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Adaptive and Array Signal Processing

NB: Those tasks highlighted **bold-faced** can be solved independently from the previous ones. Label the axis of all your graphs properly. Please show intermediate steps to get partial credits.

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1. Consider a real-valued function

   (a)
   $$f(z) = |z|^4 \tag{1}$$

   (b)
   $$f(z) = \frac{1}{z} \tag{2}$$

   (c)
   $$f(z) = \ln(z) \tag{3}$$

   (d)
   $$f(z) = z^2 \tag{4}$$

   (e)
   $$f(z) = z^* \tag{5}$$

   (f)
   $$f(z) = z^2 \tag{6}$$

   (g)
   $$f(z) = z^3 + \jmath 2z + (z^*)^2 \tag{7}$$

   (h)
   $$f(z) = \left(z^2 + z^*\right)^3 \tag{8}$$

   Check if the Cauchy-Riemann equations hold and determine analytic functions. Calculate derivative of each function using Wirtinger calculus and using method of splitting variables.

2. Calculate gradients for the following vector valued functions

   (a)
   $$f(\boldsymbol{z}) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{z} \tag{9}$$

   (b)
   $$f(\boldsymbol{z}) = \boldsymbol{z}^{\mathrm{H}}\boldsymbol{z} \tag{10}$$

   (c)
   $$f(\boldsymbol{z}) = \boldsymbol{z}^{\mathrm{H}}\boldsymbol{R}\boldsymbol{z} \tag{11}$$

   (d)
   $$f(\boldsymbol{z}) = \boldsymbol{z}^{\mathrm{H}}\boldsymbol{R}\boldsymbol{z} + |\boldsymbol{z}|^2 \tag{12}$$

3. Consider the function
   $$J(\boldsymbol{w}, \boldsymbol{w}^*) = \boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}\boldsymbol{w} - 2Re\left\{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{p}\right\} \tag{13}$$
   with $\boldsymbol{w}, \boldsymbol{p} \in \mathbb{C}^n$ and $\boldsymbol{R} = \boldsymbol{R}^{\mathrm{H}} \in \mathbb{C}^{n\times n}$

   (a) Show whether $J(\boldsymbol{w}, \boldsymbol{w}^*)$ is a real or complex valued function?

(b) Find a $\boldsymbol{w}$ that minimizes $J(\boldsymbol{w}, \boldsymbol{w}^*)$ by solving $\frac{\partial J}{\partial \boldsymbol{w}^*} = \mathbf{0}$

(c) Find a $\boldsymbol{w}$ that minimizes $J(\boldsymbol{w}, \boldsymbol{w}^*)$ by solving $\frac{\partial J}{\partial \boldsymbol{w}} = \mathbf{0}$

(d) compare the results from (b) and (c)

4. The random process $V(t)$ is defined as

$$V(t) = X\cos(2\pi f_c t) - Y\sin(2\pi f_c t) \tag{14}$$

where X and Y are random variables. Show that $V(t)$ is wide-sense stationary (WSS) if and only if $E\{X\} = E\{Y\} = 0, E\{X^2\} = E\{Y^2\}$, and $E\{XY\} = 0$.

5. Consider the sample mean and sample variance in a random sample of scalar random variables X_n: $\mathcal{N}(m, \sigma^2)$

$$\hat{m} = \frac{1}{M}\sum_{n=0}^{M-1} x_n$$
$$\hat{\sigma}^2 = \frac{1}{M}\sum_{n=0}^{M-1} (x_n - \hat{m})^2$$

Then, is $\hat{m}$ unbiased? Is $\hat{\sigma}^2$ unbiased? Show explicitly why.

6. Use the method of Lagrange multipliers to find the maximum value of the following function

$$f(x, y) = 9x^2 + 36xy - 4y^2 - 18x - 8y \tag{15}$$

subject to the constraint $3x + 4y = 32$.

7. Find the minimum of the function

$$f(\boldsymbol{w}) = \boldsymbol{w}^{\mathrm{H}} \cdot \boldsymbol{R} \cdot \boldsymbol{w} \tag{16}$$

subject to $\boldsymbol{S}^{\mathrm{H}} \cdot \boldsymbol{w} = \boldsymbol{g}$.

8. Find the minimum of the function

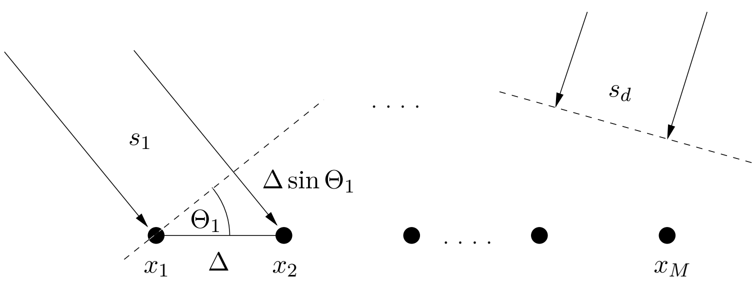
$$f(\boldsymbol{w}) = \boldsymbol{w}^{\mathrm{H}} \cdot \boldsymbol{R} \cdot \boldsymbol{w} + \alpha\,|\boldsymbol{w}|^2 \tag{17}$$

subject to $\boldsymbol{S}^{\mathrm{H}} \cdot \boldsymbol{w} = \boldsymbol{g}$.

9. Solve the following constrained real-valued minimization problem by means of Lagrangian multipliers. Minimize $f(x_1, x_2)$ subject to $g(x_1, x_2)$.

$$f(x_1, x_2) = 1 + 2x_1x_2 + x_2^2 + 3x_2^2 \tag{18}$$
$$g(x_1, x_2) = 1 + x_1 - 2x_2 = 0 \tag{19}$$

10. Consider the following uniform linear array (ULA) consisting of M identical antennas separated by a distance Δ.



Assume a single planar wave impinging from azimuth Θ_i propagating at speed c carrying a narrowband complex signal $s_i(t)$. The output vector $\boldsymbol{x}(t)$ of the antenna array is given by

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_M(t) \end{bmatrix}^{\mathrm{T}} = \boldsymbol{a}_i(\mu_i)s_i(t) \tag{20}$$

with the array steering vector

$$\boldsymbol{a}_i(\mu_i) = \begin{bmatrix} 1 & e^{j\mu_i} & \dots & e^{j(M-1)\mu_i} \end{bmatrix}^{\mathrm{T}} \tag{21}$$

the spatial frequency

$$\mu_i = -\frac{2\pi f_c}{c}\Delta \sin\Theta_i = -2\pi\frac{\Delta}{\lambda}\sin\Theta_i \tag{22}$$

and the wavelength $\lambda = c/f_c$.

(a) The antennas are separated by $\Delta = \lambda/2$. Calculate the array steering vectors $\boldsymbol{a}_1(\mu_1)$ and $\boldsymbol{a}_2(\mu_2)$ for a $M = 3$ element ULA corresponding to $\Theta_1 = -30^o$ and $\Theta_2 = 30^o$, respectively.

(b) If d narrowband planar waves are impinging on the array, the output is the superposition of the individual signals. Hence the output vector may be written as

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{a}_1(\mu_1) & \dots & \boldsymbol{a}_d(\mu_d) \end{bmatrix} \begin{bmatrix} s_1(t) \\ \vdots \\ s_d(t) \end{bmatrix} + \boldsymbol{n}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{n}(t) \tag{23}$$

where $\boldsymbol{A}$ is the matrix containing all steering vectors, and $\boldsymbol{s}$ is the vector of all narrowband signals $s_i(t)$. Note that we have included an additional, zero mean noise vector $\boldsymbol{n}(t)$. The array outputs are now weighted and added to form an output signal

$$y = \boldsymbol{w}^{\mathrm{H}}\boldsymbol{x} \tag{24}$$

where $\boldsymbol{w}$ represents the weight vector.
Show that the output power $E\{yy^*\}$ can be written as

$$E\{yy^*\} = \boldsymbol{w}^{\mathrm{H}}\boldsymbol{A}\boldsymbol{R}_{ss}\boldsymbol{A}^{\mathrm{H}}\boldsymbol{w} + \boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}_{nn}\boldsymbol{w} \tag{25}$$

assuming the noise is uncorrelated with the signals $\boldsymbol{s}(t)$. Here $\boldsymbol{R}_{ss} = E\{\boldsymbol{s}(t)\boldsymbol{s}^{\mathrm{H}}(t)\}$ ($\boldsymbol{s}(t)$ zero mean) and $\boldsymbol{R}_{nn} = E\{\boldsymbol{n}(t)\boldsymbol{n}^{\mathrm{H}}(t)\}$ are the covariance matrices of the signal and the noise vectors, respectively.

(c) Assume the two waves from 2(a) impinge on the same $M = 3$ element ULA. Calculate the weight vector $\boldsymbol{w}$ which will minimize the contribution of the noise in the output y, while still giving a unity antenna gain of $\boldsymbol{a}_i^{\mathrm{H}}(\mu_i)\boldsymbol{w} = 1$ in the directions of Θ_i of the two impinging wavefronts $\Theta_1 = -30^o$ and $\Theta_2 = 30^o$. Use the method of Lagrange multipliers and assume

$$\boldsymbol{R}_{ss} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Consider the following two cases

i.

$$\boldsymbol{R}_{nn} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ii.

$$\boldsymbol{R}_{nn} = \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 1 \end{bmatrix}$$

11. Solve the following constrained complex minimization problem. Minimize $f(\boldsymbol{w})$ subject to $\boldsymbol{g}(\boldsymbol{w})$.

$$f(\boldsymbol{w}) = \boldsymbol{w}^{\mathrm{H}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \boldsymbol{w} \tag{26}$$

$$\boldsymbol{g}(\boldsymbol{w}) = \begin{bmatrix} 1 + \boldsymbol{w}^{\mathrm{H}}\begin{bmatrix} 1 & -j & 1 \end{bmatrix}^{\mathrm{H}} \\ 1 + \boldsymbol{w}^{\mathrm{H}}\begin{bmatrix} j & 2 & -j \end{bmatrix}^{\mathrm{H}} \end{bmatrix} = \boldsymbol{0} \tag{27}$$

with $\boldsymbol{w} \in \mathbb{C}^3, f \in \mathbb{R}, \boldsymbol{g} \in \mathbb{C}^2$ by means of complex Lagrangian multipliers.

