Knowledge-Based Systems 144 (2018) 21–31

Contents lists available at ScienceDirect

Knowledge-Based Systems

journal homepage: www.elsevier.com/locate/knosys

Portfolio selection and risk investment under the hesitant fuzzy environment

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ARTICLE INFO

Article history:
Received 5 October 2016
Revised 16 December 2017
Accepted 19 December 2017
Available online 19 December 2017

Keywords:
Portfolio selection
Risk investment
Qualitative portfolio model
Optimal investment ratio
Hesitant fuzzy element

ABSTRACT

The optimal investment ratios for a set of stocks and other financial products can be obtained by the conventional portfolio theory based on quantitative data such as returns and risks. However, quantitative data are sometimes unavailable, thus qualitative information provided by experts or decision makers should be used. Based on the foregoing, we propose new portfolio selection approaches based on such qualitative information which is represented herein as hesitant fuzzy elements. For general investors and risk investors, we develop two qualitative portfolio models based on the max-score rule and the score-deviation trade-off rule, respectively. Furthermore, the deviation and score trisection approaches are developed to distinguish the three types of risk investors, which also help to construct the corresponding qualitative portfolio models. In addition, we investigate the investment opportunities and efficient frontiers of these proposed qualitative portfolio models. Also, the specific portfolio selection processes are provided. Finally, an example of selecting the optimal portfolio of risk investment is provided. On the basis of the above study and example, we can conclude that the proposed qualitative portfolio models used for the three types of risk investors are effective. The given portfolio selection processes can be reasonably used in practical qualitative risk investment.

1. Introduction

Markowitz's [20] modern portfolio theory, which led the author to won the Nobel Memorial Prize for the economic sciences in 1990, examines how the elements such like asset returns, risks, correlation, and diversification affect investment portfolio returns. Markowitz [20] also proposed the critical line algorithm to identify the optimal mean-variance portfolios and therefore the Markowits frontier was defined. [20,21]. Based on Markowitz's theory, a number of related researches have been presented. For examples, Merton [22] investigated the lifetime portfolio selection under uncertainty and provided a continuous-time case, Elton et al. [8] gave some simple criteria for optimal portfolio selection, Eun and Resnick [9] presented a complex empirical study of portfolio selection by using exchange rate and forward contracts, Park et al. [23] developed a mini-max portfolio selection rule with a linear programming solution, Liu [17] analyzed the portfolio selection in stochastic environments, Liu et al. [19] studied the multi-period portfolio selection by using interval analysis. Recently, Detemple [6] reviewed portfolio models and provided perspectives on some open issues, Levy and Kaplanski [14] proposed an improved portfolio model by introducing a mixed-normal distribution, and Ghaoui et al. [10] developed a robust portfolio model based on the worst-case value-at-risk. Obviously, we can obtain the optimal portfolios for a set of stocks and financial products by using these portfolio models.

A shortcoming of the above portfolio models is that a large amount of data is needed to calculate the statistical indexes (e.g., mean, variance, correlation, and diversification). Since time series data can be unavailable for newly listed stocks or products, some non-statistical approaches have been developed. For example, with respective to the conflicting objectives such as rate of returns, liquidity and risks, Abdelaziz et al. [1] designed a multi-objective stochastic programming for portfolio selection, which is a linear programming and similar to the above portfolio models. From the nonlinear fitting perspective, Fernandez and Gomez [9] proposed a neural-network approach and Sinha et al. [27] gave a genetic algorithm to construct the optimal portfolio, which can be used to calculate the nonlinear and mixed stock portfolio. To construct the optimal portfolio based on the efficiency evaluation, Lim et al. [16] gave the DEA cross-efficiency evaluation model which is an interesting research and can be extended into the fuzzy environ-

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https://doi.org/10.1016/j.knosys.2017.12.020

ment. Similarly, Chen et al. [5] empirically studied the portfolio efficiency in different risk measures via three kinds of DEA. Note that these methods should be only used in the real number environment. Therefore, some approaches based on fuzzy theory have been proposed to address the above limitations of portfolio selection approaches. From the programming modeling perspective, Parra et al. [24] constructed a fuzzy goal programming approach for portfolio selection, which is an early research and provides some basic principles to this model. Then, Ammar and Khalifa [2] introduced the formulation of fuzzy portfolio optimization problems by using the quadratic programming approach, and Vercher et al. [32] analyzed the fuzzy portfolio optimization methods under downside risk measures. Furthermore, by introducing other relative factors, some synthetical optimal portfolio models are presented. For the security returns which contain both randomness and fuzziness, Huang [13] developed a random and fuzzy optimal portfolio model and Yue and Wang [38] constructed a fuzzy higher order moment portfolio selection model. Furthermore, with respect to uncertain or vague returns which also present the investor risk attitude, Tsaur [30] gave a new fuzzy portfolio model. Additionally, some extended fuzzy portfolio models were developed such as the general nonlinear binary multi-objective mathematical model with fuzzy constraints [25], the multi-period fuzzy portfolio optimization model with minimum transaction lost [18], and the portfolio selection problem with fuzzy value-at-risk constraints under non-extensive statistical mechanics [40]. Although few quantitative data are needed, the above approaches also need such data to model and then obtain the optimal investment ratio and portfolio. However, in many real-world risk investments, quantitative data are sometimes unavailable. Thus, in this study, in order to select the optimal portfolio, the qualitative data provided by experts or decision makers are introduced and used for the portfolio selection approaches.

Fuzzy theory [39] is an effective approach to depict qualitative information, including qualitative financial data. This theory is utilized to explain the vagueness and uncertainty of real information by presenting objective mathematical symbols. Based on fuzzy theory, several extended fuzzy forms and fuzzy sets have been developed to depict fuzzy qualitative information in real life. For example, the Atanassov's intuitionistic fuzzy set [3] was developed to simultaneously describe the "good" and "bad" fuzzy information. The interval-valued fuzzy set [4,31] was given to express the fuzzy information using the interval-valued number which is more flexible than the real number. The fuzzy multi-sets [35] and the type-2 fuzzy set [7] were proposed to further develop the fuzzy set by extending its presentation forms. Moreover, a prominent development in this field is the hesitant fuzzy set [11,28,36]. According to Rodríguez et al. [26], the hesitant fuzzy set (HFS) is a novel extension of fuzzy sets that aims to model the uncertainty which originates from the hesitation and might arise in the assignment of fuzzy information. Additionally, a prominent advantage of the HFS is that it can describe the hesitant and uncertain information by a set of possible values. Thus, it is more convenient to use the HFS to represent the subjective evaluation information than other fuzzy sets [37]. Therefore, in this study, we apply the HFS to depict the qualitative financial information provided by investors and decision makers (DMs).

Because hesitant fuzzy information is different from quantitative data, the aforementioned portfolio selection approaches cannot be used under the hesitant fuzzy environment. Therefore, in this study, we propose a qualitative portfolio model under the hesitant fuzzy environment and focus on its application to calculate the optimal investment ratios. Additionally, we develop the extended qualitative portfolio models for risk investors and provide specific portfolio selection processes. To demonstrate the effectiveness of the proposed portfolio models, we further analyze their investment opportunities and efficient frontiers. Lastly, a qualitative risk investment example is presented to show the proposed portfolio selection approaches.

The remainder of this paper is organized as follows. Some preliminary notions such as the hesitant fuzzy element (HFE) and qualitative returns and risks are defined in Section 2. In Section 3, a qualitative portfolio model for the general investor based on the max-score rule is proposed. In Section 4, a qualitative portfolio model for risk investors based on the score-deviation trade-off rule is constructed. In Section 5, the portfolio selection processes under the hesitant fuzzy environment are illustrated. An example is then provided to demonstrate the application of the proposed approaches in Section 6. The paper ends with conclusions in Section 7.

2. Preliminary notions

In this section, we first introduce the HFS and its operations, which form the basis of calculating the investment ratios and selecting the optimal portfolio under the hesitant fuzzy environment. Then, we discuss the presentations of returns and risks under this fuzzy environment. Based on which, the qualitative portfolio models and the corresponding portfolio selection approaches can be presented.

2.1. Hesitant fuzzy set and its operations

Torra [28] pointed out that the difficulty of establishing the membership of an element is not caused by the existing margin of error or the distribution of possible values but having a set of possible values. Thus, Torra and Narukawa [29] proposed the HFS to depict membership with a set of possible values.

Definition 1 [29]**.** Let X be a fixed set, a HFS on X is in terms of a function that when applied to X returns a subset of [0, 1].

To make it easily understandable, Xia and Xu [33] expressed the HFS by a mathematical symbol:

$$E = \{< x,\ h_E(x) > | x \in X\} \tag{1}$$

where $h_E(x)$ is a set of values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to the set E. For convenience, Xia and Xu [33] called $h = h_E(x)$ a HFE and H the set of all HFEs. In addition, the following score function and comparison laws were defined:

Definition 2 ([33])**.** For a HFE $h = \cup_{\gamma \in h}\{\gamma\}$, $s(h) = \frac{1}{\#h}\sum_{\gamma \in h}\gamma$ is called the score function of h, where $\#h$ is the number of the elements in h. For two HFEs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

However, since the HFEs cannot be further distinguished when they have the same score values, $s(h_1) = s(h_2)$, similar to the accuracy function of intuitionistic fuzzy set [12,34] and the variance function of the HFS [15], Zhou and Xu [41] developed the deviation function and improved the comparison laws as follows:

Definition 3 ([41])**.** Let $h = \cup_{\gamma \in h}\{\gamma\}$ be a HFE, γ be the possible membership degrees of h in [0,1], $s(h) = \frac{1}{\#h}\sum_{\gamma \in h}\gamma$ be the score function of h, and $\#h$ be the number of the elements in h, then a deviation function $d(h)$ of h is defined as follows

$$d(h) = \frac{\sum_{\gamma \in h} |\gamma - s(h)|}{\#h} = \frac{\sum_{\gamma \in h} \sqrt{(\gamma - s(h))^2}}{\#h} \tag{2}$$

then, $d(h) \in [0, 1]$. According to the order relation between two intuitionistic fuzzy numbers [12,33], the comparison laws between two HFEs h_1 and h_2 are provided as follows:

If $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then (1) if $d(h_1) < d(h_2)$, then $h_1 > h_2$; (2) if $d(h_1) = d(h_2)$, then $h_1 \sim h_2$; (3) if $d(h_1) > d(h_2)$, then $h_1 < h_2$.

