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Arbitrary Polynomial Chaos for Uncertainty Propagation of Correlated Random Variables in Dynamic Systems

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Abstract: Dynamic simulation of stochastic systems requires uncertainty propagation. Traditional sample-based uncertainty propagation methods are often computationally intractable for online optimization-based estimation and control applications. Generalized polynomial chaos (gPC) is an efficient uncertainty propagation method that has been used for solving various nonlinear estimation and optimal control problems. However, gPC requires knowledge of the exact probability distribution of the random variables, and does not explicitly account for correlations between these random variables. This paper demonstrates the use of arbitrary polynomial chaos (aPC) for propagation of correlated multivariate random variables. aPC constructs orthogonal polynomial basis functions from only the raw moments of the random variables. Thus, aPC can be used for propagation of uncertainties with arbitrary probability distributions, even if their functional forms are unknown. The main contributions of this paper consist of presenting an algorithm for generating a set of orthogonal polynomial basis functions for correlated multivariate random variables and applying the Galerkin projection to compute closed-form expressions for the dynamics of the aPC expansion coefficients. An algorithm is also presented for efficient calculation of inner products between polynomial basis functions needed in the Galerkin projection. The error convergence properties of aPC are investigated and compared to that of gPC and Monte Carlo using a dynamic simulation case study.

Keywords: Uncertainty propagation, Arbitrary polynomial chaos, Correlated random variables

1. INTRODUCTION

Mathematical models are used for design, analysis, and control of systems in a wide range of engineering applications. A key consideration in mathematical modeling is characterization and propagation of model uncertainties that can arise from, for example, inadequate knowledge of model structure, parameters, initial conditions, and noise. In addition, some systems exhibit intrinsically uncertain dynamics and/or are under the effect of uncertain exogenous disturbances, which must also be described by the model. When the model and/or uncertainties are of probabilistic nature, uncertainty propagation poses a key challenge, particularly in online optimization-based estimation and control applications. The goal of this paper is to present a computationally efficient method for propagation of correlated random variables in dynamic systems described by general nonlinear differential equations.

Traditional sample-based uncertainty propagation methods such as Monte Carlo (MC) and Latin hypercube sampling techniques repeatedly simulate the system model for a large number of realizations to predict the time evolution of the probability density function (pdf) of system states (Metropolis and Ulam, 1949; Helton and Davis, 2003). However, these techniques can often be computationally expensive. Recently, generalized polynomial chaos (gPC) has been proposed for propagation of uncorrelated uncertainties in stochastic differential equations (Xiu and Karniadakis, 2002). In gPC, each state variable is approximated by an expansion of orthogonal polynomial basis functions (selected based on the known pdfs of the uncertainties) while the coefficients of the basis functions capture the dynamic evolution of the system. Polynomial chaos provides an efficient machinery for uncertainty propagation since the expansions can be used as a surrogate for the nonlinear model in order to perform MC simulations efficiently via evaluating the basis functions at the MC samples once offline. Alternatively, expansion coefficients can be used for computing the moments of stochastic states efficiently owing to the orthogonality of basis functions. However, the convergence of moments is guaranteed only for mean and variance (Ernst et al., 2012).

The gPC framework has been applied to various estimation (Li and Xiu, 2009; Blanchard et al., 2010; Dutta and Bhattacharya, 2010; Konda et al., 2011; Madankan et al., 2013; Bavdekar and Mesbah, 2016a) and optimal control problems (Fisher and Bhattacharya, 2009; Fagiano and Khammash, 2012; Mesbah et al., 2014; Paulson et al., 2014, 2015; Bavdekar and Mesbah, 2016b). However, gPC has two main shortcomings. First, one must have full knowledge of the pdfs of uncertainties to select the proper set of basis functions (often impractical in real systems). Second, the system uncertainties must be treated as *independent* random variables, which is often not true since the

coupled dynamic equations naturally lead to correlations between the system states and parameters. To address these shortcomings, this paper presents an arbitrary polynomial chaos (aPC) approach (Oladyshkin and Nowak, 2012; Navarro et al., 2014) for uncertainty propagation in nonlinear stochastic differential equations. In aPC, the polynomial basis functions are defined directly in terms of moments of the random variables, instead of based on their full pdfs. It is shown that aPC can explicitly account for correlations between the uncertainties. In the remainder of the paper, the aPC framework is presented and an algorithm is proposed for its efficient implementation for nonlinear dynamical systems with correlated uncertainties. The convergence properties of aPC are then compared to that of gPC and MC using a continuous-stirred-tank reactor case study.

