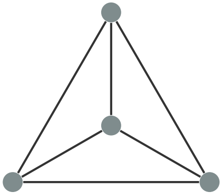
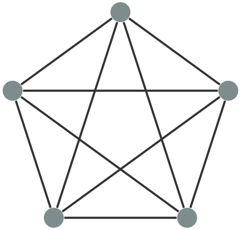
On Planar Graphs of Uniform Polynomial Growth

Farzam Ebrahimnejad

joint work with James R. Lee

Planarity





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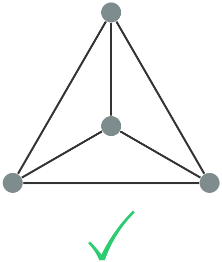

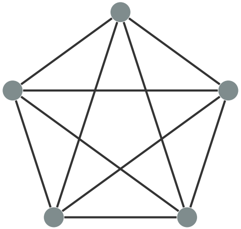

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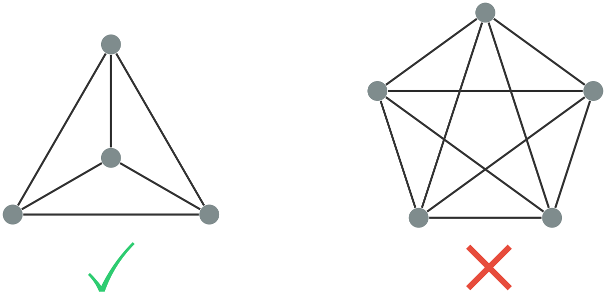

Volume Growth

Definition (Graph distance ball)

For a graph G, $v \in V(G)$ and $R \geq 0$ we define

$$B_G(v, R) = \{u : d_G(v, u) \leq R\},$$

where $d_G(u, v)$ is the shortest path distance between u and v.

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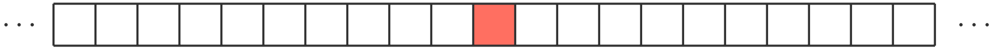
  - Let's see some examples!

Volume Growth in $\mathbb{Z}$



$$|B(v,0)| = 1$$

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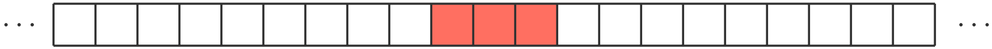

$$|B(v,1)| = 3$$

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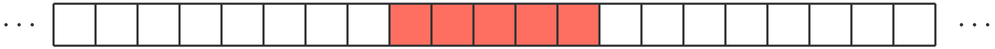

$$|B(v,R)| = 2R + 1$$

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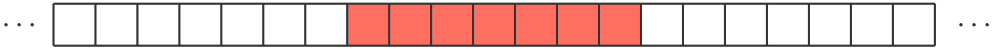

$$\begin{aligned}|B(v,R)| &= 2R+1\\ &= \Theta(R).\end{aligned}$$

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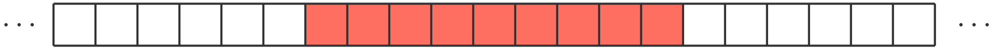

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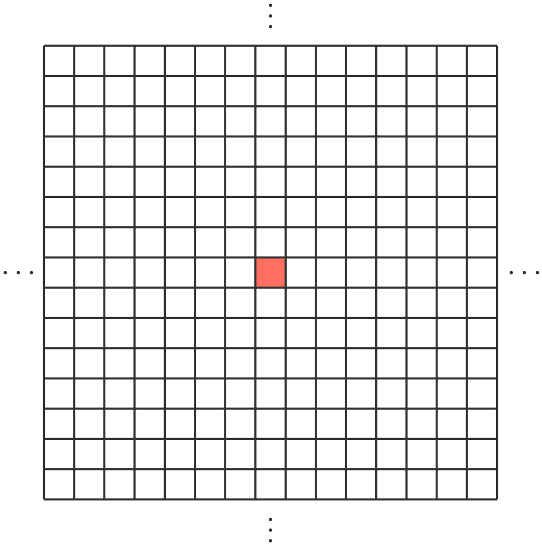

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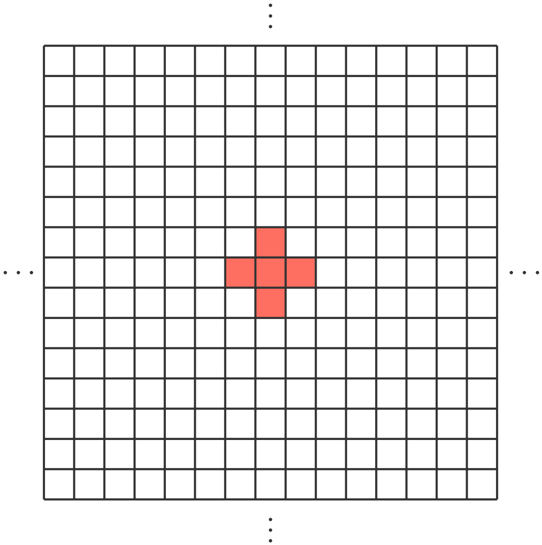
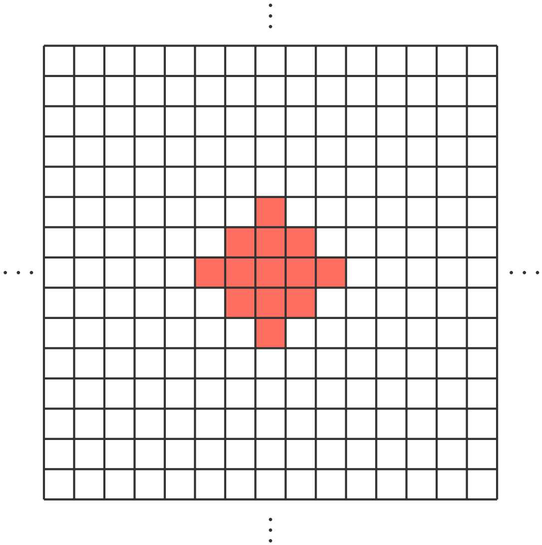


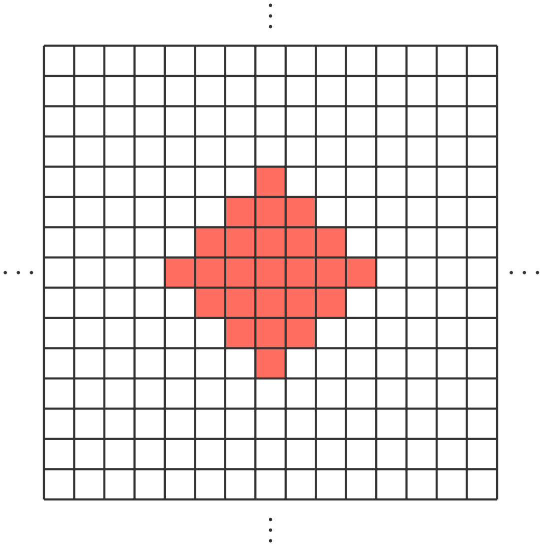
$|B(v,1)| = 5$

Volume Growth in $\mathbb{Z}^2$



$|B(v,2)| = 13$

Volume Growth in $\mathbb{Z}^2$



$$|B(v,R)| = 2R^2 - 2R + 1$$

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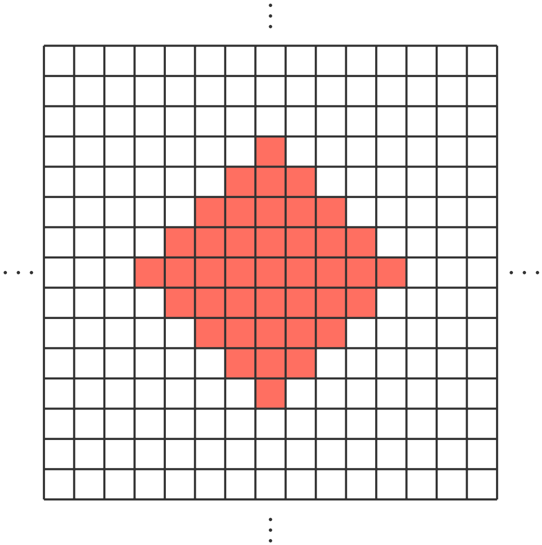

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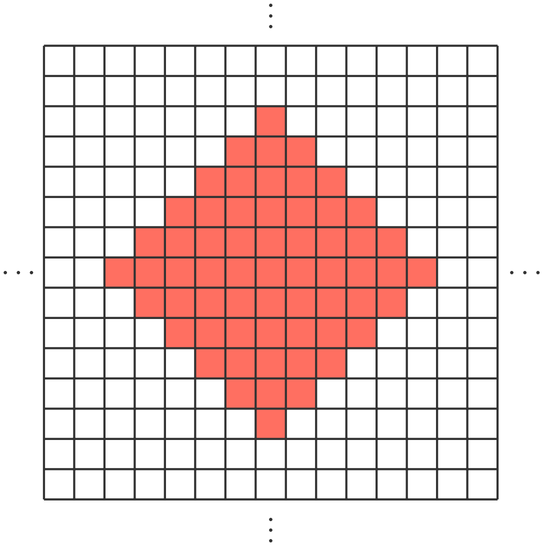

$$|B(v,R)| = 2R^2 - 2R + 1 = \Theta(R^2).$$

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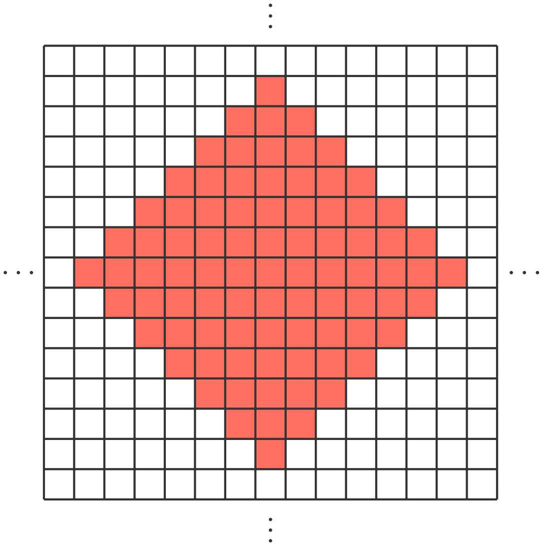

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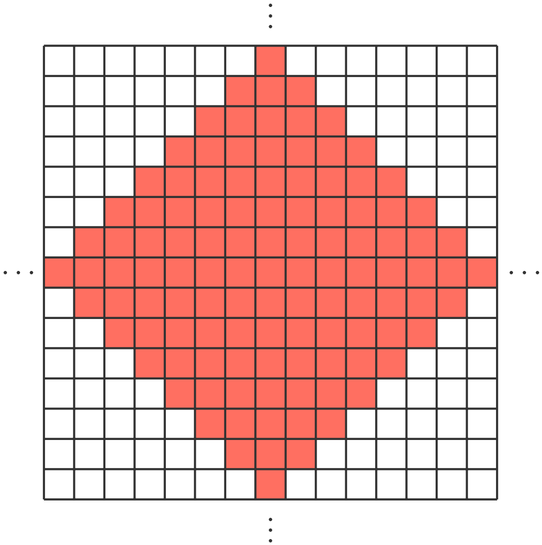

$|B(v, R)|$

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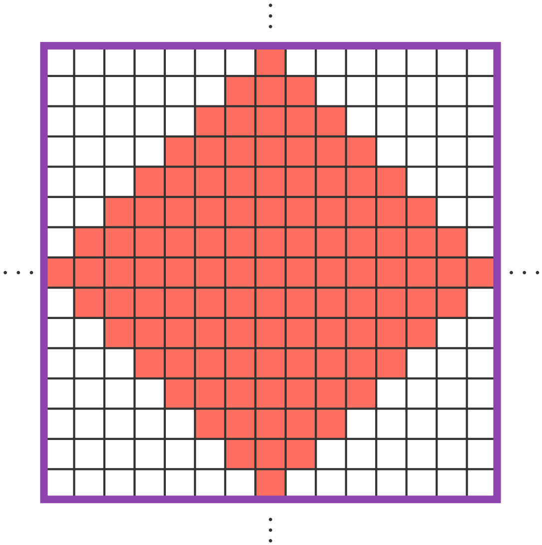

$$|B(v,R)| \leq (2R+1)^2$$

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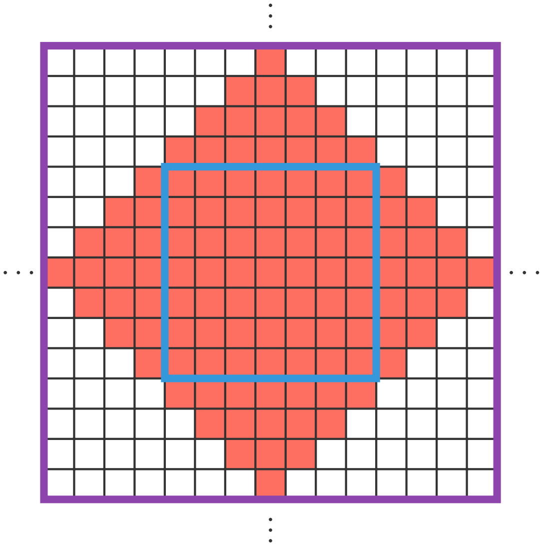
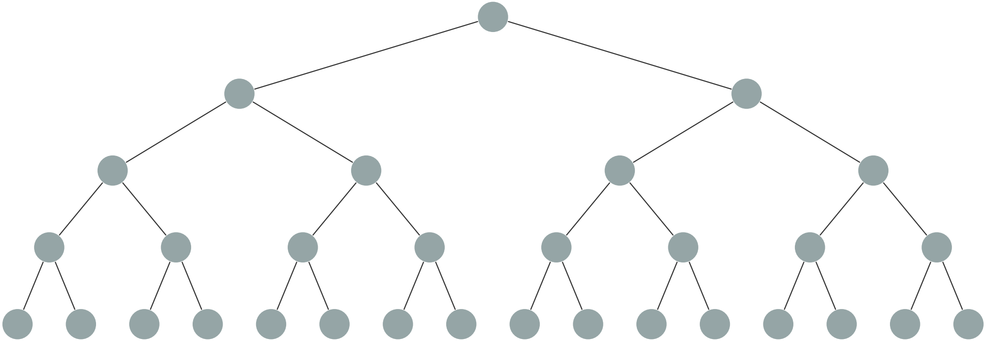


$$(R+1)^2 \leq |B(v,R)| \leq (2R+1)^2$$

Volume Growth in the Binary Tree



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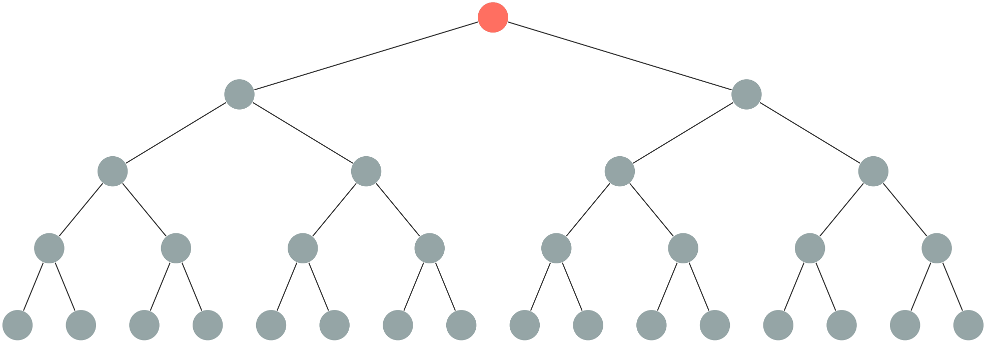

$$|B(v, 0)| = 1$$

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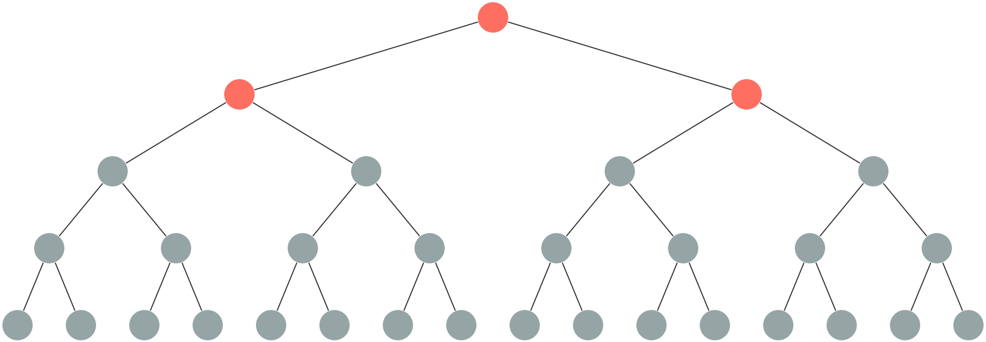

