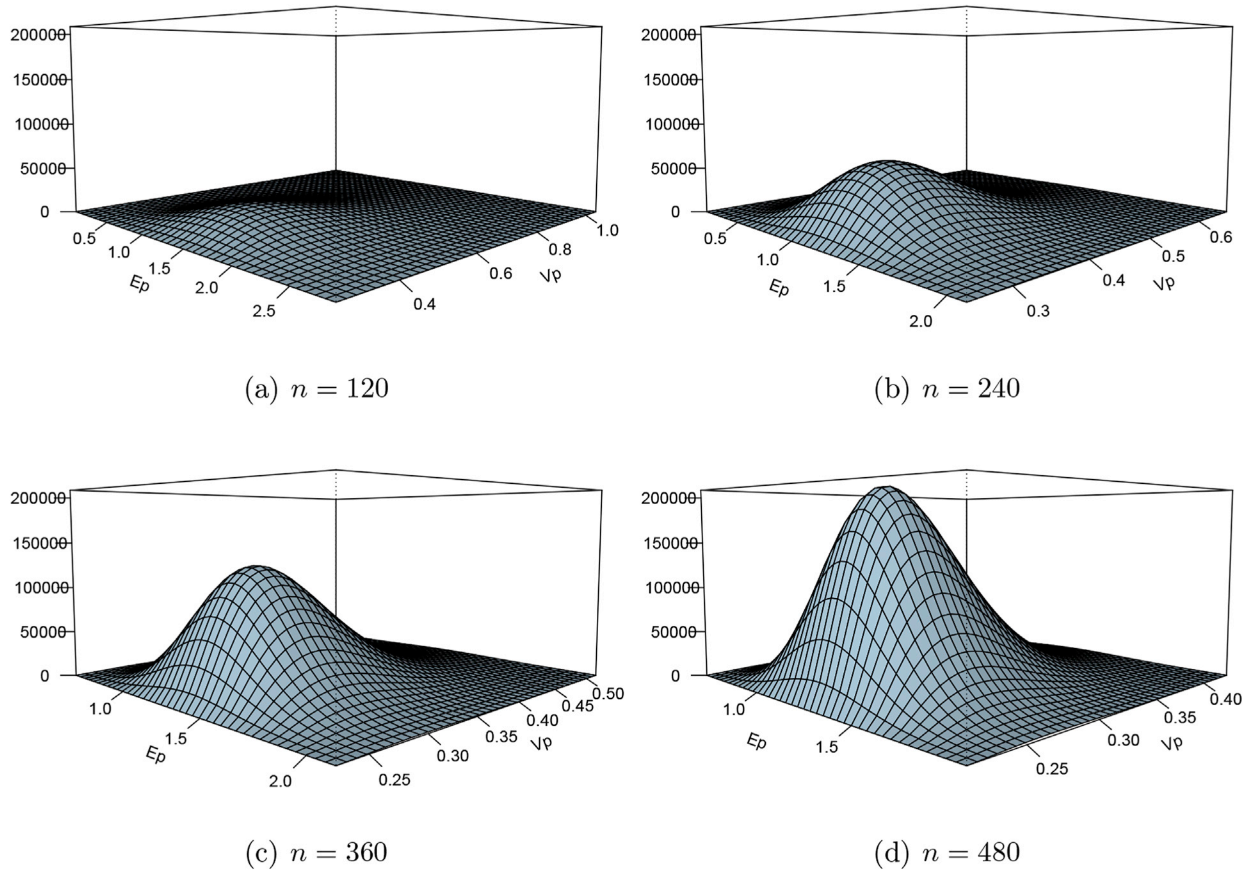


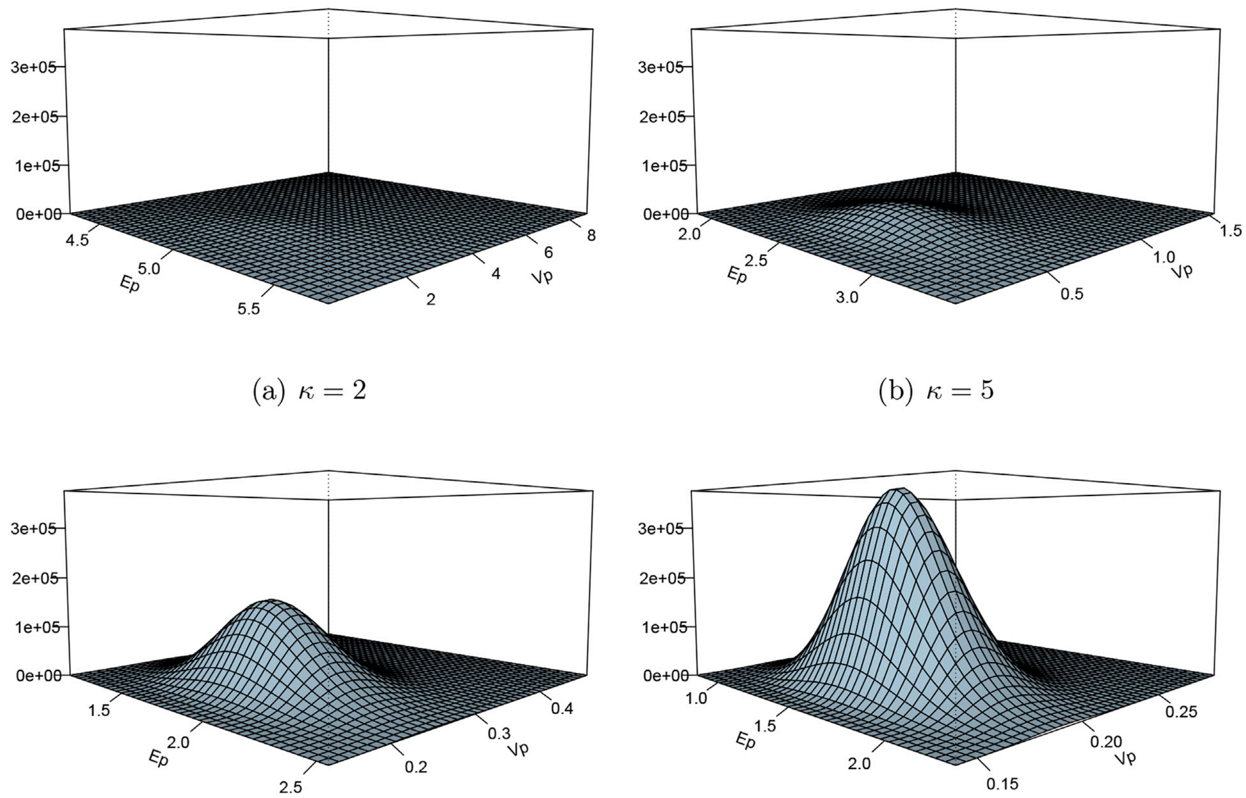

**Fig. 1. The sampling distribution of the MVE frontier with respect to different number of assets.**
This figure plots the joint sampling distribution of $\tilde{E}_p$ and $\tilde{v}_p$ from Equations (3.13) and (3.14), respectively. The distribution is illustrated given the closed form probability density function (PDF) from Proposition 4 Equation (3.16) and historical monthly returns of the Fama-French 48 value-weighted industry portfolios between Jan 1970 and Dec 2015. The panels correspond to different number of assets (i.e., industries) with fixed level of risk aversion, $\kappa = 3$, and sample size, $n = 240$.




(a) $n = 120$ (b) $n = 240$

(c) $n = 360$ (d) $n = 480$

**Fig. 2. The sampling distribution of the MVE frontier with respect to different sample sizes.**
This figure plots the joint sampling distribution of $\tilde{E}_p$ and $\tilde{v}_p$ from Equations (3.13) and (3.14), respectively. The distribution is illustrated given the closed form probability density function (PDF) from Proposition 4 Equation (3.16) and historical monthly returns of the Fama-French 48 value-weighted industry portfolios between Jan 1970 and Dec 2015. The panels correspond to different sample sizes with fixed level of risk aversion, $\kappa = 5$, and number of assets (i.e. industry portfolios), $d = 6$.




(a) $\kappa = 2$ (b) $\kappa = 5$

(c) $\kappa = 10$ (d) $\kappa = 15$

**Fig. 3. The sampling distribution of the MVE frontier with respect to different levels of risk aversion**.
This figure plots the joint sampling distribution of $\tilde{E}_p$ and $\tilde{v}_p$ from Equations (3.13) and (3.14), respectively. The distribution is illustrated given the closed form probability density function (PDF) from Proposition 4 Equation (3.16) and historical monthly returns of the Fama-French 48 value-weighted industry portfolios between Jan 1970 and Dec 2015. The panels correspond to different levels of risk aversion with fixed sample size, $n = 240$, and number of assets (i.e. industry portfolios), $d = 10$.

