CSE565-HW2

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Collaborators

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Reference

1. item-1

1 Standard Form

Write the following linear program in the standard form given in the lecture notes.

$$\begin{aligned} \max\ x + y \\ 2x + y \leq 3 \\ x + 3y \leq 5 \\ x \geq 0 \end{aligned}$$

Solution:

min - x - y
2x + y + s1 = 3
x + 3y + s2 = 5
x, s1, s2 $\geq$ 0

2 Simplex Algorithm

Given the following linear program, compute the optimum value via the Simplex Algorithm. Use the variant from class and choose the axis with the highest positive coefficient each iteration. Show your work by demonstrating the whole transformed linear system at each intermediate vertex.

$$\begin{aligned} \max\ 3x_1 - 2x_2 - x_4 \\ x_1 + 3x_3 - 2x_4 \leq 6 \\ 2x_2 - x_3 + x_4 \leq 4 \\ x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Solution: max 3x1 2x2 x4
x1 +3x3 2x4 $\leq$ 6
2x2 x3 +x4 $\leq$ 4
x1, x2, x3, x4 $\geq$ 0

Let's start from x1,x2,x3,x4 = 0, the starting point respects each boundary, so it's ok.
1. Increase x1 until boundary, x1=6, x2,x3,x4=0
y1 = 6 - x1 - 3.x3 + 2.x4
y2 = x2, y3 = x3, y4 = x4
x1 = y1- 3.y3 + 2.y4 + 6
Result:
max 3.y1 - 9.y3 + 6.y4 + 18 - 2.y2 - y4 simplify
max 3.y1 - 2.y2 - 9.y3 + 5.y4 + 18
y1 - 3.y3 + 2.y4 + 6 + 3.y3 - 2.y4 $\leq$ 6 simplify
y1 $\leq$ 0
2.y2 - y3 + y4 $\leq$ 4

max 3.y1 - 2.y2 - 9.y3 + 5.y4 + 18
2.y2 - y3 + y4 $\leq$ 4
y1 $\leq$ 0
y1, y2, y3, y4 $\geq$ 0
2. Which means y1 = 0, let's increase y4 until boundary, y4 = 4, y1,y2,y3 = 0
z1 = y1, z2 = y2, z3 = y3
z4 = 4 - 2.y2 + y3 - y4
y4 = 4 - 2.z2 + z3 - z4
max 3.z1 - 2.z2 - 9.z3 + 20 - 10.z2 + 5.z3 - 5.z4 + 18 − simplify
max 38 + 3.z1 - 12.z2 - 4.z3 - 5.z4
2.z2 - z3 + 4 - 2.z2 + z3 - z4 $\leq$ 4 − simplify
z4 $\geq$ 0

max 38 + 3.z1 - 12.z2 - 4.z3 - 5.z4
z1 = 0 ; z2,z3, z4 $\geq$ 0

We found the max value, there is no way to increase it more.
$z1,z2,z3,z4 = 0$
$y2, y3, x2, x3 = 0$
$z4 = 0 = 4 - 2.y2 + y3 - y4 = 4 -y4 = 0$
$y4 = 4, x4 = 4$
$z1 = 0 = y1 = 6 - x1 - 3.x3 + 2.x4 = 14 - x1$
$x1 = 14$

Result: $x1 = 14$, $x2=x3=0$, $x4 = 4$,
Obj: $3x1 - 2x2 - x4 = 10$

3 Duality

1 Write the dual of the following linear program.

$$\begin{aligned} \max\ & x + y \\ 2x + y &\leq 3 \\ x + 3y &\leq 5 \\ x, y &\geq 0 \end{aligned}$$

Solution: max x+y
z1: 2x+y $\leq$ 3
z2: x+3y $\leq$ 5
x, y $\geq$ 0
(2.z1+z2).x + (z1+3.z2).y $\leq$ 3.z1 + 5.z2
Dual: min 3.z1 + 5.z2
2.z1+z2 $\geq$ 1
z1 + 3.z2 $\geq$ 1
z1, z2 $\geq$ 0