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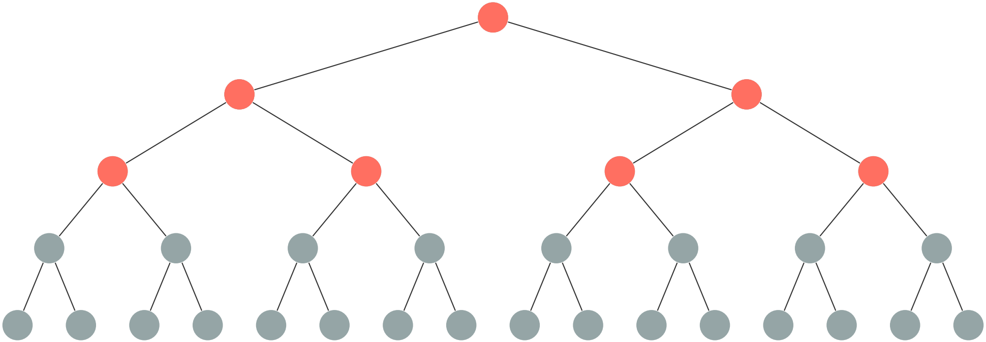

$$|B(v, 2)| = 7$$

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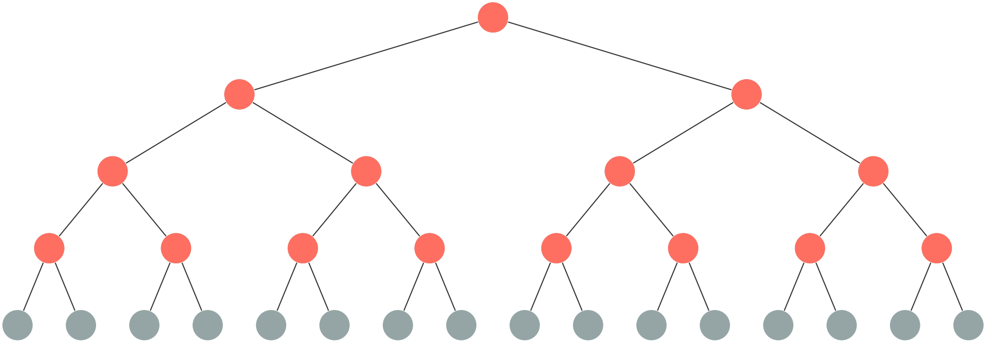

$$|B(v, R)| = 2^{R+1} - 1$$

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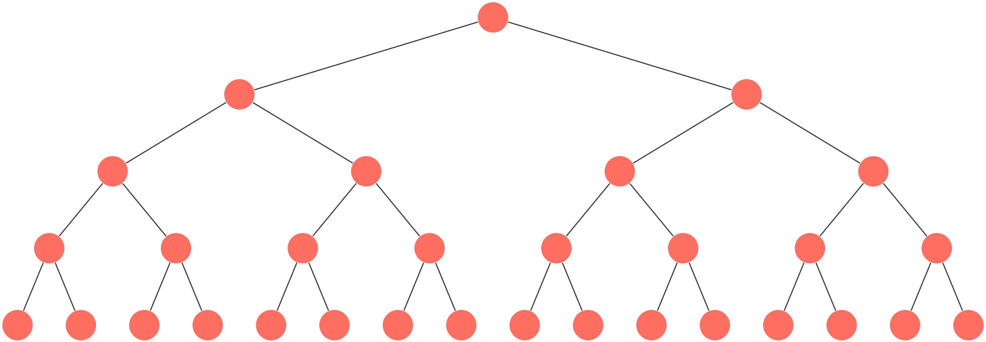

$$\begin{aligned} |B(v,R)| &= 2^{R+1} - 1 \\ &= \Theta(2^R). \end{aligned}$$

Uniform Growth

We say a graph G has *uniform volume growth* $f(R)$ if for all $v \in V(G)$ and $R \geq 0$ we have

$$c f(R) \leq |B(v,R)| \leq C f(R)$$

for some constants $0 < c \leq C$.

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Examples

- The *d-dimensional grid* has $f(R) = R^d$ uniform growth, which is polynomial.
- The *infinite binary tree* has $f(R) = 2^R$ uniform growth, which is exponential.
- Not all graphs have uniform growth rate.

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Planar Graphs with Uniform Polynomial Growth > 2

Thm. [Babai '97]

If G is a *vertex-transitive* planar graph with uniform growth, then the growth is either linear, quadratic, or exponential.

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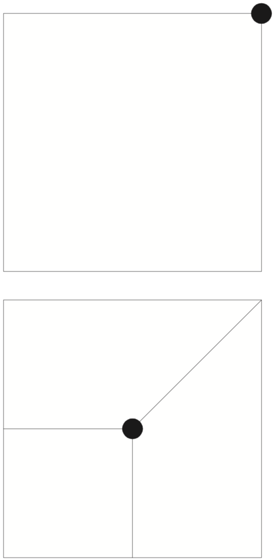
Thm. [Benjamini - Schramm '01]

For any $\alpha \geq 1$, there exists a planar graph with uniform growth R^{α}.

A Construction by Benjamini-Schramm



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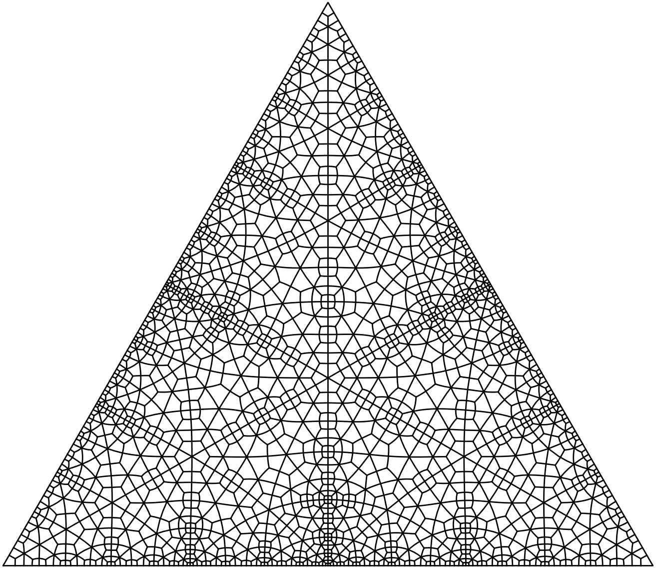
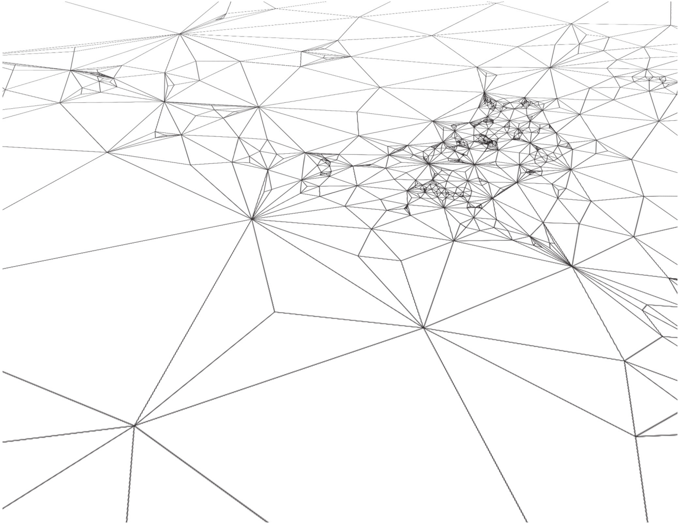


Uniform Infinite Planar Triangulation (UIPT)



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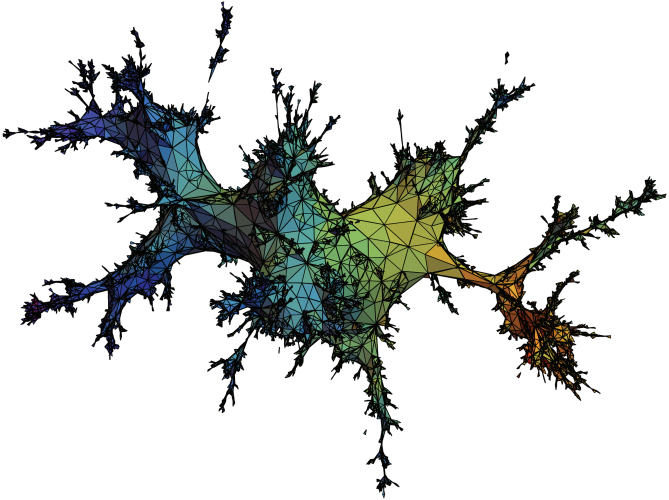

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Problem 1 [Benjamini - Papasoglu '10]

Is there a graph with uniform polynomial growth $\alpha > 2$ in which the complements of balls are all connected?

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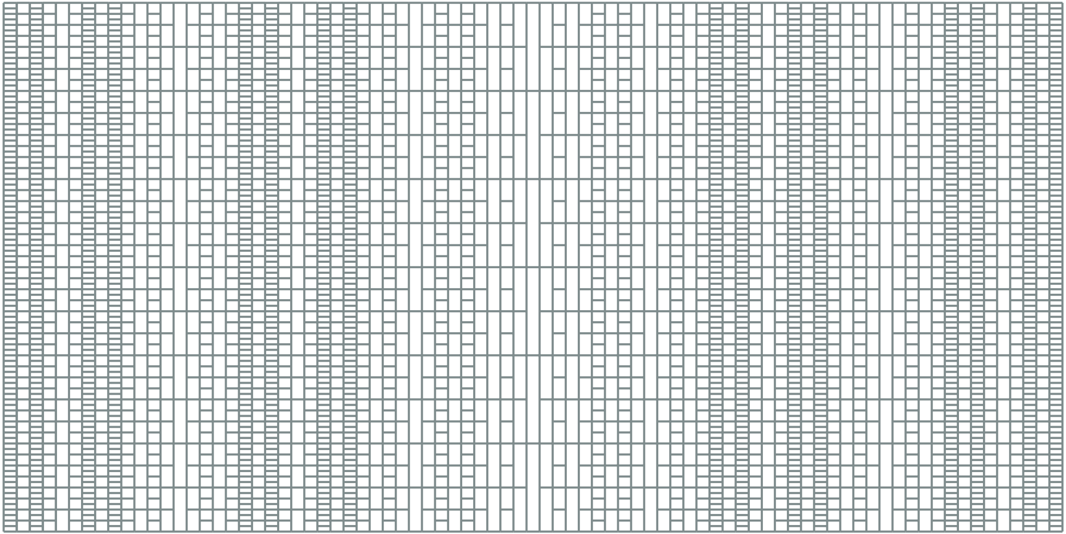
Thm. [E - Lee '20]

For any $\alpha > 2$, there exists a planar graph with uniform growth R^α in which the complements of balls are all connected.

Sneak Peek



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  Up next . . .
  - ☐ *Random walks* and *effective resistances* in these graphs.
  - ☐ Cannon's conjecture.
  - ☐ Our construction.

Random Walks

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Start at some $v_0 \in V(G)$, move to a *uniformly random neighbor* at every step.

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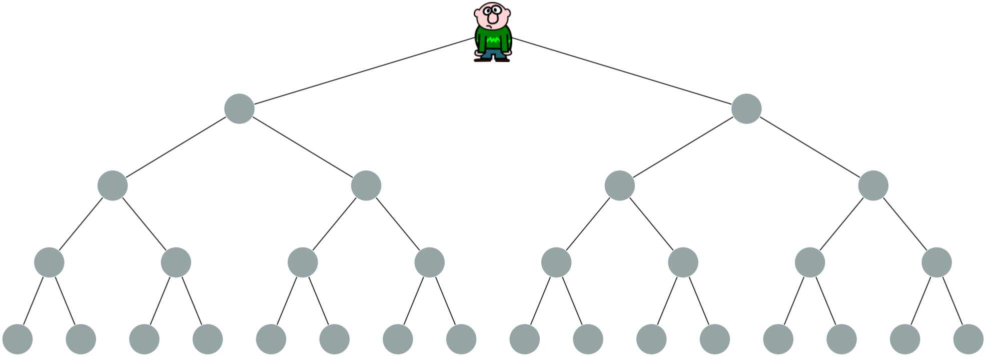

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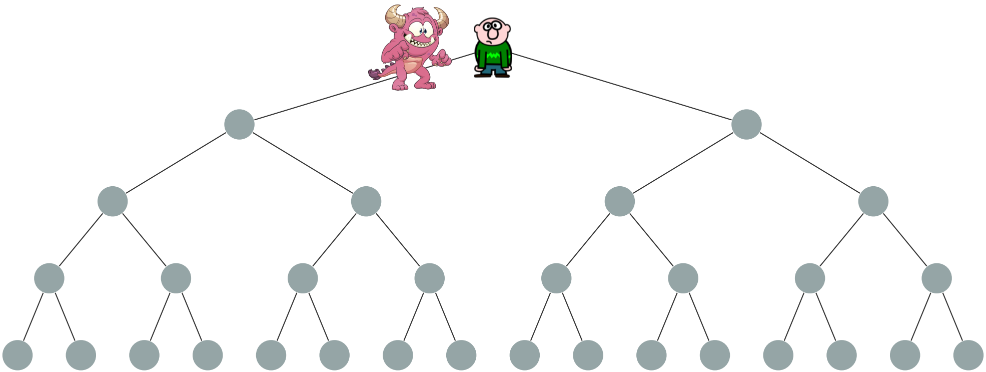