12. Determine range of the column space for the following matrix

    (a)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 0 & 1 \end{bmatrix} \tag{28}$$

    (b)

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \tag{29}$$

    Find basis vectors for the remaining fundamental subspaces as well.

13. Find all range of possible solutions for the following system of linear equations

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \tag{30}$$

    Find the solution with the smallest possible norm.

14. Compute the column space and null space for the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 3 & 2 \\ 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 3 & 3 \end{bmatrix}. \tag{31}$$

15. The SVD of the matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

    (a) Give orthonormal bases for the four subspaces $\text{im}\{\boldsymbol{A}\}$, $\text{im}\{\boldsymbol{A}^{\text{H}}\}$, $\text{null}\{\boldsymbol{A}\}$ and $\text{null}\{\boldsymbol{A}^{\text{H}}\}$.
    (b) Compute the projections of $\boldsymbol{v} = \begin{bmatrix} 3 & 2 \end{bmatrix}^{\text{T}}$ onto $\text{im}\{\boldsymbol{A}\}$ and $\text{null}\{\boldsymbol{A}^{\text{H}}\}$.
    (c) Compute the projections of $\boldsymbol{v} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}^{\text{T}}$ onto $\text{im}\{\boldsymbol{A}^{\text{H}}\}$ and $\text{null}\{\boldsymbol{A}\}$.
    (d) Consider the linear system $\boldsymbol{A}\boldsymbol{w} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \boldsymbol{w} \in \mathbb{C}^4$.
    Find one particular solution $\boldsymbol{w}_p$ to this system.
    (e) If $\boldsymbol{w}_p$ is a solution and $\boldsymbol{w}_n \in \text{null}\{\boldsymbol{A}\}$ then also is $\boldsymbol{w}_p + \boldsymbol{w}_n$. Find the complete solution to the system in the form $\boldsymbol{w} = \boldsymbol{w}_p + \boldsymbol{B}\boldsymbol{x}$ with $\boldsymbol{x} \in \mathbb{C}^3$ and $\boldsymbol{B} \in \mathbb{C}^{4\times 3}$.

16. The singular value decomposition is the appropriate tool for analyzing a mapping from one vector space into another vector space, possibly with a different dimension. Most systems of simultaneous linear equations fall into this second category.
    Any m by n matrix $\boldsymbol{A}$ can be factored into

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^{\text{H}}$$

    Where $\boldsymbol{U} \in \mathbb{C}^{m\times m}$ is a unitary[1] matrix and the columns of $\boldsymbol{U}$ are the eigenvectors of $\boldsymbol{A}\boldsymbol{A}^{\text{H}}$. Likewise, $\boldsymbol{V} \in \mathbb{C}^{n\times n}$ is a unitary matrix and the columns of $\boldsymbol{V}$ are the eigenvectors[2] of $\boldsymbol{A}^{\text{H}}\boldsymbol{A}$. The matrix $\boldsymbol{\Sigma}$ has the following form:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_S & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \qquad \text{with} \qquad \boldsymbol{\Sigma} \in \mathbb{R}^{m\times n}, \boldsymbol{\Sigma}_S \in \mathbb{R}^{r\times r} \tag{32}$$

[1] A unitary matrix is a square matrix with columns built out of the orthonormal vectors. the vectors are orthonormal when their lengths are all 1 and their dot products are zero. If the matrix $\boldsymbol{Q}$ is unitary then $\boldsymbol{Q}^{\text{H}}\boldsymbol{Q} = \boldsymbol{I}$ and $\boldsymbol{Q}^{\text{H}} = \boldsymbol{Q}^{-1}$.

[2] The number λ is an eigenvalue of the matrix $\boldsymbol{M}$ if and only if: $\det(\boldsymbol{M} - \lambda\boldsymbol{I}) = 0$. This is the characteristic equation, and each solution λ has a corresponding eigenvector $\boldsymbol{x}$: $(\boldsymbol{M} - \lambda\boldsymbol{I})\boldsymbol{x} = 0$ or $\boldsymbol{M}\boldsymbol{x} = \lambda\boldsymbol{x}$.

The matrix $\boldsymbol{\Sigma}$ has the same size as $\boldsymbol{A}$. The diagonal entries of $\boldsymbol{\Sigma}_S$, also called singular values, $\sigma_1, \dots, \sigma_r$, are the square roots of the nonzero eigenvalues of both $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}$ and $\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}$, where r is the rank of $\boldsymbol{A}$.

(a) Find the singular value decomposition of

$$\boldsymbol{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

i. Compute $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}$, find its eigenvalues $\lambda_{1,2}$(it is generally preferred to put them into decreasing order) and then find the corresponding unit eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. The matrix $\boldsymbol{U}$ is then: $\boldsymbol{U} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}$.

ii. The eigenvalues of $\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}$ are the same as the eigenvalues of $\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}$. Calculate the eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ of $\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}$. The matrix $\boldsymbol{V}$ is then: $\boldsymbol{V} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}$.

iii. Write down the matrix $\boldsymbol{\Sigma}$ and finally the SVD of $\boldsymbol{A}$.

17. Consider the following matrix $\boldsymbol{X} \in \mathbb{R}^{2\times 4}$:

$$\boldsymbol{X} = \frac{1}{4} \cdot \begin{bmatrix} 3 & 1 \\ 3 & 1 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}.$$

Its singular value decomposition $\boldsymbol{X} = \boldsymbol{U} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{V}^{\mathrm{H}}$ is given by

$$\boldsymbol{U} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2}/2 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{V} = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(a) What are the dimensions of the column space, the null space, the row space, and the left null space, respectively?

(b) Let $\tilde{\boldsymbol{X}}$ be the best rank-one approximation of $\boldsymbol{X}$ in the Frobenius norm sense. How can we find $\tilde{\boldsymbol{X}}$ from the SVD of $\boldsymbol{X}$?

(c) Compute $\tilde{\boldsymbol{X}}$.

(d) Provide a basis for the column space of $\tilde{\boldsymbol{X}}$ and a basis for the null space of $\tilde{\boldsymbol{X}}$.

(e) Compute the projection matrix $\boldsymbol{P}$ onto the column space of $\tilde{\boldsymbol{X}}$.

(10 pt) 18. Consider the following 4×3 matrix $\boldsymbol{X}$:

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ -1 & 0 & 1 \\ 0 & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}.$$

Its SVD is given by $\boldsymbol{X} = \boldsymbol{U} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{V}^{\mathrm{H}}$, where

$$\boldsymbol{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) What is the rank of the matrix $\boldsymbol{X}$?

(b) What are the dimensions of the column space, the row space, the null space, and the left null space of $\boldsymbol{X}$?

(c) Find an orthonormal basis for the left null space of $\boldsymbol{X}$.

(d) Compute the projection of the vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ onto the left null space, where

$$\boldsymbol{b}_1 = \begin{bmatrix} 4 \\ 2 \\ -4 \\ 2 \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} -1 \\ 3 \\ -1 \\ -3 \end{bmatrix}$$

We now compute a new matrix $\boldsymbol{X}_1$ via $\boldsymbol{X}_1 = \boldsymbol{U} \cdot \boldsymbol{\Sigma}_1 \cdot \boldsymbol{V}^{\mathrm{H}}$, where $\boldsymbol{U}$ and $\boldsymbol{V}$ are the same as before and $\boldsymbol{\Sigma}_1$ is

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(e) How do the dimensions of the column space, the row space, the null space, and the left null space change?

(f) Show that $\boldsymbol{X}_1$ is left-unitary, i.e., $\boldsymbol{X}_1^{\mathrm{H}} \cdot \boldsymbol{X}_1 = \boldsymbol{I}_3$.

Hint: Part (e) and (f) can be solved without explicitly computing $\boldsymbol{X}_1$.

19. We are given the following matrix $\boldsymbol{X}$: (13 pt)

$$\boldsymbol{X} = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

(a) Determine the rank of $\boldsymbol{X}$.

(b) What are the dimensions of the column space, the row space, the left null space, and the null space of $\boldsymbol{X}$, respectively?

We would like to determine the economy-size singular value decomposition (SVD) of $\boldsymbol{X}$, i.e., $\boldsymbol{X} = \boldsymbol{U}_{\mathrm{s}} \cdot \boldsymbol{\Sigma}_{\mathrm{s}} \cdot \boldsymbol{V}_{\mathrm{s}}^{\mathrm{H}}$.

(c) Determine the size of the matrices $\boldsymbol{U}_{\mathrm{s}}$, $\boldsymbol{\Sigma}_{\mathrm{s}}$, and $\boldsymbol{V}_{\mathrm{s}}$.

Let $\boldsymbol{u}_k$ be the k-th column of $\boldsymbol{U}_{\mathrm{s}}$, $\boldsymbol{v}_k$ be the k-th column of $\boldsymbol{V}_{\mathrm{s}}$, and σ_k be the k-th singular value. You are given the following:

- $\boldsymbol{u}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, $\boldsymbol{u}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$,
- $\boldsymbol{v}_1 = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, $\boldsymbol{v}_3 = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}^{\mathrm{T}}$,
- $\sigma_1 = 3\sqrt{2}$, $\sigma_2 = 2\sqrt{2}$.

(d) Complete the economy-size SVD, i.e., determine the matrices $\boldsymbol{U}_{\mathrm{s}}$, $\boldsymbol{\Sigma}_{\mathrm{s}}$, and $\boldsymbol{V}_{\mathrm{s}}$ completely.

We now consider the Hermitian matrix $\boldsymbol{R}_1 = \boldsymbol{X}^{\mathrm{H}} \cdot \boldsymbol{X}$.

