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COMBINED OPTIMUM CONTROL AND ESTIMATION THEORY

by Lewis Meier

Prepared under Contract No. NAS 2-2457 *by*
STANFORD RESEARCH INSTITUTE
Menlo Park, Calif.
for Ames Research Center

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ABSTRACT

This report presents the theory of the combined optimization problem, which is the problem of determining optimal controls on the basis of noisy measurements on a randomly disturbed plant. Solution of this problem involves elements of optimal control and optimal estimation and may be reduced to the solution of two recursion relations, the control equation and the estimation equation. Detailed discussion of an important special case, the linear case is given and examples of this and the general case are presented.

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SUMMARY

The present report describes part of the work performed by Stanford Research Institute for the NASA Ames Research Center, Mountain View, under Contract NAS 2-2457.

Combined optimal control and estimation is synthesis of control systems that maximize performance when only noisy and incomplete information about the state of the plant to be controlled is available. This problem contains elements of both optimum control and optimum estimation theory. An exact solution to this combined problem is derived, and it is shown that, with the proper definitions, the optimum system consists of simply cascading an optimum estimator with an optimum controller.

A complete literal solution of the problem is given for the special case of linear systems, Gaussian noises, and quadradic performance measures. Examples are presented for this case illustrating how the theory can be used to investigate the relation between the characteristics of information-handling components and systems performance.

For the nonlinear case, where literal solutions do not exist, a simple example is presented to demonstrate the numerical solution of the recursive equations governing control and estimation. This example also illustrates Feldbaum's "dual-control" concept; that is, the idea that control action may be used to gain information as well as control the plant.

I INTRODUCTION

Since October 1964, Stanford Research Institute has been performing research for the NASA Ames Research Center, Moffett Field, California, under Contract NAS 2-2457.

The central objective of this study is to relate the performance of a control or guidance system to the information-handling characteristics of its key components, notably its measurement subsystem, with the aim of developing improved exact and approximate synthesis techniques. In the course of the study, it became apparent that a unified theoretical framework was needed to handle such questions (ref. 1). One such framework is the theory of optimal control and estimation, which is discussed in the present report.

A major drawback to optimal control theory is that it requires knowledge of the complete state of the plant. In classical control theory—and, more often than not, in actual practice—only some of the state variables are measured. Optimal control theory can be generalized to consider such situations—the result is the theory of combined optimal control and estimation (see fig. I-1). In this problem, a plant is given and measurements on this plant are available. It is desired to determine the inputs to the plant on the basis of the measurements in a manner that optimizes system performance. The stochastic and deterministic optimal control problems, as well as the optimal estimation problem, are special cases of the combined optimal control and estimation problem.

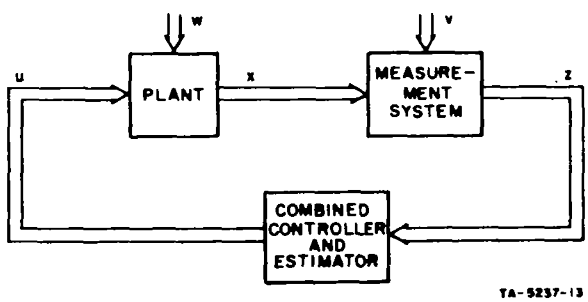

FIG. I-1 COMBINED CONTROL AND ESTIMATION

Since the conditional probability density of the state summarizes all information presently available about the future behavior of the system, the solution of the combined optimal estimation and control problem can be divided into two parts—control, which is the selection of the optimum input to the plant as a function of the conditional probability density of the state of the plant, and estimation, which is the computation of this conditional probability density. These two aspects of the problem may be solved by means of two recursive equations: the control equation generated by dynamic programming, and the estimation equation generated by application of Bayes' rule.

If the system is linear, the random effects Gaussian, and the performance quadratic, then the problem greatly simplifies and the solution is well known (refs. 2, 3, and 4). In this case, the optimal control is just a linear function of the conditional mean of the state of the plant. This linear function may be found by solving the optimal control problem under the assumption that the state is known. The conditional mean may be generated by a linear system which is, in fact, the Kalman filter (ref. 4). In the general case, unfortunately, all moments of the conditional probability distribution are needed for the computation of the control at a given time; however, approximations may be made and will be investigated in detail in future work.

The basic reason feedback is used in control systems is because there exists uncertainty about the plant to be controlled. Therefore, sensors are used to gain this missing information. Even with sensors, uncertainty is present because of sensor imperfections; however, by use of filtering techniques and control action, this uncertainty can be reduced and performance can be increased.

Because of the cost of measurement and computation, one wants to know how complex the techniques used to gain formation about the plant have to be to give adequate control. Intuitively, the answer depends upon two factors—how large the uncertainty about the plant is without measurement (*i.e.*, how little *a priori* information there is) and how much value is gained by reducing this uncertainty. By comparing optimal performances when ideal and when real (nonideal) measurement system components are used, these qualitative concepts may be made quantitative. The linear case, for which the solution is simple, provides an excellent example for illustrating the value and the uncertainty aspects of control problems.

Since, in general, there are many ways of getting the same information for a given control problem, there will also be many configurations that give the same level of performance. Combined optimization theory provides a method of comparing alternative configurations. Furthermore, in many cases it will be possible to simplify the controller and estimator without degrading performance. Because a fixed-structure system can be looked upon as a plant to be controlled that has no control inputs, the equations developed for solving the optimal control and estimation problem may be used to calculate the performance of systems containing simplified and suboptimal versions of the optimal controller-estimator. In general, the presence of *a priori* information reduces the need for gaining information about the system and thus allows simplification of the measurement system and controller estimator.

The author would like to express his gratitude for the many fruitful discussions with his colleagues, John Peschon, Robert Larson, Phil Merritt, Edward Fraser, and Wade Foy at Stanford Research Institute and with Brain Doolin, Elwood Stewart, Rodney Peery and Gerald Smith of Ames Research Center.

II GENERAL THEORY

In this section, the general theory of combined optimal control and estimation is presented. After stating the problem, we derive two recursion relations—the control equation and the estimation equation, which together provide the complete theoretical solution of the problem.

A. PROBLEM STATEMENT

The purpose of this part is to state mathematically the combined optimal control and estimation problem discussed in general terms in Sec. I.

1. Notation

Before proceeding further, it is desirable to describe briefly the notation used in this report. The symbol $p(x/y)$ denotes the probability density of x given the value of y. Similarly, $E\{x/y\}$ represents the expected value of x given y. Subscripts indicate time, *i.e.*, x_k is the value of x at the kth time instant. Lower-case letters are used for vectors and upper-case letters for matrices. Components of vectors and matrices are denoted by superscripts in parentheses, *i.e.*, $x^{(k)}$ is the kth component of the vector x. The capital letter U_k represents the set $(u_k, \ldots, u_0)$ for any time-dependent vector u_i. Finally, in the investigation of the linear case, circumflexes ($^\wedge$) are used to distinguish quantities related to estimation from similar quantities related to control.

2. Statement of the Combined Optimal Estimation and Control Problem

The statement of the problem is given

(1) The plant, described by

$$x_{k+1} = f_k(x_k, u_k, w_k) \tag{2-1}$$

where

x_k is the state vector

u_k is the input vector

w_k is the disturbance vector.

(2) The measurement system, described by

$$z_k = h_k(x_k, v_k) \quad , \tag{2-2}$$

where

z_k is the measurement vector

v_k is the measurement noise vector

(3) The probability distributions

$$\text{(a)} \quad p(x_0) \tag{2-3a}$$

$$\text{(b)} \quad p(w_i) \qquad i = 0, \ldots, N \tag{2-3b}$$

$$\text{(c)} \quad p(v_i) \qquad i = 0, \ldots, N \tag{2-3c}$$

(4) The assumption that w_i and v_i are independent and white and that x_0 is independent of both w_i and v_i, that is

$$p(x_0, w_i, v_j) = p(x_0)p(w_i)p(v_j) \tag{2-4}$$

(5) The performance index

$$J = E \sum_{i=0}^{N} l(x_i, u_i, i) \tag{2-5}$$

(6) The admissibility constraints

$$u_i \in \Omega_i \tag{2-6}$$

Find:

The admissible combined controller and estimator that minimizes J, where

(1) A combined controller and estimator is defined as any algorithm which at time k generates u_k as a function of the present and all past measurements $(z_k, \ldots, z_0)$.

(2) An admissible controller and estimator is defined as any controller and estimator which, when used in the closed-loop system shown in Fig. I-1, yields admissible u_i.

For simplicity the combined optimal control and estimation problem are referred to as the *combined optimization problem* in what follows.

3. Discussion

a. EXISTENCE OF SOLUTION

The stated problem has a solution only if there exist admissible combined controllers and estimators. From the state and measurement equations (2-1) and (2-2) and the given probability distributions, it is possible to calculate all of the conditional probabilities needed to evaluate J for any admissible combined controller and estimator.

b. OTHER TREATMENTS

Several special cases of the combined optimization problem have been treated extensively in the past. If $z_k \equiv x_k$, then the problem reduces to the stochastic optimal control problem (ref. 5); and, if in addition $w_k \equiv 0$, then the problem reduces to the conventional optimal control problem. The optimal estimation problem results when f is independent of u; in this case u_k, the computer output, is the best estimate of x_k under the criteria provided by J.

Gunckel (ref. 2), Tou and Joseph (ref. 3), and Kalman (ref. 4) have solved the combined problem under the conditions that (2-1) and (2-2) are linear; that x_0, v_k, and w_k are Gaussian; and that $l(x_k, u_k, k)$ is quadratic. Sussman (ref. 6) and Aoki (ref. 7) have considered the general problem as stated above; the development given here was obtained concurrently and independently and is similar to theirs. Feldbaum (refs. 8, 9) treats the same problem in a different, but entirely equivalent, formulation. (Section V contains a demonstration that Feldbaum's dual-control problem and the combined optimization problems are the same.) Kushner and Wonham have done considerable work on the continuous time variant of this problem; refs. 10 and 19 contain bibliographies listing their work as well as related work by other authors.

c. RELATION TO THE INFORMATION REQUIREMENTS IN CONTROL SYSTEMS

The typical control system, Fig. II-1, will contain such information handling components as sensors, communications channels, and actuators. In general, not only the plant, but these devices also, include dynamics (ref. 1). It is a simple matter to treat such a situation as a special case of the combined optimization problem by augmenting the state vector of the given plant with the state variables governing the information handling components. Similarly, adaptive and learning control problems may be treated by augmenting the state to include incompletely specified parameters.

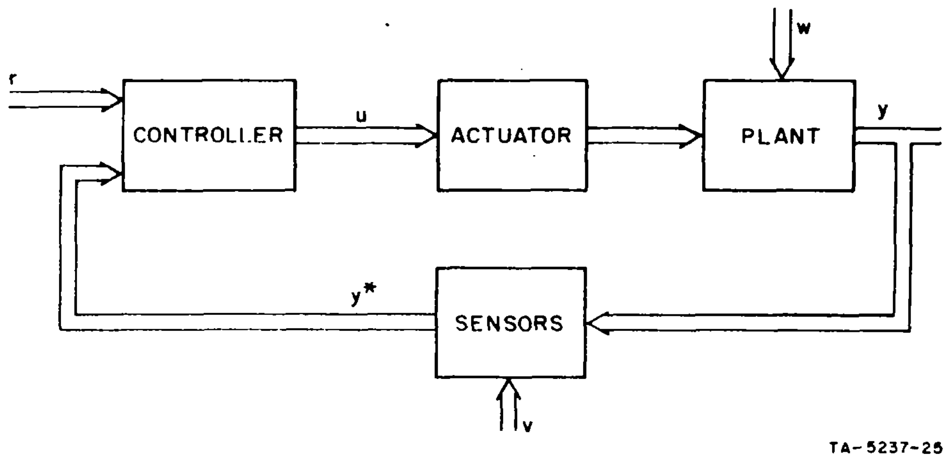

FIG. II-1 CONTROL SYSTEM CONFIGURATION
r = command input, y = sensor input, y* = sensor output,
w = perturbation, u = control decision, v = measurement noise

The combined optimization problem provides an excellent framework for evaluating the effects of information-handling components on system performance. The optimum combined controller and estimator is first synthesized for ideal information-handling components (*i.e.*, for $z_k = x_k$) and next for the actual components. The degradation in performance is then obtained by comparing the performances for the two cases. Sometimes, it is possible to get an analytic expression for performance as a function of sensor characteristics, while in other cases, simulation techniques are needed. By fixing the desired performance, one can specify the sensor characteristics as well as the trade-offs between sensor configurations.

In Sec. IV we will consider, in detail, the analysis of the effect of information-handling components on system performance.

B. CONTROL EQUATION

The solution to the combined optimization problem is a set of functions $u_0(Z_0)$, $u_1(Z_1)$, ..., $u_N(Z_N)$. Rather than finding these functions, we prefer to find their values for each possible argument, thus reducing a minimization over functions to a minimization over values. This process is accomplished by application of dynamic programming. Two steps are necessary—first, the given problem is embedded in a series of similar problems in which the initial time is not zero but varies from 0 to N, and second, a recursion relation is developed for solution of the series.

1. EMBEDDING

To perform the embedding process, consider the series of modified combined optimization problems with the following cost functions:

$$J_k \stackrel{\triangle}{=} E\left\{\sum_{i=k}^{N} l(x_i, u_i, i)/Z_{k-1}, U_{k-1}\right\} \qquad 0 < k \leq N$$

$$J_0 \stackrel{\triangle}{=} J = E\left\{\sum_{i=0}^{N} l(x_i, u_i, i)\right\} \quad . \tag{2-7}$$

The new problems may be interpreted as combined optimization problems in which the first $k - 1$ controls have been picked arbitrarily and for which we must select the remaining controls in an optimal fashion.

Consider $k = N$ with

$$J_N = E\{l(x_N, u_N, N)/Z_{N-1}, U_{N-1}\} \quad . \tag{2-8}$$

To solve this problem, we must find

$$\min_{u_N(z_N)} J_N \quad . \tag{2-9}$$

Note that since J_N contains Z_{N-1} and U_{N-1} as parameters the result will be a specification of u_N as a function of Z_N and U_{N-1}.

We wish to replace the minimization over functions (2-9) by a minimization over values. In Appendix A, it is shown that

$$\underset{x}{E}\{F(x)\} = \underset{y}{E}\{\underset{x}{E}[F(x)/y]\} \quad . \tag{2-10}$$

where the symbol under the E indicates which variable the expectation is taken with respect to. To make use of this formula, the expectation is expressed as an integral:

$$\min_{u_N(z_N)} J_N = \min_{u_N(z_N)} \underset{z_N}{E}\{\underset{x_N}{E}[l(x_N,u_N,N)/Z_N,U_{N-1}]/Z_{N-1},U_{N-1}\}$$

$$= \min_{u_N(z_N)} \int_{z_N} \underset{x_N}{E}\{l(x_N,u_N,N)/z_N,Z_{N-1},U_{N-1}\}p(z_N/Z_{N-1},U_{N-1})dz_N$$

$$= \int_{z_N} \min_{u_N} \underset{x_N}{E}\{l(x_N,u_N,N)/z_N,Z_{N-1},U_{N-1}\}p(z_N/Z_{N-1},U_{N-1})dz_N$$

$$= \underset{z_N}{E}\{\min_{u_N} \underset{x_N}{E}[l(x_N,u_N,N)/Z_N,U_{N-1}]Z_{N-1},U_{N-1}\} \quad . \tag{2-11}$$

The minimization over functions has thus been replaced by a minimization over values.

If $I^*(Z_k,U_{k-1},k)$ is defined by

$$I^*(Z_k,U_{k-1},k) = \min_{u_k,u_{k+1}(z_{k+1}),\ldots,u_N(z_N,\ldots,z_{k+1})} E\left\{\sum_{i=k}^{N} l(x_i,u_i,i)/Z_k,U_{k-1}\right\} \quad , \tag{2-12}$$

then (2-11) becomes

$$\min_{u_N(z_N)} J_N = \underset{z_N}{E}\{I^*(Z_N,U_{N-1},N)/Z_{N-1},U_{N-1}\} \quad . \tag{2-13}$$

If the analysis given above for $k = N$ is repeated for general k, the result is

$$\min_{u_k(z_k),u_{k+1}(z_{k+1},z_k),\ldots,u_N(z_N,\ldots,z_k)} J_k = \underset{z_k}{E}\{I^*(Z_k,U_{k-1},k)/Z_{k-1},U_{k-1}\} \quad . \tag{2-14}$$

In particular, if we can find $I^*(Z_0,0)$, then we will have solved the original problem.

2. Recursion Relation

Since the first term of the summation in (2-12) does not depend upon u_i for $i > k$, (2-12) may be rewritten [with the aid of (2-7)]:

$$I^*(\cdot,k) = \min_{u_k}\left[E[l(x_k,u_k,k)/Z_k,U_{k-1}] + \min_{u_{k+1}(\cdot),\ldots,u_N(\cdot)} E\left\{\sum_{i=k+1}^{N} l(x_i,u_i,i)/Z_k,U_k\right\}\right]$$

$$= \min_{u_k}\left\{E[l(x_k,u_k,k)/Z_k,U_{k-1}] + \min_{u_{k+1}(\cdot),\ldots,u_n(\cdot)} J_{k+1}\right\} , \qquad (2\text{-}15)$$

where the arguments have been suppressed for simplicity. Note that because the minimization has been split into two parts, u_k must be added to the conditioning variables, *i.e.*,

$$\min_{x,y} E\{F\} \qquad \min_{x} [\min_{y} E\{F/x\}] \qquad (2\text{-}16)$$

If (2-14) is substituted into (2-15) the following recursion relation finally results:

$$I^*(Z_k,U_{k-1},k) = \min_{u_k}\left\{\underset{x_k}{E}[l(x_k,u_k,k)/Z_k,U_{k-1}] + \underset{x_{k+1}}{E}[I^*(Z_{k+1},U_k,k+1)/Z_k,U_k]\right\}$$

$$= \min_{u_k}\left\{\underset{x_k}{E}\left[l(x_k,u_k,k) + \underset{w_k,v_{k+1}}{E}\left(I^*\{h_{k+1}[f_k(x_k,u_k,w_k),v_{k+1}],Z_k,U_k,k+1\}\right)/Z_k,U_k\right]\right\}$$

$$0 \leq k < N \qquad (2\text{-}17)$$

where the last step is performed by use of (2-1), (2-2) and (2-14).