2 Find the optimal value and vertex of both the primal and dual LPs.
Solution:
Primal problem: x = 4/5 , y = 7/5, z = 11/5
Dual problem: z1 = 2/5, x2 = 1/5, z = 11/5

3 Calculate the dual of the dual LP from sub-problem (1), showing your work along the way. Note: you will receive 0 points for the whole question if the LP resulting from this calculation is incorrect.
Solution: max - 3.z1 - 5.z2
y1: - 2.z1 - z2 $\leq$ -1
y2: - z1 - 3.z2 $\leq$ -1

(- 2.y1 - y2).z1 + (- y1 - 3.y2).z2 $\leq$ - y1 - y2

min - y1 - y2
- 2.y1 - y2 $\geq$ - 3
- y1 - 3.y2 $\geq$ - 5
y1, y2 $\geq$ 0 simplify

Dual of Dual:
max y1 + y2
2.y1 + y2 $\leq$ 3
1.y1 + 3.y2 $\leq$ 5
y1, y2 $\geq$ 0

4 Proven Flows

Consider the 4 node flow graph with the following edge capacities:

- $c((s,a)) = 2$
- $c((s,b)) = 2$
- $c((a,b)) = 1$
- $c((a,t)) = 1$
- $c((b,t)) = 1$

The source and the sink are s and t respectively.

1 Write an LP that solves the max flow problem for this graph. Label each of your constraints. State an optimal value and variable assignment for this LP. Hint: consider the max flow and min cut of the graph.

Solution:
$\mathrm{s}_a = a_b + a_t$
$s_b + a_b = b_t$

max $\mathrm{s}_a + s_b$
1: $\mathrm{s}_a \leq 2$
2: $\mathrm{s}_b \leq 2$
3: $\mathrm{s}_a - a_t \leq 1$
4: $\mathrm{a}_t \leq 1$
5: $\mathrm{s}_a + s_b - a_t \leq 1$
6: $\mathrm{s}_a, s_b, a_t \geq 0$

This is a relatively simple graph, and finding its min-cut it trivial. Values and flows at the optimal max-flow:

$$flow = 3, s_a = 2, a_b = 1, a_t = 1, b_t = 1, s_t = 1, s_b = 0, s_t = 12$$

2 Prove that the value and variable assignment above are optimal for your LP. Hint: You can change your variable assignment and the formulation of your LP to make your proof easier.

Solution:

Lets put values into the equations.

- s_a reached to its boundary from 1, it cannot increase more.

- when $s_a = 2$, we get $a_t \geq 1$ from 3, and we have $a_t \leq 1$ from 4, which means a_t is tightly bounded as well.

- when $s_a = 2$ and $a_t = 1$, we get $s_b \leq 0$ from 5, and $s_b \geq 0$ from 6, hence s_b is tightly bounded as well.

There is no way that we can go to increase the objective function with respect to boundaries, each variable is tight on boundaries, hence we are at the optimal value.

5 More Flow Problems

Following are two variations on the maximum flow problem. Write a poly-time solvable linear program for each problem. Assume a directed flow graph (V, E) with a source $s \in V$, a sink $t \in V$, and positive real edge capacities $d(e_{ij})$ $\forall e_{ij} \in E$.

1 **Max Flow Min Cost**: Each edge also has a positive real cost $c(e_{ij})$. Note that there may be multiple flow paths that achieve the maximum possible flow value for this graph. Suppose that you want to find the flow f through the graph that has the minimum total cost $\sum\limits_{i,j\in[n]} f(e_{ij})c(e_{ij})$ while achieving the maximum possible flow value.

Please also explain how to calculate the maximum flow and the total cost from the optimal variable assignment of your LP.

Solution:

First formulate the max flow problem like in 4 and find the max flow by solving that LP.

let f(s,x) be flow between s and x.