To aggregate hesitant fuzzy information, given two HFEs h_1 and h_2, Xia and Xu [33] defined the following six basic operations:

(1) $h_1^\lambda = \cup_{\gamma_1 \in h_1}\{\gamma_1^\lambda\}$, $\lambda > 0$; (2) $\lambda h_1 = \cup_{\gamma_1 \in h_1}\{1-(1-\gamma_1)^\lambda\}$, $\lambda > 0$;
(3) $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1\gamma_2\}$; (4) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1+\gamma_2-\gamma_1\gamma_2\}$;
(5) $\lambda(h_1 \oplus h_2) = \lambda h_1 \oplus \lambda h_2$; (6) $(h_1 \otimes h_2)^\lambda = h_1^\lambda \otimes h_2^\lambda$.

Thus, based on these definitions, comparison laws, and operations, it is sufficient to calculate the returns and risks of a portfolio and solve the optimal investment ratios under the hesitant fuzzy environment.

2.2. Returns and risks under the hesitant fuzzy environment

According to portfolio selection theory, statistical and econometric methods are used to calculate optimal investment ratios based on returns and risks, which are described as the mean and variance of quantitative data, respectively. Then, the optimal portfolio can be obtained based on the max-return rule or the return-risk trade-off rule. However, quantitative data are often infeasible due to their finiteness and unavailability. For example, an investor wants to place a fund to a stock market and considers some newly listed companies to be promising. As these companies are newly listed, reliable financial data about them are limited, which means returns and risks cannot be obtained, so the above quantitative approaches are meaningless. Thus, qualitative decision-making approaches based on subjective information given by experts and decision makers may be more suitable and applicable. Therefore, in this study, we introduce a general fuzzy form namely the HFS and apply the corresponding hesitant fuzzy aggregation technique to address this issue.

Definitions 2 and 3 suggest that the score and deviation of the HFE are similar to the mean and variance of quantitative data under portfolio theory. Therefore, we can take the score and deviation values to measure the return and risk values of the portfolio under the hesitant fuzzy environment, which can be presented as follows:

(1) If the evolution value of a portfolio is represented by HFE h, then its return value can be measured by its score value $s(h) = \frac{1}{\#h}\sum_{\gamma \in h}\gamma$.
(2) If the evolution value of a portfolio is represented by HFE h, then its risk value can be measured by its deviation value $d(h) = \frac{\sum_{\gamma \in h}|\gamma - s(h)|}{\#h} = \frac{\sum_{\gamma \in h}\sqrt{(\gamma - s(h))^2}}{\#h}$.

In the following two sections, similar to the conventional portfolio selection approaches, we construct two qualitative portfolio models from the perspectives of max-score and score-deviation trade-off. These two models can be respectively used by general and risk investors to obtain the optimal investment ratio and the optimal portfolio. Next, the investment opportunities and efficient frontiers of the proposed qualitative portfolio models are investigated in the corresponding section.

3. Portfolio selection for the general investor under the hesitant fuzzy environment

In this section, based on the descriptions of returns and risks under the hesitant fuzzy environment in Section 2, we propose a qualitative portfolio model and investigate the corresponding portfolio selection approach based on the max-score rule. Furthermore, we analyze the investment opportunities and efficient frontier to show the effectiveness of the proposed approach. It is also pointed out that this approach is suitable for general investors to construct the optimal portfolio.

3.1. Qualitative portfolio model based on the max-score rule

Because the max-score rule, which is used when the investor simply wants to obtain the maximum return, is direct and simple, it is suitable for depicting the investment objects of general investors. To develop the corresponding portfolio model, we set the hesitant fuzzy risk investment scenario as follows:

An investor or a fund manager wants to place a fund on n newly listed stocks (or other financial products) $\{x_1, x_2, \cdots, x_n\}$. Because of the limited data, he/she wants to choose the portfolio on the basis of the set of m qualitative criteria $\{y_1, y_2, \cdots, y_m\}$ described by the HFEs h_{ij} $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m)$, which can be represented as a hesitant fuzzy matrix $H = [h_{ij}]_{n\times m}$. This matrix can be transformed into a collective column vector $\bar{H} = [\bar{h}_i]_{n\times 1}$ by aggregating all the values on one line.

To calculate the optimal investment ratios and the optimal portfolio, the following qualitative portfolio model is proposed and used by the general investor:

Model 1:

$$F(W) = \max s\left(\oplus_{i=1}^n w_i\bar{h}_i\right)$$
$$s.t.\begin{cases}\oplus_{i=1}^n w_i\bar{h}_i = \cup_{\bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n}\left\{1 - \prod_{i=1}^n (1-\bar{\gamma}_i)^{w_i}\right\}\\ s(h) = \frac{1}{\#h}\sum_{\gamma \in h}\gamma \\ \sum_{i=1}^n w_i \le 1, w_i \ge 0\end{cases} \tag{3}$$

where $s(h)$ is the score function of the HFE h, $\bar{h}_i$ is the aggregated HFE based on $\bar{h}_i = \oplus_{j=1}^m h_{ij}$, h_{ij} is the hesitant fuzzy information of the alternative x_i with respect to the qualitative criterion y_j, $W = \{w_1, w_2, \cdots, w_n\}$ and w_i are the optimal investment ratios of this fund on these stocks, $i = 1, 2, \cdots, n$, and $j = 1, 2, \cdots, m$.

Theorem 1. *The inequality constraint in Model 1, $\sum_{i=1}^n w_i \le 1$, is equal to the equality constraint $\sum_{i=1}^n w_i = 1$. In other words, Model 1 can be transformed into Model 2 as follows:*

Model 2:

$$F(W) = \max s\left(\oplus_{i=1}^n w_i\bar{h}_i\right)$$
$$s.t.\begin{cases}\oplus_{i=1}^n w_i\bar{h}_i = \cup_{\bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n}\left\{1 - \prod_{i=1}^n (1-\bar{\gamma}_i)^{w_i}\right\}\\ s(h) = \frac{1}{\#h}\sum_{\gamma \in h}\gamma \\ \sum_{i=1}^n w_i = 1, w_i \ge 0\end{cases} \tag{4}$$

Proof. If $W^* = (w_1^*, w_2^*, \cdots, w_N^*)$ is the optimal solution of Model 1, where $\sum_{i=1}^n w_i^* < 1$ and $w_i^* \ge 0$, then

$$\begin{aligned}F(W^*) &= \max s\left(\oplus_{i=1}^n w_i^*\bar{h}_i\right)\\ &= \max s\left(\cup_{\bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n}\left\{1-(1-\bar{\gamma}_1)^{w_1^*}\prod_{i=2}^n(1-\bar{\gamma}_i)^{w_i^*}\right\}\right).\end{aligned} \tag{5}$$

If $\bar{W} = ((1-\sum_{i=2}^n w_i^*), w_2^*, \cdots, w_N^*)$, then

$$\begin{aligned}F(\bar{W}) &= \max s\left((1-\sum_{i=2}^n w_i^*)\bar{h}_1 \oplus (\oplus_{i=2}^n w_i^*\bar{h}_i)\right)\\ &= \max s\left(\cup_{\bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n}\left\{1-(1-\bar{\gamma}_1)^{(1-\sum_{i=2}^n w_i^*)}\prod_{i=2}^n(1-\bar{\gamma}_i)^{w_i^*}\right\}\right).\end{aligned} \tag{6}$$

For $\sum_{i=1}^n w_i^* < 1$, then $1-\sum_{i=2}^n w_i^* > w_1^*$ and $(1-\bar{\gamma}_1)^{(1-\sum_{i=2}^n w_i^*)} < (1-\bar{\gamma}_1)^{w_1^*}$.

Then, we can get $F(\bar{W}) > F(W^*)$, and the optimal solution of Model 1 should be $\bar{W}$, which also indicates that the optimal solution should satisfy $\sum_{i=1}^n w_i = 1$ where $w_i \ge 0$. Thus, we complete the proof of this theorem. □

If the optimal investment ratios w_i $(i = 1, 2, \cdots, n)$ are obtained, then we can derive the optimal portfolio which can be a set of stocks and other financial products. It is can be found that Model 1 and Model 2 are two nonlinear optimization models. The maximum score is similar to the maximum returns of the conventional portfolio theory. Therefore, Model 1 and Model 2 can be called the max-return portfolio approaches. To demonstrate the effectiveness of the above models, Example 1 is given as follows.

Example 1. An investor wants to place a sum of money into two stocks $\{x_1, x_2\}$ based on one criterion y_1 which is a quantified index without any quantitative data. The investor gives his/her subjective evaluations represented by the HFEs h_{ij} $(i = 1, 2; j = 1)$. Here, h_{11} denotes the evaluation value for x_1 with respect to y_1 and h_{21} represents the evaluation value for x_2 with respect to y_1, which are provided by this investor. If $h_{11} = \{0.3, 0.9\}$ and $h_{21} = \{0.5, 0.6\}$, then $H = \bar{H} = [\bar{h}_i]_{2\times 1} = [\{0.3, 0.9\}, \{0.5, 0.6\}]^T$.

By introducing the above comparison laws of HFEs, we have $s(h_{11}) = 0.6$ and $s(h_{21}) = 0.55$. Then, $h_{11} \succ h_{21}$ and the stock x_1 should be selected. It is found that this general calculation cannot provide the optimal portfolio ratios and guide the investor to place his/her money.