2. PROBLEM STATEMENT

Consider a class of nonlinear dynamical systems described by a set of stochastic ordinary differential equations (ODEs)

$$\dot{x}(t,\theta) = f(x(t,\theta), u(t), \theta), \quad x(0,\theta) = x_0(\theta), \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the vector of states with initial values x_0, $u \in \mathscr{U} \subseteq \mathbb{R}^{n_u}$ is the vector of control inputs, and $\theta \in \mathscr{P} \subseteq \mathbb{R}^n$ is the vector of uncertain parameters. The set $\mathscr{U}$ represents constraints on the control inputs and $\mathscr{P}$ is the uncertainty set. The vector function $f : \mathbb{R}^{n_x} \times \mathscr{U} \times \mathscr{P} \to \mathbb{R}^{n_x}$ describes the dynamics of the system, and is assumed to be known.

Uncertainties θ can arise from a variety of sources and can be used, for example, to represent errors in the estimates of the initial conditions or model parameters. When measurement noise and disturbances are treated as stochastic, the majority of estimation methods will provide a pdf describing θ. Let f_θ denote the pdf of θ, which is characterized from data. In the context of stochastic optimization and stochastic optimal control, methods for efficient propagation of f_θ through dynamics (1) are required since traditional uncertainty propagation methods are generally considered to be computationally intractable for online optimization and control applications, and scale poorly with respect to problem size.

2.1 Generalized polynomial chaos

When $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ is a vector of *independent* random variables with finite variance, gPC can be used to represent the states as an infinite weighted sum of orthogonal polynomials (chosen from the Askey scheme)

$$x(t,\theta) = \sum_{i=0}^{\infty} a_i(t)\Phi_i(\theta), \tag{2}$$

where a_i denotes the time-dependent expansion coefficients and Φ_i denotes the corresponding basis functions. The function Φ_i is simplified notation for the multivariate polynomial basis for θ and, due to the independence of the elements $\theta_1, \theta_2, \ldots, \theta_n$, can be constructed as a product of their corresponding univariate polynomial basis

$$\Phi_i(\theta) = \prod_{j=1}^{n} \phi_{\alpha_j^{(i)}}(\theta_1, \theta_2, \ldots, \theta_n), \tag{3}$$

where ϕ_k is the univariate polynomial basis function of order k and $\alpha_j^{(i)}$ is a multivariate index that contains all possible product combinations of the univariate basis functions. The index $\alpha_j^{(i)} \in \{0, 1, \ldots\}$ can be thought of as the degree of parameter j in the expansion term i.

The functions ϕ are chosen from the Weiner-Askey scheme based on the distribution of each element of θ (Xiu and Karniadakis, 2002). By combining orthogonality conditions for each univariate basis, orthogonality conditions can be derived for the multivariate basis functions

$$\langle \Phi_i, \Phi_j \rangle = \langle \Phi_i^2 \rangle \delta_{ij}, \tag{4}$$

where $\delta_{ij} = 1$ if $i = j$ or $\delta_{ij} = 0$ if $i \neq j$ and the inner product $\langle \cdot, \cdot \rangle$ is defined as

$$\langle F(\theta), G(\theta) \rangle = \int F(\theta) G(\theta) f_\theta(\theta) d\theta.$$

In order to implement the gPC method numerically, the infinite expansion (2) must be truncated to a finite expansion. When the highest order polynomial allowed in the expansion (2) is d, then the total number of terms in the truncated expansion is given by the combinatorial formula

$$N + 1 = \frac{(n+d)!}{n!d!}. \tag{5}$$

This imposes a constraint on the multi-index

$$\sum_{j=1}^{n} \alpha_j^{(i)} \leq d, \quad i = 0, \ldots, N,$$

since all the remaining terms in (2) must have order less than or equal to d.

