CSE 565: Algorithm Design and Analysis **Due 2022-10-07T23:59**

Problem Set 2 Solutions

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Notice: Type your answers using LaTeX and make sure to upload the answer file on Gradescope before the deadline. Recall that for any problem or part of a problem, you can use the "I'll take 20%" option. For more details and the instructions read the syllabus.

Problem 1. Standard Form

Write the following linear program in the standard form given in the lecture notes.

$$\begin{aligned} \max\ x+y \\ 2x+y \leq 3 \\ x+3y \leq 5 \\ x \geq 0 \end{aligned}$$

Solution

$$\begin{aligned} \min\ -x-y+z \\ 2x+y-z+s_1 = 3 \\ x+3y-3z+s_2 = 5 \\ x,y,z,s_1,s_2 \geq 0 \end{aligned}$$

Problem 2. Simplex Algorithm

Given the following linear program, compute the optimum value via the Simplex Algorithm. Use the variant from class and choose the axis with the highest positive coefficient each iteration. Show your work by demonstrating the whole transformed linear system at each intermediate vertex.

$$\begin{aligned} \max\ 3x_1-2x_2-x_4 \\ x_1+3x_3-2x_4 \leq 6 \\ 2x_2-x_3+x_4 \leq 4 \\ x_1,x_2,x_3,x_4 \geq 0 \end{aligned}$$

Solution

Rewrite the LP (negate the $\geq$-constraints)

$$\begin{aligned}
\max\ 3x_1 - 2x_2 - x_4 \\
x_1 + 3x_3 - 2x_4 \leq 6 \\
2x_2 - x_3 + x_4 \leq 4 \\
-x_1, -x_2, -x_3, -x_4 \leq 0
\end{aligned}$$

We start from vertex $x = (0,0,0,0)^T$ which is a feasible solution for the LP (satisfies all the constraints).

The value of the objective function at point x is 0. Now, we increase x_1 (with the highest positive coefficient) to 6 (highest that we can get satisfying the constraints; we stop when one of the constraints is tight).

Set:

$$y_1 = 6 - x_1 - 3x_3 - 2x_4, \tag{0.1}$$
$$y_2 = 0 - (-x_2) = x_2, \tag{0.2}$$
$$y_3 = 0 - (-x_3) = x_3, \tag{0.3}$$
$$y_4 = 0 - (-x_4) = x_4, \tag{0.4}$$
$$x^T = (6,0,0,0), \quad y^T = (0,0,0,0). \tag{0.5}$$

LP becomes:

$$\begin{aligned}
\max\ 18 - 3y_1 - 2y_2 - 9y_3 + y_4 \\
-y_1 \leq 0 \\
2y_2 - y_3 + y_4 \leq 4 \\
y_1 + 3y_3 - 2y_4 \leq 6 \\
-y_2 \leq 0 \\
-y_3 \leq 0 \\
-y_4 \leq 0
\end{aligned}$$

Here we pick y_4 (the variable with positive coefficient) and increase it until one of the constraints becomes tight. Hence, increase y_4 to 4 (vertex: $(0,0,0,4)^T$).

Set:

$$z_1 = 0 - (-y1) = y_1 \tag{0.6}$$
$$z_2 = 0 - (-y_2) = y_2 \tag{0.7}$$
$$z_3 = 0 - (-y_3) = y_3 \tag{0.8}$$
$$z_4 = 4 - 2y_2 + y_3 - y_4 (\Rightarrow y_4 = 4 - 2z_2 + z_3 - z_4) \tag{0.9}$$

We rewrite the LP in terms of z.

$$\begin{aligned}
\max\ 38 - 12z_1 - 4z_2 - 3z_3 - 5z_4 \\
-z_1 \leq 0 \\
-z_2 \leq 0 \\
-z_3 \leq 0 \\
-z_4 \leq 0 \\
z_1 + 4z_2 + z_3 - 2z_4 \leq 14 \\
2z_2 - z_3 + z_4 \leq 4
\end{aligned}$$

Here, all the variables should be positive and they all appear with negative coefficients in the objective function. So, increasing them is gonna drop the value and hence we stop here. This vertex ($z = (0,0,0,0)^T$) gives us value 38 which is the optimal value of this LP which is equivalent to the original LP.