To calculate the investment ratios and then obtain the optimal portfolio, Model 2 can be used as follows:

Firstly, we have

$$\begin{aligned}\oplus_{i=1}^{2} w_i \bar{h}_i &= \cup_{\bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2} \left\{1 - \prod_{i=1}^{2} (1-\bar{\gamma}_i)^{w_i}\right\} \\ &= \Big\{(1-0.7^{w_1}\cdot 0.5^{w_2}), (1-0.7^{w_1}\cdot 0.4^{w_2}), \\ &\quad (1-0.1^{w_1}\cdot 0.5^{w_2}), (1-0.1^{w_1}\cdot 0.4^{w_2})\Big\}. \end{aligned} \tag{7}$$

and then, $s(\oplus_{i=1}^{2} w_i \bar{h}_i) = 1 - 0.25(0.7^{w_1}\cdot 0.5^{w_2} + 0.7^{w_1}\cdot 0.4^{w_2} + 0.1^{w_1}\cdot 0.5^{w_2} + 0.1^{w_1}\cdot 0.4^{w_2})$. Model 2 can be transformed into the following form:

$$F(W) = \max s\big(\oplus_{i=1}^{2} w_i \bar{h}_i\big)$$
$$s.t. \begin{cases} s\big(\oplus_{i=1}^{2} w_i \bar{h}_i\big) = 1 - 0.25\big(0.7^{w_1}\cdot 0.5^{w_2} + 0.7^{w_1}\cdot 0.4^{w_2} \\ +0.1^{w_1}\cdot 0.5^{w_2} + 0.1^{w_1}\cdot 0.4^{w_2}\big) \\ w_1 + w_2 = 1 \\ w_1, w_2 \geq 0 \end{cases}. \tag{8}$$

Thus, by solving the above nonlinear optimization calculation, we can obtain the optimal investment ratios of this fund which are $w_1 = 0.6272$ and $w_2 = 0.3728$. Therefore, if the investor wants to place \$10,000 into the two stocks $\{x_1, x_2\}$, he/she should place \$6272 into x_1 and \$3728 into x_2. Here, when the amount of money is variable, the optimal investment ratios are invariable. Hence, we obtain the optimal portfolio of the two stocks under the hesitant fuzzy environment.

3.2. Investment opportunity and efficient frontier analysis

Suppose that we characterize every investment opportunity by the portfolio score under the hesitant fuzzy environment. Once the score for individual investment is identified, it is natural for us to consider what will happen when we combine investments to form a portfolio and which one can obtain the highest score. Therefore, we should analyze all the available investment opportunities based on the returns function $F(W)$ and the corresponding efficient frontier based on Model 2.

As mentioned before, Example 1 is used to demonstrate the investment opportunities and the efficient frontier. Fig. 1 illustrates all the available investment opportunities of Example 1 based on Model 2. It is found that the blue curve represents all investment opportunities of Example 1, where $w_1 \in [0, 1]$, $w_2 \in [0, 1]$, $w_1 + w_2 = 1$, and $F(W) = \max s(\oplus_{i=1}^{2} w_i \bar{h}_i)$. The point at the top of this curve represents the optimal portfolio and the efficient frontier of the investment. Fig. 2 is a plane graph based on w_1 and $F(W)$, and the efficient frontier is clearly indicated, which is a point. Furthermore, we found that the distribution shape of the investment opportunities and the efficient frontier in Example 1 based

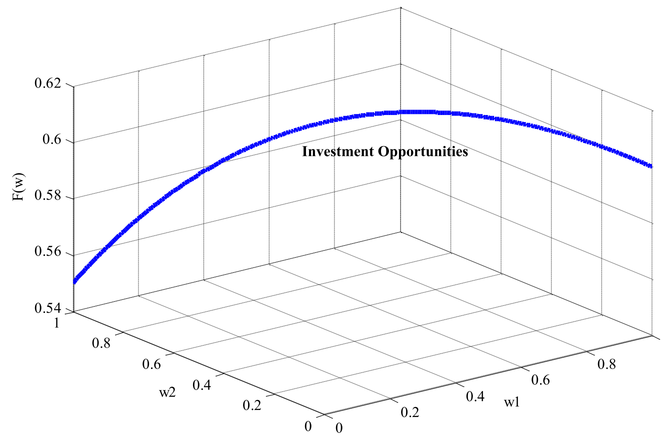

Fig. 1. Investment opportunities of Example 1.

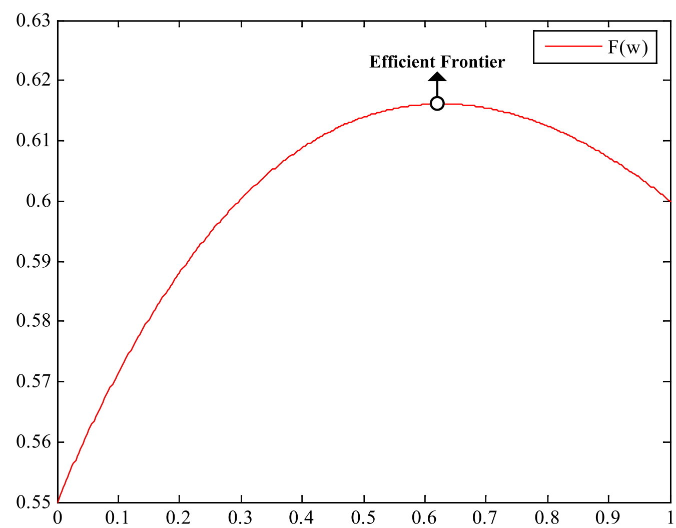

Fig. 2. Efficient frontier of Example 1.

on the max-score rule are in a curve and a point, respectively. The efficient frontier of Example 1 can be analyzed as follows.

Base on Eq. (8), we have

$$\begin{aligned} s\big(\oplus_{i=1}^{2} w_i \bar{h}_i\big) &= 1 - 0.25\big(0.7^{w_1}\cdot 0.5^{w_2} + 0.7^{w_1}\cdot 0.4^{w_2} \\ &\quad + 0.1^{w_1}\cdot 0.5^{w_2} + 0.1^{w_1}\cdot 0.4^{w_2}\big) \\ &= 1 - \frac{1}{4}\cdot\left(\frac{1}{2}\cdot\left(\frac{7}{5}\right)^{w_1} + \frac{2}{5}\cdot\left(\frac{7}{4}\right)^{w_1} + \frac{1}{2}\cdot\left(\frac{1}{5}\right)^{w_1} \right. \\ &\quad \left. + \frac{2}{5}\cdot\left(\frac{1}{4}\right)^{w_1}\right) \end{aligned}$$

Thus, the first-order partial derivative of $s(\oplus_{i=1}^{2} w_i \bar{h}_i)$ for w_1 can be represented as

$$\begin{aligned} \frac{ds\big(\oplus_{i=1}^{2} w_i \bar{h}_i\big)}{dw_1} &= -\frac{1}{4}\cdot\left(\frac{1}{2}\cdot\ln\left(\frac{7}{5}\right)\cdot\left(\frac{7}{5}\right)^{w_1} + \frac{2}{5}\cdot\ln\left(\frac{7}{4}\right)\cdot\left(\frac{7}{4}\right)^{w_1} \right. \\ &\quad \left. + \frac{1}{2}\cdot\ln\left(\frac{1}{5}\right)\cdot\left(\frac{1}{5}\right)^{w_1} + \frac{2}{5}\cdot\ln\left(\frac{1}{4}\right)\cdot\left(\frac{1}{4}\right)^{w_1}\right) \end{aligned}$$

Then, we have $\frac{ds(\oplus_{i=1}^{2} w_i \bar{h}_i)}{dw_1} = -0.0819 < 0$ when $w_1 = 1$, and $\frac{ds(\oplus_{i=1}^{2} w_i \bar{h}_i)}{dw_1} = 0.2418 > 0$ when $w_1 = 0$.

Furthermore, the second-order partial derivative of $s(\oplus_{i=1}^{2} w_i \bar{h}_i)$ for w_1 can be represented as

$$\frac{d^2 s\big(\oplus_{i=1}^{2} w_i \bar{h}_i\big)}{dw_1{}^2} = -\frac{1}{4}\cdot\begin{pmatrix} \frac{1}{2}\cdot\left(\ln\left(\frac{7}{5}\right)\right)^2\cdot\left(\frac{7}{5}\right)^{w_1} + \frac{2}{5}\cdot\left(\ln\left(\frac{7}{4}\right)\right)^2\cdot\left(\frac{7}{4}\right)^{w_1} \\ +\frac{1}{2}\cdot\left(\ln\left(\frac{1}{5}\right)\right)^2\cdot\left(\frac{1}{5}\right)^{w_1} + \frac{2}{5}\cdot\left(\ln\left(\frac{1}{4}\right)\right)^2\cdot\left(\frac{1}{4}\right)^{w_1} \end{pmatrix}$$

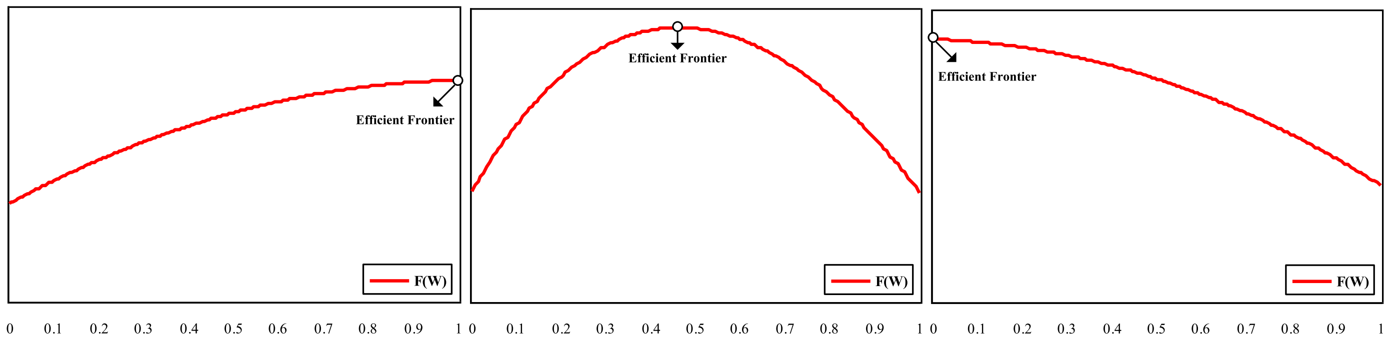

Fig. 3. Three approximate distributions of $ds(\oplus_{i=1}^{n} w_i\bar{h}_i)/dw_1$.