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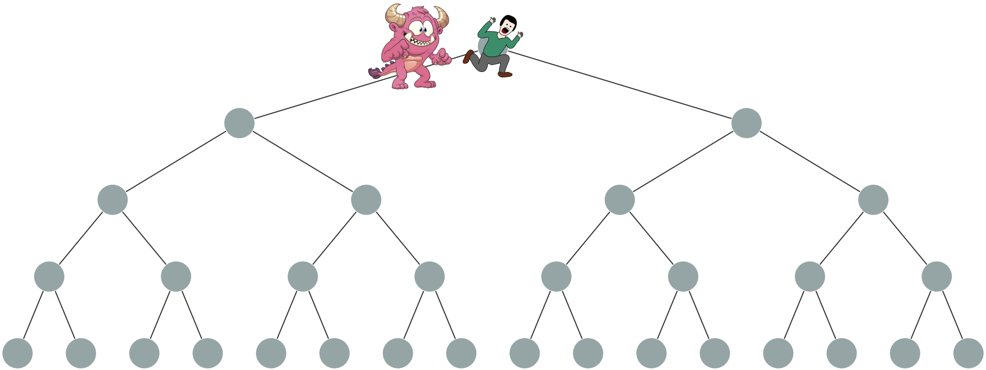

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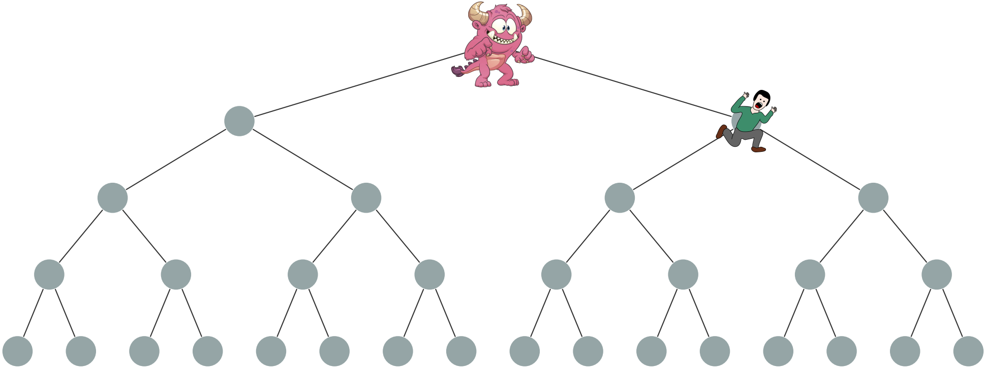

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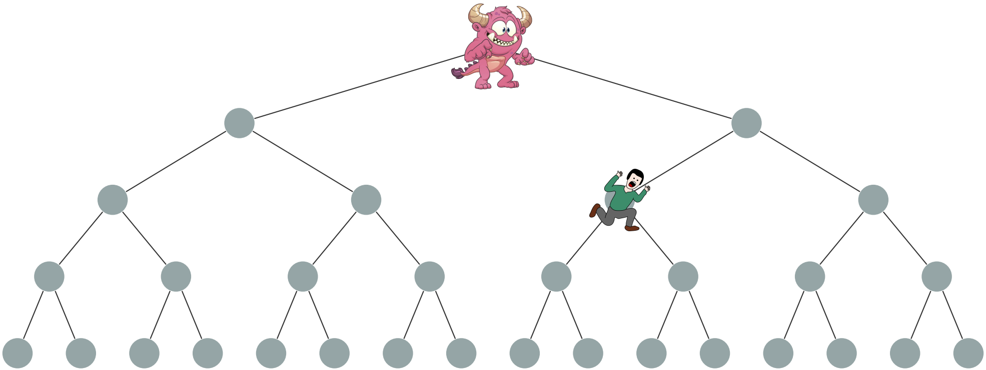

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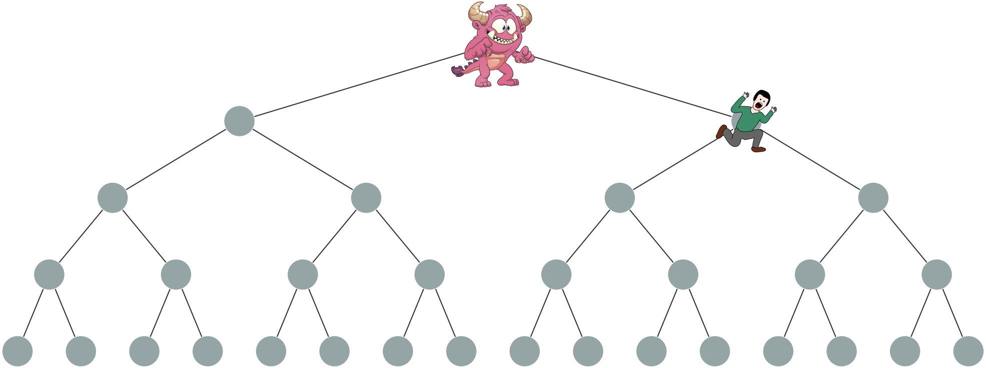

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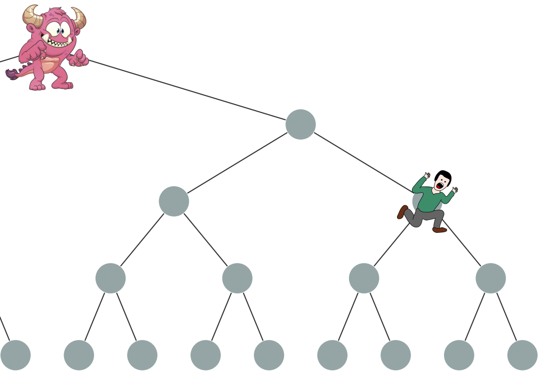

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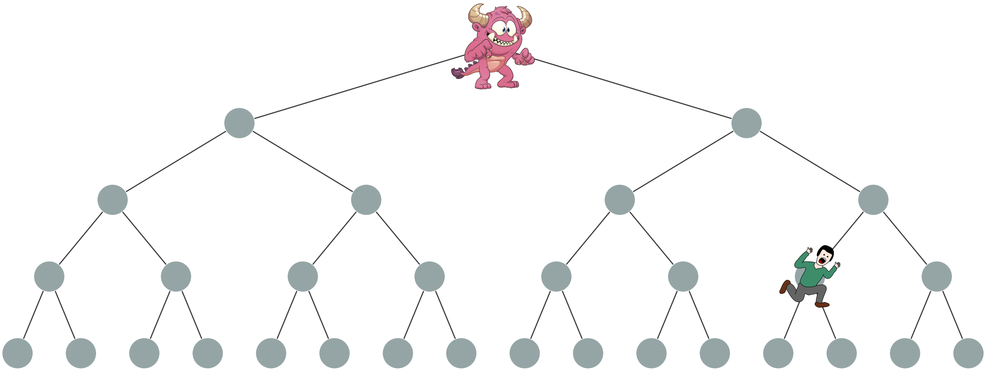

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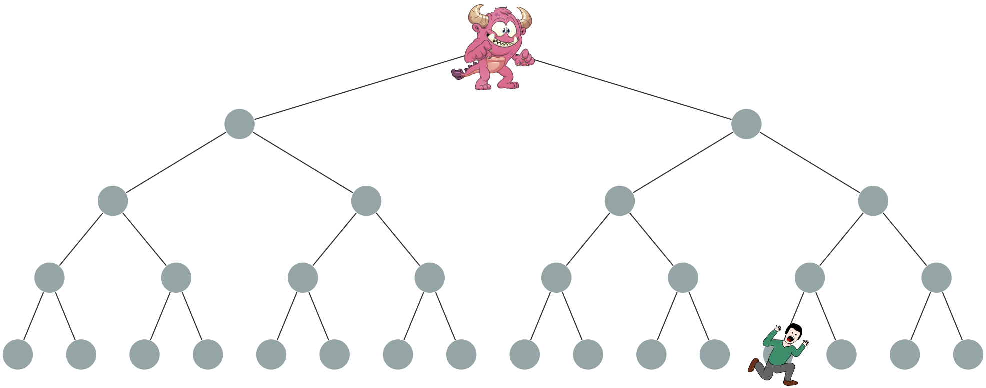

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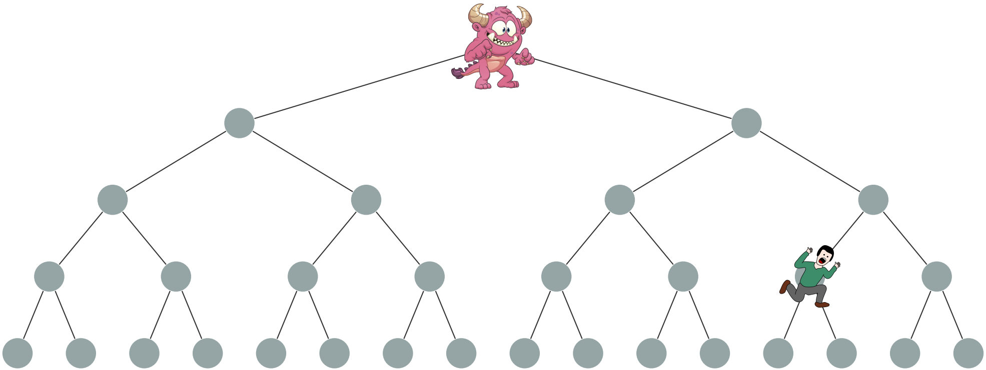

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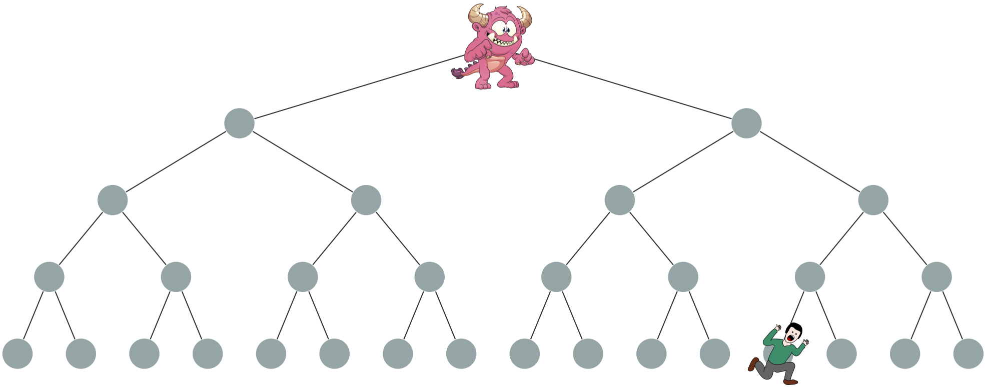

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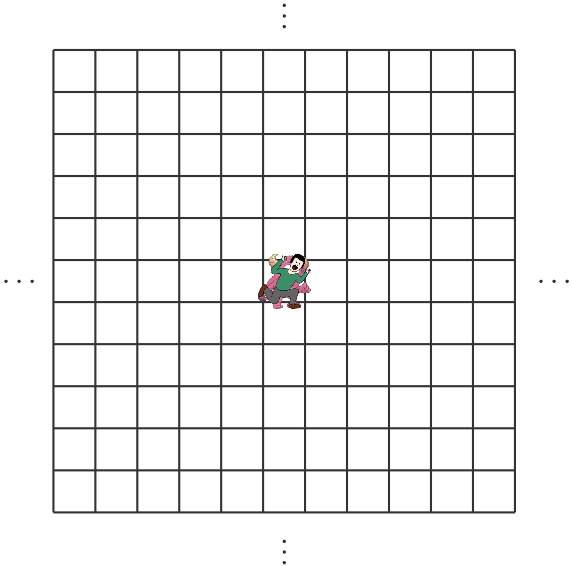

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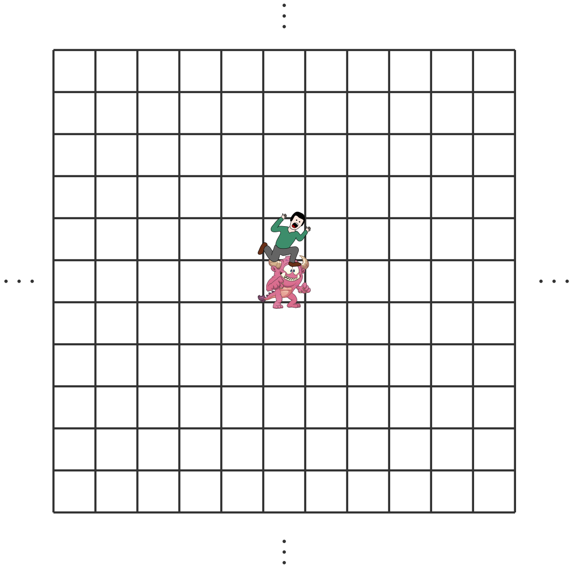

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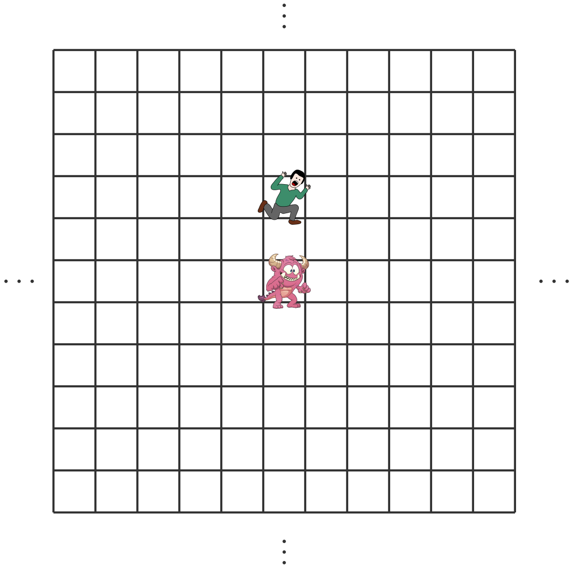

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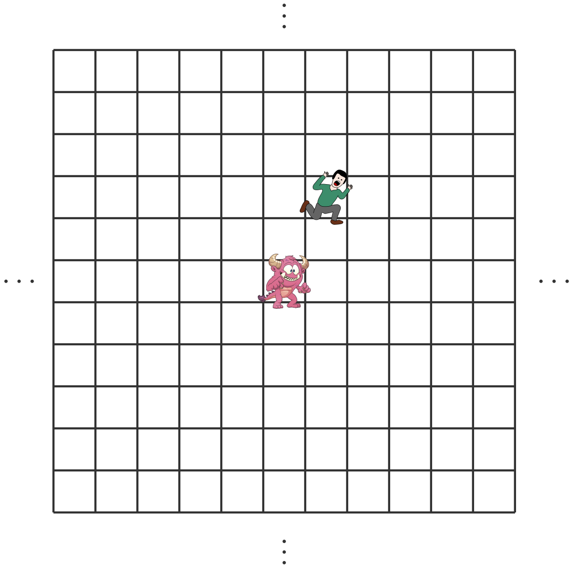