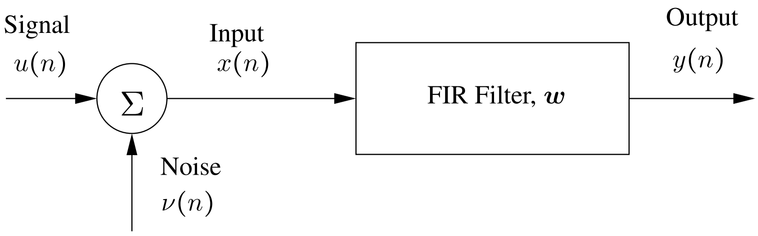
(e) Determine the rank of $\boldsymbol{R}_1$.

(f) Find all the eigenvectors and the eigenvalues of $\boldsymbol{R}_1$. (Note that an $N \times N$ matrix has N eigenvalues and eigenvectors).

(g) Find the projection of the vector $\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}^{\mathrm{T}}$ onto the the null space of $\boldsymbol{X}$ and the projection of the vector $\boldsymbol{b} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^{\mathrm{T}}$ onto the left null space of $\boldsymbol{X}$.

20. Let $x(n) = u(n) + v(n)$ be the input signal to an FIR filter with the weight vector $\boldsymbol{w}$. The signal vector $\boldsymbol{u}(n) = \begin{bmatrix} u(n) & u(n-1) & u(n-2) \end{bmatrix}^{\mathrm{T}} \in \mathbb{C}^3$ has zero mean and the correlation matrix $\boldsymbol{R} = E\left\{\boldsymbol{u}(n)\boldsymbol{u}^{\mathrm{H}}(n)\right\} \in \mathbb{C}^{3\times 3}$. The additive white noise $\boldsymbol{\nu}(n) = \begin{bmatrix} \nu(n) & \nu(n-1) & \nu(n-2) \end{bmatrix}^{\mathrm{T}}$ is uncorrelated with the signal and has the correlation matrix $\sigma^2 \boldsymbol{I} \in \mathbb{C}^{3\times 3}$. The filter output is denoted by $y(n)$. The task of the filter is to maximize the signal-to-noise ratio at its output.

(a) Since the filter is a linear system, the principle of superposition applies. Therefore the effects of signal and noise can be considered separately. Let P_o be the average output power of the signal component at the filter output. Show that $P_o = \boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}\boldsymbol{w}$.



(b) The average power of the noise component at the filter output is $N_o = \sigma^2 \boldsymbol{w}^{\mathrm{H}}\boldsymbol{w}$. Therefore the signal-to-noise ratio is

$$(SNR)_o = \frac{P_o}{N_o} = \frac{\boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}\boldsymbol{w}}{\sigma^2 \boldsymbol{w}^{\mathrm{H}}\boldsymbol{w}}$$

Under the assumption that $\boldsymbol{w} \neq \boldsymbol{0}$ the maximization of $(SNR)_o$ leads to an eigenvalue problem. The largest eigenvalue $\lambda_{\max}$ of the correlation matrix $\boldsymbol{R}$ maximizes the signal-to-noise ratio. In this case the coefficient vector $\boldsymbol{w}_o$ equals the associated eigenvector $\boldsymbol{q}_{\max}$.

$$(SNR)_{o,max} = \frac{\lambda_{\max}}{\sigma^2} \qquad \boldsymbol{w}_o = \boldsymbol{q}_{\max}$$

Perform an eigenvalue decomposition of the correlation Matrix $\boldsymbol{R}$.

$$\boldsymbol{R} = \begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) What is the maximum signal-to-noise ratio in dB if $\sigma^2 = 0.01$?

(d) What is the weight vector $\boldsymbol{w}_o$ for the maximum signal-to-noise ratio?

21. Consider an input signal with correlation matrix

$$\boldsymbol{R} = \begin{bmatrix} 2 & 0.3 & 0.2 \\ 0.3 & 4 & 0.1 \\ 0.2 & 0.1 & 6 \end{bmatrix}$$

(a) Determine the eigenfilter coefficients that maximize the SNR at the output of the filter and sketch it.

(b) Plot the frequency response of this filter

(c) What is the maximum signal-to-noise ratio in dB if $\sigma^2 = 0.01$?

22. Consider a Wiener filtering problem that is characterized as follows:

The correlation Matrix $\boldsymbol{R} = E\{\boldsymbol{u}(n)\boldsymbol{u}^*(n)\}$ of the tap-input vector $\boldsymbol{u}(n) = \begin{bmatrix} u(n) & u(n-1) \end{bmatrix}^{\mathrm{T}}$ is given by

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

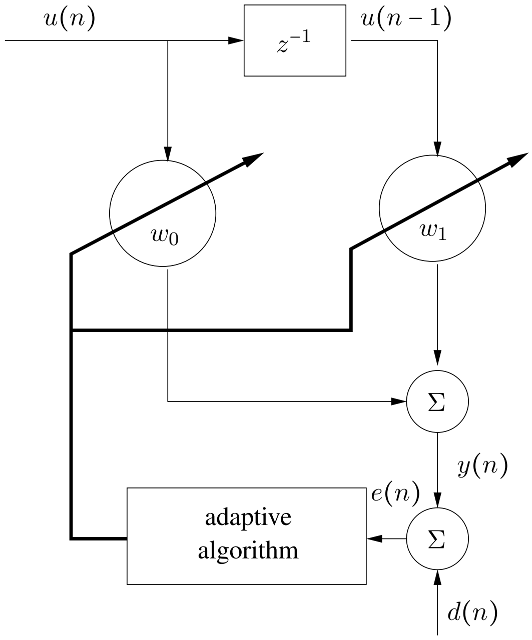
The cross-correlation vector $\boldsymbol{p} = E\{\boldsymbol{u}(n)d^*(n)\}$ between the tap-input vector $\boldsymbol{u}(n)$ and the desired response $d(n)$ is

$$\boldsymbol{p} = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}$$

Assume that the variance of the desired responses is $\sigma_d^2 = 1$.

(a) Evaluate the weight vector $\boldsymbol{w}_{opt} = \begin{bmatrix} w_0 & w_1 \end{bmatrix}^{\mathrm{T}}$ of the Wiener filter.

(b) What is the minimum mean-squared error (MMSE) J_{min} produced by this Wiener filter?



23. The following cost function $J(n)$ shall be minimized:

$$J(n) = \sigma_d^2 - \boldsymbol{w}^{\mathrm{H}}(n)\boldsymbol{p} - \boldsymbol{p}^{\mathrm{H}}\boldsymbol{w}(n) + \boldsymbol{w}^{\mathrm{H}}(n)\boldsymbol{R}\boldsymbol{w}(n)$$

with

$$\boldsymbol{p} = \begin{bmatrix} 2.5 \\ 0.5 \end{bmatrix}$$
$$\boldsymbol{R} = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$$

(a) Compute the optimum Wiener solution $\boldsymbol{w}_{opt}$

(b) Compute approximations of the Wiener solution. Start with an initial value for $\boldsymbol{w}(0) = \begin{bmatrix} 3 & 5 \end{bmatrix}^{\mathrm{T}}$ and execute three steps of the steepest descent algorithm, i.e. $\boldsymbol{w}(1), \boldsymbol{w}(2)$ and $\boldsymbol{w}(3)$ using

- the maximum step size $\mu_{\max}$
- the step size $\mu = 0.9\mu_{\max}$
- the trace dependent step size $\mu = \frac{2}{\mathrm{trace}\{R\}}$
- the optimized step size $\mu = \mu_{opt}(n)$

Consider a multi-user Direct Sequence Code Division Multiple Access (DS-CDMA) system. The received signal $\boldsymbol{y}(n)$ at time slot $n = 1, 2, \ldots, T$ can be represented by (10 pt)

$$\boldsymbol{y}(n) = \boldsymbol{h} \cdot x(n) + \boldsymbol{z}(n) + \boldsymbol{\nu}(n),$$

where $\boldsymbol{h} \in \mathbb{C}^M$ is the signature of the desired user, $x(n) \in \mathbb{C}$ is the desired data symbol, $\boldsymbol{z}(n) \in \mathbb{C}^M$ and $\boldsymbol{\nu}(n) \in \mathbb{C}^M$ represent multi-user interference and additive White Gaussian noise. respectively. Furthermore, $\boldsymbol{z}(n)$ can follow any distribution and $\boldsymbol{\nu}(n)$ is assumed to contain zero mean circularly symmetric complex Gaussian random variables with covariance matrix $\boldsymbol{R}_{\nu\nu} = \mathbb{E}\left\{\boldsymbol{\nu}(n)\boldsymbol{\nu}^{\mathrm{H}}(n)\right\} = \sigma_\nu^2 \cdot \boldsymbol{I}_M$. It is also assumed that $x(n), \boldsymbol{z}(n)$ and $\boldsymbol{\nu}(n)$ are uncorrelated.

The task is to design a receive filter which estimates the unknown symbols $x(n)$ in the presence of interference and noise. The estimated symbol $\hat{\boldsymbol{x}}(n)$ is expressed as

$$\hat{\boldsymbol{x}}(n) = \boldsymbol{w}^{\mathrm{H}}\boldsymbol{y}(n),$$

where $\boldsymbol{w} = [w_0, w_1, \ldots, w_M]^{\mathrm{T}}$ consists of filter coefficients.

The goal is to design $\boldsymbol{w}$ by solving the following constrained optimization problem

$$\begin{aligned}&\text{minimize } J(\boldsymbol{w}) = \text{minimize } \mathbb{E}\left\{|\hat{x}(n)|^2\right\}\\&\text{subject to } \boldsymbol{w}^{\mathrm{H}}\boldsymbol{h} = \gamma,\end{aligned}$$

where γ is a constant. A filter based on the above design is called constrained minimum variance (CMV) filter.

24. (a) Express the receive covariance matrix $\boldsymbol{R}_{\mathrm{yy}} = \mathbb{E}\left\{\boldsymbol{y}(n)\boldsymbol{y}^{\mathrm{H}}(n)\right\}$ in terms of the desired signature $\boldsymbol{h}$, the noise variance σ_ν^2, the covariance matrix of the interference $\boldsymbol{R}_{zz} = \mathbb{E}\left\{\boldsymbol{z}(n)\boldsymbol{z}^{\mathrm{H}}(n)\right\}$, and the transmit power $\sigma_x^2 = \mathbb{E}\left\{|x(n)|^2\right\}$.