3. Discussion

By use of (2-12) and (2-17), the combined optimal control and estimation problem may be solved as follows: From (2-12)

$$I^*(Z_N,U_{N-1},N) = \min_{u_N} \underset{x_{N-1}}{E}[l(x_N,u_N,N)/Z_N,U_{N-1}] \qquad (2\text{-}18)$$

If the minimization is performed for each value of the argument, u_N can be found as a function of Z_N and U_{N-1}. (This will of course usually require quantization of the variables.) The resulting $I^*(\cdot,N)$ may be substituted into (2-17) and $I^*(\cdot,N-1)$ and $u_{N-1}(\cdot)$ may be found, and so on. The result is the series of functions

$$u_0 = u_0(z_0)$$

$$u_1 = u_1(z_1, z_0, u_0)$$

$$u_2 = u_2(z_2, z_1, z_0, u_1, u_0)$$

$$\vdots \qquad \vdots$$

$$u_N = u_N(z_N, \ldots, z_0, u_{N-1}, \ldots, u_0) \tag{2-19}$$

This set of optimal controls depends upon the past controls as well as the past and present measurements; however, by substitution of the first equation into the remainder, u_0 can be eliminated and so on; the result is a set of optimal controls, each based on the past and present measurements at the time of application.

At each step, the minimization is performed over only those controls leading to admissible values of u_i and x_{i+1}. If at any step no such control exists, then the problem is unsolvable as formulated.

4. Optimal Control

If $z_k = x_k$, then we have the optimal control problem. Since the future behavior of the system is governed only by x_k, and future inputs u_i and disturbances w_i, $i \geq k$; the recursion relation (2-17) becomes

$$I^*(x_k, k) = \min_{u_k} \left(l(x_k, u_k, k) + \underset{w_k}{E}\{I^*[f_k(x_k, u_k, w_k), k+1]\}\right) \quad , \tag{2-20}$$

which is the functional equation for the stochastic optimal control problem derived by Bellman (ref. 5). If no disturbance inputs exist, then (2-20) reduces to

$$I^*(x_k, k) = \min_{u_k} \{l(x_k, u_k, k) + I^*[f_k(x_k, u_k), k+1]\} \quad , \tag{2-21}$$

which is the recursion relation for the optimal control problem.

C. ESTIMATION EQUATION

Calculation of the expectations appearing in (2-17) and (2-18) requires knowledge of $p(x_k/Z_k, U_{k-1})$. In this part we will show how this probability distribution may be calculated in a recursive manner by application of Bayes's rule.

1. Recursion Relation

We assume that $p(x_k/Z_k, U_{k-1})$ is known and compute $p(x_{k+1}/Z_{k+1}, U_k)$, from it and the given probability distributions. From Bayes's rule

$$p(x_{k+1}/Z_{k+1}, U_k) = \frac{p(z_{k+1}/x_{k+1}, Z_k, U_k)\, p(x_{k+1}/Z_k, U_k)}{p(z_{k+1}/Z_k, U_k)} \quad . \tag{2-22}$$

Because v_i is white and because of the absence of dynamics in the measurement equation (2-2), z_{k+1} is independent of past measurements and inputs if x_{k+1} is given:

$$p(z_{k+1}/x_{k+1}, Z_k, U_k) = p(z_{k+1}/x_{k+1}) \quad . \tag{2-23}$$

This probability density may be calculated from knowledge of $p(v_{k+1})$ and the measurement equation (2-2). From the properties of marginal distributions

$$p(x_{k+1}/Z_k, U_k) = \int_{x_k} p(x_{k+1}, x_k/Z_k, U_k)\, dx_k \quad . \tag{2-24}$$

Again, from Bayes's rule,

$$\begin{aligned} p(x_{k+1}, x_k/Z_k, U_k) &= p(x_{k+1}/x_k, Z_k, U_k)\, p(x_k/Z_k, U_k) \\ &= p(x_{k+1}/x_k, u_k)\, p(x_k/Z_k, U_{k-1}) \end{aligned} \tag{2-25}$$

since the present state is independent of the present input and the next state is independent of past inputs and measurements if the present state and input are given. From the state equation (2-1) and from the known $p(w_k)$, $p(x_{k+1}/x_k, u_k)$ can be calculated. The denominator in (2-22) is the integral of the numerator; hence from (2-22), (2-23), (2-24) and (2-25), the following recursive relation is finally obtained:

$$p(x_{k+1}/Z_{k+1}, U_k) = \frac{p(z_{k+1}/x_{k+1}) \int_{x_k} p(x_{k+1}/x_k, u_k)\, p(x_k/Z_k, U_{k-1})\, dx_k}{\int_{x_{k+1}} p(z_{k+1}/x_{k+1}) \int_{x_k} p(x_{k+1}/x_k, u_k)\, p(x_k/Z_k, U_{k-1})\, dx_{k+1}}$$

$$0 \leq k < N \quad . \tag{2-26}$$

where

$$p(x_0/Z_0,U_{-1}) \triangleq p(x_0/z_0) = \frac{p(z_0/x_0)p(x_0)}{\int_{x_0} p(z_0/x_0)p(x_0)dx_0} \quad . \tag{2-27}$$

2. Information State

Suppose there exists a quantity Y_k such that

$$p(x_k/Z_k,U_{k-1}) = p(x_k/Y_k) \quad . \tag{2-28}$$

Then it is clear that the control u_k can be based upon Y_k rather than Z_k and U_{k-1}.

Any vector Y_k which satisfies (2-28) will be referred to as the *information state* of the system. The relation between the information state and the concept of sufficient statistics is apparent; a sufficient statistic for predicting future behavior of the system is a possible information state.

In some special cases, such as the optimal control problem and the linear, Gaussian combined optimization problems, the dimension of Y_k may be reasonable; however, in the general case, the minimal-dimension information state is Z_k, U_{k-1}; hence its dimension grows in time. For this reason, approximation techniques for solving the general problem will be required. Possible approximations will be considered in Sec. V-C.

3. Optimal Estimation

Suppose f is independent of u_k, then the future cost is independent of the choice of u_k (see fig. II-2). In this case the recursion relation (2-17) reduces to

$$I^*(Z_k,k) = \min_{u_k} E\{l(x_k,u_k,k)/Z_k\} \tag{2-29}$$

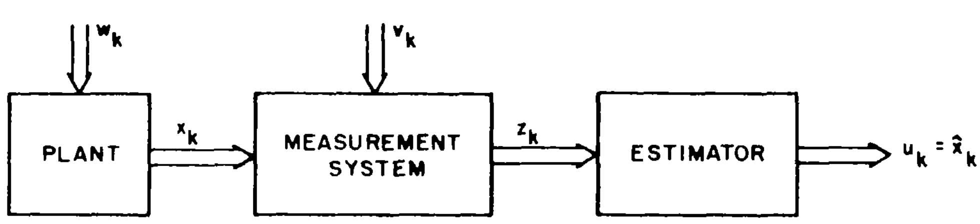

FIG. II-2 OPTIMAL ESTIMATION

If we identify u_k as being the estimate $\hat{x}_k$ of x_k; then the problem is seen to be the optimal estimation problem. For example, x_k might represent a signal and the measurement system a communication channel. Typically the loss function would take the form

$$l(x_k, u_k, k) = l^*(x_k - u_k, k) \quad . \tag{2-30}$$

Knowledge of the conditional probability distribution $p(x_k/Z_k)$ is sufficient to determine the optimal estimate given any loss function. The approach to optimal estimation based on updating the conditional probability distribution as was done in Part C-1 above is known as the Bayesian approach and has been studied by Lee (ref. 11), (ref. 12), and Peschon (ref. 13).

D. AN ALTERNATIVE VIEWPOINT

In the above formulation, we have calculated the control as a function of past and present measurements. However, the conditional probability $p(x_k/Z_k, U_{k-1})$ summarizes all the information about the future behavior of the system contained in these measurements; therefore,

$$\mathcal{P}_k \triangleq p(x_k/Z_k, U_{k-1}) \tag{2-31}$$

is a suitable information state. In terms of $\mathcal{P}_k$, the control equations (2-17) (2-18) become

$$I^*(\mathcal{P}_k, k) = \min_{u_k} \{L(\mathcal{P}_k, u_k, k) + E[I^*(\mathcal{P}_{k+1}, k + 1)/\mathcal{P}_k, u_k]\}$$

$$0 \leq k < N$$

$$I^*(\mathcal{P}_N, N) = \min_{u_N} L(\mathcal{P}_N, u_N, N) \tag{2-32}$$

where

$$L(\mathcal{P}_k, u_k, k) \triangleq E[l(x_k, u_k, k)/Z_k, U_{k-1}]$$

$$= \int_{x_k} l(x_k, u_k, k) p(x_k/Z_k, U_{k-1}) dx_k \quad . \tag{2-33}$$

Since $\mathcal{P}_k$ is a function, both $I^*(\cdot, k)$ and $U(\cdot, k)$ will be functions of functions, *i.e.*, functionals. It may appear silly to replace functions of a finite (albeit growing) number of variables with functionals, which are in essence functions of an infinite number of variables. However, in considering approximate techniques such a procedure may prove useful; furthermore, it results in separation of the combined optimization problem into two parts:

> *Control*—The optimum inputs are found as a function of the conditional probability distribution $p(x_k/Z_k, U_{k-1})$ by solution of the recursion relation (2-32). In general, this process is carried out *a priori*.
>
> *Estimation*—The conditional probability distribution is updated by use of recursion relation (2-26). In general, this process will be done in real time.

Note that (2-26) will usually be required to calculate $E[I^*(\mathbb{P}_{k+1}, k + 1)/\mathbb{P}_k, U_k]$. In the linear case discussed in the next section literal solutions may be obtained to the two problems. Section V presents a simple example of how the calculation proceeds when no such literal solution exists.

III LINEAR CASE

In this section, we consider the important linear case of the combined optimization problem, in which a complete analytic solution exists.

A. DEFINITION

The linear case of the combined optimization problem refers to the situation in which the following assumptions hold:

(1) The plant and measurement systems are linear, *i.e.*,

(a) $$x_{k+1} = A_k x_k + B_k u_k + w_k \tag{3-1}$$

(b) $$z_k = C_k x_k + v_k \tag{3-2}$$

(2) The performance index is quadratic, *i.e.*,

$$l(x_i, u_i, i) = x_i^T Q_i x_i + u_i^T R_i u_i \tag{3-3}$$

(3) The probability distributions are Gaussian, *i.e.*,

(a) $$p(x_0) = c_1 \exp \left[(x_0 - \bar{x}_0)^T (\hat{Q}_{-1})^{-1} (x_0 - \bar{x}_0) \right] \tag{3-4a}$$

(b) $$p(w_k) = c_2 \exp \left[w_k^T \hat{Q}_k^{-1} w_k \right] \tag{3-4b}$$

(c) $$p(v_k) = c_3 \exp \left[v_k^T \hat{R}_k^{-1} v_k \right] \tag{3-4c}$$

where c_1, c_2, c_3 are constants of no consequence here and where:

$\hat{Q}_{-1}$ = *a priori* covariance of x_0

$\hat{Q}_k$ = covariance of the disturbance at time k

$\hat{R}_k$ = covariance of the measurement noise at time k

$\bar{x}_0$ = *a priori* mean of x_0.

B. SOLUTION

The solution to the combined optimization problem in the linear case is well known (refs. 2, 3, and 4). In this section, we present the solution; Appendix B gives the detailed derivation. Figure III-1 illustrates the optimal solution.

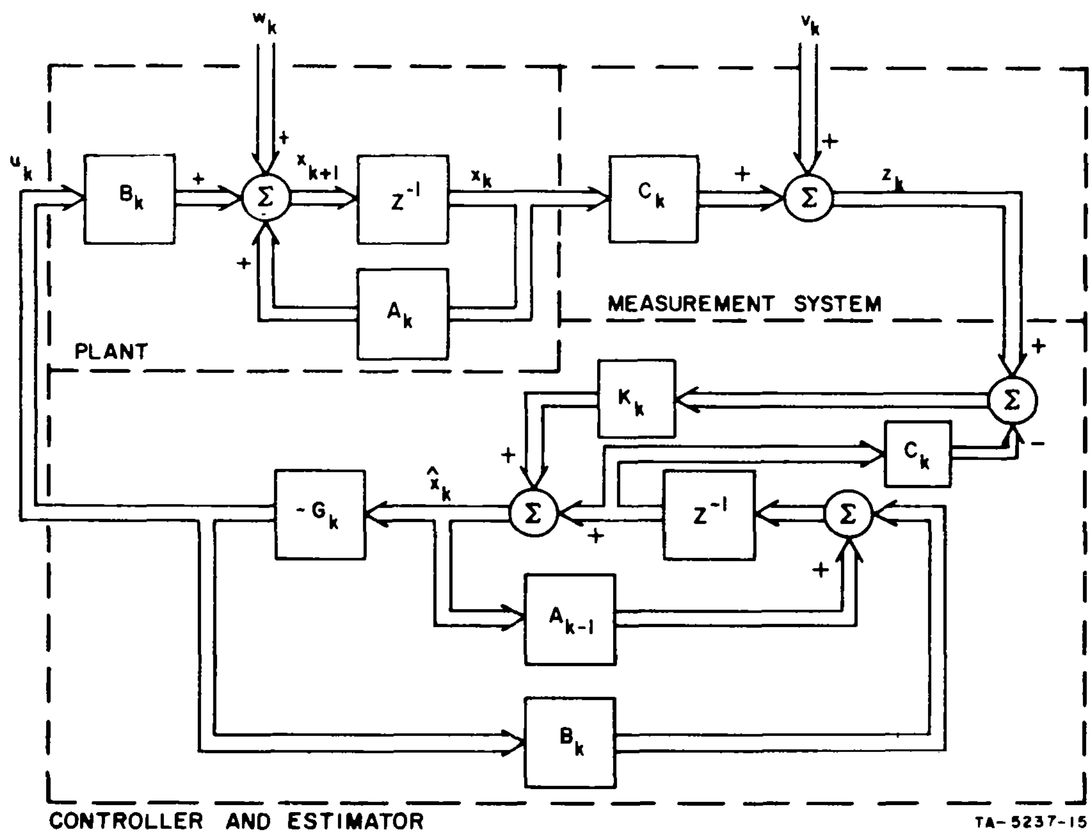

FIG. III-1 LINEAR COMBINED CONTROL AND ESTIMATION

1. Estimation

The assumptions given in Part A imply that $p(x_k/Z_k, U_{k-1})$ is Gaussian; hence it is sufficient to give equations for updating the conditional mean and conditional variance. In Appendix B these equations are obtained by application of the estimation equation (2-26). The key to solution of this equation is completion of squares and the results are

$$\bar{x}_{k+1/k+1} = A_k x_{k/k} + B_k u_k + K_{k+1}[z_{k+1} - C_{k+1}(A_k x_{k/k} + B_k u_k)] \quad 0 < k \leq N$$

$$\bar{x}_{0/0} = \bar{x}_0 + K_0(z_0 - C_0\bar{x}_0) \tag{3-5}$$

$$V_{k+1}^{-1} = [A_k V_k A_k^T + \hat{Q}_k]^{-1} + C_{k+1}^T \hat{R}_{k+1} C_{k+1} \qquad 0 < k \leq N$$

$$V_0 = [\hat{Q}_{-1}]^{-1} + C_0^T \hat{R}_0^{-1} C_0 \tag{3-6}$$

$$K_k = V_k C_k \hat{R}_k^{-1} \tag{3-7}$$

where

$$\bar{x}_{k/k} \triangleq E\{x_k/Z_k,\ U_{k-1}\} \tag{3-8}$$

$$V_k \triangleq E\{(x_k - \bar{x}_{k/k})^T(x_k - \bar{x}_{k/k})Z_k, U_{k-1}\} \quad . \tag{3-9}$$

The first two terms on the right of (3-5) represent the prediction of $\bar{x}_{k+1/k+1}$ based on the estimate $\bar{x}_{k/k}$ of the present state; the last term represents a correction due to the difference between the actual measurement z_{k+1} and predicted measurement $C_k(A_k\bar{x}_{k/k} + B_k u_k)$. Note that V_k is independent of Z_k and U_k; hence, it may be calculated *a priori* and thus $x_{k/k}$ is a suitable information state for the system.

2. CONTROL

Assumptions (1) and (2) in Part A imply that $I^*(\cdot,k)$ is quadratic in the conditional expectation of x_k given all measurements and that the control is a linear function of this conditional expectation:

$$I^*(Z_k, U_{k-1}, k) = \bar{x}_{k/k}^T P_k \bar{x}_{k/k} + b_k \tag{3-10}$$

$$u_k(Z_k, U_{k-1}, k) = -G_k \bar{x}_{k/k} \tag{3-11}$$

If (3-11) is substituted into the control equation (2.17), then by completion of squares, iterative equations for P_k and b_k may be obtained and in addition G_k found as a function of P_k and the given quantities (for details see Appendix B).

$$P_k = Q_k + A_k^T P_{k+1} A_k - A_k^T P_{k+1} B_k (B_k^T P_{k+1} B_k + R_k)^{-1} B_k^T P_{k+1} A_k$$

$$0 \leq k < N$$

$$P_N = Q_N \tag{3-12}$$

$$b_{k+1} = \mathrm{tr}[Q_k V_k + P_{k+1}(A_k V_k A_k^T - V_{k+1} + \hat{Q}_k)] + b_k \qquad 0 \leq k < N$$

$$b_N = \mathrm{tr}[Q_N V_N] \tag{3-13}$$

$$G_k = (B_k^T P_{k+1} B_k + R_k)^{-1} B_k^T P_{k+1} A_k \tag{3-14}$$

and where tr $[A]$ of a matrix A means the sum of the diagonal terms.

P_k and G_k are identical to the matrices resulting from the solution of the linear, quadratic, deterministic optimal control problem [*i.e.*, from solution of (2-21) under (3-1) and (3-3)].