Given the moment generating function in Proposition 4, one can easily derive the covariance between the portfolio mean and variance, and hence, the correlation coefficient between the two, such that the correlation coefficient between $\tilde{E}_p$ and $\tilde{v}_p$ is given by[11]:

$$\rho_{\tilde{E}_p,\tilde{v}_p} = \sqrt{\frac{Var(\tilde{E}_1)}{n\kappa^2\boldsymbol{\sigma}_0^2 + Var(\tilde{E}_1)}}, \tag{3.17}$$

where $Var(\tilde{E}_1) = 2(d-1) + 4n\boldsymbol{\mu}\prime\mathbf{B}\boldsymbol{\mu}$.

Equation (3.17) indicates that the correlation coefficient decreases with the level of risk aversion. This result is consistent with Markowitz, Reid, and Tew (1994) who document that investors with extreme level of risk aversion exhibit an obsession to avoid any portfolio risk that exceeds that of the GMV, regardless of the portfolio's mean return. Markowitz et al. (1994) further show that exponential utility function with $\kappa > 1$ leads to a pathological risk aversion behavior in which investor rejects a gamble with 50% probability of zero return and 50% probability of infinite wealth (a blank check gamble) in favor of trivial safe return. The fact that the correlation between the portfolio mean and variance converges to zero when the risk aversion of the DM increases can provide more evidence in support of this pathological risk aversion behavior, in which the trade-off between risk and mean return is not relevant any more.

The PDF function from (3.16) is proposed in a form that is convenient to derive using common statistical softwares. In order to draw conclusions about the sampling distribution of the MVE frontier, we visualize the joint PDF with respect to different number of assets, sample sizes, and levels of risk aversion. Similar to data used in the Empirical Validation 3.2.1 part, we consider the Fama-French 48 industry value-weighted portfolios for the period January 1970 – December 2015.

In Fig. 1, we plot the sampling distribution of the MVE frontier with respect to different number of assets. In doing so, we set $\kappa = 3$ and $n = 240$, and draw the sampling distribution of the MVE frontier for $d = 6, 12, 24, 48$. For each given $d$ and $n$, we randomly pick $d$ assets along with $n$ months by setting a fixed seed. Comparing among the panels in Fig. 1, we observe that for small number of assets the distribution is more concentrated and exhibits lower dispersion than the cases for larger number of assets. This is consistent with findings from Propositions 2 and 3, where estimation error increases with number of assets. Additionally, we observe that when the DM allocates her portfolio among larger number of assets, she faces an opportunity cost.[12] In this regard, the increase in $\tilde{E}_p$ comes at the expense of increase in $\tilde{v}_p$ and greater estimation error, where the distribution becomes more dispersed as the DM allocates among larger number of assets.

Looking at change with respect to sample size, we set $\kappa = 5$ and $d = 6$ while letting $n = 120, 240, 360, 480$. According to the panels from Fig. 2, we observe that as the sample size increases, the sampling distribution of the MVE becomes more concentrated and less dispersed. Since $n$ resembles the information available to the DM, the estimates converge to the true parameters as the DM possess more information. In this regard, as $n$ increases, the joint distribution eventually converges to a single point $(\boldsymbol{\eta}_p, \boldsymbol{\sigma}_p)$, which corresponds to the MVE frontier under full information.[13] Moreover, by increasing the sample size in our data, the sample covers additional periods that are not included in the smaller one. These periods could include times of high volatility and market downturns, and, hence, greater uncertainty. Nonetheless, as $n$ increases, we still observe decrease in estimation error, where the distribution generally becomes less dispersed.

In Fig. 3, we set $d = 10$ and $n = 240$ and draw the joint PDF for $k = 2, 5, 10, 15$. When $k = 2$, the DM, under full information, allocates an aggressive portfolio, where she is highly exposed to the arbitrage portfolio, $\boldsymbol{\alpha}_1$, and less to the conservative GMV portfolio. In the presence of estimation error, small $\kappa$ also implies high estimation error as the MV portfolio becomes more prone to estimation error in the mean vector. In panel (a), we observe that the distribution is almost flat, implying high uncertainty and dispersion. However, as $\kappa$ increases, i.e., panel (d), the distribution becomes less dispersed and more concentrated. In this regard, the investor is more risk averse, such that she shifts her allocation further in the direction of the GMV portfolio, which is associated with lower estimation error.

The joint sampling distribution described in (3.16) from Proposition 4 is only applicable to the conditional case, i.e. the sample covariance matrix, $\mathbf{S}$, is equal to the population covariance matrix, $\boldsymbol{\Sigma}$. We should note that Bodnar and Schmid (2009b) document a similar result. In Theorem 3.1, Bodnar and Schmid (2009b) derive the unconditional marginal distribution for $\tilde{E}_p$ and $\tilde{v}_p$ when both the mean vector and the covariance matrix are estimated. They show that the distributions of $\tilde{E}_p$ and $\tilde{v}_p$ can be represented by three mutually independent stochastic components. While Bodnar and Schmid (2009b) do not derive the joint sampling distribution of $\tilde{E}_p$ and $\tilde{v}_p$, we expect the same dynamics discussed above to follow suit.