Objective: max $\sum_{x\in V} f(s,x)$

Constraints:

- $\forall i \in n$ (except s and t) $\sum_{j\in n} f(e_{ij})= \sum_{j\in n} f(e_{ji})$
- $\forall i, j \in n$ $f(e_{ij}) \leq c(e_{ij})$
- $\forall i, j \in n$ $f(e_{ij}) \geq 0$

1. Solve the max-flow and add the max-flow as a constraint to a new LP problem(add $\sum_{x\in V} f(s,x) = MaxFlow$ to the constraints)
2. Set the objective to $min \sum_{i,j\in n} f(e_{ij})c(e_{ij})$, and solve it using Simplex.

Can solve max-flow and LP both in poly-times.

2 **Leaky Flow**: The outgoing flow from each node v_i is not the same as the incoming flow, but is smaller by a factor of (1 - ε_i), where ε_i is a real loss coefficient associated with node v_i.

Solution:

Quite similar to the problem above, but one of the constraints will be different and objective might differ as well.

Objective: Depending on calculating the flow at source or sink either

- max $\sum_{x\in V} f(s,x) or$
- max $\sum_{x\in V} f(t,x)$

Constraints:

- $\forall i \in n$ (except s and t) $\sum_{j\in n} f(e_{ij})= (1-\epsilon_i). \sum_{j\in n} f(e_{ji})$
- $\forall i, j \in n$ $f(e_{ij}) \leq c(e_{ij})$
- $\forall i, j \in n$ $f(e_{ij}) \geq 0$

6 Zero Sum Game

Two players A and B are playing a game. Player A can play $\{a_1, a_2, a_3\}$ and player B can play $\{b_1, b_2, b_3\}$. The table below indicates the competitive advantage (payoff) player A would gain (and player B would lose).

For example, if player B chooses b_1 and player A chooses a_3 the payoff is 6.

	b_1	b_2	b_3
a_1	-10	3	3
a_2	4	-1	-3
a_3	6	-9	2

1 Write an LP to find the optimal strategy for player A. What is the optimal strategy and expected payoff?

Solution:

B will try a strategy like this:

w = min $(-10.x1 + 4.x2 + 6.x3, 3.x1 - x2 - 9.x3, 3.x1 - 3.x2 + 2.x3)$

And A will try to maximize this target with these constraints.

max w

w $\leq -10.x1 + 4.x2 + 6.x3$

w $\leq 3.x1 - x2 - 9.x3$

w $\leq 3.x1 - 3.x2 + 2.x3$

$x1 + x2 + x3 = 1$ $x1, x2, x3 \geq 0$

The expected payoff is w to be negative, meaning A player to lose.

2 Now do the same for player B. What is the optimal strategy and expected payoff?

Solution:

A will try a strategy like this:

z = max $(-10.y1 + 3.y2 + 3.y3, 4.y1 - y2 - 3.y3, 6.y1 - 9.y2 + 2.y3)$

And B will try to minimize this

min z

z $\geq -10.y1 + 3.y2 + 3.y3$

z $\geq 4.y1 - y2 - 3.y3$

z $\geq 6.y1 - 9.y2 + 2.y3$

y1 + y2 + y3 = 1

y1, y2, y3 $\geq$ 0

This is strong dual of the LP in 1, and will have the same result as 1.

7 Oxygen Included

A spaceship uses some *oxidizer* units that produce oxygen for three different compartments. However, these units have some failure probabilities. Because of differing requirements for the three compartments, the units needed for each have somewhat different characteristics.

A decision must now be made on just *how many* units to provide for each compartment, taking into account design limitations on the *total* amount of *space*, *weight* and *cost* that can be allocated to these units for the entire ship. Specifically, the total space for all units in the spaceship should not exceed 500 cubic inches, the total weight should not exceed 200 lbs and the total cost should not exceed $400,000$ dollars.