2.2 Galerkin projection

Several methods exist for determining the time evolution of gPC expansion coefficients $a_i(t)$. A commonly used method is collocation wherein a set of grid points is generated over the support space for θ, the ODEs (1) are solved at each grid point, and then the resulting solutions are fit to the expansion (2) using a least-squares method. Also, quadrature methods exist for approximating the integral form of (1). In control and optimization applications, the coefficients are a function of the input $u(t)$, which can result in high computational cost for collocation or quadrature methods due to the fact that many simulations/grid points are required for achieving high accuracy.

An attractive alternative to these grid-based approaches is the Galerkin projection method that directly uses the stochastic ODEs (1) by projecting the system along each of the orthogonal basis functions (Ghanem and Spanos, 1991). As such, the stochastic ODEs can be transformed *offline* into a set of deterministic ODEs. To this end, the truncated expansion (2) is substituted into (1)

$$\dot{a}_j(t) = \frac{1}{\langle \Phi_j^2 \rangle} \left\langle f\left(\sum_{i=0}^{N} a_i(t)\Phi_i(\theta), u(t), \theta \right), \Phi_j(\theta) \right\rangle, \tag{6}$$

for all $j = 0, \ldots, N$. Once the expansion coefficients $a_i(t)$ are solved for over the time window of interest, the gPC expansions for $x(t,\theta)$ can be used as a surrogate model for approximating the state pdfs via evaluating the basis functions at several realizations of the random variables

θ. This also allows for efficient computation of the time evolution of the moments of $x(t,\theta)$.

One key shortcoming of gPC is that the pdfs of the random variables θ must be known (and of a standard type such as normal or beta) for calculating the inner products. Fitting standard distributions to measurements may introduce a large source of error in the gPC basis functions. Furthermore, standard gPC requires the random variables to be independent in order to construct the multivariate basis using the product of univariate basis functions (3). However, correlations in parameters can often occur, for example, in parameter estimation since measurements typically contain information on multiple states and parameters. Even though correlated random variables can generally be transformed to independent random variables (Rosenblatt, 1952), gPC still requires knowledge of the functional form of the pdfs, which can limit its applicability.

3. ARBITRARY POLYNOMIAL CHAOS

Arbitrary polynomial chaos (aPC) addresses the shortcomings of gPC (Oladyshkin and Nowak, 2012; Navarro et al., 2014). aPC approximates the stochastic states similarly to (2), except that the polynomials $\{\Phi_i\}$ are constructed differently so that they are orthogonal for any arbitrary (possibly unknown) distribution. For notational clarity, we denote the aPC expansion by

$$x(t,\theta) = \sum_{i=0}^{\infty} b_i(t)\Psi_i(\theta) \approx \sum_{i=0}^{N} b_i(t)\Psi_i(\theta), \tag{7}$$

where b_i denotes the aPC expansion coefficients, Ψ_i denotes the aPC polynomial basis functions, and the number of terms in the truncated expansion N is given by (5). The goal is to construct $\{\Psi_i\}$ such that they are orthogonal with respect to a *correlated* multivariate pdf f_θ. Since N is finite in practice, the basis set can be generated from merely the raw moments of θ. These moments can be calculated, for example, using quadrature methods applied to empirical data, thus requiring no knowledge of the explicit form of the pdf f_θ. The aPC multivariate basis functions directly account for correlations in the uncertain random variables via the cross moments, as shown next.