Bodnar and Schmid (2009b) further estimate the MVE efficient frontier by referring to the frontier's slope. In our analysis, this would be equivalent to:

$$\frac{\left(\tilde{E}_p - \tilde{E}_0\right)^2}{\tilde{v}_p - \sigma_0^2} = \frac{1}{n}\tilde{E}_1 = \mathbf{m}\prime\mathbf{B}\mathbf{m} \tag{3.18}$$

Since we consider the conditional case in this paper, the ratio has a non-central Chi-squared distribution (See Lemma 2). On the other hand, Bodnar and Schmid (2009b) show that the ratio for the unconditional case follows a non-central *F*-distribution.

[11] This can also be derived directly from Equations (3.13) and (3.14) and using the properties derived in Lemma 2 and Lemma 3.

[12] This follows from the observation that the $\tilde{E}_p$ and $\tilde{v}_p$ axes of the joint distribution shift to the right as $d$ increases.

[13] Recall that under full information, for each $\kappa < \infty$, there is a distinct MV portfolio with single point on the MVE frontier, where the DM achieves maximum mean return $\boldsymbol{\eta}_p$ for a given level of risk, $\boldsymbol{\sigma}_p^2$.

## 4. Robustness

Bodnar and Schmid (2011) derive the distribution of a linear combination of the estimated portfolio weights that allows the decision maker to characterize the distribution of specific positions in the portfolio, which we refer to as the unconditional case. In this paper, we focus on a conditional case in which the sample covariance matrix is equal to the population covariance matrix. In this section, we explore the differential implications of Bodnar and Schmid's (2011) and our model.

Let $\mathbf{l}$ be a $d \times 1$ deterministic vector. For a given $\mathbf{S} = \mathbf{\Sigma}$, Proposition 1 yields

$$\mathbf{l}\prime \sim N\left(\mathbf{l}\prime\boldsymbol{\xi}, n^{-1}\kappa^{-2}\mathbf{l}'\mathbf{B}\mathbf{l}\right).$$

where

$$\mathbf{x} = \boldsymbol{\xi}(\mathbf{m}, \mathbf{\Sigma}) \tag{4.1}$$

and

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{\mu}, \mathbf{\Sigma}) \tag{4.2}$$

Bodnar and Schmid (2011) show that the unconditional distribution of $\mathbf{l}\prime\boldsymbol{\xi}(\mathbf{m}, \mathbf{S})$ can be decomposed into four independent random variables (including a Gaussian component). The unconditional distribution is expected to have more dispersion than the conditional one proposed in this paper, where the latter neglects estimation error induced by the sample covariance matrix.

For comparison, we sample 10 random industries from the Fama-French 48 value-weighted industry portfolios including the financial industry (see data description in the Empirical Validation 3.2.1 part). Similar to Bodnar and Schmid (2011), we look at both the conditional and unconditional distribution of the portfolio weights allocated to one of the assets - the financial industry in our case. We demonstrate the results in Fig. 4.

Fig. 4 shows that when the sample size is small, i.e. $n = 30$, there is a greater discrepancy between the conditional and the unconditional distributions. However, when we increase the sample size, i.e. $n = 120$, there is more overlap and consistency between the two distributions. This evidence indicates that estimation error induced by the mean vector plays a stronger role than that of the covariance matrix, which is consistent with past studies, (see e.g., Chopra and Ziemba (1993)).

In addition, as the level of risk aversion increases, a greater proportion of the portfolio weights is determined by the global minimum variance portfolio. The impact of estimation error induced by the mean vector is expected to become less pronounced when $\kappa$ increases. For instance, comparing between $\kappa = 2$ and $\kappa = 10$, we observe from Fig. 4 that both distributions become more concentrated when $\kappa$ increases. In addition, for the same sample size, we discern that the overlap between the conditional and the unconditional distributions is unaffected.

Overall, Fig. 4 illustrates that estimation error is dominated by the sample mean vector. While the portfolio manager can not ignore the importance of estimation error induced by the covariance matrix, such estimation error is substantially attenuated as the sample size increases.

## 5. Conclusions

In Markowitz's (1952) paradigm, the objective of the decision maker is to choose a well diversified portfolio on the efficient set. However, in the real world, she faces extreme difficulty generating reliable estimates of the model inputs, which is amplified as more assets added to the portfolio. Hence, the decision maker generally faces a tradeoff between gains from holding a diversified portfolio and losses induced by estimation errors. In this paper, we provide a theoretical framework to characterize the effect of estimation errors on portfolio theory by tying estimation errors to the estimated moments of the asset return distribution. With this framework, we are able to model analytically such tradeoff and identify numerically the critical number of assets for a given level of estimation errors.

