Ordinary Differential Equations With Applications

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Contents

*Sections marked with an asterisk may easily be omitted for a shorter course.

3 Applications of First-Order Equations, 55

4 Linear Equations of Higher Order, 82

5 Applications Involving Initial Value Problems, 126

Preface

This book is intended as a one-semester, one-quarter, or two-quarter introductory course in ordinary differential equations and their applications. It is designed for students in mathematics, engineering, or science who have already successfully completed a basic course in calculus.

Applications in the physical sciences as well as some in the social and life sciences are prominent throughout the text, but kept at a fairly elementary level so that little background in the various sciences is required to understand them. And although applications are discussed throughout the text, I have attempted to maintain a close relationship between mathematical theories and applications. In this regard, I have tried to avoid the temptation of introducing a multitude of applications in many diverse areas of application. My experience is that exposing the student to too many applications proves distracting, at least to the average student. (More of something good is not always better.)

For the most part, the text contains the standard material that is usually found in introductory one-term courses. There are, however, some distinctions in the text that are perhaps noteworthy. For example, in addition to the integrating factor technique for solving first-order linear equations, a separate discussion of the general theory of these differential equations is presented (in Chapter 2) from the same point of view that is used in Chapter 4 in developing the theory of higher-order linear equations. Thus, not only is a more unified approach to the theory of all linear equations possible, but the simpler theory for the first-order equation can be used for motivation in developing the corresponding theory for higher-order equations.

Another feature of the text is the introduction of Green's functions (Chapter 5) for handling nonhomogeneous equations in a systematic and physically meaningful fashion. I believe an early exposure to Green's functions involving ordinary differential equations enhances the student's understanding and use of these functions when required in more advanced courses involving partial differential equations.

A brief discussion of the qualitative methods used in oscillation theory is presented in the chapter on initial value problems (Chapter 5). Because many students take differential equations prior to a course in linear algebra or matrix theory, an elementary (nonmatrix) treatment of linear systems of differential equations is presented in the early sections of Chapter 7. Hence, using techniques already familiar to the student, a natural transition from higher-order linear equations to linear systems of equations can be attained. A separate section on the use of matrix methods in solving linear systems of differential equations is also included for those students who are already versed in matrix techniques.

The text contains more than enough material for a one-semester course, and most of the chapters after Chapter 4 are sufficiently independent of each other so that various arrangements of topics can be made to suit individual needs. Also, sections that can easily be omitted for a shorter course are marked [O]. Hence, the text should be flexible enough to lend itself to several different types of courses. For example, a course may consist of selected material from Chapters 1–5, 7, and 9, or from Chapters 1–6, 8, and (possibly) 9. If applications are not stressed, a course could be based upon Chapters 1, 2, 4, 6–9, and so forth.

With respect to the physical layout of the text, I have numbered sections and subsections in decimal form. The subsections do not contain separate exercise sets. Various equations, theorems, figures, and tables are numbered consecutively in each chapter, but not according to section. For instance, Theorem 4.5 is the fifth theorem in Chapter 4 without regards to any particular section. Since worked examples usually play an important role in learning new material, I have included a large number of them (over 170) of varying difficulty throughout the text. Each example is generally indicative of typical problems to be found in the exercise sets at the end of each section (which consist of more than 1100 problems). Furthermore, all examples are set apart from the textual discussion by the use of horizontal lines in the hopes of making them easy for the student to find. The exercise sets usually contain a blend of drill-like problems, some more difficult, and some that extend the theory and applications beyond that discussed in the exposition. Problems that are considered to be more challenging and those used to extend the theory are generally separated from the more routine or drill-like problems by placing them either at the end of the exercise sets, or identifying them by a star (★), or both. References are listed at the end of each chapter.

I am grateful to Gerald Bradley, William Fitzgibbons, John Gregory, Samuel Rankin, Burton Rodin, and Chester Scott, who served as reviewers, and to Leslie Rochlin for suggesting Scott, Foresman and Company to me. Also, I would like to record my appreciation to the editorial and production staff of Scott, Foresman and Company for their helpfulness and cooperation with this project. And finally, I wish to thank Jack Pritchard, editorial vice-president, who became closely acquainted with this project near its end, but who was very helpful in working out the final details and seeing it through to completion.

L. C. Andrews

Ordinary Differential Equations

1 Basic Concepts

Differential equations play a fundamental role in engineering, mathematical sciences, and the life sciences because they can be used in the formulation of many physical laws and relations. The development of the theory of differential equations is closely interlaced with the development of mathematics in general, and it is indeed difficult to separate the two. In fact, most of the famous mathematicians from the time of Newton and Leibniz had some part in the cultivation of this fascinating subject.

In developing the theory of differential equations systematically, it is helpful to *classify* the various types of differential equations, since equations in a particular class can often be solved by the same method. Therefore, in Section 1.2 we discuss some of the various classification schemes, emphasizing *order* and *linearity*.

In Section 1.3 we carefully define what we mean by a *solution* of a differential equation. Here we first discover that differential equations are peculiar in that they generally possess many different solutions, and for this reason we usually seek a specific function, called a *general solution*, with the property that all solutions can be obtained from it.

We introduce the notions of *initial value problems* and *boundary value problems* in the last section of the chapter. These are the names attached to those problems occurring in applications wherein the solution of a differential equation must satisfy certain additional *auxiliary conditions*. The study of these problems is so important in applications that we have devoted a significant portion of the text to developing the theory associated with them.

1.1 INTRODUCTION

The theory of differential equations (DEs) has played an important role in science and engineering since the introduction of the calculus by Newton* and Leibniz.† Problems in the physical sciences have long since been investigated primarily by formulating them as DEs. Differential equations were first used in the early eighteenth century to solve problems in mechanics, but more recently the theory of DEs has found its way into the social and life sciences.

1.2 CLASSIFICATION OF DEs

By a *differential equation* we mean simply an equation that is composed of a single unknown function and a finite number of its derivatives.‡ One of the simplest examples that occurs early in the calculus is to find all functions $y = \phi(x)$ for which

$$y' = f(x). \tag{1}$$

For instance, if $f(x) = x^2$, the unknown function y is obtained through a simple integration to yield

$$y = \frac{x^3}{3} + C, \tag{2}$$

where C is a constant of the integration which can assume any value. Other examples of DEs are

$$y'' + k^2 y = \sin x, \qquad \text{LINEAR} \tag{3}$$

*Sir ISAAC NEWTON (1642–1727) was born on Christmas Day. Although mathematics began as a recreation for Newton, he was soon known as a great mathematician after his invention of the calculus, discovery of the law of universal gravitation, and experimental proof that white light is composed of light of all colors. He accomplished all this before the age of 24! A controversy erupted between Newton and Leibniz over the invention of the calculus, brought on primarily by friends of Newton who accused Leibniz of plagiarism. Eventually both Newton and Leibniz themselves became occupied with the controversy, even though both were convinced that they had reached their results independently. Newton spent the last two years of his life in constant pain, and he finally died in his sleep on March 20, 1727.

†GOTTFRIED WILHELM VON LEIBNIZ (1646–1716) is considered one of the great thinkers of modern times. Known as both a philosopher and mathematician, he is credited with building a remarkable system of modern philosophy and was greatly responsible for the development of the calculus. The last years of his life were saddened by ill health, controversy, and neglect. A bitter controversy had been started by friends of Newton over the legitimacy of Leibniz's discovery of the calculus. Newton had also discovered the calculus a few years earlier than Leibniz but failed to publish his results until after Leibniz had published his. Leibniz died alone on November 14, 1716, and only his faithful servant attended the funeral. Many members of the scientific community were not even aware of his death until almost a year later.

‡Derivative identities such as $d/dx \sin x = \cos x$ are not included in our basic definition of DEs.

$$(1 - x^2)y'' - 2xy' + 6y = 0, \quad \text{LINEAR} \tag{4}$$

$$y'' + K \sin y = 0, \quad \text{NON-LIN.} \tag{5}$$

$$y''' + xy'' + y^2 = e^x, \quad \text{NON-LIN.} \tag{6}$$

$$(y')^2 + 3xy = 1, \quad \text{NON-LIN.} \tag{7}$$

$$a^2 u_{xx} = u_{tt} - ku_t, \tag{8}$$

$$u_{xy} = 0, \tag{9}$$

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0. \tag{10}$$

Remark. Various notations for derivatives are commonly employed, depending upon which is convenient at the time. For instance, it is customary to make the association $y' = dy/dx$, $y'' = d^2y/dx^2$, . . . , $y^{(n)} = d^ny/dx^n$. For partial derivatives comparable notation is $u_x = \partial u/\partial x$, $u_{xy} = \partial^2 u/\partial y\, \partial x$, and so forth.

In solving DEs it is important to classify them. If the unknown function appearing in a DE has only a single independent variable, the equation is said to be an *ordinary differential equation*. All chapters in this text are concerned with ordinary DEs. When the unknown function depends upon more than one independent variable, the derivatives will be partial derivatives and the equation is called a *partial differential equation*. Examples of ordinary DEs are given by (3) through (7) above, while (8) through (10) are partial DEs.

Another classification is according to the *order* of the DE.

Definition 1.1 *The* ***order*** *of a DE is the order of the highest derivative appearing in the equation.*

According to Definition 1.1, Equations (3), (4), (5), (8), and (9) are second-order, (6) is third-order, (10) is fourth-order, and (7) is first-order, even though the highest derivative, y', is squared.

DEs are further divided into two large classes—*linear* and *nonlinear*—for which we have the following definition.

Definition 1.2 *A* ***linear*** *(ordinary) DE of order n is any equation that can be expressed in the form*

$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \cdots + A_1(x)y' + A_0(x)y = F(x),$$

where $A_0(x)$, $A_1(x)$, . . . , $A_n(x)$, and $F(x)$ are specified functions. Except for $F(x)$, these functions are called the coefficients of the DE.

The essential features of linear DEs as defined above are (1) that the unknown function y and all its derivatives are of the first degree (algebraically) and (2) that each coefficient depends only upon the independent variable x.

Since DEs must be either linear or nonlinear, any DE that is not linear (i.e., cannot be put down in the form given by Definition 1.2) is *nonlinear*. Thus we see that Equations (3) and (4) are linear, and (5), (6), and (7) are nonlinear.

In the next two chapters we will deal with first-order DEs, which can be expressed either in the form

$$\frac{dy}{dx} = F(x, y) \tag{11}$$

or

$$M(x, y)\,dx + N(x, y)\,dy = 0. \tag{12}$$

For such equations we can interpret either x or y as the independent variable, so when checking for linearity we should consider both possibilities. For example, the DE

$$3\,dx + xy\,dy = 0$$

is *nonlinear in y* but *linear in x*.

1.2.1 Origin and Application of DEs

As mentioned in the introduction, DEs originated out of a study of certain kinds of problems in mechanics. Today, however, their use is far more widespread. They occur in various branches of engineering and the sciences, and are used in

1. the study of particle motion,
2. the analysis of electric circuits and servomechanisms,
3. continuum and quantum mechanics,
4. the theory of diffusion processes and heat flow,
5. electromagnetic theory, and
6. the theory of vibrations and sounds.

Disciplines such as economics and the biological sciences are also now making use of DEs to investigate problems in

7. interest rates,
8. population growth, and
9. the ecological balance of systems,

among other types of problems.

The mathematical formulation of problems like those listed above gives rise to a DE. This occurs because the various scientific laws employed in the formulation of these problems involve certain *rates of change* of one or more quantities with respect to other quantities, and such rates of change are expressed mathematically by *derivatives*. Hence, the ensuing mathematical equations involve derivatives in the unknown quantities, and this is what we mean by a differential equation.

In formulating the DE, one must normally make certain simplifying assumptions so that the resulting DE is tractable. Determining just what assumptions are reasonable is often the most critical part of a problem. Sometimes certain aspects of a problem seem relatively unimportant and can be modified by assuming an approximate situation; sometimes an aspect of a problem may even be entirely eliminated. The DE resulting from any such assumptions will actually be that of an idealized situation.

Even after making a number of simplifying assumptions, the mathematical formulation of a problem can still lead to a DE that is troublesome to solve. This is particularly true when the resulting DE is *nonlinear,* since they are generally difficult or impossible to solve exactly. *Linear* DEs are much easier to handle in many ways, mostly because various properties of their solutions can be characterized in a general sort of way and standard solution techniques have been developed for solving many linear DEs. For this reason, linear equations occupy a more prominent place in the theory and applications of DEs.

Because of the difficulties encountered in solving most nonlinear DEs, they are often approximated by linear DEs when the resulting linear equations can describe certain fundamental characteristics of the nonlinear systems. For instance, the angle θ that an oscillating pendulum of length b makes with the vertical direction (Figure 1.1) is governed by the nonlinear DE

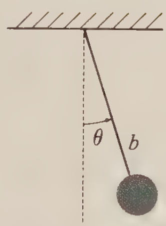


Figure 1.1 Pendulum.

$$\frac{d^2\theta}{dt^2} + \frac{g}{b}\sin\theta = 0,^* \tag{13}$$

which is nonlinear due to the term $\sin\theta$. If we write

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots,$$

then for small angular displacements θ, it may be reasonable to set $\sin\theta \cong \theta$, and thus (13) can be replaced by the linear equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{b}\theta = 0. \tag{14}$$

In this particular example the linear equation (14) gives fairly accurate information about the behavior of the pendulum when θ is small. There are a number of applications, however, for which this approach is fruitless, since the phenomenon

*The constant g is the gravitational constant.

being studied does not lend itself to any reasonable linear approximation. In such cases we try to solve the nonlinear DE itself. Techniques for handling such DEs will be discussed somewhat in Chapters 2 and 7.

EXERCISES 1.2

For each of the following DEs, state whether the equation is linear *or* nonlinear, *and give its* order.

1. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + y = 0$

2. $y' + xy^2 = 0$

3. $(x^2 + y^2)\,dx = 2xy\,dy$

4. $y' + a_0(x)y = f(x)$

5. $y''' - 2y'' + 6y' - y = x$

6. $\dfrac{dP}{dt} = aP - bP^2$

7. $\dfrac{d^4y}{dx^4} = q(x)$

8. $y' = \dfrac{x}{y}$

9. $L\dfrac{di}{dt} + Ri = E(t)$

10. $xy''' - 2(y')^4 + y = 0$

11. $2x^2y'' + 3xy' - y = 0$

12. $x^2\,dy + (y - xy - xe^x)\,dx = 0$

13. $y^2\,dx + x\,dy = 0$

14. $\left|\dfrac{dy}{dx}\right| + y = 0$

15. $yy''' + y'' - y' + 15xy = 0$

16. $\sqrt{1 + x^2}\,y'' + (\sin x)y' = e^x$

17. $y' + 1 = (y')^2 - \sin x$

18. $\sqrt{1 + y''} + y' = x^2 + 3$

19. $(x^2 - y^2)y' + 3xy = 0$

20. $x^2y'' + xy' - y = 3\cos^2 x$

1.3 SOLUTIONS OF DEs

One obvious requirement of a solution function of a DE is that it be differentiable. More precisely, we define *solution* in the following way.

Definition 1.3 *A **solution** of a DE on a given interval $x_1 < x < x_2$ is a continuous function possessing all derivatives occurring in the equation that, when substituted into the DE, reduces it to an identity for all x in the interval.*

EXAMPLE 1 Verify that $y = e^{-x}$ is a solution of $y' + y = 0$, and state the interval of validity.

Solution The function $y = e^{-x}$ is continuous and has derivatives of all orders for all x. Furthermore,

$$y' + y = -e^{-x} + e^{-x} = 0$$

for all values of x.

It should be pointed out that a given DE will usually possess many solutions. For example, it is easily verified that $y = Ce^{-x}$ satisfies the DE in Example 1 for any value of the constant C. Thus we obtain the family of solutions shown in Figure 1.2. Even the *trivial solution* $y = 0$, obtained by setting $C = 0$, is a solution of $y' + y = 0$.

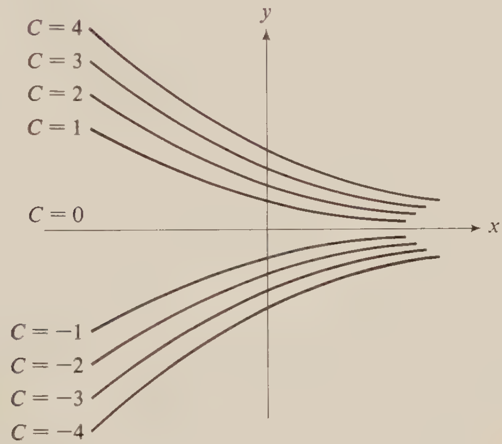


Figure 1.2 Solutions of $y' + y = 0$.

EXAMPLE 2 Verify that $y_1 = C_1 \cos 2x$ and $y_2 = C_2 \sin 2x$, where C_1 and C_2 are any constants, are both solutions of

$$y'' + 4y = 0.$$

Solution For $y_1 = C_1 \cos 2x$, we find

$$y_1' = -2C_1 \sin 2x, \quad y_1'' = -4C_1 \cos 2x,$$

so that

$$y_1'' + 4y_1 = -4C_1 \cos 2x + 4C_1 \cos 2x = 0.$$

Similarly, it follows that

$$y_2'' + 4y_2 = -4C_2 \sin 2x + 4C_2 \sin 2x = 0.$$

We might observe here that the functions

$$y = C_1 \cos 2x + C_2 \sin 2x$$

and

$$y = C_1 \sin x \cos x$$

are also solutions of the DE, the verification of which is left to the reader.

Not all solutions of a DE can be written in the explicit form $y = \phi(x)$. In some cases the solution must be defined *implicitly* by a relation of the form $g(x, y) = 0$.

EXAMPLE 3 Verify that $g(x, y) = 0$ is a solution of $y' = x/y$, where $g(x, y) = x^2 - y^2 - 1$.

Solution By implicit differentiation of $g(x, y) = 0$, we obtain

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) = 0 \quad \text{or} \quad 2x - 2yy' = 0,$$

and solving for y' yields

$$y' = \frac{x}{y}.$$

Once we are provided with a particular function, verifying whether it is a solution of a given DE is rather routine. The same is not true, however, for finding such solution functions except for a few special cases. But even prior to seeking the solution of a DE, we should inquire as to whether any solution, trivial or nontrivial, *exists*. Not all DEs have solutions! For example, the simple-looking DE

$$(y')^2 = -1$$

clearly does not have a (real) solution. The question concerning the existence of a solution is very important in applications where the DE is our mathematical model of some physical situation. If our model does not have a solution, it is probably a poor model, since meaningful physical problems should have solutions.

Another question about solutions concerns *uniqueness*. In the examples we have discussed, the solutions are not unique. Nonetheless, certain additional requirements can be imposed in most instances so that a unique solution can be singled out for those problems having solutions. More will be said about this situation later.

The existence and uniqueness questions are generally difficult to answer. Usually it is preferable to discuss these questions with respect to rather narrow classifications of equations, and this we do to some extent as we go along. A more practical question that concerns us is, How do we find solutions? We will devote most of our efforts in succeeding chapters to answering this question. Observe, however, that once a solution has been found, we have answered the question of existence. But if no solution is found, it is not clear whether this is because a solution doesn't exist or simply because we are unable to find it. It is embarrassing at the least to use, say, a computer to produce a numerical approximation to the solution of some problem only to find out later that the problem has no solution.

EXAMPLE 4 Determine the values of m such that $y = e^{mx}$ is a solution of $y'' - y' - 12y = 0$.

Solution The direct substitution of $y = e^{mx}$ into the DE yields

$$\begin{aligned} y'' - y' - 12y &= m^2e^{mx} - me^{mx} - 12e^{mx} \\ &= (m^2 - m - 12)e^{mx}, \end{aligned}$$

which equals zero only if ($e^{mx} \neq 0$)

$$m^2 - m - 12 = (m - 4)(m + 3) = 0.$$

Thus, $m = 4$ or $m = -3$, and we find the two solutions

$$y_1 = e^{4x} \quad \text{and} \quad y_2 = e^{-3x}.$$

$y' + y = 2x + y$

The reader should verify that $y = C_1e^{4x} + C_2e^{-3x}$ is also a solution of the DE.

EXAMPLE 5 Find a first-order DE involving both y' and y for which $y = x^2 - 1$ is a solution.

Solution Computing $y' = 2x$, it becomes clear that $xy' - 2y = 2$. Also, we observe that $\frac{1}{4}(y')^2 - y = 1$. Hence, we conclude that given a particular solution function, we can find several different DEs that it will solve.

1.3.1 General Solutions

From the examples considered thus far, it is apparent that DEs generally have more than one solution—in fact, infinitely many solutions. For this reason we often seek a solution function involving some arbitrary constants such that all *particular solutions* can be obtained from this one solution function by appropriately specializing the arbitrary constants. Such a solution function is then called a *general solution*.

A general solution of a DE must contain the same number of arbitrary constants as the order of the DE. It is important, however, that these be essential constants. That is to say, the set of constants cannot be reduced to a fewer number by some algebraic manipulation. Consider the following examples.

EXAMPLE 6 Both $y = C_1 \cos 2x + C_2 \sin 2x$ and $y = C_1 \sin x \cos x + C_2 \sin 2x$ are two-parameter solutions of $y'' + 4y = 0$. Are both general solutions?

Solution The first solution, $y = C_1 \cos 2x + C_2 \sin 2x$, is a general solution, since no algebraic manipulation will allow us to combine the two constants into a single constant. On the other hand, we find that

$$\begin{aligned} y &= C_1 \sin x \cos x + C_2 \sin 2x \\ &= \frac{1}{2} C_1 \sin 2x + C_2 \sin 2x \\ &= \left(\frac{1}{2} C_1 + C_2\right) \sin 2x \\ &= C_3 \sin 2x, \end{aligned}$$

so that this solution cannot be a general solution.

EXAMPLE 7 Both $y = C_1 e^x + C_2 e^{-x}$ and $y = C_1 \cosh x + C_2 \sinh x$ satisfy the DE $y'' - y = 0$. Are both general solutions?

Solution In this case both functions do represent general solutions, since neither can be reduced to a single-parameter solution. This shows that general solutions can assume different forms for the same equation.

Using the definition of hyperbolic functions, we also note that

$$\begin{aligned} y &= C_1 \cosh x + C_2 \sinh x \\ &= \frac{1}{2} C_1 (e^x + e^{-x}) + \frac{1}{2} C_2 (e^x - e^{-x}) \\ &= \frac{1}{2} (C_1 + C_2) e^x + \frac{1}{2} (C_1 - C_2) e^{-x} \\ &= C_3 e^x + C_4 e^{-x}, \end{aligned}$$

illustrating that we can obtain one general solution from the other.

Usually we think of a general solution as containing all solutions of a particular DE. However, a solution of an equation may exist that cannot be obtained from a general solution by specializing the constants. Such solutions, referred to as *singular solutions*, arise only in the case of certain nonlinear DEs; they do not arise in solving linear DEs with appropriate restrictions. For instance, we can easily verify that

$$(x - C)^2 + y^2 - 1 = 0 \tag{15}$$

is a solution of the DE

$$(y')^2 + 1 = \frac{1}{y^2} \tag{16}$$

for any value of the constant C. Moreover, it can likewise be shown that $y = 1$ and $y = -1$ are also solutions of (16) but not members of the family (15), since no value can be assigned to C to obtain these solutions. They are singular solutions (see Figure 1.3).

Remark. Because of the possibility of singular solutions, some authors do not consider (15) to be a general solution, since it does not contain all solutions of (16).

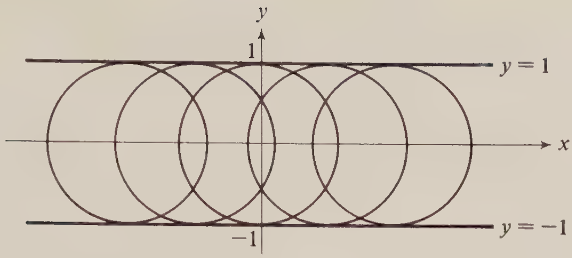


Figure 1.3 Solutions of $(y')^2 + 1 = 1/y^2$.

Since this situation arises only in connection with nonlinear DEs, and since most of our analysis concerns linear DEs, such a distinction need not worry us for the most part.

Another significant difference between linear and nonlinear DEs is that a linear combination of two or more solutions is also a solution in the case of many linear equations (specifically those for which $F(x) \equiv 0$ in Definition 1.2). This property, however, is generally not true for nonlinear equations, as illustrated in the following example. This is one of the reasons why nonlinear DEs are usually much harder to deal with than linear DEs.

Remark. The concept of adding two or more solutions to produce new solutions is called the *superposition principle*. This principle, which plays an important role throughout our discussion of linear DEs, is first introduced in Chapter 4.

EXAMPLE 8 Show that although $y_1 = e^{-x}$ and $y_2 = e^{2x}$ are solutions of both the linear equation

$$y'' - y' - 2y = 0$$

and the nonlinear equation

$$yy'' - (y')^2 = 0,$$

the expression $y = C_1e^{-x} + C_2e^{2x}$ is a solution of only the linear equation for arbitrary nonzero values of C_1 and C_2.

Solution By direct substitution of $y = C_1e^{-x} + C_2e^{2x}$ into the linear DE, we see that

$$\begin{aligned} y'' - y' - 2y &= (C_1e^{-x} + 4C_2e^{2x}) - (-C_1e^{-x} + 2C_2e^{2x}) - 2(C_1e^{-x} + C_2e^{2x}) \\ &= (C_1 + C_1 - 2C_1)e^{-x} + (4C_2 - 2C_2 - 2C_2)e^{2x} \\ &= 0 \cdot e^{-x} + 0 \cdot e^{2x} = 0, \end{aligned}$$

whereas the substitution of this expression into the nonlinear equation leads to

$$\begin{aligned} yy'' - (y')^2 &= (C_1e^{-x} + C_2e^{2x})(C_1e^{-x} + 4C_2e^{2x}) - (-C_1e^{-x} + 2C_2e^{2x})^2 \\ &= (C_1^2 - C_1^2)e^{-2x} + (5C_1C_2 + 4C_1C_2)e^{x} + (4C_2^2 - 4C_2^2)e^{4x} \\ &= 9C_1C_2e^{x}, \end{aligned}$$

which is clearly not identically zero for arbitrary values of C_1 and C_2. In fact, it is zero only when either C_1 or C_2 itself is zero.

Throughout the calculus we have become accustomed to the idea that the "solution" of a problem is an exact, closed-form, analytical expression in terms of elementary functions from which numerical values can be generated when needed. In practice, however, this is rarely the case, especially when solving a DE. That is to say, comparatively few DEs occur in applications for which simple exact solutions can be found by known methods. Hence, in practice the "solution" often consists of some type of approximation function or a set of numerical values approximating the true numerical values of the solution over some interval. Several such approximation methods are available for this purpose, and they are becoming increasingly more common due to the complexities of the DEs arising in modern science and engineering problems, and also due to the widespread availability of modern high-speed computers.

In spite of what we have just said, most of the problems encountered in this text will lead to exact solutions in much the same way as we have come to expect in the calculus. The reason is that studying problems with exact simple solutions aids the student in understanding the general theory and solution methods available; for example, they provide an answer that can often be easily checked. Some approximation methods are briefly discussed in this text, but these techniques should be taken up in more detail, such as in a course in numerical analysis.

EXERCISES 1.3

In problems 1–14, verify that the given function is a solution of the specified DE. Assume C_1 and C_2 denote constants.

1. $y' + 2xy = 0; \quad y = C_1e^{-x^2}$

2. $xy' + y = 0; \quad y = \dfrac{C_1}{x}, \quad x \neq 0$

3. $xy' - y = x; \quad y = x\log x, \quad x > 0$

***4.** $(1 + y^2)\,dx + (1 + x^2)\,dy = 0; \quad xy + x + y - 1 = 0$

5. $3y\,dx = 2x\,dy; \quad x^3 - C_1y^2 = 0$

6. $(y')^2 = \dfrac{y}{x}; \quad \sqrt{y} = \sqrt{x} + 3, \quad x > 0$

7. $\dfrac{dP}{dt} = aP - bP^2; \quad P = \dfrac{ae^{at}}{(1 + be^{at})} \quad (a, b \text{ constants})$

8. $xy' + (1 - x)y = xe^x; \quad y = \dfrac{1}{2}xe^x + \dfrac{C_1e^x}{x}, \quad x \neq 0$

9. $y'' + 2y' - 3y = 0; \quad y = C_1e^x + C_2e^{-3x}$

10. $y'' - 6y' + 9y = 0; \quad y = (C_1 + C_2x)e^{3x}$

11. $y'' - 2y' + 5y = 0;\quad y = C_1e^x\cos 2x + C_2e^x\sin 2x$

12. $y'' + y' - 2y = 2x - 40\cos 2x;$

$$y = C_1e^x + C_2e^{-2x} - \frac{1}{2} - x + 6\cos 2x - 2\sin 2x$$

n **13.** $x^2y'' + 5xy' + 4y = 0;\quad y = C_1x^{-2} + C_2x^{-2}\log x,\quad x > 0$

***14.** $y' - 2xy = 1;\quad y = e^{x^2}\int_0^x e^{-t^2}\,dt + e^{x^2}$

Each of problems 15–20 has a solution of the form $y = e^{mx}$ for some value of the constant m. Determine for what values of m the following linear DEs have such a solution.