3. DUALITY

Kalman (ref. 4) was first to notice that, mathematically, control and estimation are equivalent in the linear case. This equivalence is not apparent from the results presented above; however, if the estimation problem is restated in terms of

$$\begin{aligned} \hat{P}_k &\triangleq E\{(x_{k+1} - x_{k+1/k})^T (x_{k+1} - \bar{x}_{k+1/k})/Z_k, U_k\} \\ &= A_k V_k A_k^T + \hat{Q}_k \quad , \end{aligned} \tag{3-15}$$

where

$$\bar{x}_{k+1/k} = E\{x_{k+1}/Z_k, U_k\} \quad , \tag{3-16}$$

instead of V_k, we find that the equation for $\hat{P}_k$ is

$$\hat{P}_{k+1} = \hat{Q}_{k+1} + A_{k+1}\hat{P}_k A_{k+1}^T - A_{k+1}\hat{P}_k C_{k+1}^T (C_{k+1}\hat{P}_k C_{k+1}^T + \hat{R}_{k+1})^{-1} C_{k+1}\hat{P}_k A_{k+1}$$

$$0 < k \leq N$$

$$\hat{P}_0 = \hat{Q}_{-1} \tag{3-17}$$

and that

$$K_{k+1} = \hat{P}_k C_{k+1}^T (C_{k+1}\hat{P}_k C_{k+1}^T + \hat{R}_k)^{-1} \tag{3-18}$$

The detailed derivation of these equations is presented in Appendix B.

If the following identifications are made

$$
\begin{aligned}
A_{k+1}^{T} &\rightarrow A_k \\
C_{k+1}^{T} &\rightarrow B_k \\
\hat{Q}_{k+1} &\rightarrow Q_k \\
\hat{R}_{k+1} &\rightarrow R_k \\
\tilde{V}_{k+1} &\rightarrow P_k \\
\tilde{V}_k &\rightarrow P_{k+1} \\
K_{k+1}^{T} A_{k+1}^{T} &\rightarrow G_k \\
N &\rightarrow 0
\end{aligned}
\tag{3-19}
$$

then (3-17) becomes identical with (3-12) and (3-18) becomes identical with (3-14); therefore, the control and estimation problems are indeed mathematically equivalent in the linear case.

This last derivation is not an idle exercise. Equations (3-6) and (3-7) require inversion of $n \times n$ matrices, where n is the order of the system, while (3-17) and (3-18) require inversion of $k \times k$ matrices, where k is the dimension of the input. Since, in general, k will be less than n, solution of (3-17) and (3-18) is simpler than solution of (3-6) and (3-7) even though (3-6) and (3-7) appear to be simpler in form.

We thus see that in the linear case, control and estimation can be solved by solving essentially the same equation. The automatic design by computer of linear control systems based on similar equations for the continuous time case has been treated quite extensively by Kalman (ref. 14).

4. Costs

In Appendix B it is shown that the cost J using optimal control and estimation is given by

$$
J = \bar{x}_0^T P_0 \bar{x}_0 + \mathrm{tr}\,[P_0 \hat{Q}_{-1}] + \sum_{k=1}^{N} \Delta\beta_k \quad , \tag{3-20}
$$

where

$$\Delta\beta_k = \mathrm{tr}[P_{k+1}\hat{Q}_k + P^*_{k+1}V_k] \tag{3-21}$$

$$P^*_{k+1} \triangleq Q_k + A_k^T P_{k+1} A_k - P_k \quad . \tag{3-22}$$

Consider (3-20) in greater detail. The first two terms represent the expected cost due to initial conditions, the third term the cost due to disturbances and uncertainty about the state of the plant. The quantity $\Delta\beta_k$ represents the incremental cost of operating the system for kth interval because of these latter effects. From (3-21) it is clear that, ignoring initial conditions, the cost of operating during the kth increment is the sum of two terms: the first resulting from the disturbances and the second resulting from uncertainty about the state of the plant.

Equation (3-21) may be interpreted in an alternative manner as follows: From (3-12)

$$P^*_{k+1} = A_k^T P_{k+1} B_k (B_k^T P_{k+1} B_k + E_k)^{-1} B_k^T P_{k+1} A_k \tag{3-23}$$

and by use of the definition of G_k

$$P^*_{k+1} = G_k^T (B_k^T P_{k+1} B_k + E_k) G_k \tag{3-24}$$

and hence

$$\begin{aligned} \mathrm{tr}[P^*_{k+1}V_k] &= \mathrm{tr}[G_k^T(B_k^T P_{k+1} B_k + E_k) G_k V_k] \\ &= \mathrm{tr}[(B_k^T P_{k+1} B_k + E_k) G_k V_k G_k^T] \quad . \end{aligned} \tag{3-25}$$

The term $G_k V_k G_k^T$is, by the definition of V_k, given by

$$\begin{aligned} G_k V_k G_k^T &= G_k E\{(x_k - \bar{x}_{k/k})^T (x_k - \bar{x}_{k/k})/Z_k, U_{k-1}\} G_k^T \\ &= E\{(G_k x_k - G_k x_{k/k})(G_k x_k - G_k x_{k/k})^T/Z_k, U_{k-1}\} \\ &= E\{(u_k - u_k^o)(u_k - u_k^o)^T\} \quad , \end{aligned} \tag{3-26}$$

where

$u_k \triangleq G_k \bar{x}_{k/k}$ = applied control

$u_k^{o'} \triangleq G_k x_k$ = optimal control given the state .

Thus (3-25) gives the performance cost due to the variation of the applied control from that which is truly optimal in terms of the actual state of the system.

C. NON-OPTIMAL PERFORMANCE

In the proceeding part, we have seen how to find the optimal controller and estimator and how to calculate optimal performance. Suppose, however, that we want to calculate the performance of a suboptimal system in order to compare it with the optimal system. In this part, we will show how this may be done using the optimal theory and we will give analytical results for an important class of suboptimal controller estimators.

Consider the situation shown in fig. III-2. The operation of this system is governed by

$$x_{k+1} = (A_k - B_k G_k)x_k + w_k \quad . \tag{3-27}$$

We wish to calculate the performance of such a system assuming a cost function of the form

$$\begin{aligned} l &= x_k^T Q_k x_k + u_k^T R_k u_k \\ &= x_k^T (Q_k + G_k^T R_k G_k) x_k \quad . \end{aligned} \tag{3-28}$$

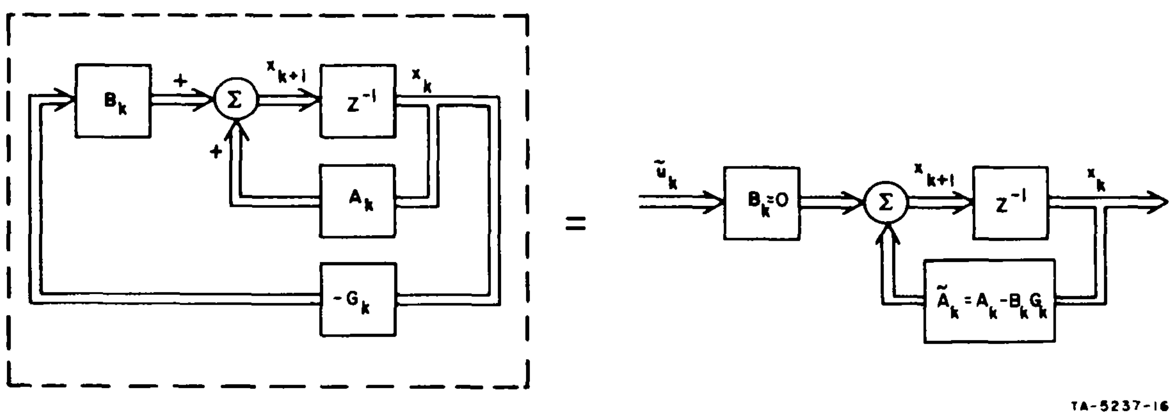

FIG. III-2 SUBOPTIMAL CONTROL

Optimal control theory may be used to calculate the performance of the suboptimal control system of fig. III-2 as follows: Consider the plant, also shown in fig. III-2 and described by

$$x_{k+1} = \tilde{A}_k x_k + \tilde{B}_k \tilde{u}_k \tag{3-29}$$

where

$$\tilde{A}_k = A_k - B_k G_k \tag{3-30}$$

$$\tilde{B}_k = 0 \tag{3-31}$$

Find the controller that optimizes the performance when the cost is

$$\tilde{J} = \sum_{k=0}^{N} (x_k^T \tilde{Q}_k x_k + \tilde{u}_k^T \tilde{R}_k \tilde{u}_k) \tag{3-32}$$

where

$$\tilde{Q}_k = Q_k + G_k^T R_k G_k$$
$$\tilde{R}_k = I \tag{3-33}$$

For the tilded system of fig. III-4 we have

$$\tilde{J} = x_0^T \tilde{P}_0 x_0 \tag{3-34}$$

where by use of (3-12) and (3-30), (3-37), (3-32) and (3-33)

$$\begin{aligned}\tilde{P}_k &= \tilde{Q}_k + \tilde{A}_k^T \tilde{P}_{k+1} \tilde{A}_k + \tilde{A}_k^T \tilde{P}_{k+1} \tilde{B}_k (\tilde{B}_k^T \tilde{P}_{k+1} \tilde{B}_k + \tilde{R}_k)^{-1} \tilde{B}_k^T \tilde{P}_{k+1} \tilde{A}_k^T \\ &= Q_k + G_k^T R_k G_k + (A_k - B_k G_k)^T \tilde{P}_{k+1} (A_k - B_k G_k) \quad .\end{aligned} \tag{3-35}$$

But since $\tilde{u}_k$ cannot affect the plant, it is obvious that the optimum $\tilde{u}_k$ is zero; therefore, the performance of this optimal system is identical with the performance of the original suboptimal system. Thus, by utilizing the identity between the systems shown in fig. III-2, it has been possible to derive easily (3-35) for the suboptimal system and to put it in a format such that the resulting performance degradation can be analyzed conveniently. Furthermore, if

$$\begin{aligned}\tilde{P}^*_{k+1} &\triangleq Q_k + A_k^T \tilde{P}_{k+1} A_k - \tilde{P}_k \\ &= G_k^T B_k^T \tilde{P}_{k+1} A_k + A_k^T \tilde{P}_{k+1} B_k G_k \\ &- G_k^T (B_k^T \tilde{P}_{k+1} B_k - R_k) G_k\end{aligned} \tag{3-36}$$

and if the optimal G_k is used, then (3-35) and (3-36) reduce (except for the tildes on P and P^*) to the previously derived equations (3-12) and (3-22).

The same procedure can be applied to calculate the performance of any suboptimal control system. Of particular interest is a system which has the same form as the optimal system given in fig. III-1, but for which G_k and K_k are not optimal.

To calculate the performance of such a system, we view it (in a manner similar to suboptimal control) as a combined optimization problem in which there exist no measurements or controls. Since the derivation involves considerable unenlightening algebra, only the result is produced here (with details relegated to Appendix B).

$$J = \bar{x}^T P_0 \bar{x}_0 + \mathrm{tr}[\tilde{P}_0 \tilde{V}_0] + \sum_{k=1}^{N} \Delta\tilde{\beta}_k \tag{3-37}$$

$$\Delta\tilde{\beta}_k = \mathrm{tr}[\tilde{P}_{k+1} \hat{Q}_k + \tilde{P}^*_{k+1} \tilde{V}_k + 2K_k^T P_k' K_k \hat{R}_k - 2P_k' K_k C_k \tilde{V}_k] \tag{3-38}$$

where P'_{k+1} is defined in Appendix B, and where

$$\tilde{V}_{k+1} = (I - K_{k+1}C_{k+1})A_k \tilde{V}_k A_k^T (I - K_{k+1}C_{k+1})^T + \tilde{Q}_k \tag{3-39}$$

$$\tilde{V}_{-1} = \tilde{Q}_{-1}$$

$$\tilde{Q}_k = (I - K_{k+1}C_{k+1})\hat{Q}_k (I - K_{k+1}C_{k+1})^T + K_{k+1}\hat{R}_{k+1}K_{k+1}^T \quad . \tag{3-40}$$

Note that (3-37) is the same as (3-20), the analogous equation for optimal systems; furthermore, if optimum control is used, but not necessarily optimum estimation, P'_{k+1} is zero and (3-38) reduces to (3-21).

The last two terms of (3-38) may be rewritten

$$\mathrm{tr}[2K_k^T P_k' K_k \hat{R}_k - 2P_k' K_k C_k \tilde{V}_k] = \mathrm{tr}[2P_k' K_k (\hat{R}_k K_k - C_k \tilde{V}_k)] \tag{3-41}$$

For optimum estimation the term $(\hat{R}_k K_k - C_k \tilde{V}_k)$ is zero; hence (3-21) is valid if either optimum estimation or optimum control, but not necessarily both, is used. In any case, (3-37) and (3-38) together with (3-35), (3-36), (3-39), and (3-40), provide a method of calculating the performance of any combined controller estimator of the form given in fig. III-1.

IV INFORMATION AND CONTROL

The purpose of this section is to show how combined optimization theory can be applied to answer various practical questions about control systems. A brief summary of the purpose of feedback control is given, followed by a linear example, which illustrates the application of combined optimization.

A. UNCERTAINTY AND CONTROL SYSTEMS

To see how the framework of the combined optimization problem may be used to investigate the effect of sensors on control system performance, we will take a brief look at the philosophy behind control systems. Feldbaum (ref. 9) presents an excellent summary of this philosophy in his first paper on dual control. Many of the concepts discussed here were inspired by that paper.

1. FEEDBACK CONTROL

If we knew the exact state of the system and if we knew all future inputs, we could specify the control to be applied to the system as a time function: that is, we could use open-loop control. But, in general, the initial state of the system is unknown *a priori*; furthermore the plant may be subject to unknown disturbances; and finally, the plant parameters may be partially unknown and varying. Note that if such parameters are considered as state variables, this last source of uncertainty is equivalent to the first two. Since sufficient *a priori* knowledge about the state of the plant is seldom available, some measurements must be made on the plant in real time to gain information. Thus, the need for feedback control is a result of the lack of *a priori* information about the state of the plant.

In a feedback control system, several methods exist for improving the knowledge of the state of the plant. Naturally some sensors must be used to make measurements on the plant, but due to sensor defects it will not in general be possible to determine the state exactly from these measurements. In Sec. II, estimation was defined as the determination of the conditional probability density of the state. The processing of the raw data to produce a reasonable estimate of the state is usually referred to as *filtering*. In addition to this straightforward approach, we also have the possibility of using control action to improve the estimate (*i.e.*, make the conditional probability distribution narrower). The concept of using control to improve estimation is Feldbaum's dual control (refs. 8 and 9), and is considered further in Sec. V.

2. Information Gathering and Processing

As mentioned before, the estimate of the state of the plant produced—either explicitly or implicitly—in any feedback control system will not be exact because of sensor imperfections. Furthermore, as a result of computational approximations, the estimate will not usually be as good as attainable for the given measurements. A cost will be associated with improving the estimate of the state. Removing sensor imperfections is costly and the use of complicated filtering and dual-control schemes may be even more costly. However, we know from experience that adequate control systems can be designed without using the best possible sensors along with the exact optimal control and estimation schemes; hence, there are two questions we would like to answer:

(1) Which state variables should be measured and how well?

(2) How complicated need the controller estimator be?

3. Value and Uncertainty

Intuitively, the answer to the two questions posed above depends upon the amount of *a priori* uncertainty present and the value of reducing that uncertainty in the following manner:

(1) The less the uncertainty about the plant costs in degraded performance, the more the uncertainty may be tolerated.

(2) The more *a priori* information present about the plant, the less information need be gathered.

Several possible mathematical approaches exist to answer the questions given in Part B. Information theory is the quantitative study of uncertainty. Unfortunately, information theory does not tell us much about how difficult it is to get information or how useful this information will be in terms of reducing operating cost once we have it. Hence, information theory by itself will not solve the whole problem, although it may be useful in considering some aspects.

Ideally, one would like to put the cost of estimation into the performance index and then compute the optimum system, taking into account the cost of sensors and of computing. Most performance indices do not include such costs for the simple reason that it is not clear mathematically how to include them, and it is possible that, if they could be included, the problem would be unsolvable. Even if solvable, such a formulation would not give the true optimum since it does not include the cost of design.

The approach we will take is to consider the solution of the combined optimization problem ignoring the cost of sensing and computing. In simple cases, this will provide quantitative answers about what performance can be expected from sensors of a given quality. In more complex cases the formulation will indicate reasonable approximations.

In the linear case, the importance of both value and uncertainty stand out clearly in the second term of $\Delta\beta_k$ in (3-21):

$$\mathrm{tr}\ [P^*_{k+1} V_k] \quad . \tag{4-1}$$

If P^*_{k+1} is large (*i.e.*, if the value of information is high) or if the *a priori* uncertainty about the state is large, then it is profitable to use very good sensors in an attempt to improve performance. On the other hand, if P^*_{k+1} is small and the *a priori* variance of the state is small, the use of very good sensors will not improve performance significantly. In the next section, some examples are presented to illustrate this fact and to demonstrate how the general theory of Sec. II may be used to answer the questions posed in Sec. IV 2.

B. EXAMPLES

In this part, we apply the concepts developed in the previous section to some illustrative examples. The examples considered are all linear; however, since the solution of the combined optimization problem for the non-linear case can be used to treat similar and more general nonlinear situations, the results serve not only to show the application of the combined optimization in the linear case, but to indicate its usefulness in the general case.

In keeping with previous work, the discrete time case is treated. However, entirely analogous results can be obtained for the continuous time case, either directly or by use of limiting arguments on the discrete time case. The continuous time case is extremely important in the linear situation, since it is practical to build linear, continuous-time controllers; in nonlinear situations, however, it is likely that digital controllers will be required, and hence continuous time results are less important in these situations.