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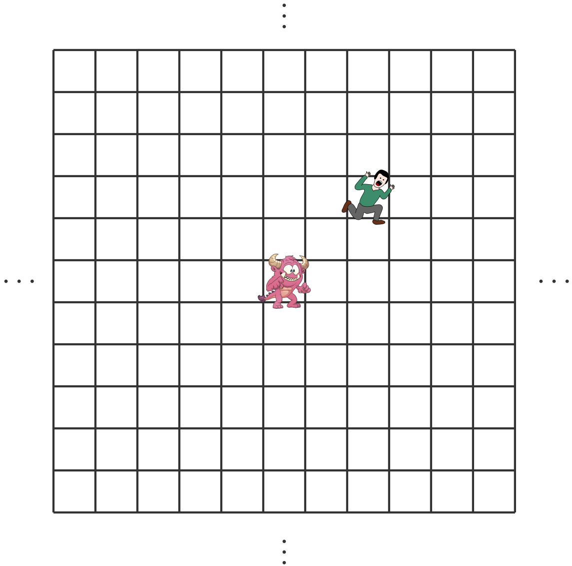

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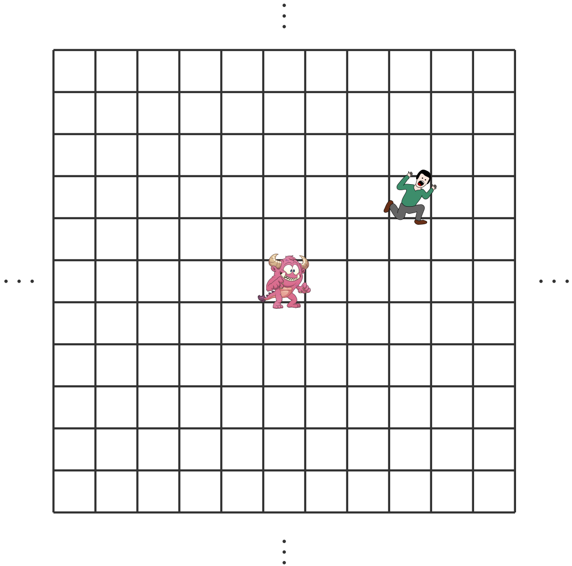

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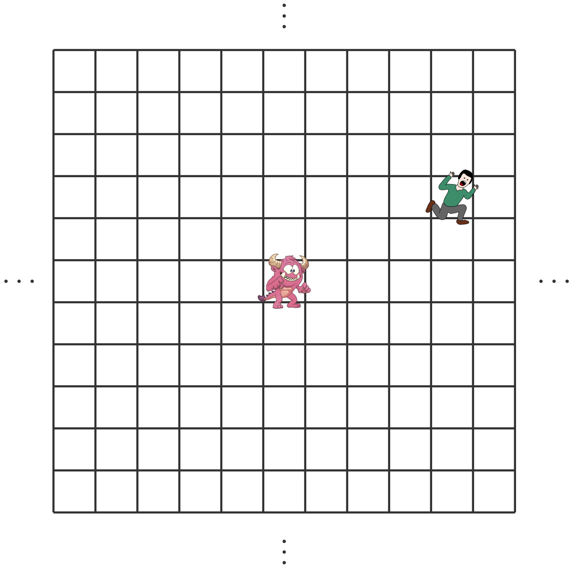

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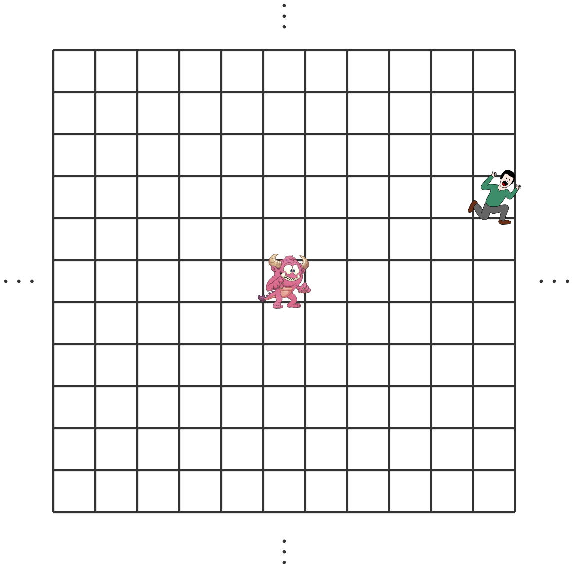

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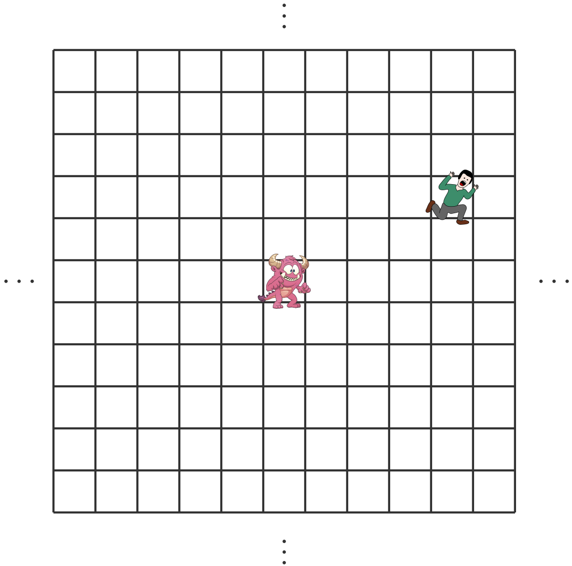

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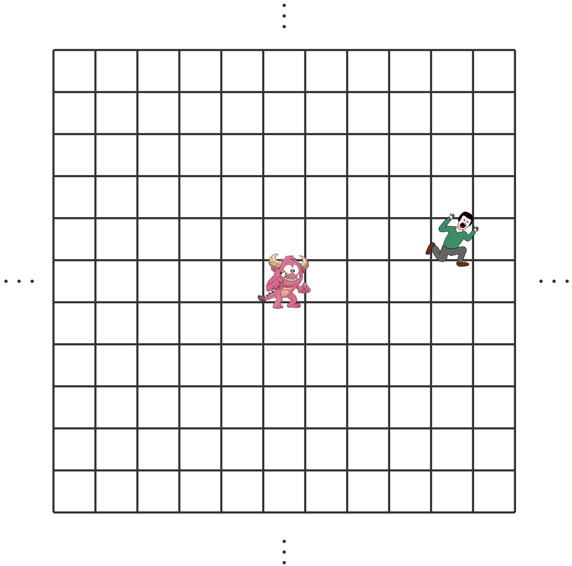

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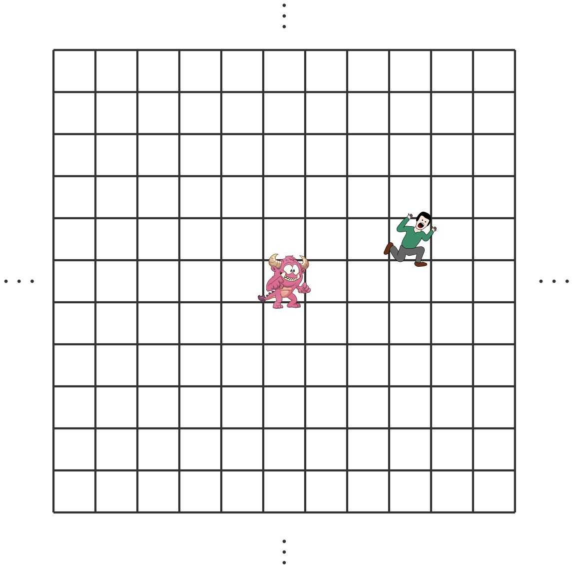

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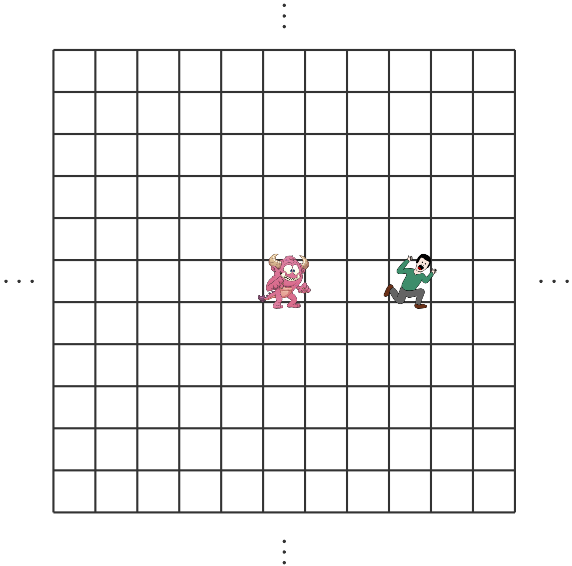

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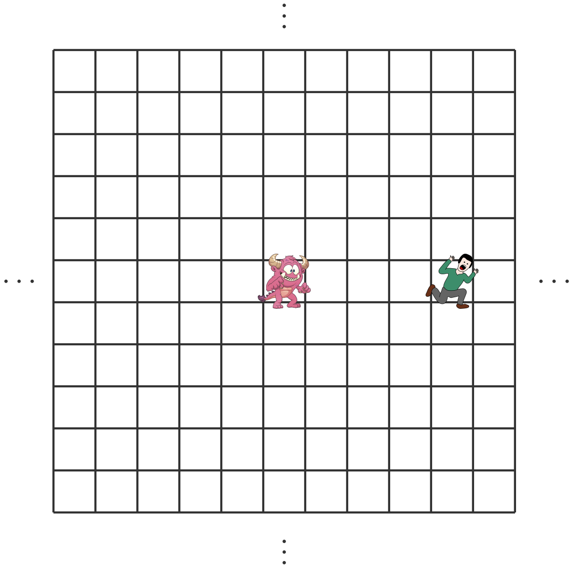

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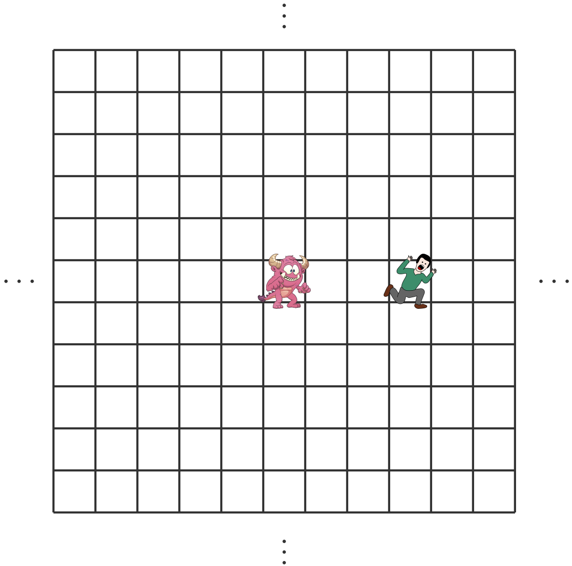

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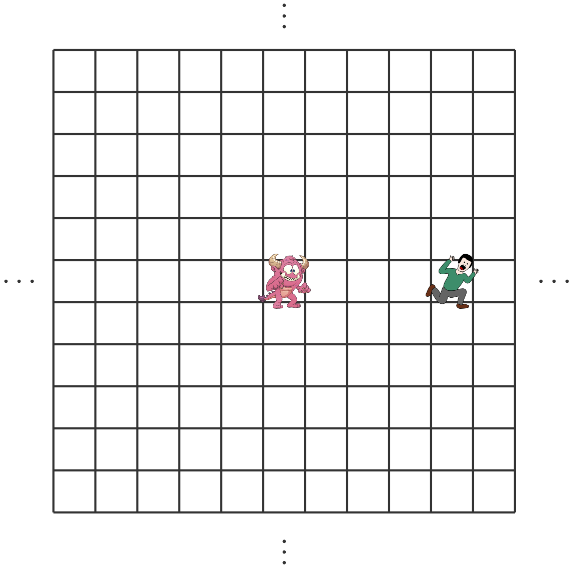

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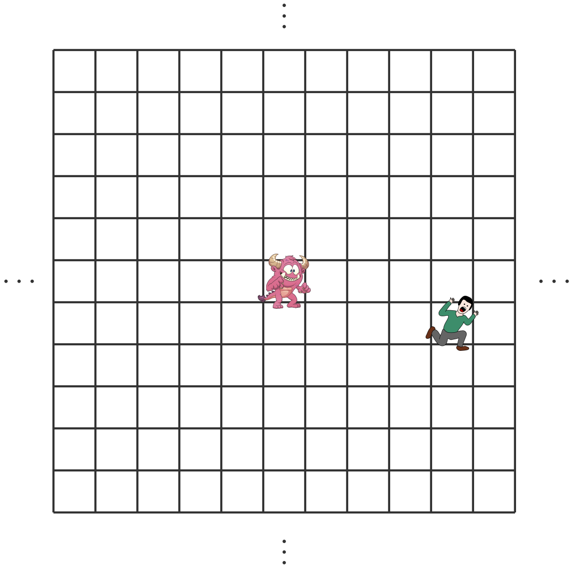

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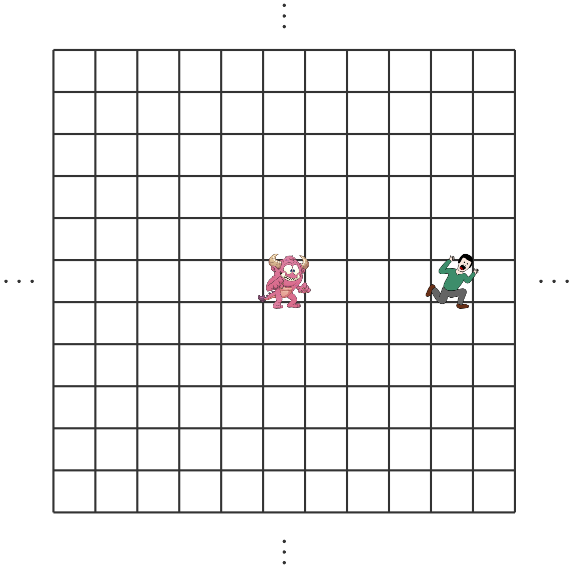

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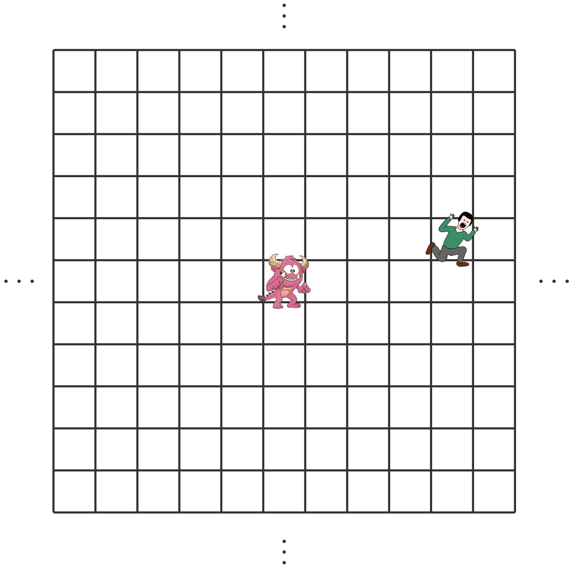