3.1 Generation of basis functions

The set of basis functions $\{\Psi_i\}$ are constructed such that they are mutually orthogonal with respect to any arbitrary pdf f_θ. Starting from an arbitrary basis, the Gram-Schmidt orthogonalization method can be used to construct basis functions that are orthogonal with respect to the given inner product. To this end, a set of linearly independent polynomials must first be constructed. We use the set of monic polynomials $\{p_j(\theta)\}_{j=0}^N$ for simplicity (Navarro et al., 2014)

$$p_j(\theta) = \prod_{k=1}^{n} \theta_k^{\alpha_k^{(j)}}, \quad j = 0,\dots,N, \quad \sum_{k=1}^{n} \alpha_k^{(j)} \le d, \tag{8}$$

where $\alpha_k^{(j)}$ is the multi-index and d is the maximum order of the polynomials (as defined previously). We consider a simple example for illustrative purpose. When $n = d = 2$, the number of elements in the basis is $N + 1 = 6$. Hence, enumerating (8) yields a set of linearly independent monic polynomials $\{1, \theta_1, \theta_2, \theta_1^2, \theta_1\theta_2, \theta_2^2\}$.

According to the Gram-Schmidt method, the aPC polynomial basis functions are constructed as

$$\Psi_0(\theta) = 1, \tag{9}$$
$$\Psi_j(\theta) = p_j(\theta) - \sum_{k=0}^{j-1} c_{jk}\Psi_k(\theta), \quad 1 \le j \le N,$$

where the coefficients c_{jk} are defined by

$$c_{jk} = \frac{\langle p_j(\theta), \Psi_k(\theta)\rangle}{\langle \Psi_k^2(\theta)\rangle}. \tag{10}$$

Remark 1. The basis functions generated in (9) are not unique, as they are dependent on the choice and ordering of the polynomials $\{p_j(\theta)\}_{j=0}^N$.

3.2 Efficient calculation of c_{jk}

The main computational burden in the Gram-Schmidt method (9) is computing the coefficients c_{jk} using (10) for all $j = 1,\dots,N$ and $0 \le k < j$. Calculating the inner products in (10) typically hinges on the knowledge of the functional form of f_θ. However, the coefficients c_{jk} can also be computed via direct substitution of the moments of θ, as described below.

Let $\boldsymbol{\alpha}^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, \dots, \alpha_n^{(j)})$ be the index vector associated with the monic polynomial $p_j(\theta)$ in (8), whose entries are the integer exponents of $\theta = (\theta_1, \theta_2, \dots, \theta_n)$. Define the multivariate raw (cross-correlated) moment $\mu_\theta(\boldsymbol{\alpha}^{(j)})$ for a given $\boldsymbol{\alpha}^{(j)}$ as

$$\mu_\theta(\boldsymbol{\alpha}^{(j)}) = \int \theta_1^{\alpha_1^{(j)}} \theta_2^{\alpha_2^{(j)}} \cdots \theta_n^{\alpha_n^{(j)}} f_\theta(\theta) d\theta. \tag{11}$$

These raw moments for any multi-index $\boldsymbol{\alpha}^{(j)}$ can be determined completely offline using either samples (data binned into histograms to form an empirical pdf) or the functional form of the pdf $f_\theta(\theta)$.

Proposition 1. The inner product between polynomials $p_i(\theta)$ and $p_j(\theta)$ is given by

$$\langle p_i(\theta), p_j(\theta)\rangle = \mu_\theta(\boldsymbol{\alpha}^{(i)} + \boldsymbol{\alpha}^{(j)}). \tag{12}$$

Proof. From the definitions of the inner product and the monic polynomials $\{p_j(\theta)\}_{j=0}^N$ in (8), we have

$$\begin{aligned}\langle p_i(\theta), p_j(\theta)\rangle &= \int p_i(\theta)p_j(\theta)f_\theta(\theta)d\theta,\\ &= \int \prod_{k=1}^{n} \theta_k^{\alpha_k^{(i)}} \prod_{k=1}^{n} \theta_k^{\alpha_k^{(j)}} f_\theta(\theta)d\theta,\\ &= \int \prod_{k=1}^{n} \theta_k^{\alpha_k^{(i)}+\alpha_k^{(j)}} f_\theta(\theta)d\theta.\end{aligned}$$