Our framework is built based on the case when the sample covariance matrix is equal to the population covariance matrix, which is partly motivated by previous findings that estimation errors in the covariance matrix are much less severe than the vector of expected returns (see e.g., Chopra and Ziemba (1993)). This simplifies the results and allows us to inspect the implications of estimation errors for portfolio performance in a more tractable manner. Nonetheless, we discern that the unconditional case should be considered to achieve more accurate results for future work.

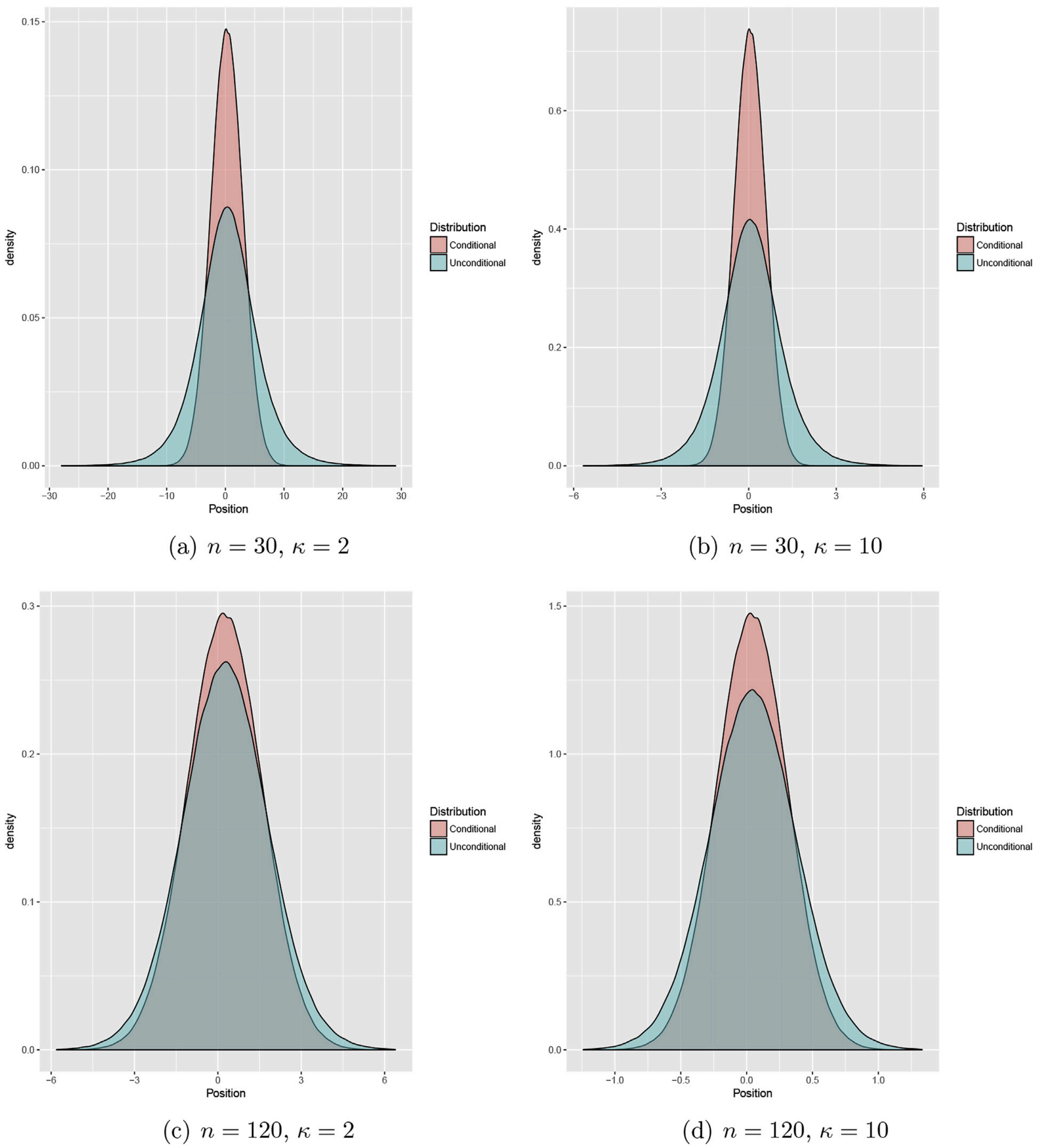
## Proofs of Lemmas and Propositions

### A.1 Proof of Lemma 1

The properties of MVE frontier are well-established in the literature. For instance, see Bodnar and Schmid (2009a) and the references therein.