15. $y' - 2y = 0$ me' $e^{mx}(m-2)$

16. $y'' - y = 0$

n **17.** $y'' + y' - 6y = 0$

18. $y'' + 2y' = 0$

19. $y''' + 3y'' - 4y' = 0$ $m=2$

***20.** $y''' + 6y'' + 11y' + 6y = 0$

In problems 21–25, not all constants that appear in the given expression are essential. Rewrite the expression using only essential constants.

n **21.** $y = C_1e^{x+a}$ $= Ae^x$

22. $y = C_1\cos(x + a) + C_2\cos(x - a)$

23. $y = \dfrac{ax + b}{cx + d}$

24. $y = C_1 + C_2\log(ax^3)$

n **25.** $y = C_1\sin x + C_2\sin 3x + C_3\sin^3 x$

26. Show that $g(x, y) = x^3 + 3xy^2 - 1 = 0$ is an implicit solution of $2xyy' + x^2 + y^2 = 0$ on the interval $0 < x < 1$.

***27.** Show that $g(x, y) = 5x^2y^2 - 2x^3y^2 - 1 = 0$ is an implicit solution of $xy' + y = x^3y^3$ on the interval $0 < x < \frac{5}{2}$.

In problems 28–35, find a first-order DE involving both y' and y for which the given function is a solution.

28. $y = e^{-2x}$ $y' = -4e^{2x} \cdot 2$

29. $y = x^3 - 4$

30. $y = \frac{1}{2}xe^x$

31. $y = 2e^{2x} - 2x - 1$

32. $y = \cos x$

n **33.** $y = \sinh 3x$

34. $(x - 1)^2 + y^2 - 1 = 0$

35. $y^2 = x^2 + y$

***36.** Verify that $y = Cx - C^2$ is a solution of

$$(y')^2 - xy' + y = 0$$

for all values of the constant C. Determine a value of K such that $y = Kx^2$ is a *singular solution* of the DE.

***37.** Verify that $y = (\frac{1}{4}x^2 + C)^2$ is a solution of $y' = xy^{1/2}$. Also find a singular solution by inspection.

38. The function $y = C_1e^{2x+C_2}$ is a solution of the DE

$$y'' - y' - y = 0$$

for any choice of the constants C_1 and C_2. Explain why this is not a general solution.

***39.** Show that the piecewise-defined function

$$y = \begin{cases} -x^4, & x < 0 \\ x^4, & x \geqslant 0 \end{cases}$$

is a solution of $xy' = 4y$ as is $y = Cx^4$. Can the solution defined piecewise be obtained from $y = Cx^4$ by an appropriate choice of C? Explain.

***40.** The functions $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$ are both solutions of $yy' + x = 0$ on the interval $-1 < x < 1$. Is the function

$$y = \begin{cases} -\sqrt{1 - x^2}, & -1 < x < 0 \\ \sqrt{1 - x^2}, & 0 \leqslant x < 1 \end{cases}$$

also a solution? Explain.

41. Verify that $y_1 = 1$ and $y_2 = x^2$ satisfy each of the DEs

$$xy'' - y' = 0 \quad \text{and} \quad 2yy'' - (y')^2 = 0,$$

but that $y = C_1 + C_2x^2$ only satisfies the first equation for arbitrary values of C_1 and C_2. Explain.

42. Verify that $y_1 = 1$ and $y_2 = x^{1/2}$ satisfy each of the DEs

$$2xy'' + y' = 0 \quad \text{and} \quad 8x^3(y'')^2 - yy' = 0,$$

but that $y = C_1 + C_2x^{1/2}$ only satisfies the first equation for arbitrary values of C_1 and C_2. Explain.

1.4 INITIAL AND BOUNDARY VALUE PROBLEMS

In most applications the unknown function y must satisfy certain restraints, or *auxiliary conditions*, in addition to satisfying the DE. These conditions imposed upon the problem determine which of an infinite collection of solutions are peculiar to the given problem, and the number of these conditions is usually equal to the order of the differential equation.

EXAMPLE 9 Solve $y' - y = 0$, $y(0) = 2$, given that $y = Ce^x$ is a general solution.

Solution The general solution $y = Ce^x$ is a family of curves in the xy-plane (Figure 1.4). By specifying $y(0) = 2$, we are seeking that one particular curve which passes through the point $(0, 2)$. Thus, by substituting $x = 0$ in the general solution, we obtain $2 = Ce^0$, or $C = 2$. The solution we seek is therefore

$$y = 2e^x.$$

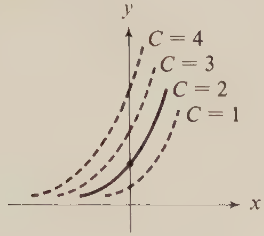


Figure 1.4

EXAMPLE 10 Solve $y'' + y = 0$, $y(0) = 2$, $y'(0) = 0$, given that $y = C_1 \cos x + C_2 \sin x$ is a general solution.

Solution The auxiliary conditions require that the solution we seek pass through the point (0, 2) of the xy-plane with zero slope at that point. Imposing these conditions on the general solution leads to

$$y(0) = C_1 \cos 0 + C_2 \sin 0 = 2,$$
$$y'(0) = -C_1 \sin 0 + C_2 \cos 0 = 0,$$

which, upon simplifying, yields $C_1 = 2$, $C_2 = 0$. Hence, our solution becomes

$$y = 2\cos x.$$

EXAMPLE 11 Solve $y'' + y = 0$, $y(0) = 0$, $y(\pi/2) = 1$, given that $y = C_1 \cos x + C_2 \sin x$ is a general solution.

Solution Applying the first condition $y(0) = 0$ to the general solution leads to

$$y(0) = C_1 \cos 0 + C_2 \sin 0 = 0,$$

from which we deduce $C_1 = 0$. The second condition yields

$$y\left(\frac{\pi}{2}\right) = C_2 \sin \frac{\pi}{2} = 1,$$

or $C_2 = 1$. The solution we seek is therefore

$$y = \sin x.$$

When the auxiliary conditions are all specified at a single value of x, such as in Examples 9 and 10, we refer to the problem as an *initial value problem*. Although we could choose the single value of x as any value, it is customary in practice to use $x = 0$. If the auxiliary conditions are specified at more than one point on the interval of interest, the resulting problem is called a *boundary value problem* (see

Example 11). The boundary conditions are usually specified at only two points of the interval, and we refer to these problems as *two-point boundary value problems*. Such problems involve DEs that are at least second-order, since first-order DEs have but a single auxiliary condition and as such are classified as initial value problems.

In general, initial value problems are well behaved in that they almost always lead to unique solutions. Unfortunately, the same is not true of boundary value problems, and so the general theory of these problems is more complicated. The following examples give some hint of the difficulties that arise in solving boundary value problems.

EXAMPLE 12 Given the general solution $y = C_1 \cos x + C_2 \sin x$, solve the boundary value problem

$$y'' + y = 0, \quad y(0) = 0, \quad y(\pi) = 2.$$

Solution Imposing the prescribed boundary conditions, we find

$$y(0) = C_1 = 0$$

and

$$y(\pi) = C_2 \sin \pi = 2.$$

But since $\sin \pi = 0$, we conclude that no member in the general family of solutions satisfies both boundary conditions. Since the equation is linear, we know that the DE does not have any singular solutions, so we must accept the fact that this boundary value problem has *no solution*.

EXAMPLE 13 Solve the boundary value problem in Example 12 if the boundary conditions are changed to $y(0) = 0, \quad y(\pi) = 0$.

Solution This time the boundary conditions demand that

$$y(0) = C_1 = 0$$

and

$$y(\pi) = C_2 \sin \pi = 0,$$

which are satisfied for $C_1 = 0$ and any value assigned to C_2. Hence, we obtain the family of solutions

$$y = C_2 \sin x,$$

where C_2 is an arbitrary constant.

EXERCISES 1.4

Solve the initial value problems 1–10, for which the general solution is specified.

1. $y' - 2y = 0, \quad y(0) = 1; \quad y = C_1 e^{2x}$

2. $xy' + 2y = 0, \quad y(1) = 4; \quad y = \dfrac{C_1}{x^2}$

3. $y' + y = e^x, \quad y(0) = 0; \quad y = \dfrac{1}{2}e^x + C_1e^{-x}$

4. $y'' + y = 0, \quad y(0) = 1, \quad y'(0) = -1; \quad y = C_1 \cos x + C_2 \sin x$

5. $y'' = 1, \quad y(0) = 0, \quad y'(0) = 0; \quad y = \dfrac{1}{2}x^2 + C_1 + C_2x$

6. $y'' = e^x, \quad y(0) = 2, \quad y'(0) = -1; \quad y = e^x + C_1 + C_2x$

7. $x^2y'' - 2xy' + 2y = 0, \quad y(1) = 0, \quad y'(1) = 1; \quad y = C_1x + C_2x^2$

8. $y'' - y = 0, \quad y(0) = 0, \quad y'(0) = 0; \quad y = C_1e^x + C_2e^{-x}$

9. $y'' - 2y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 0; \quad y = e^x(C_1 \cos x + C_2 \sin x)$

10. $x^2y'' - xy' + y = 0, \quad y(1) = 3, \quad y'(1) = -1; \quad y = C_1x + C_2x \log x$

Find solutions (if they exist) for each of the boundary value problems 11–15. The general solution is provided.

11. $y'' = 0, \quad y(0) = 0, \quad y(1) = 1; \quad y = C_1 + C_2x$

12. $y'' = 0, \quad y(0) = 0, \quad y(1) = 0; \quad y = C_1 + C_2x$

13. $y'' + y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0; \quad y = C_1 \cos x + C_2 \sin x$

14. $y'' + \pi^2y = 0, \quad y(0) = 1, \quad y(1) = 0; \quad y = C_1 \cos \pi x + C_2 \sin \pi x$

15. $y'' - y = 0, \quad y(-1) = 0, \quad y'(1) = 0; \quad y = C_1e^x + C_2e^{-x}$

16. Verify that $y = C_1 + C_2x^2$ is a solution of the linear DE

$$xy'' - y' = 0.$$

Can constants C_1 and C_2 be found such that $y(0) = 0$ and $y'(0) = 1$?

*17. For which values of k (if any) does the boundary value problem

$$y'' + k^2y = 0, \quad y(0) = 0, \quad y(\pi) = 0,$$

have nonzero solutions? The general solution is $y = C_1 \cos kx + C_2 \sin kx$.

REFERENCES

Bell, E. T. 1937. *Men of mathematics*. New York: Simon & Schuster.

Boyce, W. E., and DiPrima, R. C. 1977. *Elementary differential equations*. 3rd ed. New York: Wiley.

Eves, H. 1964. *An introduction to the history of mathematics*. 3rd ed. New York: Holt, Rinehart & Winston.

Ince, E. L. 1956. *Ordinary differential equations*. New York: Dover.

Ross, S. L. 1980. *Introduction to ordinary differential equations*. 3rd ed. New York: Wiley.

Zill, D. G. 1979. *A first course in differential equations with applications*. Boston: Prindle, Weber & Schmidt.

Solution Techniques for First-Order Equations

2

In this chapter we consider certain basic types of first-order DEs for which exact solutions may be obtained by clearly designated techniques. Proficiency in solving such equations rests heavily on the ability to recognize the various types of DEs and apply the corresponding method of solution, which often consists of certain "devices" or "tricks." This is particularly true when solving nonlinear DEs, since no clear underlying theory can be applied to them.

In Section 2.2 we discuss equations that can be solved by the method of *separation of variables*. This is the easiest technique to apply and comes up most often in practice. Other equations that permit exact solutions in closed form are discussed in Section 2.3, although some of them occur infrequently in applications. These include *exact equations*, equations solvable by the use of *integrating factors*, *linear equations*, and *homogeneous equations* which are solved by reduction to variables separable by a change of variable.

In Sections 2.4 and 2.5 we again discuss linear DEs of the first order, but in the context of the general theory of linear equations of any order so as to provide a more comprehensive approach. Thus we introduce the notions of a *homogeneous solution* and a *particular solution* of a DE, the sum of which forms a *general solution.* A physical interpretation of a general solution is given, and the basic *existence-uniqueness theorem* concerning linear initial value problems of the first order is discussed.

A special nonlinear equation, called *Bernoulli's equation,* is solved in the final section of the chapter by first reducing it to a linear equation through an appropriate change of variable.

For a shorter course, Sections 2.4 through 2.6 can be omitted.

2.1 INTRODUCTION

First-order DEs arise in a variety of problems, including the determination of the velocity of free-falling bodies subject to a resistive force; finding curves of population growth, radioactive decay, and the pursuit of a predator tracking its prey; and finding the current or charge in an electrical circuit. The types of DEs involved in these applications fall into several classifications, each of which demands a different method of solution. We will discuss the various solution techniques in the present chapter and take up the applications in Chapter 3.

The DEs to be studied may be expressed either in the derivative form

$$y' = F(x, y) \tag{1}$$

or in the differential form

$$M(x, y)\,dx + N(x, y)\,dy = 0, \tag{2}$$

depending upon the type of DE and the solution treatment. The equivalence of these two forms can be seen in writing (2) as

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}.$$

In applications the solutions of these DEs are usually required to satisfy an auxiliary condition of the form

$$y(x_0) = y_0, \tag{3}$$

which specifies that the solution function pass through the point (x_0, y_0) of the xy-plane.

Nonlinear DEs do not lend themselves to the development of general solution formulas, even in the case of first-order equations. Except for the few special cases treated in this chapter, finding the solution of a nonlinear DE is almost always very difficult or impossible by known techniques. Therefore, approximation and qualitative methods (see Chapters 7 and 8) often play a marked role in the analysis of a nonlinear DE, and for that reason it is important to be able to answer the existence and uniqueness questions in advance. We state the following relevant *existence-uniqueness theorem* without proof.

Theorem 2.1 If F and $\partial F/\partial y$ are both continuous functions in some domain of the xy-plane containing the point (x_0, y_0), then there exists a unique solution of the initial value problem

$$y' = F(x, y), \quad y(x_0) = y_0,$$

defined on some interval $|x - x_0| \leq h$, where h is "sufficiently small."*

*For a proof of Theorem 2.1, the reader should consult E. I. Ince, *Ordinary Differential Equations* (New York: Dover, 1956).

In order to understand the significance of the continuity requirement in Theorem 2.1, consider the initial value problem

$$y' = y^{1/3}, \quad y(0) = 0.$$

It is easily verified that

$$y_1 = \left(\frac{2x}{3}\right)^{3/2}, \quad x \geqslant 0$$

is a solution, but it is also obvious from inspection that

$$y_2 = 0$$

is another solution (called a *singular solution*). A unique solution does not exist because

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(y^{1/3}) = \frac{1}{3}y^{-2/3}$$

is not a continuous function in any region containing the x-axis ($y = 0$). If we prescribe the initial condition at any other point (x_0, y_0) not on the x-axis, a unique solution can then be found.

2.2 SEPARATION OF VARIABLES

The first-order DE

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{4}$$

is said to be *separable* if it can be put in the form

$$f(x)\,dx = g(y)\,dy \tag{5}$$

through algebraic manipulations. Observe that the left-hand side is a function of x alone, while the right-hand side is a function of y alone. The general solution of (5) can then be obtained by integrating each side,

$$\int f(x)\,dx = \int g(y)\,dy + C$$

or

$$F(x) = G(y) + C, \tag{6}$$

where C is an arbitrary constant of integration and where

$$\frac{d}{dx}F(x) = f(x) \quad \text{and} \quad \frac{d}{dx}G(y) = g(y)\frac{dy}{dx}.$$

This general technique, called the method of *separation of variables*, is the most important method that we will discuss for solving first-order DEs.

EXAMPLE 1 Solve $(1 - x)\,dy + y\,dx = 0$.

$(1-x)\,dy = -y\,dx$

Solution Division by $(1 - x)y$ leads to

$$\frac{dy}{y} = \frac{dx}{1 - x}.$$

$\frac{dy}{-y} = \frac{dx}{1-x}$

Thus, integrating both sides, we get

$$\log y = \log(1 - x) + C_1,$$

$\ln y = +\ln|1-x|$

which can also be expressed in the form

$$y = e^{C_1}(1 - x).$$

Since C_1 is arbitrary, so is e^{C_1}, and we can redefine it as C so that the solution reads $y = C(1 - x)$.

In Example 1 we are using the symbol $\log x$ to denote the natural logarithm, also commonly denoted by $\ln x$. Since integrals leading to logarithmic terms are quite prevalent when employing this solution technique, some care should be exercised in evaluating these integrals. Recall from the calculus that

$$\int \frac{du}{u} = \log|u| + C, \quad u \neq 0,$$

where the absolute value is usually retained unless we know in advance that $u > 0$. As a general rule, however, we will retain the absolute value only in those cases for which a logarithmic term remains in the final solution form of the DE.

EXAMPLE 2 Solve $y' = (3y + 1)/x^2, \quad x > 0, \quad y > -\frac{1}{3}$.

Solution Separating the variables gives

$$\frac{dy}{3y + 1} = \frac{dx}{x^2},$$

$\int \frac{dy}{3y+1}$

$\frac{1}{3}\ln|3y+1| = \frac{1}{x}$

and integrating yields

$$\frac{1}{3}\log(3y + 1) = -\frac{1}{x} + \frac{1}{3}\log C, \quad C > 0.*$$

exp

Simplifying the algebra finally leads to

$3y+1 = e^{-\frac{3}{x}}$

$y+1 = Ce^{-3/x}$

*Writing the arbitrary constant as $\frac{1}{3}\log C$ is a simplifying device suggested by the form of the term involving y in the general solution. Since the constant is arbitrary, it can assume any form that seems convenient for the problem.

$$3y + 1 = Ce^{-3/x}$$

or

$$y = \frac{1}{3}(Ce^{-3/x} - 1).$$

Although we technically have the general solution of the DE once the integration has been completed, it is usually advantageous to simplify the algebra in this solution when possible. The simplicity of the final solution in the next example clearly illustrates the gain resulting from a little algebraic maneuvering.

EXAMPLE 3 Solve the initial value problem

$$(x^2 + 1)y' + y^2 + 1 = 0, \quad y(0) = 1.$$

Solution Rearranging terms, we have

$$\frac{dy}{1 + y^2} = -\frac{dx}{1 + x^2}.$$

Integrating this equation gives the general solution

$$\arctan y = -\arctan x + C$$

or

$$\arctan y + \arctan x = C.$$

From the trigonometric identity

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B},$$

we find

$$\tan(\arctan y + \arctan x) = \frac{y + x}{1 - xy},$$

and consequently our general solution takes the form

$$\frac{y + x}{1 - xy} = \tan C.$$

Applying the initial condition $y(0) = 1$ reveals that $\tan C = 1$, and hence

$$\frac{y + x}{1 - xy} = 1$$

or

$$y = \frac{1 - x}{1 + x},$$

which is the intended solution.

EXAMPLE 4 Solve $(x - 4)y^4\,dx - x^3(y^2 - 3)\,dy = 0$.

Solution We can separate the variables by dividing by x^3y^4, finding

$$\frac{(x - 4)}{x^3}dx - \frac{(y^2 - 3)}{y^4}dy = 0$$

or

$$(x^{-2} - 4x^{-3})\,dx - (y^{-2} - 3y^{-4})\,dy = 0.$$

Upon integrating, we get

$$-\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = C.$$

In separating the variables, it was necessary to assume that $x \neq 0$ and $y \neq 0$. Note, however, that $y = 0$ is also a solution of the DE, which cannot be obtained from the general solution by selecting an appropriate value of C. It is a *singular solution* of the problem.

This last example illustrates one of the frustrations encountered in solving nonlinear DEs, namely, the possibility of singular solutions. These solutions may be lost in the solution process as in the above example, and yet in some instances they may be just the solution we seek. In practice, therefore, the conditions of Theorem 2.1 should be carefully checked with each problem to determine whether the problem has a unique solution.

EXERCISES 2.2

In problems 1–25, obtain the general solution by separating the variables.

1. $2y\,dx = 3x\,dy$ **2.** $y' = xy^2$ **3.** $y' = -\dfrac{x}{y}$

4. $y' = \dfrac{x - 4}{x - 3}$ n **5.** $y' = \dfrac{y^2 - 1}{x^2 - 1}$

6. $\sin x \sin y\,dx + \cos x \cos y\,dy = 0$

n **7.** $\sec^2 x\,dy + \csc y\,dx = 0$

8. $x^2(1 + y^3)\,dx + y^2(1 + x^3)\,dy = 0$ n **9.** $\dfrac{dP}{dt} = aP - bP^2$

10. $\dfrac{dV}{dP} = -\dfrac{V}{P}$

11. $\dfrac{dN}{dt} + N = Nte^{t+2}$

12. $e^x(y-1)\,dx + 2(e^x + 4)\,dy = 0$

n ***13.** $(xy + x)\,dx - (x^2y^2 + x^2 + y^2 + 1)\,dy = 0$

14. $x\cos^2 y\,dx + \tan y\,dy = 0$

n **15.** $(1 + \log x)\,dx + (1 + \log y)\,dy = 0$

16. $x^2yy' = e^y$

WRONG **17.** $2yy' - 1 = 2x$

18. $y' = e^{2x+3y}$

19. $(y + 1)\,dy - y\log x\,dx = 0$

20. $dy = y\sqrt{1 - y^2}\,dx$

21. $(x^2 - 1)y' = 2xy\log y$

***22.** $(y')^2 = y^3 - y^2$

n ***23.** $x(y')^2 = y$

***24.** $y(y')^2 = 1$

***25.** $y(y')^2 = e^{2x}$

In problems 26–30, obtain the particular solution satisfying the prescribed auxiliary condition.

26. $v\dfrac{dv}{dt} = g, \quad v(t_0) = v_0$

27. $y' = -2xy, \quad y(0) = y_0$

28. $xyy' = 1 + y^2, \quad y(2) = 3$

29. $x\sin x e^{-y}\,dx - y\,dy = 0, \quad y(0) = 1$

30. $y' = \dfrac{2x}{1 + 2y}, \quad y(2) = 0$

***31.** Given $y' = y^2 - 4$:

(a) Derive the general solution

$$y = \frac{-2(1 - Ce^{-4x})}{1 + Ce^{-4x}}.$$

(b) Show that the initial condition $y(0) = 2$ cannot be satisfied by the solution in (a). (Can you explain?)

(c) By inspection, obtain two singular solutions and verify that one of these satisfies the initial condition in (b).

***32.** Given the initial value problem $y' = xy^{1/2}, \quad y(0) = 0$:

(a) Find two particular solutions of the DE satisfying the prescribed initial condition. *Hint*: One is a singular solution.

(b) Verify that the piecewise-defined function

$$y = \begin{cases} 0, & x < 1 \\ \dfrac{(x^2 - 1)^2}{16}, & x \geq 1 \end{cases}$$

is also a solution. Are there any others?

n ***33.** Solve $xy' - y^2 + 1 = 0$ subject to

(a) $y(0) = 1$. (b) $y(0) = -1$. (c) $y(0) = 2$.

***34.** Solve $y' - x^k(y^2 + 1) = 0$ when

(a) $k \neq -1$. (b) $k = -1$.

***35.** Find all solutions of $(y')^2 + y^2 = a^2, \quad a \neq 0$.

Although most of the first-order DEs that permit exact solutions are of the separable variables type, several other types occasionally arise for which exact solutions can be obtained.

2.3.1 Exact Equations

Recall from the calculus that if $f(x, y)$ is a function of two variables, then its *total differential* is given by the expression

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy. \tag{7}$$

EXAMPLE 5 Find the total differential of $f(x, y) = x^2y^3$.

Solution We have $\partial f/\partial x = 2xy^3$ and $\partial f/\partial y = 3x^2y^2$, so that

$$df = 2xy^3\,dx + 3x^2y^2\,dy.$$

If the DE

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{8}$$

is such that the left-hand side is the total differential of some function f, then we call $M(x, y)\,dx + N(x, y)\,dy$ an *exact differential* and write (8) as

$$df = 0. \tag{9}$$

In this case it follows that the solution of (9) is given by the family of curves

$$f(x, y) = C, \tag{10}$$

where C can assume any constant value.

For example, if

$$2xy^3\,dx + 3x^2y^2\,dy = 0, \tag{11}$$

then we recognize the left-hand side of this expression as being the total differential of the function $f(x, y) = x^2y^3$ (see Example 5). Hence, (11) has the solution

$$x^2y^3 = C \tag{12}$$

for any constant C.

Equations like (8) for which the left-hand side is an exact differential are called *exact differential equations*. The difficulty in solving such equations is that it is not always easy to determine by inspection whether a given differential expression is exact, as we determined in the above example. Therefore, the following theorem provides us with a test for deciding whether the expression $M(x, y)\,dx + N(x, y)\,dy$ is an exact differential, and the proof provides us with a scheme for finding the function $f(x, y)$ in those cases for which it is an exact differential.

Theorem 2.2 If $M(x, y)$ and $N(x, y)$ are continuous functions and have continuous first partial derivatives in some region of the xy-plane, then a necessary and sufficient condition that

$$M(x, y)\,dx + N(x, y)\,dy$$

be an exact differential in this region is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Proof: To prove the necessity part of the theorem, we merely observe that if the expression is exact, then

$$M(x, y)\,dx + N(x, y)\,dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

and hence

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{dy} = N.$$

Taking mixed partials of f, we obtain

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial M}{\partial y}, \quad \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial N}{\partial x},$$

from which we deduce $\partial M/\partial y = \partial N/\partial x$.

To prove the sufficiency part, let us assume that $\partial M/\partial y = \partial N/\partial x$ and

$$\frac{\partial f}{\partial x} = M(x, y).$$

If we formally integrate this latter expression with respect to x, we find

$$f(x, y) = \int M(x, y)\,dx + g(y)$$

where the function g is the "constant" of integration (constant with respect to x). Differentiating this last expression now with respect to y yields

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x, y)\,dx + g'(y) = N(x, y).$$

The function g is then a solution of the first-order DE

$$g'(y) = N(x, y) - \frac{\partial}{\partial y}\int M(x, y)\,dx,$$

the right-hand side of which is independent of x. To see this, observe that

$$\frac{\partial}{\partial x}\left(N - \frac{\partial}{\partial y}\int M\,dx\right) = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\int M\,dx\right) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0,$$

and the theorem is proved. □

Remark. In the sufficiency proof of Theorem 2.2, we could just as easily have started with $\partial f/\partial y = N(x, y)$ and integrated to find

$$f(x, y) = \int N(x, y)\,dy + h(x).$$

Then $h(x)$ is determined by

$$h'(x) = M(x, y) - \frac{\partial}{\partial x}\int N(x, y)\,dy.$$

The essence of Theorem 2.2 is that if $\partial M/\partial y = \partial N/\partial x$, then $M\,dx + N\,dy = 0$ is an exact DE, but if $\partial M/\partial y \neq \partial N/\partial x$, the DE is *not* exact. Hence the condition

$$\frac{\partial M}{\partial y} \overset{?}{=} \frac{\partial N}{\partial x} \tag{13}$$

is a *conclusive test for exactness*.

EXAMPLE 6 Show that the DE $(4x - 2y + 5)\,dx - (2x - 2y)\,dy = 0$ is exact and find a general solution.

Solution We identify $M = 4x - 2y + 5$ and $N = -(2x - 2y)$, from which we calculate

$$\frac{\partial M}{\partial y} = -2 = \frac{\partial N}{\partial x}.$$

Hence the equation is exact. By Theorem 2.2 a function $f(x, y)$ exists such that

$$\frac{\partial f}{\partial x} = 4x - 2y + 5 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y - 2x.$$

Integrating the first of these expressions with respect to x gives

$$f(x, y) = 2x^2 - 2xy + 5x + g(y).$$

Differentiating it with respect to y and setting the result equal to N, we find $\frac{d(\int M\,dx)}{dy}$

$$\frac{\partial f}{\partial y} = \cancel{-2x} + g'(y) = 2y \cancel{-2x}.$$

$g'(y) = 2y$

It now follows that

$$g'(y) = 2y \quad \text{or} \quad g(y) = y^2,$$

where a constant of integration is not needed since one appears in the final solution. Therefore,

$$f(x, y) = 2x^2 - 2xy + 5x + y^2,$$

and our solution is

$$2x^2 + y^2 - 2xy + 5x = C.$$

Remark. Observe in Example 6 that the solution is *not* $f(x, y) = 2x^2 + y^2 - 2xy + 5x$, but $f(x, y) = C$ where C is any constant.

EXAMPLE 7 Solve $(y^2 - 1)\,dx + (2xy - \sin y)\,dy = 0$.

Solution We first test for exactness and find

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}.$$

Thus,

$$\frac{\partial f}{\partial x} = y^2 - 1 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2xy - \sin y.$$

For the sake of variety, let us integrate the second of these expressions this time to get

$$f(x, y) = xy^2 + \cos y + h(x),$$

from which we obtain

$$\frac{\partial f}{\partial x} = y^2 + h'(x) = y^2 - 1.$$

Hence, $h'(x) = -1$ or $h(x) = -x$, and our solution is

$$xy^2 - x + \cos y = C.$$

2.3.2 Integrating Factors

Occasionally we find that while the DE of interest

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{14}$$

is not exact, it can be made so by multiplying it by a suitable function $\mu(x, y)$, called an *integrating factor*. The resulting exact DE is then of the form

$$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0, \tag{15}$$

which we consider to be "essentially equivalent" to (14) in that it has the same general solution. We do caution, however, that the use of integrating factors may result in the loss or gain of solutions to the original equation. Furthermore, except for a few special cases where the integrating factor is found by inspection, this method of solution can be very difficult.

EXAMPLE 8 Solve $(3x + 2y)\,dx + x\,dy = 0$.