The assertion follows from the definition of μ_θ in (11). □

The result of Proposition 1 can now be used to present a recursive algorithm for computing c_{jk}, as summarized in Algorithm 1. Within Algorithm 1, we take advantage of the fact that every Ψ_j can be represented as a linear combination of the monic polynomials

$$\Psi_j(\theta) = \sum_{k=0}^{j} w_{jk} p_k(\theta), \quad j = 0,\dots,N, \tag{13}$$

where w_{jk} are coefficients that must be determined.

Algorithm 1 Computation of c_{jk} from raw moments.

1: **Initialize:** $w_{00} = \langle \Psi_0^2 \rangle = 1$ and set counter $j = 1$.
2: **while** $j \leq N$ **do**
3: Extract coefficients $\{w_{jk}\}_{k=0}^{j-1}$ using the monic polynomials and (9).
4: For every $k = 0, 1, \ldots, j-1$, calculate

$$c_{jk} = \frac{1}{\langle \Psi_k^2 \rangle} \sum_{l=0}^{k} w_{kl} \mu_\theta(\boldsymbol{\alpha}^{(j)} + \boldsymbol{\alpha}^{(l)}).$$

5: Calculate the inner product using

$$\langle \Psi_j^2 \rangle = \mu_\theta(2\boldsymbol{\alpha}^{(j)}) - \sum_{k=0}^{j-1} c_{jk}^2 \langle \Psi_k^2 \rangle.$$

6: Increment index $j \leftarrow j + 1$.
7: **end while**

3.3 Galerkin projection for aPC

Once the aPC orthogonal polynomial basis set $\{\Psi_j(\theta)\}_{j=0}^N$ is constructed, it can be used for uncertainty propagation of dynamic systems of the form (1) in a similar manner to gPC. The Galerkin projection for aPC will have the same form as (6), however, with $\Phi_i(\theta) \leftarrow \Psi_i(\theta)$ replaced for all $i = 0, \ldots, N$. As in gPC, a key consideration is how to compute the inner products that appear in the Galerkin projection. When the system equations f have a polynomial (or rational) form, these inner products can be rewritten in terms of the raw moments μ_θ. In this case, the right-hand side of (6) will be a polynomial function of θ. Hence, the Galerkin projection would involve computation of the inner product of multiple basis functions.

For example, if $f(x(t,\theta), u(t), \theta) = A(\theta)x(t,\theta) + B(\theta)u(t)$ is a linear function, then all of the triple products $\langle \Psi_i(\theta)\Psi_j(\theta)\Psi_k(\theta) \rangle$ must be computed (e.g., see (Paulson et al., 2014) for the details). Instead of performing the integration directly, we can take advantage of (13) to rewrite the inner products in terms of the known raw moments of θ, which greatly reduces their computation. The triple product of the basis functions is explicitly defined by

$$\langle \Psi_i \Psi_j \Psi_k \rangle = \sum_{l=0}^{i} \sum_{m=0}^{j} \sum_{q=0}^{k} w_{il} w_{jm} w_{kq} \langle p_l p_m p_q \rangle, \qquad (14)$$

where, by Proposition 1, the monomial inner product $\langle p_l p_m p_q \rangle = \mu_\theta(\boldsymbol{\alpha}^{(l)} + \boldsymbol{\alpha}^{(m)} + \boldsymbol{\alpha}^{(q)})$. Higher-order products are calculated in a similar manner. All the inner products (of any finite order) can be computed offline when the aPC basis set is constructed using Algorithm 1. Note that all the inner product computations in aPC rely on the raw moments of θ only, as opposed to gPC that requires the exact knowledge of f_θ.

