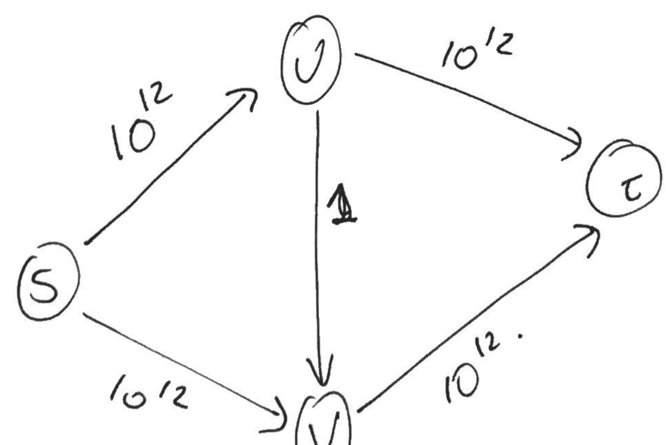
A better algorithm:

⇒ Recall we show that our implementation of requires $O(C)$ iterations where

$$C = \sum_{e \text{ out of } s} c_e$$

⇒ Is this tight?

Example:



⇒ The maximum flow has value $2 \cdot 10^{12}$.

⇒ Let $P_1 = s \to u \to v - t$ and $P_2 = s \to v \to u \to t$ in the residual grah. The flow values increases by exactly 1 in each iteration. Hence,

$2 \cdot 10^{12}$ iterations are indeed needed.

Better algorithm : better choice of augmenting paths.

Idea: Select the path with the largest bottleneck.

Exercise: How?

(Each iteration is costly now).

Idea: Avoid slowdown by not worrying about finding the largest bottleneck exactly, but one with sufficiently large bottleneck.

$\Rightarrow$ Specifically we would look for paths that have bottleneck capacity $\geq \Delta$, where Δ is a scaling parameter.

Scaling Max-Flow

$f(e) = 0 \quad \forall e$ in G

// Initialize Δ

Δ = largest power 2 that is no larger than the maximum capacity out of s: $\Delta \leq \max_{e \text{ out of } s} c_e$.

While $(\Delta \geq 1)$

While ($\exists$ an s-t path P in G_f with bottleneck $\geq \Delta$)

$f' = \text{augment}(f, P)$.

Update f to f'

Update G_f to $G_{f'}$.

$\Delta = \Delta/2$.

⇒ All properties about correctness we have proved about Max-Flow hold for this variant.

⇒ Note in particular that when $\Delta = 1$, the algorithm terminates with a maximum flow.

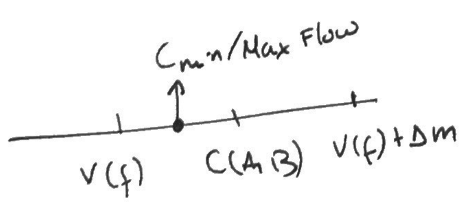
Running Time:

Fact 1 The number of iteration of the outer loop is $O(\log_2 C)$.

Fact 2 During the Δ-scaling phase, each augmentation increases the value by at least Δ.

Key Insight: At the end of Δ-scaling phase the flow f cannot be too far from the max value.

Lemma 3: Let f be the flow at the end of the Δ-scaling phase. There is an s-t cut (A,B) in G such that:



$$V(f) \le C(A,B) \le V(f) + \Delta m.$$

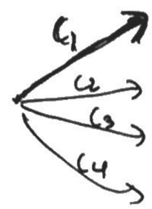
$m = |E|$

Lemma 4: The number of augmentations in a scaling phase is at most $2m$.

Proof of Lemma 4

⇒ In first scaling phase, we can use each edge out of S for at most one augmentation in that phase

· Δ largest power of $2 \le \max c_i$



⇒ Consider the scaling phase for Δ.

⇒ Let f_p be the flow at the end of previous scaling phase.

⇒ In previous phase, we use (2Δ) as the scaling parameter.

⇒ By Lemma 3,

$$V(f_p) \le \underbrace{V(f^*)}_{\text{max flow}} \le V(f_p) + 2\Delta m$$

⇒ In each augmentation the value of the flow increases by Δ, so at most $2m$ iterations required.

∎

Proof of Lemma 3

f flow at the end of Δ-scaling phase,

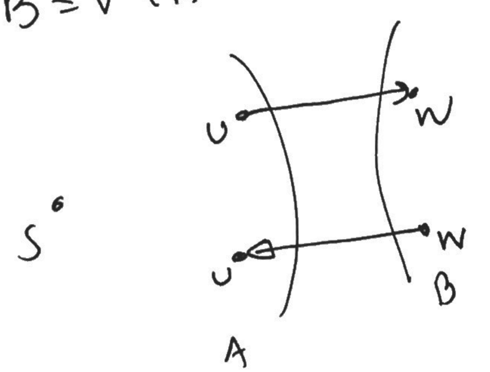
$$\Downarrow$$

$\exists$ an s-t cut (A,B) s.t.:

$$V(f) \le C(A,B) \le V(f) + m\Delta.$$

$\Rightarrow$ Let **A** be the set of edges reachables from s using edges with residual capacity $\ge \Delta$.

$\Rightarrow$ Let $B = V \setminus A$



$\rightarrow C_{(u,w)} - f((u,w)) \le \Delta$

$f((u,w)) \le \Delta$

$$V(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A}' f(e)$$

$$\ge \sum_{e \text{ out of } A} C(u,w) - \sum_{e \text{ out of } A}' \Delta - \sum_{e \text{ into } A}' \Delta$$

$$\ge C(A,B) - \Delta m$$

∎