Solution Testing for exactness reveals that

$$\frac{\partial M}{\partial y} = 2 \quad \text{and} \quad \frac{\partial N}{\partial x} = 1,$$

and thus the equation is not exact. However, if we multiply the DE by x, the new equation

$$(3x^2 + 2xy)\,dx + x^2\,dy = 0$$

is exact, i.e., $\partial M/\partial y = 2x = \partial N/\partial x$. Now solving by the method of Section 2.3.1, we have that $f(x, y) = x^3 + x^2 y$, and hence a general solution is

$$x^3 + x^2 y = C.$$

Mostly we determine integrating factors by recognizing certain groups as differentials of known expressions. The following formulas may be helpful in this regard:

$$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right) \tag{16}$$

$$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right) \tag{17}$$

$$y\,dx + x\,dy = d(xy) \tag{18}$$

$$\frac{y\,dx - x\,dy}{xy} = d\left(\log\frac{x}{y}\right) \tag{19}$$

$$\frac{y\,dx - x\,dy}{x^2 + y^2} = d\left(\tan^{-1}\frac{x}{y}\right) \tag{20}$$

$$\frac{2x\,dx + 2y\,dy}{x^2 + y^2} = d[\log(x^2 + y^2)] \tag{21}$$

For instance, from (16) it appears that $\mu(x, y) = 1/x^2$ is an integrating factor of

$$x\,dy - y\,dx + a(x)\,dx = 0.$$

There are no general rules for finding integrating factors. The procedure is one primarily of trial and error and educated guesses.

EXAMPLE 9 Solve $y^2\,dx - x(x\,dy - y\,dx) = 0$.

Solution The combination $x\,dy - y\,dx$ suggests division by either y^2 or x^2. Here division by y^2, accompanied by a division by x, leads to the expression

$$\frac{dx}{x} + \frac{y\,dx - x\,dy}{y^2} = 0 \quad \text{or} \quad d(\log|x|) + d\left(\frac{x}{y}\right) = 0,$$

which is an exact differential. Direct integration now gives

$$\log|x| + \frac{x}{y} = C$$

as a general solution. In this case the integrating factor is $\mu(x, y) = 1/xy^2$.

2.3.3 Linear Equations

This section can be omitted if Sections 2.4 and 2.5 are discussed.

In Chapter 1 we defined the general form of a linear DE. For first-order equations, this form reduces to

$$A_1(x)y' + A_0(x)y = F(x). \tag{22}$$

It is customary to divide (22) by $A_1(x)$ to get the more useful form

$$y' + a_0(x)y = f(x). \tag{23}$$

To solve (23), we wish to find an integrating factor $\mu(x)$ such that the left-hand member of (23) can be expressed as the derivative of a single function $\mu(x)y$. That is, we want

$$\mu(x)[y' + a_0(x)y] = \frac{d}{dx}[\mu(x)y] = \mu(x)y' + \mu'(x)y. \tag{24}$$

This condition is satisfied for some $\mu(x)$ such that

$$\mu(x)a_0(x)y = \mu'(x)y,$$

or

$$\frac{\mu'(x)}{\mu(x)} = a_0(x). \tag{25}$$

If we assume that $\mu(x) > 0$, integrating (25) gives

$$\log \mu(x) = \int a_0(x)\,dx,$$

and finally, upon exponentiation,

$$\mu(x) = \exp\left[\int a_0(x)\,dx\right]. \tag{26}$$

Of course, $\mu(x)$ is not unique, since any constant multiple of it also does the job.

Returning now to (23) multiplied by $\mu(x)$, we write

$$\frac{d}{dx}[\mu(x)y] = \mu(x)f(x).$$

Therefore, it follows that

$$\mu(x)y = \int \mu(x)f(x)\,dx + C_1,$$

or

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)f(x)\,dx + C_1\right]. \tag{27}$$

The function y given by (27) represents a general solution of Equation (23).

Remark. This solution technique has the disadvantage of not being adaptable to higher-order linear DEs. A more encompassing technique is discussed in Section 2.5.

EXAMPLE 10 Solve $y' + 2xy = x$.

Solution The integrating factor is

$$\mu(x) = \exp\left[2\int x\,dx\right] = e^{x^2}.$$

Therefore, from (27) we have

$$\begin{aligned} y &= e^{-x^2}\left[\int xe^{x^2}\,dx + C_1\right] \\ &= \frac{1}{2} + C_1e^{-x^2}. \end{aligned}$$

2.3.4 Homogeneous Equations

Let us now consider a class of DEs that can be reduced to the separable variables type by a change of variable.

If the DE

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{28}$$

has the property that

$$\begin{aligned} M(tx, ty) &= t^nM(x, y), \\ N(tx, ty) &= t^nN(x, y), \end{aligned} \tag{29}$$

then we say the functions M and N are *homogeneous functions* of degree n, and (28) is called a *homogeneous DE*.* Because of the homogeneous nature of the coefficients M and N, we can always express them as

*The term *homogeneous* as used here should not be confused with the meaning of this term as used in subsequent sections and chapters.

$$M(x, y) = x^n M\left(1, \frac{y}{x}\right),$$

$$N(x, y) = x^n N\left(1, \frac{y}{x}\right),$$

and hence (28) can be written in the form

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)} = -\frac{x^n M(1, y/x)}{x^n N(1, y/x)}$$

or, equivalently,

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right). \tag{30}$$

This last form of the DE suggests either the substitution $y = vx$ or $x = vy$. Setting $y = vx$, we get

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

through application of the chain rule. Equation (30) now becomes

$$x\frac{dv}{dx} + v = F(v),$$

which is the same as

$$\frac{dx}{x} = \frac{dv}{F(v) - v}. \tag{31}$$

Therefore, regardless of the form of the function F, we can always reduce a homogeneous equation to one in which the variables can be separated by the substitution $y = vx$ or $x = vy$.

EXAMPLE 11 Solve $(x^4 + y^4)\,dx + 2x^3y\,dy = 0$.

Solution We note first that M and N are both homogeneous functions of degree 4. If we try the substitution $y = vx$, there follows

$$(x^4 + x^4v^4)\,dx + 2x^4v(v\,dx + x\,dv) = 0,$$

or, if $x \neq 0$,

$$(1 + 2v^2 + v^4)\,dx + 2xv\,dv = 0.$$

Separating variables, we find

$$\frac{dx}{x} + \frac{2v}{(v^2 + 1)^2}dv = 0$$

which yields the solution

$$\log|x| - (v^2 + 1)^{-1} = C.$$

Replacing v with y/x now gives

$$\log|x| - \frac{x^2}{x^2 + y^2} = C, \quad x \neq 0.$$

EXAMPLE 12 Solve $y^2\,dx + (y^2 - xy + x^2)\,dy = 0$.

Solution Since the coefficient of dx is simpler than that of dy, it is somewhat easier to use $x = vy$ rather than $y = vx$. Making this substitution, we obtain

$$\frac{dv}{1 + v^2} + \frac{dy}{y} = 0,$$

which leads to

$$\arctan v + \log|y| = C$$

or

$$\arctan\left(\frac{x}{y}\right) + \log|y| = C, \quad y \neq 0.$$

EXERCISES 2.3

In problems 1–10, test for exactness and solve the equation.

1. $(3x^2 - 6xy)\,dx - (3x^2 + 2y)\,dy = 0$
2. $(2xy - \cos x)\,dx + (x^2 - 1)\,dy = 0$
3. $2xt\,dx + (x^2 - 1)\,dt = 0$
4. $(2y^2x - 3)\,dx + (2yx^2 + 4)\,dy = 0$
5. $(\cos\theta\sin\theta - \theta r^2)\,d\theta + r(1 - \theta^2)\,dr = 0$
6. $\left(1 + \log x + \dfrac{y}{x}\right)dx - (1 - \log x)\,dy = 0$
7. $(2u - e^{3v})\,du - 3(ue^{3v} - \cos 3v)\,dv = 0$
8. $3x(xy - 2)\,dx + (x^3 + 2y)\,dy = 0$
9. $(\tan x - \sin x\sin y)\,dx + \cos x\cos y\,dy = 0$

*10. $[x^{-1} + x^{-2} - y(x^2 + y^2)^{-1}]\,dx + [ye^{-y} + x(x^2 + y^2)^{-1}]\,dy = 0$

In problems 11–15, use the suggested integrating factor to solve the equation.

11. $2y(y - 1)\,dx + x(2y - 1)\,dy = 0; \quad \mu = x$
12. $(x + y)\,dx + x\log x\,dy = 0; \quad \mu = \dfrac{1}{x}$

13. $y(x + y + 1)\,dx + (x + 2y)\,dy = 0; \quad \mu = e^x$

*14. $(x^2 + 2xy - y^2)\,dx + (y^2 + 2xy - x^2)\,dy = 0; \quad \mu = \dfrac{1}{(x + y)^2}$

15. $(x^2 + y^2 - x)\,dx - y\,dy = 0; \quad \mu = \dfrac{1}{(x^2 + y^2)}$

In problems 16–22, find an integrating factor by inspection and solve the equation.

16. $x\,dx + y\,dy = y^2(x^2 + y^2)\,dy$
17. $y\,dx - x\,dy = x^3\,dx$
18. $(x + 3y)\,dx - (y - 3x)\,dy = 0$
19. $(x^4 - y)\,dx + (x^2y^2 + x)\,dy = 0$
*20. $(x^3y + 3x^2 + y)\,dx + (2xy^4 + 6y^3 + x)\,dy = 0$
*21. $(3x^5y^4 + 4y)\,dx + (2x^6y^3 + 3x)\,dy = 0$ *Hint*: Multiply by x^3y^2.
22. $(x + y)\,dx + (y - x)\,dy = 0$

In problems 23–32, verify that the given DE is linear and find its solution. (Check for linearity in both x and y.)

23. $y' + 2y = 0$
24. $y' + (\cos x)y = 0$
25. $4xy^3\,dy + dx = 0$
26. $(x^2 + 9)\,dy + xy\,dx = 0$
27. $y' + 2xy = xe^{-x^2}$
28. $y' + (\cos x)y = e^{-\sin x}$
29. $x\,dy + y\,dx = (x \sin x)\,dx$
30. $y^2\,dx + x\,dy = 5\,dy$
31. $(y^2 + 1)\,dx + xy\,dy = dy$
*32. $(x^2 - 1)y' + 2xy = \cos x$

In problems 33–40, show that the equation has homogeneous coefficients and solve the equation.

33. $(x^2 + y^2)\,dx + (x^2 - xy)\,dy = 0$
34. $xy' = y + xe^{y/x}$
35. $(u + v)\,du + (v - u)\,dv = 0$
36. $x\,dx + (y - 2x)\,dy = 0$
37. $xy\,dx + (x^2 + y^2)\,dy = 0$
*38. $(s - t)(4s + t)\,ds + s(5s - t)\,dt = 0$
*39. $(x - y\log y + y\log x)\,dx + x(\log y - \log x)\,dy = 0$
*40. $\left[x - y\,\mathrm{Arctan}\left(\dfrac{y}{x}\right)\right]dx + x\,\mathrm{Arctan}\left(\dfrac{y}{x}\right)dy = 0$

In problems 41–52, solve the equation by any method.

41. $(x - y)\,dx + (3x + y)\,dy = 0, \quad y(2) = -1$
42. $2x(y + 1)\,dx - y\,dy = 0$
43. $y(2xy^2 - 3)\,dx + (3x^2y^2 - 3x + 4y)\,dy = 0$

44. $(x - 2y)\,dx + 2(y - x)\,dy = 0$

45. $x\,dy + 3y\,dx = (x - 2)\,dx$ (Solve two ways.)

46. $(y - x)\,dx + dy = 0$ (Solve two ways.)

47. $(xy^2 + x - 2y + 3)\,dx + x^2y\,dy = 2(x + y)\,dy, \quad y(1) = 1$

48. $x^2y' - y - xy = 0$ (Solve two ways.)

49. $x\,dx - y\,dy = y^2(x^2 - y^2)\,dy$

50. $2xyy' = 1 + y^2, \quad y(2) = 3$

***51.** $3y\,dx + 2x\,dy = xy^2(x\,dy + y\,dx)$

52. $2xy\,dx + (y^2 - x^2)\,dy = 0$

***53.** Prove that

$$y' = \frac{ax + by}{cx + dy}, \quad ad - bc \neq 0$$

is an exact DE if and only if $b + c = 0$, and find a general solution in this case.

***54.** Prove that $\mu = \mu(x, y)$ is an integrating factor for $M\,dx + N\,dy = 0$ if and only if μ satisfies the relation

$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} + \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu = 0.$$

***55.** Using the result of problem 54, show that if $\mu(x, y) = x^py^q$ is an integrating factor of $M\,dx + N\,dy = 0$, then p and q must satisfy the relation

$$pyN - qxM = xy\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right).$$

56. Use the result of problem 55 to obtain an integrating factor of the form $\mu(x, y) = x^py^q$ and solve the equation.

(a) $y(xy + 1)\,dx - x\,dy = 0$ (b) $(x + 2y)\,dx - x\,dy = 0$

(c) $2y\,dx + 3x\,dy = 3x^{-1}\,dy$

***57.** Any DE of the form

$$y = xy' + f(y')$$

is called an *equation of Clairaut*.

(a) Show that a general solution of this equation is the family of straight lines

$$y = mx + f(m)$$

where m is an arbitrary constant.

(b) Show that this DE also has the *singular solution* given by the parametric equations

$$x = -f'(t), \quad y = -tf'(t) + f(t).$$

Hint: Differentiate both sides of the Clairaut equation with respect to x to obtain $[x + f'(y')]y'' = 0$.

58. Referring to problem 57, solve the following Clairaut equations:

(a) $y = xy' + \frac{1}{2}(y')^2$ (b) $y = xy' + (y')^3$ (c) $xy' - y = e^{y'}$

The general theory of linear DEs is discussed in Chapter 4. However, one can achieve a great amount of insight into this theory by first developing the corresponding theory for first-order linear equations. Although we have presented a solution technique for this type of DE in Section 2.3.3, it cannot be adapted to higher-order equations and therefore cannot provide the kind of insight into the general theory that we are seeking.

A first-order DE is said to be *linear* if it can be arranged in the form

$$A_1(x)y' + A_0(x)y = F(x). \tag{32}$$

The functions $A_0(x)$, $A_1(x)$, and $F(x)$ are known functions generally assumed to be continuous on some specified interval, and in most cases it is also assumed that $A_1(x) \neq 0$ on this same interval. Applications involving (32) usually require the solution function y to satisfy the additional condition

$$y(x_0) = k_0, \tag{33}$$

where k_0 is a known value. Although it doesn't have to be, we frequently think of the number x_0 as the initial point in the interval of interest, and thus (33) is called an *initial condition*.

We say that (32) is *homogeneous** when $F(x) \equiv 0$; similarly, when $k_0 = 0$, the initial condition (33) is said to be *homogeneous*. Any other specialization of either (32) or (33) is called *nonhomogeneous*.

If we divide each member of (32) by the function $A_1(x)$, we get

$$y' + a_0(x)y = f(x), \tag{34}$$

where $a_0(x) = A_0(x)/A_1(x)$ and $f(x) = F(x)/A_1(x)$. We will henceforth refer to (34) as the *normal form* of the linear equation.

In addition to writing the equation in normal form, notation is simplified by introducing the concept of a *differential operator* in certain situations. For instance, the symbol D is an example of a differential operator that is defined by the rule

$$Dy = y'. \tag{35}$$

A more general example of a differential operator is

$$M = D + a_0(x), \tag{36}$$

where $a_0(x)$ is any function. We interpret this operator such that

$$[D + a_0(x)]y = y' + a_0(x)y, \tag{37}$$

and thus the operator M can be used to abbreviate (34) as simply $M[y] = f(x)$. We refer to M as a *normal linear operator* on any interval for which $a_0(x)$ is con-

*The use of the term *homogeneous* as given here and in succeeding chapters is not the same as implied in Section 2.3.4.

tinuous.* (When using operator notation, it is customary to use brackets to separate the operator M from the function y.)

2.4.1 The Homogeneous Equation

If we set $f(x) \equiv 0$ in Equation (34), the resulting expression

$$y' + a_0(x)y = 0 \tag{38}$$

SOLUTION BELOW

is called the *associated homogeneous equation.* One feature of a homogeneous linear DE (common to all homogeneous linear DEs of any order) is that $y = 0$ is a solution, called the *trivial solution.* Of course, what concerns us most in practice is the possibility of nontrivial solutions.

For the special case when $a_0(x)$ is a constant, i.e., $a_0(x) = a$, then (38) becomes $y' + ay = 0$ or, equivalently,

$$y' = -ay. \tag{39}$$

The solution of (39) must therefore be a function whose derivative is a constant multiple of itself. Clearly, $y = e^{-ax}$ is a function with this property, as is

$$y = C_1 e^{-ax} \tag{40}$$

for any value of the constant C_1. Hence (40) is a general solution of (39).

In the more general case when $a_0(x)$ is not a constant, we solve (38) by separating terms involving the variables x and y (i.e., separation of variables). Hence

$$\frac{dy}{y} = -a_0(x)\,dx, \quad y \neq 0, \tag{41}$$

direct integration of which yields the solution function

$$\log|y| = -\int a_0(x)\,dx + C, \tag{42}$$

where C is a constant of integration and $\log|y|$ denotes the natural logarithm. Exponentiation of this expression leads to the explicit solution function

$$y = C_1 \exp\left(-\int a_0(x)\,dx\right), \tag{43}$$

where $\exp(x) = e^x$ and $C_1 = e^C$. Notation is simplified by writing

$$y_1(x) = \exp\left(-\int a_0(x)\,dx\right), \tag{44}$$

and then our general solution (43) becomes simply

$$y = C_1 y_1(x). \tag{45}$$

*In general, a *linear operator* M is one for which $M[C_1 f(x) + C_2 g(x) = C_1 M[f(x)] + C_2 M[g(x)]$ (see problem 27).

The function y_1 represents the only nontrivial solution (to within a multiplicative constant) of Equation (38). Thus (45) is truly a *general solution*, since it contains all solutions of (38), including the trivial solution $y = 0$ obtained by setting $C_1 = 0$. Also observe that (45) reduces to (40) when $a_0(x) = a$.

In Section 2.5 it will be necessary to distinguish the solution function given by (45) from another part of the general solution required in solving nonhomogeneous equations. Therefore, the symbol y_H will be used in that section to identify (45), while here we will continue to use the symbol y alone since it is the only solution that presently concerns us.

Remark. In obtaining the general solution (45), it was necessary in (41) to assume $y \neq 0$. Now that we have solved (38), however, it is clear that this restriction can be removed.

EXAMPLE 13 Solve $y' + 2xy = 0$.

Solution Separating the variables x and y, we find

$$\frac{dy}{y} = -2x\,dx,$$

integration of which yields

$$\log|y| = -x^2 + C$$

or, equivalently,

$$y = C_1 e^{-x^2}, \quad C_1 = e^C.$$

Of course, we could have used (43) as a solution formula, which requires only that we identify $a_0(x) = 2x$. Completing the integration in (43) then leads to the same result.

EXAMPLE 14 Solve the initial value problem

$$xy' + y = 0, \quad y(1) = 3.$$

Solution Assuming $x \neq 0$, we divide the DE by x to get the *normal form*

$$y' + \left(\frac{1}{x}\right)y = 0.$$

Thus, $a_0(x) = 1/x$, and substituting this expression for $a_0(x)$ into (43) gives us

$$y = C_1 \exp\left(-\int\frac{dx}{x}\right) = \frac{C_1}{x}, \quad x \neq 0.$$

By imposing the initial condition $y(1) = 3$ on this solution function, we obtain the value

$$y(1) = C_1 = 3.$$

Hence, the solution we seek is

$$y = \frac{3}{x}, \quad x \neq 0,$$

which is valid on any interval not containing $x = 0$.

If we wish, we can use the general solution (45) to produce an explicit solution formula for the initial value problem

$$y' + a_0(x)y = 0, \quad y(x_0) = k_0. \tag{46}$$

That is, imposing the initial condition in (46) upon the general solution $y = C_1 y_1(x)$, we find

$$y(x_0) = C_1 y_1(x_0) = k_0.$$

Assuming $y_1(x_0) \neq 0$, we obtain the value

$$C_1 = \frac{k_0}{y_1(x_0)}, \tag{47}$$

and therefore the solution of (46) can be put in the form

$$y = \frac{k_0 y_1(x)}{y_1(x_0)}, \quad y_1(x_0) \neq 0. \tag{48}$$

If $a_0(x)$ is continuous throughout an interval containing the point $x = x_0$, then it cannot happen that $y_1(x_0) = 0$ (see problem 26). In such a case, (48) is the unique solution of (46). For those cases where $a_0(x)$ is not continuous, the initial value problem (46) may still have a unique solution, but the problem may also have no solution or more than one solution. Consider the following example.

EXAMPLE 15 Solve the initial value problem $xy' = 4y$, $y(0) = 0$. $y' - \frac{4}{x}y = 0$

Solution Dividing the equation by x, we see that $a_0(x) = -4/x$ has a discontinuity at $x = 0$. Nonetheless, we can still produce the general solution

$$y = C_1 x^4,$$

which is a well-behaved function for all x. Imposing the prescribed initial condition on y, however, does not require C_1 to assume any particular value, and hence the given problem has infinitely many solutions.

We should observe that if a nonzero value was prescribed for $y(0)$, the problem would have no solution.

EXERCISES 2.4

1. Show that

(a) $e^{-\log x} = \dfrac{1}{x}, \quad x > 0$

(b) $e^{(1/2)\log x} = \sqrt{x}, \quad x > 0$

(c) $e^{-\log|\sec x|} = |\cos x|$

(d) $e^{-(x^2+2\log x)} = \dfrac{e^{-x^2}}{x^2}, \quad x > 0$

In problems 2–12, find a general solution.

2. $y' + 2y = 0$

3. $a\dfrac{dy}{dx} + by = 0 \quad (a, b \text{ constants})$

4. $y' + (\tan x)y = 0$

5. $\dfrac{dy}{dx} = xy$

6. $\dfrac{dy}{dx} = \dfrac{y}{x}$

7. $x^2y' + y = 0$

8. $4x^3y\,dx + dy = 0$

***9.** $(\log x)y\,dx + dy = 0$

10. $t\,ds + s(2t + 1)\,dt = 0$

11. $(1 + t)y' + y = 0$

12. $(x^2 + 9)y' + xy = 0$

In problems 13–24, find a solution of the given initial value problem and state the interval for which the solution is valid.

13. $y' + (\cos x)y = 0, \quad y(0) = 2$

14. $y' + (\sin x)y = 0, \quad y\left(\dfrac{\pi}{2}\right) = 1$

15. $\dfrac{dw}{dx} = 5x^4w, \quad w(0) = -7$

16. $\dfrac{dy}{dx} = -(3\sin x)y, \quad y(0) = \dfrac{1}{2}$

17. $y' + e^xy = 0, \quad y(0) = -1$

18. $z' + 4x^3z = 0, \quad z(1) = 2$

19. $y' + (6x^2 + 2x)y = 0, \quad y(1) = 1$

20. $y' + ky = 0, \quad y(0) = y_0 \quad (k, y_0 \text{ constants})$

21. $xy' - y = 0, \quad y(1) = 1$

22. $x^2y' + y = 0, \quad y(-1) = 3$

23. $(1 + x)y' + y = 0, \quad y(0) = 2$

24. $y' - (\cot x)y = 0, \quad y\left(\dfrac{\pi}{2}\right) = 1$

***25.** Given the DE $y' + a_0(x)y = 0$, it can be shown that if $a_0(x)$ has a point of discontinuity, the solutions may or may not be discontinuous at this point. In problems 21–24, the function $a_0(x)$ has certain points of discontinuity. Discuss whether the solutions of these equations are continuous or discontinuous at these points.

***26.** If $a_0(x)$ is continuous on some interval I, prove that the solution function

$$y_1(x) = \exp\left(-\int a_0(x)\,dx\right)$$

cannot vanish anywhere on I.

27. Given that f and g are differentiable functions and C_1 and C_2 are any constants, demonstrate that the linear operator $M = D + a_0(x)$ has the property

$$M[C_1f(x) + C_2g(x)] = C_1M[f(x)] + C_2M[g(x)].$$

We now consider the *nonhomogeneous* linear equation

$$M[y] \equiv y' + a_0(x)y = f(x). \tag{49}$$

The prescribed function $f(x)$ is called the nonhomogeneous term of the DE and corresponds in physical problems to an externally applied force or stimulus. Thus it is often called the *forcing function*.

Suppose y_P is any *particular solution* (not necessarily involving any arbitrary constant) of the DE. Then if $y_H = C_1y_1(x)$ denotes a general solution of the associated homogeneous equation $M[y] = 0$, it follows that

$$y = y_P + y_H = y_P + C_1y_1(x) \tag{50}$$

is a *general solution* of (49). To see this, we will assume Y is any solution of (49) and define $y = Y - y_P$. The direct substitution of $y = Y - y_P$ into (49) gives

$$\begin{aligned} M[y] &= (Y - y_P)' + a_0(x)(Y - y_P) \\ &= Y' + a_0(x)Y - [y_P + a_0(x)y_P] \\ &= f(x) - f(x) = 0. \end{aligned}$$

Hence, the function $y = Y - y_P$ is a solution of the homogeneous equation $M[y] = 0$ and as such must be contained in y_H, i.e.,

$$y = Y - y_P = C_1y_1(x).$$

It now follows that

$$Y = y_P + C_1y_1(x)$$

and therefore belongs to the general family of solutions described by (50). Since Y could be any solution of (49), we conclude that (50) is a general solution. Summarizing, we have the following theorem.

Theorem 2.3 If y_P is any particular solution of the nonhomogeneous equation

$$y' + a_0(x)y = f(x),$$

and if $y_H = C_1y_1(x)$ is a general solution of the associated homogeneous equation $y' + a_0(x)y = 0$, then $y = y_P + y_H$ is a general solution of the nonhomogeneous equation.

EXAMPLE 16 Verify that $y_P = -2x - 1$ and $z_P = e^{2x} - 2x - 1$ are both particular solutions of $y' - 2y = 4x$, and find a general solution in each case.

Solution For y_P, we have

$$y_P' - 2y_P = -2 + 4x + 2 = 4x,$$

and for z_P it follows that

$$z_P' - 2z_P = 2e^{2x} - 2 - 2e^{2x} + 4x + 2 = 4x,$$

so that both functions are indeed particular solutions.

The associated homogeneous equation is $y' - 2y = 0$ with solution $y_H = C_1e^{2x}$. Thus the two general solutions obtained from this result are

$$y = y_P + y_H = -2x - 1 + C_1e^{2x}$$

and

$$y = z_p + y_H = e^{2x} - 2x - 1 + C_1e^{2x}.$$

However, in the second case we note that

$$y = -2x - 1 + (1 + C_1)e^{2x} = -2x - 1 + C_2e^{2x},$$

and so the two general solutions are actually equivalent.

From Example 16 it is clear that the particular solution of a nonhomogeneous DE is not unique. Nonetheless, any particular solution added to a general solution of the associated homogeneous equation will *always* lead to the same (or equivalent) general solution of the nonhomogeneous DE upon algebraic simplification.

Remark. The difference of two particular solutions of a linear nonhomogeneous DE is always a solution of the associated homogeneous DE, as we have already demonstrated.

Since the homogeneous solution y_H can be found by the method discussed in Section 2.4, we now concentrate on finding a particular solution of (49). In the method of constructing y_P that we employ, called *variation of parameters*, we assume the form

$$y_P = u(x)y_1(x) \tag{51}$$

for some function $u(x)$ to be determined, and where $y_1(x)$ is defined by (44) in Section 2.4. The technique derives its name from the fact that the arbitrary constant in the homogeneous solution $y = C_1y_1(x)$ is replaced by the unknown function $u(x)$.

The operator M applied to y_P leads to

$$\begin{aligned} M[y_P] &= (uy_1)' + a_0(x)uy_1 \\ &= u'y_1 + \underbrace{u[y_1' + a_0(x)y_1]}_{\text{Zero}}, \end{aligned}$$

which reduces to $M[y_P] = u'y_1$ since y_1 satisfies the homogeneous equation. Now if also $M[y_P] = f(x)$, then clearly the function $u(x)$ must be chosen such that

$$u'(x)y_1(x) = f(x),$$

or

$$u(x) = \int \frac{f(x)}{y_1(x)}\,dx + C. \tag{52}$$

Technically any constant C can appear in (52), and the resulting function will lead to a particular solution, so it is customary to set $C = 0$ and write the particular solution as

$$y_P = y_1(x) \int \frac{f(x)}{y_1(x)}\,dx. \tag{53}$$

Combining y_P with the homogeneous solution function y_H leads to the formula

$$y = y_P + y_H = y_1(x) \int \frac{f(x)}{y_1(x)}\,dx + C_1 y_1(x) \tag{54}$$

as a general solution of (49).

Remark. Motivation for the technique used in constructing y_P is difficult to give. In fact, many of the techniques commonly used in solving DEs are the result of someone having tried various "schemes," "tricks," or "devices" until something worked.

EXAMPLE 17 Find a general solution of $xy' + (1 - x)y = xe^x$.

Solution We first rewrite the equation in *normal form*,

$$y' + \left(\frac{1}{x} - 1\right)y = e^x, \quad x \neq 0,$$

and hence

$$y_1(x) = \exp\left[-\int\left(\frac{1}{x} - 1\right)dx\right] = \frac{e^x}{x}.$$

From (53) we obtain the particular solution

$$y_P = \frac{e^x}{x}\int \frac{e^x}{\frac{e^x}{x}}\,dx = \frac{e^x}{x}\int x\,dx = \frac{1}{2}xe^x,$$

which, added to the homogeneous solution $y_H = C_1 e^x/x$, gives

$$y = \frac{1}{2}xe^x + \frac{C_1 e^x}{x}, \quad x \neq 0.$$

EXAMPLE 18 Solve the initial value problem

$$dy = (\sin 2x - y\tan x)\,dx, \quad y(0) = -1.$$

Solution Rearranging terms, we have

$$y' + (\tan x)y = \sin 2x,$$

from which we immediately deduce

$$y_1(x) = \exp\left(-\int \tan x\, dx\right) = \cos x.$$

Therefore, the general solution is

$$\begin{aligned} y &= \cos x \int \frac{\sin 2x}{\cos x}\, dx + C_1 \cos x \\ &= 2\cos x \int \sin x\, dx + C_1 \cos x \\ &= -2\cos^2 x + C_1 \cos x, \end{aligned}$$

where we have made use of the identity $\sin 2x = 2 \sin x \cos x$. According to the initial condition, we find

$$-2 + C_1 = -1,$$

or $C_1 = 1$. The solution of the initial value problem is then

$$y = \cos x - 2\cos^2 x.$$

When checking the linearity of a given DE, checking for linearity in just one variable is not always enough, as our next example illustrates.