### A.2 Proof of Proposition 1

Recall that $\mathbf{m}$ is the sample mean vector, where, $\mathbf{m} = (\overline{r}_1, \ldots, \overline{r}_d)'$ and $\overline{r}_i = \sum_{t=1}^{n} r_{i,t}/n, \ \forall i = 1, .., d$ and $t = 1, \ldots, n$ such that $\overline{r}_i \sim N(\mu_i, n^{-1}\sigma_i^2)$. It is known that the sample mean, $\mathbf{m}$, is an unbiased and consistent estimator of $\boldsymbol{\mu}$ and follows a joint normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $n^{-1}\mathbf{\Sigma}$.




(a) $n = 30$, $\kappa = 2$ (b) $n = 30$, $\kappa = 10$

(c) $n = 120$, $\kappa = 2$ (d) $n = 120$, $\kappa = 10$

**Fig. 4. Comparing between the conditional and unconditional distribution of MV portfolio weights**.
This figure compares between the conditional distribution of the portfolio weights from Proposition 1 (pink colored) and the unconditional one proposed by Bodnar and Schmid (2011) (green colored). Each distribution is illustrated using historical monthly returns of the Fama-French 48 value-weighted industry portfolios between Jan 1970 and Dec 2015. 10 industry portfolios are randomly picked, from the which the MV portfolio is constructed. Each distribution focuses on the portfolio allocation to one asset (in our case it is the financial industry). The panels correspond to different levels of risk aversion and sample sizes.

Under multivariate normal distribution, both the sample mean and the sample covariance matrix are independent, such that the conditional distribution of $\boldsymbol{\xi}$ given $\mathbf{S} = \boldsymbol{\Sigma}$ is equal to the distribution of the random vector

$$\mathbf{x} = \boldsymbol{\xi}(\mathbf{m}, \boldsymbol{\Sigma}) = \boldsymbol{\alpha}_0 + \frac{1}{\kappa}\mathbf{Bm}.$$

Given that $\mathbf{m} \sim N(\boldsymbol{\mu}, n^{-1}\boldsymbol{\Sigma})$ and that $\mathbf{x}$ is a linear function of $\mathbf{m}$, then $\mathbf{x}$ follows a joint normal distribution as well. Since the multivariate normal distribution is defined by its first two moments, it is sufficient to derive the expected value and the variance of $\mathbf{x}$. Hence, it follows that $E(\mathbf{x}) = \boldsymbol{\alpha}_0 + \kappa^{-1}\mathbf{B}E(\mathbf{m}) = \boldsymbol{\alpha}_0 + \kappa^{-1}\mathbf{B}\boldsymbol{\mu} = \boldsymbol{\xi}$ and that $Var(\mathbf{x}) = \kappa^{-2}\mathbf{B}Var(\mathbf{m})\mathbf{B} = n^{-1}\kappa^{-2}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}$. Since $\mathbf{B}\boldsymbol{\Sigma}$ is an idempotent matrix, (Property (1) from Lemma 1), then the variance of $\mathbf{x}$ is equal to $n^{-1}\kappa^{-2}\mathbf{B}$. Therefore, it holds true that $\mathbf{x} \sim N(\boldsymbol{\xi}, n^{-1}\kappa^{-2}\mathbf{B})$. Finally, since $\mathbf{x}$ is an unbiased estimator, its MSE is equal to its variance, which obviously converges to zero as the sample size goes to infinity.

### *A.3 Proof of Lemma 2*

Let $A = n\mathbf{B}$ and recall that $\mathbf{m} \sim N(\boldsymbol{\mu}, n^{-1}\boldsymbol{\Sigma})$. According to Corollary 3.3.3c of Wang and Chow (1994), it holds true that $n\mathbf{m}'\mathbf{Bm} \sim \chi^2_{d-1,n\boldsymbol{\mu}'\mathbf{B}\boldsymbol{\mu}}$ if (i) $\mathbf{B}\boldsymbol{\Sigma}$ is an idempotent matrix and (ii) $rank(\mathbf{B}) = d-1$. According to Property (1) from Lemma 1, the first condition is satisfied. For the second condition, we know that $\boldsymbol{\Sigma}$ is a non-singular matrix such that $rank(\mathbf{B}) = rank(\mathbf{B}\boldsymbol{\Sigma})$. Since $\mathbf{B}\boldsymbol{\Sigma}$ is idempotent, then $rank(\mathbf{B}) = rank(\boldsymbol{\Sigma}\mathbf{B}) = trace(\boldsymbol{\Sigma}\mathbf{B}) = trace(\mathbf{I} - \mathbf{e}\alpha_0') = d - \alpha_0'\mathbf{e} = d - 1$.