    (b) Specify the Lagrangian $L(\boldsymbol{w}, \lambda)$ for this constrained optimization problem in terms of $\boldsymbol{R}_{yy}$ an γ.

    (c) Compute the CMV filter $\boldsymbol{w}_{\mathrm{opt}}$.

    (d) The output Signal-to-Interference and-Noise-Ratio (SINR) can be calculated by

$$\mathrm{SINR} = \frac{\mathbb{E}\left\{|\boldsymbol{w}^{\mathrm{H}}(\boldsymbol{h} \cdot x(n))|^2\right\}}{\mathbb{E}\left\{|\boldsymbol{w}^{\mathrm{H}}(\boldsymbol{z}(n) + \boldsymbol{\nu}(n))|^2\right\}}.$$

    What would be the SINR expression when $\boldsymbol{w} = \boldsymbol{w}_{\mathrm{opt}}$? Please specify the SINR only as a function of $\boldsymbol{h}$, σ_x^2 and $\boldsymbol{R}_{\mathrm{yy}}$.

(11 pt) 25. We consider an M-tap linear FIR filter with weight vector $\boldsymbol{w} \in \mathbb{C}^{M\times 1}$ that operates on a received signal $\boldsymbol{x}[n] \in \mathbb{C}^{M\times 1} = \begin{bmatrix} x[n], & x[n-1], & \ldots, & x[n-M+1] \end{bmatrix}^{\mathrm{T}}$ and produces a scalar output given by

$$y[n] = \boldsymbol{w}^{\mathrm{H}}\boldsymbol{x}[n] = \sum_{m=1}^{M} w_m^* \cdot x[n-m+1]$$

The following quantities for such a filter are given:

$$\boldsymbol{R} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad \boldsymbol{p} = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix},$$

where $\boldsymbol{R} = \mathbb{E}\left\{\boldsymbol{x}[n] \cdot \boldsymbol{x}^{\mathrm{H}}[n]\right\}$ is the autocovariance matrix of the zero mean random process $\boldsymbol{x}[n]$ and $\boldsymbol{p} = \mathbb{E}\left\{\boldsymbol{x}[n] \cdot d[n]^*\right\}$ is the cross-correlation vector between $\boldsymbol{x}[n]$ and the desired signal $d[n]$. We compute the filter weights $\boldsymbol{w}$ iteratively by applying the steepest descent method.

(a) What is the maximum step size $\mu_{\max}$ that still ensures the convergence?

(b) What are the filter weights after two iterations for a fixed step size $\mu = 0.25$? You may start with an initial weight vector $\boldsymbol{w}_0 = \boldsymbol{0}_{M\times 1}$, i.e., the zero vector.

Now we consider a time varying cost function given as

$$J[n] = |e[n]|^2 + \alpha\|\boldsymbol{w}[n]\|^2, \tag{33}$$

where $\alpha \geq 0$ is a constant user selectable parameter and $e[n]$ is the estimation error given as

$$e[n] = d[n] - \boldsymbol{w}^{\mathrm{H}}[n]\boldsymbol{x}[n] \tag{34}$$

We apply the iterative algorithm least mean squares (LMS) to compute the filter weights $\boldsymbol{w}$. The goal is to minimize the cost function $J[n]$ with respect to the weight vector $\boldsymbol{w}[n]$. Such an algorithm is known as leaky LMS algorithm.

(c) Show that the time update for the tap-weight vector $\boldsymbol{w}[n]$ using $\frac{\partial J[n]}{\partial \boldsymbol{w}^*[n]}$ is calculated by

$$\boldsymbol{w}[n+1] = (1 - \mu\alpha)\boldsymbol{w}[n] + \mu\boldsymbol{x}[n]e^*[n]$$

Now we consider an adaptive LMS filter in which the step size is adaptively controlled as opposed to the conventional LMS filter where a fixed step-size is used. The new time varying cost function for such a filter is given as

$$J[n] = |e[n]|^2,$$

where $e[n]$ is the estimation error as defined in equation (34). The gradient vector is defined as

$$\gamma[n] = \frac{\partial \boldsymbol{w}[n]}{\partial \mu[n]}, \tag{35}$$

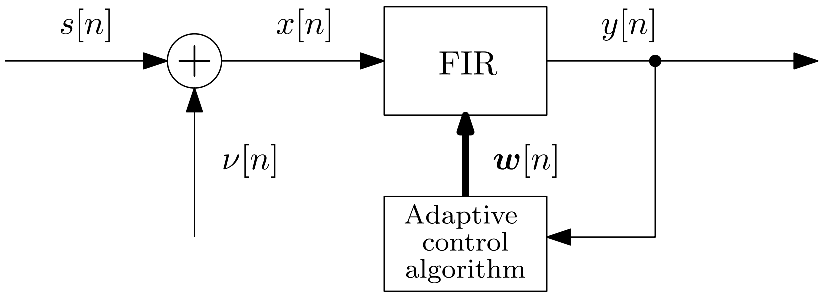
where $\boldsymbol{w}[n]$ is the weight vector estimated by an LMS filter with time-varying step-size parameter $\mu[n]$.

(d) Show that

$$\gamma[n] = \Big(\boldsymbol{I}_M - \mu[n]\boldsymbol{x}[n-1]\boldsymbol{x}^{\mathrm{H}}[n-1]\Big)\gamma[n-1] + \boldsymbol{x}[n-1]e^*[n-1],$$

where $\boldsymbol{I}_M$ is the identity matrix of size $M \times M$.

26. We consider a temporal adaptive system depicted in the figure. An adaptive FIR filter with a complex valued weight vector $\boldsymbol{w} \in \mathbb{C}^M$ operates on a zero-mean input signal $\boldsymbol{s}[n] \in \mathbb{C}^M$ with correlation matrix $\boldsymbol{R}_{ss} = \mathbb{E}\left\{\boldsymbol{s}[n]\boldsymbol{s}^{\mathrm{H}}[n]\right\}$. The output signal at the time snapshot $n = 1, 2 \ldots, T$ is given as $y[n] = \boldsymbol{w}^{\mathrm{H}}\boldsymbol{x}[n]$, where $\boldsymbol{x}[n] = \boldsymbol{s}[n] + \boldsymbol{\nu}[n]$, where $\boldsymbol{\nu}[n] \in \mathbb{C}^M$ is a zero-mean noise vector with temporal correlation matrix $\boldsymbol{R}_{\nu\nu} = \mathbb{E}\left\{\boldsymbol{\nu}[n]\boldsymbol{\nu}^{\mathrm{H}}[n]\right\} = \sigma^2\boldsymbol{I}_M$. The noise and the input signal are assumed to be statistically independent. Therefore, $\mathbb{E}\left\{\nu[n]s^*[m]\right\} = 0 \ \forall m, n$. (10 pt)



(a) Find the output power of the signal $y[n]$ and provide the expression for the signal to noise ratio at the output of the filter as a function of the filter coefficients $\boldsymbol{w}$.

Let $M = 3$ such that $\boldsymbol{w} = [w_0, w_1, w_2]^{\mathrm{T}}$. Now we add two additional constraints on $\boldsymbol{w}$:

$$w_0 + w_1 = 0$$

$$w_1 - w_2 = 1$$

(b) Write the constraints in the form $c(\boldsymbol{w}) = \boldsymbol{C}^{\mathrm{H}}\boldsymbol{w} - \boldsymbol{g} = \boldsymbol{0}$, i.e., find $\boldsymbol{C}$ and $\boldsymbol{g}$.

(c) What is the rank of the matrix $\boldsymbol{C}$?

(d) Find an orthonormal basis for the column space and the left null space of $\boldsymbol{C}$. Please provide your solution explicitly.

Now we consider a Generalized Sidelobe Canceler, where the vector of filter coefficients $\boldsymbol{w}$ is decomposed into $\boldsymbol{w} = \boldsymbol{w}_q + \boldsymbol{C}_a\boldsymbol{w}_a$, such that $\boldsymbol{C}^{\mathrm{H}}\boldsymbol{C}_a$ is a matrix of zeros.

(e) Compute a vector $\boldsymbol{w}_q$ such that $\boldsymbol{C}^{\mathrm{H}}\boldsymbol{w} = \boldsymbol{g}$. Is there a unique solution $\boldsymbol{w}_q$? If this is not the case, provide all possible solutions. Please provide your solution explicitly.

(f) How is $\boldsymbol{C}_a$ related to the four fundamental subspaces of $\boldsymbol{C}$?

(g) Determine the matrix $\boldsymbol{C}_a$.

27. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two positive-definite matrices of size $M \times M$ which are related by (10 pt)

$$\boldsymbol{A} = \boldsymbol{B}^{-1} + \boldsymbol{C}\boldsymbol{D}^{-1}\boldsymbol{C}^{\mathrm{H}}, \tag{36}$$

where $\boldsymbol{D}$ is a positive-definite matrix of size $N \times N$ and $\boldsymbol{C}$ is an $M \times N$ matrix. The inverse of the matrix $\boldsymbol{A}$ may be expressed by matrix inversion lemma

$$\boldsymbol{A}^{-1} = \boldsymbol{B} - \boldsymbol{B}\boldsymbol{C}(\boldsymbol{D} + \boldsymbol{C}^{\mathrm{H}}\boldsymbol{B}\boldsymbol{C})^{-1}\boldsymbol{C}^{\mathrm{H}}\boldsymbol{B}. \tag{37}$$

(a) Establish the validity of the matrix inverse lemma. Elaborate all intermediate steps.
Hint: Proof that $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_M$. Use the identity $(\boldsymbol{D} + \boldsymbol{C}^{\mathrm{H}}\boldsymbol{B}\boldsymbol{C})(\boldsymbol{D} + \boldsymbol{C}^{\mathrm{H}}\boldsymbol{B}\boldsymbol{C})^{-1} = \boldsymbol{I}$.