EXAMPLE 19 Find the solution of $dy/dx = 1/(x + y^2)$, which passes through the point $(-3, 0)$ of the xy-plane.

Solution The DE is not linear in y due to the presence of the term y^2. However, if we invert the equation, we get

$$\frac{dx}{dy} = x + y^2 \quad \text{or} \quad \frac{dx}{dy} - x = y^2,$$

which is *linear in x*. Thus, interchanging the roles of x and y in (54), we first calculate

$$x_1(y) = \exp\left(\int dy\right) = e^y,$$

and hence

$$x(y) = e^y \int y^2 e^{-y}\, dy + C_1 e^y = -y^2 - 2y - 2 + C_1 e^y.$$

To satisfy the prescribed condition $x(0) = -3$, we find that $C_1 = -1$, and so

$$x(y) = -y^2 - 2y - 2 - e^y.$$

2.5.1 Physical Interpretations

When an initial condition is prescribed, the problem is characterized by

$$M[y] = f(x), \quad y(x_0) = k_0, \tag{55}$$

where $M = D + a_0(x)$. As before, we assume the solution can be expressed in the form $y = y_P + y_H$, where $y_H = C_1y_1(x)$. In order to derive a general solution formula, and also to anticipate physical interpretations, it is advantageous to select a "special" particular solution function y_P. For example, if we choose to write y_P in the form

$$y_P = y_1(x)\int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt,^* \tag{56}$$

then we obtain the important property that when $x = x_0$,

$$y_P(x_0) = y_1(x_0)\int_{x_0}^{x_0} \frac{f(t)}{y_1(t)}\,dt = 0.$$

Hence we write

$$\begin{aligned} y &= y_P + y_H \\ &= y_1(x)\int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt + C_1y_1(x), \end{aligned}$$

and imposing the prescribed auxiliary condition, we find

$$y(x_0) = 0 + C_1y_1(x_0) = k_0,$$

from which we deduce $C_1 = k_0/y_1(x_0)$. Our solution formula for the initial value problem (55) can therefore be represented in the form

$$y = y_1(x)\int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt + \frac{k_0y_1(x)}{y_1(x_0)}, \tag{57}$$

provided $y_1(x_0) \neq 0$.

Remark. It may be interesting to note that the solution formula (57) for the initial value problem described by (55) depends only upon the solution function $y_1(x)$ of the associated homogeneous DE and the input parameters k_0 and $f(x)$. This is a fundamental characteristic of all linear nonhomogeneous DEs and is the primary reason why so much effort in succeeding chapters on linear DEs is directed at finding solutions of homogeneous equations.

Not only have we derived the solution formula (57) for the initial value problem (55), but our choice of y_P and y_H now leads to important physical interpretations. For instance, since $y_P(x_0) = 0$, we can think of y_P [defined by (56)] as the solution of the initial value problem

*It is customary to introduce a dummy variable of integration such as in (56) when one or both limits of integration are variable.

$$M[y] = f(x), \quad y(x_0) = 0, \tag{58}$$

wherein the initial condition is homogeneous. The physical implication of y_P, then, is that it represents the response of a system at rest which at some time $x = x_0$ is subjected to the external disturbance $f(x)$. Moreover, since y_H must now satisfy the auxiliary condition $y_H(x_0) = k_0$, it is a solution of the initial value problem

$$M[y] = 0, \quad y(x_0) = k_0, \tag{59}$$

which physically describes the behavior of a system due entirely to the initial condition, without any external disturbance.

We find, not only here but also in studying higher-order linear DEs, that it is often convenient (but not necessary) to think of nonhomogeneous problems as composed of two simpler problems—one having a nonhomogeneous DE and homogeneous auxiliary conditions [such as (58)] and the other having a homogeneous DE but nonhomogeneous auxiliary conditions [like (59)]. In this way, each solution function will describe important physical characteristics of the system being studied that may not be as discernible once the two solutions are superimposed.

EXAMPLE 20 Solve the initial value problem

$$y' - 2y = 4x, \quad y(0) = 5.$$

Solution Since $a_0(x) = -2$, we readily calculate

$$y_1(x) = \exp\left(-\int a_0(x)\,dx\right) = \exp\left(2\int dx\right) = e^{2x}.$$

Therefore the homogeneous solution is $y_H = C_1e^{2x}$. If we utilize the solution formula (59), then we can apply the initial condition directly to this solution function without first finding the general solution of the nonhomogeneous DE. This action leads to

$$y_H = 5e^{2x}.$$

[We should take note of the fact that if we were using (54) to solve this problem, we could not impose the initial condition at this step of the solution process.] Substituting y_1 into (56), we find

$$y_P = e^{2x}\int_0^x 4te^{-2t}\,dt = -2x - 1 + e^{2x},$$

and thus the solution we seek is

$$y = y_P + y_H = -2x - 1 + 6e^{2x}.$$

One of the difficulties that arises from time to time in solving initial value problems is that the integrals that need to be evaluated are not elementary. Consider the next example.

EXAMPLE 21 Solve the initial value problem

$$y' - 2xy = 1, \quad y(0) = 3.$$

Solution Direct integration of $-a_0(x) = 2x$ yields

$$y_1(x) = \exp\left(2\int x\,dx\right) = e^{x^2}.$$

Substituting this function into (57), we find

$$y = e^{x^2}\int_0^x e^{-t^2}\,dt + 3e^{x^2},$$

but the integral to be evaluated is no longer an elementary integral. When this happens, we generally leave the solution in integral form. If numerical values of y are required, they can be obtained by numerical integration techniques, such as Simpson's rule. In this instance, however, the integral is related to a function known as the *error function*, which has been extensively tabulated. This function is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt,$$

so that we can express our solution of the initial value problem as

$$y = \left[\frac{1}{2}\sqrt{\pi}\operatorname{erf}(x) + 3\right]e^{x^2}.$$

This last example illustrates that even simple-looking DEs may lead to non-elementary solution functions such as the error function. A host of other functions of this type falls under the category of *special functions*. Many of these other special functions also arise in the process of solving a DE that does not possess solutions expressible as elementary functions. As long as these new functions are tabulated, we can treat them essentially as known functions. In computing numerical values of functions, there is little difference between consulting a table to evaluate an exponential or trigonometric function and consulting a different table to evaluate, say, the error function.

2.5.2 Existence-Uniqueness Theorem

To begin, we prove the following lemma.

Lemma 2.1 The homogeneous initial value problem

$$y' + a_0(x)y = 0, \quad y(x_0) = 0$$

has only the trivial solution $y = 0$ on any interval containing $x = x_0$ for which $a_0(x)$ is a continuous function.

Proof: The general solution of the DE has been shown to be

$$y = C_1 y_1(x)$$

where

$$y_1(x) = \exp\left[-\int a_0(x)\,dx\right].$$

Since $a_0(x)$ is assumed to be continuous, its integral is also continuous, and therefore y_1 is always positive. Imposing the initial condition now yields

$$y(x_0) = C_1 y_1(x_0) = 0,$$

from which we deduce that $C_1 = 0$. Hence the only solution of the initial value problem is $y = 0$. □

We are now prepared to state and prove the following important *existence-uniqueness theorem* for linear initial value problems of the first order.

Theorem 2.4 If $M = D + a_0(x)$ is a normal differential operator on the interval $x_1 < x < x_2$ containing the point $x = x_0$, and $f(x)$ is continuous on this same interval, then the initial value problem

$$M[y] \equiv y' + a_0(x)y = f(x), \quad y(x_0) = k_0$$

has exactly one solution on this interval, given by

$$y = y_1(x)\int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt + \frac{k_0 y_1(x)}{y_1(x_0)},$$

where

$$y_1(x) = \exp\left[-\int a_0(x)\,dx\right].$$

Proof: To verify that y is a solution, we observe that

$$y' = y_1(x)\frac{f(x)}{y_1(x)} + y_1'(x)\int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt + \frac{k_0 y_1'(x)}{y_1(x_0)},$$

and so it follows that

$$\begin{aligned} y' + a_0(x)y &= f(x) + \underbrace{[y_1' + a_0(x)y_1]}_{\text{Zero}}\int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt + \underbrace{[y_1' + a_0(x)y_1]}_{\text{Zero}}\frac{k_0}{y_1(x_0)} \\ &= f(x). \end{aligned}$$

To show that y is a unique solution, let us assume that u_1 and u_2 are both solutions of the initial value problem and define $z = u_1 - u_2$. The function z then satisfies the homogeneous equation

$$M[z] = M[u_1] - M[u_2] = f(x) - f(x) = 0$$

and the homogeneous initial condition

$$z(x_0) = u_1(x_0) - u_2(x_0) = k_0 - k_0 = 0.$$

Based upon Lemma 2.1, z is the trivial solution and $u_1 = u_2$ for all x in the interval of interest. □

Remark. Observe that the existence theorem in Section 2.1 (Theorem 2.1) also applies to the linear equation discussed here. The advantage of Theorem 2.4 over Theorem 2.1 for linear DEs is that it not only informs us of the existence of a solution but also provides us with a solution formula.

Some comments about Theorem 2.4 should be made at this point. First, the theorem not only guarantees the *existence* of a solution but also states that there is only one solution, i.e., that the solution is *unique*. Furthermore, the solution is differentiable and has a continuous derivative on the specified interval. This can be observed by writing the equation in the form

$$y' = -a_0(x)y + f(x),$$

where the right-hand side is composed of continuous functions. However, the condition that $a_0(x)$ and $f(x)$ be continuous on the interval of interest is not a necessary condition, but a sufficient condition of Theorem 2.4. For instance, in Example 17, the solution $y = \frac{1}{2}xe^x + C_1e^x/x$ is not valid at $x = 0$. Yet if we impose the initial condition $y(0) = 0$, then we can select $C_1 = 0$, and the solution reduces to the well-behaved function $y = \frac{1}{2}xe^x$ for all values of x. In other words, the situation here is similar to that discussed in Section 2.4. That is, if $a_0(x)$ is not continuous, then a unique solution may still exist, but it is also possible that either no solution exists or more than one solution exist.

When the function $f(x)$ is not continuous, there is also no guarantee that a unique solution of the initial value problem exists. And in certain applications the input function $f(x)$ commonly exhibits a discontinuity at some point due to, say, a switch being (instantaneously) turned on or off.

EXAMPLE 22 Solve the initial value problem

$$y' + y = f(x), \quad y(0) = 1,$$

where

$$f(x) = \begin{cases} 0, & 0 \leq x \leq 1 \\ 1, & x > 1. \end{cases}$$

Solution Here the input function has a jump discontinuity at $x = 1$, so we solve the problem in two parts,

$$y' + y = 0, \quad y(0) = 1, \quad 0 \leq x \leq 1$$

and

$$y' + y = 1, \quad x > 1.$$

The solution of the first problem is found to be

$$y = e^{-x}, \quad 0 \leq x \leq 1.$$

For $x > 1$, we have $y = 1 + C_2e^{-x}$, and hence we write

$$y = \begin{cases} e^{-x}, & 0 \leq x \leq 1 \\ 1 + C_2e^{-x}, & x > 1. \end{cases}$$

In order that our solution be continuous for all x, we want to select the constant C_2 such that

$$\lim_{x \to 1^+} (1 + C_2e^{-x}) = \lim_{x \to 1^-} e^{-x}.$$

Thus, $C_2 = 1 - e$, and our solution takes the form

$$y = \begin{cases} e^{-x}, & 0 \leq x \leq 1 \\ 1 + (1 - e)e^{-x}, & x > 1. \end{cases}$$

Although this solution is continuous, it is not differentiable at $x = 1$ (see Figure 2.1).

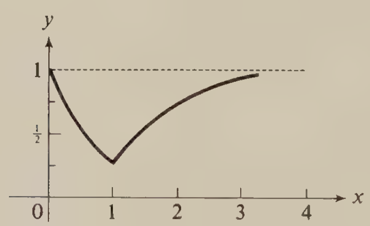


Figure 2.1

EXERCISES 2.5

In problems 1–20, find a general solution.

1. $\dfrac{dy}{dx} = 1 - 5y$ **2.** $y' - 2y = 8x$

3. $xy' + y = x\sin x$

4. $xy' - y = x^2 \sin x$

5. $xy' = 4y + x^6 e^x$

6. $(x^5 + 3y)\,dx - x\,dy = 0$

7. $z' = x - 4xz$

8. $\dfrac{dy}{dx} = \csc x + y\cot x$

9. $\dfrac{dy}{dx} = \csc x - y\cot x$

***10.** $u\,dt + (3t - tu + 2)\,du = 0$

11. $(x^2 + 9)y' + xy = x$

12. $(1 + e^y)\dfrac{dx}{dy} + e^y x = y$

13. $y' - my = e^{mx}$

14. $y' - my = e^{kx} \quad (k \neq m)$

15. $y' + 2xy = xe^{-x^2}$

16. $y' + (\cos x)y = e^{-\sin x}$

17. $xy' - ay = bx^k \quad (x > 0, \quad a \neq k)$

18. $xy' - ky = bx^k \quad (x > 0)$

19. $x^2 y' + y = 5$

***20.** $(x^2 - 1)y' + 2xy = \cos x$

In problems 21–26, solve the given initial value problem.

21. $L\dfrac{di}{dt} + Ri = E, \quad i(0) = 0 \quad (L, R, E \text{ constants})$

22. $L\dfrac{di}{dt} + Ri = A\sin\omega t, \quad i(0) = 0$

23. $(2x + 3)y' - y = (2x + 3)^{1/2}, \quad y(-1) = 0$

24. $(D - 1)y = 2xe^{2x}, \quad y(0) = 1$

25. $xy' + 2y = \sin x, \quad y\left(\dfrac{\pi}{2}\right) = 1$

***26.** $y' = x^3 - 2xy, \quad y(1) = 1$

27. Solve the initial value problem

$$y' + y = f(x), \quad y(0) = 0, \quad \text{where} \quad f(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1 \\ 0, & x > 1. \end{cases}$$

28. Solve the initial value problem

$$y' + 2xy = f(x), \quad y(0) = 2, \quad \text{where} \quad f(x) = \begin{cases} x, & 0 \leqslant x < 1 \\ 0, & x \geqslant 1. \end{cases}$$

29. Solve the initial value problem

$$(1 + x^2)y' + 2xy = f(x), \quad y(0) = 0, \quad \text{where} \quad f(x) = \begin{cases} x, & 0 \leqslant x < 1 \\ -x, & x \geqslant 1. \end{cases}$$

***30.** Given the initial value problem

$$y' + ky = A\sin\omega t, \quad y(0) = y_0 \quad (\omega \neq k),$$

(a) find its solution.

(b) The part of the solution in (a) that persists as $t \to \infty$ is called the *steady-state solution*. Find the steady-state solution.

(c) What is the steady-state solution when $\omega = k$?

31. If $y = \phi(x)$ is a solution of

$$y' + a_0(x)y = f(x)$$

and $y = \psi(x)$ is a solution of

$$y' + a_0(x)y = g(x),$$

show that $y = \phi(x) + \psi(x)$ satisfies

$$y' + a_0(x)y = f(x) + g(x).$$

32. Use the result of problem 31 to solve

$$y' + y = 2\sin x + 3e^x.$$

***33.** Show that the substitution $z = \log y$ transforms the DE

$$y' + a_0(x)y = f(x)y\log y$$

into a linear DE in the variable z.

***34.** The nonlinear DE

$$y' = P(x)y^2 + Q(x)y + R(x)$$

is called a *Riccati equation*. If $y = y_P$ is a particular solution of the equation, show that $y = y_P + 1/z$ is a general solution where z is a general solution of the first-order linear DE

$$z' + [2y_PP(x) + Q(x)]z = -P(x).$$

***35.** Referring to problem 34, find a general solution of

$$y' = y^2 + (1 - 2x)y + x^2 - x + 1,$$

given that $y_P = x$ is a particular solution.

***36.** When $P(x) = -1$, the Riccati equation (problem 34) becomes

$$y' = -y^2 + Q(x)y + R(x).$$

Show that the substitution $y = u'/u$ reduces this equation to the second-order linear DE

$$u'' - Q(x)u' - R(x)u = 0.$$

[O] 2.6 BERNOULLI'S EQUATION

The *nonlinear* DE

$$y' + a_0(x)y = f(x)y^n, \quad n \neq 0, 1, \tag{60}$$

is called *Bernoulli's equation* after the Swiss mathematician Jakob Bernoulli.* It is assumed that n is any real number except as noted. The values $n = 0$ and $n = 1$ are excluded from the discussion, since a linear equation results in each case.

*JAKOB BERNOULLI (1654–1705) was a member of the famous Bernoulli family, which produced eight mathematicians in three generations. Besides studying the equation named in his honor, Jakob's research included probability theory, the integral and differential calculus, and the calculus of variations wherein he solved the *isoperimetric problem* and, along with several other prominent mathematicians, the famous *brachistochrone problem* proposed by his younger brother Johann.

Bernoulli's equation is an example of a nonlinear equation that can be reduced to a linear equation by means of a suitable change of variable. For example, if we make the substitution

$$z = y^{1-n},$$

then

$$z' = (1-n)y^{-n}y',$$

and (60) can be reduced to the linear DE in z

$$z' + (1-n)a_0(x)z = (1-n)f(x), \tag{61}$$

which is solvable by our previous technique.

EXAMPLE 23 Solve $dy + 2xy\,dx = xe^{-x^2}y^3\,dx$.

Solution Dividing by dx, we find

$$y' + 2xy = xe^{-x^2}y^3,$$

which is a Bernoulli equation with $n = 3$. Therefore we set

$$z = y^{1-n} = y^{-2},$$

from which we obtain the linear DE

$$z' - 4xz = -2xe^{-x^2}.$$

Here we calculate

$$\begin{aligned} z &= e^{2x^2}\int(-2xe^{-3x^2})\,dx + C_1e^{2x^2} \\ &= \frac{1}{3}e^{-x^2} + C_1e^{2x^2} \end{aligned}$$

or, changing back to the dependent variable y,

$$3y^{-2} = e^{-x^2} + Ce^{2x^2}.$$

EXAMPLE 24 Solve the initial value problem

$$\frac{dP}{dt} = aP - bP^2, \quad P(0) = P_0 \quad \left(P_0 \neq \frac{a}{b}\right),$$

where a and b are fixed constants.

Solution Rearranging terms, we have

$$\frac{dP}{dt} - aP = -bP^2,$$

which is recognized as a Bernoulli equation with $n = 2$. Making the substitution $z = P^{-1}$ leads to the linear DE in z and initial condition

$$\frac{dz}{dt} + az = b, \quad z(0) = \frac{1}{P_0}.$$

Here the general solution is

$$z(t) = e^{-at} \int be^{at}\, dt + C_1 e^{-at}$$
$$= \frac{b}{a} + C_1 e^{-at}.$$

Imposing the initial condition $z(0) = 1/P_0$ on $z(t)$, we find $C_1 = 1/P_0 - b/a$, and hence

$$z(t) = \frac{bP_0 + (a - bP_0)e^{-at}}{aP_0}.$$

Since $z(t)$ and $P(t)$ are reciprocals, the solution we seek is

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}.$$

We should also observe that $P = a/b$ is a *singular solution* of the Bernoulli equation, but it does not satisfy the prescribed initial condition.

EXERCISES 2.6

Solve the given Bernoulli equation.

1. $xy' + y = x^2y^{-1}$

2. $y' = ay - by^3$ (a, b constants)

3. $x^2y' + 2xy = y^3$

4. $xy' - (1 + x)y = xy^2$

***5.** $6y^2\, dx - x(2x^3 + y)\, dy = 0$

***6.** $y' + y = (xy)^2, \quad y(0) = \frac{1}{3}$

REFERENCES

Boyce, W. E., and DiPrima, R. C. 1977. *Elementary differential equations*. 3rd ed. New York: Wiley.

Coddington, E. A. 1961. *An introduction to ordinary differential equations*. Englewood Cliffs, N. J.: Prentice-Hall.

Finney, R. L., and Ostberg, D. R. 1976. *Elementary differential equations with linear algebra*. Reading, Mass.: Addison-Wesley.

Ince, E. L. 1956. *Ordinary differential equations*. New York: Dover.

Reiss, E. L., Callegari, A. J., and Ahluwalia, D. S. 1976. *Ordinary differential equations with applications*. New York: Holt, Rinehart & Winston.

Applications of First-Order Equations

3

First-order DEs can be applied in many diverse areas of the physical and life sciences. In spite of this, we often find the same DE evolving out of our mathematical formulations of problems, regardless of the field of application. That is, the DE together with an initial condition is simply a mathematical model that someone has developed of a problem under study, and the same model can often be used to describe different phenomena, such as the motion of a free-falling body and the growth rate of a certain type of bacteria. Once the problem has been properly formulated mathematically, the origin of the problem makes little difference until we need to interpret the solution.

In Section 3.2 we discuss some general elementary applications wherein the governing DE may be either linear or nonlinear. These applications include finding *orthogonal trajectories* of a given family of curves, finding the velocity of *free-falling bodies*, applying *Torricelli's law* to the flow of water through an orifice, and finding *curves of pursuit* that describe the path of a pursuer tracking its prey.

We restrict the applications in Section 3.3 to those involving linear equations. Here we discuss problems of *growth and decay*, *free-falling bodies*, *electric circuits*, *cooling bodies*, and the *mixing of two solutions*. Again we point out that all of these areas of application have basically the same governing DE.

3.1 INTRODUCTION

Differential equations were first applied in science problems, viz., problems in mechanics. Most of these problems involve quantities that change in a *continuous* manner, such as distance, velocity, acceleration, and force. This is not the case, however, in problems in the life sciences where the quantity of interest may be a particular population size. Clearly, the total population of a community changes by discrete amounts rather than continuously; for that reason we might not expect to describe such changes by derivatives, or DEs, since these concepts are meaningful only for continuous variables. This is indeed the case for small populations, but if a population size is sufficiently large, it can often be *modeled as a continuous system*. That is to say, the continuous system may describe the general characteristics of the problem being studied and even predict certain results that can be verified experimentally. The justification for using such a model then depends simply on whether it works! Because continuous models have proven effective in a number of different problems in the life sciences, the use of DEs in this area has grown significantly in recent years.

In the application of DEs, we are concerned with more than just solving a particular initial value or boundary value problem. The complete solution process consists of the following three steps.

1. *Construction of a mathematical model*. The variables involved must be carefully defined and the governing physical laws identified. The mathematical model is then some equation(s) representing an idealization of the physical laws, taking into account some simplifying assumptions in order to make the model tractable.
2. *Solution of the mathematical equation(s)*. When permitted, exact solutions are usually desired, but in many cases one must rely on approximate solutions; in this case it is reassuring to be able to establish the existence and uniqueness of a solution of the model.
3. *Interpretation of the results*. The solutions obtained should be consistent with physical intuition and physical evidence. If a good model has been constructed, the solution should describe many of the essential characteristics of the system under study.

3.2 GENERAL APPLICATIONS

The examples in this section are typical of applications that lead to first-order DEs.

3.2.1 Orthogonal Trajectories

It is well known in analytic geometry that the slopes of perpendicular lines are negative reciprocals; i.e., $m_1 = -1/m_2$. In a more general setting, we say that two intersecting curves are *orthogonal* if and only if their tangent lines are perpendicular at the point of intersection.

Suppose we have the one-parameter family of curves defined by

$$f(x, y, c) = 0, \tag{1}$$

where each member of the family corresponds to a particular value of the parameter c. In certain applications it is important to be able to obtain a second family of curves given by

$$g(x, y, k) = 0 \tag{2}$$

with the property that all intersections of the two families are orthogonal.* The two families are then said to be *orthogonal trajectories* of each other. This means that their slopes at the points of intersection are negative reciprocals (see Figure 3.1).

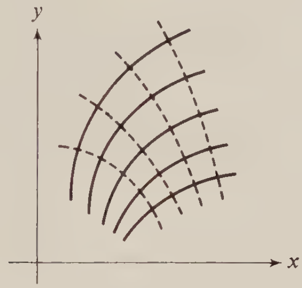


Figure 3.1 Curves and their orthogonal trajectories.

Suppose we imagine (1) to be the general solution of a DE having the form

$$y' = F(x, y). \tag{3}$$

It then follows that the DE whose general solution is (2) must be

$$y' = -\frac{1}{F(x, y)}. \tag{4}$$

The procedure is therefore to find a DE (3) for which family (1) is a general solution, and then obtain the orthogonal trajectories as solutions of (4).

EXAMPLE 1 Find the orthogonal trajectories of the family of circles

$$x^2 + y^2 = c^2.$$

Solution The constant in the equation $x^2 + y^2 = c^2$ can be eliminated by implicit differentiation, leading to

*For instance, in an electric field the lines of force are orthogonal to the equipotential curves (i.e., curves of constant potential).

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0,$$

or

$$2x + 2yy' = 0.$$

Solving for y', we find

$$y' = -\frac{x}{y},$$

and thus the DE for the family of orthogonal trajectories is

$$y' = \frac{y}{x}$$

with general solution

$$y = kx.$$

Hence, the orthogonal trajectories are straight lines passing through the origin (see Figure 3.2).

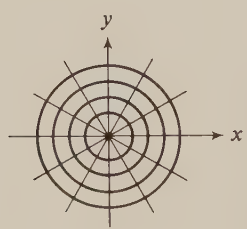


Figure 3.2

EXAMPLE 2 Find the orthogonal trajectories of the family of parabolas $y^2 = 4px$.

Solution If we rewrite the equation as $y^2/x = 4p$, then differentiation leads to the DE

$$y' = \frac{y}{2x}.$$

The orthogonal trajectories must therefore satisfy the DE

$$y' = -\frac{2x}{y},$$

leading to the family of ellipses given by $2x^2 + y^2 = k^2$ (Figure 3.3).

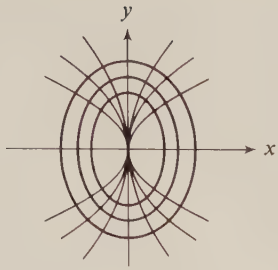


Figure 3.3

3.2.2 Free-Falling Bodies

The basic principle of mechanics used in studying particle motion is *Newton's second law of motion*, which states that "the time rate change of momentum of a body is equal to the resultant force acting on the body." The *momentum* of a body is defined by the product mv, where m denotes the mass of the body and v its velocity, so that in symbols Newton's law reads

$$\frac{d}{dt}(mv) = F.$$

If the mass is considered constant, this equation becomes

$$m\frac{dv}{dt} = F.$$

Since $a = dv/dt$ is the acceleration of the body, Newton's second law is often formulated

$$F = ma.$$

Consider the problem of a free-falling body that is acted upon only by the force of gravity. If the body is close to the surface of the earth, the weight of the body is essentially the constant mg, where g is the *gravitational constant*. It has been determined experimentally that at sea level $g \cong 32$ ft/s^2 (English system) or $g \cong 980$ cm/s^2 (metric system). Newton's law in this case is simply

$$m\frac{dv}{dt} = mg,$$

where our sign convention is such that the *positive* direction is *downward*.

If the body encounters air resistance as it falls, then the weight mg must be offset somewhat by a resistive force F_R. In such a case the governing equation is

$$m\frac{dv}{dt} = mg - F_R. \tag{5}$$

The amount of air resistance depends upon the velocity of the body, but a general law expressing this dependency is not known. We often make the assumption that $F_R = cv$ or $F_R = cv^2$, where c is some positive constant.

EXAMPLE 3 A body of mass m is dropped from a height of 5000 feet. If the air resistance is described by $F_R = mv^2/40$, find the velocity of the body for any time t.

Solution At time $t = 0$, the velocity of the body is assumed to be $v = 0$. At any later time, the velocity is described by the solution of the DE

$$\cancel{m}\frac{dv}{dt} = \cancel{m}g - \frac{\cancel{m}v^2}{40}.$$

Separating the variables, we find

$$\frac{dv}{40g - v^2} = \frac{1}{40}dt,$$

which leads to

$$\frac{1}{4\sqrt{10g}} \log\left(\frac{2\sqrt{10g} + v}{2\sqrt{10g} - v}\right) = \frac{1}{40}t + C.$$

Imposing the auxiliary condition $v = 0$ when $t = 0$, we see that $C = 0$. Hence we rewrite this last expression as

$$\frac{2\sqrt{10g} + v}{2\sqrt{10g} - v} = e^{\sqrt{g/10}\,t},$$

and, solving for v, we get the desired result

$$v = 2\sqrt{10g}\left(\frac{e^{\sqrt{g/10}\,t} - 1}{e^{\sqrt{g/10}\,t} + 1}\right)$$

$$= 2\sqrt{10g}\left(\frac{1 - e^{-\sqrt{g/10}\,t}}{1 + e^{-\sqrt{g/10}\,t}}\right).$$

Observe that as $t \to \infty$, we find $v \to 2\sqrt{10g}$. This is called the *terminal velocity*, or *limiting velocity*, of the body.