It can be found that $\frac{d^2 s(\oplus_{i=1}^{2} w_i\bar{h}_i)}{d{w_1}^2} < 0$.

According to the above results, we can derive that there is one and only one optimal solution for Eq. (8) based on the max-score rule, and this point is the efficient frontier of Example 1.

We can further analyze the efficient frontier of Model 2. Since the score value of the portfolio is represented as $F(W) = \max s(\cup_{\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n}\{1-\prod_{i=1}^{n}(1-\bar{\gamma}_i)^{w_i}\})$, then

$$F(W) = \max s\Big(\cup_{\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n}\Big\{1-(1-\bar{\gamma}_1)^{w_1} \times (1-\bar{\gamma}_2)^{w_2}\cdots(1-\bar{\gamma}_{n-1})^{w_{n-1}}\cdot(1-\bar{\gamma}_n)^{1-\sum_{i=1}^{n-1}w_i}\Big\}\Big).$$

After that, we can get the first-order and second-order derivatives of $s(\oplus_{i=1}^{n} w_i\bar{h}_i)$ for w_1 as follows:

$$\begin{aligned}\frac{ds\left(\oplus_{i=1}^{n} w_i\bar{h}_i\right)}{dw_1} &= s\Big(\cup_{\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n}\Big\{-\ln(1-\bar{\gamma}_1)\prod_{i=1}^{n}(1-\bar{\gamma}_i)^{w_i} \\ &\quad + \ln(1-\bar{\gamma}_n)\prod_{i=1}^{n}(1-\bar{\gamma}_i)^{w_i}\Big\}\Big) \\ &= s\Big(\cup_{\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n}\{(\ln(1-\bar{\gamma}_n) \\ &\quad - \ln(1-\bar{\gamma}_1))\prod_{i=1}^{n}(1-\bar{\gamma}_i)^{w_i}\Big\}\Big)\end{aligned}$$

$$\begin{aligned}&\frac{d^2 s\left(\oplus_{i=1}^{n} w_i\bar{h}_i\right)}{d{w_1}^2} \\ &= s\Big(\cup_{\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n}\Big\{\big(-\ln^2(1-\bar{\gamma}_n)+2\ln(1-\bar{\gamma}_1) \\ &\quad \times \ln(1-\bar{\gamma}_n)-\ln^2(1-\bar{\gamma}_n)\big)\prod_{i=1}^{n}(1-\bar{\gamma}_i)^{w_i}\Big\}\Big) \\ &= s\Big(\cup_{\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n}\Big\{-(\ln(1-\bar{\gamma}_1)-\ln(1-\bar{\gamma}_n))^2 \\ &\quad \times \prod_{i=1}^{n}(1-\bar{\gamma}_i)^{w_i}\Big\}\Big)\end{aligned}$$

Therefore, $\frac{d^2 s(\oplus_{i=1}^{n} w_i\bar{h}_i)}{d{w_1}^2} < 0$ and $\frac{ds(\oplus_{i=1}^{n} w_i\bar{h}_i)}{dw_1}$ can be depicted as the three approximate distributions shown in Fig. 3.

Thus, w_1 in Model 2 only has one optimal solution ${w_1}^*$. Similarly, we can obtain the only optimal investment ratios $W^* = (w_1^*, w_2^*, \cdots, w_n^*)$, which presents the efficient frontier of the risk investment based on Model 2.

These also show that the qualitative portfolio model based on the max-score rule, i.e., Model 2, can be used to construct the optimal portfolio for a general investor under the hesitant fuzzy environment. It is found that the construction principle of this model is the max-score rule; then, this model can be further developed by changing this rule into the score-deviation trade-off rule. To do so, in the next section, the developed qualitative portfolio model based on the new rule is proposed.

4. Portfolio selection for the risk investor under the hesitant fuzzy environment

In Section 3, we provide a model for the general investor to calculate the optimal investment ratios based on the max-score rule. As aforementioned, this model can be further developed by introducing the score-deviation trade-off rule. It is obvious that this developed model can be used by risk investors to construct the optimal portfolio. Therefore, in this section, we focus on the qualitative portfolio model for the risk investors under the hesitant fuzzy environment and its investment opportunity and efficient frontier.

4.1. Qualitative portfolio model based on the score-deviation trade-off rule

Compared with general investors, the score-deviation trade-off rule is more suitable for risk investors as they want to obtain the maximum returns with limited risks or bear the minimum risks on the condition of obtaining acceptable returns. To develop this qualitative portfolio model, we first set the following hesitant fuzzy risk investment scenario, which is similar to that proposed in Section 3.1.

A risk investor wants to place a fund on n newly listed stocks and other financial products $\{x_1, x_2, \cdots, x_n\}$. Because of limited data, he/she wants to choose the portfolio on the basis of the set of m qualitative criteria $\{y_1, y_2, \cdots, y_m\}$ described by the HFEs h_{ij} $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m)$, which can be represented as a hesitant fuzzy matrix $H = [h_{ij}]_{n\times m}$. This matrix can be transformed into a collective column vector $\bar{H} = [\bar{h}_i]_{n\times 1}$ by aggregating all the values on one line.

To calculate the optimal investment ratios and obtain the best portfolio, the following qualitative portfolio model is proposed for use by the risk investor.

Model 3:

$$F(W) = \max s\left(\oplus_{i=1}^{n} w_i\bar{h}_i\right) + k\cdot\min d\left(\oplus_{i=1}^{n} w_i\bar{h}_i\right)$$
$$s.t.\begin{cases}\oplus_{i=1}^{n} w_i\bar{h}_i = \cup_{\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\cdots,\bar{\gamma}_n\in\bar{h}_n}\left\{1-\prod_{i=1}^{n}(1-\bar{\gamma}_i)^{w_i}\right\} \\ s(h) = \frac{1}{\#h}\sum_{\gamma\in h}\gamma \\ d(h) = \frac{1}{\#h}\sum_{\gamma\in h}\sqrt{(\gamma-s(h))^2} \\ \sum_{i=1}^{n} w_i = 1, w_i \ge 0\end{cases} \tag{9}$$

where $s(h)$ and $d(h)$ are the score function and the deviation function of the HFE h, $\bar{h}_i$ is the aggregated HFE based on $\bar{h}_i = \oplus_{j=1}^{m} h_{ij}$, h_{ij} is the hesitant fuzzy information of the alternative x_i with respect to the qualitative criterion x_j, $W = \{w_1, w_2, \cdots, w_n\}$ and w_i are the optimal investment ratios of this fund on these stocks, $i = 1, 2, \cdots, n$, and $j = 1, 2, \cdots, m$.

It is found that Model 3 is a multi-objective programming model. According to selection principle of the investor, we can transform Model 3 into an objective programming model, namely

Model 4 or Model 5, by setting the deviation function or the score function. These new models can be used to choose the optimal portfolio for the risk investor. First, we introduce Model 4 as follows:

Model 4:

$$F(W) = \max s\left(\oplus_{i=1}^{n} w_i \bar{h}_i\right)$$
$$s.t.\begin{cases} d\left(\oplus_{i=1}^{n} w_i \bar{h}_i\right) \le D \\ \oplus_{i=1}^{n} w_i \bar{h}_i = \cup_{\bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n} \left\{1 - \prod_{i=1}^{n} (1-\bar{\gamma}_i)^{w_i}\right\} \\ s(h) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma \\ d(h) = \frac{1}{\#h} \sum_{\gamma \in h} \sqrt{(\gamma - s(h))^2} \\ \sum_{i=1}^{n} w_i = 1, w_i \ge 0 \end{cases} \tag{10}$$

where D is the max-deviation degree set by the selection principle of the investor, and the other parameters are as above.

In Model 4, the max-deviation degree D can be used to represent investors with different risk appetites. For example, for a risk seeker, D can be set as a large value, for a risk neutral investor, D can be set as an intermediate value, while for the risk averter, D can be set as a small value. The obvious question is how to calculate the max-deviation degree D. To address this issue, a so-called deviation trisection approach is defined as follows:

Definition 4. Assume that the range of the max-deviation degree D is $[\min D, \max D]$. Then, the max-deviation degree D_1 of a risk seeker can be set as $D_1 = \max D$, the max-deviation degree D_2 of a risk neutral investor can be set as $D_2 = \min D + \frac{2}{3}(\max D - \min D)$, and the max-deviation degree D_3 of a risk averter can be set as $D_3 = \min D + \frac{1}{3}(\max D - \min D)$. This process is called the deviation trisection approach.

Furthermore, the following model is provided to calculate $\min D$ and $\max D$.

$$D(W) = \max d\left(\oplus_{i=1}^{n} w_i \bar{h}_i\right) or D(W) = \min d\left(\oplus_{i=1}^{n} w_i \bar{h}_i\right)$$
$$s.t.\begin{cases} d(h) = \frac{1}{\#h} \sum_{\gamma \in h} \sqrt{(\gamma - s(h))^2} \\ \oplus_{i=1}^{n} w_i \bar{h}_i = \cup_{\bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n} \left\{1 - \prod_{i=1}^{n} (1-\bar{\gamma}_i)^{w_i}\right\} \\ \sum_{i=1}^{n} w_i = 1, w_i \ge 0 \end{cases} \tag{11}$$

where the parameters are set as those in Model 4.

Thus, we can obtain the max-deviation degree of the risk investor based on Eq. (11). The deviation trisection approach is used to find the optimal investment ratios based on Model 4, which completes the portfolio selection process for the risk investor under the hesitant fuzzy environment.

Similar to Model 4, we can transform Model 3 into Model 5 by setting the score function which is shown as follows:

Model 5:

$$F(W) = \min d\left(\oplus_{i=1}^{n} w_i \bar{h}_i\right)$$
$$s.t.\begin{cases} s\left(\oplus_{i=1}^{n} w_i \bar{h}_i\right) \ge S \\ \oplus_{i=1}^{n} w_i \bar{h}_i = \cup_{\bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n} \left\{1 - \prod_{i=1}^{n} (1-\bar{\gamma}_i)^{w_i}\right\} \\ s(h) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma \\ d(h) = \frac{1}{\#h} \sum_{\gamma \in h} \sqrt{(\gamma - s(h))^2} \\ \sum_{i=1}^{n} w_i = 1, w_i \ge 0 \end{cases} \tag{12}$$

where S is the min-score degree set by the selection principle of the investor, and the other parameters are as above.