Now consider a correlation matrix

$$\boldsymbol{\Phi}(n) = \boldsymbol{u}(n)\boldsymbol{u}^{\mathrm{H}}(n) + \delta\boldsymbol{I}_M, \tag{38}$$

where $\boldsymbol{u}(n)$ is tap-input vector of size M and δ is a small positive constraint.

(b) Use the matrix inverse lemma to evaluate $\boldsymbol{P}(n) = \boldsymbol{\Phi}^{-1}(n)$ in terms of $\boldsymbol{u}(n)$.

Now we consider another correlation matrix of the tap-inputs which is used in the RLS algorithm

$$\boldsymbol{\Phi}(n) = \lambda\boldsymbol{\Phi}(n-1) + \boldsymbol{u}(n)\boldsymbol{u}^{\mathrm{H}}(n), \tag{39}$$

where $\boldsymbol{\Phi}(n-1)$ is the old value of correlation matrix and λ is the exponential forgetting factor. The cross-correlation vector $\boldsymbol{z}(n)$ between the tap inputs and the desired response $d(n)$ is

$$\boldsymbol{z}(n) = \lambda\boldsymbol{z}(n-1) + \boldsymbol{u}(n)d^*(n).$$

(c) Use the matrix inverse lemma to evaluate $\boldsymbol{P}(n) = \boldsymbol{\Phi}^{-1}(n)$. Elaborate all intermediate steps.

We define a new vector $\boldsymbol{k}(n)$ of size $M \times 1$ which is also referred to as a gain vector as

$$\boldsymbol{k}(n) = \frac{\lambda^{-1}\boldsymbol{P}(n-1)\boldsymbol{u}(n)}{1 + \lambda^{-1}\boldsymbol{u}^{\mathrm{H}}(n)\boldsymbol{P}(n-1)\boldsymbol{u}(n)}.$$

(d) Show that $\boldsymbol{k}(n) = \boldsymbol{P}(n)\boldsymbol{u}(n)$

(e) What will be the estimate of the tap-weight vector $\boldsymbol{w}(n)$ at iteration n according to the Wiener-Hof equation?

(7 pt) 28. We consider an M-tap linear FIR filter with weight vector $\boldsymbol{w} \in \mathbb{C}^{M\times 1}$ that operates on a received signal $\boldsymbol{x}[n] \in \mathbb{C}^{M\times 1} = [x[n], x[n-1], \ldots, x[n-M+1]]^{\mathrm{T}}$ and produces a scalar output $y[n] = \boldsymbol{w}^{\mathrm{H}} \cdot \boldsymbol{x}[n] = \sum_{m=1}^{M} w_m^* \cdot x[n-m+1]$. The goal is to design the filter such that $y[n]$ follows a desired signal $d[n]$, i.e., the squared error signal $|e[n]|^2 = |y[n] - d[n]|^2$ is minimized.

Moreover, the following quantities are given:

$$\boldsymbol{R} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad \boldsymbol{R}^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \quad \boldsymbol{p} = \begin{bmatrix} 4 \\ -4 \\ 8 \end{bmatrix} \tag{40}$$

where $\boldsymbol{R} = \mathbb{E}\left\{\boldsymbol{x}[n] \cdot \boldsymbol{x}[n]^{\mathrm{H}}\right\}$ is the autocovariance matrix of the zero mean random process $\boldsymbol{x}[n]$ and $\boldsymbol{p} = \mathbb{E}\left\{\boldsymbol{x}[n] \cdot d[n]^*\right\}$ is the cross-correlation vector between $\boldsymbol{x}[n]$ and the desired signal $d[n]$.

The eigenvalues of $\boldsymbol{R}$ are given by $\lambda_1 = 4$, $\lambda_2 = 1$, and $\lambda_3 = 1$. The variance of the desired signal $d[n]$ is given by $\mathbb{E}\left\{|d(n)|^2\right\} = 100$.

(a) Compute the filter weight vector $\boldsymbol{w}_{\mathrm{opt}}$ which minimizes the mean square error $J(\boldsymbol{w}) = \mathbb{E}\left\{|e(n)|^2\right\}$.

(b) Determine the resulting mean square error $J_{\min} = J(\boldsymbol{w}_{\mathrm{opt}})$.

Alternatively, we can compute $\boldsymbol{w}$ iteratively. We examine the method of steepest descent where we take small steps in the direction of the negative gradient, starting from an initial weight vector $\boldsymbol{w}_0$.

(c) Provide an explicit expression for the gradient of the cost function $\boldsymbol{\gamma}(\boldsymbol{w}) = \frac{\partial J(\boldsymbol{w})}{\partial \boldsymbol{w}^*}$.

(d) For the given covariance matrix, what is the maximum step size $\mu_{\max}$ that still allows the algorithm to converge?

(e) Starting with the initial weight vector $\boldsymbol{w}_0 = \mathbf{0}_{M\times 1}$, i.e., the zero vector, perform two steps of the method of steepest descent using the step size $\mu = 0.25$, i.e.,

i. Compute the gradient $\boldsymbol{\gamma}(\boldsymbol{w}_0)$.

ii. Perform one step of steepest descent to compute $\boldsymbol{w}_1$.

iii. Compute the new gradient using $\boldsymbol{\gamma}(\boldsymbol{w}_1)$.

iv. Perform a second step of steepest descent to compute $\boldsymbol{w}_2$.

29. Consider a device with M antennas which receives a signal $\boldsymbol{y}[n]$ at time snapshot $n = 1, 2, \ldots, T$ and assume that $\boldsymbol{y}[n]$ obeys the following model (8+2 pt)

$$\boldsymbol{y}[n] = \boldsymbol{h} \cdot s[n] + \boldsymbol{\nu}[n]$$

where $s[n] \in \mathbb{C}$ is the desired information symbol and $\boldsymbol{\nu}[n] \in \mathbb{C}^M$ represents an additive noise process, which are assumed to be zero mean circularly symmetric complex Gaussian random variables with covariance matrix $\boldsymbol{R}_{\nu\nu} = \mathbb{E}\{\boldsymbol{\nu}[n] \cdot \boldsymbol{\nu}^{\mathrm{H}}[n]\} = \sigma_\nu^2 \cdot \boldsymbol{I}_M$. Moreover, $s[n]$ and $\boldsymbol{\nu}[n]$ are uncorrelated.

Moreover, $\boldsymbol{h} \in \mathbb{C}^M$ is a vector of complex coefficients. For instance, such a model is found when information symbols $s[n]$ are transmitted over a SIMO channel where $\boldsymbol{h}$ contains the channel coefficients. Our task is to design a linear receive filter which recovers the unknown symbols $s[n]$. The linear filtering operation can be expressed as

$$\hat{s}[n] = \sum_{m=1}^{M} w_m^* \cdot y_m[n] = \boldsymbol{w}^{\mathrm{H}} \cdot \boldsymbol{y}[n],$$

where w_m are the filter coefficients of the receive filter and $\hat{s}[n]$ is the estimate of our symbol $s[n]$.

Our goal is to design $\boldsymbol{w}$ such that the mean square error between $s[n]$ and $\hat{s}[n]$

$$\mathrm{MSE}(\boldsymbol{w}) = \mathbb{E}\{|\hat{s}[n] - s[n]|^2\} = \mathbb{E}\{|\boldsymbol{w}^{\mathrm{H}} \cdot \boldsymbol{y}[n] - s[n]|^2\}$$

is minimized.

(a) Express the receive covariance matrix $\boldsymbol{R}_{yy} = \mathbb{E}\{\boldsymbol{y}[n] \cdot \boldsymbol{y}^{\mathrm{H}}[n]\}$ as a function of the channel vector $\boldsymbol{h}$, the noise variance σ_ν^2 and the transmit power $\sigma_{\mathrm{s}}^2 = \mathbb{E}\{|s[n]|^2\}$.

(b) Show that the cost function $\mathrm{MSE}(\boldsymbol{w})$ can be expressed as $\mathrm{MSE}(\boldsymbol{w}) = \boldsymbol{w}^{\mathrm{H}}\boldsymbol{R}_{yy}\boldsymbol{w} - \boldsymbol{w}^{\mathrm{H}} \cdot \boldsymbol{z} - \boldsymbol{z}^{\mathrm{H}} \cdot \boldsymbol{w} + c$. Express $\boldsymbol{z}$ and c in terms of the given quantities.

(c) Is $\mathrm{MSE}(\boldsymbol{w})$ an analytic (holomorphic) function? Explain why / why not.

(d) Find the vector $\boldsymbol{w}_{\mathrm{opt}}$ which minimizes the cost function, i.e., $\boldsymbol{w}_{\mathrm{opt}} = \arg\min_{\boldsymbol{w}} \mathrm{MSE}(\boldsymbol{w})$.

(e) Simplify $\boldsymbol{w}_{\mathrm{opt}}$ into a form that does not involve any matrix inversion any more. Use the following rule

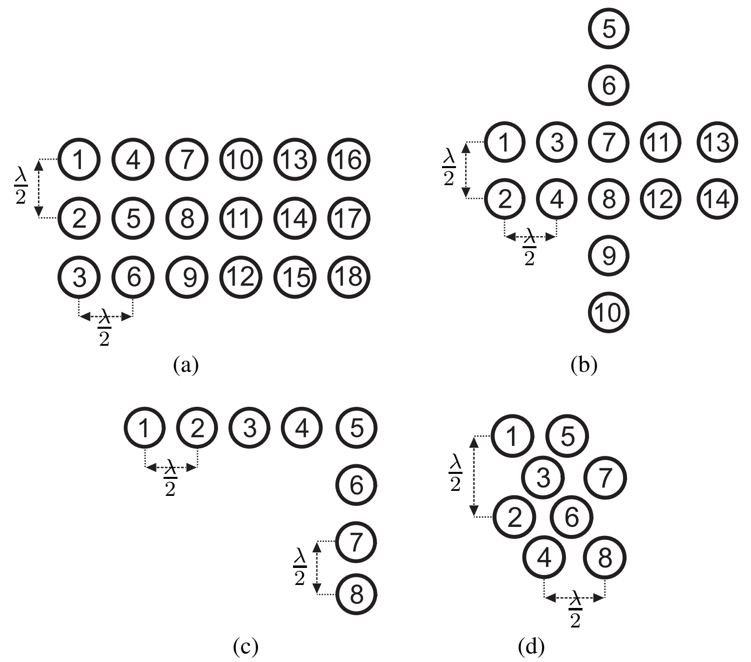
$$\left(\boldsymbol{I}_M + \boldsymbol{u} \cdot \boldsymbol{u}^{\mathrm{H}}\right)^{-1} = \boldsymbol{I}_M - \frac{\boldsymbol{u} \cdot \boldsymbol{u}^{\mathrm{H}}}{1 + \boldsymbol{u}^{\mathrm{H}} \cdot \boldsymbol{u}}, \quad \boldsymbol{u} \in \mathbb{C}^M.$$

and the short-hand notation $\rho = \sigma_s^2/\sigma_\nu^2$.