At large radial distances r from the center of the earth (distances beyond sea level), the weight of a body differs from the weight at sea level, since the acceleration a is not equal to the constant g. According to *Newton's law of gravitation*, the acceleration of a body is inversely proportional to the square of the distance from the center of the earth. In symbols we write

$$a = \frac{dv}{dt} = \frac{k}{r^2},$$

where k is a proportionality constant. If the body is falling to earth, the constant k is positive, since the velocity is increasing; if the body is leaving the earth, k is negative. Suppose we denote the radius of the earth by R and consider a body projected upward (Figure 3.4). Then $a = -g$ when $r = R$, and thus $-g = k/R^2$, or

$$k = -gR^2.$$

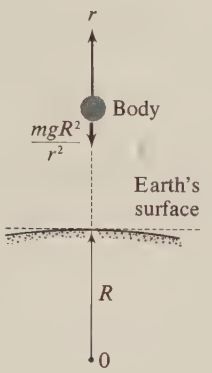


Figure 3.4

Invoking Newton's second law, we get the equation of motion

$$m\frac{dv}{dt} = -\frac{mgR^2}{r^2}, \quad r > R. \tag{6}$$

For solution purposes we wish to express the velocity v in terms of the distance variable r. To do so, we observe that

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr},$$

and thus (6) becomes

$$v\frac{dv}{dr} = -\frac{gR^2}{r^2}, \quad r > R. \tag{7}$$

If initially the body left the earth's surface with velocity v_0, then we have the auxiliary condition $v(R) = v_0$. Hence, a particle projected radially outward from the earth's surface with velocity v_0 will have the velocity given by

$$v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR, \quad r > R \tag{8}$$

when the body is r units from the center of the earth.

To calculate whether the velocity v_0 of the body is sufficient for the body to escape the earth's gravity, we note that the velocity v must remain positive, for otherwise the body will stop and then fall back to earth with negative velocity. The critical initial velocity v_0 is such that $v_0^2 - 2gR \geqslant 0$, or $v_0 \geqslant \sqrt{2gR}$. The minimum value of v_o is

$$v_e = \sqrt{2gR}, \tag{9}$$

called the *velocity of escape.*

Remark. Observe that air resistance was neglected in obtaining (9).

EXAMPLE 4 Given that $R = 3960$ miles and $g = 32$ ft/s^2, determine the escape velocity of the earth.

Solution Converting units, we have that $g = 32$ ft/s^2 corresponds roughly to $g = 6 \times 10^{-3}$ mi/s^2. Therefore,

$$v_e = \sqrt{(2)(6 \times 10^{-3})(3960)} \cong 6.9 \text{ mi/s}.$$

Of course, the escape velocity for other heavenly bodies, such as the moon or Mars, will be different, since both g and R are different for these bodies.

3.2.3 Flow of Water Through an Orifice

Consider a tank filled with water that is pouring out near the bottom of the tank through an orifice. *Torricelli's law** states that the velocity with which water issues from an orifice is

$$v = 0.6\sqrt{2gh},$$

where g is the gravitational constant and h is the instantaneous height of water above the orifice (Figure 3.5). Hence, if V is the instantaneous volume of water in the tank, it follows that

$$\frac{dV}{dt} = -Av = -0.6A\sqrt{2gh}, \tag{10}$$

where A is the cross-sectional area of the orifice.

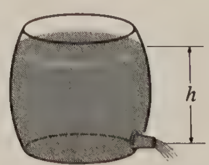


Figure 3.5

*Named in honor of the Italian physicist Evangelista Torricelli (1608–1647).

For solution purposes it may be advantageous to express V in terms of the height h and use the chain rule

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}.$$

If $B(h)$ denotes the cross-sectional area of the tank at height h, then $dV/dh = B$, and (10) becomes

$$B\frac{dh}{dt} = -0.6A\sqrt{2gh}. \tag{11}$$

EXAMPLE 5 How long will it take to empty a cylindrical tank of radius $\frac{1}{2}$ foot and vertical axis 2 feet if the tank is initially full of water and the orifice is a $\frac{1}{3}$-inch hole in the bottom of the tank?

Solution The volume of the water in the tank at a height of h units is $V = \pi(\frac{1}{2})^2 h$. Therefore

$$\frac{dV}{dh} = \frac{1}{4}\pi,$$

and (11) becomes

$$\frac{1}{4}\pi\frac{dh}{dt} = -0.6\pi\left(\frac{1}{72}\right)^2\sqrt{2(32)h}.$$

Separating variables, we obtain the general solution

$$2h^{1/2} = C - \frac{t}{270}.$$

The initial condition is $h(0) = 2$, so that $C = 2\sqrt{2}$. Solving for t when $h = 0$ then leads to the result

$$t = 270(2\sqrt{2}) \cong 764 \text{ seconds},$$

or approximately 12 minutes and 44 seconds to empty the tank.

3.2.4 Curves of Linear Pursuit

A *curve of pursuit* is a path generated by a point P that is always moving in the direction of a second point Q constrained to move along a prescribed path. In other words, it is the path of a pursuer tracking its prey. The problem of finding such a curve seems to have originated with Leonardo da Vinci in the fifteenth century, but its curious difficulties still intrigue modern mathematicians.

The general problem of determining pursuit curves is very difficult, but certain special cases lend themselves to solution methods that we have already discussed. For simplicity, let us assume the "prey" is located at Q and constrained to move

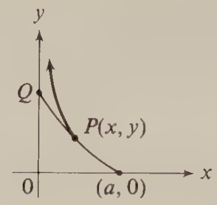


Figure 3.6 Pursuit curve.

along the y-axis (see Figure 3.6). The pursuer is assumed to be at the point $(a, 0)$ when Q is at the origin. It is further assumed that the speeds of the pursuer and prey are always in the same constant ratio.

To illustrate, let us imagine a large field in which a fox is located at the point $(a, 0)$. He spots a rabbit at $(0, 0)$ running along the y-axis in the positive direction with constant speed v. The fox immediately runs toward the rabbit with speed w, and after t seconds the rabbit is now at $Q(0, vt)$ and the fox at $P(x, y)$. The line $\overline{PQ}$ is always tangent to the path of the fox and therefore has the slope

$$y' = \frac{y - vt}{x}. \tag{12}$$

In order to solve (12), we need to eliminate the parameter t. To do so, we observe that the length of the path traveled by the fox at speed w can be computed by the arc length formula

$$wt = \int_x^a \sqrt{1 + (y')^2}\, dx. \tag{13}$$

Solving (12) and (13) for t and equating the resulting expressions, we find

$$\frac{y - xy'}{v} = \frac{1}{w}\int_x^a \sqrt{1 + (y')^2}\, dx. \tag{14}$$

Although (14) at first appears to be formidable, let us differentiate it with respect to x to get (after simplification)

$$xy'' = \frac{v}{w}\sqrt{1 + (y')^2} = k\sqrt{1 + (y')^2},$$

where k is the ratio of the two speeds. Now setting $p = y'$ leads to

$$xp' = k\sqrt{1 + p^2},$$

which can be separated according to

$$\frac{dp}{\sqrt{1 + p^2}} = \frac{k}{x}\, dx.$$

Integrating both sides of this expression, we have

$$\log\left(p + \sqrt{1 + p^2}\right) = k \log x - C_1. \tag{15}$$

At time $t = 0$, the slope of the pursuit curve is zero, so that we write $p = 0$ when $x = a$; hence, $C_1 = k \log a$. Using properties of logarithms and some algebra, we can rewrite (15) in the form

$$p = y' = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{k} - \left(\frac{x}{a}\right)^{-k}\right].$$

If we assume that the fox runs faster than the rabbit, i.e., $k < 1$, the integral of this last expression yields the general solution

$$y = \frac{1}{2}a\left[\frac{(x/a)^{1+k}}{1+k} - \frac{(x/a)^{1-k}}{1-k}\right] + C_2.$$

Since $y = 0$ when $x = a$, we deduce that $C_2 = ak/(1 - k^2)$, and thus

$$y = \frac{1}{2}a\left[\frac{(x/a)^{1+k}}{1+k} - \frac{(x/a)^{1-k}}{1-k}\right] + \frac{ak}{1-k^2}. \tag{16}$$

The fox will catch the rabbit when $x = 0$, and this happens when $y = ak/(1 - k^2)$. Those cases for which the two speeds are equal or the rabbit runs faster than the fox are taken up in the exercises.

EXERCISES 3.2

In problems 1–12, find the orthogonal trajectories of the given family of curves.

1. $xy = c$

2. $y^2 = cx^3$

3. $y^2 = x + c$

4. $x - 2y = c$

5. $x^2 - y^2 = c$

6. $e^y - e^{-x} = c$

7. $x^2 - y^2 = cx$

8. Circles through the origin with centers on the y-axis

9. $y^2 = 4x^2(1 - cx)$

10. $y = 3x - 1 + ce^{-3x}$

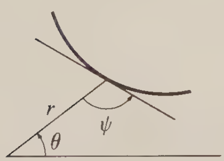
11. $x^{1/3} + y^{1/3} = c$

12. $y = \dfrac{1 + cx}{1 - cx}$

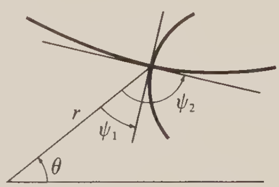
***13.** In the calculus it is shown that the angle ψ measured positive in the counterclockwise direction from the radius vector to the tangent line at a point satisfies the relation (Figure a)

$$\tan \psi = r\frac{d\theta}{dr},$$

where r and θ are polar coordinates. If two curves in polar coordinates are orthogonal, show that (Figure b)



Problem 13(a)



Problem 13(b)

$$\tan \psi_1 = -\frac{1}{\tan \psi_2}.$$

In problems 14–18, find the orthogonal trajectories of the following families of curves using the result of problem 13.

14. $r = c(1 + \cos \theta)$ **15.** $r = c(1 - \sin \theta)$ **16.** $r = 2c \cos \theta$

17. $r = c \cos^2 \theta$ **18.** $r^2 = c \sin 2\theta$

***19.** A family of curves that intersect another family of curves at a constant angle $\alpha \neq 90°$ are called *isogonal trajectories*. If $y' = F(x, y)$ is the DE of the given family of curves, show that

$$y' = \frac{F(x, y) \pm \tan \alpha}{1 \mp F(x, y) \tan \alpha}$$

is the DE of the isogonal family.

***20.** Referring to problem 19, find the isogonal families of the following families of curves.

(a) $y(x + c) = 1; \quad \alpha = 45°$

(b) $y = cx; \quad \alpha = 45°$

(c) $y = cx; \quad \alpha = 30°$

(d) $y^2 = x + c; \quad \alpha = \tan^{-1}(4)$

21. A given family of curves is said to be *self-orthogonal* if it has the property that its family of orthogonal trajectories is the same as that of the given family. Verify that the family of parabolas $y^2 = 2cx + c^2$ is self-orthogonal.

22. Solve Example 3 when the resistive force is given by $F_R = mv/40$. Find the limiting velocity of the body in this case.

23. Solve Example 3 when the resistive force is cv^2. What is the limiting velocity?

24. Suppose a parachutist falls from rest toward the earth. Assume the chutist weighs 160 pounds and his speed is 30 ft/s at the instant the chute opens ($t = 0$). If the air resistance is $F_R = cv^2$,

(a) determine the velocity v at any later time t.

(b) What is the skydiver's limiting velocity as $t \to \infty$?

(c) Calculate the limiting velocity if the air resistance is cv instead of cv^2.

25. A parachutist weighing 160 pounds falls from rest toward the earth. Before the parachute opens, the air resistance is equal to $\frac{1}{2}v$. The chute opens 5 s later, and the air resistance changes to $5v^2/8$. Find the velocity of the skydiver

(a) before the parachute opens.

(b) after the parachute opens.

(c) If the parachute never opened, what would be the limiting velocity of the skydiver? Compare this value with the value obtained after the chute opens.

***26.** A man and his parachute together weigh 192 pounds. Assume a safe landing velocity is 16 ft/s and that the air resistance is known to be proportional to the square of the velocity, equaling $\frac{1}{2}$ pound for each square foot of cross-sectional area of the parachute when it is moving 20 ft/s. What is the cross-sectional area of the parachute necessary for the chutist to make a safe landing?

27. If it takes time T for a ball thrown upward to reach its highest point ($F_R = 0$), show that the return time is also T. What is the velocity of the ball upon return if the initial velocity is v_0?

28. Determine the escape velocity from the moon given that the moon's radius is roughly 1080 miles and the acceleration of gravity is $0.165g$, where g is the acceleration of gravity on the earth's surface.

29. Determine the escape velocity from Mars given that its radius is 2100 miles and the acceleration of gravity is $0.38g$.

30. Given that the force of gravity on Venus is about 85% of the earth's gravity and the radius of Venus is roughly 3800 miles, determine the escape velocity.

31. At 200 miles above the earth's surface, the atmosphere offers almost no resistance. What velocity should a rocket have at this altitude in order to reach a height of 4000 miles if all its fuel is exhausted at this point?

***32.** If a body is shot straight up from the earth's surface with an initial velocity v_0 and no air resistance is assumed, show that the *rising time* t as a function of the distance r of the body from the center of the earth is given by

$$t = C - \frac{1}{B}\sqrt{Ar - Br^2} + \frac{A}{2B^{3/2}} \operatorname{Arcsin}\left(\frac{2Br - A}{A}\right),$$

where $A = 2gR^2$, $B = 2gR - v_0^2 > 0$, and C is a constant to be determined by the initial condition $t = 0$, $r = R$.
Hint: Show that $dr/dt = 1/r\sqrt{Ar - Br^2}$.

33. How long does it take to empty a cylindrical tank of height 20 feet and radius 10 feet initially full of water if the cross-sectional area of the orifice in the bottom is 1 ft^2?

34. A conical tank of circular cross section standing on its apex whose angle is 60° has an outlet of cross-sectional area 0.5 cm^2. The tank is initially full of water, and at time $t = 0$ the outlet is opened and the water flows out. Assuming the height of the tank is 1 meter, how long will it take for the tank to empty?

35. A water tank in the shape of a paraboloid of revolution measures 6 feet in diameter at the top and contains water 3 feet deep. When will the tank empty if a hole at the bottom has a 1-inch diameter?

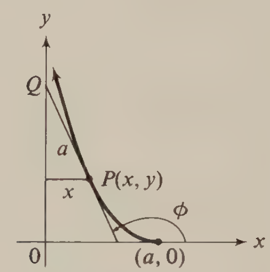
***36.** A large hemispherical cistern with a 25-foot radius is filled with water. A circular hole of radius 1 foot is cut into the bottom of the bowl. How long will it take for the cistern to empty?

37. In the fox-rabbit problem discussed in Section 3.2.4, find the path of the fox when $k = 1$; i.e., when $v = w$. Will the fox ever catch up with the rabbit?

38. In problem 37, show that the fox will never get even as close as $\frac{1}{2}a$ to the rabbit.

39. Referring to the fox-rabbit problem in Section 3.2.4, find the fox's path when $v > w$, or $k > 1$. Compute the distance between the fox and the rabbit in terms of the variable x.

***40.** A man standing at O holds a rope of length a to which a weight is attached, initially at the point $(a, 0)$. The man walks along the positive y-axis, dragging the weight after him (see figure).



Problem 40

(a) Show that the slope of the path along which the weight moves is

$$y' = -\frac{\sqrt{a^2 - x^2}}{x}.$$

Hint: Find $\tan \phi$ directly from the figure.

(b) Solve the DE in (a) to find the path of the weight. This particular curve is called a *tractrix*.

(c) Show that the solution in (b) can be expressed both in terms of a logarithm and in terms of the inverse hyperbolic secant.

41. Two skaters are located on the x-axis, Q at the origin and P at the point $(36, 0)$. Suppose that Q skates along the positive y-axis and that P skates directly toward Q at all times. If P skates twice as fast as Q, how far will Q travel before being caught by P? Answer the question if P skates 3 times as fast as Q.

***42.** A pilot always keeps the nose of his plane pointed toward a city C due west of his starting point at $(a, 0)$. If the plane's speed is v mi/h, and a wind is blowing from the south at the rate of w mi/h, show that the equation of the plane's path is

$$y = \frac{1}{2}a\left[\left(\frac{x}{a}\right)^{1-k} - \left(\frac{x}{a}\right)^{1+k}\right],$$

where $k = w/v$.

43. In problem 42, if the wind speed and plane speed are equal, show that the path is that of a parabola. Will the pilot ever reach city C?

44. Find the equation of a curve that passes through the point $(4, 1)$ and has slope $-y/(x - 3)$ at any point (x, y) on the curve.

45. Find the shape of a curved mirror such that light from a distant source will be reflected to the origin.
Hint: The slope of such a curve in the xy-plane is given by

$$y' = \frac{-x \pm \sqrt{x^2 + y^2}}{y}.$$

***46.** On a winter day it began snowing early in the morning, and the snow continued falling at a constant rate. The speed at which a snowplow can clear a road is inversely proportional to the height of the accumulated snow. The snowplow started at 11:00 A.M. and had cleared 4 miles of road by 2 P.M. Another 2 miles was cleared by 5 P.M. At what time did the snow begin?

***47.** A cable of constant density hanging from two pegs (such as a telephone line) assumes a shape determined by the DE

$$y + k\sqrt{1 + (y')^2} = 0$$

where k is a constant. Show that the cable assumes the shape of a hyperbolic cosine, called a *catenary*.

***48.** (*Brachistochrone problem*) One of the most famous problems in mechanics is called the brachistochrone problem. The problem is to determine the curve along which a particle will slide (without friction) from point O to point P in the shortest time where gravity is the only acting force. Point P is below O, but not directly beneath it. The curve that solves the problem is a solution of the nonlinear DE (derived through principles of the calculus of variations)

$$[1 + (y')^2]y = 2a,$$

where a is a constant.

(a) Solve the DE for y' using the negative square root. (Why use the negative square root?)

(b) Introduce a new variable t through the relation

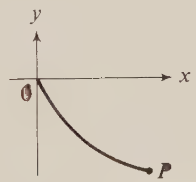
$$y = -2a\sin^2 t$$

and solve the resulting DE.

(c) By writing $\theta = 2t$, show that the solution of the original DE can be expressed in the parametric form

$$x = a(\theta - \sin\theta), \quad y = -a(1 - \cos\theta),$$

which satisfies the auxiliary condition $y(0) = 0$. These last equations are the well-known parametric equations of a *cycloid*.



Problem 48

3.3 APPLICATIONS INVOLVING LINEAR EQUATIONS

First order linear DEs occur in a wide variety of applications, some of which are discussed in the following subsections.

3.3.1 Growth and Decay Problems

The simple linear DE

$$\frac{dy}{dt} = ky \tag{17}$$

arises in numerous physical theories concerning either growth or decay of some entity. For instance, Equation (17) might describe the rate at which a radioactive substance decomposes or the rate at which temperature changes in a cooling body. This same equation might be used to predict the population growth of certain small animals over short intervals of time or to describe the growth rate of certain bacteria.

The graphs of solutions of (17) are shown in Figure 3.7 for cases when $k > 0$ and $k < 0$. Since dy/dt represents the slope of y, the sign of k gives an indication of whether the function y is increasing ($k > 0$) or decreasing ($k < 0$).

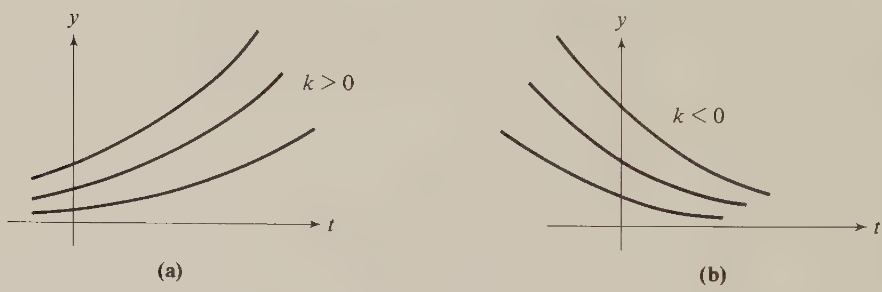


Figure 3.7 Solutions of $dy/dt = ky$.

EXAMPLE 6 (*Radioactive decay problem*) Experimental evidence indicates that a radioactive substance decays at a rate directly proportional to the amount present. Starting at time $t = 0$ with y_0 grams of undecayed matter, find the amount present at some later time.

Solution If y denotes the amount of undecayed matter at any time t, then the experimental evidence is mathematically described by the initial value problem

$$\frac{dy}{dt} = -ky, \quad y(0) = y_0, \quad k > 0$$

where the negative sign indicates that y is a decreasing function. This is a homogeneous equation with general solution

$$y = C_1 e^{-kt},$$

and application of the initial condition determines $C_1 = y_0$. Thus

$$y(t) = y_0e^{-kt}.$$

In physics, the stability of a radioactive substance is measured in terms of its *half-life*, i.e., the time it takes for one-half of the atoms in an initial sample of the substance to decompose. The longer the half-life is, the more stable the substance. For instance, radium that decomposes quite rapidly has a half-life of approximately 1700 years, whereas the isotope uranium 238 has a half-life of about 4.5×10^9 years.

An interesting application involving radioactive decay is in approximating the age of fossils. It is known that cosmic radiation interacting with nitrogen produces the isotope carbon 14 in the atmosphere. Curiously, the ratio of carbon 14 to ordinary carbon in the atmosphere is roughly constant. Thus, the same proportion of carbon 14 is found in the bodies of all breathing creatures as in the atmosphere. By comparing the amount of carbon 14 present in a fossil with the constant ratio found in the atmosphere, the age of the fossil (i.e., the time when the organism stopped breathing) can be reasonably approximated. The method, which is based upon knowledge of the half-life of carbon 14 (see problem 3 in this section), was devised by Willard Libby and won him the Nobel Prize for chemistry in 1960.

EXAMPLE 7 (*Population growth problem*) The population of a particular community is observed to increase at a rate proportional to the number of people present at any one time. In the last 5 years the population has doubled. How many years will it take for the population to triple?

Solution The governing equation is

$$\frac{dP}{dt} = aP,$$

where $P(t)$ denotes the population at any time and a is a positive constant (since P is increasing). Let us assume the population is P_0 at time $t = 0$, although we don't know its numerical value. The solution of this initial value problem is therefore

$$P(t) = P_0e^{at}.$$

Now when $t = 5$ years, it is known that $P(5) = 2P_0$, which leads to

$$2P_0 = P_0e^{5a} \quad \text{or} \quad e^{5a} = 2,$$

and solving for a yields the value

$$a = \frac{1}{5}\log 2 \cong 0.1386.$$

Thus

$$P(t) = P_0e^{0.1386t}.*$$

To find the time at which the population has tripled, we solve

$$3P_0 = P_0e^{0.1386t}$$

for t. Taking logarithms of each side of this expression, it follows that

$$0.1386t = \log 3$$

or

$$t \cong 7.9 \text{ years.}$$

The simple equation

$$\frac{dP}{dt} = aP \tag{18}$$

used in Example 7 to describe population growth is open to criticism, since prolonged exponential growth is unrealistic for most populations. Thus, Equation (18) is generally assumed valid only in the initial stages of growth but not over a long period of time.

To offset the rapid growth predicted by (18), an inhibitive factor, which is sometimes referred to as the "death rate," is usually introduced into the model. One of the earliest mathematical formulations of the problem employing such a term was suggested in 1844 by the Belgian mathematician P. F. Verhulst. He assumed that the inhibiting factor was proportional to $-P^2$, resulting in the DE

$$\frac{dP}{dt} = aP - bP^2, \tag{19}$$

where both a and b are positive constants. This equation is now known as the *logistic equation*, and its solutions are referred to as *logistic curves*.

Although (19) still generally does not provide a very accurate model for human population growth, it has proven quite effective in predicting the growth patterns (in a limited space) of fruit flies, for example, and certain types of bacteria.

The solution of (19) subject to the initial condition $P(0) = P_0$ was given in Example 24 in Section 2.6,† viz.,

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}. \tag{20}$$

*This solution can also be expressed in the equivalent form $P(t) = P_0 2^{t/5}$, which more clearly reveals that $P(t)$ doubles every 5 years.

†Equation (19) is also solvable by separation of variables.

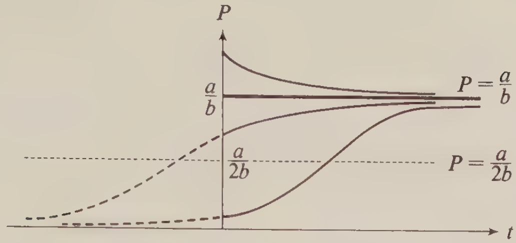


Figure 3.8 The logistic curve.

The general shape of (20) is illustrated in Figure 3.8 for the cases when $0 < P_0 < a/2b$, $a/2b < P_0 < a/b$, and $P_0 > a/b$. In the first two cases, the value $P = a/2b$ cuts the graph at a point of inflection, as can be verified by examining the second derivative $P''(t)$. Regardless of the value of P_0, we observe that

$$\lim_{t \to \infty} P(t) = \frac{a}{b} \qquad \text{(saturation level).}$$

When the initial population P_0 exceeds the value a/b, then $P(t)$ approaches this value asymptotically from above rather than from below. Finally, if the initial value of the population is a/b, it has this population value for all t.

3.3.2 Motion of a Particle and Simple Electric Circuits

Let us imagine a particle of mass m moving in a straight line. *Newton's second law of motion* states that the total force acting on the particle is proportional to the acceleration a of the particle (see also Section 3.2.2). In symbols, we write

$$F = ma, \tag{21}$$

where the mass m is the constant of proportionality. But since acceleration is the rate of change of velocity, Equation (21) can also be expressed as a first-order DE,

$$m\frac{dv}{dt} = F. \tag{22}$$

In those problems for which the force is due entirely to gravity, then $F = mg$, and (6) becomes simply

$$m\frac{dv}{dt} = mg, \tag{23}$$

where g is the gravitational constant (approximately 32 ft/s^2, or 980 cm/s^2). Sometimes the particle may also encounter a resistive force F_R due to the medium in which the motion takes place (see Figure 3.9). For example, the air resistance encountered by a parachutist is such a force, as is the resistance offered by a viscous

Figure 3.9 Free-falling body.

liquid into which a ball is dropped. Such a resistive force is frequently proportional to the velocity of the moving mass, i.e., $F_R = cv$, so that the governing equation is then modified to

$$m\frac{dv}{dt} = mg - cv, \quad c > 0,$$

or

$$\frac{dv}{dt} + \frac{c}{m}v = g. \tag{24}$$

The value of the positive constant c is determined by the nature of the resistive force.

EXAMPLE 8 (*Particle motion problem*) A parachutist weighing 150 pounds opens his chute when his downward velocity is 100 ft/s.
(a) If the force of air resistance is $25v$, find the velocity of the parachutist at any later time (prior to hitting the ground).
(b) What is the limiting velocity of the parachutist?
(c) If distance s is related to the velocity by $ds/dt = -v$ and the parachutist opens his parachute at 2000 feet, how close to the ground will he be after 5 minutes?

Solution The governing equation is

$$m\frac{dv}{dt} + 25v = mg, \quad v(0) = 100.$$

Since $g \cong 32$ ft/s^2 and the mass is $m = 150/32$ slugs,* the equation reduces to

$$\frac{dv}{dt} + \frac{16}{3}v = 32, \quad v(0) = 100.$$

(a) The solution of this initial value problem is readily found to be

$$v(t) = 6 + 94e^{-16t/3},$$

*A "slug" is a unit of mass equal to 1 lb/(ft/s^2).

which gives the velocity at all later times.
(b) The limiting velocity* is found by taking the limit of v as $t \to \infty$, i.e.,

$$v_\infty = 6 \text{ ft/s}.$$

(c) Since the position s of the parachutist is determined from the initial value problem

$$\frac{ds}{dt} = -v(t), \quad s(0) = 2000,$$

a single integration yields the general solution

$$s(t) = -6t + 17.6e^{-16t/3} + C_1.$$

Setting $t = 0$, we see that $C_1 = 1982.4$, and since 5 minutes correspond to 300 seconds, the position of the parachutist at this time is

$$s(300) \cong -1800 + 1982.4 = 182.4 \text{ feet}$$

above the ground.

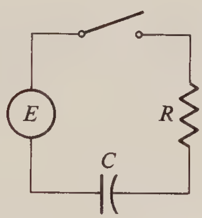


Figure 3.10 *RC* circuit.

Consider the single-loop electric circuit containing a resistor R and capacitor C connected in series with a voltage source (battery) $E(t)$, as shown in Figure 3.10. *Kirchhoff's second law*† states that the sum of instantaneous-voltage drops across each part of the circuit is equal to $E(t)$. From experimental observations, we get

$$\text{voltage drop across a resistor} = Ri,$$

$$\text{voltage drop across a capacitor} = \frac{q}{C},$$

where q denotes the electric charge on the capacitor and is related to the current i by $i = dq/dt$. Thus, the governing equation is

*From a practical point of view, the velocity approaches the limiting velocity in a relatively short interval of time and remains at this value from that point on.

†Named in honor of the German physicist, GUSTAV R. KIRCHHOFF (1824–1887).

$$Ri + \frac{q}{C} = E(t),$$

or, in terms of charge q, we find

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{25}$$

For the special case when $E(t) = E_0$, a constant, we find that (25) has the general solution

$$q(t) = e^{-t/RC}\int \frac{E_0}{R}e^{t/RC}dt + C_1e^{-t/RC} = E_0C + C_1e^{-t/RC}. \tag{26}$$

Also, since $i(t) = dq/dt$, it follows that the current is

$$i(t) = -\frac{C_1}{RC}e^{-t/RC} = i_0\, e^{-t/RC}, \tag{27}$$

where i_0 is the initial current in the circuit.

EXAMPLE 9 (*Electric circuit problem*) If a resistance of 2000 ohms and a capacitance of 5×10^{-6} farad are connected in series with a voltage source of 100 volts as shown in Figure 3.10, what is the current at any time, given that $i = 10^{-2}$ ampere at the time the switch is closed ($t = 0$). Also compute the initial charge on the capacitor.

