CSE 565: Algorithm Design and Analysis **Due 2022-12-08T23:59**

Problem Set 6

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Notice: Type your answers using LaTeX and make sure to upload the answer file on Gradescope before the deadline. Recall that for any problem or part of a problem, you can use the "I'll take 20%" option. For more details and the instructions read the syllabus.

Problem 1. MAX 4COLOR

ou are given an undirected graph $G = (V,E)$, and you need to color each vertex with one of the given four colors. We say an edge $(u,v) \in E$ is satisfied if u and v are assigned different colors. Given $G = (V,E)$, the problem is to color all vertices so that the number of satisfied edges is maximized.

Design a randomized 4/3-approximation algorithm for this problem, that is, the expected number of satisfied edges returned by your algorithm should be at least 3/4 fraction of the number of the satisfied edges in the optimal solution. Prove your algorithm can achieve such randomized approximation ratio. Your algorithm should run in polynomial-time.

Problem 2. Close Games

Consider a balls-and-bins experiment with $2n$ balls but only two bins. As usual, each ball independently selects one of the two bins, both bins equally likely. The expected number of balls in each bin is n. In this problem, we explore the question of how big their difference is likely to be. Let the random variables X_1 and X_2 denote the number of balls in the two bins, respectively. Prove that for any $\varepsilon > 0$ there is a constant $c > 0$ such that the probability $\Pr[X_1 - X_2 \geq c\sqrt{n}] \leq \varepsilon$.

Hint: use Chebyshev's inequality.

Problem 3. Speeding on the Grass

Consider the following problem: given an unsorted array A of n elements, we wish to sample from the middle half of the array. i.e. we want a procedure that returns an element a of A such that $a = A'[i]$ where A' is a sorted version of A and $i \in \mathbb{N}$ where $n/4 \leq i \leq 3n/4$.

Consider the following algorithm for this problem: choose $k = 10\log n$ elements from A u.a.r., sort them, and return the median of the sorted elements. What is the time complexity of this algorithm? What is the probability that one iteration of this algorithm returns an element that is not from the middle half of the array?

You can use these inequalities: $\binom{k}{k/2} \leq 4^{k/2}$, $\sum_{i=k/2}^{k}(1/3)^i \leq (1/3)^{k/2}\frac{3}{2}$, $(3/4)^{k/2} \leq (1/2)^{k/5}$. Assume that it takes $O(\log n)$ time to compare two elements and $O(k\log n)$ time to select k random elements from an array of length n. Note that for the error probability an exact calculation is difficult and a reasonable upper bound is fine.

Problem 4. Cliche Cliques

Let $G \sim G(n,p)$ and let X be the random variable corresponding to the number of cliques of size 4 in $G = (V,E)$. Let Ω_4 be the set of all the subsets of size 4 of V. That is, $\Omega_4 = \{C \subset V : |C| = 4\}$ and so $|\Omega_4| = \binom{n}{4}$. Here $G(n,p)$ is a distribution of n node graphs generated by including each possible edge independently with probability p.

1. Show that if $pn^{2/3} \to \infty$, then $E[X] \to \infty$ and that if $pn^{2/3} \to 0$, then $E[X] \to 0$.

   Hint: Observe that $X = \sum_{C \in \Omega_4} X_C$ *where* X_C *is the* $0/1$ *random variable for whether C is a clique or not in G.*

2. Use part (a) and Markov's inequality to show that $\Pr[G$ has a 4 clique$] \to 0$ when $pn^{2/3} \to 0$ (here $n \to \infty$).

3. Show that the variance of X satisfies:

$$\mathrm{Var}(X) = \sum_{C \in \Omega_4} \mathrm{Var}(X_C) + \sum_{C,D \in \Omega_4 : D \neq C} \mathrm{Cov}(X_C, X_D),$$

   where the covariance is defined as $\mathrm{Cov}(X_C, X_D) = E[X_C X_D] - E[X_C]E[X_D]$.

4. Show that $\sum_{C \in \Omega_4} \mathrm{Var}(X_C) = O(n^4 p^6)$.

5. Show that $\sum_{C,D \in \Omega_4 : D \neq C} \mathrm{Cov}(X_C, X_D) = O(n^6 p^{11}) + O(n^5 p^9)$.

6. Use part 4., 5. and Chebyshev's inequality to show that $\Pr[G$ has a 4 clique$] \to 1$ when $pn^{2/3} \to \infty$ (here $n \to \infty$).

   Hint: Use Chebyshev's inequality to prove that it is sufficient that $\frac{\mathrm{Var}(X)}{E[X]^2} \to 0$.