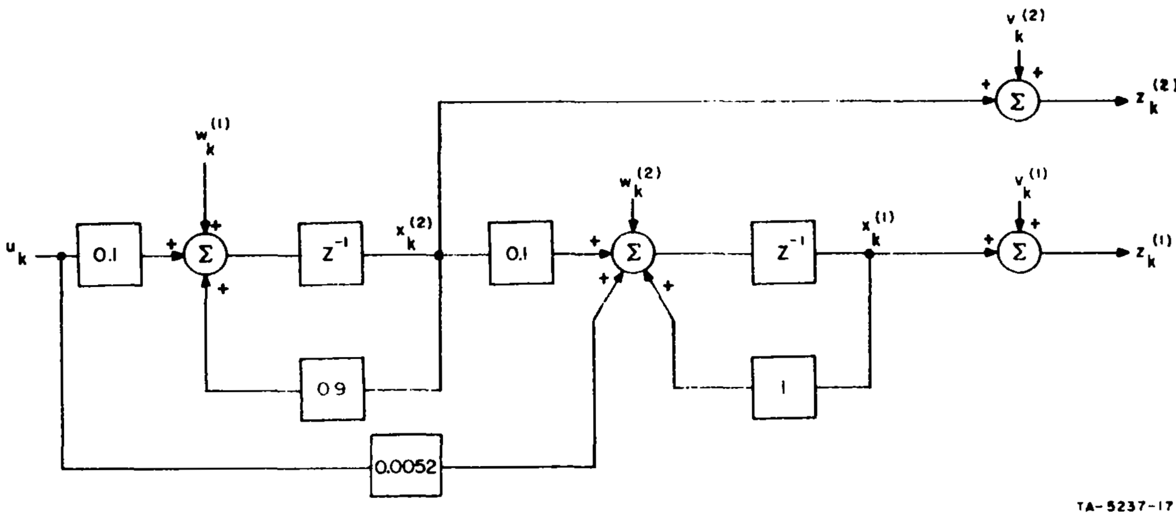

FIG. IV-1 DISCRETIZED POSITION SERVO
The symbol Z^{-1} stands for a unit delay in accordance with customary notation

We consider the problem of controlling a plant (see fig.IV-1) described by

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{4-2}$$

with

$$A = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.9 \end{bmatrix} \qquad B = \begin{bmatrix} 0.0052 \\ 0.1 \end{bmatrix} \tag{4-3}$$

This plant is just the discretized version of the simple position controller illustrated in fig.IV-2. We wish to minimize the cost

$$J = \sum_{k=0}^{\infty} (x_k^T Q x_k + u_k^T R u_k) \quad , \tag{4-4}$$

where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix} \qquad R = [0] \tag{4-5}$$

The problem is considered for alternative measurement systems:

$$z_k = x_k + v_k \quad , \tag{4-6}$$

with statistics

$$\hat{Q} = E(w_k w_k^T) \tag{4-7}$$

$$\hat{R} = E(v_k v_k^T) \quad . \tag{4-8}$$

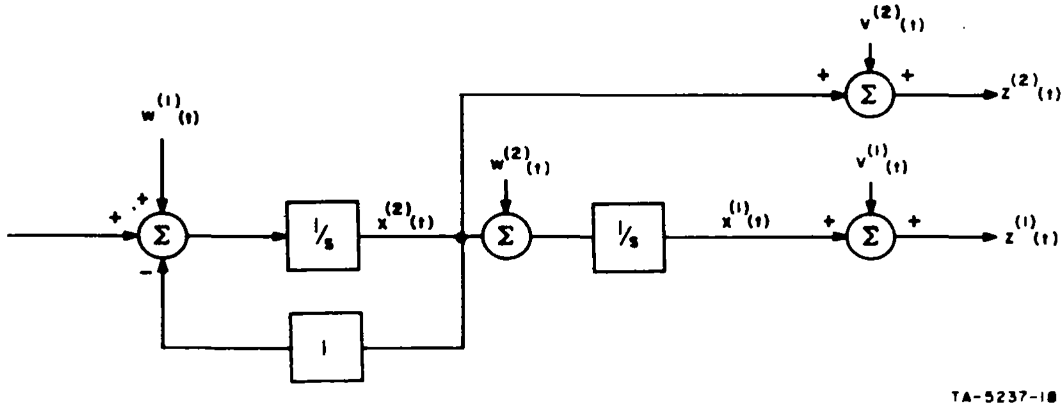

FIG. IV-2 POSITION SERVO

We will assume that the system operates forever; hence, to find the optimal control, we must find the steady-state value of P by solving

$$P = M + A^TPA - A^TPB(B^TPB)^{-1}BPA \quad . \tag{4-9}$$

The solution is

$$P = \begin{bmatrix} 4 & 0.16 \\ 0.16 & 0.1 \end{bmatrix} \quad . \tag{4-10}$$

Also, we have

$$G = (B^TPB^{-1})BPA = [27 \quad 10.5] \tag{4-11}$$

$$P^* = M + A^TPA - P = \begin{bmatrix} 1 & 0.38 \\ 0.38 & 0.15 \end{bmatrix} \quad . \tag{4-12}$$

To find the optimum estimator, we must solve the variance equation for V_k. Knowing V_k, $\Delta\beta_k$ can be found by solving (3-21). The results of such calculations for several different C, $\hat{Q}$, and $\hat{R}$ are given in table IV-1. It is assumed that the system has been operating indefinitely so that steady-state values of V_k are found.

Now consider these examples in more detail. Suppose that originally we try to control the plant using the measurement system of Case 1 and that the performance attained is inadequate. Several possible methods exist for improving performance. A rate sensor may be added (Case 2), the system may be "insulated" to reduce the level of rate disturbance (Case 3) or the position sensor may be made more accurate (Case 4).

Table IV-1

EXAMPLES

CASE NO.	$\hat{R}$	$\hat{Q}$	V	tr $[PQ]$	tr $[P^{*}V]$	$\Delta\beta$
1	$\begin{bmatrix}1 & 0\\ 0 & \infty\end{bmatrix}$	$\begin{bmatrix}0.1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}0.38 & 0.45\\ 0.45 & 0.35\end{bmatrix}$	0.5	1.24	1.74
2	$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}0.1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}0.28 & 0.02\\ 0.02 & 0.6\end{bmatrix}$	0.5	0.38	0.88
3	$\begin{bmatrix}1 & 0\\ 0 & \infty\end{bmatrix}$	$\begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix}$	$\begin{bmatrix}0.29 & 0.09\\ 0.09 & 0.48\end{bmatrix}$	0.41	0.42	0.83
4	$\begin{bmatrix}0.1 & 0\\ 0 & \infty\end{bmatrix}$	$\begin{bmatrix}0.1 & 0\\ 0 & 1\end{bmatrix}$	$\begin{bmatrix}0.07 & 0.12\\ 0.12 & 3.0\end{bmatrix}$	0.5	0.60	1.10

Comparing Case 3 with Case 4, we see that to a certain extent *a priori* information (gained by reducing disturbance inputs) can be traded off with measured information (gained by using more accurate sensors). We also see from comparison of Case 2 and Case 4 that we can find trade-offs between the methods of gathering more information; for example, we may add sensors or we may make the present sensor more accurate. In any case, it is clear that by use of combined optimization theory, we can investigate alternative sensing systems for obtaining the same performance. Combined optimization provides a "figure of merit" for comparing sensing systems, the standard for comparison being the optimum performance.

Consider now the possibility of simplifying the system by using a suboptimal controller estimator. This may be accomplished in two ways: simplification of the time variation of the controller estimator or reduction of the number of dynamics in the controller estimator. We treat the latter possibility first.

In Case 4, the mean square error V_{11} of 0.07 in the position $x_k^{(1)}$ with the optimal estimator* is not much smaller than the mean square error of 0.1, which would result from using the unfiltered output of the position sensor as an estimate. This observation suggests that the state variable of the estimator associated with $x_k^{(1)}$ might be eliminated without degrading performance significantly. Such a procedure amounts to replacing the Kalman filter, which is the optimal estimator by a Luenberger observer (ref. 15). The equations governing the optimal estimator for Case 4 are

$$\hat{x}_{k+1}^{(1)} = 0.311\hat{x}_k^{(1)} + 0.0311\hat{x}_k^{(2)} - 0.689z_{k+1}^{(1)} + 0.00162u_k \tag{4-13}$$

$$\hat{x}_{k+1}^{(2)} = -1.16\hat{x}_k^{(1)} + 0.784\hat{x}_k^{(2)} + 1.16z_{k+1}^{(1)} + 0.094u_k \quad . \tag{4-14}$$

If we set

$$\hat{x}_{k+1}^{(1)} = z_{k+1}^{(1)} \quad ,$$

the remaining equation becomes

$$\hat{x}_{k+1}^{(2)} = 0.784x_k^{(2)} - 1.16z_k^{(1)} + 1.16z_{k+1}^{(1)} + 0.094u_k \quad . \tag{4-15}$$

Hence, for this suboptimal estimator

$$(I - KC)A = \begin{bmatrix} 0 & 0 \\ -1.16 & 0.784 \end{bmatrix} \tag{4-16}$$

and

$$K^T = \begin{bmatrix} 1 \\ 1.16 \end{bmatrix} \quad \tilde{Q} = \begin{bmatrix} -1 & 0.116 \\ 0.116 & 1.27 \end{bmatrix} \quad . \tag{4-17}$$

Note that, to delete the state variable associated with $x_k^{(1)}$, we must define

$$y_k = \hat{x}_k^{(1)} - 1.48z_k \quad , \tag{4-18}$$

which obeys

$$y_{k+1} = 0.748y_k - 0.32z_{k+1} + 0.094u_k \quad . \tag{4-19}$$

---

* Superscripts in parentheses are used to indicate components of vectors.

In terms of this new state variable, the suboptimum estimator is first order. Both the optimal and this suboptimal estimator are shown in Fig. IV-3.

By use of the theory presented in Sec. III-C and in particular (3-38) and (3-39), we obtain

$$\tilde{V} = \begin{bmatrix} 0.1 & 0.12 \\ 0.12 & 3.1 \end{bmatrix} \tag{4-20}$$

and

$$\Delta\beta = P\hat{Q} + P^*\tilde{V} = 0.5 + 0.64 = 1.14 \quad . \tag{4-21}$$

These results compare with

$$V = \begin{bmatrix} 0.07 & 0.12 \\ 0.12 & 3.0 \end{bmatrix} \qquad \Delta\beta = 1.10 \quad , \tag{4-22}$$

which were previously obtained for optimal estimation. Note that $\hat{V}$ for the suboptimal estimator is not much different from V for the optimal estimator; furthermore, the cost associated with using the suboptimal estimator is increased only slightly from 1.10 to 1.14.

If, in Case 3, the optimal estimator is replaced by a Luenberger observer, as was done for Case 4, we get

$$V = \begin{bmatrix} 1 & 0.453 \\ 0.453 & 0.67 \end{bmatrix} \tag{4-23}$$

$$\Delta\beta = 0.41 + 1.45 \doteq 1.86 \tag{4-24}$$

as compared to

$$V = \begin{bmatrix} 0.29 & 0.12 \\ 0.12 & 0.48 \end{bmatrix}$$

$$\Delta\beta = 0.83 \tag{4-25}$$

for the optimal estimator.

In this case, there is a significant degradation in performance. Thus, while gaining *a priori* information (by reducing disturbances) may allow improved performance for a given sensing system, to achieve this improved performance may require an optimal estimator.

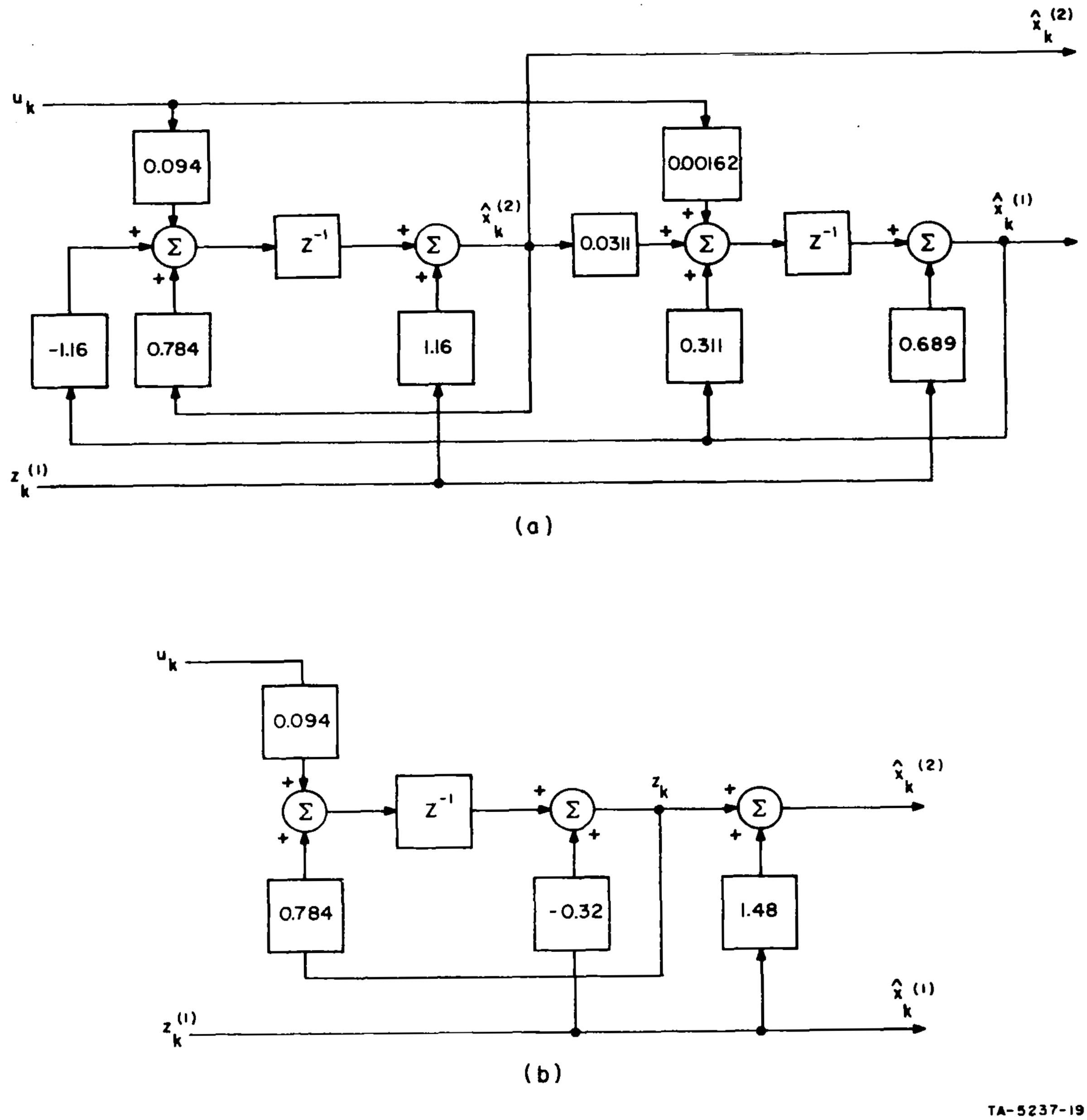

FIG. IV-3 ESTIMATORS FOR CASE 4
(a) Optimal
(b) Suboptimal

We now turn our attention from the steady-state to transient performance and give examples in which *a priori* information or measured information may simplify the controller estimator by approximating its time variation. In general, the optimum estimator, which is a Kalman filter, is time varying. We would like to know under what conditions the Kalman filter may be replaced by its steady-state limit which, for stationary plants and noise, is the Wiener filter.

The Wiener and Kalman filters are identical, if the initial uncertainty about the state, as measured by its variance, is the same as the steady-state uncertainty. In this situation, we start in steady-state operation. Since it is reasonable to assume that the *a priori* uncertainty is at least as great as the steady-state uncertainty, this situation provides one example where *a priori* information simplifies time variation of the optimal system from what would be optimum if little *a priori* information about the initial state existed.

Table IV-II presents the results of replacing the Kalman filter, which is optimal, with the Wiener filter, which is suboptimal, for Cases 1 and 4 with an *a priori* uncertainty about the state given by

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \tag{4-26}$$

Notice that the percentage degradation is somewhat worse in Case 4, the case with the better position sensor. This result may be explained by the fact that in Case 1 the optimal estimator gain K changes only from [0.5 0] initially to [0.376 0.453] for steady-state, whereas for Case 2 the optimal K changes from [0.9 0] to [0.69 1.16].

Thus, Case 4 presents an example where the use of better measurements improves performance, but where the optimal estimator is required to get the full improvement. However, in both examples, the cost from using the Weiner filter is minor. Needless to say, as time of operation increases, the effect of initial non-optimability becomes less significant.

Table IV-2

TRANSIENT RESPONSE $V = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$

	k	CASE 1		CASE 4	
		Optimal	Suboptimal	Optimal	Suboptimal
	1	2.50	2.52	2.09	2.14
	2	2.60	2.63	1.79	2.09
	3	2.53	2.60	1.46	1.77
	4	2.35	2.47	1.28	1.48
	5	2.17	2.28	1.19	1.29
$\Delta\beta_k$	6	2.02	2.12	1.15	1.20
	7	1.92	1.99	1.12	1.15
	8	1.85	1.89	1.11	1.13
	9	1.81	1.82	1.11	1.12
	10	1.78	1.78	1.11	1.11
	Σ	21.53	22.10	13.41	14.48

V. GENERAL CASE

In this section, the solution of the combined optimization problem in the general case is considered. Dual control, *i.e.*, the possibility of using control action to gain information, now arises. A simple example is presented to illustrate both the concept of dual control and the nature of the computations involved in solving the nonlinear case. Finally, we discuss some possible approximation techniques to get a practically feasible solution of the combined optimization problem.

A. DUAL-CONTROL THEORY

It is intuitively reasonable to expect that test signals may be used to gain information about a plant. Therefore, in the control of a plant with imperfect knowledge of its state, the input serves two purposes: as a control signal, to drive the plant toward its desired state and, as a test signal, to gain information about the state of the plant. We would not expect in general that the input which provides the most information would be identical with the input which is optimum from a control standpoint. The problem of trading off these two uses of an input is known as the *dual-control problem* (refs. 8, 9).