3.4 aPC algorithm for uncertainty propagation

We can now present the aPC algorithm for propagation of correlated random variables through the nonlinear system (1). Using (7), the uncertainty propagation reduces to determining the time evolution of $b_i(t)$. This involves deriving a set of deterministic ODEs for $b_i(t)$ obtained via aPC and the Galerkin projection, as summarized in Algorithm 2.

Algorithm 2 aPC algorithm for uncertainty propagation.

1: Calculate $\mu_\theta(\boldsymbol{\alpha})$ for all required multi-indices $\boldsymbol{\alpha}$, which is a function of n and d.
2: Generate a list of monic polynomials $\{p_j(\theta)\}_{j=0}^N$ according to procedure defined by (8).
3: Calculate a set of coefficients $\{c_{jk}\}$ using Algorithm 1. Store $\{w_{jk}\}$ obtained during these calculations.
4: Construct the aPC orthogonal basis set $\{\Psi_j(\theta)\}_{j=0}^N$ using the coefficients $\{c_{jk}\}$ as shown in (9).
5: Substitute the aPC expansion (7) into stochastic ODEs (1). Perform the Galerkin projection to derive (6) with $a_i(t) \leftarrow b_i(t)$ and $\Phi_i(\theta) \leftarrow \Psi_i(\theta)$.
6: Substitute raw moments μ_θ for the inner products on the right-hand side of (6) using, e.g., (14) (as well as higher-order versions of this expression).

Several aspects of aPC remain open research questions. For example, the convergence properties of aPC have not been theoretically established thus far. In addition, the construction of the aPC basis functions and inner products for the Galerkin projection may require a large number of moments, which can be difficult to determine accurately when computed numerically from data. This may introduce some error into the uncertainty propagation, which must be accounted for.

4. CASE STUDY

To demonstrate the accuracy of aPC for uncertainty propagation, Algorithm 2 is applied to a continuous-stirred-tank reactor (CSTR) with correlated uncertain kinetic parameters. The chemical reactions in the CSTR involve a reactant species A being converted to a desired product B and an undesired byproduct C

$$\mathrm{A} \xrightarrow{k_1} \mathrm{B}, \quad 2\mathrm{A} \xrightarrow{k_2} \mathrm{C}.$$

The dynamics of the CSTR are described by

$$\begin{bmatrix} \dot{C}_A \\ \dot{C}_B \\ \dot{C}_C \end{bmatrix} = \begin{bmatrix} F(C_{A,in} - C_A) - k_1 C_A - k_2 C_A^2 \\ -F C_B + k_1 C_A \\ -F C_C + k_2 C_A^2 \end{bmatrix},$$

where C denotes concentration, $C_{A,in}$ is the inlet concentration of A, k_1 and k_2 are kinetic rate constants, and F is the inlet flow rate to the reactor (control input $u(t)$). The initial conditions for the CSTR are $C_A(0) = 0.8$ mol/L, $C_B(0) = 0.5$ mol/L, and $C_C(0) = 0$ mol/L. The kinetic parameters k_1 and k_2 are normally distributed

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \bar{k}_1 \\ \bar{k}_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \right),$$

where $\bar{k}_1 = 50.61$ h^{-1}, $\bar{k}_2 = 55.15$ $\frac{\mathrm{L}}{\mathrm{mol\ h}}$, $\sigma_1^2 = 25.62$ h^{-2}, $\sigma_2^2 = 30.41$ $\frac{\mathrm{L}^2}{\mathrm{mol}^2\ \mathrm{h}^2}$, and $\rho = 0.95$.

