Formula Sheet, Signals and Systems Qualifier Exam, Fall 2021

Euler's Formula:

- $e^{j\theta} = \cos\theta + j\sin\theta$
- $\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$
- $\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$

Table 1: PROPERTIES OF THE FOURIER TRANSFORM

Property	Aperiodic signal	Fourier Transform
	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{jwt}d\omega$	$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-jwt}$
	$y(t)$	$Y(j\omega)$
Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X\left(j(\omega-\omega_0\right)$
مزدوج Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{j\omega}{a})$
Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega)*Y(j\omega)$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int\limits_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$

Parsavel's Relation for Aperiodic Signals

$$\int\limits_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Table 2: BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier Transform
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2sin\omega T_1}{\omega}$
$\frac{sinWT}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$
$u(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a+j\omega}$

Finite-Sum Formula: $\sum_{n=0}^{M-1} a^n = \frac{1-a^M}{1-a}$

Table 3: PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Property	**Aperiodic signal**	**Fourier Transform**
	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{jwn}dw$	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$
	$y[n]$	$Y(e^{j\omega})$
Time Shifting	$x[n-n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time Reversal	$x[-n]$	$X(e^{-j\omega})$
Time Expansion	$x_{(k)}[n] = \begin{cases} x[\frac{n}{k}], & \text{if n is multiple of k} \\ 0, & \text{if n is not multiple of k} \end{cases}$	$X(e^{jk\omega})$
Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Differencing in Time	$x[n]-x[n-1]$	$(1-e^{-j\omega})X(e^{j\omega})$
Accumulation	$\sum\limits_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$
Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$

Parsavel's Relation for Aperiodic Signals

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Table 4: BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	**Fourier Transform**
$e^{j\omega_0 n}$	$2\pi \sum\limits_{l=-\infty}^{+\infty} \delta(\omega-\omega_0-2\pi l)$
$x[n] = \begin{cases} 1, & \lvert n\rvert < N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{sin[\omega(N_1+\frac{1}{2})]}{sin(\frac{\omega}{2})}$
$\frac{W}{\pi}sinc(\frac{Wn}{\pi}), 0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum\limits_{k=-\infty}^{+\infty} \pi\delta(\omega-2\pi k)$
$a^n u[n], \lvert a\rvert < 1$	$\frac{1}{1-ae^{-j\omega}}$

Table 5: PROPERTIES OF THE LAPLACE TRANSFORM

Property	Signal	Laplace Transform	ROC
	$x(t)$	$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Time Shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s-s_0)$	Shifted version of R (i.e s is in the ROC if $s-s_0$ is in R)
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{s}{a})$	Scaled ROC (i.e, s is in the ROC if s/a is in R
Conjugation	$x^*(-t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the Time Domain	$\int\limits_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re e\{s\} > 0\}$

Table 6: LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

Signal	Transform	ROC
$u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
$-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$

Table 7: PROPERTIES OF THE z-TRANSFORM

Property	Signal	z-Transform	ROC
$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$	$x[n]$	$X(z)$	R
	$x_1[n]$	$X_1(z)$	R_1
	$x_2[n]$	$X_2(z)$	R_2
Time Shifting	$x[n-n_0]$	$z^{-n_0}X(z)$	R,except, for the possible addition or deletion of the origin
Scaling in the z-Domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
	$z_0^n x[n]$	$X(\frac{z}{z_0})$	$z_0 R$
	$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e,$\|a\|R$= the set of points $\{\|a\|z\}$ for z in R)
Time Reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e, R^{-1}=the set of points z^{-1}, where z is in R)
Time Expansion	$x_{(k)}[n]=\begin{cases} x[r], & if n=rk \\ 0, & if n\neq rk \end{cases}$	$X(z^k)$	$R^{\frac{1}{k}}$ (i.e, the set of points $z^{\frac{1}{k}}$, where z is in R
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-z^{-1}}X(z)$	At least the intersection of R and $\|z\|>1$
Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R

Table 8: SOME COMMON z-TRANSFORM PAIRS

Signal	Transform	ROC
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\|>1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\|<1$
$\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\|z\|>\alpha$
$-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\|z\|<\alpha$