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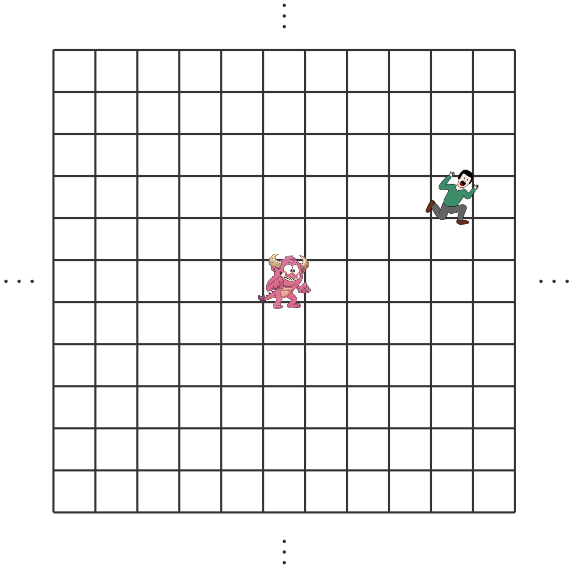

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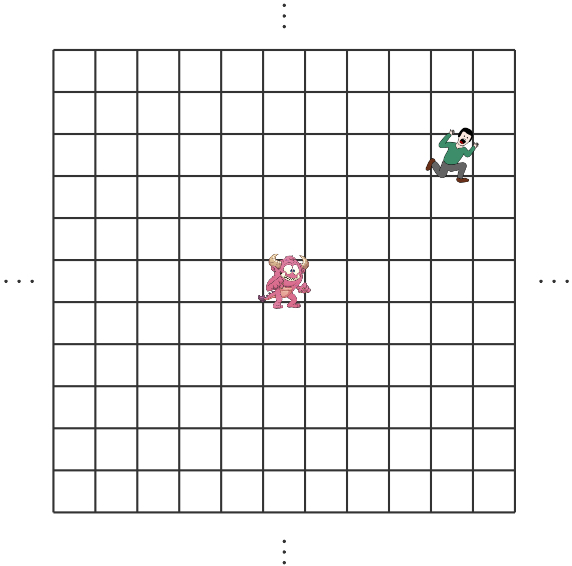

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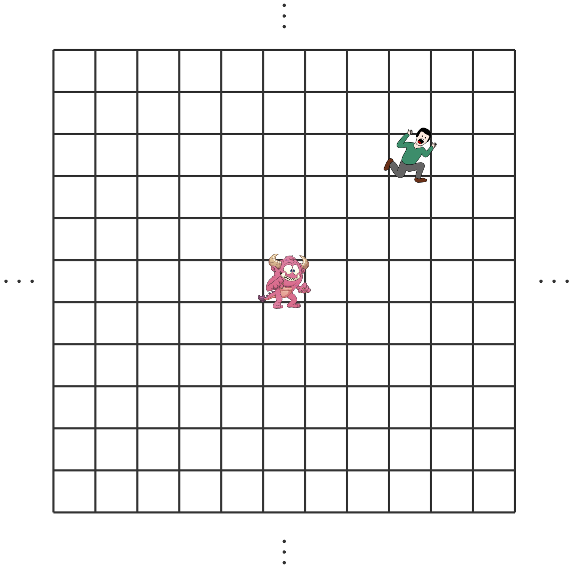

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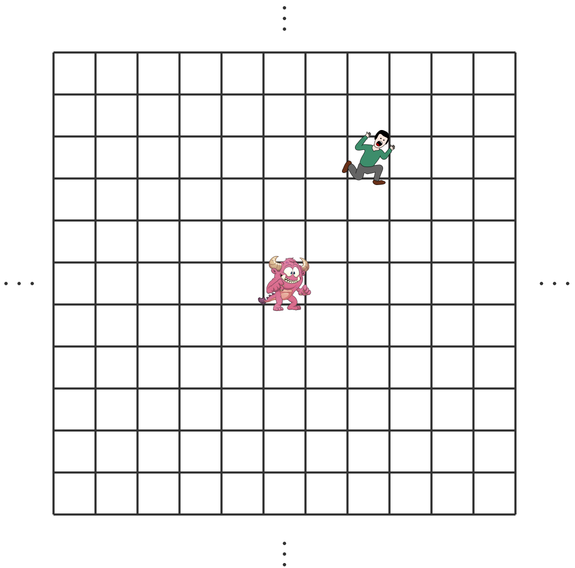

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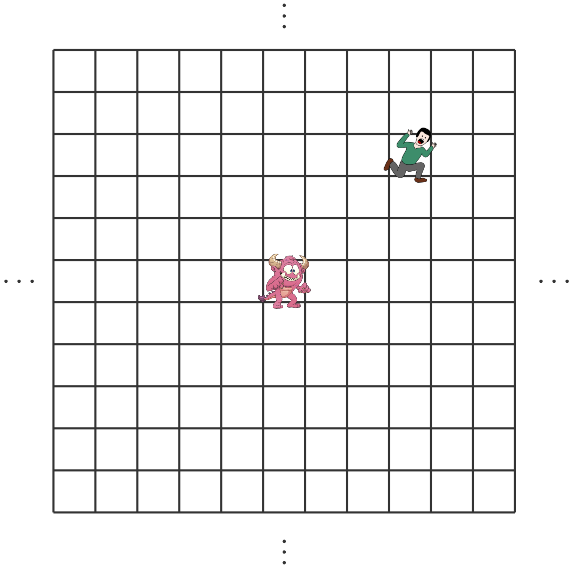

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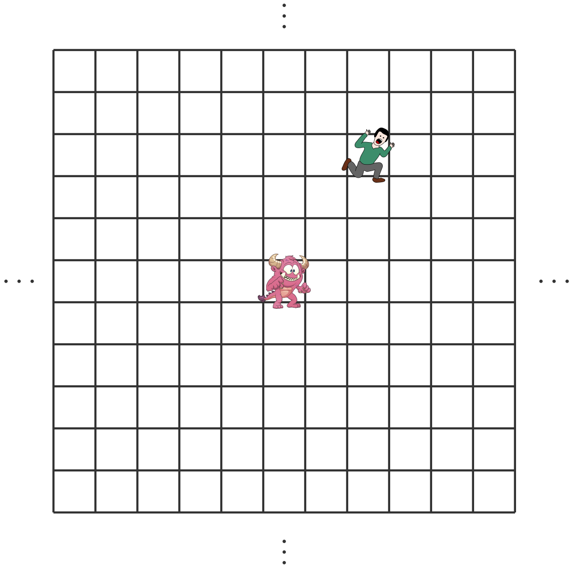

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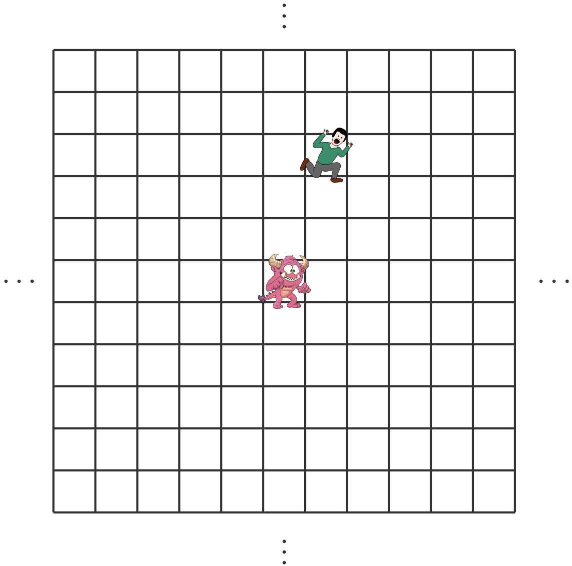

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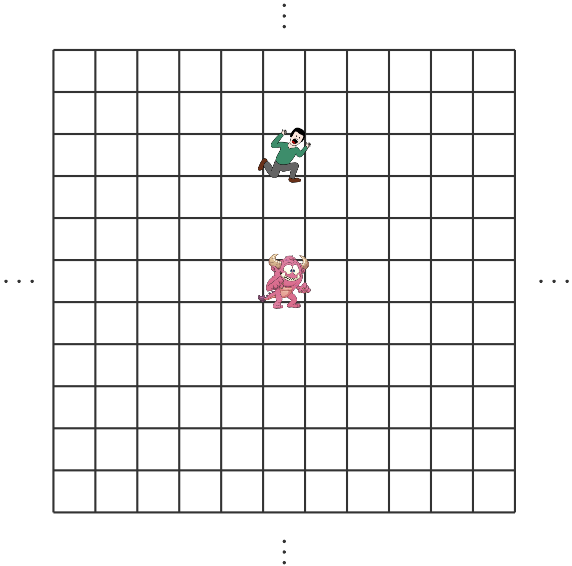

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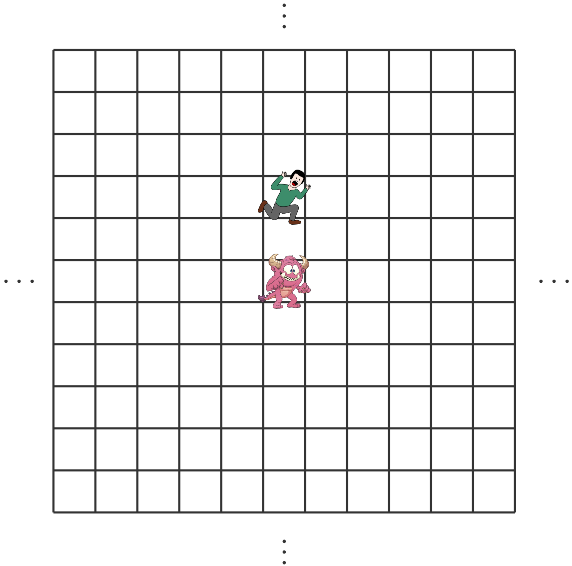

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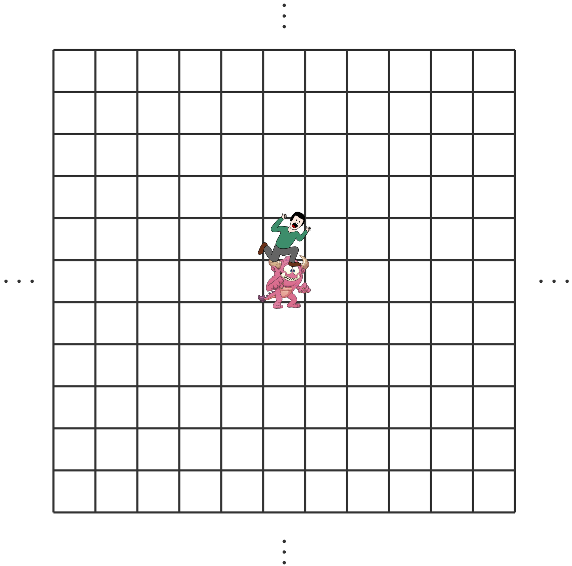

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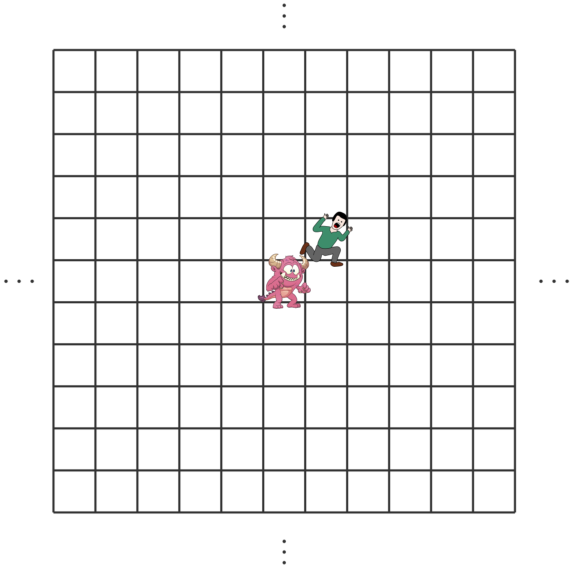

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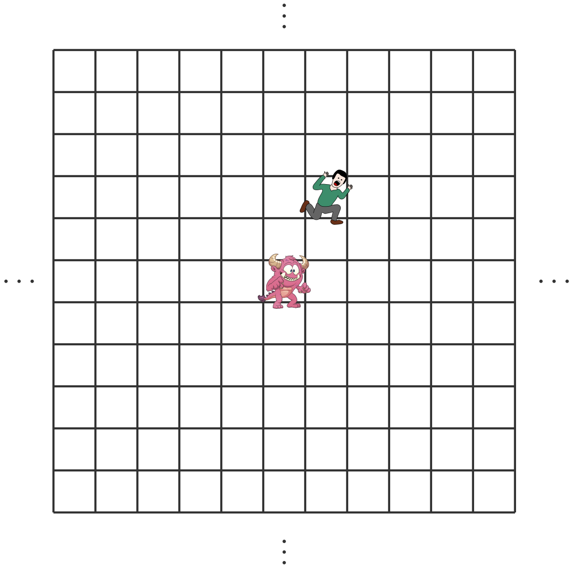

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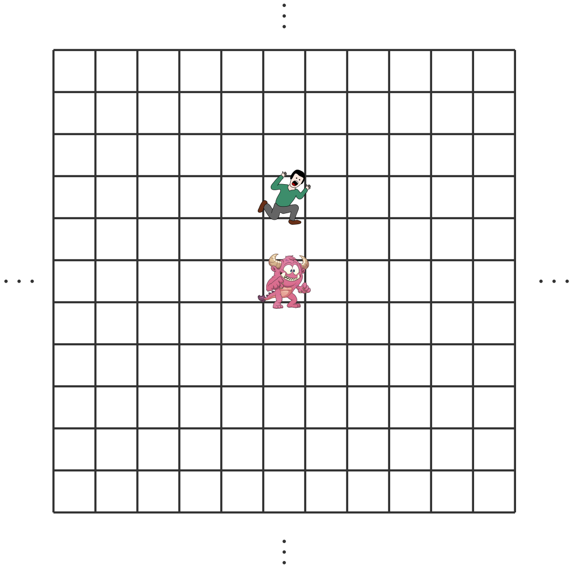

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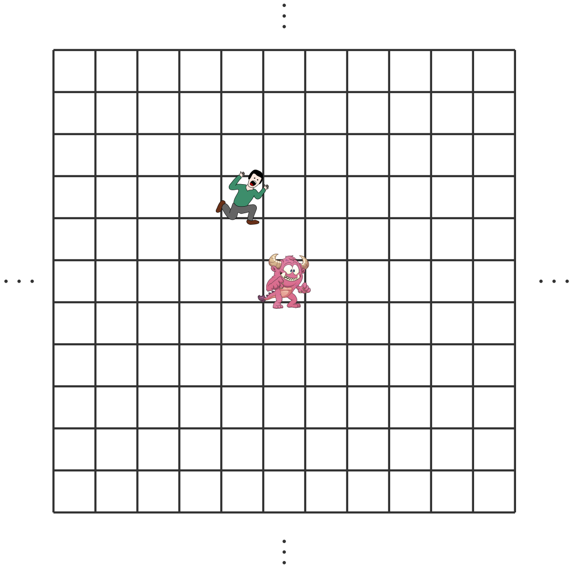