(f) (**Bonus +2p**) Show that the optimal receive filter $\boldsymbol{w}_{\mathrm{opt}}$ is a scaled version of $\boldsymbol{h}$, i.e., $\boldsymbol{w}_{\mathrm{opt}} = c(\boldsymbol{h}, \sigma_s^2, \sigma_\nu^2) \cdot \boldsymbol{h}$ and find $\boldsymbol{w}_{\mathrm{opt}}$ in the limiting case $\sigma_v^2 \to 0$ (high SNR). Interpret your result.

30. Consider the following four 2-D arrays: (10 pt)

Each of these arrays has a double shift-invariance structure, i.e., it is shift-invariant in horizontal (x) and in vertical (y) direction.



(a) (b)

(c) (d)

(a) For each array find the largest possible subarrays (i.e., with the most number of sensors M_{sub} in x- and y-direction). Document your findings in a table like this

Array	Subarray 1, x	Subarray 2, x	Subarray 1, y	Subarray 2, y
(a)	...	...	...	...
(b)	...	...	...	...
(c)	...	...	...	...
(d)	...	...	...	...

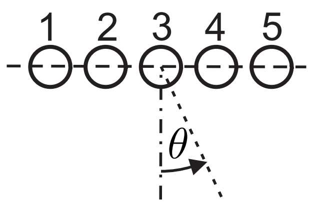
by filling in the indices of the sensors that belong to the respective subarrays. Note that the displacement between the two subarrays must not exceed $\lambda/2$.

(b) For each array find the largest possible number of wavefronts $d_{\max}$ that can be resolved via 2-D ESPRIT if all wavefronts are non-coherent and a sufficient number of snapshots is available ($N > M$).

(c) For array (c) provide the selection matrices $\boldsymbol{J}_{\mu,1}, \boldsymbol{J}_{\mu,2}, \boldsymbol{J}_{\nu,1}, \boldsymbol{J}_{\nu,2}$ that are needed for the 2-D shift invariance equations explicitly. Here, μ corresponds to the horizontal and ν corresponds to the vertical direction.

(d) On which of the arrays can 2-D Unitary ESPRIT be applied?

(13 pt) 31. We consider a uniform linear array (ULA) with $M = 5$ elements and $\lambda/2$ inter-element spacing.



Its array steering vector $\boldsymbol{a}(\theta) = [a_1(\theta), a_2(\theta), a_3(\theta), a_4(\theta), a_5(\theta)]^{\mathrm{T}}$ satisfies $a_3(\theta) = 1\,\forall\theta$, i.e., the phase reference of the array is chosen in the middle.

(a) Provide expressions for $a_1(\theta), a_2(\theta), a_4(\theta)$, and $a_5(\theta)$, where θ is the azimuth angle.

Now assume that d wavefronts are impinging from distinct directions $\theta_1, \theta_2, \ldots, \theta_d$.

(b) Show that the array steering matrix $\boldsymbol{A} \in \mathbb{C}^{M\times d}$ for d impinging wavefronts is left-$\boldsymbol{\Pi}$-real.

(c) We would like to estimate the directions of arrival θ_i via ESPRIT. If all wavefronts are non-coherent and enough snapshots are available ($N > M$), what is the maximum number of wavefronts $d_{\max}$ that can be resolved with this array?

(d) If all wavefronts are coherent, what is the maximum number of wavefronts $d_{\max}$ that can be resolved via ESPRIT by applying

i. Forward-backward averaging

ii. Spatial smoothing

iii. Both forward-backward averaging and spatial smoothing

as preprocessing steps?

Now we consider the special case $d = 2$ and ignore the contribution of the noise for clarity. Consequently, the covariance matrix of the received signal $\boldsymbol{x}[n]$ is given by $\boldsymbol{R}_{\text{xx}} = \mathbb{E}\left\{\boldsymbol{x}[n] \cdot \boldsymbol{x}[n]^{\text{H}}\right\} = \boldsymbol{A} \cdot \boldsymbol{R}_{\text{ss}} \cdot \boldsymbol{A}^{\text{H}}$, where $\boldsymbol{A} \in \mathbb{C}^{M \times 2}$ and the source covariance matrix $\boldsymbol{R}_{\text{ss}} \in \mathbb{C}^{2 \times 2}$ can be expressed as

$$\boldsymbol{R}_{\text{ss}} = \mathbb{E}\left\{\boldsymbol{s}[n] \cdot \boldsymbol{s}[n]^{\text{H}}\right\} = \begin{bmatrix} 1 & \rho \\ \rho^* & 1 \end{bmatrix}. \tag{41}$$

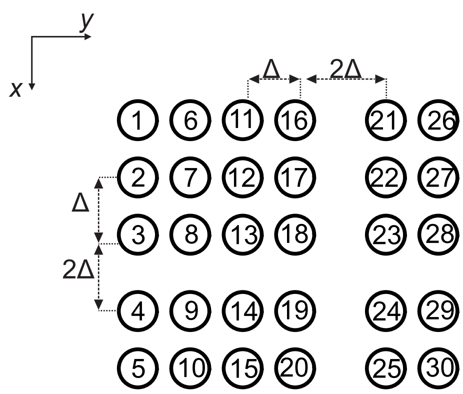
Here $\rho = |\rho| e^{\jmath \varphi_\rho} \in \mathbb{C}$ denotes the correlation coefficient.

(e) Show that $\operatorname{rank}\{\boldsymbol{R}_{\text{ss}}\} = 1$ if $|\rho| = 1$.

(f) Prove that after applying forward-backward averaging, the corresponding covariance matrix $\boldsymbol{R}_{\text{xx}}^{(\text{fba})}$ can be written as $\boldsymbol{R}_{\text{xx}}^{(\text{fba})} = \boldsymbol{A} \cdot \tilde{\boldsymbol{R}}_{\text{ss}} \cdot \boldsymbol{A}^{\text{H}}$, where $\tilde{\boldsymbol{R}}_{\text{ss}} = \boldsymbol{R}_{\text{ss}} + \boldsymbol{R}_{\text{ss}}^*$.

(g) Show under which conditions the rank of $\boldsymbol{R}_{\text{xx}}^{(\text{fba})}$ is 2.

32. We would like to apply 2-D ESPRIT on the following array of omnidirectional sensor elements (13 pt)



(a) For $\Delta = \lambda/2$, find the sensor elements belonging to the largest possible subarrays in the vertical (x) and the horizontal (y) direction, respectively. Note that the displacement between the corresponding elements of the first and the second subarray must be less than or equal to $\lambda/2$.

(b) What is the maximum number of incoherent wavefronts $d_{\max}$ which can be resolved if enough snapshots N are available?

(c) Can Forward-Backward Averaging be applied to the entire matrix of observations from this array? Please explain your reason.

Now, consider a uniform rectangular array of $M_x \times M_y$ sensors. There are $d = 3$ incoherent impinging planar wavefronts and we would like to apply 2-D Unitary ESPRIT to estimate the spatial frequencies μ_i and ν_i, $i = 1, 2, 3$. To this end, we need to solve the two shift invariance equations in the x- and y- directions to obtain $\boldsymbol{\Gamma}_\mu$ and $\boldsymbol{\Gamma}_\nu$.

(d) What are the two shift invariance equations? Define all the matrices that you use and specify their dimensions.

(e) Describe how to obtain correctly paired parameter estimates μ_i and ν_i, $i = 1, 2, 3$ via one eigenvalue decomposition (EVD) based on $\boldsymbol{\Gamma}_\mu$ and $\boldsymbol{\Gamma}_\nu$.

(f) As a result of this eigenvalue decomposition, we have obtained the three eigenvalues $w_1 = 1-j$, $w_2 = 1+j$, and $w_3 = 0 + j$. Find the spatial frequencies μ_i and ν_i, $i = 1, 2, 3$.

If the sources are coherent, we need to apply proper preprocessing to decorrelate them. To this end, we apply 2-D spatial smoothing by dividing the array into $L_x \times L_y$ subarrays (L_x in the x-direction and L_y in the y-direction) where $1 \le L_x < M_x$ and $1 \le L_y < M_y$. In total, this decorrelates $2 \cdot L = 2 \cdot L_x \cdot L_y$ sources.

(g) What is the size of the virtual array after 2-D spatial smoothing has been applied (as a function of M_x, L_x, M_y, and L_y)?

(h) How many coherent sources can be estimated via 2-D Unitary ESPRIT for a given choice of L_x and L_y?

(i) Is it possible to detect $d_{\max} = 6$ coherent sources using a uniform rectangular array with $M_x = 5$ and $M_y = 3$ and by choosing the best possible L_x and L_y? If yes, what are the corresponding values of L_x and L_y? If no, please explain your reason.

(10 pt) 33. Consider a uniform rectangular array of $M = M_x \times M_y$ sensors. Without loss of generality, we assume $M_x \le M_y$ (for $M_x > M_y$ we can simply flip the dimensions).

(a) You are given a total of $M = 64$ sensors. Find all arrangements of the M sensors in form of an $M_x \times M_y$ URA where $1 < M_x \le M_y$.