The nominal CSTR model, with $k_1 = \bar{k}_1$ and $k_2 = \bar{k}_2$, was first used to design an optimal piecewise linear inlet flow rate profile by solving a dynamic optimization problem aimed at maximizing C_B while ensuring $C_C < 0.17$ mol/L. The optimal input profile was then used to compare the accuracy of aPC for uncertainty propagation (i.e., correlated random variables k_1 and k_2) to that of gPC and MC simulations with 10,000 realizations of the random variables. aPC and gPC expansions were used as

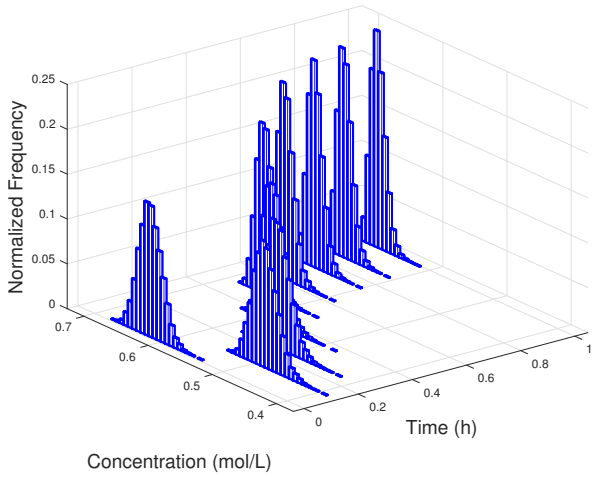
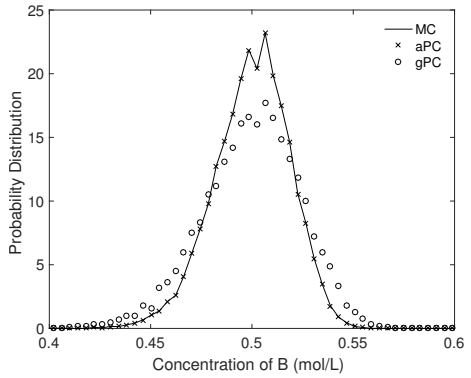


Fig. 1. Time evolution of the pdf of C_B, determined using 10,000 MC simulations. The control inputs are specified as the solution to an open-loop optimal control problem based on the nominal system.

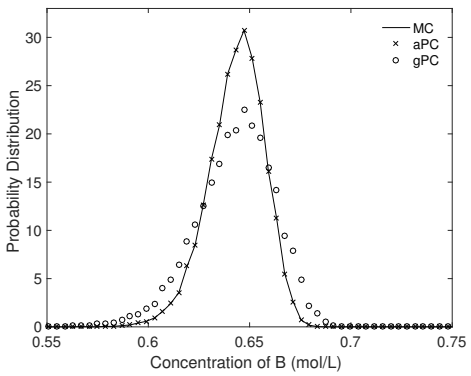
surrogates for the solution to the stochastic ODEs with state pdfs constructed through sampling. For consistency, the same uncertainty realizations were used in the three cases. Note that since the random variables k_1 and k_2 are correlated, the multivariate polynomial basis functions are defined from the cross-correlated moments of k_1 and k_2 in aPC. In gPC, however, Hermite polynomials were used to define the basis functions (based on the marginal distributions of k_1 and k_2, neglecting the correlation) since the random variables are normally distributed.

The time evolution of the pdf of C_B using the open-loop optimal control inputs is shown in Fig. 1. Fig. 2 shows a comparison of the pdf of C_B constructed by the MC simulations, aPC, and gPC at time points 0.1 h, 0.5 h, and 1.0 h. The pdfs obtained from aPC almost exactly match those predicted by the MC simulations, while the pdfs obtained from gPC do not accurately represent the MC simulation results. The convergence properties of these two methods were also evaluated against the MC simulation results. The normalized errors in the mean and variance of C_B were computed at multiple time points corresponding to the dynamic and steady-state behavior of the CSTR. In addition, the Kolmogorov distance (Kolmogorov, 1950), which provides a measure for the degree of overlap between pdfs, was used to compare the pdf convergence for aPC and gPC with respect to the MC simulation results.