Problem 3. Duality

1 Write the dual of the following linear program.

$$\begin{aligned}
\max\ x+y \\
2x+y \leq 3 \\
x+3y \leq 5 \\
x,y \geq 0
\end{aligned}$$

(2) Find the optimal value and vertex of both the primal and dual LPs.

(3) Calculate the dual of the dual LP from sub-problem (1), showing your work along the way. Note: you will receive 0 points for the whole question if the LP resulting from this calculation is incorrect.

Solution

(1) The dual LP is:

$$\begin{aligned}
\min\ 3u+5w \\
2u+w \geq 1 \\
u+3w \geq 1 \\
u,w \geq 0
\end{aligned}$$

(2) The optimal solution for the primal is $\frac{11}{5}$ given by $(x,y) = \left(\frac{4}{5}, \frac{7}{5}\right)$. The corresponding dual optimum is given by $(u,w) = \left(\frac{2}{5}, \frac{1}{5}\right)$.

(3) Coefficients of goal function are the constants of the constraints: $1,1,0,0$. Let's call the two variables x and y. Goal is to minimize since the previous goal was to maximize. So the goal is:

$$\max\ x+y$$

The constraint coefficients of the dual are the matrix transpose of the primal. So the constraints coefficients are given by:

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

The constraint constants are given by the coefficients of the goal which are $3,5$. Finally we also need to flip the direction of the inequality, which gives us these constraints:

$$\begin{aligned}
2x+y \leq 3 \\
x+3y \leq 5
\end{aligned}$$

We also add the constraint that variables are non-negative $x, y \geq 0$. The resulting LP is

$$\begin{aligned} \max\ x+y \\ 2x+y \leq 3 \\ x+3y \leq 5 \\ x, y \geq 0 \end{aligned}$$

Problem 4. Proven Flows

Consider the 4 node flow graph with the following edge capacities:

- $c((s,a)) = 2$
- $c((s,b)) = 2$
- $c((a,b)) = 1$
- $c((a,t)) = 1$
- $c((b,t)) = 2$ NOTE: this was incorrectly set to 1 in the question text.

The source and the sink are s and t respectively.

(1) Write an LP that solves the max flow problem for this graph. Label each of your constraints. State an optimal value and variable assignment for this LP. Hint: consider the max flow and min cut of the graph.

(2) Prove that the value and variable assignment above are optimal for your LP. Hint: You can change your variable assignment and the formulation of your LP to make your proof easier.

Solution

(1)

$$\begin{array}{rc} \max & f_{at} + f_{bt} \\ & \\ 1) & f_{sa} \leq 2 \\ 2) & f_{sb} \leq 2 \\ 3) & f_{ab} \leq 1 \\ 4) & f_{at} \leq 1 \\ 5) & f_{bt} \leq 2 \\ 6) & f_{sa} = f_{ab} + f_{at} \\ 7) & f_{sb} + f_{ab} = f_{bt} \\ 8) & 0 \leq f_{ij}\ \forall (i,j) \in E \end{array}$$

Max Flow is 3.

Variable assignment:

$$f_{sa} = 1$$
$$f_{sb} = 2$$
$$f_{ab} = 0$$
$$f_{at} = 1$$
$$f_{bt} = 2$$

(2) Easy way: by the definition of the goal, constraints 4 and 5 limit the maximum value to 3 by themselves. Our variable assignment gives us a value of 3, so it is optimal. Done!

Hard way (assuming you used a goal of max $f_{sa} + f_{sb}$ instead):

Our vertex is tight on constraints 2, 4, 5, 6, 7, and 8 (for f_{ab} specifically). Due to convexity, it is sufficient to show that relaxing any of these constraints results in a non-positive gradient.

- 2) Gradient is negative by definition of the goal
- 4) By 6, decreasing f_{at} means decreasing f_{sa} (negative gradient) or increasing f_{ab}. By the tightness of 5 and 7, increasing f_{ab} means decreasing f_{sb} which would result in a negative gradient.
- 5) By the tightness of 7 and 8, decreasing f_{bt} means decreasing f_{sb} which has a negative gradient.
- 6) and 7) are equalities
- 8) By the tightness of 5 and 7, increasing f_{ab} by ε requires decreasing f_{sb} by ε, while by the tightness of 6 and 4 it must increase f_{sa} by ε. So the total change to the goal is $\varepsilon - \varepsilon = 0$, which is non-positie.

