Zero-sum games: games where if one player loses x, the other player wins x.

Example: Rock, Paper, Scissors

$$\begin{array}{c|ccc} & R & P & S \\ \hline R & 0 & -1 & 1 \\ P & 1 & 0 & -1 \\ S & -1 & 1 & 0 \end{array} = P \rightarrow \text{payoff matrix}$$

A strategy for row player: (x_1, x_2, x_3) where each x_i is a probability: $\sum_i x_i = 1$, $x_i \geq 0$

A strategy for column player: (y_1, y_2, y_3) where each y_i is a probability: $\sum_i y_i = 1$, $y_i \geq 0$

A strategy is called pure if $x_i = 1$ for some i, or mixed otherwise

Expected payoff:

$$V = \sum_i \sum_j P_{ij} \cdot x_i \cdot x_j$$

(Assuming independence)

$\Rightarrow$ Row player wants to maximize expected payoff but column wants to minimize it!

Strategy Example:

Row: $(x_1, x_2, x_3) = (1/3, 1/3, 1/3)$

Then:

$$V = \sum_i \sum_j P_{ij} \cdot \frac{1}{3} \cdot y_j = \sum_j \frac{1}{3} \cdot y_j \cdot \sum_i P_{ij} = 0$$

$\Rightarrow$ Expected payoff is 0, regardless of column strategy.

Another example:

$$P = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$$

$\Rightarrow$ Suppose row announces $(1/2, 1/2)$ as the strategy.

If column plays $(1,0)$ $\Rightarrow$ $V = 3\cdot\frac{1}{2} - 2\cdot\frac{1}{2} = 1/2$

If column plays $(0,1)$ $\Rightarrow$ $V = -\frac{1}{2} + \frac{1}{2} = 0$

So column play $(0,1)$!

What about for a generic mixed row strategy (x_1, x_2)?

If column plays $(1,0)$ $\Rightarrow$ $V = 3\cdot x_1 - 2x_2$

If column plays $(0,1)$ $\Rightarrow$ $V = -x_1 + y_2$

So column will play the strategy that corresponds to:

$$\min\{3x_1 - 2x_2, -x_1 + y_2\}$$

So row will choose (x_1, x_2) that corresponds to:

$$\boxed{\begin{array}{ll} \max & \min\{3x_1 - 2x_2, -x_1 + x_2\} \\ \text{s.t.} & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{array}}$$

Can be rewritten as an LP

$$z = \min\{3x_1 - 2x_2, -x_1 + x_2\} \iff \begin{array}{l} \max z \\ z \leq 3x_1 - 2x_2 \\ z \leq -x_1 + x_2. \end{array}$$

So,

$$LP1 = \boxed{\begin{array}{ll} \max & z \\ \text{s.t.} & z \leq 3x_1 - 2x_2 \\ & z \leq -x_1 + x_2 \\ & x_1 + x_2 = 0 \\ & x_1, x_2 \geq 0 \end{array}}$$

What about from column point of view?

(y_1, y_2) - strategy

If row picks $(1,0)$ $\Rightarrow$ $V = 3y_1 - y_2$

If row picks $(0,1)$ $\Rightarrow$ $V = -2y_1 + y_2$

$$
\min_{y_1, y_2} \; \max \{3y_1 - y_2, \, -2y_1 + y_2\}
$$
$$
y_1 + y_2 = 1
$$
$$
y_1, y_2 \geq 1
$$

$$\Updownarrow$$

$$
LP2 = \boxed{\begin{array}{ll} \min & w \\ \text{s.t.} & w \geq 3y_1 - y_2 \\ & w \geq -2y_1 + y_2 \\ & y_1 + y_2 = 1 \\ & y_1, y_2 \geq 0 \end{array}}
$$

Question: Does it make a difference who announces strategy first?

Lemma No!

Proof. Check that the dual of LP1 is LP2.

Then by "Strong Duality" they have the same optimum, say V

Strong Duality: If an LP has bounded optimum, the so does its dual, and the two optimum values coincide

Row can ensure payoff at least V (no matter what column does)

Column can ensure payoff at most V (" row does).

Dual of LP1:

$$
\begin{aligned}
&\max \quad z \\
&-3x_1 + 2x_2 + z \leq 0 \\
&x_1 - x_2 + z \leq 0 \\
&\quad x_1 + x_2 = 1 \\
&\quad x_1, x_2 \geq 0
\end{aligned}
$$

$$
z \leq (-3x_1 + 2x_2 + z) y_1 + (x_1 - x_2 + z) y_2 + (x_1 + x_2) \cdot y_3 \leq y_3
$$

$$
z \leq (-3y_1 + y_2 + y_3) \cdot x_1 + (2y_1 - y_2 + y_3) x_2 + (y_1 + y_2) \cdot z \leq y_3
$$

$$
\begin{array}{ll} \min & y_3 \\ \text{s.t} & -3y_1 + y_2 + y_3 \geq 0 \\ & 2y_1 - y_2 + y_3 \geq 0 \end{array} \quad \Rightarrow LP2 \quad (y_3 = z)
$$

$$y_1 + y_2 = 1$$

$$y_1, y_2 \geq 0$$

General Form for Dual

Primal: $\max \quad c_1 x_1 + \dots + c_n x_n$ $\qquad\qquad m = |I| + |E|$

$$a_{i_1} x_1 + \dots + a_{i_n} x_n \leq b_i \qquad i \in I$$

$$a_{i_1} x_1 + \dots + a_{i_n} x_n = b_i \qquad i \in E$$

$$x_j \geq 0 \qquad j \in N$$

Dual: $\min \quad b_1 y_1 + b_2 y_2 + \dots + b_m y_m$ $\qquad\qquad$ Coefficient of x_j:

$$\sum_{i \in I \cup E} a_{ij} y_i \geq c_j \qquad j \in N$$

$$\sum_{i \in I \cup E} a_{ij} y_i = c_j \qquad j \notin N$$

$$y_j \geq 0 \qquad j \in I$$