In his papers on dual control (refs. 8, 9) **Feldbaum** considers a situation similar to that of this memorandum. However, he assumes that the disturbance and noise input probability distributions contain unknown parameters.

It is clear that the problem considered here is a special case of **Feldbaum's** problem, since the distribution $p(x_0)$, $p(w_k)$, $p(v_k)$ are assumed known. On the other hand, if the state of the plant is augmented to include the unknown parameters, **Feldbaum's** problem becomes a special case of the combined optimization problem. Thus the two problems are slightly different formulations of the same basic problem. The approach to adaptive control, in which the unknowns are treated as parameters, appears to be more systematic. Furthermore the use of a recursion relation to calculate conditional probabilities is a significant improvement over the method used by **Feldbaum**.

Mathematically, these concepts can be illustrated as follows: Let $I(x_k,k)$ be the performance of a plant when controlled optimally, and using exact knowledge of the state. Then it is reasonable to define the input u_k^C, which is optimal from the control point of view, as

$$u_k^C(Y_k) = \underset{u_k}{\operatorname{argmin}}\, E\{ l(x_k,u_k,k) + I(x_{k+1},k+1)/Y_k,u_k\} \tag{5-1}$$

where $\underset{u}{\operatorname{argmin}}\, F(u)$ is that value of u which minimizes F and where Y_k is the information state as defined in Part II-C-2. On the other hand, if $U(u_k,Y_k,k)$ is the expected uncertainty about x_{k+1} when the input u_k is applied, then the input which is optimal from an information point of view is given by

$$u_k^I(Y_k) = \underset{u_k}{\operatorname{argmin}}\, U(u_k,Y_k,k) \quad . \tag{5-2}$$

In general

$$u_k^I(Y_k) \neq u_k^C(Y_k) \quad . \tag{5-3}$$

Since improvement in information about the plant may allow us to make better control choices in the future, the input u_k^O which is truly optimal, will be a compromise between u_k^I and u_k^C. Combined optimization, which includes both control through the recursion relation derived by dynamic programming, and estimation through the recursion relation derived by use of Bayes' rule, will provide that input which constitutes the optimal trade-off between the control and informational aspects.

We have not yet defined the function U, which is used to measure the expected uncertainty. One possibility is to use the entropy $H(x_k,Y_k,k)$ of the conditional probability distribution and to define U as

$$U(u_k,Y_k,k) = E\{H(x_{k+1},Y_{k+1},k+1)/Y_k,u_k\} \quad , \tag{5-4}$$

where

$$H(x_k,Y_k,k) \overset{\Delta}{=} \int_{x_k} p(x_k/Y_k)\log p(x_k/Y_k)dx_k \quad . \tag{5-5}$$

A second possibility is to define

$$U(u_k,Y_k,k) = E\{I^*(Y_{k+1},k+1) - I(x_{k+1},k+1)/Y_k,u_k\} \quad , \tag{5-6}$$

where I^* is the minimum cost in the case of incomplete knowledge about the state. If there is no uncertainty about the state, then $U(u_k,Y_k,k)$ reduces to zero for both definitions of U; furthermore, if $u^I = u^C$ for the u defined in

(5.5), then u^C is the optimal input, since

$$I^*(Y_k,k) = \min_{u_k} E\{[l(x_k,u_k,k) + I^*(Y_{k+1},k+1)]/Y_k,u_k\}$$

$$= \min_{u_k}\Big(E\{[l(x_k,u_k,k) + I(x_{k+1},k+1)]/Y_k,u_k\} + U(u_k,Y_k,k)\Big) . \tag{5-7}$$

This property does not necessarily hold for expected entropy. On the other hand, the entropy can be calculated without solving the combined optimization problem. This is significant, since we wish to calculate u_k^I and u_k^C as an aid to solving the combined problem.

For a linear system

$$U(u_k,Y_k,k) = \sum_{i=k+1}^{N} \mathrm{tr}(P_i^* V_i) \tag{5-8}$$

is independent of u_k, which affects only the conditional mean of x_i for $i > k$. Thus, in the linear case, the optimum control u_k^O is u_k^C. We will now present a simple example in which u_k^O does not equal u_k^C.

B. AN EXAMPLE

Since we are forced to consider a nonlinear example to illustrate dual control, we will consider discrete state systems (*i.e.*, systems which at any time are in one of a finite number of discrete states). In such systems, the conditional probability density consists of impulses at the various states. This density can be replaced by a probability vector giving the conditional probabilities that the system is in a given state. The formulation given in Sec. II is still valid except that summations replace integrations in the appropriate places. The discrete state case has also been considered by Astrom (ref. 16) who presents other examples. The advantage of discrete state systems is that the probability vector is a nongrowing finite dimensional information state.

Consider a plant with two states (0 and 1) and two inputs (0 and 1) and whose state equation is

$$x_{k+1} = x_k \oplus u_k \tag{5-9}$$

where $\oplus$ is the exclusive OR function described in table V-1. Note that a zero input leaves the state of the plant unchanged; a unity input changes the state to the other state.

Table V-1

LOGIC FUNCTIONS

$$x_{k+1} = \left\{ \begin{array}{c|c|c} x_k \;\; u_k & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline 1 & 1 & 0 \end{array} \right. \qquad Z_k = \left\{ \begin{array}{c|c|c} x_k \;\; v_k & 0 & 1 \\ \hline 0 & 0 & 0 \\ \hline 1 & 0 & 1 \end{array} \right.$$

Exclusive OR AND

The measurement systems is given by

$$z_k = x_k \cdot v_k \quad , \tag{5-10}$$

where $\cdot$ is the logical "AND" function shown in Table V-1, and where

$$Pr(v_k = 1) = a \quad . \tag{5-11}$$

In this example, the conditional probability distribution can be described by two numbers: the conditional probability that x_k is zero and the conditional probability that x_k is one. Since these two numbers must add up to one, a suitable information state is

$$p_k \triangleq Pr(x_k = 1/Z_k, U_{k-1}) \quad . \tag{5-12}$$

The recursion relation for p_k obtained from the estimation equation (2-27) takes the form (5-13). The calculations may be divided into three cases:

$$p_{k+1} = Pr(x_{k+1} = 1/z_{k+1}, \ldots, z_0, u_k, \ldots, u_0)$$

$$= \frac{Pr(z_{k+1}/x_{k+1} = 1)[Pr(x_{k+1} = 1/x_k = 1, u_k)p_k + Pr(x_{k+1} = 1/x_k = 0, u_k)(1 - p_k)}{Pr(z_{k+1}/x_{k+1} = 1)[Pr(x_{k+1} = 1/x_k = 1, u_k)p_k + Pr(x_{k+1} = 1/x_k = 0, u_k)(1 - p_k)] + Pr(z_{k+1}/x_{k+1} = 0)[Pr(x_{k+1} = 0/x_k = 1, u_k)p_k + Pr(x_{k+1} = 0/x_k = 0, u_k)(1 - p_k)]} \tag{5-13}$$

If $u_k = 0$, $z_{k+1} = 0$, then

$$Pr\ (z_{k+1}/x_{k+1} = 1) = (v_k = 0) = 1 - a \tag{5-14}$$

$$Pr\ (x_{k+1} = 1/x_k = 1, u_k) = 1 \tag{5-15}$$

$$Pr\ (x_{k+1} = 0/x_k = 1, u_k) = 0 \tag{5-16}$$

$$Pr\ (x_{k+1} = 1/x_k = 0, u_k) = 0 \tag{5-17}$$

$$Pr\ (x_{k+1} = 0/x_k = 0, u_k) = 1 \tag{5-18}$$

$$Pr\ (z_{k+1}/x_{k+1} = 0) = 1 \quad ; \tag{5-19}$$

hence,

$$p_{k+1} = \frac{(1 - a)[1 \cdot p_k + 0 \cdot (1 - p_k)]}{(1 - a)[1 \cdot p_k + 0 \cdot (1 - p_k)] + 1[0 \cdot p_k + 1 \cdot (1 - p_k)]}$$

$$= \frac{(1 - a)p_k}{(1 - a)p_k + (1 - p_k)} = \frac{(1 - a)p_k}{1 - ap_k} \quad . \tag{5-20}$$

If $u_k = 1$, $z_{k+1} = 0$, then by similar calculations (or by symmetry)

$$p_{k+1} = \frac{(1 - a)(1 - p_k)}{1 - a(1 - p_k)} \quad . \tag{5-21}$$

Finally, by inspection of 5-10 and table V-1; if $z_{k+1} = 1$, then

$$p_{k+1} = 1 \quad . \tag{5-22}$$

Note that, if the output is ever 1, then we have perfect knowledge of the state of the system.

Now we proceed to calculate the expected entropy at time $k + 1$ for $u_k = 0$ and for $u_k = 1$. If $u_k = 0$ and $z_{k+1} = 0$ then

$$-H(x_{k+1}, p_{k+1}, k) = p_{k+1} \log p_{k+1} + (1 - p_{k+1}) \log (1 - p_{k+1})$$

$$= \frac{(1 - a)p_k}{1 - ap_k} \log \frac{(1 - a)p_k}{1 - ap_k} + \frac{(1 - p_k)}{1 - ap_k} \log \frac{(1 - p_k)}{1 - ap_k} \tag{5-23}$$

If $u_k = 0$ and $z_{k+1} = 1$

$$H(x_{k+1}, p_{k+1}, k) = 0 \quad . \tag{5-24}$$

Therefore, if $u_k = 0$, U may be found by multiplying the right side of (5-22) by the $Pr\,(z_{k+1} = 0/u_k = 0, p_k)$, which is $(1 - ap_k)$:

$$-U(o,p_k,k) = (1-a)p_k \log (1-a)p_k + (1-p_k) \log (1-p_k) - (1-ap_k) \log (1-ap_k) \quad . \tag{5-25}$$

By symmetry, we have

$$\begin{aligned} -U(1,p_k,k) &= -U(o,1-p_k,k) \\ &= (1-a)(1-p_k) \log (1-a)(1-p_k) + p_k \log p_k \\ &\quad - [1-a(1-p_k)] \log [1-a(1-p_k)] \quad . \end{aligned} \tag{5-26}$$

In fig. V-1 $U(0,p_k,k)$ and $U(1,p_k,k)$ are plotted for $a = 1/2$. Note that for $p_k < 1/2$.

$$U(0,p_k,k) < U(1,p_k,k) \tag{5-27}$$

and vice versa for $p_k > 1/2$; therefore

$$\begin{aligned} u_k^I &= 0 \qquad p_k < \frac{1}{2} \\ u_k^I &= 1 \qquad p_k > \frac{1}{2} \end{aligned} \tag{5-28}$$

This relation holds for any $0 < a < 1$. For the boundary cases of no measurement $(a = 0)$ and perfect measurement $(a = 1)$, we find $U(0,p_k,k) = U(1,p_k,k)$. In fig. V-1, the entropy $H(p_k,k)$ is also plotted; note that the expected entropy at the next state is less than the present entropy for both controls. Again this relationship holds for any $0 < a < 1$, with the three curves coinciding for $a = 0$ and $a = 1$.

If our desire is to have the system in state zero, then it is reasonable to pick

$$\begin{aligned} u_k^C &= 0 \qquad p_k < \frac{1}{2} \\ u_k^C &= 1 \qquad p_k > \frac{1}{2} \quad . \end{aligned} \tag{5-29}$$

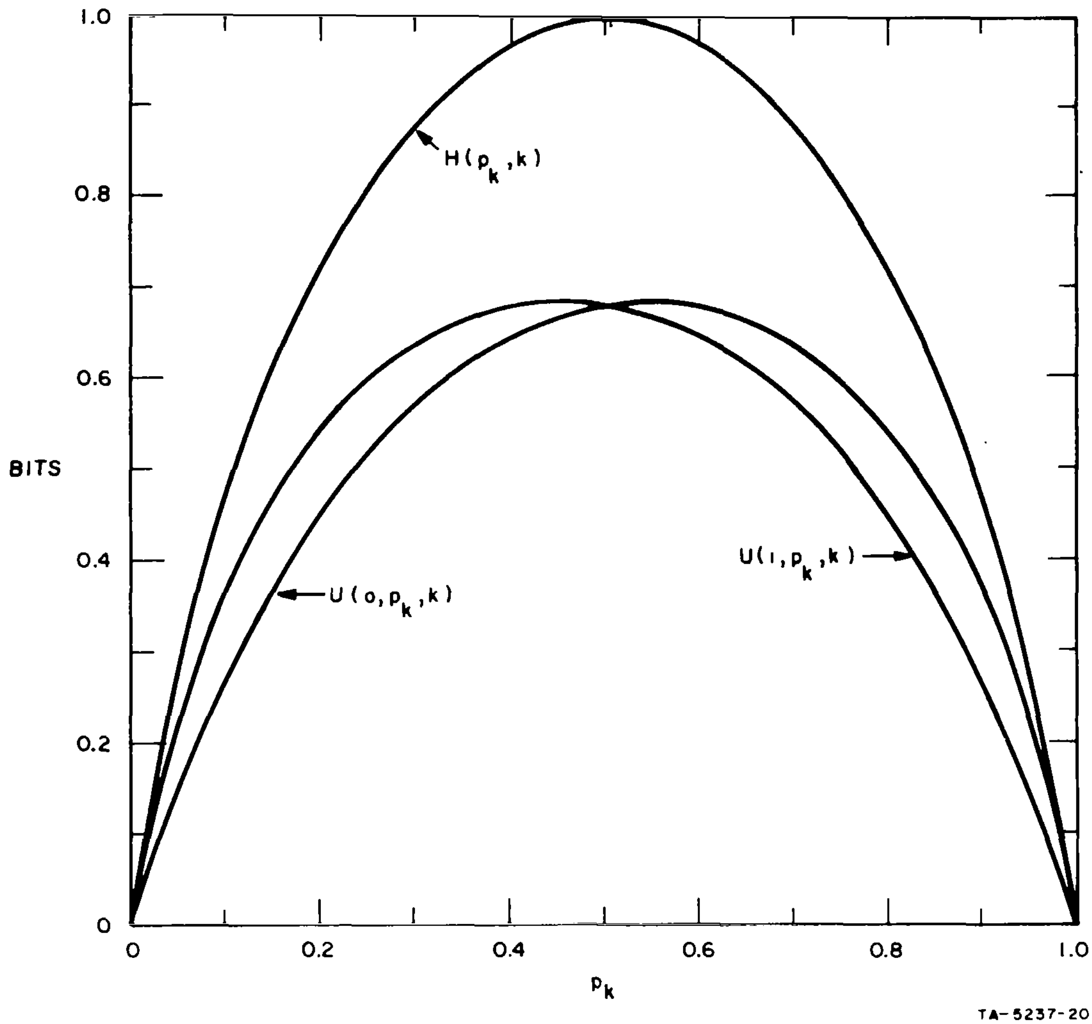

FIG. V-1 EXPECTED ENTROPY

Comparing (5-28) and (5-29), we see that in this situation $u_k^C = u_k^I$ and the problem of dual control does not exist. On the other hand, if our desire is to have the system in state one, then

$$u_k^C = 1 \qquad p_k < \frac{1}{2}$$

$$u_k^C = 0 \qquad p_k > \frac{1}{2} \quad . \tag{5-30}$$

In this situation $u_k^C \neq u_k^I$; therefore, we would expect that there exist cost functions and information states p_k for which u_k^I is used rather than u_k^C.

To illustrate these situations, we will set a equal to 1/2 and N equal to 2 and find the optimal policy when the cost of being in the undesired state is 1 for $k < N$ and 10 for $k = N$. Consider first the case where the desired state is zero. For this case the cost is

$$\begin{aligned} l(u_k, x_k, k) &= x_k \qquad k < 2 \\ &= 10x_k \qquad k = 2 \quad . \end{aligned} \tag{5-31}$$

The optimal policy is found by applying the control equation (2-31).

$$I^*(p_2, 2) = \min_{u_2} E\{10x_2\} = 10p_2 \quad . \tag{5-32}$$

The choice of u_2 is arbitrary: a reasonable choice is $u_2^0 = u^C$. A second application of the control equation yields

$$\begin{aligned} I^*(p_1, 1) &= \min_{u_1} E\{x_1 + 10p_2/p_1, u_1\} \\ &= p_1 + 10 \min_{u_1} E\{p_2/p_1, u_1\} \end{aligned} \tag{5-33}$$

Making use of (5-19), and (5-20), and (5-21), we find

$$\begin{aligned} E\{p_2/p_1, u_1 = 0\} &= \frac{(1 - a)p_1}{1 - ap_1} Pr\ (z_2 = 0/p_1, u_1 = 0) + Pr\ (z_2 = 1/p_1, u_1 = 0) \\ &= \frac{(1 - a)p_1}{(1 - ap_1)} (1 - ap_1) + ap_1 = p_1 \end{aligned} \tag{5-34}$$

$$E\{p_2/p_1, u_1 = 1\} = 1 - p_1 \tag{5-35}$$

Therefore the optimal input u_1^0

$$\begin{aligned} u_1 &= 0 \qquad p_1 < \frac{1}{2} \\ u_1 &= 1 \qquad p_1 > \frac{1}{2} \end{aligned} \tag{5-36}$$

and

$$\begin{aligned} I^*(p_1, 1) &= 11p_1 \qquad p_1 < \frac{1}{2} \\ &= p_1 + 10(1 - p_1) \qquad p_1 > \frac{1}{2} \quad . \end{aligned} \tag{5-37}$$

Table V- 2 presents the results of completing the calculations of the optimal policy in a similar manner. Notice that in all cases

$$u_k^0 = u_k^I = u_k^C \qquad (5\text{-}38)$$

Table V-2

OPTIMAL POLICY FOR DESIRED STATE 0

k	0	1	2
$u_k^0(p_k)$	$0 \quad p_0 < \frac{1}{2}$ $1 \quad p_0 > \frac{1}{2}$	$0 \quad p_1 < \frac{1}{2}$ $1 \quad p_1 > \frac{1}{2}$	$0 \quad p_2 < \frac{1}{2}$ $1 \quad p_2 > \frac{1}{2}$
$I^*(p_k, k)$	$7p_0, p_0 < \frac{1}{2}$ $p_0 + 6(1 - p_0), p_0 > \frac{1}{2}$	$11p_1, p_1 < \frac{1}{2}$ $p_0 = 10(1 - p_1), p_1 > \frac{1}{2}$	$10p_2$

These results are presented graphically in fig. V-2.