Similarly, the min-score degree S which is calculated in the following score trisection approach can be used to depict investors with different risk appetites.

Definition 5. Assume that the range of the min-score degree S is $[\min S, \max S]$. Then, the min- score degree S_1 of a risk seeker is set as $S_3 = \max S - \frac{1}{3}(\max S - \min S)$, the min-score degree S_2 of a risk neutral investor is set as $S_2 = \max S - \frac{2}{3}(\max S - \min S)$, and the min- score degree S_3 of a risk seeker is set as $S_2 = \min S$. This process is called the score trisection approach.

Furthermore, the following model is provided to calculate $\min S$ and $\max S$.

$$S(W) = \max s\left(\oplus_{i=1}^{n} w_i \bar{h}_i\right) or S(W) = \min s\left(\oplus_{i=1}^{n} w_i \bar{h}_i\right)$$
$$s.t.\begin{cases} s(h) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma \\ \oplus_{i=1}^{n} w_i \bar{h}_i = \cup_{\bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \cdots, \bar{\gamma}_n \in \bar{h}_n} \left\{1 - \prod_{i=1}^{n} (1-\bar{\gamma}_i)^{w_i}\right\} \\ \sum_{i=1}^{n} w_i = 1, w_i \ge 0 \end{cases} \tag{13}$$

where the parameters are set as those in Model 5.

Thus, we construct two computable portfolio models, namely Model 4 and Model 5, based on the score-deviation trade-off rule, which can be used to obtain the optimal portfolio for the three types of risk investors under the hesitant fuzzy environment. We find that Model 4 and Model 5 are two different programming models based on two different principles. Model 4 is suitable for the investor who wants to obtain maximum returns based on limited risks, whereas Model 5 is suitable for the investor who wants to bear minimum risks under the condition of obtaining acceptable returns. Example 2 is provided to demonstrate these models and their application processes.

Example 2. An investor wants to place a sum of money into three stocks $\{x_1, x_2, x_3\}$ based on the criterion y_1 which is a quantified index without any quantitative data. Therefore, the investor offers his/her subjective evaluations represented by the HFEs h_{ij} $(i = 1, 2, 3; j = 1)$. Here, h_{11} denotes the evaluation value for x_1 with respect to y_1, h_{21} represents the evaluation value for x_2 with respect to y_1, and h_{31} represents the evaluation value for x_3 with respect to y_1. If $h_{11} = \{0.3, 0.9\}$, $h_{21} = \{0.5, 0.6\}$, and $h_{31} = \{0.45\}$, we have $H = \bar{H} = [\bar{h}_i]_{3\times 1} = [\{0.3, 0.9\}, \{0.5, 0.6\}, \{0.45\}]^T$. To obtain the optimal portfolio, Model 4 and Model 5 can be applied as follows:

First, we have $h_{11} = \{0.3, 0.9\}$, $h_{21} = \{0.5, 0.6\}$, and $h_{31} = \{0.45\}$. We can conclude that there are $2 \times 2 \times 1 = 4$ elements in the aggregated values $\oplus_{i=1}^{3} w_i \bar{h}$, which can be presented as follows:

$$\begin{aligned} \oplus_{i=1}^{3} w_i \bar{h}_i &= \cup_{\bar{\gamma}_1 \in \bar{h}_1, \bar{\gamma}_2 \in \bar{h}_2, \bar{\gamma}_3 \in \bar{h}_3} \left\{1 - \prod_{i=1}^{3} (1-\bar{\gamma}_i)^{w_i}\right\} \\ &= \left\{\begin{aligned} &(1-(1-0.3)^{w_1}\cdot(1-0.5)^{w_2}\cdot(1-0.55)^{w_3}), \\ &(1-(1-0.3)^{w_1}\cdot(1-0.6)^{w_2}\cdot(1-0.55)^{w_3}), \\ &(1-(1-0.9)^{w_1}\cdot(1-0.5)^{w_2}\cdot(1-0.55)^{w_3}), \\ &(1-(1-0.9)^{w_1}\cdot(1-0.6)^{w_2}\cdot(1-0.55)^{w_3}) \end{aligned}\right\} \\ &= \left\{\begin{aligned} &(1-0.7^{w_1}\cdot 0.5^{w_2}\cdot 0.45^{w_3}), \\ &(1-0.7^{w_1}\cdot 0.4^{w_2}\cdot 0.45^{w_3}), \\ &(1-0.1^{w_1}\cdot 0.5^{w_2}\cdot 0.45^{w_3}), \\ &(1-0.1^{w_1}\cdot 0.4^{w_2}\cdot 0.45^{w_3}) \end{aligned}\right\} \end{aligned} \tag{14}$$

Furthermore, we can get Eq. (15) based on Definition 3 and Eq. (14).

$$\begin{aligned} s\left(\oplus_{i=1}^{3} w_i \bar{h}_i\right) = 1 - 0.25 \cdot \big(&0.7^{w_1}\cdot 0.5^{w_2}\cdot 0.45^{w_3} + 0.7^{w_1}\cdot 0.4^{w_2}\cdot 0.45^{w_3} \\ &+ 0.1^{w_1}\cdot 0.5^{w_2}\cdot 0.45^{w_3} + 0.1^{w_1}\cdot 0.4^{w_2}\cdot 0.45^{w_3}\big) \end{aligned} \tag{15}$$

Then based on Model 4, the following steps are illustrated to obtain the optimal investment ratios.

According to Eq. (11), we can get $\min D = 0$ when $w_1 = w_2 = 0$ and $w_3 = 1$ and $\max D = 0.3$ when $w_1 = 1$ and $w_2 = w_3 = 0$. Then, based on the deviation trisection approach, the following conclusions can be derived.

(1) For the risk seeker, we have $D_1 = 0.3$ and

$$\begin{aligned} F(W) &= \max s\left(\oplus_{i=1}^{3} w_i \bar{h}_i\right) \\ &= \max \big\{1 - 0.25 \cdot \big(0.7^{w_1}\cdot 0.5^{w_2}\cdot 0.45^{w_3} \\ &\quad + 0.7^{w_1}\cdot 0.4^{w_2}\cdot 0.45^{w_3} + 0.1^{w_1}\cdot 0.5^{w_2}\cdot 0.45^{w_3} \end{aligned}$$

$$+0.1^{w_1}\cdot 0.4^{w_2}\cdot 0.45^{w_3}\big)\Big\}$$

$$s.t.\begin{cases} d\left(\oplus_{i=1}^{3} w_i\bar{h}_i\right)\le 0.3\\ \sum_{i=1}^{3} w_i = 1, w_i\ge 0\end{cases}$$

By solving the above nonlinear optimization problem, we obtain $F(W) = 0.6161$, and the optimal investment ratios of this fund are $w_1 = 0.6272$, $w_2 = 0.3728$, and $w_3 = 0$. It is found that the results are similar to those in Example 1. Indeed, because the first constrain in the above programming model always holds, the two results are the same. Therefore, if the risk seeker wants to place \$10,000 into the three stocks $\{x_1, x_2, x_3\}$, then the investor should place \$6272, \$3728, and \$0 into x_1, x_2, and x_3, respectively.

(2) For the risk neutral investor, we have $D_2 = 0.2$ and

$$\begin{aligned}F(W) &= \max s\left(\oplus_{i=1}^{3} w_i\bar{h}_i\right)\\ &= \max\Big\{1-0.25\cdot\big(0.7^{w_1}\cdot 0.5^{w_2}\cdot 0.45^{w_3}\\ &\quad +0.7^{w_1}\cdot 0.4^{w_2}\cdot 0.45^{w_3}+0.1^{w_1}\cdot 0.5^{w_2}\cdot 0.45^{w_3}\\ &\quad +0.1^{w_1}\cdot 0.4^{w_2}\cdot 0.45^{w_3}\big)\Big\}\end{aligned}$$

$$s.t.\begin{cases} d\left(\oplus_{i=1}^{3} w_i\bar{h}_i\right)\le 0.2\\ \sum_{i=1}^{3} w_i = 1, w_i\ge 0\end{cases}$$

By solving the above nonlinear optimization problem, we obtain $F(W) = 0.6159$, and the optimal investment ratios of this fund are $w_1 = 0.5935$, $w_2 = 0.4065$, and $w_3 = 0$. Therefore, if the risk neutral investor wants to place \$10,000 into the three stocks $\{x_1, x_2, x_3\}$, then the investor should place \$5935, \$4065, and \$0 into x_1, x_2, and x_3, respectively.

(3) For the risk averter, we have $D_3 = 0.1$ and

$$\begin{aligned}F(W) &= \max s\left(\oplus_{i=1}^{3} w_i\bar{h}_i\right)\\ &= \max\Big\{1-0.25\cdot\big(0.7^{w_1}\cdot 0.5^{w_2}\cdot 0.45^{w_3}\\ &\quad +0.7^{w_1}\cdot 0.4^{w_2}\cdot 0.45^{w_3}+0.1^{w_1}\cdot 0.5^{w_2}\cdot 0.45^{w_3}\\ &\quad +0.1^{w_1}\cdot 0.4^{w_2}\cdot 0.45^{w_3}\big)\Big\}\end{aligned}$$

$$s.t.\begin{cases} d\left(\oplus_{i=1}^{3} w_i\bar{h}_i\right)\le 0.1\\ \sum_{i=1}^{3} w_i = 1, w_i\ge 0\end{cases}$$

By solving the above nonlinear optimization calculation, we obtain $F(W) = 0.5960$, and the optimal investment ratios of this fund are $w_1 = 0.2598$, $w_2 = 0.4995$, and $w_3 = 0.2407$. Therefore, if the risk averter wants to place \$10,000 into the three stocks $\{x_1, x_2, x_3\}$, then the investor should place \$2598, \$4995, and \$2407 into x_1, x_2, and x_3, respectively.