Solution Substituting $R = 200$, $C = 5 \times 10^{-6}$, $E_0 = 100$, and $i_0 = 10^{-2}$ into (27) we get

$$i(t) = 10^{-2}\, e^{-100t}.$$

To obtain the initial charge $q(0)$, we observe that the constant C_1 appearing in (26) is related to the initial current i_0 by [see (27)]

$$C_1 = -RCi_0 = -10^{-4},$$

and therefore

$$q(t) = 5 \times 10^{-4} - 10^{-4}e^{-100t},$$

which yields the initial value $q(0) = 4 \times 10^{-4}$ volt.

3.3.3 Cooling and Mixing Problems

Newton's law of cooling states that "the rate of change of temperature u in a cooling body is proportional to the difference between u and the temperature T_0 of the surrounding medium." In symbols, this law reads

$$\frac{du}{dt} = -k(u - T_0), \tag{28}$$

where k is the proportionality constant.

EXAMPLE 10 (*Cooling problem*) A metal ball is heated to a temperature of 100°C and then immersed in water at temperature 30°C at time $t = 0$. After 3 minutes the temperature of the ball is reduced to 70°C. Find its temperature at all later times.

Solution The governing equation is (28) with $T_0 = 30$°C. (It is assumed that the water can be maintained at 30°C even with the metal ball immersed).* The solution of (28) satisfying the initial condition $u(0) = 100$ is

$$u(t) = 30 + 70e^{-kt}.$$

The additional information $u(3) = 70$ allows us to determine a value for k, i.e.,

$$70 = 30 + 70e^{-3k},$$

or

$$k = \frac{1}{3}\log\left(\frac{7}{4}\right) = 0.1865.$$

Using this value of k, we see that subsequent temperatures are given by

$$u(t) = 30 + 70e^{-0.1865t}.$$

Observe that if we wait long enough ($t \to \infty$), the temperature of the metal ball will eventually approach 30°C, the temperature of the water. From a practical point of view, this will happen in approximately 15 minutes (see Figure 3.11).

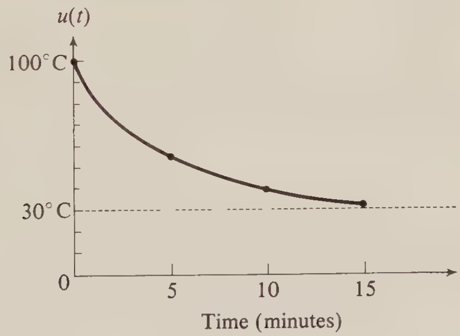


Figure 3.11

The mixing of two solutions can also give rise to a first-order linear DE. In the next example we will consider the mixing of pure water with a salt brine solution.

EXAMPLE 11 (*Mixing problem*) A certain tank contains 100 gallons of a solution of dissolved salt and water, the mixture being kept uniform by stirring. If pure water is now allowed

*This will be true if the volume of water is relatively large compared with the size of the metal ball.

to flow into the tank at the rate of 4 gal/min, and the mixture flows out at the rate of 3 gal/min (Figure 3.12), how much salt will remain in the tank after t minutes if 15 pounds of salt is initially in the mixture?

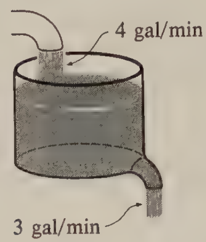


Figure 3.12

Solution Let us denote the amount of salt present in the tank at any one time by $x(t)$. The net rate at which $x(t)$ changes is given by

$$\frac{dx}{dt} = \text{rate of salt in} - \text{rate of salt out}.$$

Since pure water is coming in, the rate of salt entering the tank is zero. The rate at which salt is leaving the tank is the product of the amount of salt per gallon and the number of gallons per minute leaving the tank, i.e.,

$$\text{rate of salt out} = (3 \text{ gal/min})\left(\frac{x}{V} \text{ lb/gal}\right).$$

The governing equation is therefore

$$\frac{dx}{dt} = -3x/V, \quad x(0) = 15.$$

However, the volume V of the mixture is not constant but is determined by

$$\frac{dV}{dt} = \text{rate of liquid in} - \text{rate of liquid out} = 4 - 3 = 1,$$

or

$$V(t) = t + C_1.$$

Because at time $t = 0$ the volume is known to be 100, we find that $C_1 = 100$. Thus

$$\frac{dx}{dt} + \frac{3x}{t + 100} = 0, \quad x(0) = 15,$$

the solution of which is readily found to be

$$x(t) = \frac{15 \times 10^6}{(t + 100)^3}.$$

EXERCISES 3.3

1. The half-life of a certain radioactive substance is 1620 years. If 10 grams are initially present in a given sample, how much will be left after 162 years?

2. A certain breeder reactor converts uranium 238 into the isotope plutonium 239. After 15 years, 0.043% of the initial amount of plutonium has decayed. What is the half-life of this isotope?

3. By comparing the amount of carbon 14 found in a fossil with the constant ratio found in the atmosphere, the age of the fossil can be estimated. Assuming the half-life of carbon 14 is 5600 years, determine the approximate age of a fossil that is found to contain 0.1% of the original amount of carbon 14.

4. A certain chemical is converted into another substance at a rate proportional to the square of the amount of unconverted chemical. Starting at time $t = 0$ with an amount y_0 of unconverted chemical, determine the amount of unconverted chemical for all $t \geq 0$.

5. The population of a certain country was 1 million in the year 1950, and the instantaneous growth rate since that time is observed to be 3% of the current population. Assuming this trend continues, what population is expected by the year 2000?

6. Suppose that a certain population has grown to 10,000 after 3 years and that after 4 years the population will be double the original amount. What was the original population, and what will be the population after 10 years if it continues to grow at a rate proportional to the number of people present at any time?

*7. If we allow a population to change by either immigration into or emigration out of, then the governing equation is modified to

$$\frac{dP}{dt} = aP + f(t),$$

where $f(t)$ is the rate that members of the population are being added or subtracted from outside the system. Determine the population growth if $a = -3$ and the immigration is governed by the periodic function

$$f(t) = 1000(1 + b \sin t).$$

Assume the initial population is P_0, and discuss separately the cases $|b| < 1$ and $|b| > 1$; i.e., which case represents immigration and which emigration? Finally, determine the steady-state population by considering the limit of $P(t)$ as $t \to \infty$.

8. The infusion of glucose into the bloodstream is an important medical technique. Let us imagine glucose is infused into the bloodstream at the constant rate of b grams per minute. At the same time, the glucose is converted and removed from the bloodstream at a rate proportional to the amount of glucose present.

   (a) Show that the amount of glucose $G(t)$ present at any time is governed by the DE

   $$\frac{dG}{dt} = b - kG \quad (k \text{ constant}).$$

   (b) What concentration of glucose is attained after a sufficiently long period of time ($t \to \infty$)?

9. Suppose a sum of money M_0 is deposited into a bank that pays 6% interest.

   (a) Show that the value $M(t)$ of the investment at the end of t years is given by the expression

$$M(t) = M_0\left(1 + \frac{0.06}{k}\right)^{kt},$$

where k is the frequency (number of times each year) at which the interest is compounded.

(b) Determine the amount of money in the bank after 10 years if $k = 1$, $k = 4$, and $k = 365$.

(c) If the interest is *compounded continuously*, write the DE for $M(t)$ illustrating this growth of investment, and show that its solution is the same as obtained by allowing $k \to \infty$ in the formula given in (a). Determine the amount of money in the bank at the end of 10 years as predicted by this model.

10. In problem 9, how long will it take to double the original investment if the interest is compounded continuously? If $k = 1$?

11. A stone weighing $\frac{1}{4}$ pound falls from rest from a tall building. If the air resistance is known to be $v/160$, where v is the velocity of the stone, determine v at any time.

***12.** If the building in problem 11 is 200 feet high, what is the velocity of the stone when it hits the ground? (Approximate your answer.)

13. An object is thrown upward with an initial velocity v_0, and the air resistance is cv.

(a) Find the time required for the object to reach its maximum height.

(b) Find the maximum height if $c = \frac{1}{160}$, $mg = \frac{1}{4}$, and $v_0 = 8$.

14. In the RC circuit shown in Figure 3.10, how long will it take the current $i(t)$ to decrease to one-half its original value if the voltage source is the constant E_0?

15. Find the steady-state current in an RC circuit when the voltage source is $E(t) = E_0 \sin \omega t$. *Hint*: Let $t \to \infty$ in the solution.

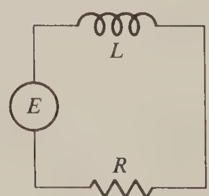
***16.** A variable resistance $R = (5 + t)^{-1}$ ohm and a capacitance of 5×10^{-6} farad are connected in series with a voltage source of 100 volts. If the initial charge q is zero, what is the charge on the capacitor after 60 seconds?

17. The current $i(t)$ in a circuit containing only a resistance R and an inductance L in series with a voltage source $E(t)$ is governed by

$$L\frac{di}{dt} + Ri = E(t), \quad i(0) = i_0,$$

where L and R are known constants (see figure). Solve this initial value problem when

(a) $E(t) = E_0$ (constant).



Problem 17

(b) $E(t) = E_0 \sin \omega t$.

(c) $R = 1$ ohm, $L = 10$ henrys, $i_0 = 6$ amperes, $E(t) = 6$ volts for $0 \leq t \leq 10$, $E(t) = 0$ for $t > 10$.

18. A thermometer reading 20°F is brought into a room kept at 72°F. Two minutes later the thermometer reads 46°F. Find the temperature reading for any time. What is the temperature reading of the thermometer after 6 minutes?

†19. A thermometer is taken from an inside room to the outside, where the temperature is 7°F. After 1 minute the thermometer reads 50°F, and after 4 minutes the reading is 32°F. What is the temperature of the inside room?

20. A pie is removed from a 350°F baking oven. If the room temperature is 75°F, how long will it take the pie to cool to 100°F if it cooled 150° in the first 4 minutes? How long will it take the pie to reach 76°? How long to reach room temperature?

21. Fifty pounds of salt are initially dissolved in a tank holding 300 gallons of water. A brine solution is pumped into the tank at the rate of 2 gal/min, and the (well-stirred) solution is allowed to flow out of the tank at the same rate. If the salt concentration entering the tank is 2 lb/gal, determine the amount of salt in the tank at any time. How much salt is present after 60 minutes? How much salt will remain after a long period of time?

22. Solve problem 21 if pure water instead of a brine mixture is pumped into the tank at the rate of 2 gal/min.

23. Solve problem 21 if the mixture is allowed to flow out of the tank

(a) at the slower rate of 1 gal/min.

(b) at the faster rate of 3 gal/min.

24. A particular tank contains 200 liters of a dye solution with a concentration of 1 g/l. Fresh water is entering the tank at the rate of 2 l/min, and the (well-stirred) solution is flowing out at the same rate. Find how much time will elapse before the dye concentration in the tank reaches 1% of its original value.

REFERENCES

Boyce, W. E., and DiPrima, R. C. 1977. *Elementary differential equations*. 3rd ed. New York: Wiley.

Coddington, E. A. 1961. *An introduction to ordinary differential equations*. Englewood Cliffs, N.J.: Prentice-Hall.

Derrick, W. R., and Grossman, S. I. 1981. *Elementary differential equations with applications*. 2nd ed. Reading, Mass.: Addison-Wesley.

Rainville, E. D., and Bedient, P. E. 1980. *Elementary differential equations*. 6th ed. New York: Macmillan.

Ross, S. L. 1980. *Introduction to ordinary differential equations*. 3rd ed. New York: Wiley.

Linear Equations of Higher Order 4

The study of linear DEs is of both theoretical and practical importance. In Chapter 2 we investigated the general theory concerning first-order linear equations and presented various applications of these equations in Chapter 3. Here we wish to build upon that theory and in some instances use it for motivational purposes for developing the higher-order linear theory. Applications of higher-order linear DEs will be taken up in subsequent chapters.

The notion of *linear independence*, which is so critical in the development of a *general solution*, is introduced in Section 4.2 for second-order equations. The question of linear independence is later reduced to evaluating a special determinant called the *Wronskian*, the nonvanishing of which is established as both a necessary and sufficient condition for linear independence of a set of solutions of a homogeneous linear DE. In Section 4.3 a method is discussed for producing a second linearly independent solution of a second-order equation given that one solution is known. Although this technique is often considered to be mostly of theoretical importance, it is used in practice from time to time.

The solution of *second-order, homogeneous, constant-coefficient equations* is discussed in Section 4.4. The significant feature here is that such equations can be solved entirely by algebraic methods. In Section 4.5 we generalize the theory and solution techniques to equations of order greater than 2.

We turn our attention to *nonhomogeneous* DEs in Sections 4.6 and 4.7. The *method of undetermined coefficients*, which requires us to guess at the structural form of the *particular solution*, is introduced first. Although this method is fairly easy to apply, its application is restricted to a narrow class of DEs. A more general method called *variation of parameters* is then introduced, which theoretically is applicable to all linear equations.

In the last section we extend the solution techniques for constant-coefficient equations to a special variable-coefficient equation called the *Cauchy-Euler equation.*

4.1 INTRODUCTION

Most nonlinear equations of order greater than 1 are very difficult to solve. This is because there is no general theory concerning the solution of these equations, and even the techniques discussed in Chapter 2 are not generally applicable. We will therefore restrict our attention in the remainder of the text almost exclusively to *linear equations.*

Linear DEs of order greater than 1 can be applied in numerous areas. Of these equations, the most prominent are those of the second order, and so we will direct most of our theory development and applications at them. For example, the forced and free oscillations of a spring-mass system are governed by the second-order linear DE

$$my'' + cy' + ky = F(t), \tag{1}$$

where m, c, and k are system parameters and $F(t)$ is a prescribed function. A similar equation arises in electric circuit problems where the circuit components may include resistors, coils, and capacitors. The static displacements of a string or wire supporting a distributed load are described by a particular second-order linear DE, as are the steady-state temperatures in a rod. Problems involving cylindrical symmetry often lead to *Bessel's equation*

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \tag{2}$$

whereas *Legendre's equation*

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \tag{3}$$

frequently occurs in physical situations featuring spherical geometry.

In the chapters that follow, we will discuss all of the above phenomena in terms of their mathematical models, and we will also discuss applications that sometimes require a DE of order greater than 2. The purpose of the present chapter, however, is to discuss appropriate solution techniques for some of these equations and to develop the general underlying theory of linear DEs.

Recall from Chapter 1 that a linear DE of order n has the form

$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \cdots + A_1(x)y' + A_0(x)y = F(x), \tag{4}$$

where the coefficients of y and its derivatives are functions of x alone. When $F(x) \neq 0$, we refer to (4) as being *nonhomogeneous*, and when $F(x)$ vanishes identically on some interval, the equation reduces to the *associated homogeneous equation*

$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \cdots + A_1(x)y' + A_0(x)y = 0. \tag{5}$$

An important property of homogeneous linear DEs is contained in the following theorem, sometimes called the *linearity property* or the *superposition principle.*

Theorem 4.1 If $y = y_1(x)$ is a solution of the homogeneous DE (5), then so is $y = C_1y_1(x)$ for any constant C_1. More generally, if $y_1, y_2, \ldots, y_k$ are all solutions of (5) on some specified interval, then

$$y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ky_k(x)$$

is also a solution on this interval for arbitrary constants $C_1, C_2, \ldots, C_k$.

Proof: For notational simplicity we will prove only the case when $n = k = 2$. Let y_1 and y_2 be two solutions of

$$A_2(x)y'' + A_1(x)y' + A_0(x)y = 0.$$

The substitution of $y = C_1y_1(x) + C_2y_2(x)$ into this DE gives

$$\begin{aligned} &A_2(x)(C_1y_1'' + C_2y_2'') + A_1(x)(C_1y_1' + C_2y_2') + A_0(x)(C_1y_1 + C_2y_2) \\ &\quad = C_1\underbrace{[A_2(x)y_1'' + A_1(x)y_1' + A_0(x)y_1]}_{\text{Zero}} + C_2\underbrace{[A_2(x)y_2'' + A_1(x)y_2' + A_0(x)y_2]}_{\text{Zero}} \\ &\quad = C_1\cdot 0 + C_2\cdot 0 = 0. \square \end{aligned}$$

In the development of the theory of linear DEs that follows, we will generally present the theory for second-order equations first and later generalize the results for equations of order n. Unfortunately, even for second-order linear DEs no general solution techniques are available, unlike the case for first-order linear equations. Only for certain narrow classes of equations can formulas for general solutions be found.

In order to develop some of the theory for second-order DEs, it is helpful to put the equation in *normal form*

$$y'' + a_1(x)y' + a_0(x)y = f(x), \tag{6}$$

which is obtained from the general linear form

$$A_2(x)y'' + A_1(x)y' + A_0(x)y = F(x) \tag{7}$$

through division by $A_2(x)$. Hence we identify $a_1(x) = A_1(x)/A_2(x)$, $a_0(x) = A_0(x)/A_2(x)$, and $f(x) = F(x)/A_2(x)$.

4.2 LINEAR DEPENDENCE AND INDEPENDENCE

In Chapter 1 we defined a *general solution* of an nth-order DE as one containing n essential constants. In the case of a second-order linear DE, a general solution turns out to be a superposition of two *linearly independent solutions*, y_1 and y_2.

Two nonzero functions, y_1 and y_2, are said to be *linearly dependent* on some interval I if they are proportional on I; that is, if

$$y_2(x) = ky_1(x) \quad (k \text{ constant}) \tag{8}$$

or, equivalently, if

$$\frac{y_2(x)}{y_1(x)} = k \quad (k \text{ constant}) \tag{9}$$

for all x in I. If y_1 and y_2 are not proportional, they are said to be *linearly independent* on I.

EXAMPLE 1 Show that $y_1 = x$ and $y_2 = 5x$ are linearly dependent on any interval while y_1 and $y_3 = x^2$ are linearly independent on any interval.

Solution In the first case, we merely observe that

$$\frac{y_2(x)}{y_1(x)} = 5 \quad \text{for all } x;$$

hence they are linearly dependent. Likewise,

$$\frac{y_3(x)}{y_1(x)} = x \neq \text{ constant},$$

which implies that y_1 and y_3 are linearly independent.

EXAMPLE 2 Discuss the linear independence of $y_1 = x$ and $y_2 = |x|$ over the intervals $0 \le x < \infty$ and $-\infty < x < \infty$ (see Figure 4.1).

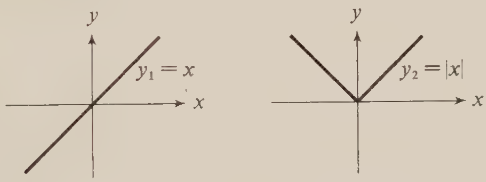


Figure 4.1

Solution Here we see that

$$\frac{y_2(x)}{y_1(x)} = \begin{cases} 1, & 0 \le x < \infty \\ -1, & -\infty < x < 0. \end{cases}$$

Thus, since the ratio y_2/y_1 remains constant over the interval $0 \le x < \infty$, these functions are clearly linearly dependent on this interval. However, over the larger interval $-\infty < x < \infty$, the ratio y_2/y_1 changes values (from -1 to $+1$), which implies linear independence on this interval.

The significance of linear independence is apparent from the following theorem, which we state without proof.

Theorem 4.2 If y_1 and y_2 are linearly independent solutions of the homogeneous linear equation

$$y'' + a_1(x)y' + a_0(x)y = 0$$

on some interval I where $a_1(x)$ and $a_0(x)$ are continuous, then the general solution of this equation is

$$y = C_1 y_1(x) + C_2 y_2(x),$$

where C_1 and C_2 are any constants. Furthermore, a set of linearly independent solutions always exists.*

EXAMPLE 3 Both $y_1 = e^x$ and $y_2 = e^{-2x}$ are solutions of

$$y'' + y' - 2y = 0.$$

What is the general solution?

Solution Note that y_1 and y_2 are linearly independent on any interval, since

$$\frac{y_2(x)}{y_1(x)} = e^{-3x} \neq \text{constant},$$

and hence the general solution is

$$y = C_1 e^x + C_2 e^{-2x},$$

where C_1 and C_2 are arbitrary constants.

4.2.1 Wronskians

When more than two functions are involved, our definitions of linear dependence and independence do not apply. Thus we seek a more general definition of these terms which can be applied to situations involving more than two functions.

Observe that if constants C_1 and C_2 (not both zero) can be found such that

$$C_1 y_1(x) + C_2 y_2(x) = 0 \tag{10}$$

for all x in some interval I, then y_1 and y_2 are linearly dependent on I. That is to say, if (10) is true, then ($C_1 \neq 0$)

$$y_1(x) = -\frac{C_2}{C_1} y_2(x),$$

which shows that y_1 and y_2 are proportional and hence linearly dependent.

Now suppose we consider the expression

$$C_1 y_1(x) + C_2 y_2(x) = 0 \tag{11a}$$

and its derivative

$$C_1 y_1'(x) + C_2 y_2'(x) = 0. \tag{11b}$$

*For a proof of Theorem 4.2, see E. L. Ince, *Ordinary Differential Equations* (New York: Dover, 1956).

If we think of C_1 and C_2 as the unknowns in (11a) and (11b), then using Cramer's rule we see that nonzero values for C_1 and C_2 are possible only when the coefficient determinant of these two equations is zero, i.e., when

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = 0. \tag{12}$$

This coefficient determinant is called the *Wronskian*, named after the Polish mathematician J. Wronski* and denoted by

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x). \tag{13}$$

The *nonvanishing* of this determinant implies the linear independence of the functions y_1 and y_2.

Theorem 4.3 If y_1 and y_2 possess first derivatives on some interval I, and

(a) if $W(y_1, y_2)(x) \neq 0$ for at least one point in I, then y_1 and y_2 are linearly independent.
(b) if y_1 and y_2 are linearly dependent on I, then $W(y_1, y_2)(x) = 0$ for all x in I.

EXAMPLE 4 Show that $y_1 = \cos x$ and $y_2 = \sin x$ are linearly independent.

Solution Computing the Wronskian of y_1 and y_2, we find

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \neq 0$$

for every value of x. Hence $\cos x$ and $\sin x$ are linearly independent.

EXAMPLE 5 Show that $y_1 = e^{m_1x}$ and $y_2 = e^{m_2x}$ are linearly independent if $m_1 \neq m_2$.

Solution Here we find the Wronskian to be

$$W(e^{m_1x}, e^{m_2x}) = \begin{vmatrix} e^{m_1x} & e^{m_2x} \\ m_1e^{m_1x} & m_2e^{m_2x} \end{vmatrix} = (m_2 - m_1)e^{(m_1+m_2)x} \neq 0$$

for all x. Thus the functions are linearly independent.

We should be aware that the conditions listed in Theorem 4.3 are only sufficient, not necessary. That is to say, the Wronskian may vanish in some cases even when the functions are linearly independent. Consider the next example.

*JOZEF M. H. WRONSKI (1778–1853) studied mathematics in Germany but lived most of his life in France. His sole lasting contribution to mathematics appears to be the Wronskian determinant.

EXAMPLE 6 Show that the functions $y_1 = x^3$ and $y_2 = |x|^3$ are linearly independent on the interval $-\infty < x < \infty$, but that the Wronskian of y_1 and y_2 is identically zero (see Figure 4.2).

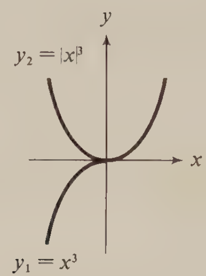


Figure 4.2

Solution By setting $x = 1$ and $x = -1$ in the expression

$$C_1 x^3 + C_2 |x|^3 = 0,$$

we find

$$C_1 + C_2 = 0,$$
$$-C_1 + C_2 = 0,$$

from which we deduce $C_1 = C_2 = 0$. Hence the functions are linearly independent.

On the other hand, if $x \geqslant 0$, then

$$W(y_1, y_2)(x) = \begin{vmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{vmatrix} = 0,$$

and if $x < 0$, then

$$W(y_1, y_2)(x) = \begin{vmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{vmatrix} = 0.$$

This shows that the converse of Theorem 4.3 is *false*.

Because our interest is confined to only those sets of functions that are solutions of linear DEs, an additional restriction can be imposed on any given set of functions, which, together with the nonvanishing of their Wronskian, imply linear independence. The only requirement is that the functions involved be solutions of a homogeneous linear DE in order that the nonvanishing of the Wronskian becomes both necessary and sufficient for the functions to be linearly independent. The proof of this result relies on the following important lemma.

Lemma 4.1 (*Abel's formula*)* If y_1 and y_2 are linearly independent solutions of

$$y'' + a_1(x)y' + a_0(x)y = 0$$

on some interval I where $a_1(x)$ and $a_0(x)$ are continuous, then the Wronskian of y_1 and y_2 is given by

$$W(y_1, y_2)(x) = C\exp\left(-\int a_1(x)\,dx\right)$$

for an appropriate value of the constant C.

Proof: Taking the derivative of the Wronskian leads to

$$\frac{d}{dx}W(y_1, y_2)(x) = \frac{d}{dx}(y_1y_2' - y_1'y_2) = y_1y_2'' - y_1''y_2,$$

and since $y'' = -a_1(x)y' - a_0(x)y$, we can rewrite this expression in the form

plug y'' into eqn. + solve for y'' - do again for y''

$$\begin{aligned}\frac{d}{dx}W(y_1, y_2) &= y_1[-a_1(x)y_2' - a_0(x)y_2] - y_2[-a_1(x)y_1' - a_0(x)y_1]\\ &= -a_1(x)(y_1y_2' - y_1'y_2)\\ &= -a_1(x)W(y_1, y_2).\end{aligned}$$

Thus the Wronskian satisfies the first-order linear DE

$$W' + a_1(x)W = 0$$

with general solution (solution of homo eqn)

$$W(y_1, y_2)(x) = C\exp\left(-\int a_1(x)\,dx\right). \square$$

EXAMPLE 7 Find the Wronskian of the solutions of

$$xy'' + y' + xy = 0$$

to within a multiplicative constant.

Solution We first rewrite the DE in normal form to get

$$y'' + \frac{1}{x}y' + y = 0.$$

Using Abel's formula, we then have

*NEILS H. ABEL (1802–1829) was one of six children born into a poor Norwegian family. One of his early accomplishments was proving that the general fifth-degree algebraic equation has no radical solution. Stricken with tuberculosis in 1827, he died two years later at the age of 26.

$$W(y_1, y_2)(x) = C\exp\left(-\int \frac{dx}{x}\right) = \frac{C}{x}.$$

Theorem 4.4 If y_1 and y_2 are solutions of

$$y'' + a_1(x)y' + a_0(x)y = 0$$

on some interval I where $a_1(x)$ and $a_0(x)$ are continuous, then y_1 and y_2 are linearly independent on I if and only if $W(y_1, y_2)(x) \neq 0$ for every x in I.

Proof: First, if $W(y_1, y_2)(x) \neq 0$ for every x in I, it follows from Theorem 4.3 that y_1 and y_2 are linearly independent. Conversely, if y_1 and y_2 are known to be linearly independent, then from Lemma 4.1 their Wronskian is given by

$$W(y_1, y_2)(x) = C\exp\left(-\int a_1(x)\,dx\right),$$

which is clearly nonzero for $C \neq 0$, and the theorem is proved. □

From Theorem 4.4 it is clear that when y_1 and y_2 are solutions of a second-order homogeneous linear DE on some interval I, either their Wronskian is identically zero or is never zero on I. Moreover, if the Wronskian is identically zero on I, it now follows that the functions y_1 and y_2 are linearly dependent.

EXERCISES 4.2

In problems 1–5, determine whether the given functions are linearly dependent or independent.

1. $y_1 = 1, \quad y_2 = x$

2. $y_1 = x^n, \quad y_2 = x^{n+1}$

3. $y_1 = \log x, \quad y_2 = \log x^2$

4. $y_1 = \sin 2x, \quad y_2 = \sin x \cos x$

5. $y_1 = e^x + e^{-x}, \quad y_2 = e^x - e^{-x}$

6. Show that the functions $y_1 = x^2$ and $y_2 = x|x|$ are linearly dependent on any interval for which either $x > 0$ or $x < 0$, but are linearly independent on $-\infty < x < \infty$.

7. Verify that $y_1 = \sin 3x$ and $y_2 = \cos 3x$ are solutions of

$$y'' + 9y = 0,$$

and show that their Wronskian satisfies Abel's formula (Lemma 4.1).

In problems 8–11, use Abel's formula (Lemma 4.1) to determine the Wronskian to within a multiplicative constant of the solutions of the given DEs.

8. $y'' - 4y' + 4y = 0$

9. $y'' - 3y' + 2y = 0$

10. $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$

11. $x^2y'' + xy' + (x^2 - n^2)y = 0$

*12. Prove Theorem 4.3.

*13. Show that if y_1 and y_2 are linearly independent functions, then

$$y_3 = y_1 + y_2 \quad \text{and} \quad y_4 = y_1 - y_2$$

are linearly independent.

*14. Prove that if y_1 and y_2 vanish at the same point in the interval $a \leq x \leq b$, they cannot form a set of linearly independent solutions of a second-order equation.

15. Given the DE

$$y'' - 6y' + 9y = 0,$$

verify that $y_1 = e^{3x}$ and $y_2 = xe^{3x}$ are linearly independent solutions and write the general solution.

16. Given the DE

$$x^2y'' + xy' - 4y = 0,$$

verify that $y_1 = x^{-2}$ and $y_2 = x^2$ are linearly independent solutions and write the general solution.

4.3 CONSTRUCTING A SECOND SOLUTION FROM A KNOWN SOLUTION

It is a curious fact that given one nontrivial solution of a second-order linear DE, a second linearly independent solution can be constructed from it.