Now consider the case where the desired state is one; in this case

$$\begin{aligned} l(x_k, u_k, k) &= (1 \oplus x_k) \qquad k < 2 \\ &= 10(1 \oplus x_k) \qquad k = 2 \end{aligned} \qquad (5\text{-}39)$$

Again the control equation is used to find the optimal policy, which is given in table V- 3 As an example of these calculations, consider $k = 0$.

$$\begin{aligned} I^*(p_0, 0) &= \min_{u_0} E\{[(1 + x_0) + I^*(p_1, 1)]/p_0, u_0\} \\ &= 1 - p_0 + \min_{u_0} E\{I^*(p_1, 1)/p_0, u_0\} \quad . \end{aligned} \qquad (5\text{-}40)$$

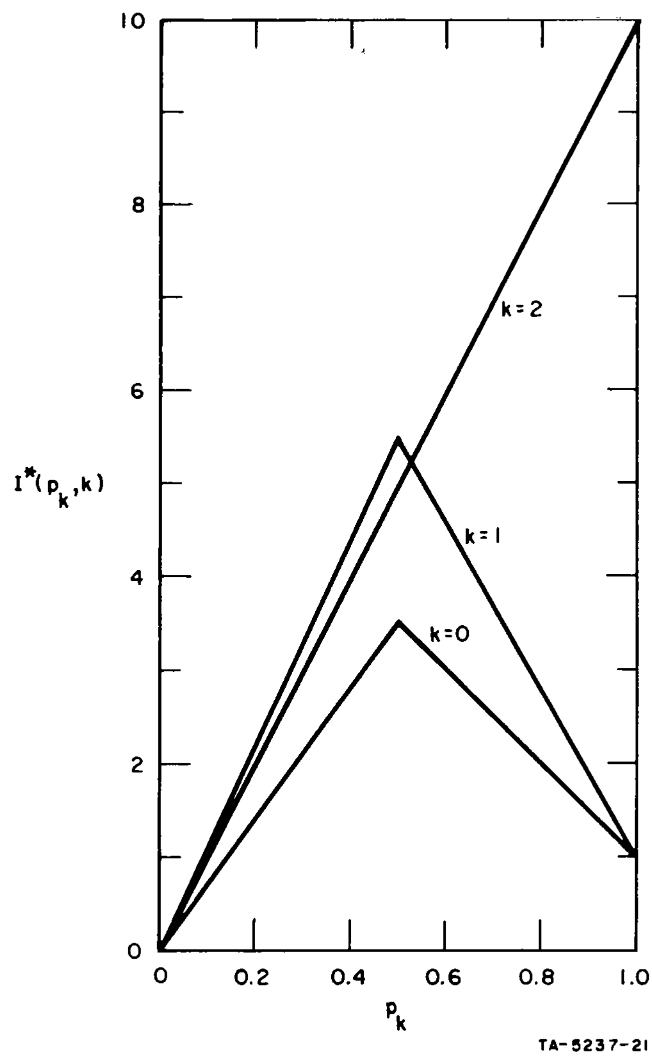

FIG. V-2 DESIRED STATE ZERO

Table V-3

OPTIMAL POLICY FOR DESIRED STATE ONE

k	0	1	2
$u_k^0(p_k)$	1 $\quad p_0 < \frac{1}{7}$ 0 $\quad \frac{1}{7} < p_0 < \frac{1}{2}$ 1 $\quad \frac{1}{2} < p_0 < \frac{6}{7}$ 6 $\quad \frac{6}{7} < p_0$	1 $\quad p_1 < \frac{1}{2}$ 0 $\quad p_1 > \frac{1}{2}$	1 $p_2 < \frac{1}{2}$ 0 $p_2 > \frac{1}{2}$
$I^*(p_k,k)$	$1 + 10p_0 \quad p_0 < \frac{1}{7}$ $2 + 3p_0 \quad \frac{1}{7} < p_0 < \frac{1}{2}$ $6 - 5p_0 \quad \frac{1}{2} < p_0 < \frac{6}{7}$ $12(1 - p_0) \quad \frac{6}{7} < p_0$	$1 + 9p_1, p_1 < \frac{1}{2}$ $11(1 - p_1), p_1 > \frac{1}{2}$	$10(1 - p_2)$

Since for $u_0 = 0$ and $z_0 = 0$, $p_1 < 1/2$ if $p_0 < 1/3$, we have from (5-10) and (5-22):

$$
\begin{aligned}
E\{I^*(p_1,1)/p_0,u_0 = 0\} &= \left(1 + 9\,\frac{\frac{1}{2}\,p_0}{1 - \frac{1}{2}\,p_0}\right)\left(1 - \frac{1}{2}\,p_0\right) + 0\,\frac{1}{2}\,p_0 \\
&= 1 + 4p_0 \qquad\qquad p_0 < \frac{2}{3} \\
&= \left(11 - 11\,\frac{\frac{1}{2}\,p_0}{1 - \frac{1}{2}\,p_0}\right)\left(1 - \frac{1}{2}\,p_0\right) \\
&= 11(1 - p_0) \qquad\qquad p_0 > \frac{2}{3}\ .
\end{aligned}
\tag{5-41}
$$

By symmetry

$$E\{I^*(p_1,1)/p_0,u_0 = 1\} = 11p_0 \qquad p_0 < \frac{1}{3}$$

$$= 5 - 4p_0 \qquad p_0 > \frac{1}{3} \qquad (5\text{-}42)$$

These two functions are plotted in fig. V-3, from which it is obvious that there exists a region about the origin in which u^I is applied rather than u^C. In fact

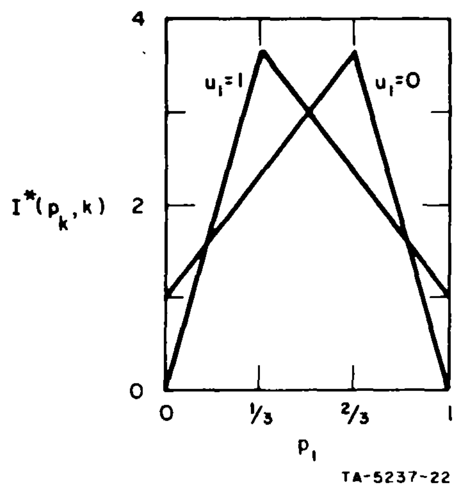

FIG. V-3 CALCULATION OF MINIMUM

$$\begin{aligned} u_1^0 &= 1 & p_0 &< \frac{1}{7} \\ &= 0 & \frac{1}{7} < p_0 &< \frac{1}{2} \\ &= 1 & \frac{1}{2} < p_0 &< \frac{6}{7} \\ &= 0 & \frac{6}{7} &< p_0 \end{aligned} \qquad (5\text{-}43)$$

The explanation for these results is simple. Because there is a very high cost for being in the wrong state at Time 2, it is important that the proper input is used at Time 1. Therefore, if at Time 0 the uncertainty is large (p_k is near 1/2) then it pays to use the control that reduces this uncertainty most.

Figure V-4 presents these results graphically with the scale reversed for easy comparison with fig. V-2. Such comparison indicates that in general, when $u^C \neq u^I$, we can expect a higher cost than when $u^C = u^I$.

Based upon this example, we can conclude that there exist cases where the input that provides most information differs from the input that is optimal from an immediate control point of view. In these cases, the optimal policy is a compromise between gaining information and bringing the plant to its desired state. In general when such a compromise must be made, the cost is higher than for analogous cases where the compromise is not necessary.

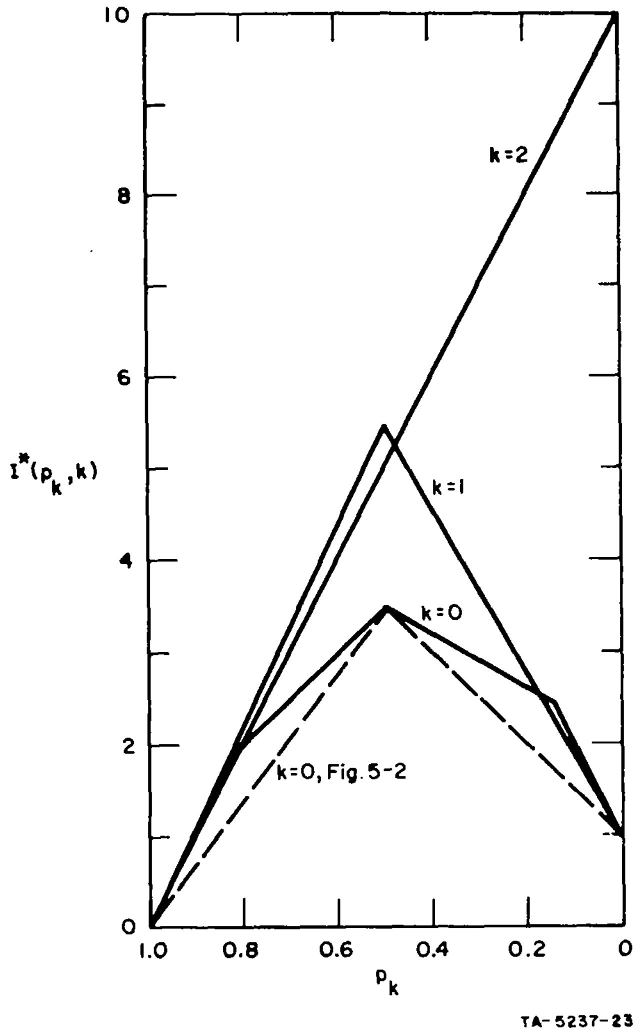

FIG. V-4 DESIRED STATE ONE

C. APPROXIMATIONS

To solve the two recursion relations (2-17) and (2-26) for the combined optimization, a digital computer will usually be required in the nonlinear case, in which case the variables will have to be quantized. To get some idea of the computational difficulty involved in performing the minimization to solve the control equation (2-17), consider a system with one input, one output, and three state variables, with each variable quantized into ten levels.

If we choose to specify the control as a function of the previous inputs and outputs, then at time $k = 15$ we must select one of ten levels for each of the $10^{15}\ 10^{16} = 10^{31}$ values of the argument.

On the other hand, suppose we choose to specify the control as a function of the probability density. At a given time, the system may be in any one of 10^3 states. Now suppose the probabilities are quantized into eleven uniform levels between 0 and 1. Since the total probability must be 1, there are ten lumps of probability 0.1, each lump of which can be assigned to one of the 10^3 states. Therefore, there are $(10^3)^9/9! = 10^{21.5}$ different possible probability distributions.

Thus, we have come up against the "curse of dimensionality", which plagues all numerical optimizations (ref. 17). Even if the states are exactly measurable, we must specify a control for each of 10^3 possible states; but in the combined problem, we appear to be cursed twice. Because of the obvious impossibility of calculating exact solutions, approximate techniques must be found. Two broad possibilities exist: to divide the problem into several smaller problems (*i.e.*, to partition the states), or to reduce the number of state variables used in computation.

If we base the controls on the previous outputs and inputs, a reasonable approach is to use only the past few outputs and inputs. In the above example, if we restrict our computations to the past three time increments, the number of arguments for which a control must be calculated is $10^3 - 10^2 \times 10^6$.

A reasonable approximation to control based upon the conditional probability distribution is control based upon the first few moments of this distribution. If the control is based on the conditional mean, then there are only 10^3 arguments for which controls must be calculated at each time in our example. However, the conditional mean contains no information about the uncertainty; thus, we would not expect the dual-control aspect to arise for this approximation. The conditional variance contains information about the uncertainty; in our example, use of conditional mean and variance yields

$10^3 \cdot 10^6 = 10^9$ arguments. Conditional entropy is a measure of uncertainty which may be easily calculated from the conditional probabilities; with the conditional mean and uncertainty, we have $10^3 \cdot 10 = 10^4$ arguments for the example.

D. BOUNDS ON PERFORMANCE

Upper and lower bounds upon the minimum cost in a given combined optimization problem may be calculated by solving related but simpler combined problems. For the given plant and disturbances, if J_{min} is the minimum cost with a perfect measurement system, J_{OL} is the minimum cost with no measurement system, and J the cost with the actual measurement system then

$$J_{min} \leq J \leq J_{OL} \quad . \tag{5-43}$$

These inequalities hold because the smaller member of each relation is obtained when information disregarded by the larger member is used; use of additional information optimally cannot degrade performance. (Of course, nonoptimal use of additional information may degrade performance).

VI CONCLUSIONS

The combined optimization problem—the problem of controlling a randomly disturbed plant on the basis of incomplete knowledge of the state of the plant—requires the solution of two iterative equations: the estimation equation which updates the conditional probability density of the state of the plant and the control equation which yields the optimal input as a function of this density.

A complete solution to the combined optimization problem exists if the system is linear, if the costs are quadratic, and if the random disturbances are Gaussian. In this case, the solution of the control and estimation equations reduces to finding two matrices by solving dual matrix difference equations. Optimum estimation reduces to the calculation of the conditional mean of the state, and optimum control is a linear function of this conditional mean. The cost is the sum of two parts: a transient cost due to initial conditions and an operating cost. The cost $\Delta\beta_k$ of operating for kth time interval further breaks down to the sum of a cost due to disturbances and a cost due to uncertainty about the state:

$$\Delta\beta_k = \mathrm{tr}\ [P\hat{Q} + P^* V] \tag{6-1}$$

where $\hat{Q}$ is the covariance matrix of the disturbance, V is the covariance matrix of the estimator error and P and P^* are matrices found in solution of the control equation.

Because it provides the optimal method of using information gathered by sensors and because it gives the optimal expected performance; the combined optimization problem is a natural framework for considering a variety of important control problems—in particular the problem of determining the effects of information-handling components on system performance. Feedback control is necessary because of incomplete *a priori* knowledge about the state of the plant. Since reduction of such uncertainty by means of measurement, filtering, and control action is not without cost, it is desirable to determine how complicated information gathering and processing must be for adequate control. Combined optimization theory considers both the difficulty of gathering information and the value, in terms of improved performance, of the information gathered; it therefore provides a standard for comparing alternative systems.

In the linear case P^* tells which state variables need to be known accurately and P indicates which state variables are most sensitive to disturbances. Furthermore the optimum estimator for this case, the Kalman filter, is no more complex than the plant to be controlled.

The general nonlinear combined optimization problem is considerably more complicated than the linear case because of the necessity to compromise the control action between the purposes of gaining information and taking the plant to a desired state. Exact solution of the problem in all but simple cases is impossible because of the extremely large dimension of the problem.

Suitable approximations can sometimes reduce the effective dimension of a combined optimization problem to the dimension of the corresponding optimal control problem in which the state is assumed to be exactly measurable. Research on these suitable approximations is an important task which should be taken in the immediate future because exorcizing the demon behind the "curse of dimensionality" is necessary if practical realization not only of combined optimal schemes, but even of classical optimal control schemes, are to be found for systems of large dimension.

APPENDIX A

IDENTITY INVOLVING EXPECTATION

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IDENTITY INVOLVING EXPECTATION

Suppose that we wish to find the expected value of some function $F(x)$ of the random variable x and that y is a second random variable. Then the following identity holds (ref. 18):

$$\underset{x}{E}\{F(x)\} = \underset{y}{E}\{\underset{x}{E}\{F(x)/y\}\} \tag{A-1}$$

If the necessary probability distributions exist, this relation may be proved as follows:

$$\underset{x}{E}\{F(x)\} = \int_x F(x)p(x)dx \quad . \tag{A-2}$$

But

$$p(x) = \int_y p(x,y)dy \tag{A-3}$$

and from Bayes's rule

$$p(x,y) = p(x/y)p(y) \quad . \tag{A-4}$$

Substitution of (A-3) and (A-4) in (A-2) yields

$$\underset{x}{E}\{F(x)\} = \int_x F(x) \int_y p(x/y)p(y)dydx \quad . \tag{A-5}$$

Finally, on interchange of integrations

$$\begin{aligned} \underset{x}{E}\{F(x)\} &= \int_y \int_x F(x)p(x/y)dxp(y)dy \\ &= \underset{y}{E}\{\underset{x}{E}\{F(x)/y\}\} \end{aligned} \tag{A-6}$$

Q.E.D.

APPENDIX B

LINEAR CASE

APPENDIX B

LINEAR CASE

1. LINEAR CONTROL

Suppose assumptions 1 and 2 given in Sec. III-A are true. Then

$$I^*(Z_N, U_{N-1}, N) = \min_{u_N} E\{(x_N^T Q_N x_N + u_N^T R_N u_N)/Z_N, U_N\} \quad . \tag{B-1}$$

We digress for a moment to evaluate $E(x^T Qx)$, where Q is any symmetric $n \times n$ matrix:

$$x^T Qx = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i Q_{ij} x_j \quad . \tag{B-2}$$

Hence

$$E\{x^T Qx\} = \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} E\{x_i x_j\} \quad , \tag{B-3}$$

but

$$E\{x_i, x_j\} = E\{x_i\}E\{x_j\} + E[(x_i - E\{x_i\})(x_j - E\{x_j\})] \quad ; \tag{B-4}$$

therefore, if

$$\bar{x} \triangleq E\{x\}$$

$$V \triangleq E\{(x - \bar{x})(x - \bar{x})^T\} \quad , \tag{B-5}$$

(B-4) becomes

$$E\{x^T Qx\} = \bar{x}^T Qx + \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} V_{ij}$$

$$= \bar{x}^T Qx + \mathrm{tr}\ [QV] \quad , \tag{B-6}$$

where tr $[A]$ stands for trace of A (*i.e.*, sum of diagonal terms).

Making use of (B-6), we may rewrite (B-1) as

$$I^*(\cdot, N) = \min_{u_N} \{\bar{x}_{N/N}^T Q_N \bar{x}_{N/N} + \mathrm{tr}\ [Q_N V_N] + u_N^T R_N u_N\} \quad ; \tag{B-7}$$

where $\bar{x}_{k/k}$ and V_k are as defined by (3-8) and (3-9).