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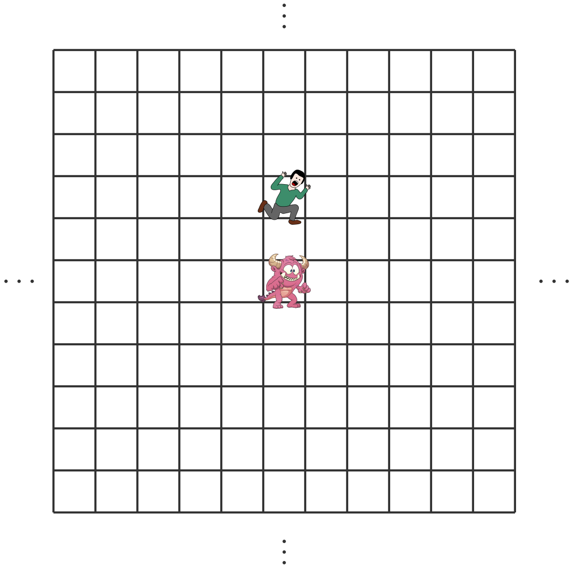

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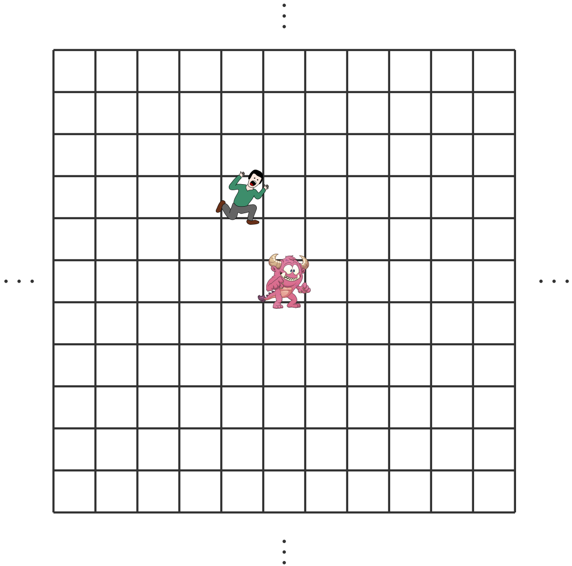

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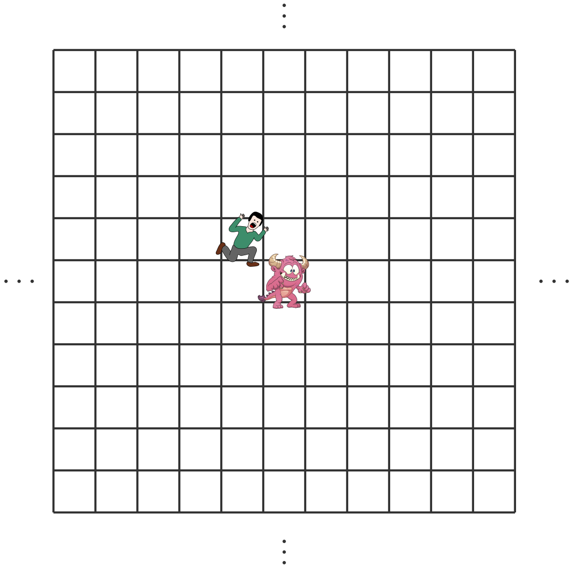

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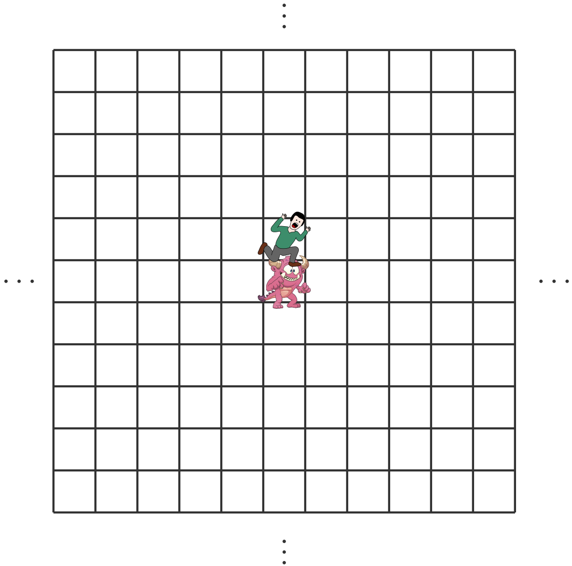

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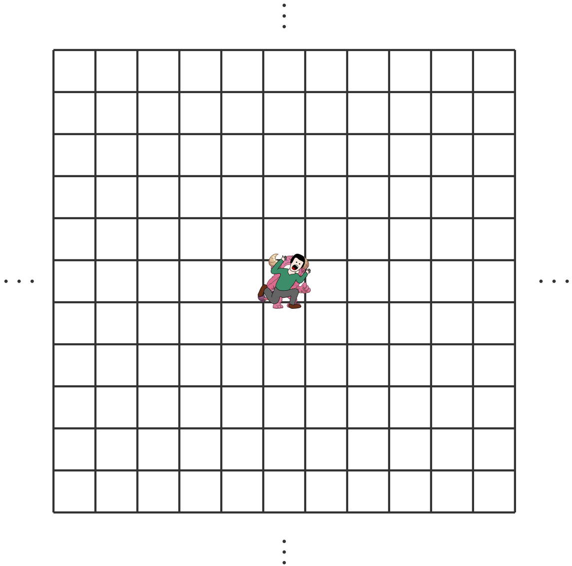

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- All previously-known constructions of planar graphs with uniform polynomial growth are *recurrent*.

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Thm. [E - Lee '20]

For any $\alpha > 2$ we can construct a *transient* planar graph with growth R^α.

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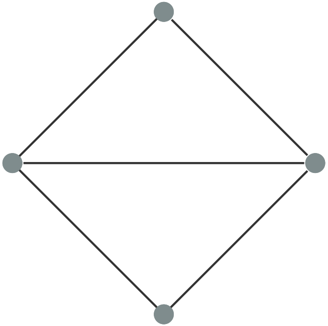

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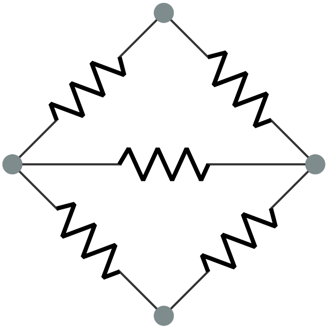

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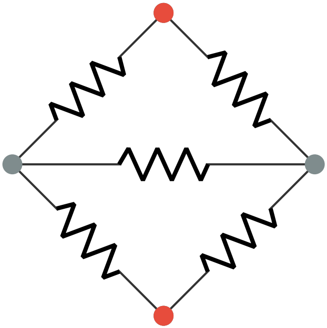

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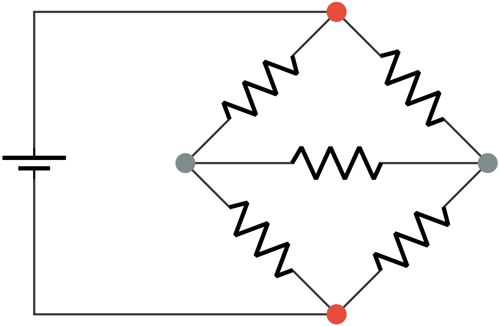

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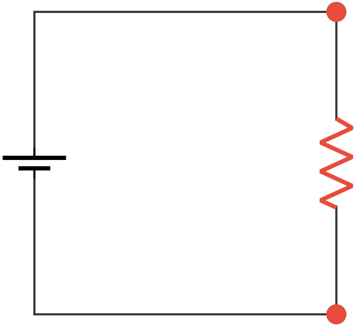


$R_{\text{eff}} = \frac{1}{2} \cdot 2 = 1$

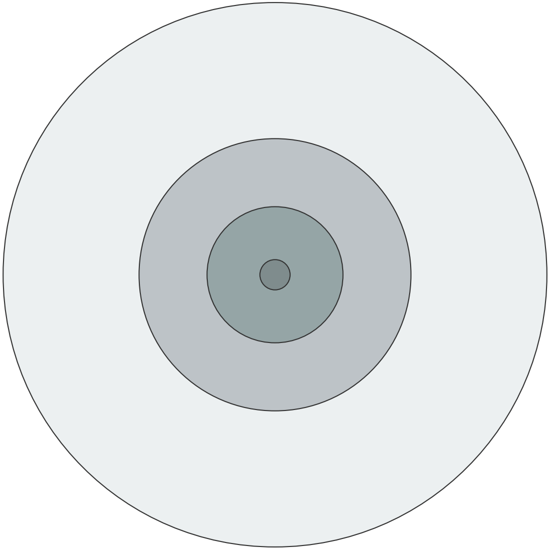
Effective Resistance and Recurrence

Thm. [Doyle - Snell '84]

A graph G is recurrent if and only if $\mathrm{Reff}(v, \infty) = \infty$ for $v \in V(G)$. Where

$$\mathrm{Reff}(v, \infty) = \lim_{R \to \infty} \mathrm{Reff}\left(v, V(G) \setminus B(v, R)\right).$$

One Approach…



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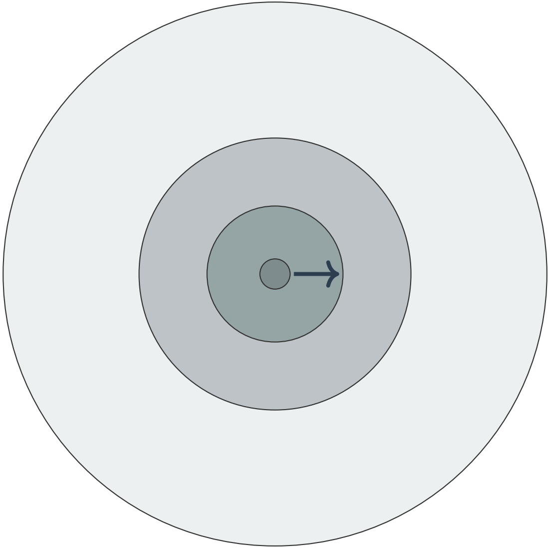

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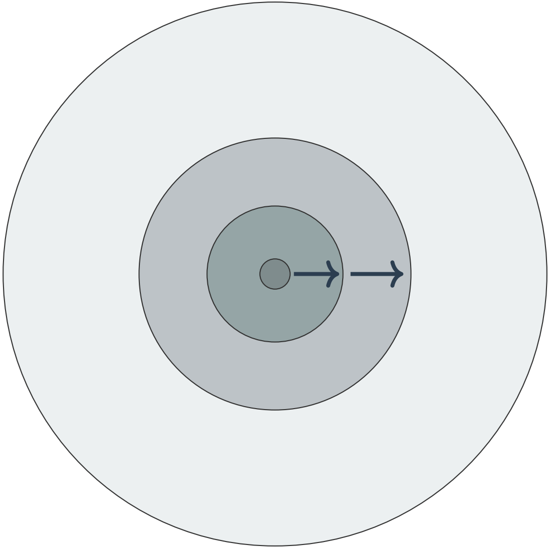

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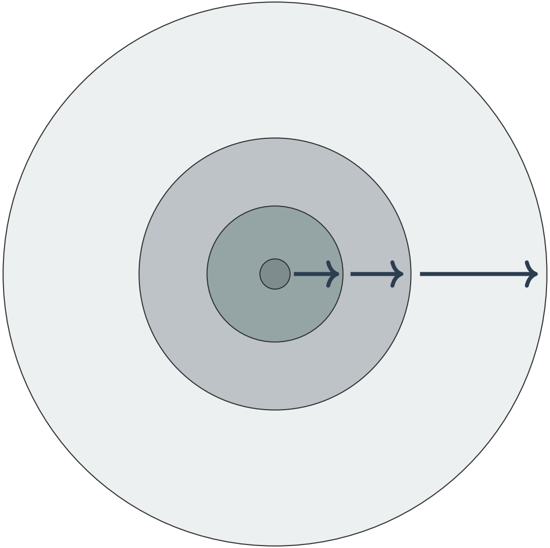

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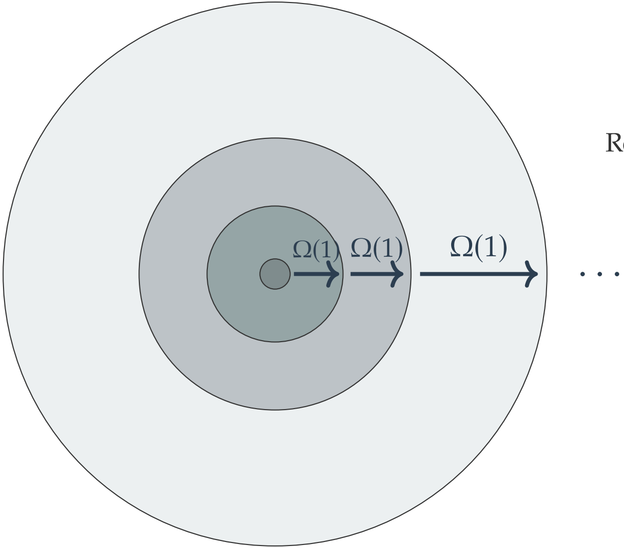

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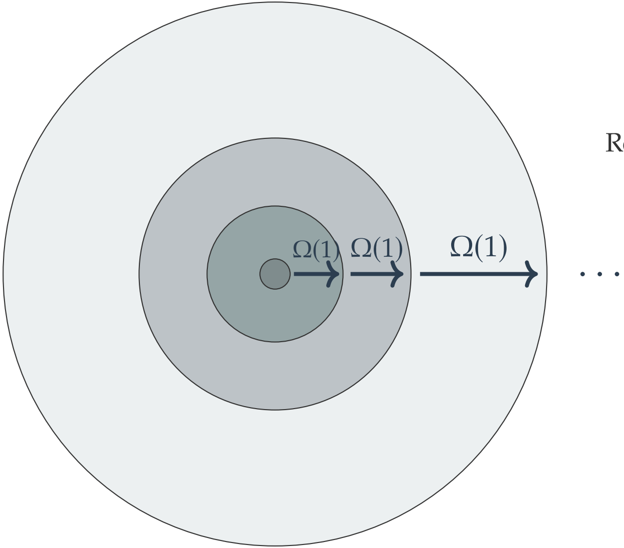

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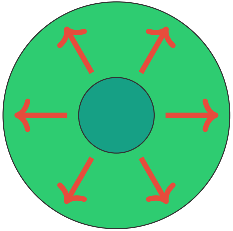

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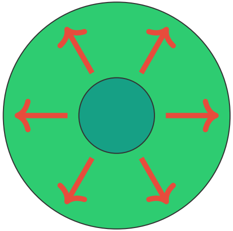

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- Thm. [Benjamini - Papasoglu '10]: $\geq \Omega(\frac{1}{R})$.
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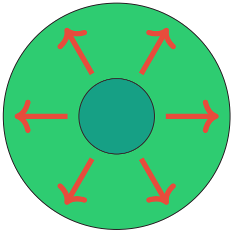
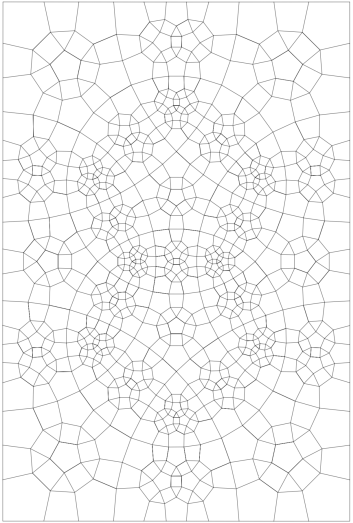
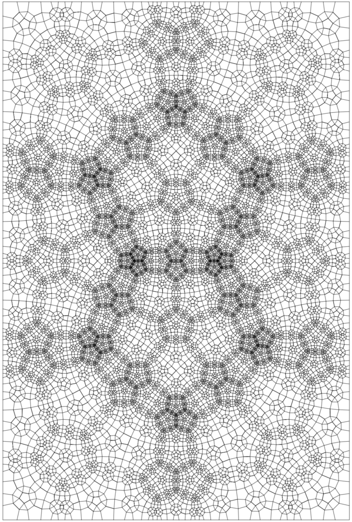


Cannon's Conjecture





Source: Cannon - Floyd - Parry '01

Conjecture 2 [Lee '17]

For any planar graph with uniform polynomial growth we have

$$\mathrm{Reff}\,(B(v,R), V(G) \setminus B(v,10R)) \geq \Omega(1).$$

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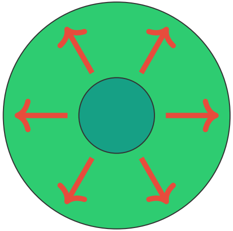

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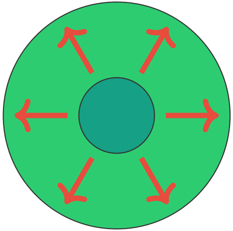

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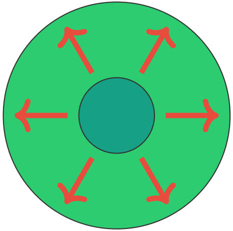

Thm. [E - Lee '20]

For any $\epsilon > 0$ we can construct a planar graph with uniform polynomial growth having

$$\mathrm{Reff}\,(B(v,R), V(G) \setminus B(v,10R)) \leq O(\frac{1}{R^{1-\epsilon}}).$$

So far I have covered ...