(b) For each arrangement find the maximum number of incoherent wavefronts $d_{\max}$ we can estimate for 2-D Unitary ESPRIT. What is the minimum number of snapshots that are required to achieve these results?

If sources are coherent, we need to apply proper preprocessing to decorrelate them. Since the array is centro-symmetric we use forward-backward averaging. Additionally, we apply 2-D spatial smoothing by dividing the array into $L_x \times L_y$ subarrays (L_x in x-direction and L_y in y-direction) where $1 \le L_x < M_x$ and $1 \le L_y < M_y$. In total, this decorrelates $2 \cdot L = 2 \cdot L_x \cdot L_y$ sources.

(c) What is the size of the virtual array after 2-D spatial smoothing has been applied (as a function of M_x, L_x, M_y, and L_y)?

(d) How many coherent sources can be estimated via 2-D Unitary ESPRIT for a given choice of L_x and L_y if only one snapshot ($N = 1$) is available?

(e) Find the maximum possible number of coherent sources (i.e., choose the best possible L_x and L_y) for $M_x = 5$, $M_y = 3$, and $N = 1$. What are the corresponding values of L_x and L_y? Hint: Try out all combinations of L_x and L_y for $1 \le L_x < M_x$ and $1 \le L_y < M_y$.

(f) Consider a URA with $M = M_x \times M_y$ elements where M is a square number, i.e., $M = m^2$ for an integer m. Show that in this case, for incoherent wavefronts the best distribution of M sensors in form of a URA which maximizes $d_{\max}$ is given by setting $M_x = \sqrt{M} = m$.

34. Consider an omni-directional uniform linear array (ULA) of $M = 3$ sensor elements with half wavelength inter-element spacing $\Delta = \frac{\lambda}{2}$. Assume that the covariance matrix of the noise corrupted measurements is known. It is defined as

$$\boldsymbol{R}_{xx} = \mathrm{E}\left\{\boldsymbol{x}(t)\boldsymbol{x}^{\mathrm{H}}(t)\right\} = \boldsymbol{A}\boldsymbol{R}_{ss}\boldsymbol{A}^{\mathrm{H}} + \sigma_N^2 \boldsymbol{I}_3 \in \mathbb{C}^{M\times M}$$

and given by

$$\boldsymbol{R}_{xx} = \frac{1}{2}\begin{bmatrix} 4 & -1+j & 0 \\ -1-j & 4 & -1+j \\ 0 & -1-j & 4 \end{bmatrix}.$$

Here, $\boldsymbol{R}_{ss} \in \mathbb{C}^{d\times d}$ denotes the unknown signal covariance matrix, $\boldsymbol{A} \in \mathbb{C}^{M\times d}$ is the array steering matrix, and the variance of the additive noise is given by $\sigma_n^2 = 1$. We are interested in finding the desired directions of arrival of the impinging wavefronts using 1-D Unitary ESPRIT.

(a) Specify the relationship between the forward-backward averaged covariance matrix $\boldsymbol{R}_{xx}^{\mathrm{fb}}$ and the conventional covariance matrix $\boldsymbol{R}_{xx}$.

(b) Provide an explicit expression for the left $\boldsymbol{\Pi}$-real transformation matrix $\boldsymbol{Q}_3^{(\mathrm{s})}$.

(c) Calculate the covariance matrix $\boldsymbol{R}_{xx}^{\mathrm{fb}}$ explicitly.

(d) Show that the forward-backward averaged covariance matrix $\boldsymbol{R}_{xx}^{\text{fb}}$ is centro-Hermitian.

(e) Show how the transformed real-valued matrix $\varphi(\boldsymbol{R}_{xx}^{\text{fb}})$ is calculated from the covariance matrix $\boldsymbol{R}_{xx}$.

(f) Calculate the transformed real-valued matrix $\varphi(\boldsymbol{R}_{xx}^{\text{fb}})$.
Hint: $\boldsymbol{Q}_M^{(\text{s})^{-1}} = \boldsymbol{Q}_M^{(\text{s})^{\text{H}}}$

Next, assume that the eigenvalue decomposition (EVD) of $\varphi(\boldsymbol{R}_{xx}^{\text{fb}})$ is given by

$$\varphi(\boldsymbol{R}_{xx}^{\text{fb}}) = \frac{1}{2}\begin{bmatrix} -1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & \sqrt{2} & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} -1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & \sqrt{2} & -1 \end{bmatrix}.$$

(g) What is the number of the impinging wavefronts d? Please explain your answer.

(h) Find a basis for the real-valued (transformed) signal subspace which is given by $\text{im}\left\{\boldsymbol{Q}_M^{(\text{s})^{\text{H}}}\boldsymbol{A}\right\} = \text{im}\left\{\boldsymbol{E}_\text{s}\right\}$.

Next, assume that the subarrays are defined as depicted in Fig. 1.

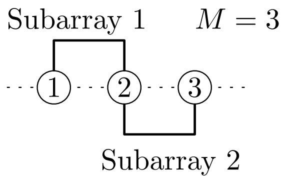


Figure 1: Definition of the subarrays for a ULA of size $M = 3$.

(i) Provide the selection matrices, $\boldsymbol{J}_1, \boldsymbol{J}_2$ according to the sub-array choice depicted in Fig. 1.

(j) Provide the matrix expressions of how to find the transformed selection matrices $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$.
Hint: You do not need to calculate these matrices explicitly.

Next, the following transformed selection matrices, $\boldsymbol{K}_1, \boldsymbol{K}_2$ are given

$$\boldsymbol{K}_1 = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{K}_2 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & -\sqrt{2} & 0 \end{bmatrix}.$$

(k) Provide an expression for the real-valued shift-invariance equation.

(l) (**Bonus +6p**) Solve the above equation using the least square (LS) solution and calculate the estimated spatial frequencies, μ_i, $i = 1, \ldots, d$.

35. Let a tensor $\boldsymbol{\mathcal{X}}$ be given by (11 pt)

$$\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{I}}_2 \times_1 \boldsymbol{A} \times_2 \boldsymbol{B} \times_3 \boldsymbol{C}$$

where $\boldsymbol{\mathcal{I}}_2$ is the $2 \times 2 \times 2$ identity tensor (with elements $[\boldsymbol{\mathcal{I}}_2]_{(1,1,1)} = [\boldsymbol{\mathcal{I}}_2]_{(2,2,2)} = 1$ and all other elements zero) and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are the loading matrices given by

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

(a) What is the (tensor) rank $r = \text{rank}\{\boldsymbol{\mathcal{X}}\}$?

(b) Demonstrate a) by explicitly computing r rank-one tensors $\boldsymbol{\mathcal{X}}_1 \ldots \boldsymbol{\mathcal{X}}_r$ such that $\boldsymbol{\mathcal{X}} = \sum_{i=1}^{r} \boldsymbol{\mathcal{X}}_r$.

(c) Compute the unfoldings $[\boldsymbol{\mathcal{X}}]_{(1)}$, $[\boldsymbol{\mathcal{X}}]_{(2)}$, $[\boldsymbol{\mathcal{X}}]_{(3)}$ in forward (MATLAB) column ordering.

(d) What are the n-ranks of $\boldsymbol{\mathcal{X}}$ for $n = 1, 2, 3$?

(e) Find a basis for the space spanned by the one-mode vectors of $\boldsymbol{\mathcal{X}}$, i.e., the column space of $[\boldsymbol{\mathcal{X}}]_{(1)}$.

(f) Show that for an arbitrary tensor $\boldsymbol{\mathcal{X}} \in \mathbb{C}^{M_1 \times M_2 \times M_3}$ and matrices $\boldsymbol{A} \in \mathbb{C}^{N_1 \times M_1}$, $\boldsymbol{B} \in \mathbb{C}^{P_1 \times N_1}$, the following identity holds: $\boldsymbol{\mathcal{X}} \times_1 \boldsymbol{A} \times_1 \boldsymbol{B} = \boldsymbol{\mathcal{X}} \times_1 (\boldsymbol{B} \cdot \boldsymbol{A})$. Hint: Use a suitable n-mode unfolding.

36. We are given the one-mode unfolding of a $2 \times 2 \times 2$ tensor $\boldsymbol{\mathcal{X}}$ in forward (MATLAB) column ordering (9 pt)

$$[\boldsymbol{\mathcal{X}}]_{(1)} = \begin{bmatrix} 3 & 3 & 3 & 3 \\ -1 & 3 & -1 & 3 \end{bmatrix}$$

(a) Find the two-mode and the three-mode unfoldings $[\boldsymbol{\mathcal{X}}]_{(2)}$, $[\boldsymbol{\mathcal{X}}]_{(3)}$ in forward (MATLAB) column ordering.

(b) What are the n-ranks of $\boldsymbol{\mathcal{X}}$ for $n = 1, 2, 3$?

Now we compute a new tensor $\boldsymbol{\mathcal{Y}}$ via $\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{X}} \times_1 \boldsymbol{U}$, where the matrix $\boldsymbol{U}$ is given by

$$\boldsymbol{U} = \begin{bmatrix} 2 & -1 \\ -2 & 1 \\ 2 & -1 \end{bmatrix}$$

(c) Compute the one-mode unfolding $[\boldsymbol{\mathcal{Y}}]_{(1)}$ of $\boldsymbol{\mathcal{Y}}$ in forward (MATLAB) column ordering.

(d) Using the previous result, find the two-mode and the three-mode unfoldings $[\boldsymbol{\mathcal{Y}}]_{(2)}$, $[\boldsymbol{\mathcal{Y}}]_{(3)}$ in forward (MATLAB) column ordering.

(e) What are the new n-ranks of $\boldsymbol{\mathcal{Y}}$ for $n = 1, 2, 3$?

(9 pt) 37. We are given the one-mode unfolding of a $2 \times 2 \times 2$ tensor $\boldsymbol{\mathcal{X}}$ in forward (MATLAB) column ordering

$$[\boldsymbol{\mathcal{X}}]_{(1)} = \begin{bmatrix} 4 & 4 & -2 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(a) Find the two-mode and the three-mode unfoldings $[\boldsymbol{\mathcal{X}}]_{(2)}$, $[\boldsymbol{\mathcal{X}}]_{(3)}$ in forward (MATLAB) column ordering.