Similarly, we can calculate the optimal investment ratios for the three types of risk investors based on Model 5 and Definition 5. Note that the optimal results based on Model 4 and Model 5 could differ in various modeling rules.

4.2. Investment opportunity and efficient frontier analysis

In Section 4.1, we provide a portfolio selection approach and develop two qualitative portfolio models to select the optimal portfolio for the risk investor based on the score-deviation trade-off rule. Furthermore, we try to evaluate what will happen when we combine investments to form a portfolio and which portfolio yields maximum returns with limited risks or minimum risks on the condition of obtaining acceptable returns. Below, we analyze all the available investment opportunities for the three types of investors and present an efficient frontier analysis based on Model 4.

Example 2 is used again to demonstrate these investment opportunities and the efficient frontiers. Figs. 4–6 show all the available investment opportunities for the risk seeker, the risk neutral investor, and the risk averter in Example 2. Based on these figures, the following conclusions are derived.

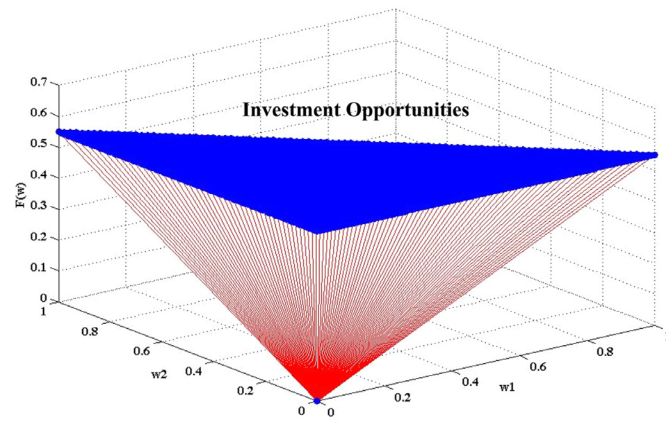

Fig. 4. Investment opportunities of the risk seeker.

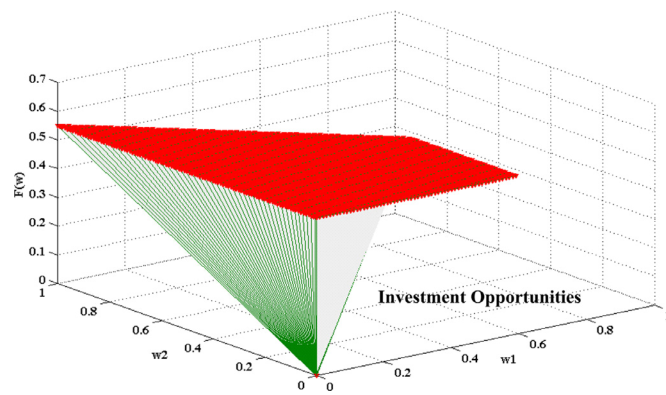

Fig. 5. Investment opportunities of the risk neutral investor.

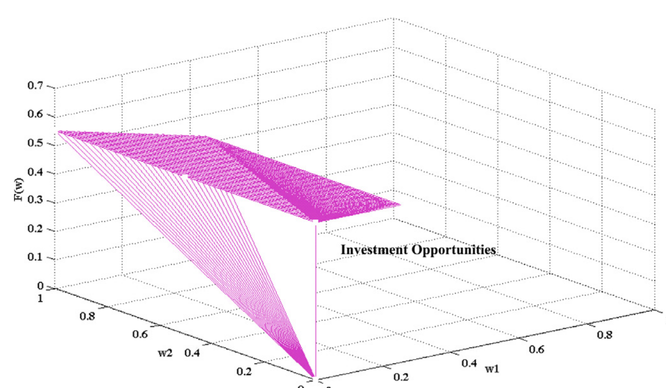

Fig. 6. Investment opportunities of the risk averter.

(1) All of the three figures of the available investment opportunities are three-dimensional, which are consistent with the number of selectable objects. In this example, the objects are three stocks. Therefore, if the number of selectable objects is less than or equal to three, we can depict the distribution of all the available investment opportunities. For this reason, there are two and three objects are provided in Examples 1 and 2.

(2) The investment opportunities that the risk neutral investor and the risk averter have are included as a part of alternatives for the risk seeker. It is reasonable that the three corresponding constraints D satisfy the inclusion relations. The risk seeker obtains more investment opportunities than the other investors owing to his/her larger risk tolerance.

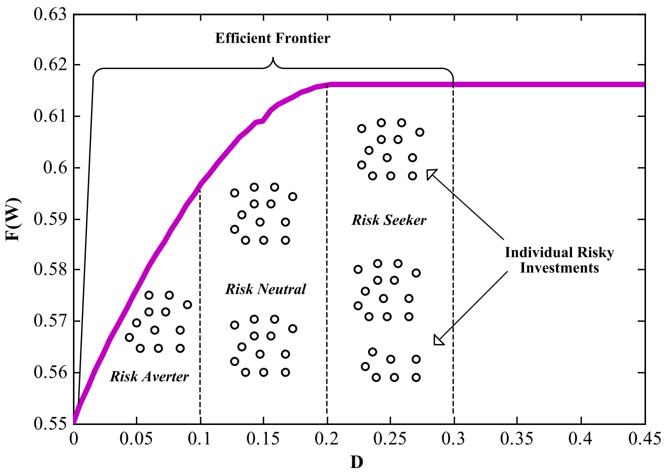

Fig. 7. The efficient frontiers of Example 2.

(3) If the max-deviation degree D is larger than $\max D$, then the investment opportunities remain unchanged. Thus, Figure 4 presents all the investment opportunities when $D \le \max D$. Of course, the set max-deviation degree D cannot be less than $\min D$.

In addition, we provide the efficient frontiers of Example 2 in Fig. 7, where the abscissa axis is the range of all the possible max-deviation degrees D and $D \in [\min D, \max D]$, and the ordinate axis is the corresponding $F(W)$ based on Model 4. The dots show all the possible individual risk investments and the pink curve represents their efficient frontier.

The following conclusions can be derived from Fig. 7.

(a) The curve of the efficient frontiers in Example 2 is different from that in Example 1. We can further infer that the distribution of the efficient frontiers is a curved surface if there are three selectable objectives.
(b) The efficient frontiers' values of the risk seeker are larger than those of the risk neutral investor and the risk averter, which is similar to the aforementioned conclusion (2).
(c) The max-deviation degree D is a decisive element of the optimal portfolio. If the investor bears a larger D, then he/she can obtain a larger return; meanwhile, if the investor wants to obtain a larger return, then he/she should bear a larger D.
(d) There is one and only one point as the optimal investment selection of Example 2 based on the score-deviation trade-off rule.

Thus, we respectively propose two qualitative portfolio models for the general and risk investors in Sections 3 and 4. Furthermore, their investment opportunities and efficient frontiers are analyzed. Also, the deviation and score trisection approaches are provided. By integrating these two portfolio models and the above approaches, the portfolio selection processes for practical risk investment under the hesitant fuzzy environment are summarized in the next section.

5. Portfolio selection processes of risk investment under the hesitant fuzzy environment

To apply the proposed portfolio selection approaches, such as the qualitative portfolio model for general investors (Model 2) and the qualitative portfolio models for risk investors (Models 4 and 5), the following two portfolio selection processes under the hesitant fuzzy environment are established for qualitative risk investment.

Assume that an investor wants to place fund K on n newly listed stocks $\{x_1, x_2, \cdots, x_n\}$. Because quantitative data are limited (or even unavailable), he/she wants to select the portfolio on the basis of the set of m qualitative criteria $\{y_1, y_2, \cdots, y_m\}$ described by the HFEs h_{ij} $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m)$. These qualitative criteria can be provided by this investor, his/her investment advisers, or the investment experts. These HFEs can be presented as a hesitant fuzzy matrix $H = [h_{ij}]_{n\times m}$. Then, the following portfolio selection processes, namely Process A and Process B, can be used to calculate the optimal investment ratios and construct the optimal portfolio for the general investor and the risk investor, respectively.

Process A

For a general investor who simply wants to obtain the maximum returns, the following portfolio selection process can be used:

Step 1. Transform the hesitant fuzzy matrix $H = [h_{ij}]_{n\times m}$ $(i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$ into a collective column vector $\bar{H} = [\bar{h}_i]_{n\times 1}$ $(i = 1, 2, \cdots, n)$ by aggregating all the values on one line.
Step 2. Construct the portfolio model based on Model 2.
Step 3. Solve the portfolio model and obtain the optimal investment ratios w_i $(i = 1, 2, \cdots, n)$.
Step 4. Calculate the optimal invested amounts k_i $(i = 1, 2, \cdots, n)$ of the stocks x_i $(i = 1, 2, \cdots, n)$ according to $k_i = K \cdot w_i$ $(i = 1, 2, \cdots, n)$.

Hereby, we obtain the optimal investment amount k_i and construct the optimal portfolio of these stocks for the general investor.

Process B

For a risk investor who wants to construct the portfolio based on the score-deviation trade-off rule, the following portfolio selection process can be used:

Step 1. Transform the hesitant fuzzy matrix $H = [h_{ij}]_{n\times m}$ $(i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$ into the collective column vector $\bar{H} = [\bar{h}_i]_{n\times 1}$ $(i = 1, 2, \cdots, n)$ by aggregating all the values on one line.
Step 2. Construct the qualitative portfolio model based on Model 4 or Model 5. Here, if the investor wants to obtain the maximum returns with limited risks, then Model 4 should be applied. However, if the investor wants to bear the minimum risks on the condition of obtaining the acceptable returns, then Model 5 should be applied.
Step 3. If Model 4 is selected, then we calculate the range of the max-deviation degree D based on Eq. (11) and obtain $[\min D, \max D]$. If Model 5 is selected, then calculate the range of the min-score degree S based on Eq. (13) and obtain $[\min S, \max S]$.
Step 4. According to the deviation trisection approach or the score trisection approach, set the max-deviation degree D or the min-score degree S.
Step 5. Solve the portfolio model and obtain the optimal investment ratios w_i $(i = 1, 2, \cdots, n)$.
Step 6. Calculate the optimal invested amounts k_i $(i = 1, 2, \cdots, n)$ of the stocks x_i $(i = 1, 2, \cdots, n)$ according to $k_i = K \cdot w_i$ $(i = 1, 2, \cdots, n)$.