Problem 5. More Flow More Problems

Following are two variations on the maximum flow problem. Write a poly-time solvable linear program for each problem. Assume a directed flow graph (V,E) with a source $s \in V$, a sink $t \in V$, and positive real edge capacities $d(e_{ij})\ \forall e_{ij} \in E$.

(1) Max Flow Min Cost: Each edge also has a positive real cost $c(e_{ij})$. Note that there may be multiple flow paths that achieve the maximum possible flow value for this graph. Suppose that you want to find the flow f through the graph that has the minimum total cost $\sum_{i,j\in[n]} f(e_{ij})c(e_{ij})$ while achieving the maximum possible flow value.

Please also explain how to calculate the maximum flow and the total cost from the optimal variable assignment of your LP.

(2) Leaky Flow: The outgoing flow from each node v_i is not the same as the incoming flow, but is smaller by a factor of $(1 - \varepsilon_i)$, where ε_i is a real loss coefficient associated with node v_i.

Solution

(1) Idea: add an edge from t to s with unlimited capacity and "large" negative cost. Then if we optimize for cost we will push the maximum flow with minimized cost.

Let $E' = E \cup e_{ts}$ and $c(e_{ts}) = -2 \times \sum_{e_{ij}\in E} c(e_{ij})$

Variables:

$$f(e_{ij}) \forall e_{ij} \in E'$$

Goal:

$$\min \quad \sum_{e_{ij} \in E'} f(e_{ij}) c(e_{ij})$$

Constraints:

$$\begin{aligned} 0 \leq f(e_{ts}) & \\ 0 \leq f(e_{ij}) \leq d(e_{ij}) & \quad \forall e_{ij} \in E \\ \sum_{e_{vj} \in E'} f(e_{vj}) = \sum_{e_{iv} \in E'} f(e_{iv}) & \quad \forall v \in V \end{aligned}$$

If the optimal values of $f(e_{ij})$ are known, then the minimized total cost is $\sum_{e_{ij} \in E} f(e_{ij}) c(e_{ij})$ and the maximum flow value is $\sum_{e_{sj} \in Out(s)} f(e_{sj}))$ where $Out(s)$ is the out edges of s.

(2)

$$\begin{array}{rll} \max & \sum_{(s,u) \in E} f_{su} & \\ \forall e \in E & 0 \leq f_e \leq c_e & \\ \forall v \in V - \{s,t\} & \sum_{(v,z) \in E} f_{vz} & = (1 - \varepsilon_v) \sum_{(w,v) \in E} f_{wv} \end{array}$$

Problem 6. Zero Sum Game

Two players A and B are playing a game. Player A can play $\{a_1, a_2, a_3\}$ and player B can play $\{b_1, b_2, b_3\}$. The table below indicates the competitive advantage (payoff) player A would gain (and player B would lose).

For example, if player B chooses b_1 and player A chooses a_3 the payoff is 6.

	b_1	b_2	b_3
a_1	-10	3	3
a_2	4	-1	-3
a_3	6	-9	2

(1) Write an LP to find the optimal strategy for player A. What is the optimal strategy and expected payoff?

(2) Now do the same for player B. What is the optimal strategy and expected payoff?

Solution

(1) Player A should try to maximize her payoff in worst case scenario which is max-min approach. Let p_i be weight of playing a_i, for $i \in \{1,2,3\}$. Hence, the maximum payoff for player A is the result of the following optimization.

$$\max \{-10p_1 + 4p_2 + 6p_3, 3p_1 - p_2 - 9p_3, 3p_1 - 3p_2 + 2p_3\}.$$

Let X be the payoff of player A and the goal is to maximize X given regardless of how player B plays (consider the worst case scenario). So, the LP is:

$$\max X \tag{0.10}$$
$$-10p_1+4p_2+6p_3 \geq X \tag{0.11}$$
$$3p_1-p_2-9p_3 \geq X \tag{0.12}$$
$$3p_1-3p_2+2p_3 \geq X \tag{0.13}$$
$$p_1+p_2+p_3 = 1 \tag{0.14}$$
$$0 \leq p_i \leq 1 \tag{0.15}$$

Solving this LP, we get the following optimal answer happens when $(p_1,p_2,p_3) = (\frac{85}{254}, \frac{143}{254}, \frac{26}{254})$ and the corresponding (expected) pay off value is $-\frac{122}{254} = -\frac{61}{127}$.