It is obvious that $u_N = 0$ and

$$I^*(Z_N, U_{N-1}, N) = \bar{x}^T_{N/N} Q_N \bar{x}_{N/N} + \text{tr}\,[R_N V_N] \tag{B-8}$$

Since $I^*(\cdot, N)$ is quadratic in $\bar{x}_{N/N}$ and since the system is linear and cost quadratic, $I(\cdot, k)$ will be quadratic in $\bar{x}_{k/k}$. Hence, we may write

$$I^*(Z_k, U_{k-1}, k) = \bar{x}^T_{k/k} P_k \bar{x}_{k/k} + b_k \quad , \tag{B-9}$$

where from (B-8)

$$P_N = Q_N \tag{B-10}$$

$$b_N = \text{tr}\,[Q_N V_N] \quad . \tag{B-11}$$

When (B-9) is substituted in recursion relation (2-17):

$$\bar{x}^T_{k/k} P_k \bar{x}_{k/k} + b_k = \min_{u_k} E\{[x_k^T Q_k x_k + u_k^T R_k u_k + \bar{x}^T_{k+1/k+1} P_{k+1} \bar{x}_{k+1/k+1} + b_{k+1}]/Z_k, U_{k-1}\} \quad . \tag{B-12}$$

But from (B-6),

$$E\{x^T_{k+1} P_{k+1} x_{k+1}/Z_{k+1}, U_k\} = \bar{x}^T_{k+1/k+1} P_{k+1} \bar{x}_{k+1/k+1} + \text{tr}\,[P_{k+1} V_{k+1}] \quad , \tag{B-13}$$

and from (A-6), (3-1), and (3-4b)

$$\begin{aligned} \underset{z_{k+1}}{E}\,[E\{\bar{x}^T_{k+1} P_{k+1} \bar{x}_{k+1}/Z_{k+1}, U_k\}/Z_k, U_k] &= E\{x^T_{k+1} P_{k+1} x_{k+1}/Z_k, U_k\} \\ &= (A_k \bar{x}_{k/k} + B_k u_k)^T P_{k+1} (A_k \bar{x}_{k/k} + B_k u_k) \\ &\quad + \text{tr}\,[P_{k+1}(A_k V_k A_k^T + \hat{Q}_k)] \quad . \end{aligned} \tag{B-14}$$

Therefore,

$$\begin{aligned} E\{\bar{x}^T_{k+1/k+1} P_{k+1} \bar{x}_{k+1/k+1}/Z_k, U_k\} &= (A_k \bar{x}_{k+k} + B_k u_k)^T P_{k+1} (A_k \bar{x}_{k/k} + B_k u_k) \\ &\quad + \text{tr}\,[P_{k+1}(A_k V_k A_k^T \doteq V_{k+1} + \hat{Q}_k)] \end{aligned} \tag{B-15}$$

Finally substitution of (B-15) into (B-12) yields

$$\bar{x}^T_{k/k}P_k\bar{x}_{k/k} + b_k = \min_{u_k}\Big(\bar{x}^T_{k/k}Q_k\bar{x}_{k/k} + \mathrm{tr}\,[Q_kV_k] + u_k^TR_ku_k$$

$$+ (A_k\bar{x}_{k/k} + B_ku_k)^TP_{k+1}(A_k\bar{x}_{k/k} + B_ku_k)$$

$$+ \mathrm{tr}\,[P_{k+1}(A_kV_kA_k^T - V_{k+1} + \hat{Q}_k)] + E\{b_{k+1}/Z_k, U_k\}\Big) \quad . \qquad \text{(B-16)}$$

If the square on u_k is completed, (B-16) reduces to

$$\bar{x}^T_{k/k}P_k\bar{x}_{k/k} + b_k = \min_{u_k}\Big((u_k + G_kx_{k/k-1})^T(B_k^TP_{k+1}B_k + R_k)(u_k + G_kx_{k/k-1})$$

$$+ \bar{x}^T_{k/k-1}[Q_k + A_k^TP_{k+1}A_k + G_k^T(B_k^TP_{k+1}B_k + R_k)G_k]\bar{x}_{k/k-1}$$

$$+ \mathrm{tr}\,[Q_kV_k + P_{k+1}(A_kV_kA_k^T - V_{k+1} + \hat{Q}_k)] + E\{b_{k+1}/Z_k, U_k\}\Big)$$

(B-17)

where G_k is

$$\boxed{G_k = (B_k^TP_{k+1}B_k + R_k)^{-1}B_k^TP_{k+1}A_k} \quad . \qquad \text{(B-18)}$$

From (B-17), it is clear that the optimum control is given by

$$\boxed{u_k = -G_kx_{k/k}} \qquad \text{(B-19)}$$

and that

$$P_k = Q_k + A_k^TP_{k+1}A_k - G_k^T(B_k^TP_{k+1}B_k + R_k)G_k$$

$$\boxed{P_k = Q_k + A_k^TP_{k+1}A_k - A_k^TP_{k+1}B_k(B_k^TP_{k+1}B_k + R_k)^{-1}B_k^TP_{k+1}A_k} \qquad \text{(B-20)}$$

and

$$\boxed{b_k = \mathrm{tr}\,[Q_kV_k + P_{k+1}(A_kV_kA_k^T - V_{k+1} + \hat{Q}_k)] + E\{b_{k+1}/Z_k, U_k\}} \quad . \qquad \text{(B-21)}$$

2. LINEAR ESTIMATION

Suppose assumptions 1 and 3 of Sec. III-A hold, then all conditional probabilities will be Gaussian. From the state equation (3-1)

$$\bar{x}_{k+1/k} \triangleq E\{x_{k+1}/Z_k, U_k, u_0\} = A_k\bar{x}_{k/k} + B_ku_k \qquad \text{(B-22)}$$

and from (3-4b)

$$E\{(x_{k+1} - \bar{x}_{k+1/k})(x_{k+1} - \bar{x}_{k+1/k})^T/Z_k, U_k\} = A_k V_{k/k} A_k^T + \hat{Q}_k \quad , \qquad \text{(B-23)}$$

since w_k is independent of x_k. Therefore,

$$p(x_{k+1}/Z_k, U_k) = c_4 \exp \{(x_{k+1} - A_k\bar{x}_{k/k} - B_k u_k)[A_k V_k A_k^T + \hat{Q}_k]^{-1}(x_{k+1} - A_k\bar{x}_{k/k} - B_k u_k)\} \quad . \qquad \text{(B-24)}$$

From the measurement equation (3-2) and from (3-4c)

$$p(z_{k+1}/x_{k+1}) = c_5 \exp [(z_{k+1} - C_{k+1}x_{k+1})^T \hat{R}_{k+1}^{-1}(z_{k+1} - C_{k+1}x_k)] \quad . \qquad \text{(B-25)}$$

Hence, the numerator α of (2-26) is

$$\alpha \overset{\Delta}{=} c_6 \exp \{(x_{k+1} - A_k\bar{x}_{k/k} - B_k u_k)^T[A_k V_k A_k^T + \hat{Q}_k]^{-1}(x_{k+1} - A_k\bar{x}_{k/k} - B_k u_k)$$

$$+ (z_{k+1} - C_{k+1}x_{k+1})^T \hat{R}_{k+1}^{-1}(z_{k+1} - C_{k+1}x_{k+1})\} \qquad \text{(B-26)}$$

Completion of the square on x_{k+1} yields

$$\alpha = c_7 \exp (\bar{x}_{k+1} - \bar{x}_{k+1/k+1})^T V_{k+1}^{-1}(x_{k+1} - \bar{x}_{k+1/k+1}) \cdot \exp [\text{other terms}] \qquad \text{(B-27)}$$

where

$$\boxed{V_{k+1}^{-1} = [A_k V_k A_k^T + \hat{Q}_k]^{-1} + C_{k+1}^T \hat{R}_{k+1}^{-1} C_{k+1}} \qquad \text{(B-28)}$$

$$\bar{x}_{k+1/k+1} = V_{k+1}\{[A_k V_k A_k^T + \hat{Q}_k]^{-1}(A_k\bar{x}_{k/k} - B_k u_k) + C_{k+1}^T R_{k+1}^{-1} z_{k+1}\}$$

$$\boxed{\bar{x}_{k+1/k+1} = A_k\bar{x}_{k/k} - B_k u_k + K_{k+1}[z_{k+1} - C_{k+1}(A_k\bar{x}_{k/k} - B_k u_k)]} \qquad \text{(B-29)}$$

and

$$\boxed{K_{k+1} = V_{k+1} C_{k+1}^T \hat{R}_{k+1}^{-1}} \qquad \text{(B-30)}$$

From (2-26)

$$p(x_{k+1}/x_k, \ldots, z_0, u_k, \ldots, u_0) = \frac{\alpha}{\int_{x_{k+1}} \alpha dx_{k+1}} \qquad \text{(B-31)}$$

But since the first exponential in (B-27) is just a probability distribution except for a constant and since the other exponent is independent of x_{k+1}:

$$\int_{x_{k+1}} \alpha dx_{k+1} = c_8 \exp \text{[other terms]} \tag{B-32}$$

and

$$p(x_{k+1}/Z_k, U_k) = c_9 \exp (x_{k+1} - \bar{x}_{k+1/k+1})^T V_{k+1}^{-1} (x_{k+1} - \bar{x}_{k+1/k+1}) \tag{B-33}$$

The initial conditions $\bar{x}_{0/0}$ and V_0 are calculated as follows: From (2-27) and (3-49)

$$p(x_0/Z_0, U_{-1}) = \frac{p(z_0/x_0)p(x_0)}{\int_{x_0} p(z_0/x_0)p(x_0)}$$

$$= \frac{\exp\ [(z_0 - C_0x_0)^T\hat{R}_0^{-1}(z_0 - C_0x_0) + (x_0 - \bar{x}_0)^T(\hat{Q}_{-1})^{-1}(x_0 - \bar{x}_0)]}{\int_{x_0} \exp\ [(z_0 - C_0x_0)^T\hat{R}_0^{-1}(z_0 - C_0x_0) + (x_0 - \bar{x}_0)^T(\hat{Q}_{-1})^{-1}(x_0 - \bar{x}_0)]dx_0}$$

$$= c_{10} \exp \{[x_0 - \bar{x}_0 - K_0(z_0 - C_0\bar{x}_0)]^T[C_0^T\hat{R}^{-1}C_0 + (\hat{Q}_{-1})^{-1}]$$

$$\cdot\ [x_0 - \bar{x}_0 - K_0(z_0 - C_0x_0)]\} \quad . \tag{B-34}$$

Therefore,

$$\bar{x}_{0/0} = \bar{x}_0 + K_0(z_0 - C_0\bar{x}_0) \tag{B-35}$$

$$V_0^{-1} = (\hat{Q}_{-1})^{-1} + C_0^T\hat{R}_0^{-1}C_0 \quad . \tag{B-36}$$

Therefore by use of (B-28), (B-29), and (B-30), the conditional probability distribution may be updated. Note that V_k and K_k can be calculated *a priori*, and only $\bar{x}_{k/k}$ need be generated in real time. Equation (B-29) represents a linear system for updating the conditional mean $\bar{x}_{k/k}$, which in fact, is the Kalman Filter.

3. DUALITY PRINCIPLE

The duality principle (ref. 4) is just recognition that in the linear case the control problem is equivalent mathematically to the estimation problem. Equations (B-29) and (B-30), which govern estimation, do not appear similar to (B-20) and (B-19), which govern control because of the formulation of the combined optimization chosen.

In the given formulation, it was assumed that the present measurement is available for use in determining the present control; an alternate formulation would have the assumption that the present measurement is not available until the next control. Such a formulation would have given the estimation result for the linear case in dual form; however, the given formulation was chosen because the alternate formulation is a special case of the given formulation. The assumption that the measurement is not available until the next time can be handled in the given formulation by augmenting the state variables and using the additional state variables to delay the measurements.

The cost of greater generality in formulation is greater difficulty in deriving the duality principle. The above discussion suggests that we replace V_k, the covariance of the present state given all information up to the present, by $\hat{P}_k$ defined to be the covariance of the next state given all information up to the present. $\hat{P}_k$ and V_k are related by

$$\hat{P}_k = A_k V_k A_k^T + \hat{Q}_k \tag{B-37}$$

If (B-37) is substituted into (B-28):

$$V_{k+1}^{-1} = \hat{P}_k^{-1} + C_{k+1}^T \hat{R}_{k+1}^{-1} C_{k+1} \tag{B-38}$$

which when inverted* yields

$$V_{k+1} = \hat{P}_k - \hat{P}_k C_{k+1}^T (C_{k+1} \hat{P}_k C_{k+1}^T + R_{k+1})^{-1} C_{k+1} \hat{P}_k \tag{B-39}$$

Use of (B-37) a second time yields

$$\hat{P}_{k+1} = \hat{Q}_{k+1} + A_{k+1} \hat{P}_k A_{k+1}^T - A_{k+1} \hat{P}_k C_{k+1}^T (C_{k+1} \hat{P}_k C_{k+1}^T + \hat{R}_{k+1})^{-1} C_{k+1} \hat{P}_k A_{k+1}^T \tag{B-40}$$

If (B-39) is substituted into (B-30)

$$K_{k+1} = \hat{P}_k C_{k+1}^T \hat{R}_{k+1}^{-1} - \hat{P}_k C_{k+1} (C_{k+1} \hat{P}_k C_{k+1}^T + \hat{R}_{k+1})^{-1} C_{k+1} \hat{P}_k C_{k+1}^T \hat{R}_{k+1}^{-1} \tag{B-41}$$

---

* This result may be verified by multiplying (B-38) times (B-39):

$$\begin{aligned} V_{k+1}^{-1} V_{k+1} &= I + C_{k+1}^T \hat{R}_{k+1}^{-1} C_{k+1} \hat{P}_k - C_{k+1}^T (C_{k+1} \hat{P}_k C_{k+1}^T + \hat{R}_{k+1})^{-1} C_{k+1} \hat{P}_k - C_{k+1}^T R_{k+1}^{-1} C_{k+1} \hat{P}_k C_{k+1}^T (C_{k+1} \hat{P}_k C_{k+1}^T + \hat{R}_{k+1})^{-1} C_{k+1} \hat{P}_k \\ &= I + C_{k+1}^T \hat{R}_{k+1}^{-1} C_{k+1} - C_{k+1}^T \hat{R}_{k+1}^{-1} (\hat{R}_{k+1} + C_{k+1} \hat{P}_k C_{k+1}^T)(C_{k+1} \hat{P}_k C_{k+1} + \hat{R}_{k+1})^{-1} C_{k+1} \hat{P}_k = I \quad . \end{aligned}$$

The quantity

$$I = (C_{k+1}\hat{P}_k C_{k+1}^T + \hat{R}_{k+1})^{-1}(C_{k+1}\hat{P}_k C_{k+1}^T + \hat{R}_{k+1}) \qquad \text{(B-42)}$$

may be substituted between C_{k+1}^T and $\hat{R}_{k+1}^{-1}$ in the first term of (B-41) to yield

$$K_{k+1} = \hat{P}_k C_{k+1}^T (C_{k+1}\hat{P}_k C_{k+1}^T + \hat{R}_{k+1})^{-1}(C_{k+1}\hat{P}_k C_{k+1}^T + \hat{R}_{k+1})\hat{R}_{k+1}^{-1}$$

$$- \hat{P}_k C_{k+1}^T (C_{k+1}\hat{P}_k C_{k+1}^T + \hat{R}_{k+1})^{-1} C_{k+1}\hat{P}_k C_{k+1}^T \hat{R}_{k+1}^{-1}$$

$$K_{k+1} = \hat{P}_k C_{k+1}^T (C_{k+1}\hat{P}_k C_{k+1}^T + \hat{R}_k)^{-1} \quad . \qquad \text{(B-43)}$$

This last trick is nothing more than putting expression (B-41) over a common denominator.

4. CALCULATION OF J

From (2-14) and (B-9) the minimum cost J^0 in the linear case is given by

$$J^0 = E\{I(\bar{x}_{0/0}, 0)\} = E\{\bar{x}_{0/0}^T P_0 \bar{x}_{0/0} + b_0\} \qquad \text{(B-44)}$$

where

$$\bar{x}_{0/0} = E\{x_0/z_0\}$$

and where P_k and b_k are given by (B-20) and (B-21).

Since Q_k, V_k, P_k, and $\hat{Q}_k$ are independent of Z_k and U_k then (B-21) b_k will be independent of these quantities if b_{k+1} is. But this is true for b_N; hence, it is true by induction for all b_k, and (B-21) may be rewritten

$$b_k = \mathrm{tr}\ [Q_k V_k + P_{k+1}(A_k V_k A_k^T - V_{k+1} + \hat{Q}_k)] + b_{k+1} \quad . \qquad \text{(B-45)}$$

Now let

$$\beta_k \triangleq b_k - \mathrm{tr}\ [P_k V_k] \qquad k \le N$$

$$= \mathrm{tr}\ [(Q_k - P_k)V_k + P_{k+1}(A_k V_k A_k^T + \hat{Q}_k)] + \beta_{k+1} \qquad \text{(B-46)}$$

and

$$\Delta\beta_{k+1} \triangleq \beta_k - \beta_{k+1} = \mathrm{tr}\ [(M_k - P_k)V_k + P_{k+1}(A_k V_k A_k^T + \hat{Q}_k)] \quad ;$$

$$k \le N \qquad \text{(B-47)}$$

then

$$J^0 = E\{x^T_{0/0}Px_{0/0} + \mathrm{tr}\ [P_0V_0] + \sum_{k=1}^{N} \Delta\beta_k\} \quad . \tag{B-48}$$

Use of the identity (B-6) on the expected value of a quadratic form A, results in

$$J^0 = E[E\{x_0^TP_0x_0/z_0\}] + \sum_{k=1}^{N} \Delta\beta_k \tag{B-49}$$

and use of the identity on conditioned expectations proven in A yields

$$J^0 = E\{x_0^TP_0x_0\} + \sum_{k=1}^{N} \Delta\beta_k = \bar{x}_0^TP_0\bar{x}_0 + \mathrm{tr}\ [P_0V] + \sum_{k=1}^{N} \Delta\beta_k \tag{B-50}$$

where following the notation given in Sec. III

$$\bar{x}_0 = E\{x_0\}$$

$$V = E\{x_0^Tx_0\} \quad .$$

Rewriting (B-47) with the aid of the identity

$$\mathrm{tr}\ [AB] = \mathrm{tr}\ [BA] \quad , \tag{B-51}$$

we have

$$\Delta\beta_k = \mathrm{tr}\ [P_{k+1}\hat{Q}_k + P^*_{k+1}V_k] \quad , \tag{B-52}$$

where

$$P^*_{k+1} \triangleq Q_k + A_k^TP_{k+1}A_k - P_k \quad . \tag{B-53}$$

Now let us calculate the cost J for the situation pictures in fig. III-1, but where G_k and K_k are not necessarily optimum. Before actually calculating J, estimation is considered.