### *Proof of Lemma 3*

Since $\mathbf{m}$ is normally distributed with mean vector $\boldsymbol{\mu}$ and covariance matrix $n^{-1}\boldsymbol{\Sigma}$, then, according to Corollary 3.4.1 of Wang and Chow (1994), $\tilde{E}_1 = n\mathbf{m}'\mathbf{Bm}$ and $\tilde{E}_0 = \mathbf{m}'\boldsymbol{\alpha}_0$ are mutually independent, if the condition $\alpha_0'[n^{-1}\boldsymbol{\Sigma}]n\mathbf{B} = 0'$ holds true. Given Property (1) from Lemma 1, we know that $\alpha_0'\boldsymbol{\Sigma}\mathbf{B} = 0'$, which completes the proof and yields that $\tilde{E}_1 \perp \tilde{E}_0$.

### *A.4 Proof of Proposition 2*

Given Lemma 2 and Lemma 3, the proof of Proposition 2 is straightforward:

(a) According to (3.3) and Lemma 2, $\tilde{E}_p$ is a combination of non-central Chi-squared and normal distributions:

$$\tilde{E}_p = \mathbf{m}'\boldsymbol{\alpha}_0 + (n\kappa)^{-1}n\mathbf{m}'\mathbf{Bm} = \tilde{E}_0 + (n\kappa)^{-1}\tilde{E}_1 \tag{A.1}$$

where

$$\tilde{E}_1 = n\mathbf{m}'\mathbf{Bm} \sim \chi^2_{d-1,n\boldsymbol{\mu}'\mathbf{B}\boldsymbol{\mu}}$$
$$\tilde{E}_0 = \mathbf{m}'\boldsymbol{\alpha}_0 \sim N\left(\boldsymbol{\eta}_0, n^{-1}\boldsymbol{\sigma}_0^2\right)$$

Hence, utilizing the expectation of $\tilde{E}_p$, it follow that

$$E\left[\tilde{E}_p\right] = \boldsymbol{\eta}_p + \frac{d-1}{n\kappa} \tag{A.2}$$

(b) Since $\tilde{E}_1 \perp \tilde{E}_0$, (Lemma 3), it follows that $Var[\tilde{E}_p] = Var[\tilde{E}_0] + (n\kappa)^{-2}Var[\tilde{E}_1]$. Simplifying this, we have

$$Var\left[\tilde{E}_p\right] = \frac{1}{n}\left[\boldsymbol{\sigma}_p^2 + 3\kappa^{-2}\boldsymbol{\sigma}_1^2\right] + \frac{2(d-1)}{n^2\kappa^2}, \tag{A.3}$$

such that

$$MSE\left[\tilde{E}_p\right] = \frac{1}{n}\left[\boldsymbol{\sigma}_p^2 + 3\kappa^{-2}\boldsymbol{\sigma}_1^2\right] + \frac{d^2-1}{n^2\kappa^2}. \tag{A.4}$$

(c) As $\tilde{E}_1 \perp \tilde{E}_0$, (Lemma 3), the moment generating function of the estimated portfolio's mean return is

$$M_{\tilde{E}_p}(t) = M_{\tilde{E}_0}(t) \cdot M_{\tilde{E}_1}\left(\frac{t}{n\kappa}\right), \tag{A.5}$$

where

$$M_{\tilde{E}_0}(t) = exp\left(\boldsymbol{\eta}_0 t + \frac{t^2}{2n}\boldsymbol{\sigma}_0^2\right)$$
$$M_{\tilde{E}_1}(s) = (1-2s)^{-\frac{d-1}{2}} exp\left(\frac{ns}{1-2s}\boldsymbol{\mu}'\mathbf{B}\boldsymbol{\mu}\right)$$

Hence, we establish that

$$M_{\widetilde{E_p}}(t) = \left(1 - \frac{t}{n\kappa}\right)^{-\frac{d-1}{2}} exp\left(t\left(\boldsymbol{\eta}_0 + \frac{n}{n\kappa - 2t}\boldsymbol{\mu}\prime\mathbf{B}\boldsymbol{\mu}\right) + \frac{t^2}{2n}\boldsymbol{\sigma}_0^2\right). \tag{A.6}$$

*A.5 Proof of Proposition 3*

Given Equation (3.4), Lemma 2, and the properties of the efficient set from Lemma 1, we note that $\tilde{v}_p$ is a linear function of $\tilde{E}_1$ and that it follows a shifted, scaled non-central Chi-squared distribution, where

$$\tilde{v}_p = \boldsymbol{\sigma}_0^2 + n^{-1}\kappa^{-2}\tilde{E}_1 \tag{A.7}$$

with

$$\tilde{E}_1 = n\mathbf{m}\prime\mathbf{B}\mathbf{m} \sim \chi^2_{d-1, n\boldsymbol{\mu}\prime\mathbf{B}\boldsymbol{\mu}}.$$

Similar to the proof of Proposition 2, the proof of the Proposition 3 is straightforward.