Theorem 4.5 If $y = y_1(x)$ is a nontrivial solution of the homogeneous second-order DE

$$y'' + a_1(x)y' + a_0(x)y = 0,$$

then

$$y_2 = y_1(x) \int \frac{\exp\left(-\int a_1(x)\,dx\right)}{y_1^2(x)}\,dx$$

is a second linearly independent solution.

Proof: If y_1 and y_2 are linearly independent solutions of the DE, then

$$y_1(x)y_2' - y_1'(x)y_2 = W(y_1, y_2)(x)$$

where the right-hand side is nonzero. We wish to interpret this equation as a nonhomogeneous first-order linear DE in y_2. Thus, after dividing the equation by $y_1(x)$ to put it in normal form, we recall that a solution is given by

$$y_2 = y_1(x) \int \frac{W(y_1, y_2)(x)}{y_1^2(x)}\,dx.$$

From Lemma 4.1 it now follows that

$$y_2 = Cy_1(x) \int \frac{\exp\left(-\int a_1(x)\,dx\right)}{y_1^2(x)}\,dx.$$

Regardless of the value of C, this last expression is a solution of our second-order DE in the theorem, and so by letting $C = 1$ we obtain the desired result. $\square$

EXAMPLE 8 The function $y_1 = e^{2x}$ is a solution of the DE

$$y'' - 4y' + 4y = 0.$$

Find a second linearly independent solution.

Solution Using the result of Theorem 4.5, we obtain

$$y_2 = e^{2x} \int \frac{\exp(4 \int dx)}{e^{4x}}\,dx = e^{2x} \int \frac{e^{4x}}{e^{4x}}\,dx$$

$$(e^{2x})^2 \to e^{4x}$$

$$= e^{2x} \int dx = xe^{2x}.$$

$$y = C_1 e^{2x} + C_2 x e^{2x} = (C_1 + C_2 x) e^{2x}$$

EXAMPLE 9 Given that $y_1 = x^{-1}$ is a solution of

$$x^2y'' + 3xy' + y = 0,$$

find a general solution valid for $x > 0$.

Solution Here it is necessary to put the DE in normal form first:

$$y'' + \frac{3}{x}y' + \frac{1}{x^2}y = 0.$$

Thus $a_1(x) = 3/x$, and a second linearly independent solution is given by

$$y_2 = x^{-1} \int \frac{\exp\left(-3\int \frac{dx}{x}\right)}{x^{-2}}\,dx \qquad \Rightarrow \frac{1}{x^3}$$

$$= x^{-1} \int \frac{dx}{x}$$

$$= x^{-1} \log x.$$

Therefore, we can write

$$y = x^{-1}(C_1 + C_2 \log x), \quad x > 0$$

as a general solution.

The utility of Theorem 4.5 is clearly limited to those situations wherein one solution of the DE is known. In most situations it is just as difficult to produce one solution of a DE as it is to produce both, and so the theorem is of little help. However, occasionally one solution of a DE is obtained by "inspection," a series method, or some other means, and the theorem can then be very useful in the construction of a general solution.

EXERCISES 4.3

In problems 1–15, use Theorem 4.5 to construct a second linearly independent solution of the given DE.

1. $y'' + 2y' = 0;\quad y_1 = 1$

2. $y'' + 2y' - 3y = 0;\quad y_1 = xe^x$

3. $y'' - 6y' + 9y = 0;\quad y_1 = e^{3x}$

4. $y'' - 4y' + 4y = 0;\quad y_1 = xe^{2x}$

5. $y'' + y = 0;\quad y_1 = \sin x$

6. $y'' - y = 0;\quad y_1 = \sinh x$

7. $y'' - 2y' + 5y = 0;\quad y_1 = e^x \cos 2x$

8. $x^2y'' + 2xy' - 6y = 0;\quad y_1 = x^2$

9. $x^2y'' - 6y = 0;\quad y_1 = x^3$

10. $x^2y'' - 20y = 0;\quad y_1 = x^{-4}$

11. $4x^2y'' + y = 0;\quad y_1 = x^{1/2}\log x$

12. $x^2y'' - xy' + 2y = 0;\quad y_1 = x\sin(\log x)$

13. $(1 - x^2)y'' - 2xy' = 0;\quad y_1 = 1$

***14.** $(1 - x^2)y'' - 2xy' + 2y = 0;\quad y_1 = x$

***15.** $x^2y'' + xy' + \left(x^2 - \dfrac{1}{4}\right)y = 0;\quad y_1 = x^{-1/2}\sin x$

16. Verify that $y_1 = e^x$ is a solution of

$$xy'' - (x + n)y' + ny = 0,$$

where n is a nonnegative integer.

(a) Find a second linearly independent solution for the case when $n = 1$.

(b) Repeat (a) for the case when $n = 2$.

(c) For n a positive integer, verify the interesting result

$$y_2 = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!},$$

which is simply the first $n + 1$ terms of the Maclaurin series for e^x.

***17.** By assuming $y_2 = u(x)y_1(x)$ is a second solution of

$$y'' + a_1(x)y' + a_0(x)y = 0,$$

given that y_1 is a known solution,

(a) show that the function u satisfies

$$u'' + \left[a_1(x) + 2\frac{y_1'(x)}{y_1(x)}\right]u' = 0.$$

(b) Let $v = u'$ and solve the resulting first-order DE in v to obtain the result

$$v(x) = \frac{1}{y_1^2(x)} \exp\left[-\int a_1(x)\,dx\right].$$

(c) From (b), obtain an expression for $u(x)$ and verify that $y_2 = u(x)y_1(x)$ is the same solution as given in Theorem 4.5.

4.4 HOMOGENEOUS SECOND-ORDER EQUATIONS WITH CONSTANT COEFFICIENTS

Differential equations with constant coefficients are the easiest class of linear equations to solve by any general technique. This particular fact coupled with the fact that they occur in such a wide variety of physical applications accounts for the special place that these equations occupy in the general theory of linear equations. We have previously found that the first-order constant-coefficient linear DE

$$y' + ay = 0$$

has the exponential solution $y = C_1e^{-ax}$ valid on the interval $-\infty < x < \infty$. Because of the special property associated with derivatives of the exponential function, it may seem natural to inquire as to whether higher-order linear DEs with constant coefficients also exhibit exponential solutions.

For example, suppose we consider the second-order equation

$$ay'' + by' + cy = 0, \tag{14}$$

where a, b, and c are constants. Let us assume that (14) has an exponential solution of the form

$$y = e^{mx}$$

for some value or values of the parameter m. Direct substitution of this function into the left-hand side of (14) leads to

$$\begin{aligned} ay'' + by' + cy &= am^2e^{mx} + bme^{mx} + ce^{mx} \\ &= (am^2 + bm + c)e^{mx}, \end{aligned}$$

which can vanish if and only if (since $e^{mx} \neq 0$)

$$am^2 + bm + c = 0. \tag{15}$$

This quadratic equation in m is called the *auxiliary equation* of (14) and has the roots

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{16}$$

Clearly now, there are three separate cases to consider: whether m_1 and m_2 are real and distinct roots ($b^2 - 4ac > 0$), real but equal roots ($b^2 = 4ac$), or complex conjugate roots ($b^2 - 4ac < 0$).

Case I—Real and Distinct Roots: When the roots m_1 and m_2 of the auxiliary equation (15) are real and unequal, the solutions resulting from (14) correspond to

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}.$$

We have already shown these functions to be linearly independent in Example 5, and thus it follows that a general solution for this case is given by

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}. \tag{17}$$

EXAMPLE 10 Find a general solution of $2y'' + 5y' - 3y = 0$.

Solution The auxiliary equation is $2m^2 + 5m - 3 = 0$ or

$$(2m - 1)(m + 3) = 0.$$

Thus the roots are $m_1 = \frac{1}{2}$ and $m_2 = -3$, giving us the general solution

$$y = C_1 e^{x/2} + C_2 e^{-3x}.$$

Case II—Equal Roots: When the two roots of the auxiliary equation are equal, i.e., $m_1 = m_2 = -b/2a$, we obtain only one solution initially:

$$y_1 = e^{-bx/2a}.$$

A second linearly independent solution can be found, however, by applying Theorem 4.5, which leads to

$$y_2 = e^{-bx/2a} \int e^{bx/a} e^{-bx/a}\, dx = xe^{-bx/2a}.$$

Thus a general solution in this case is

$$y = (C_1 + C_2 x)e^{-bx/2a}. \quad (C_1 + C_2 x)e^{mx} \tag{18}$$

EXAMPLE 11 Find a general solution of $y'' - 4y' + 4y = 0$.

Solution This time the auxiliary equation is $m^2 - 4m + 4 = 0$ with the double root $m = 2$. Hence,

$$y = (C_1 + C_2 x)e^{2x}. = C_1 e^{2x} + C_2 x e^{2x}$$

Case III—Complex Conjugate Roots: If the roots m_1 and m_2 are complex, then they are complex conjugates and we can write

$$m_1 = p + iq \quad \text{and} \quad m_2 = p - iq.$$

In this case the general solution is

$$\begin{aligned} y &= C_1 e^{(p+iq)x} + C_2 e^{(p-iq)x} \\ &= e^{px}(C_1 e^{iqx} + C_2 e^{-iqx}). \end{aligned} \tag{19}$$

For physical problems, real solutions are generally preferred to (19). In order to accomplish this, we invoke the famous *Euler formulas*

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad e^{-i\theta} = \cos\theta - i\sin\theta, \tag{20}$$

to rewrite (19) as

$$\begin{aligned} y &= e^{px}[C_1(\cos qx + i\sin qx) + C_2(\cos qx - i\sin qx)] \\ &= e^{px}[(C_1 + C_2)\cos qx + i(C_1 - C_2)\sin qx] \\ &= e^{px}(C_3 \cos qx + C_4 \sin qx), \end{aligned}$$

where C_3 and C_4 are any constants. We usually relabel the constants as C_1 and C_2 once again and write

$$y = e^{px}(C_1 \cos qx + C_2 \sin qx) \tag{21}$$

as a general solution. That $y_1 = e^{px}\cos qx$ and $y_2 = e^{px}\sin qx$ are linearly independent solutions of (14) will be verified in the exercises.

EXAMPLE 12 Find a general solution of $y'' - 4y' + 13y = 0$.

Solution The auxiliary equation is $m^2 - 4m + 13 = 0$, with complex roots $m_1 = 2 + 3i$ and $m_2 = 2 - 3i$. Hence we write the general solution as

$$y = e^{2x}(C_1 \cos 3x + C_2 \sin 3x).$$

EXAMPLE 13 Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = 3, \quad y'(0) = 1.$$

Solution The auxiliary equation $m^2 - 2m + 1 = 0$ has the double root $m = 1$. The general solution is therefore

$$y = (C_1 + C_2 x)e^x.$$

By differentiating, we obtain

$$y' = (C_1 + C_2 x)e^x + C_2 e^x.$$

From these expressions for y and y' and from the initial conditions, it follows that

$$y(0) = C_1 = 3, \quad y'(0) = C_1 + C_2 = 1.$$

Hence, $C_1 = 3$ and $C_2 = -2$, leading to the desired solution

$$y = (3 - 2x)e^x.$$

EXAMPLE 14 Find general solutions of

$$y'' + k^2y = 0 \quad \text{and} \quad y'' - k^2y = 0.$$

Solution The first DE has the auxiliary equation $m^2 + k^2 = 0$ with pure imaginary roots $m_1 = ik$ and $m_2 = -ik$. Thus the general solution is

$$y = C_1 \cos kx + C_2 \sin kx.$$

In the second case the auxiliary equation is $m^2 - k^2 = 0$, with $m_1 = k$ and $m_2 = -k$ the roots. Its general solution is therefore

$$y = C_1 e^{kx} + C_2 e^{-kx}.$$

However, there is another representation of the general solution that is more convenient in particular applications. Moreover, it more closely resembles the solution of the first DE. Since the hyperbolic functions are defined by the linear combinations

$$\cosh kx = \frac{1}{2}(e^{kx} + e^{-kx}), \quad \sinh kx = \frac{1}{2}(e^{kx} - e^{-kx}),$$

and these are linearly independent functions (see problem 5, Exercises 4.2), it follows that

$$y = C_3 \cosh kx + C_4 \sinh kx$$

is also a general solution of $y'' - k^2y = 0$ (see Example 7 in Section 1.3.1).

EXAMPLE 15 Find a general solution of $y'' + y' - y = 0$.

Solution Here we get the auxiliary equation $m^2 + m - 1 = 0$, which cannot be factored. Thus we use the quadratic formula, finding that

$$m_1 = \frac{1}{2}(-1 + \sqrt{5}) \quad \text{and} \quad m_2 = \frac{1}{2}(-1 - \sqrt{5}).$$

Since these roots are distinct and real, the general solution is

$$y = C_1 \exp\left[\frac{1}{2}(-1 + \sqrt{5})x\right] + C_2 \exp\left[\frac{1}{2}(-1 - \sqrt{5})x\right].$$

However, using a device similar to that used in Case III for obtaining a general solution in terms of trigonometric functions, we can write the general solution of this problem in the alternate form

$$y = e^{-x/2}\left[C_1 \cosh\left(\frac{1}{2}\sqrt{5}x\right) + C_2 \sinh\left(\frac{1}{2}\sqrt{5}x\right)\right],$$

the verification of which is left to the reader. (See problem 20 in this section.)

4.4.1 Differential Operators

An *operator* can be defined as a function that transforms one function into another function. Familiar examples of such transformations are integration and differentiation. Since transformations of one type or another are so common in many areas of application, operator methods are playing an ever increasing role in mathematics.

Our primary purpose here is to illustrate the algebraic nature of constant-coefficient differential operators. This feature is peculiar to this class of operators and accounts for the fact that constant-coefficient DEs are solvable by purely algebraic methods.

In Chapter 2 we introduced the notion of an operator for the first time, the simplest one of which is D, where $Dy = y'$. We now consider the second-order differential operator

$$P(D) = aD^2 + bD + c \tag{22}$$

associated with the constant-coefficient equation $ay'' + by' + cy = 0$. The interesting feature of the operator $P(D)$ is that it can be manipulated according to the basic rules of algebra applied to any polynomial. To see why, let us apply the operator (22) to the exponential function $y = e^{mx}$, which leads to the relation

$$P(D)[e^{mx}] = (am^2 + bm + c)e^{mx} = P(m)e^{mx}.$$

Hence $P(D)$ and $P(m)$ have the same polynomial form and therefore factor exactly the same.

The above remarks suggest that the constant-coefficient DE

$$(2D^2 - D - 3)y = 0$$

can also be written in either of the equivalent forms

$$(D + 1)(2D - 3)y = 0$$

or $2D^2 - 3D$

$$(2D - 3)(D + 1)y = 0.$$

Thus simple inspection of the factored operator reveals that the roots of the auxiliary equation are $m_1 = -1$ and $m_2 = \frac{3}{2}$. A linear differential operator of any order with constant coefficients can also be factored in the same manner. Factorization of operators with variable coefficients, however, does not satisfy the basic laws of algebra, so one must exercise more caution when dealing with operators of this type.

EXAMPLE 16 Verify that

$$(D + 1)(2D - 3)y = (2D - 3)(D + 1)y$$

whereas

$$(xD + 1)(D - 2)y \neq (D - 2)(xD + 1)y.$$

Solution In the first case, we have

$$\begin{aligned}(D+1)(2D-3)y &= (D+1)(2y'-3y)\\ &= 2y''-y'-3y\\ &= (2D^2-D-3)y\end{aligned}$$

and

$$\begin{aligned}(2D-3)(D+1)y &= (2D-3)(y'+y)\\ &= 2y''-y'-3y\\ &= (2D^2-D-3)y,\end{aligned}$$

which shows the equivalence of the two expressions. On the other hand,

$$\begin{aligned}(xD+1)(D-2)y &= (xD+1)(y'-2y)\\ &= xy''+(1-2x)y'-2y\\ &= [xD^2+(1-2x)D-2]y\end{aligned}$$

and

$$\begin{aligned}(D-2)(xD+1)y &= (D-2)(xy'+y)\\ &= xy''+2(1-x)y'-2y\\ &= [xD^2+2(1-x)D-2]y,\end{aligned}$$

which are clearly not the same. The distinction, of course, is that in the second case the operators have variable coefficients.

EXERCISES 4.4

For problems 1–6, show that the auxiliary equation has distinct real roots and find the general solution.

1. $3y''-y'=0$
2. $y''-4y=0$
3. $y''+2y'-3y=0$
4. $2y''-5y'-3y=0$
5. $y''-10y'+17y=0$
6. $3y''-10y'+4y=0$

For problems 7–12, show that the auxiliary equation has a repeated root and find the general solution.

7. $y''-2y'+y=0$
8. $y''-10y'+25y=0$
9. $9y''+6y'+y=0$
10. $4y''+4y'+y=0$
11. $9y''-12y'+4y=0$
12. $y''-2\sqrt{2}y'+2y=0$

For problems 13–18, show that the auxiliary equation has complex roots and find the general solution.

13. $y'' + y' + y = 0$

14. $y'' + 25y = 0$

15. $y'' - 6y' + 25y = 0$

16. $y'' - 4y' + 13y = 0$

17. $2y'' - y' + y = 0$

18. $2y'' - 3y' + 10y = 0$

19. Given the two functions

$$y_1 = e^{px}\cos qx \quad \text{and} \quad y_2 = e^{px}\sin qx,$$

(a) show that they are linearly independent.

(b) Verify that $y = C_1y_1(x) + C_2y_2(x)$ satisfies the DE

$$[(D - p)^2 + q^2]y = 0.$$

20. Given the two functions

$$y_1 = e^{px}\cosh qx \quad \text{and} \quad y_2 = e^{px}\sinh qx,$$

(a) show that they are linearly independent.

(b) Verify that $y = C_1y_1(x) + C_2y_2(x)$ satisfies the DE

$$[(D - p)^2 - q^2]y = 0.$$

***21.** Show that the general solution of $y'' + y = 0$ can be expressed in the form

$$y = A\cos(x - \phi)$$

or

$$y = B\sin(x - \theta),$$

where A, ϕ, B, and θ are arbitrary constants.

***22.** Show that the solution of the initial value problem

$$y'' - 6y' + 25y = 0, \quad y(0) = -3, \quad y'(0) = -1$$

can be expressed in the form

$$y = \sqrt{13}\,e^{3x}\sin(4x - \theta),$$

where the angle θ is defined by the equations

$$\cos\theta = \frac{2}{\sqrt{13}} \quad \text{and} \quad \sin\theta = \frac{3}{\sqrt{13}}.$$

Hint: See problem 21.

In problems 23–30, perform the indicated multiplication.

23. $(D - 5)(4D + 3)$

24. $(2D + 7)(2D - 7)$

25. $(D - 1)(D^2 + D + 1)$

26. $(D + 2)(D - 1)^2$

27. $(D - x)(D + x)$

28. $(D + x)(D - x)$

29. $(xD + 3)(xD - 2)$

30. $(xD - 2)(xD + 3)$

In problems 31–35, find the general solution. Assume that x is the independent variable.

31. $(D - 5)(4D + 3)y = 0$

32. $(D^2 - 8D + 16)y = 0$

33. $(3D^2 - 14D - 5)y = 0$

34. $(4D^2 - 12D + 5)y = 0$

***35.** $(D + 1)(5D - 3)(D + 4)y = 0$

In problems 36–40, solve the DE subject to the prescribed conditions.

36. $y'' - 4y' + 13y = 0, \quad y(0) = -1, \quad y'(0) = 2$

37. $y'' - 2y' - 3y = 0, \quad y(0) = 0, \quad y'(0) = -4$

38. $y'' + k^2y = 0, \quad y(0) = y_0, \quad y'(0) = v_0$

***39.** $y'' - 3y' + 2y = 0, \quad y(1) = 0, \quad y(2) = 1$

40. $(D^2 - D - 6)y = 0, \quad y(0) = 0, \quad y(1) = e^3$

4.5 HIGHER-ORDER LINEAR EQUATIONS

Although second-order linear DEs are far more prevalent in practice than equations of higher order, there are applications wherein the mathematical model demands a DE of order greater than 2. For example, in studying the small deflections of a beam supporting a distributed load, we find that the governing DE is fourth order, as is the DE describing the buckling modes of a long, slender column under an axial compressive force.

In this section we wish to extend the theory of second-order linear equations developed in previous sections to higher-order linear equations. Much of the theory is simply a natural generalization of that developed for second-order DEs, so we will mostly state appropriate theorems without providing their proofs. In discussing the results for these higher-order equations, we find it convenient once again to put the DE in normal form, but this time we will also introduce the concept of a *differential operator*. Assuming $A_n(x)$ does not vanish anywhere on the interval of interest I, let us divide the equation

$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \cdots + A_1(x)y' + A_0(x)y = F(x) \tag{23}$$

by $A_n(x)$ to get the *normal form*

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x). \tag{24}$$

Now if the coefficients $a_0(x), a_1(x), \ldots, a_n(x)$ are continuous functions on the interval I, we say the operator M defined by

$$M = D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x), \tag{25}$$

where $D = d/dx$, is a *normal differential operator* on I. In terms of M, we can now express (24) as simply

$$M[y] = f(x). \tag{26}$$

When $f(x) = 0$ everywhere on I, we say that (26) is *homogeneous* and write

$$M[y] = 0. \tag{27}$$

Otherwise we say that (26) is a *nonhomogeneous* DE.

4.5.1 Linear Independence

Extending our notions of linear dependence and independence from Section 4.2.1, we get the following definition.

Definition 4.1 *The set of n functions $y_1, y_2, \ldots, y_n$ is said to be* ***linearly dependent*** *on the interval I if and only if there exists a set of n constants $C_1, C_2, \ldots, C_n$, not all zero, such that*

$$C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) = 0$$

for all x in I. If the set of functions is not linearly dependent, it is said to be ***linearly independent.***

Based upon Definition 4.1, the only way it can happen that

$$C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x) = 0 \tag{28}$$

when the set of functions $y_1, y_2, \cdots, y_n$ is linearly independent is for $C_1 = C_2 = \cdots = C_n = 0$. However, if the functions are linearly dependent, then at least one of them can be expressed as a linear combination of the others. For example, suppose (28) is true where at least one of the C's, say C_1, is different from zero. This being the case, we can solve for y_1, getting the result

$$y_1 = -\frac{C_2}{C_1}y_2 - \frac{C_3}{C_1}y_3 - \cdots - \frac{C_n}{C_1}y_n. \tag{29}$$

EXAMPLE 17 Show that $y_1 = 3x^2 - 8x$, $y_2 = x^2$, and $y_3 = 4x$ are linearly dependent on any interval.

Solution We see that $y_1 = 3y_2 - 2y_3$ for all x, and thus it follows that the functions are linearly dependent.

Theorem 4.6 If M is a normal differential operator of order n on an interval I, and if $y_1, y_2, \ldots, y_n$ constitute a set of linearly independent solutions of $M[y] = 0$ on I, then the general solution of this homogeneous equation on I is

$$y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x),$$

where $C_1, C_2, \ldots, C_n$ are any constants. Moreover, a set of linearly independent solutions always exists.

Remark. The set of linearly independent solutions $y_1, y_2, \cdots, y_n$ is often referred to as a *fundamental set* of solutions of $M[y] = 0$.

Next, generalizing the notion of a Wronskian to a set of n functions, we have

$$W(y_1, \ldots, y_n)(x) = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix}. \tag{30}$$

Theorem 4.7 If the set of functions $y_1, y_2, \ldots, y_n$ possesses at least $n - 1$ derivatives on some interval I, and

(a) if $W(y_1, \ldots, y_n)(x) \neq 0$ for at least one point in the interval I, then the set of functions is linearly independent.
(b) if the set of functions is linearly dependent, then it follows that $W(y_1, \ldots, y_n)(x) = 0$ for all x in I.

EXAMPLE 18 Discuss the linear dependence of $y_1 = 3x^2 - 8x$, $y_2 = x^2$, and $y_3 = 4x$, using the Wronskian.

Solution The Wronskian gives us

$$W(y_1, y_2, y_3)(x) = \begin{vmatrix} 3x^2 - 8x & x^2 & 4x \\ 6x - 8 & 2x & 4 \\ 6 & 2 & 0 \end{vmatrix} \equiv 0.$$

Nonetheless, we cannot conclude anything about the linear dependence or independence of these functions from this result and Theorem 4.7. In other words, $W(y_1, y_2, y_3) \equiv 0$ does not necessarily mean the functions are linearly dependent. (In Example 17, however, we did establish that these functions are indeed linearly dependent.)

Abel's formula (Lemma 4.1) can also be extended to nth-order equations, which in turn can be used to prove Theorem 4.8 below.

Lemma 4.2 (*Abel's formula*) If $y_1, y_2, \ldots, y_n$ are linearly independent solutions of $M[y] = 0$ on the interval I where M is the normal linear operator

$$M = D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x),$$

the associated Wronskian is given by

$$W(y_1, \ldots, y_n)(x) = C \exp\left[-\int a_{n-1}(x)\,dx\right]$$

for an appropriate value of the constant C.

Theorem 4.8 If $y_1, y_2, \ldots, y_n$ are solutions of $M[y] = 0$ on the interval I where M is a normal linear operator of order n, then the set of solutions is linearly independent on I if and only if

$$W(y_1, \ldots, y_n)(x) \neq 0$$

for every x in I.

4.5.2 Homogeneous Constant-Coefficient Equations

If we wish to solve the linear DE of order n,

$$(a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y = 0, \tag{31}$$

where the coefficients $a_0, a_1, \ldots, a_n$ are all constants, it is necessary to find the roots of the nth-degree polynomial

$$a_nm^n + a_{n-1}m^{n-1} + \cdots + a_1m + a_0 = 0, \tag{32}$$

which is the *auxiliary equation* associated with (31). We obtain (32) by putting $y = e^{mx}$ into (31) and simplifying. Because of the many possible combinations of roots of (32) depending upon n, it is difficult to identify all cases when $n > 2$. However, the following generalizations are fairly easy to establish:

1. If m_1 is a real distinct root of the auxiliary equation, there corresponds the single solution
$$y_1 = e^{m_1x}.$$
2. If m_1 is a real root of multiplicity k, there corresponds k linearly independent solutions given by
$$y_1 = e^{m_1x},\ y_2 = xe^{m_1x},\ \ldots,\ y_k = x^{k-1}e^{m_1x}.$$
3. If $p \pm iq$ are distinct complex roots of the auxiliary equation, there correspond the linearly independent solutions
$$y_1 = e^{px}\cos qx, \quad y_2 = e^{px}\sin qx.$$
4. If $p \pm iq$ are complex roots of multiplicity k, there correspond the $2k$ linearly independent solutions
$$y_1 = e^{px}\cos qx,\ y_2 = xe^{px}\cos qx,\ \ldots,\ y_k = x^{k-1}e^{px}\cos qx,$$
$$y_{k+1} = e^{px}\sin qx,\ y_{k+2} = xe^{px}\sin qx,\ \ldots,\ y_{2k} = x^{k-1}e^{px}\sin qx.$$

We leave it to the reader to verify that in each case the given functions are indeed solutions of the DE and, moreover, linearly independent solutions.

EXAMPLE 19 Find the general solution of the sixth-order DE

$$(D - 1)^3(D + 2)^2(3D - 2)y = 0.$$

Solution The auxiliary equation is obviously given by

$$(m - 1)^3(m + 2)^2(3m - 2) = 0,$$

with roots $m = 1, 1, 1, -2, -2, \frac{2}{3}$, obtained by inspection. Hence the six solutions corresponding to these values of m are

$$y_1 = e^x, \quad y_2 = xe^x, \quad y_3 = x^2e^x, \quad y_4 = e^{-2x}, \quad y_5 = xe^{-2x}, \quad y_6 = e^{2x/3},$$

providing us with the general solution

$$y = (C_1 + C_2x + C_3x^2)e^x + (C_4 + C_5x)e^{-2x} + C_6e^{2x/3}.$$

EXAMPLE 20 Find the general solution of the eighth-order DE

$$D^4(D^2 - 2D + 5)^2y = 0.$$

Solution Here we find the auxiliary equation to be

$$m^4(m^2 - 2m + 5)^2 = 0$$

with roots $m = 0, 0, 0, 0, 1 \pm 2i, 1 \pm 2i$, leading to the solutions

$$y_1 = 1, \quad y_2 = x, \quad y_3 = x^2, \quad y_4 = x^3, \quad y_5 = e^x\cos 2x,$$

$$y_6 = xe^x\cos 2x, \quad y_7 = e^x\sin 2x, \quad y_8 = xe^x\sin 2x.$$

Hence,

$$y = C_1 + C_2x + C_3x^2 + C_4x^3 + e^x[(C_5 + C_6x)\cos 2x + (C_7 + C_8x)\sin 2x].$$

When the DE does not have the operator in factored form (which is the normal situation in practice) as in Examples 19 and 20, finding the roots of the auxiliary equation can be the most difficult part of solving the DE. According to the general theory of polynomials, if (32) has a rational real root of the form $m_1 = p/q$, where p and q are integers, then p must be a factor of a_0 and q a factor of a_n. Of course, once we have found one root m_1, we can divide (32) (either directly or by using synthetic division) by the factor $(m - m_1)$ to obtain a polynomial of one less degree from which to determine the remaining roots.

EXAMPLE 21 Solve $(4D^3 - 3D + 1)y = 0$.