It is convenient to define

$$\tilde{x}_k = x_k - \hat{x}_k \quad . \tag{B-54}$$

From fig. III-1

$$\tilde{x}_{k+1} = (I - K_{k+1}C_{k+1})A_k\tilde{x}_k + K_{k+1}(C_{k+1}w_k + v_{k+1}) - w_k \quad . \tag{B-55}$$

Since v_k and w_k have zero mean, it follows that if $\hat{x}_0 = \bar{x}_{0/0}$ then $E\{\tilde{x}_0/z_0\} = 0$, and

$$E\{\tilde{x}_k/Z_k, U_{k-1}\} = 0$$

or

$$E\{\hat{x}_k/Z_k, U_{k-1}\} = x_{\bar{k}/k} \quad . \tag{B-56}$$

Hence this suboptimal estimate is, like the optimal estimate, unbiased; it will however have a greater variance;

$$\tilde{V}_k = E\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T/Z_k, U_{k-1}\} \quad , \tag{B-57}$$

than the optimal estimate.

To find $\tilde{V}_k$ we take an approach analogous (but dual) to that used for suboptimal control in Sec. III-C. Let (B-55) describe a system whose state is to be estimated but for which no output exists, *i.e.*, for which the measurement equation is given by

$$\tilde{z}_k = \tilde{C}_k \tilde{x}_k + \tilde{v}_k \quad , \tag{B-58}$$

where

$$\tilde{C}_k = 0 \quad . \tag{B-59}$$

The situation is shown in fig. B-1. Since the conditional mean of $\tilde{x}_k$ is zero, the variance equation (B-28) can be used to calculate $\hat{V}_k$:

$$\tilde{V}_{k+1} = (I_k - K_{k+1}C_{k+1})A_k\tilde{V}_kA_k^T(I - K_{k+1}C_{k+1})^T + \tilde{Q}_k \tag{B-60}$$

where

$$\tilde{Q}_k \triangleq E\{\tilde{w}_k\tilde{w}^T\} = (I - K_{k+1}C_{k+1})\hat{Q}_k(I - K_{k+1}C_k) + \hat{R}_{k+1} \quad . \tag{B-61}$$

From (B-56) and (3-5)

$$\tilde{x}_0 = x_0 - \bar{x}_0 + K_0(z_0 - C_0x_0) \tag{B-62}$$

and so

$$\tilde{V}_0 = (I - K_0C_0)\hat{Q}_{-1}(I - K_0C_0)^T + K_0\hat{R}_0K_0 = \tilde{Q}_{-1} \tag{B-63}$$

We now return to the combined problem. The system shown in fig. III-1 can be described in terms of the state vector

$$\underline{x}_k = \begin{bmatrix} \bar{x}_k \\ \\ \tilde{x}_k \end{bmatrix} = \begin{bmatrix} x_k \\ \\ \hat{x}_k - x_k \end{bmatrix} . \qquad \text{(B-64)}$$

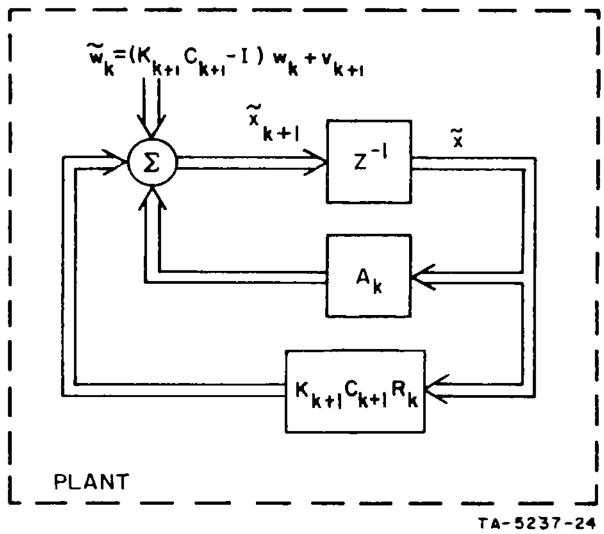

FIG. B-1 SUBOPTIMAL ESTIMATION

The state equation for the system may be found by inspection of fig. III-1 to be

$$\underline{x}_{k+1} = \underline{A}_k \underline{x}_k + \underline{w}_k \quad , \qquad \text{(B-65)}$$

where

$$\underline{A}_k = \begin{bmatrix} A_k - B_k G_k & - B_k G_k \\ \\ 0 & (I - K_{k+1} C_{k+1}) A_k \end{bmatrix} \qquad \text{(B-66)}$$

$$\underline{w}_k = \begin{bmatrix} w_k \\ \\ K_{k+1}(C_{k+1} w_k + v_{k+1}) - w_k \end{bmatrix} . \qquad \text{(B-67)}$$

The statistics of $\underline{w}_k$ are

$$\underline{Q}_k \triangleq E\{\underline{w}_k \underline{w}_k^T\} = \begin{bmatrix} \hat{Q}_k & -\hat{Q}_k (I - K_{k+1} C_{k+1})^T \\ \\ -(I - K_{k+1} C_{k+1}) \hat{Q}_k & \tilde{Q}_k \end{bmatrix} . \qquad \text{(B-68)}$$

We wish to calculate the performance of this system with cost

$$\underline{J} = \sum_{k=0}^{N} (x_k^T Q_k x_k + u_k^T R_k u_k) = \sum_{k=0}^{N} [x_k^T Q_k x_k + (x_k + \tilde{x}_k)^T G_k^T R_k G_k (x_k + \tilde{x}_k)] = \sum_{k=0}^{N} \underline{x}_k^T \underline{Q}_k \underline{x}_k \tag{B-69}$$

where

$$\underline{Q}_k = \begin{bmatrix} Q_k + G_k^T R_k G_k & G_k^T R_k G_k \\ G_k^T R_k G_k & G_k^T R_k G_k \end{bmatrix} \tag{B-70}$$

To perform this calculation, we assume the system is a plant to be controlled with no input or output. For this case the cost is

$$\underline{J} = \overline{\underline{x}}_0^T \underline{P}_0 \overline{\underline{x}}_0 + \operatorname{tr} [\underline{P}_0 \underline{V}_0] + \sum_{k=1}^{N} \Delta\underline{\beta}_k \tag{B-71}$$

where

$$\underline{P}_k = \underline{A}_k^T \underline{P}_{k+1} \underline{A}_k \qquad k < N$$

$$\underline{P}_N = \underline{Q}_N \tag{B-72}$$

and

$$\overline{\underline{x}}_0 = \begin{bmatrix} \overline{x}_0 \\ 0 \end{bmatrix} \tag{B-73}$$

and

$$\underline{V}_0 = \begin{bmatrix} \hat{Q}_{-1} & -\hat{Q}_{-1}(I - K_{-0}C_0)^T \\ -(I - K_{-0}C_0)\hat{Q}_{-1} & (I - K_{-0}C_0)\hat{Q}_{-1}(I - K_{-0}C_0)^T + K_{-0}^T R_0 K_{-0} \end{bmatrix} \tag{B-74}$$

and

$$\Delta\underline{\beta}_{k+1} = \operatorname{tr} [\underline{P}_{k+1} \underline{Q}_k] \quad . \qquad k < N \tag{B-75}$$

(Note $P_k^* \equiv 0$).

Consider first (B-72). Partitioning

$$\underline{P}_k = \begin{bmatrix} P_{1,k} & P_{12,k} \\ P_{12,k}^T & P_{2,k} \end{bmatrix} \tag{B-76}$$

and substitution into (B-72) gives (if the optimum value $G_N = 0$ is chosen)

$$P_{1,N} = Q_N \tag{B-77}$$

$$P_{12,N} = P_{2,N} = 0 \tag{B-78}$$

$$P_{1,k} = (A - B_k G_k)^T P_{1,k+1}(A_k - B_k G_k) + Q_k + G_k^T R_k G_k \tag{B-79}$$

$$P_{12,k} = (A_k - B_k G_k)^T [P_{12,k+1}(I - K_{k+1}C_{k+1})A_k - P_{1,k+1}B_k G_k] + G_k^T R_k G_k \tag{B-80}$$

$$P_{2,k} = A_k^T(I - K_{k+1}C_{k+1})^T P_{2,k+1}(I - K_{k+1}C_{k+1})A_k - A_k^T(I - K_{k+1}C_{k+1})^T P_{12,k+1}^T B_k G_k$$

$$- G_k^T B_k^T P_{12,k+1}(I - K_{k+1}C_{k+1})A_k + G_k^T(B_k^T P_{1,k+1}B_k + R_k)G_k \quad . \tag{B-81}$$

Comparing (B-79) with (3-35) we note that

$$P_{1,k} = \tilde{P}_k \quad . \tag{B-82}$$

The optimum G_k satisfies

$$A_k \tilde{P}_{k+1} B_k = G_k^T(B_k^T \tilde{P}_{k+1} B_k + R_k) \quad ; \tag{B-83}$$

hence, if optimum control is used

$$P_{12,k} = (A_k - B_k G_k)^T P_{12,k+1}(I - K_{k+1}C_{k+1}) \tag{B-84}$$

But since $P_{12,N} = 0$, (B-84) implies that for optimum control

$$P_{12,k} = 0 \quad . \tag{B-85}$$

Substitution of the partitioned $\underline{P}$ and $\underline{Q}$ from (B-76) and (B-68) into (B-75) yields

$$\Delta\mathcal{B}_{k+1} = \operatorname{tr}\, [\tilde{P}_{k+1}\hat{Q}_k - P_{12,k+1}^T \hat{Q}_k (I - K_k C_{k+1})^T - P_{12,k+1}(I - K_k C_{k+1})\hat{Q}_k + P_{2,k+1}\tilde{Q}_k] \quad . \tag{B-86}$$

From (B-60)

$$\tilde{Q}_k = \tilde{V}_{k+1} - (I - K_{k+1}C_{k+1})A_k \tilde{V}_k A_k^T (I - K_{k+1}C_{k+1})^T \tag{B-87}$$

therefore

$$\mathrm{tr}\ [P_{2,k+1}\tilde{Q}_k] = \mathrm{tr}\ [P_{2,k+1}\tilde{V}_{k+1} - P_{2;k+1}(I - K_{k+1}C_{k+1})A_k\tilde{V}_kA_k^T(I - K_{k+1}C_{k+1})^T]$$

$$= \mathrm{tr}\ [P_{2,k+1}\tilde{V}_{k+1} - A_k^T(I - K_{k+1}C_{k+1})^TP_{2,k+1}(I - K_{k+1}C_{k+1})A_k\tilde{V}_k] \quad . \text{(B-88)}$$

Use of (B-81) results in

$$\mathrm{tr}\ [\dot{P}_{2,k+1}\tilde{Q}_k] = \mathrm{tr}\ [P_{2,k+1}\tilde{V}_{k+1}] - \mathrm{tr}\ [P_{2,k}\tilde{V}_k]$$

$$+ \mathrm{tr}\ \{[G_k^T(B_k^T\tilde{P}_{k+1}B_k + R_k)G_k - A_k^T(I - K_{k+1}C_{k+1})^TP_{12,k+1}B_kG_k$$

$$- G_k^TB_k^TP_{12,k+1}(I - K_{k+1}C_{k+1})A_k]\tilde{V}_k\} \quad . \qquad \text{(B-89)}$$

By (B-80)

$$\mathrm{tr}\ [(P_{12,k} + P_{12,k}^T)\tilde{V}_k] = \mathrm{tr}\ \{[(A_k - B_kG_k)^TP_{12,k+1}(I - K_{k+1}C_{k+1})A_k$$

$$+ A_k^T(I - K_{k+1}C_{k+1})^TP_{12,k+1}^T(A_k - B_kG_k)$$

$$+ A_k^T\tilde{P}_{k+1}B_kG_k + B_k^TG_k^T\tilde{P}_{k+1}A_k$$

$$- 2G_k^T(B_k^T\tilde{P}_{k+1}B_k + R_k)G_k]\tilde{V}_k\} \qquad \text{(B-90)}$$

Subtraction of (B-90) from (B-89) with the help of (B-60) yields

$$\mathrm{tr}\ [P_{2,k+1}\tilde{Q}_k] = \mathrm{tr}\ [P_{2,k+1}\hat{V}_{k+1}] - \mathrm{tr}\ [P_{2,k}\hat{V}_k] + \mathrm{tr}\ [\tilde{P}_{k+1}^*\tilde{V}_k]$$

$$- \mathrm{tr}\ \{P_{12,k+1}^T[(I - K_{k+1}C_{k+1})A_k\hat{V}_kA_k^T]\}$$

$$- \mathrm{tr}\ \{P_{12,k+1}^T[A_k\tilde{V}_kA_k^T(I - K_{k+1}C_{k+1})^T]\} + \mathrm{tr}\ [(P_{12,k} + P_{12,k}^T)\tilde{V}_k] \quad . \text{(B-91)}$$

If

$$P_{k+1}' \triangleq [(I - K_{k+1}C_{k+1})^T]^{-1}P_{12,k+1} \qquad \text{(B-92)}$$

then, because of (B-60),

$$\text{tr}\ [P_{12,k+1}(I - K_{k+1}C_{k+1})\hat{Q}_k] = \text{tr}\ [P'_{k+1}(I - K_{k+1}C_{k+1})\hat{Q}_k(I - K_{k+1}C_{k+1})^T]$$

$$= \text{tr}\ [P'_{k+1}(\tilde{V}_{k+1} - K_{k+1}\hat{R}_{k+1}K_k^T)]$$

$$- \text{tr}\ [P_{12,k+1}(I - K_{k+1}C_{k+1})A_k\tilde{V}_kA_K^T] \quad . \qquad \text{(B-93)}$$

Note that for optimal control since $P_{12,k}$ is zero so is P'_k. Finally returning to (B 86) we have, using (B-91) and (B-93)

$$\Delta\underline{\beta}_{k+1} = \text{tr}\ [P_{2,k+1}\tilde{V}_{k+1}] - \text{tr}\ [P_{2,k}\tilde{V}_k] + \text{tr}\ [(P_{12,k} + P_{12,k}^T)\tilde{V}_k]$$

$$+ \text{tr}\ [\tilde{P}_{k+1}\hat{Q}_k + \tilde{P}^*_{k+1}\tilde{V}_k] + \text{tr}\ [(P'_{k+1} + P'^T_{k+1})\ (\tilde{V}_{k+1} - K_{k+1}\hat{R}_{k+1}K_{k+1}^T)] \quad . \qquad \text{(B-94)}$$

But since

$$\text{tr}\ [AB] = \text{tr}\ [B^TA^T] = \text{tr}\ [A^TB^T] \qquad \text{(B-95)}$$

(B-94) can be written

$$\Delta\underline{\beta}_{k+1} = \text{tr}\ [P_{2,k+1}\tilde{V}_{k+1}] - \text{tr}\ [P_{2,k}\tilde{V}_k] - 2\ \text{tr}\ [P'_{k+1}(\tilde{V}_{k+1} - K_{k+1}\hat{R}_{k+1}K_{k+1})]$$

$$+ 2\ \text{tr}\ [P'_k(\tilde{V}_k - K_k\hat{R}_kK_k)] + \Delta\beta_{k+1} \qquad \text{(B-96)}$$

where

$$\Delta\beta_{k+1} \triangleq \text{tr}\ [P_{k+1}\hat{Q}_k + \tilde{P}^*_{k+1}\tilde{V}_k + 2K_k^TP'_kK_k\hat{R}_k - 2P'_kK_kC_k\tilde{V}_k] \quad . \qquad \text{(B-97)}$$

If the $\Delta\underline{\beta}_k$'s are summed we get

$$\sum_{k=1}^{N} \Delta\underline{\beta}_k = -\text{tr}\ [P_{2,0}\tilde{V}_0] + 2\ \text{tr}\ [P'_0(\tilde{V}_0 - K_0\hat{R}_0K_0)] + \sum_{k=1}^{N} \Delta\beta_k \quad . \qquad \text{(B-98)}$$

Furthermore

$$\text{tr}\ [\underline{P}_0\underline{V}_0] = \text{tr}\ [\tilde{P}_0\hat{Q}_{-1}] - 2\ \text{tr}\ [P_{12,0}(I - K_0C_0)\hat{Q}_{-1}]$$

$$+ \text{tr}\ \{P_{2,0}[(I - K_0C_0)\hat{Q}_{-1}(I - K_0C_0)^T + K_0R_0K_0^T]\} \quad . \qquad \text{(B-99)}$$

Since

$$\tilde{V}_0 = (I - K_0 C_0) V (I - K_0 C_0)^T + K_0 R_0 K_0^T \quad , \tag{B-100}$$

we have

$$\mathrm{tr}\ [P_0 \underline{V}_0] = \mathrm{tr}\ [\tilde{P}_0 \hat{Q}_{-1}] - 2\ \mathrm{tr}\ [P_0' (\tilde{V}_0 - K_0 \hat{R}_0 K_0^T)] + \mathrm{tr}\ [P_{2,0} \tilde{V}_0] \quad . \tag{B-101}$$

Therefore

$$J = \bar{x}_0^T \tilde{P}_0 \bar{x}_0 + \mathrm{tr}\ [\tilde{P}_0 \tilde{V}_0] + \sum_{k=1}^{N} \Delta\beta_k \quad . \tag{B-102}$$

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