- ✓ Planarity.
- ✓ Uniform volume growth.
- ✓ Planar graphs with uniform polynomial growth > 2.
  - ✓ Their *structure*.
  - ✓ *Random walks* and *effective resistances* in these graphs.
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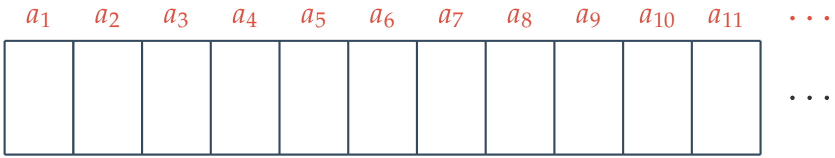
Up next . . .

- ☐ Our construction.

The Construction

A Generalization of the Grid

Given an infinite sequence $(a_1, a_2, a_3, \ldots)$, we construct the graph $\mathscr{G}$ as follows:



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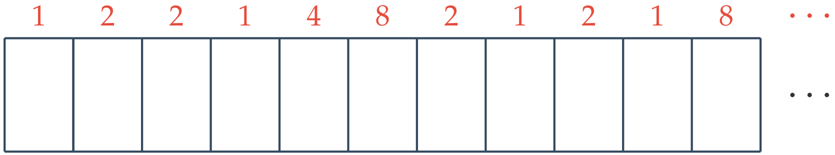

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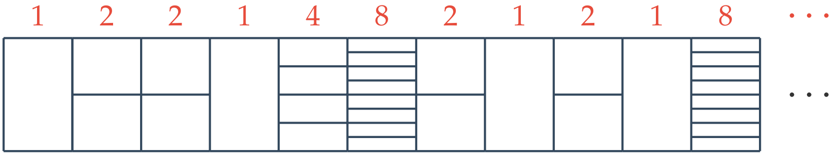

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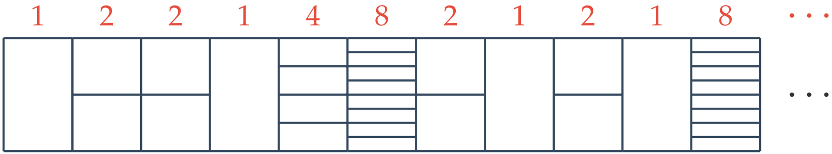

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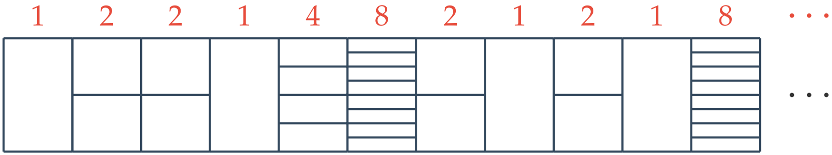

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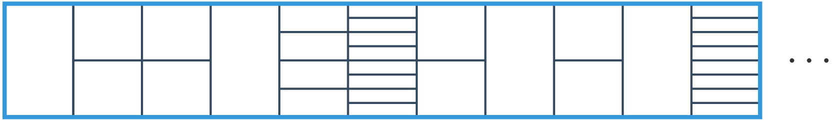

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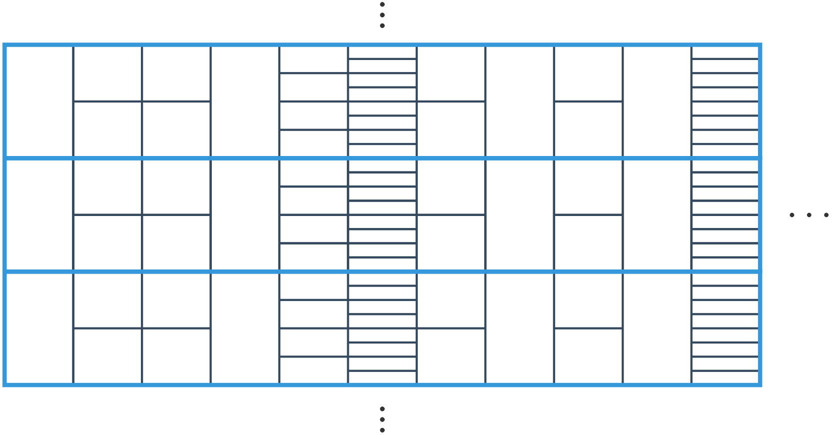

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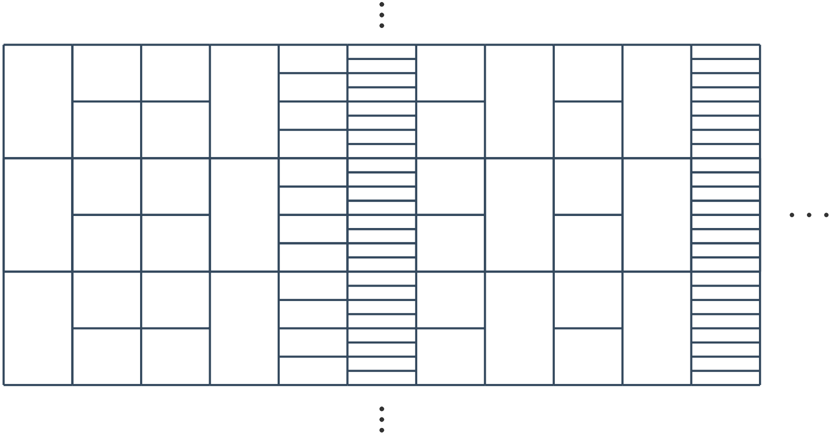

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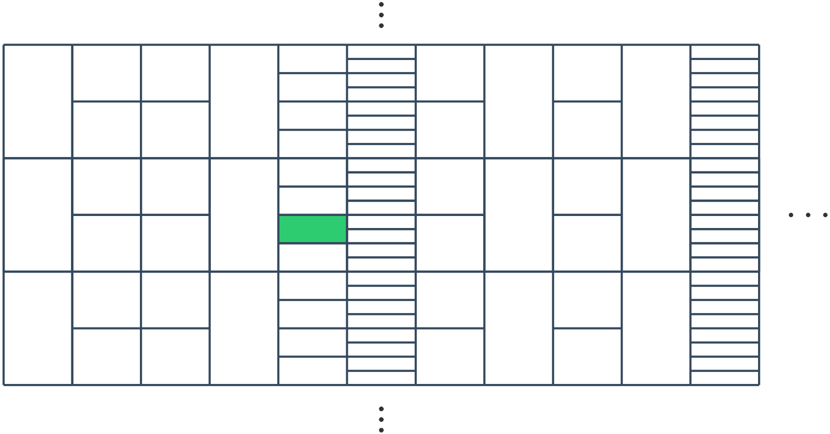

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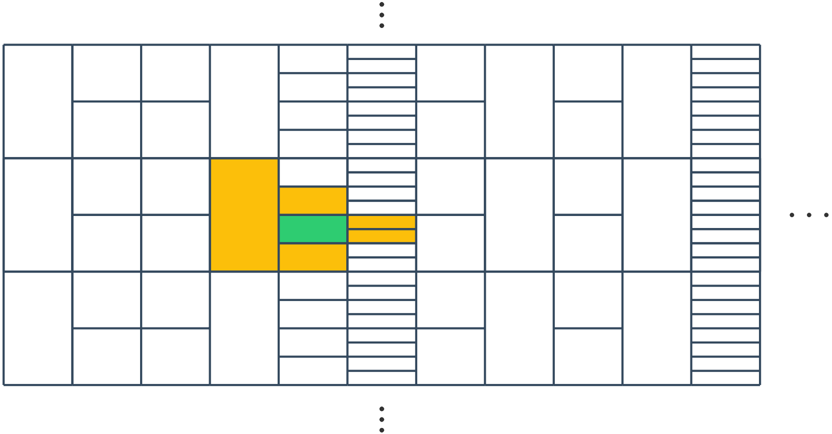


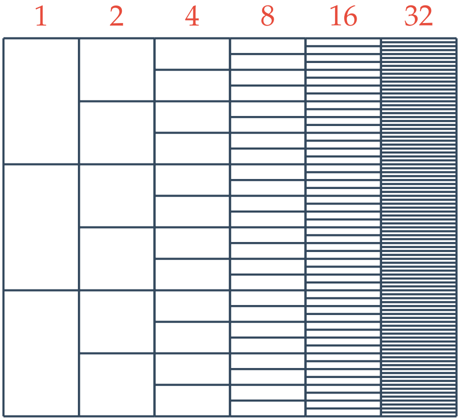
Examples – The Grid

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

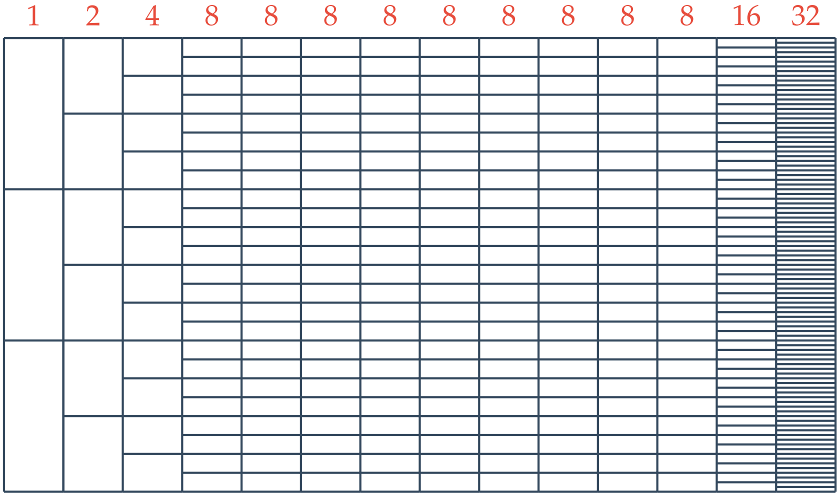
Examples – Scaling Doesn't Matter

10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10

Examples – Exponential Growth



Examples – Sub-exponential Growth?



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Definition

Let us set $\tau_0 = (1)$. We recursively define

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\tau_0 &= (1) \\
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\tau_2 &= (\underbrace{1, 2, 1}_{\tau_1}, \underbrace{2, 4, 2}_{2\tau_1}, \underbrace{1, 2, 1}_{\tau_1})
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$$\tau_3 = (\underbrace{1, 2, 1, 2, 4, 2, 1, 2, 1}_{\tau_2}, \underbrace{2, 4, 2, 4, 8, 4, 2, 4, 2}_{2\tau_2}, \underbrace{1, 2, 1, 2, 4, 2, 1, 2, 1}_{\tau_2})$$

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Finally we set $\vec{\tau} = \lim_{n\to\infty} \tau_n$.

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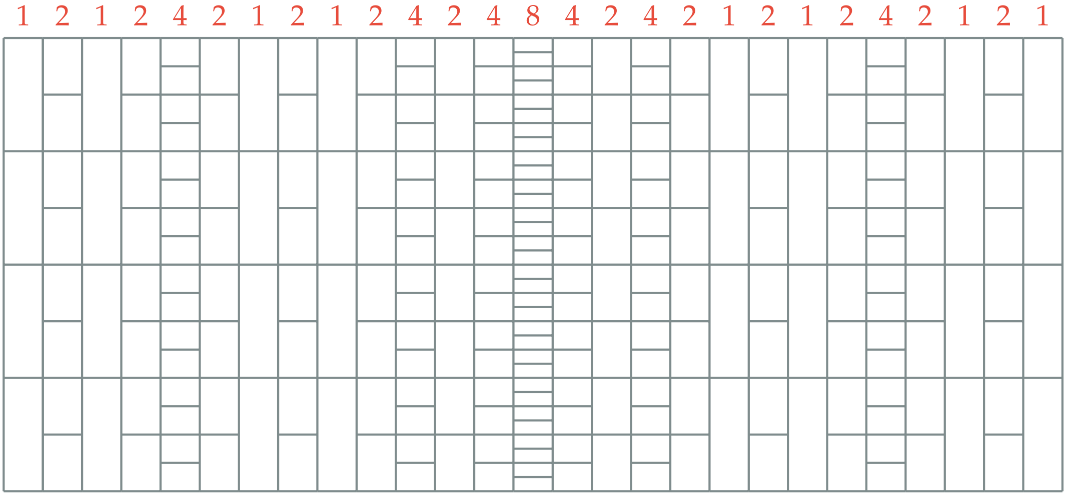
How Does It Look Like?