*A.6 Proof of Proposition 4*

(a) Let $\tilde{\Pi}_p$ be a $2 \times 1$ vector that represents the estimated mean and risk of the MV portfolio return, such that $\tilde{\Pi}_p = (\tilde{E}_p, \tilde{v}_p)$. In addition, let $\mathbf{t} = (t_1, t_2)\prime$. The moment generating function of the $\tilde{\Pi}_p$, hence, can be derived as follows

$$M_{\tilde{E}_p\tilde{v}_p}(t_1, t_2) = E\left[exp\left(\tilde{\Pi}_p{}'\mathbf{t}\right)\right] = exp\left(t_2\boldsymbol{\sigma}_0^2\right)\cdot E\left[exp\left(t_1\tilde{E}_0 + \frac{\kappa t_1 + t_2}{n\kappa^2}\cdot\tilde{E}_1\right)\right] \tag{A.8}$$

Since $\tilde{E}_0$ and $\tilde{E}_1$ are mutually independent (Lemma 3), it holds true that

$$M_{\tilde{E}_p\tilde{v}_p}(t_1, t_2) = exp\left(t_2\boldsymbol{\sigma}_0^2\right)\cdot M_{\tilde{E}_0}(t_1)\cdot M_{\tilde{E}_1}\left(\frac{\kappa t_1 + t_2}{n\kappa^2}\right) \tag{A.9}$$

Hence, given the moment generating function for each of $\tilde{E}_0$ and $\tilde{E}_1$ (see the proof of Proposition 2), we establish that

$$M_{\tilde{E}_p\tilde{v}_p}(t_1, t_2) = exp\left(t_2\boldsymbol{\sigma}_0^2 + \boldsymbol{\eta}_0 t_1 + \frac{\boldsymbol{\sigma}_0^2 t_1^2}{2n} + \boldsymbol{\mu}\prime\mathbf{B}\boldsymbol{\mu}\frac{t^*}{1 - 2t^*}\right)\left(1 - 2\frac{1}{n}t^*\right)^{-\frac{d-1}{2}} \tag{A.10}$$

with

$$t^* = \frac{\kappa t_1 + t_2}{\kappa^2}$$

(b) The joint density of $\tilde{E}_p$ and $\tilde{v}_p$ can be written as,

$$f_{\tilde{E}_p\tilde{v}_p}(E_p, v_p) = f\left(\tilde{E}_p = E_p | \tilde{v}_p = v_p\right) f_{\tilde{v}_p}(v_p) \tag{A.11}$$

Combining Equations (3.13) and (3.14), we deduce that

$$\tilde{E}_p = \tilde{E}_0 + \kappa\left(\tilde{v}_p - \boldsymbol{\sigma}_0^2\right) \tag{A.12}$$

where $\tilde{E}_0 \sim \mathrm{N}(\boldsymbol{\eta}_0, n^{-1}\boldsymbol{\sigma}_0^2)$, such that the conditional distribution of $\tilde{E}_p$ given $\tilde{v}_p$ is normal:

$$\tilde{E}_p | \tilde{v}_p = v_p \sim N\left(\boldsymbol{\eta}_0 + \kappa\left(v_p - \boldsymbol{\sigma}_0^2\right), \ n^{-1}\boldsymbol{\sigma}_0^2\right). \tag{A.13}$$

From Proposition 3, it is evident that $\tilde{v}_p = \boldsymbol{\sigma}_0^2 + n^{-1}\kappa^{-2}\chi^2_{(d-1), n\boldsymbol{\mu}'\mathbf{B}\boldsymbol{\mu}}$, hence,

$$f_{\tilde{v}_p}(v_p) = n\kappa^2 f_{\chi^2}\left(v_p; \ d-1, \ n\boldsymbol{\mu}\prime\mathbf{B}\boldsymbol{\mu}\right). \tag{A.14}$$

Putting Equations (A.11), (A.13), and (A.14) together yields the joint PDF from Equation (3.16).

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