Thus, we obtain the optimal investment amount k_i and construct the optimal portfolio of the stocks for the three types of risk investors.

6. Illustrative example

To further illustrate the application of the above proposed portfolio models, a qualitative risk investment example is presented in this section.

Table 1
The hesitant fuzzy matrix $H = [h_{ij}]_{4\times3}$ provided by the investor.

Criteria stocks	Profitability capability y_1	External reputation y_2	Technical feasibility y_3
Off-shore gas recovery company x_1	{0.45}	{0.35, 0.95}	{0.15}
Genetically modified food company x_2	{0.35, 0.65, 0.90}	{0.10}	{0.55}
UAV manufacturing enterprise x_3	{0.75}	{0.15}	{0.35}
Pilot training company x_4	{0.10, 0.70}	{0.30}	{0.65}

6.1. Example and calculations

Example 3. An investor wants to place an idle fund of $1000,000 into the New Tertiary Board as a long-term risk investment. He/she wants to obtain high returns on the condition of tolerating low risks.

As a new stock exchange market in China, the New Tertiary Board, which is also called the National Equities and Quotations, was rebuilt in 2012. It was built to activate the multiple-level capital market and services for small and medium-sized enterprises. Moreover, it is characterized by high risks and high returns. Four newly listed companies x_i $(i = 1, 2, 3, 4)$ on this stock market are considered to be promising by the investor which respectively represent four emerging industries: an off-shore gas recovery company x_1, a genetically modified food company x_2, an unmanned aerial vehicle (UAV) manufacturing enterprise x_3, and a pilot training company x_4. Furthermore, this investor prefers to use three criteria to compare these companies: the profitability capability y_1, the external reputation y_2, and the technical feasibility y_3.

Because all of the four companies have been established in recent years and are newly listed on the New Tertiary Board, useful financial data and quantified material about them are unavailable. Thus, the conventional portfolio theory is unsuitable in this case. Instead, the proposed qualitative portfolio models under the hesitant fuzzy environment can be applicable to the given situation. Therefore, the portfolio selection processes constructed in Section 5 are applied in this example.

The investor provides the characteristics of the stocks (or listed companies) x_i $(i = 1, 2, 3, 4)$ with respect to the criteria y_j $(j = 1, 2, 3)$. All the evaluations are described by the HFEs h_{ij} $(i = 1, 2, 3, 4;\ j = 1, 2, 3)$. Then, the hesitant fuzzy matrix $H = [h_{ij}]_{4\times3}$ is constructed based on the h_{ij}, as presented in Table 1.

If the investor is a general investor, then Process A can be used to calculate the optimal investment ratios of the four stocks according to the following steps.

Step 1. Transform $H = [h_{ij}]_{4\times3}$ $(i = 1, 2, 3, 4;\ j = 1, 2, 3)$ into the collective column vector $\bar{H} = [\bar{h}_i]_{4\times1}$ $(i = 1, 2, 3, 4)$ by aggregating all the values in one line, then we have

$$\begin{aligned}\bar{h}_1 &= h_{11} \oplus h_{12} \oplus h_{13}\\ &= \{0.45\} \oplus \{0.35, 0.95\} \oplus \{0.15\} = \{0.6961, 0.9766\};\\ \bar{h}_2 &= h_{21} \oplus h_{22} \oplus h_{23}\\ &= \{0.35, 0.65, 0.9\} \oplus \{0.1\} \oplus \{0.55\} = \{0.7368, 0.8583, 0.9595\};\\ \bar{h}_3 &= h_{31} \oplus h_{32} \oplus h_{33}\\ &= \{0.75\} \oplus \{0.15\} \oplus \{0.35\} = \{0.8619\};\ \text{and}\\ \bar{h}_4 &= h_{41} \oplus h_{42} \oplus h_{43}\\ &= \{0.1, 0.7\} \oplus \{0.3\} \oplus \{0.65\} = \{0.7795, 0.9265\}.\end{aligned}$$

Thus, we can get the collective column vector $\bar{H} = [\bar{h}_1, \bar{h}_2, \bar{h}_3, \bar{h}_4]^T$ as follows:

$$\bar{H} = \left[\bar{h}_i\right]_{4\times1} = [\{0.6961, 0.9766\}, \{0.7368, 0.8583, 0.9595\}, \{0.8619\}, \{0.7795, 0.9265\}]^T$$

Step 2. Construct the qualitative portfolio model based on Model 2, then we have

$$F(W) = \max s\left(\oplus_{i=1}^4 w_i\bar{h}_i\right)$$
$$s.t.\begin{cases}\oplus_{i=1}^4 w_i\bar{h}_i = \cup_{\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\bar{\gamma}_3\in\bar{h}_3,\bar{\gamma}_4\in\bar{h}_4}\left\{1 - \prod_{i=1}^n (1 - \bar{\gamma}_i)^{w_i}\right\}\\ \sum_{i=1}^4 w_i = 1, w_i \geq 0\end{cases} \tag{16}$$

where $s(h)$ is the score function of the HFE h, and

$$s\left(\oplus_{i=1}^4 w_i\bar{h}_i\right) = 1 - \frac{1}{24}\left\{\begin{array}{l}0.3039^{w_1}\cdot 0.2633^{w_2}\cdot 0.1381^{w_3}\cdot 0.2205^{w_4}\\ +0.3039^{w_1}\cdot 0.1418^{w_2}\cdot 0.1381^{w_3}\cdot 0.2205^{w_4}\\ +0.3039^{w_1}\cdot 0.0405^{w_2}\cdot 0.1381^{w_3}\cdot 0.2205^{w_4}\\ +0.3039^{w_1}\cdot 0.2633^{w_2}\cdot 0.1381^{w_3}\cdot 0.0735^{w_4}\\ +0.3039^{w_1}\cdot 0.1418^{w_2}\cdot 0.1381^{w_3}\cdot 0.0735^{w_4}\\ +0.3039^{w_1}\cdot 0.0405^{w_2}\cdot 0.1381^{w_3}\cdot 0.0735^{w_4}\\ +0.0234^{w_1}\cdot 0.2633^{w_2}\cdot 0.1381^{w_3}\cdot 0.2205^{w_4}\\ +0.0234^{w_1}\cdot 0.1418^{w_2}\cdot 0.1381^{w_3}\cdot 0.2205^{w_4}\\ +0.0234^{w_1}\cdot 0.0405^{w_2}\cdot 0.1381^{w_3}\cdot 0.2205^{w_4}\\ +0.0234^{w_1}\cdot 0.2633^{w_2}\cdot 0.1381^{w_3}\cdot 0.0735^{w_4}\\ +0.0234^{w_1}\cdot 0.1418^{w_2}\cdot 0.1381^{w_3}\cdot 0.0735^{w_4}\\ +0.0234^{w_1}\cdot 0.0405^{w_2}\cdot 0.1381^{w_3}\cdot 0.0735^{w_4}\end{array}\right\} \tag{17}$$

Step 3. Solve Eq. (17) and obtain the optimal investment ratios w_i as $w_1 = 0.3164$, $w_2 = 0.3256$, $w_3 = 0.0863$, $w_4 = 0.2717$, and $F(W) = 0.8771$.

Step 4. Calculate the optimal invested amount k_i $(i = 1, 2, 3, 4)$, and we can obtain $k_1 = 316400$, $k_2 = 325600$, $k_3 = 86300$, and $k_4 = 271700$.

Therefore, the investor should place $316,400 into the off-shore gas recovery company x_1, $325,600 into the genetically modified food company x_2, $86,300 into the UAV manufacturing enterprise x_3, and $271,700 into the pilot training company x_4, which is his/her optimal portfolio.

If the investor is a risk investor and wants to obtain the maximum returns with limited risks, then Process B and Model 4 can be used to calculate the optimal investment ratios of the four stocks according to the following steps.

Step 1. See Step 1.

Step 2. Construct the qualitative portfolio model based on Model 4, and then we can get

$$F(W) = \max s\left(\oplus_{i=1}^4 w_i\bar{h}_i\right)$$
$$s.t.\begin{cases}d\left(\oplus_{i=1}^4 w_i\bar{h}_i\right) \leq D\\ \oplus_{i=1}^4 w_i\bar{h}_i = \cup_{\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\bar{\gamma}_3\in\bar{h}_3,\bar{\gamma}_4\in\bar{h}_4}\left\{1 - \prod_{i=1}^n (1 - \bar{\gamma}_i)^{w_i}\right\}\\ \sum_{i=1}^4 w_i = 1, w_i \geq 0\end{cases} \tag{18}$$

where $s(h)$ and $d(h)$ are the score function and the deviation function of the HFE h, and $s(\oplus_{i=1}^4 w_i\bar{h}_i)$ is equal to Eq. (17).

Step 3. Calculate the range of the max-deviation degree D based on Eqs. (11) and (17), then we have

$$\max D = \max d\left(\oplus_{i=1}^4 w_i\bar{h}_i\right) or \min D = \min d\left(\oplus_{i=1}^4 w_i\bar{h}_i\right)$$
$$s.t.\begin{cases}d(h) = \frac{1}{\#h}\sum_{\gamma\in h}\sqrt{(\gamma - s(h))^2}\\ \sum_{i=1}^4 w_i = 1, w_i \geq 0\end{cases} \tag{19}$$

and $[\min D, \max D] = [0, 0.1402]$. Here, $\min D = 0$ when $w_1 = w_2 = w_4 = 0$, and $w_3 = 1$; and $\max D = 0.1402$ when $w_1 = 1$ and $w_2 = w_3 = w_4 = 0$.

Step 4. According to the deviation trisection approach, we have $D_1 = 0.1402$, which is the max-deviation degree of the risk seeker; $D_2 = 0.0935$, which is the max-deviation degree of the risk neutral investor; and $D_3 = 0.0467$, which is the max-deviation degree of the risk averter.