The following table summarizes the characteristics of units for each compartment and also the total limitation:

	Space (cu in.)	Weight (lb)	Cost ($)	Probability of failure
Units for compartment 1	40	15	30,000	0.30
Units for compartment 2	50	20	35,000	0.40
Units for compartment 3	30	10	25,000	0.20
Limitation	500	200	400,000	

The objective is to *minimize the probability* of all units failing in all three compartments, subject to the above limitations and the further restriction that each compartment have a probability of no more than 0.05 that all its units fail.

Formulate the *linear programming model* for this problem.

Solution:

Let x1, x2, x3 be number of units inside the respective compartments. Prob of all units failing: $0.3^{x1}.0.4^{x2}.0.2^{x3}$

min $0.3^{x1}.0.4^{x2}.0.2^{x3}$
40x1 + 50x2 + 30x3 $\leq$ 500
15x1 + 20x2 + 10x3 $\leq$ 200
30000x1 + 35000x2 + 25000x3 $\leq$ 400000
$0.3^{x1} \leq 0.05$
$0.4^{x2} \leq 0.05$
$0.2^{x3} \leq 0.05$
x1, x2, x3 $\geq$ 0

8 Producer Profits

A film producer wants to make a motion picture. For this, she needs to choose among n available actors. Actor i demands a payment of s_i dollars to participate in the picture.

The funding for the picture will come from m investors. The k-th investor will pay the producer p_k dollars, but under the following condition: the k^{th} investor has a list of actors $L_k \subseteq \{1, \dots, n\}$, and will only invest if *all* the actors on the list appear in the picture.

The profit of the producer is the sum of payments from the investors that she agrees to take funding from, minus the sum of payments she makes to the actors that appear in the picture. The goal is to maximize the producer's profit.

Give a linear programming based efficient algorithm for maximizing the producer's profit. (Hint: Express the problem as a linear program and then prove that an optimum solution will involve integral variable values.)

Solution:

$i_i = 1$ if investor i is investing
$a_i = 1$ if actor i is playing
$max \sum_{i=0}^{m} p_i.i_i - \sum_{i=0}^{n} s_i.a_i$, and solve it using
m many equations: for i in (1,..m) $|L_i|.i_i \leq \sum_{j \in L_i} a_j$
$a_1, ..., a_j,\ i_1, ..., i_j \leq 1$
$a_1, ..., a_j,\ i_1, ..., i_j \geq 0$
When we solve this LP we will see that all $a_i and\ i_j$ s are integral :)

9 Coverage Gap

Given a graph $G = (V, E)$, a *vertex cover* of G is a subset $S \subseteq V$ of nodes such that every edge in E is incident to *at least* one node of S. (S "covers" the edges of E.) The size of the vertex cover is $|S|$.

1 Give a linear program such that the optimal integral solution (that is, a solution where all the variables are assigned integer values) gives a vertex cover of minimum size.

Solution:

Let vertexes be in the form of $v_1 ... v_m$

$s_i = 1$ means $v_i \in S$

Let edges be in the form of e = (v_i, v_j) where e connects vertexes v_i and v_j.

LP:

min $\sum_{j=1}^{m} s_j$

For all $e = (v_i, v_j) \in E$; $1 \leq s_i + s_j$

$s_1, ..., s_j \geq 0$

In the integral optimum solution any of s_i is either 0 or 1. $|S|$ is of minimum size because LP minimizes number of vertexes inside S. Also constraints cause S to be a vertex cover.

2 If G is a general graph, the optimal solution to the LP is not necessarily integral, which means that it may not correspond to a vertex cover in G. Give an example of a graph G where the optimal solution to the LP you give in part (a) is not integral, and give the optimal solution. (This phenomenon is called the integrability gap)

Solution:

Let $v_1, v_2 \in V$ and $E = [(v_1, v_2)]$. This results with the following LP.

$min\ s_1 + s_2$

$1 \leq s_1 + s_2$

$s1, s2 \geq 0$

$s_1 = s_2 = 1/2$ is an optimal solution and is not integral.