(b) Determine n-ranks of $\boldsymbol{\mathcal{X}}$ for $n = 1, 2, 3$.

(c) Compute the frontal slices of $\boldsymbol{\mathcal{X}}$ (the matrices $[\boldsymbol{\mathcal{X}}]_{(:,:,1)}$ and $[\boldsymbol{\mathcal{X}}]_{(:,:,2)}$) and the lateral slices of $\boldsymbol{\mathcal{X}}$ (the matrices $[\boldsymbol{\mathcal{X}}]_{(:,1,:)}$ and $[\boldsymbol{\mathcal{X}}]_{(:,2,:)}$). Hint: $[\boldsymbol{\mathcal{X}}]_{(:,:,1)} = \begin{bmatrix} x_{(1,1,1)} & x_{(1,2,1)} \\ x_{(2,1,1)} & x_{(2,2,1)} \end{bmatrix}$ and $[\boldsymbol{\mathcal{X}}]_{(:,1,:)} = \begin{bmatrix} x_{(1,1,1)} & x_{(1,1,2)} \\ x_{(2,1,1)} & x_{(2,1,2)} \end{bmatrix}$, where $x_{(i,j,k)}$ denotes the (i, j, k)-th element of $\boldsymbol{\mathcal{X}}$.

It is known that $\boldsymbol{\mathcal{X}}$ can be alternatively given by

$$\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{I}}_2 \times_1 \boldsymbol{A} \times_2 \boldsymbol{B} \times_3 \boldsymbol{C}$$

where $\boldsymbol{\mathcal{I}}_2$ is the $2 \times 2 \times 2$ identity tensor (with elements $[\boldsymbol{\mathcal{I}}_2]_{(1,1,1)} = [\boldsymbol{\mathcal{I}}_2]_{(2,2,2)} = 1$ and all other elements zero). The loading matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ are given by

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(d) Find the loading matrix $\boldsymbol{B}$.

(e) What is the (tensor) rank $r = \operatorname{rank}\{\boldsymbol{\mathcal{X}}\}$?

(f) Demonstrate e) by explicitly computing r rank-one tensors $\boldsymbol{\mathcal{X}}_1 \ldots \boldsymbol{\mathcal{X}}_r$ such that $\boldsymbol{\mathcal{X}} = \sum_{l=1}^{r} \boldsymbol{\mathcal{X}}_l$.

(x pt) 38. We are given the three-mode unfolding of a $2 \times 2 \times 2$ tensor $\boldsymbol{\mathcal{X}}$ in forward (MATLAB) column ordering

$$[\boldsymbol{\mathcal{X}}]_{(3)} = \begin{bmatrix} -2 & 1 & 2 & 1 \\ 4 & 1 & 4 & 1 \end{bmatrix}$$

(a) Find the one-mode and the two-mode unfoldings $[\boldsymbol{\mathcal{X}}]_{(1)}$, $[\boldsymbol{\mathcal{X}}]_{(2)}$ in forward (MATLAB) column ordering.

(b) Determine n-ranks of $\mathcal{X}$ for $n = 1, 2, 3$.

Now let a new tensor $\mathcal{Y}$ be defined via $\begin{bmatrix} 4 \\ 1 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) What is the (tensor-) rank $r = \text{rank}\{\mathcal{Y}\}$?

(d) Compute $\mathcal{Z} = \mathcal{X} - \mathcal{Y}$.

It is known that tensor $\mathcal{Z}$ can be similarly defined as $\mathcal{Y}$, i.e., $\mathcal{Z} = \boldsymbol{a} \circ \boldsymbol{b} \circ \boldsymbol{c}$, where $\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 \end{bmatrix}^{\mathrm{T}}$, $\boldsymbol{b} = \begin{bmatrix} b_1 & b_2 \end{bmatrix}^{\mathrm{T}}$, and $\boldsymbol{c} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}^{\mathrm{T}}$.

(e) What is the rank of $\mathcal{Z}$?

(f) What is the rank of $\mathcal{X}$?

(g) What are the values of a_2 and c_2? What is the relation of the values of b_1 and b_2? Provide one possible batch of solutions of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

(h) Find the three matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, such that $\mathcal{X} = \mathcal{I} \times_1 \boldsymbol{A} \times_2 \boldsymbol{B} \times_3 \boldsymbol{C}$, where $\mathcal{I}$ is the three-way identity tensor of appropriate size.

39. Let a tensor $\mathcal{X}$ be given by (10 pt)

$$\mathcal{X} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \circ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \circ \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \circ \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \circ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \circ \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \circ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \circ \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

(a) What are the dimensions of $\mathcal{X}$?

(b) What is the (tensor-) rank $r = \text{rank}\{\mathcal{X}\}$?

(c) Find three matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ such that $\mathcal{X} = \mathcal{I} \times_1 \boldsymbol{A} \times_2 \boldsymbol{B} \times_3 \boldsymbol{C}$, where $\mathcal{I}$ is a 3-D identity tensor of appropriate size.

(d) Find the one-mode, the two-mode, and the three-mode unfoldings $[\mathcal{X}]_{(1)}$, $[\mathcal{X}]_{(2)}$, and $[\mathcal{X}]_{(3)}$ in forward (MATLAB) column ordering.

(e) Determine n-ranks of $\mathcal{X}$ for $n = 1, 2, 3$.

Given an arbitrary tensor $\mathcal{X} \in \mathbb{C}^{M_1 \times M_2 \times M_3}$ and matrices $\boldsymbol{U}_r \in \mathbb{C}^{N_r \times M_r}$, $\boldsymbol{U}_p \in \mathbb{C}^{N_p \times M_p}$, where $r, p \in \{1, 2, 3\}$.

(f) For which values of r and p does the following identity hold?

$$\mathcal{X} \times_r \boldsymbol{U}_r \times_p \boldsymbol{U}_p = \mathcal{X} \times_p \boldsymbol{U}_p \times_r \boldsymbol{U}_r$$

(g) Taking such a pair of r and $p \in \{1, 2, 3\}$ that fulfills the condition in f), show that the identity above holds.

40. Let a tensor $\mathcal{X}$ be defined via $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \circ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \circ \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ (9 pt)

(a) What is the tensor-rank $r = \text{rank}\{\mathcal{X}\}$?

(b) Find the two-mode unfolding $[\mathcal{X}]_{(2)}$ in forward (MATLAB) column ordering.

We are now given the two-mode unfolding of another $2 \times 2 \times 2$ tensor $\mathcal{W}$ in forward (MATLAB) column ordering

$$[\mathcal{W}]_{(2)} = \begin{bmatrix} 3 & 3 & 3 & 3 \\ -1 & 3 & -1 & 3 \end{bmatrix}.$$

(c) Find the one-mode and the three-mode unfoldings $[\mathcal{W}]_{(1)}$, $[\mathcal{W}]_{(3)}$ in forward (MATLAB) column ordering.

(d) Determine n-ranks of $\mathcal{W}$ for $n = 1, 2, 3$.

Many high-resolution parameter estimation schemes use **Forward-Backward Averaging** as a preprocessing step in order to enhance the estimation accuracy. The forward-backward averaged version of a tensor $\boldsymbol{\mathcal{Y}} \in \mathbb{C}^{M_1 \times M_2 \times M_3}$ is computed via

$$\boldsymbol{\mathcal{Z}} = [\boldsymbol{\mathcal{Y}} \sqcup_3 \boldsymbol{\mathcal{Y}}^* \times_1 \boldsymbol{\Pi}_{M_1} \times_2 \boldsymbol{\Pi}_{M_2} \times_3 \boldsymbol{\Pi}_{M_3}], \tag{42}$$

where $\boldsymbol{\Pi}_p$ is the $p \times p$ exchange matrix having ones on its antidiagonal and zeros elsewhere. Here $[\boldsymbol{\mathcal{A}} \sqcup_n \boldsymbol{\mathcal{B}}]$ denotes the **concatenation** of two tensors $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}} \in \mathbb{C}^{I_1 \times I_2 \ldots \times I_N}$ along the n-mode. The n-mode vectors of the resulting tensor are given by stacking the n-mode vectors of $\boldsymbol{\mathcal{A}}$ on top of the n-mode vectors of $\boldsymbol{\mathcal{B}}$.

(e) Provide the two-mode unfolding of $[\boldsymbol{\mathcal{X}} \sqcup_2 \boldsymbol{\mathcal{W}}]$ based on the definition of the concatenation operation given above.

(f) What are the dimensions of $\boldsymbol{\mathcal{Z}}$ as the forward-backward averaged version of $\boldsymbol{\mathcal{Y}}$?

A tensor $\boldsymbol{\mathcal{C}} \in \mathbb{C}^{I_1 \times I_2 \cdots \times I_N}$ is called **centro-Hermitian** if it satisfies

$$\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}^* \times_1 \boldsymbol{\Pi}_{I_1} \times_2 \boldsymbol{\Pi}_{I_2} \cdots \times_N \boldsymbol{\Pi}_{I_N}.$$

(g) Prove that $\boldsymbol{\mathcal{Z}}$ as defined in (42) is a centro-Hermitian tensor.
Hint: For two tensors $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}} \in \mathbb{C}^{I_1 \times I_2 \ldots \times I_N}$, and a matrix $\boldsymbol{U}_r \in \mathbb{C}^{I_r \times P_r}$, the following identity holds:

$$[\boldsymbol{\mathcal{A}} \sqcup_n \boldsymbol{\mathcal{B}}] \times_r \boldsymbol{U}_r = [\boldsymbol{\mathcal{A}} \times_r \boldsymbol{U}_r \sqcup_n \boldsymbol{\mathcal{B}} \times_r \boldsymbol{U}_r] \quad \text{where } n \neq r.$$

Moreover, think about the three-mode unfolding of $\boldsymbol{\mathcal{C}}$.