Step 5. Solve the qualitative portfolio model and calculate the optimal investment ratios w_i $(i = 1, 2, 3, 4)$. The details are given as follows:

(1) If the risk investor is a risk seeker, then the qualitative portfolio model is presented as

$$F(W) = \max s\left(\oplus_{i=1}^{4} w_i\bar{h}_i\right)$$
$$s.t.\begin{cases} d\left(\oplus_{i=1}^{4} w_i\bar{h}_i\right) \le 0.1402 \\ \oplus_{i=1}^{4} w_i\bar{h}_i = \cup_{\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\bar{\gamma}_3\in\bar{h}_3,\bar{\gamma}_4\in\bar{h}_4}\left\{1-\prod_{i=1}^{n}(1-\bar{\gamma}_i)^{w_i}\right\} \\ \sum_{i=1}^{4} w_i = 1, w_i \ge 0 \end{cases} \tag{20}$$

where $s(h)$ and $d(h)$ are the score function and the deviation function of the HFE h, and $s(\oplus_{i=1}^{4} w_i\bar{h}_i)$ is equal to Eq. (17).

By solving the nonlinear optimization problem in Eq. (20), we obtain $w_1 = 0.3164$, $w_2 = 0.3256$, $w_3 = 0.0863$, $w_4 = 0.2717$, and $F(W) = 0.8771$.

(2) If the risk investor is a risk neutral, then the qualitative portfolio model is presented as

$$F(W) = \max s\left(\oplus_{i=1}^{4} w_i\bar{h}_i\right)$$
$$s.t.\begin{cases} d\left(\oplus_{i=1}^{4} w_i\bar{h}_i\right) \le 0.0935 \\ \oplus_{i=1}^{4} w_i\bar{h}_i = \cup_{\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\bar{\gamma}_3\in\bar{h}_3,\bar{\gamma}_4\in\bar{h}_4}\left\{1-\prod_{i=1}^{n}(1-\bar{\gamma}_i)^{w_i}\right\} \\ \sum_{i=1}^{4} w_i = 1, w_i \ge 0 \end{cases} \tag{21}$$

where $s(h)$ and $d(h)$ are the score function and the deviation function of the HFE h, and $s(\oplus_{i=1}^{4} w_i\bar{h}_i)$ is equal to Eq. (17).

By solving the nonlinear optimization problem in Eq. (21), we obtain $w_1 = 0.3084$, $w_2 = 0.3220$, $w_3 = 0.1012$, $w_4 = 0.2684$, and $F(W) = 0.8770$.

(3) If the risk investor is a risk averter, then the qualitative portfolio model is presented as

$$F(W) = \max s\left(\oplus_{i=1}^{4} w_i\bar{h}_i\right)$$
$$s.t.\begin{cases} d\left(\oplus_{i=1}^{4} w_i\bar{h}_i\right) \le 0.0467 \\ \oplus_{i=1}^{4} w_i\bar{h}_i = \cup_{\bar{\gamma}_1\in\bar{h}_1,\bar{\gamma}_2\in\bar{h}_2,\bar{\gamma}_3\in\bar{h}_3,\bar{\gamma}_4\in\bar{h}_4}\left\{1-\prod_{i=1}^{n}(1-\bar{\gamma}_i)^{w_i}\right\} \\ \sum_{i=1}^{4} w_i = 1, w_i \ge 0 \end{cases} \tag{22}$$

where $s(h)$ and $d(h)$ are the score function and the deviation function of the HFE h, and $s(\oplus_{i=1}^{4} w_i\bar{h}_i)$ is equal to Eq. (17).

By solving the nonlinear optimization problem in Eq. (22), we can obtain $w_1 = 0.2843$, $w_2 = 0.3114$, $w_3 = 0.1456$, $w_4 = 0.2587$, and $F(W) = 0.8769$.

Step 6. Calculate the optimal invested funds k_i $(i = 1, 2, 3, 4)$ of the stocks x_i $(i = 1, 2, 3, 4)$ according to $k_i = K \cdot w_i$ $(i = 1, 2, 3, 4)$. Then, we have

(1) If the risk seeker wants to place $1000,000 into the four stocks $\{x_1, x_2, x_3, x_4\}$, then he/she should place $316,400, $325,600, $86,300, and $271,700 into x_1, x_2, x_3, and x_4 respectively.
(2) If the risk neutral investor wants to place $1000,000 into the four stocks $\{x_1, x_2, x_3, x_4\}$, then he/she should place $308,400, $322,000, $101,200, and $268,400 into x_1, x_2, x_3, and x_4 respectively.
(3) If the risk averter wants to place $1000,000 into the four stocks $\{x_1, x_2, x_3, x_4\}$, then he/she should place $284,300, $311,400, $145,600, and $258,700 into x_1, x_2, x_3, and x_4 respectively.

Thus, we obtain the optimal investment amounts k_i $(i = 1, 2, 3, 4)$, and construct the optimal portfolios of these stocks for the three types of risk investors.

6.2. Discussion of the results and further analysis

According to the calculations in Section 6.1, the optimal investment ratios are different for the three risk investors in Example 3. The optimal portfolios for the general investor and the three types of risk investors are shown in Fig. 8.

To further analyze the different results for these four investors and demonstrate the feasibility of the proposed qualitative portfolio models, the following comparisons are given:

(1) The optimal portfolios for the general investor and the risk seeker are the same in Example 3, as shown by the above calculation results and Fig. 8.. This phenomenon is inevitable based on the proposed qualitative portfolio models, because the additional constraint in Model 4 is ignorable for $D = \max D$. Thus, Model 4 is equal to Model 2, and the optimal portfolios for the general investor and the risk seeker are the same, which also can be verified by Example 2.
(2) The investment appetites of the four investors are similar: they all place the most money in x_2 and x_1, then x_4, and the least money in x_3. This phenomenon is reasonable as the maximum criteria values are {0.35, 0.65, 0.9} and {0.35, 0.95} in x_2 and x_1, respectively, compared with {0.1,0.7} and {0.75} in x_4 and x_3. Moreover, other criteria values are {0.35} and {0.65} in x_4, and {0.15} and {0.35} in x_3, suggesting that more funds should be placed in x_4 than x_3.
(3) The distributions of the optimal investment ratios for the three types of risk investors are different and the concentration level of the optimal investment ratios for the risk averter is higher than that of the others. This conclusion indicates that the risk averter tends to place his/her funds equally, whereas the risk seeker likes to place more funds into the better object. Thus, the risk seeker could obtain higher returns and bear higher risks than other risk investors, and vice versa.

Similar to the above calculation process, we can also obtain the optimal investment ratios for the three types of risk investors based on Model 5 and Definition 5. However, the optimal results based on Model 4 and Model 5 could differ owing to the different modeling rules.

The above conclusions suggest that the proposed portfolio selection approaches can be effectively used in qualitative risk investment. Some obvious conclusions are obtained, which also demonstrate the feasibility of the proposed approaches to risk investment.

7. Conclusions

To select the optimal portfolio among different stocks and other financial products when quantitative data are unavailable, this paper has proposed some qualitative portfolio models under the hesitant fuzzy environment. For the general investor, a qualitative portfolio model has been designed based on the max-score rule, which is similar to the max-returns rule under the conventional portfolio theory. For the risk investor, a qualitative portfolio model has been designed based on the score-deviation trade-off rule, which is similar to the return-risk trade-off rule under the conventional

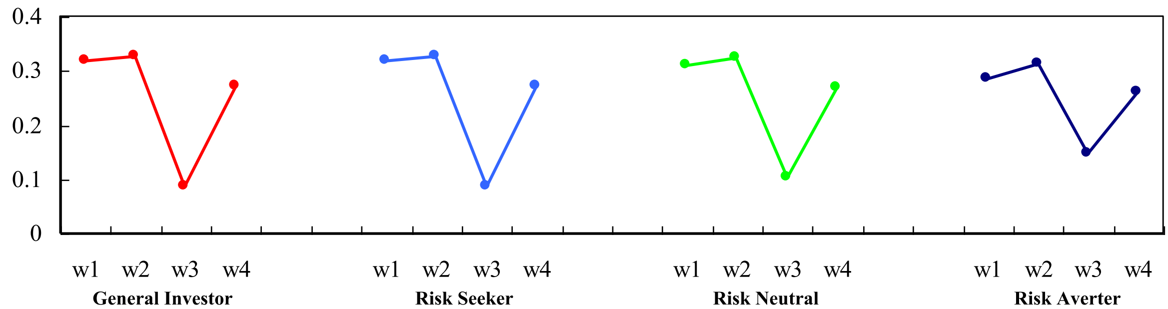

Fig. 8. Optimal portfolios for the four investors in Example 3.

portfolio theory. In consideration of the different risk appetites of the risk investors, this paper has further developed the deviation trisection approach and the score trisection approach to distinguish the risk seeker, the risk neutral investor, and the risk averter. Furthermore, three qualitative portfolio models for use by the three types of risk investors have been developed. The investment opportunities and efficient frontiers of these new portfolio models have been investigated to demonstrate their effectiveness. Also, the specific portfolio selection processes under the hesitant fuzzy environment have been summarized. Lastly, a risk investment example has been discussed to further prove the advantages of the proposed portfolio selection approaches.

These proposed approaches can be used in group qualitative portfolio selection, with the modeling process which is similar to Process A or Process B except Step 1. If these approaches are used to select the optimal portfolio based on the hesitant fuzzy evolution information provided by k decision makers, then k hesitant fuzzy matrices are obtained and the hesitant fuzzy aggregation technologies should be used to aggregate them into a collective hesitant fuzzy matrix. After that, the other modeling processes are the same in both Process A and Process B.

Research on the qualitative portfolio selection under the hesitant fuzzy environment is only at an early stage. The proposed qualitative portfolio selection approaches for the general investor and the three types of risk investors are hampered by certain limitations. Therefore, a great deal of future work remains.

Acknowledgement

This work was supported by the Natural Science Foundation of China (Nos. 71561026, 71571123, and 71771155), China Postdoctoral Science Foundation (Nos. 2015M570792 and 2016T90864), and Applied Basic Research Programs of Science and Technology Commission of Yunnan Province (No. 2017FB102).

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