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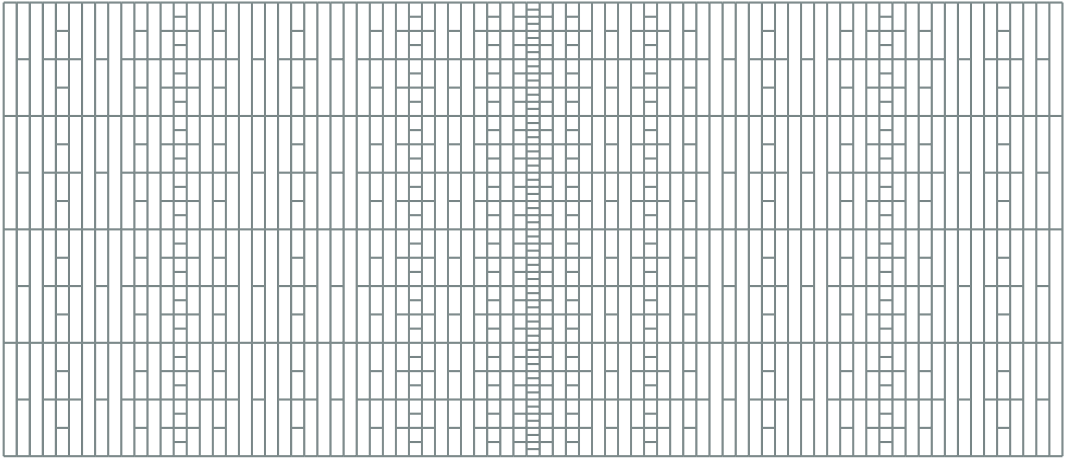
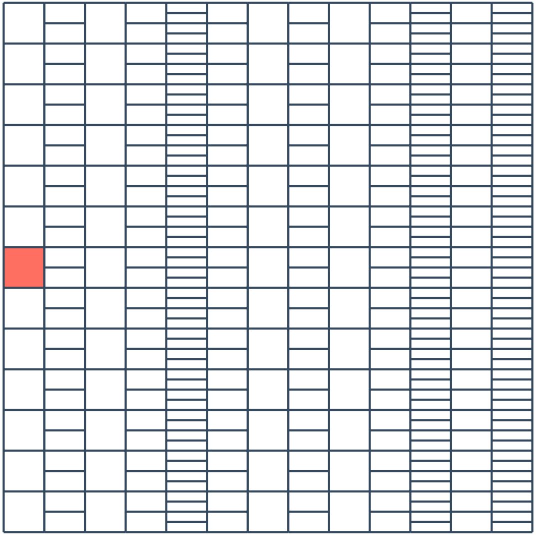


Volume Growth



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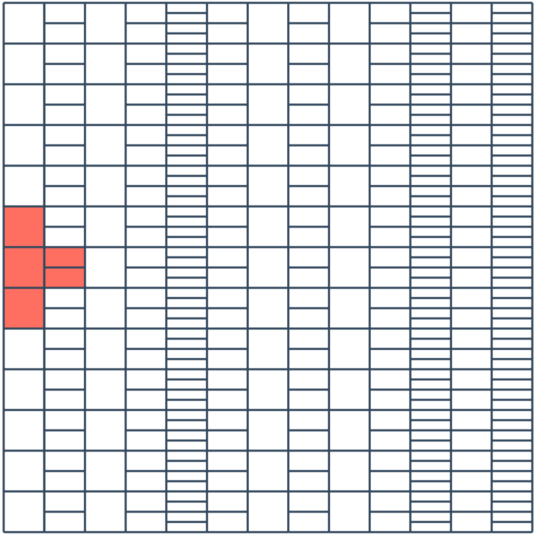

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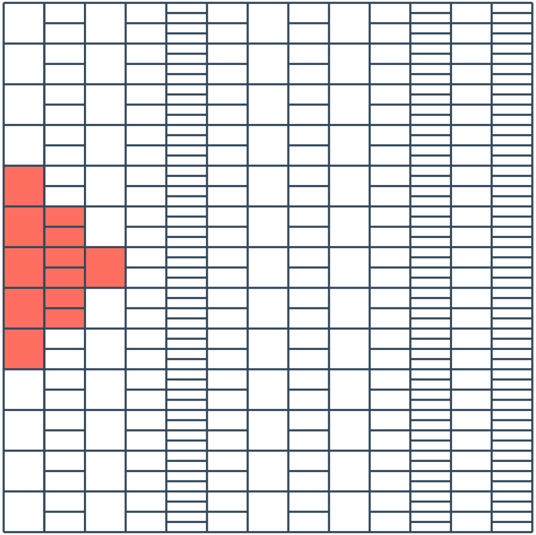

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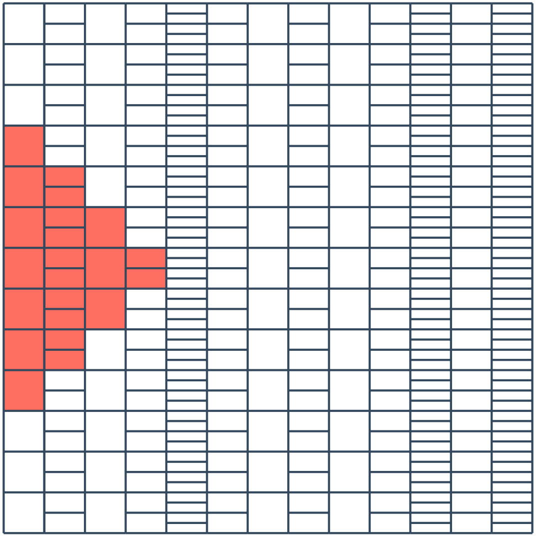

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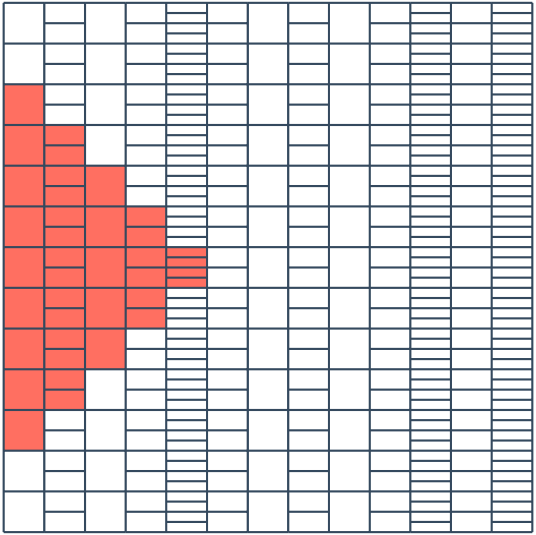

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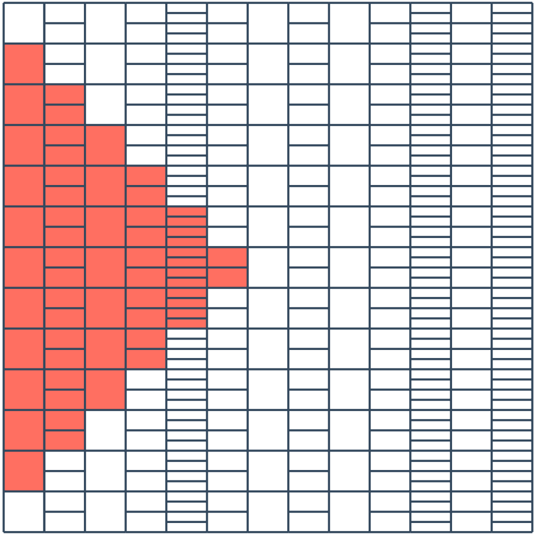

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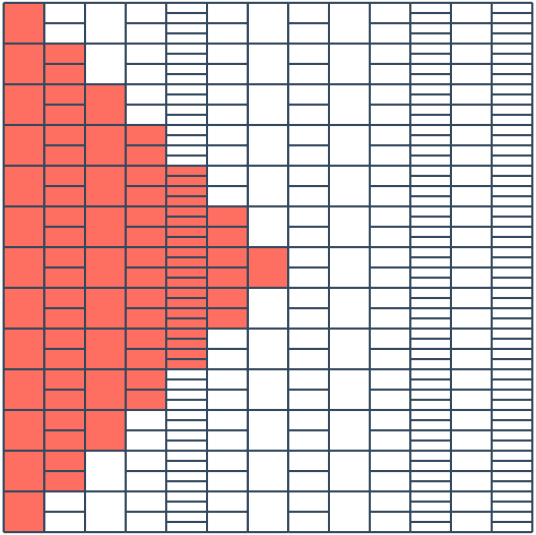

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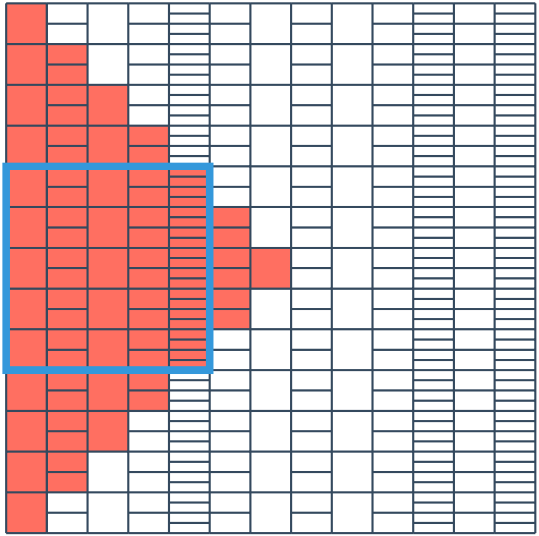

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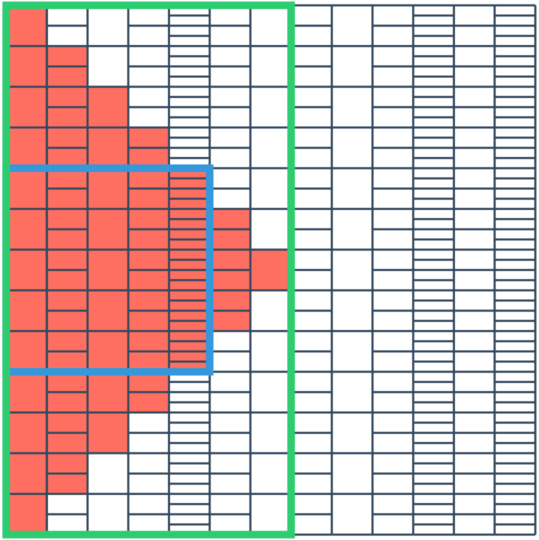

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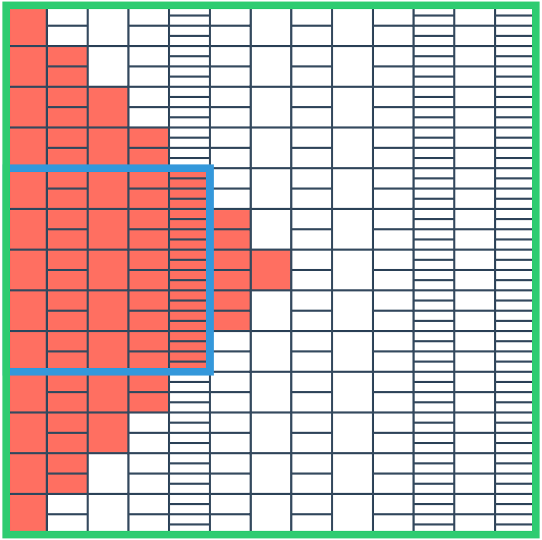

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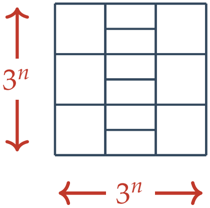

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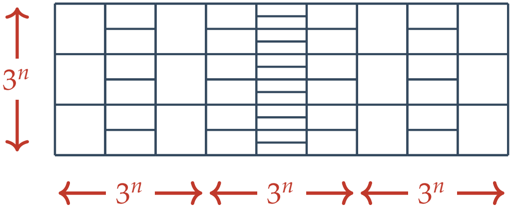

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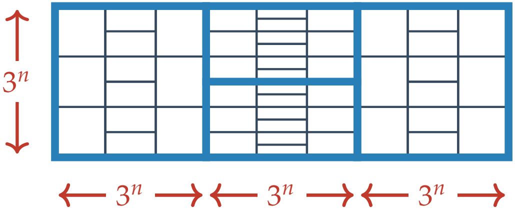

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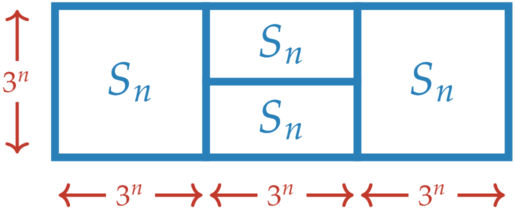

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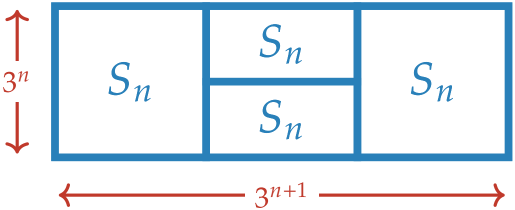

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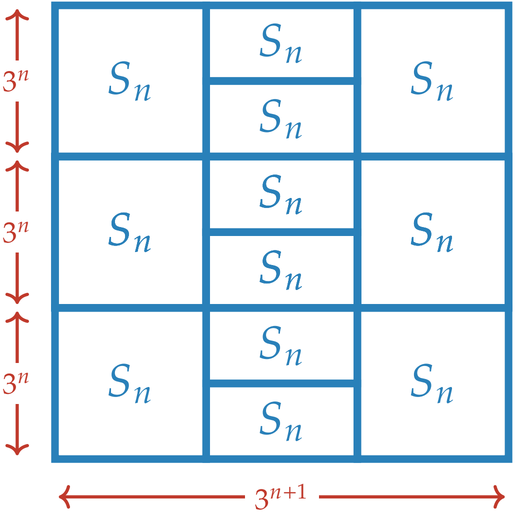

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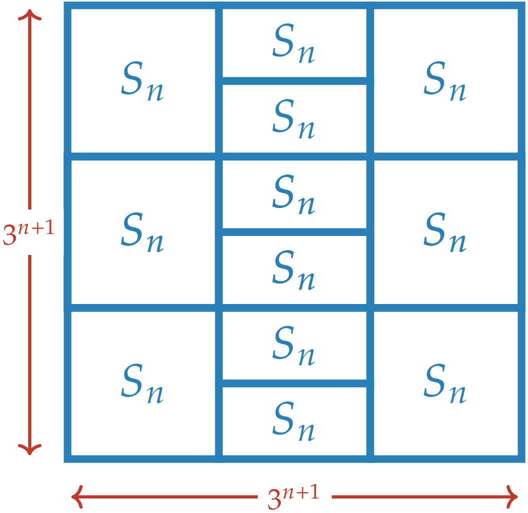

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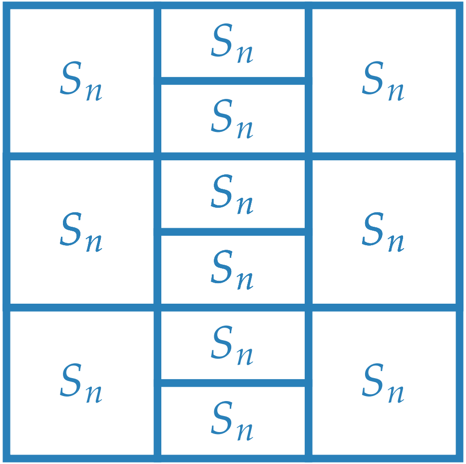
$S_{n+1} =$



The Transient Construction

$$T = \boxed{S_0}\boxed{S_1}\boxed{S_2}\boxed{S_3}\boxed{S_4}\cdots$$

So far I have covered . . .

- ✓ Planarity.
- ✓ Uniform volume growth.
- ✓ Planar graphs with uniform polynomial growth > 2.
  - ✓ Their *structure*.
  - ✓ *Random walks* and *effective resistances* in these graphs.
  - ✓ Cannon's conjecture.
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- Cannon's conjecture remains wide open.

Questions?

Thank you!