(2) We do the same thing this time for player B but here we have to minimize the pay off (the amount that B should pay A) regardless of A's strategy (worst case scenario). Let q_i be weight of playing a_i, for $i \in \{1,2,3\}$.

$$\min \{-10q_1+3q_2+3q_3, 4q_1-q_2-3q_3, 6q_1-9q_2+2q_3\}.$$

Let X be the payoff of player B and the goal is to maximize X.

$$\min X \tag{0.16}$$
$$-10q_1+3q_2+3q_3 \leq X \tag{0.17}$$
$$4q_1-q_2-3q_3 \leq X \tag{0.18}$$
$$6q_1-9q_2+2q_3 \leq X \tag{0.19}$$
$$q_1+q_2+q_3 = 1 \tag{0.20}$$
$$0 \leq p_i \leq 1 \tag{0.21}$$

Solving this LP, we get the following optimal answer happens when $(q_1,q_2,q_3) = (\frac{34}{127}, \frac{41}{127}, \frac{52}{127})$ and the corresponding (expected) pay off value is $-\frac{61}{127}$.

Problem 7. Oxygen Included

spaceship uses some *oxidizer* units that produce oxygen for three different compartments. However, these units have some failure probabilities. Because of differing requirements for the three compartments, the units needed for each have somewhat different characteristics.

A decision must now be made on just *how many* units to provide for each compartment, taking into account design limitations on the *total* amount of *space*, *weight* and *cost* that can be allocated to these units for the entire ship. Specifically, the total space for all units in the spaceship should not exceed 500 cubic inches, the total weight should not exceed 200 lbs and the total cost should not exceed 400,000 dollars.

The following table summarizes the characteristics of units for each compartment and also the total limitation:

	Space (cu in.)	Weight (lb)	Cost ($)	Probability of failure
Units for compartment 1	40	15	30,000	0.30
Units for compartment 2	50	20	35,000	0.40
Units for compartment 3	30	10	25,000	0.20
Limitation	500	200	400,000	

The objective is to *minimize the probability* of all units failing in all three compartments, subject to the above limitations and the further restriction that each compartment have a probability of no more than 0.05 that all its units fail.

Formulate the *linear programming model* for this problem.

Solution

Let x_1 be the number of oxidizer units provided for compartment 1, and define similarly x_2 for compartment 2 and x_3 for compartment 3. The probability that all units fail in compartment 1 is $(0.3)^{x_1}$, for compartment 2 is $(0.4)^{x_2}$ and for compartment 3 is $(0.2)^{x_3}$. The probability that all units fail in all compartments is $(0.3)^{x_1}(0.4)^{x_2}(0.2)^{x_3}$. The exponential function is non-linear, so we take logarithm (which preserves ordering of numbers) to get the linear program.

$$\text{Minimize } \log(0.3)x_1 + \log(0.4)x_2 + \log(0.2)x_3$$
$$\text{subject to} \begin{cases} 40x_1 + 50x_2 + 30x_3 \le 500 & \text{space constraint (cu in.)} \\ 15x_1 + 20x_2 + 10x_3 \le 200 & \text{weight constraint (lb)} \\ 30x_1 + 35x_2 + 25x_3 \le 400 & \text{cost constraint (\$1000)} \\ \log(0.3)x_1, \log(0.4)x_2, \log(0.2)x_3 \le \log(0.05) & \text{reliability constraints (log scale)} \end{cases} .$$

(Note that the reliability constraints imply the non-negativity constraints that $x_1, x_2, x_3 \ge 0$.)

Problem 8. Producer Profits

film producer wants to make a motion picture. For this, she needs to choose among n available actors. Actor i demands a payment of s_i dollars to participate in the picture.

The funding for the picture will come from m investors. The k-th investor will pay the producer p_k dollars, but under the following condition: the k^{th} investor has a list of actors $L_k \subseteq \{1, \ldots, n\}$, and will only invest if *all* the actors on the list appear in the picture.