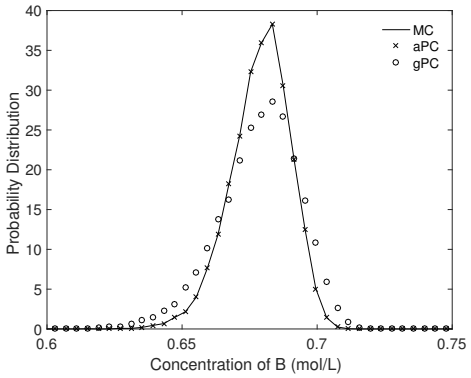
Fig. 3 shows the convergence errors for aPC and gPC as a function of the order of polynomial basis. As the order rises, the convergence errors are expected to decrease. As can be seen, aPC consistently shows significantly better error convergence than gPC (note the logarithmic scale of the y-axis). In fact, a polynomial basis of order 2 in aPC has an error approximately 3 orders of magnitude less than that of a polynomial basis of order 5 for gPC. The error in the mean and variance only slightly decreases with increasing the gPC basis order. The same error convergence trends were also observed for C_A and C_C (not shown here). Note that aPC and gPC showed comparable online computational burden (on the order of seconds), which grew as a function of the order of polynomial basis.



(a) t = 0.2 h



(b) t = 0.5 h



(c) t = 1.0 h

Fig. 2. Probability distribution of C_B obtained from MC simulations, aPC, and gPC at three time points.

5. CONCLUSIONS AND FUTURE WORK

This paper demonstrates the use of aPC for uncertainty propagation of correlated random variables through nonlinear differential equations. Unlike gPC, aPC does not require exact knowledge of the pdf of the random variables and is able to directly account for correlations between random variables. Simulation results suggest that aPC can accurately propagate the mean, variance, and pdf of the states in time. This work reveals the promise of aPC as an efficient uncertainty propagation method for optimization-based estimation and control applications, which will be investigated in our future work. The theoretical error convergence properties of aPC will also be investigated.

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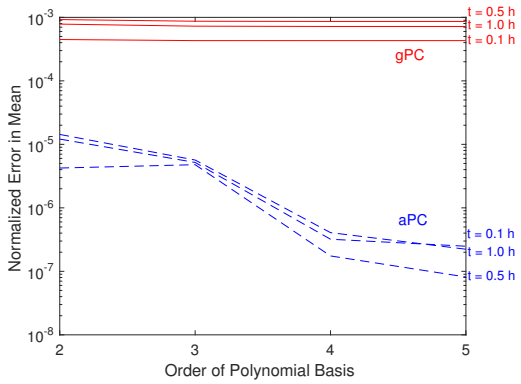
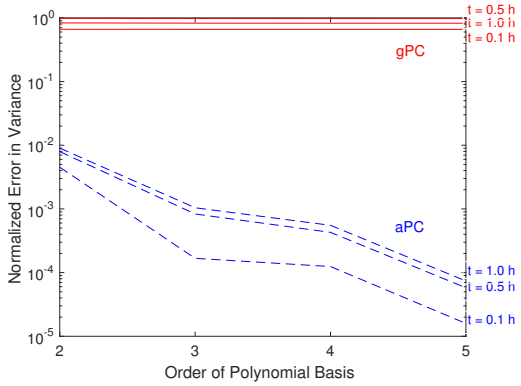
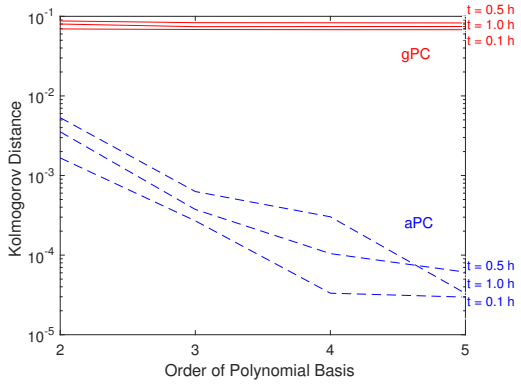


(a) Error in mean



(b) Error in variance



(c) Error in pdf based on Kolmogorov distance

Fig. 3. Convergence errors for aPC and gPC (with respect to MC) for the mean, variance, and pdf of C_B as a function of the order of the polynomial basis.

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