The profit of the producer is the sum of payments from the investors that she agrees to take funding from, minus the sum of payments she makes to the actors that appear in the picture. The goal is to maximize the producer's profit.

Give a linear programming based efficient algorithm for maximizing the producer's profit. (Hint: Express the problem as a linear program and then prove that an optimum solution will involve integral variable values.)

Solution

We write a linear program for the problem. The variables are $x_i, 1 \le i \le n$, one per actor, and $y_k, 1 \le k \le m$, one per investor. The intention is to have $x_i = 1$ if actor i plays in the movie and 0 otherwise, and $y_k = 1$ if investor k signs up for funding and 0 otherwise. With this in mind, we write the following linear program:

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_{k=1}^{m} p_k y_k - \sum_{i=1}^{n} s_i x_i \\ \text{subject to} \quad & x_i \ge y_k, \text{ for all } k \text{ and } i \in L_k \\ & x_i, y_k \le 1, \text{ for all } i, k, \\ & x_i, y_k \ge 0, \text{ for all } i, k. \end{aligned}$$

Any feasible solution of the problem is also a feasible solution of the linear program, which we obtain by choosing all x_i and y_k as intended. The first type of constraint says that whenever $i \in L_k$, choosing investor k (setting $y_k = 1$) implies choosing actor i (that is, $x_i = 1$). The other two types of constraints are satisfied trivially.

Now we want to show that any *optimal* solution of the linear program is also a feasible solution of the problem. Namely, in any feasible solution that is also optimal, it must be the case that all variables x_i and y_k are integral (zero or one). To argue this, we use the fact that the optimum value of a linear program is always attained at a vertex of the polyhedron determined by the constraints.

Consider an arbitrary vertex v of the polyhedron. This vertex is uniquely determined by a subset S of the constraints, with the inequalities replaced by equalities. The constraints in S of the first type partition the set of variables $\{x_1,\ldots,x_n,y_1,\ldots,y_k\}$ into classes $C_1,\ldots,C_m$ such that at vertex v, any two variables in the same class must be assigned the same value. The rest of the constraints in S assign either 0 or 1 to some of the variables. Note that two variables in the same class cannot be assigned inconsistent values (for this would violate constraints of the first type), and each class must contain at least one assigned variable (for otherwise v wouldn't be uniquely determined by the constraints). So all variables are assigned either zero or one.

Problem 9. Coverage Gap

Given a graph $G=(V,E)$, a *vertex cover* of G is a subset $S\subseteq V$ of nodes such that every edge in E is incident to *at least* one node of S. (S "covers" the edges of E.) The size of the vertex cover is $|S|$.

(1) Give a linear program such that the optimal integral solution (that is, a solution where all the variables are assigned integer values) gives a vertex cover of minimum size.

(2) If G is a general graph, the optimal solution to the LP is not necessarily integral, which means that it may not correspond to a vertex cover in G. Give an example of a graph G where the optimal solution to the LP you give in part (a) is not integral, and give the optimal solution. (This phenomenon is called the integrability gap)

Solution

(1)

Here is one such solution.

$$\begin{array}{lll} \text{minimize} & \sum\limits_{v\in V} x_v & \\ \text{subject to} & x_u+x_v\geq 1 & \forall (u,v)\in E \\ & x_v\geq 0 & \forall v\subseteq V \end{array}$$

Notice that an optimal integral solution must necessarily have all $x_v\in\{0,1\}$, since if $x_v>1$, then any constraint containing x_v would still be satisfied if $x_v=1$, and yet the objective function would be decreased by doing so.

Next we verify that any vertex cover S corresponds to a feasible solution by setting $x_v=1$ if $v\in S$ and 0 otherwise. The value of the objective function is precisely $|S|$. Similarly, for any feasible solution, we let S be the set $\{v : x_v=1\}$. Then S is a vertex cover, since for any edge $(u,v)\in E$, then at least one of u and v are in S.

This means the optimum feasible integral solution corresponds to a vertex cover of minimum size.

(2)

Again consider a triangular graph with 3 vertices and 3 edges. The optimal solution to the resulting LP is $x_1=x_2=x_3=1/2$ with total value $3/2$. The optimal integral solution has value 2.