Solution The auxiliary equation is

$$4m^3 - 3m + 1 = 0.$$

If there are any real rational roots, they are among the possibilities $m = \pm 1, \pm \frac{1}{2}, \pm\frac{1}{4}$. Checking, we find that $m_1 = -1$ is such a root. By division, we then obtain

$$4m^3 - 3m + 1 = (m + 1)(4m^2 - 4m + 1) = 0,$$

which shows that the remaining roots are $m_2 = m_3 = \frac{1}{2}$. Thus, the general solution is

$$y = C_1e^{-x} + (C_2 + C_3x)e^{x/2}.$$

EXAMPLE 22 Solve $(4D^4 - 15D^2 + 5D + 6)y = 0$.

Solution For the auxiliary equation

$$4m^4 - 15m^2 + 5m + 6 = 0,$$

the rational root possibilities are

$$m = \pm 1, \pm 2, \pm 3, \pm 6, \pm\tfrac{1}{4}, \pm\tfrac{1}{2}, \pm\tfrac{3}{4}, \pm\tfrac{3}{2}.$$

By checking, we find that $m_1 = 1$ is a root, so we can write

$$4m^4 - 15m^2 + 5m + 6 = (m - 1)(4m^3 + 4m^2 - 11m - 6).$$

The remaining roots must then satisfy the reduced equation

$$4m^3 + 4m^2 - 11m - 6 = 0.$$

This time we obtain $m_2 = \frac{3}{2}$ and write

$$4m^4 - 15m^2 + 5m + 6 = (m - 1)(m - \tfrac{3}{2})(4m^2 + 10m + 4) = 0,$$

and so the remaining two roots are readily found to be $m_3 = -\frac{1}{2}$ and $m_4 = -2$. The general solution is therefore given by

$$y = C_1e^{x} + C_2e^{3x/2} + C_3e^{-x/2} + C_4e^{-2x}.$$

EXERCISES 4.5

In problems 1–5, determine whether the given functions are linearly dependent or independent.

1. $y_1 = x, \quad y_2 = 3x^2, \quad y_3 = x^2 - 7x$
2. $y_1 = x, \quad y_2 = 3x^2, \quad y_3 = x^2 - 7x + 1$
3. $y_1 = e^x, \quad y_2 = xe^x, \quad y_3 = x^2e^x$
4. $y_1 = e^x, \quad y_2 = e^{-x}, \quad y_3 = \cosh x$
5. $y_1 = \cos^2 x, \quad y_2 = \sec^2 x, \quad y_3 = \sin^2 x, \quad y_4 = \tan^2 x$
6. Given the DE

$$y''' - 6y'' + 5y' + 12y = 0,$$

verify that $y_1 = e^{-x}$, $y_2 = e^{3x}$, and $y_3 = e^{4x}$ are linearly independent solutions and write the general solution.

*7. Prove Lemma 4.2 for the case $n = 3$.

*8. Prove Theorem 4.8.

In problems 9–29, find the general solution. Assume that x is the independent variable.

9. $(D - 1)^3 y = 0$

10. $D(2D + 3)^2 y = 0$

11. $(D - 4)^3(D + 2)^2 y = 0$

12. $(D^2 + 1)^3 y = 0$

13. $(D^3 + 3D^2 + 3D + 1)y = 0$

14. $(D^3 - 1)y = 0$

15. $(D^3 + 1)y = 0$

16. $D(D^2 + 3D - 4)y = 0$

17. $D^2(D^2 + 3D + 1)y = 0$

18. $(D^2 + 1)^2(D^2 - 1)^2 y = 0$

19. $(4D^3 + 4D^2 + D)y = 0$

20. $(D^3 + D^2 - 2)y = 0$

21. $(3D^3 - 19D^2 + 36D - 20)y = 0$

22. $(D^4 - 5D^3 + 6D^2 + 4D - 8)y = 0$

23. $(4D^4 - 4D^3 - 23D^2 + 12D + 36)y = 0$

24. $(D^4 - 4D^3 + D^2 + 6D)y = 0$

25. $(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0$

26. $(D^5 - D^3)y = 0$

27. $D^3(D^2 + 9)^2(D^2 - 2D + 1)^2 y = 0$

***28.** $D^4 - 4D^3 + 10D^2 - 20D + 25)y = 0$

***29.** $(D^5 + 2D^3 + D)y = 0$

In problems 30–34, solve the DE subject to the prescribed conditions.

30. $(D^3 - 3D - 2)y = 0, \quad y(0) = 0, \quad y'(0) = 9, \quad y''(0) = 0$

31. $y''' + y'' - y' - y = 0, \quad y(0) = 1, \quad y(2) = 0, \quad \lim_{x\to\infty} y = 0$

32. $y''' + 5y'' + 17y' + 13y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 6$

33. $y^{(4)} = 0, \quad y(0) = 2, \quad y'(0) = 3, \quad y''(0) = 4, \quad y'''(0) = 5$

***34.** $(D^4 + 6D^3 + 9D^2)y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 6, \quad \lim_{x\to\infty} y' = 1$

35. The roots of the auxiliary equation of some tenth-order DE are known to be 3, 3, 3, 3, $1 \pm 2i$, $1 \pm 2i$, 5, and -1. Write the general solution of the DE.

36. The roots of the auxiliary equation of a third-order DE are known to be $m_1 = -4$ and $m_2 = m_3 = \frac{1}{2}$. What is the corresponding DE?

37. Repeat problem 36 when $m_1 = 3$, $m_2 = 2 + i$, and $m_3 = 2 - i$.

***38.** Given that $y_1 = \sin x$ is one solution of

$$y^{(4)} + 2y''' + 6y'' + 2y' + 5y = 0,$$

find the general solution.

***39.** Given that $y_1 = e^{-x}\cos 2x$ is one solution of

$$(D^4 + 4D^3 + 14D^2 + 20D + 25)y = 0,$$

find the general solution.

40. Find four linearly independent solutions of

$$y^{(4)} - \lambda y = 0$$

for the case when

(a) $\lambda = 0$.

(b) $\lambda > 0$.

Hint: Set $\lambda = k^4$.

***41.** Solve problem 40 subject to the boundary conditions

$$y(0) = 0, \quad y''(0) = 0, \quad y(1) = 0, \quad y''(1) = 0.$$

4.6 THE NONHOMOGENEOUS EQUATION

In Section 2.5 we stated that every solution of the first-order linear DE

$$y' + a_0(x)y = f(x)$$

is of the form $y = y_P + y_H$, where y_P is any *particular solution* of the nonhomogeneous equation and y_H* is a general solution of the associated homogeneous equation

$$y' + a_0(x)y = 0.$$

This situation can be generalized to linear equations of any order n as the next theorem states.

Theorem 4.9 If y_P is any particular solution of the nonhomogeneous equation

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x),$$

and if $y_H = C_1y_1(x) + \cdots + C_ny_n(x)$ is a general solution of the associated homogeneous equation resulting from the deletion of $f(x)$, then $y = y_P + y_H$ is a general solution of the nonhomogeneous equation.

Proof: For simplicity, we will present the proof only for the case when $n = 2$. Suppose that y_P is a particular solution of the second-order equation

$$y'' + a_1(x)y' + a_0(x)y = f(x),$$

and let Y be any solution of the same equation. The quantity $y = Y - y_P$ then satisfies

$$\begin{aligned} y'' + a_1(x)y' + a_0(x)y &= Y'' + a_1(x)Y' + a_0(x)Y - y_P'' - a_1(x)y_P' - a_0(x)y_P \\ &= f(x) - f(x) = 0. \end{aligned}$$

Thus, since $Y - y_P$ satisfies the associated homogeneous DE, it must be expressible in the form

*The function y_H is frequently called the *complementary solution* and is also denoted by the symbol y_c in many texts.

$$Y - y_P = C_1y_1(x) + C_2y_2(x).$$

Transposing, we obtain

$$Y = y_P + C_1y_1(x) + C_2y_2(x) = y_P + y_H,$$

which shows that every solution of the nonhomogeneous equation is contained in the sum $y_P + y_H$. $\square$

4.6.1 The Method of Undetermined Coefficients

In many instances, the nonhomogeneous term $f(x)$ is composed of functions with either terminating or repeating derivatives. When this is the case, the particular solution y_P can be constructed by a relatively simple technique called the *method of undetermined coefficients* (or "method of good guesses"). It is restricted pretty much to constant-coefficient equations, and of these, only the ones for which $f(x)$ is either

1. a polynomial in x (including constants),
2. an exponential function e^{px},
3. $\cos qx$ or $\sin qx$, or
4. a finite sum and/or product of these functions.

To illustrate the gist of the method, suppose we wish to find a particular solution of

$$y'' + y = 3e^x. \tag{33}$$

Since differentiation of an exponential function merely reproduces the function again with at most a multiplicative constant, it seems natural to "guess" that a particular solution exists of the form

$$y_P = Ae^x,$$

where A is an "undetermined coefficient." We substitute y_P into (33), getting

$$Ae^x + Ae^x = 3e^x \quad \text{or} \quad 2Ae^x = 3e^x,$$

which reduces to an identity if $A = \frac{3}{2}$. Thus

$$y_P = \frac{3}{2}e^x$$

is a particular solution of (33).

Now suppose the nonhomogeneous term is changed to $3x^2$ so that the equation we wish to solve is

$$y'' + y = 3x^2. \tag{34}$$

Proceeding as before, we might guess

$$y_P = Ax^2.$$

This time the substitution of y_P into (33) yields

$$2A + Ax^2 = 3x^2,$$

which cannot be satisfied for any choice of the constant A. The problem is that the derivatives of Ax^2 produce new functions that are linearly independent of it. To get an idea of what to do, we observe that (34) can be transformed into a homogeneous DE by taking three derivatives of each side. That is, (34) becomes

$$y^{(5)} + y''' = 0 \tag{35}$$

since $d^3/dx^3(3x^2) = 0$. Now the auxiliary equation associated with (35) is $m^5 + m^3 = 0$ with roots $m = 0, 0, 0, \pm i$, so that its general solution can be expressed as

$$y = \underbrace{C_1 + C_2x + C_3x^2}_{y_P} + \underbrace{C_4\cos x + C_5\sin x}_{y_H}. \tag{36}$$

It can be argued that every solution of (34) is also a solution of (35), and since

$$y_H = C_4\cos x + C_5\sin x$$

is the homogeneous solution of (34), it follows that the particular solution of (34) gives

$$y_P = Ax^2 + Bx + C$$

for some choice of the constants A, B, and C. The substitution of this y_P into (34) gives

$$2A + Ax^2 + Bx + C = 3x^2$$

or

$$(2A + C) + Bx + Ax^2 = 3x^2.$$

By equating like coefficients in this last identity, we have

$$\begin{aligned} 2A + C &= 0 \\ B &= 0 \\ A &= 3, \end{aligned}$$

which gives the simultaneous solution

$$A = 3, \quad B = 0, \quad C = -6.$$

Thus a particular solution this time is

$$y_P = 3x^2 - 6.$$

The general rule illustrated here is that y_P *should have the basic structure of the nonhomogeneous term* $f(x)$, *plus all linearly independent derivatives of* f.

Another difficulty in the method arises when $f(x)$ is composed of a function that occurs in the homogeneous solution. For example, suppose we wish to solve

$$y'' + y = 3\cos x. \tag{37}$$

Assuming

$$y_P = A\cos x + B\sin x,$$

we find that substituting this expression into (37) leads to

$$-A\cos x - B\sin x + A\cos x + B\sin x = 0 = 3\cos x,$$

which is absurd. To correct for this situation, we must assume y_P to be a function linearly independent of any functions in the homogeneous solution. Therefore we write (since $y_H = C_1\cos x + C_2\sin x$)

$$y_P = x(A\cos x + B\sin x),$$

from which it follows that

$$y_P' = A\cos x + B\sin x + x(-A\sin x + B\cos x),$$

$$y_P'' = -2A\sin x + 2B\cos x - x(A\cos x + B\sin x),$$

and, when these expressions for y_P, y_P' and y_P'' are substituted into the DE, leads to

$$-2A\sin x + 2B\cos x = 3\cos x.$$

Equating like coefficients, we have

$$-2A = 0, \quad 2B = 3,$$

or $A = 0$ and $B = \frac{3}{2}$, and thus

$$y_P = \frac{3}{2}x\sin x.$$

The method of solution is summarized in Table 4.1. Using the table consists of the following four steps.

Step 1: If the DE is of the form (not necessarily normal form)

$$M[y] = f_1(x) + f_2(x) + \cdots + f_r(x),$$

where each $f_i(x)$, $i = 1, 2, \ldots, r$, is a different type of function occurring in one of the six categories in Table 4.1, then we replace the DE with the equivalent system of equations

$$M[y] = f_1(x),$$
$$M[y] = f_2(x),$$
$$\vdots$$
$$M[y] = f_r(x).$$

Step 2: Identify each function $f_i(x)$, $i = 1, 2, \ldots, r$, as belonging to one of the major categories in the table.

Step 3: For each $f_i(x)$, assume the form of y_P given in conditions (a) or (b), depending upon whether the specified value of m is a root of the auxiliary equation.

Step 4: Add all y_P's found in Step 3 together to form the proper y_P for the original DE.

Table 4.1 Method of Undetermined Coefficients

DE: $a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$	
Auxiliary equation: $a_n m^n + a_{n-1}m^{n-1} + \cdots + a_1 m + a_0 = 0$	
I. $f(x) = b_j x^j + \cdots + b_1 x + b_0,\quad j = 0, 1, 2, \ldots$	
a. $m \neq 0$ $y_P = A_j x^j + \cdots + A_1 x + A_0$	**b.** $m = 0$, *k times* $y_P = x^k(A_j x^j + \cdots + A_1 x + A_0)$
II. $f(x) = be^{cx}$	
a. $m \neq c$ $y_P = Ae^{cx}$	**b.** $m = c$, *k times* $y_P = Ax^k e^{cx}$
III. $f(x) = (b_j x^j + \cdots + b_1 x + b_0)e^{cx},\quad j = 0, 1, 2, \ldots$	
a. $m \neq c$ $y_P = (A_j x^j + \cdots + A_1 x + A_0)e^{cx}$	**b.** $m = c$, *k times* $y_P = x^k(A_j x^j + \cdots + A_0)e^{cx}$
IV. $f(x) = a\cos qx + b\sin qx$	
a. $m \neq \pm iq$ $y_P = A\cos qx + B\sin qx$	**b.** $m = \pm iq$, *k times* $y_P = x^k(A\cos qx + B\sin qx)$
V. $f(x) = (b_i x^i + \cdots + b_1 x + b_0)\cos qx + (c_j x^j + \cdots + c_0)\sin qx$	
a. $m \neq \pm iq$ $y_P = (A_r x^r + \cdots + A_0)\cos qx$ $+ (B_r x^r + \cdots + B_0)\sin qx,$ $r = \max(i, j)$	**b.** $m = \pm iq$, *k times* Multiply y_P in (a) by x^k.
VI. $f(x) = ae^{px}\cos qx + be^{px}\sin qx$	
a. $m \neq p \pm iq$ $y_P = Ae^{px}\cos qx + Be^{px}\sin qx$	**b.** $m = p \pm iq$, *k times* Multiply y_P in (a) by x^k.

EXAMPLE 23 Determine the form of y_P for $y'' + 2y' + y = 3x^2e^{-x}$.

Solution The roots of the auxiliary equation are $m = -1, -1$. In order to determine y_P, we first note that if $m = -1$ were not a root of the auxiliary equation, we would select

$$y_P = (Ax^2 + Bx + C)e^{-x}$$

as suggested in Category III in Table 4.1. But since $m = -1$ is a *double root* of the auxiliary equation, the proper form to assume for y_P is

$$y_P = x^2(Ax^2 + Bx + C)e^{-x} = (Ax^4 + Bx^3 + C^2x^2)e^{-x}.$$

EXAMPLE 24 Solve $D^2(D - 1)y = 2\sin x - 5e^x$.

Solution By inspection, we see that the auxiliary equation has roots $m = 0, 0, 1$. The homogeneous solution is therefore

$$y_H = C_1 + C_2x + C_3e^x.$$

Following Step 1 as outlined above, we reexpress the DE as the system of equations

$$D^2(D - 1)y = 2\sin x,$$

$$D^2(D - 1)y = -5e^x.$$

The term $2\sin x$ in the first equation occurs in Category IV of Table 4.1, and since $m \neq \pm i$, we assume

$$y_P = A\cos x + B\sin x.$$

Corresponding to the term $-5e^x$, however, we need to write

$$y_P = x(Ce^x),$$

since $m = 1$ is a root of the auxiliary equation (see Category II). Thus, we write the complete particular solution as

$$y_P = A\cos x + B\sin x + Cxe^x.$$

Computing derivatives, we have

$$y_P' = -A\sin x + B\cos x + C(xe^x + e^x),$$

$$y_P'' = -A\cos x - B\sin x + C(xe^x + 2e^x),$$

$$y_P''' = A\sin x - B\cos x + C(xe^x + 3e^x),$$

and substituting these expressions into the DE yields (after simplification)

$$(A + B)\sin x + (A - B)\cos x + Ce^x = 2\sin x - 5e^x.$$

Comparing like coefficients, we get,

$$A + B = 2, \quad A - B = 0, \quad C = -5.$$

Therefore, $A = B = 1$, $C = -5$, and our solution is

$$\begin{aligned} y &= y_P + y_H \\ &= \cos x + \sin x - 5xe^x + C_1 + C_2x + C_3e^x. \end{aligned}$$

EXAMPLE 25 Solve $y'' + y = (x - 1)\cos x$.

Solution The roots of the auxiliary equation are $m = \pm i$, and thus

$$y_H = C_1 \cos x + C_2 \sin x.$$

Since $m = \pm i$ are roots (one time), the proper form to assume for our particular solution is

$$\begin{aligned} y_P &= x[(Ax + B)\cos x + (Cx + D)\sin x] \\ &= (Ax^2 + Bx)\cos x + (Cx^2 + Dx)\sin x. \end{aligned}$$

The substitution of y_P into the DE yields

$$\begin{aligned} y_P'' + y_P &= -(Ax^2 + Bx)\cos x - (4Ax + 2B)\sin x + 2A\cos x \\ &\quad - (Cx^2 + Dx)\sin x + (4Cx + 2D)\cos x + 2C\sin x \\ &\quad + (Ax^2 + Bx)\cos x + (Cx^2 + Dx)\sin x \\ &= 4Cx\cos x + 2(A + D)\cos x - 4Ax\sin x + 2(C - B)\sin x \\ &= x\cos x - \cos x. \end{aligned}$$

Equating like coefficients gives

$$\begin{aligned} &4C = 1 \\ &2A + 2D = -1 \\ &-4A = 0 \\ &2C - 2B = 0, \end{aligned}$$

from which we deduce $A = 0$, $B = \frac{1}{4}$, $C = \frac{1}{4}$, $D = -\frac{1}{2}$. The general solution is therefore

$$\begin{aligned} y &= y_P + y_H \\ &= \tfrac{1}{4}x^2\sin x + \tfrac{1}{4}x\cos x - \tfrac{1}{2}x\sin x + C_1\cos x + C_2\sin x. \end{aligned}$$

EXERCISES 4.6

In problems 1–20, obtain a general solution by using the method of undetermined coefficients.

1. $y'' + y' = -\cos x$.
2. $y'' + 9y = 18$
3. $y'' - 6y' + 9y = e^x$
4. $y'' + 3y' + 2y = 6x^3$
5. $y'' + 8y = 2e^{-x} + 5x$
6. $y'' - y = 3e^x$
7. $y'' + 9y = x\sin 3x$
8. $y'' + 4y = 3\sin x + 4\cos x - 8$

9. $y'' + 4y = 2\sin x\cos x - 7$
10. $y'' - 3y' - 4y = 30e^{4x}$
11. $y'' - y = 8xe^{x}$
12. $y'' - y = \cosh x$
13. $y'' + y' = (x + 1)^3$
14. $y'' - 2y' + 5y = e^{x}\sin x$
15. $y'' + y' + y = 2\sin^2 x$
16. $y''' + y'' - 4y' - 4y = 3e^{-x} - 4x - 6$
17. $y''' + 4y'' + 4y' = xe^{-x}$
*18. $2y''' - 3y'' - 3y' + 2y = 4\cosh^2 x$
19. $16y^{(4)} - y = 6e^{x/2}$
20. $y^{(4)} - 4y'' = 8e^{-2x} + 3e^{x} - x + 8$

In problems 21–24, set up the correct form for y_P but do not solve for the coefficients.

21. $y'' - 2y' + y = 5x^2 - 7x + 4x^2e^{x}$
22. $y''' + 2y'' + y' = 3xe^{-x} + 5x^2$
23. $y''' - 4y'' + 4y' = 5x^3e^{2x} - x + 10$
24. $y'' + 4y = 3(x + 1)e^{x}\cos x$

In problems 25–28, solve the given initial value problem.

25. $y'' + y = \sin x, \quad y(0) = -1, \quad y'(0) = 1$
26. $y'' - 5y' = x - 2, \quad y(0) = 0, \quad y'(0) = 2$
27. $y'' + y = 8\cos 2x - 4\sin x, \quad y\left(\dfrac{\pi}{2}\right) = 1, \quad y'\left(\dfrac{\pi}{2}\right) = 0$
28. $y'' - 3y' - 4y = e^{-x}, \quad y(2) = 3, \quad y'(2) = 0$

In problems 29–34, solve the given boundary value problem.

29. $y'' + 2y' + y = x, \quad y(0) = -3, \quad y(1) = -1$
30. $y'' + y = x + 1, \quad y(0) = 1, \quad y(1) = \dfrac{1}{2}$
31. $y'' + y = 2\cos x, \quad y(0) = 0, \quad y(\pi) = 0$
32. $y'' + y = \sin x, \quad y(0) = 1, \quad y\left(\dfrac{\pi}{2}\right) = -1$
33. $y'' + 4y = e^{-x}, \quad y(0) = -1, \quad y'(2) = 3$
34. $y'' - y = x^2, \quad y(0) = 0, \quad y(1) = 0$
35. Show that when the nonhomogeneous term is a constant, the resulting DE
$$ay'' + by' + cy = K \quad (K \text{ constant})$$
has the particular solution $y_P = K/c$.

4.7 VARIATION OF PARAMETERS

When $f(x)$ is not of the form assumed in Section 4.6.1, or when the DE has variable coefficients, a more general method of constructing y_P is required. This more general method, called *variation of parameters*, is analogous to the method used in Section 2.5 for solving first-order linear DEs. For simplicity we will restrict our development to second-order equations in *normal form*

$$y'' + a_1(x)y' + a_o(x)y = f(x), \tag{38}$$

although the technique can be extended to higher-order equations.

Suppose the general solution of the associated homogenous DE

$$y'' + a_1(x)y' + a_0(x)y = 0 \tag{39}$$

is known to be

$$y_H = C_1y_1(x) + C_2y_2(x), \tag{40}$$

where y_1 and y_2 are linearly independent on an interval $a \leq x \leq b$. We then seek a particular solution y_P that is of the form

$$y_P = u(x)y_1(x) + v(x)y_2(x), \tag{41}$$

where we try to determine $u(x)$ and $v(x)$ in such a way that (41) satisfies (38). Notice that (41) has the form of (40) with the arbitrary constants C_1 and C_2 replaced by the functions $u(x)$ and $v(x)$.

To determine u and v, we need two relations that they satisfy. Only one such relation is obtained by requiring (41) to satisfy (38), and so we must come up with a second relation that will lead to an easy determination of these functions. Specifically, we will force u and v to satisfy two first-order DEs, since such equations are fairly straightforward to solve.

From (41), we get

$$y_P' = u(x)y_1'(x) + v(x)y_2'(x) + u'(x)y_1(x) + v'(x)y_2(x).$$

Let us eliminate the terms involving derivatives of u and v by equating them to zero:

$$u'(x)y_1(x) + v'(x)y_2(x) = 0.$$

Thus,

$$y_P' = u(x)y_1'(x) + v(x)y_2'(x), \tag{42}$$

and, differentiating again, we find

$$y_P'' = u(x)y_1'' + v(x)y_2'' + u'(x)y_1'(x) + v'(x)y_2'(x). \tag{43}$$

Substituting (41), (42), and (43) for y_P, y_P', and y_P'' into (38) leads to

$$\begin{aligned} y_P'' + a_1(x)y_P' + a_0(x)y_P = {} & u(x)\underbrace{[y_1'' + a_1(x)y_1' + a_0(x)y_1]}_{\text{Zero}} \\ & + v(x)\underbrace{[y_2'' + a_1(x)y_2' + a_0(x)y_2]}_{\text{Zero}} \\ & + u'(x)y_1'(x) + v'(x)y_2'(x) \end{aligned}$$

$$= u(x) \cdot 0 + v(x) \cdot 0 + u'(x)y_1'(x) + v'(x)y_2'(x)$$
$$= f(x).$$

Therefore,

$$u'(x)y_1'(x) + v'(x)y_2'(x) = f(x),$$

and so we find that the two functions u' and v' satisfy the simultaneous equations

$$u'(x)y_1(x) + v'(x)y_2(x) = 0, \tag{44}$$

$$u'(x)y_1'(x) + v'(x)y_2'(x) = f(x). \tag{45}$$

Nontrivial solutions of (44) and (45) exist provided that the coefficient determinant

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

does not vanish. But since this determinant is precisely the Wronskian of the linearly independent solutions y_1 and y_2, it can never be zero on the interval $a \le x \le b$, and so the simultaneous solution of (44) and (45) yields

$$u'(x) = -\frac{y_2(x)f(x)}{W(y_1, y_2,)(x)}, \quad v'(x) = \frac{y_1(x)f(x)}{W(y_1; y_2)(x)}. \tag{46}$$

Taking any integral of these expressions provides suitable functions for u and v, and we conclude that our particular solution is given by

$$y_P = -y_1(x)\int \frac{y_2(x)f(x)}{W(y_1, y_2)(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(y_1, y_2)(x)}dx. \tag{47}$$

In certain instances these integrals cannot be evaluated explicitly, and so we must leave y_P in the integral form suggested by (47).

EXAMPLE 26 Solve $2y'' + 18y = \csc 3x$.

Solution We first rewrite the DE in normal form

$$y'' + 9y = \frac{1}{2}\csc 3x. \qquad m^2+9=0$$

The solution of the homogeneous equation $y'' + 9y = 0$ is

$$y_H = C_1 \cos 3x + C_2 \sin 3x,$$

where we identify $y_1 = \cos 3x$ and $y_2 = \sin 3x$. Computing the Wronskian, we have

$$W(y_1, y_2)(x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3.$$

With $f(x) = \frac{1}{2}\csc 3x$, we find from (46)

$$u'(x) = -\frac{1}{6}\sin 3x \csc 3x = -\frac{1}{6}$$

and

$$v'(x) = \frac{1}{6}\cos 3x \csc 3x = \frac{\cos 3x}{6\sin 3x},$$

and thus $u(x) = -x/6$ and $v(x) = \frac{1}{18}\log|\sin 3x|$. Hence

$$\begin{aligned} y_P &= u(x)\cos 3x + v(x)\sin 3x \\ &= -\frac{x}{6}\cos 3x + \frac{1}{18}\log|\sin 3x|\sin 3x, \end{aligned}$$

and the general solution $y = y_P + y_H$ becomes

$$y = \left(C_1 - \frac{x}{6}\right)\cos 3x + \left(C_2 + \frac{1}{18}\log|\sin 3x|\right)\sin 3x.$$

EXAMPLE 27 Solve $y'' - 3y' + 2y = 1/(1 + e^{-x})$.

Solution The associated homogeneous DE has the solution

$$y_H = C_1e^x + C_2e^{2x}$$

with Wronskian $W(y_1, y_2)(x) = e^{3x}$. Hence,

$$u'(x) = -\frac{e^{2x}}{e^{3x}(1 + e^{-x})} = -\frac{e^{-x}}{1 + e^{-x}}$$

and

$$v'(x) = \frac{e^x}{e^{3x}(1 + e^{-x})} = \frac{e^{-2x}}{1 + e^{-x}}.$$

Integrating the above functions, we find

$$u(x) = -\int \frac{e^{-x}}{1 + e^{-x}}\,dx = \log(1 + e^{-x})$$

and

$$v(x) = \int \frac{e^{-2x}}{1 + e^{-x}}\,dx = \int \left(e^{-x} - \frac{e^{-x}}{1 + e^{-x}}\right)dx$$

or

$$v(x) = -e^{-x} + \log(1 + e^{-x}).$$

Then, from (47)

$$y_P = e^x \log(1 + e^{-x}) - e^x + e^{2x} \log(1 + e^{-x}),$$

which leads to the general solution

$$y = C_3 e^x + C_2 e^{2x} + (e^x + e^{2x}) \log(1 + e^{-x}),$$

where $C_3 = C_1 - 1$.

EXERCISES 4.7

In problems 1–20, use variation of parameters to find a general solution.

1. $y'' - y = e^x$

2. $y'' - y = x$

3. $y'' + 9y = \sin 3x$

4. $y'' + y = \csc x$

5. $y'' + y = \sec x$

6. $y'' + y = \sec^2 x$

7. $y'' + y = \sec^4 x$

8. $y'' + y = \tan x$

9. $y'' + y = \cot x$

10. $y'' + y = \tan^2 x$

11. $y'' + y = \sec x \csc x$

12. $y'' + y = \csc x \cot x$

13. $y'' - 4y' + 4y = (x + 1)e^{2x}$

***14.** $y'' - 3y' + 2y = \dfrac{e^{3x}}{1 + e^x}$

15. $y'' + 2y' + y = e^{-x} \log x, \quad x > 0$

***16.** $y'' - 2y' + y = \dfrac{e^{2x}}{(e^x + 1)^2}$

17. $y'' - y = e^{-2x} \sin(e^{-x})$

18. $y'' - 3y' + 2y = \sin(e^{-x})$

19. $y'' + y = \csc^3 x \cot x$

***20.** $4y'' - 4y' + y = (1 - x^2)^{1/2} e^{x/2}$

Use variation of parameters to solve the initial value problems 21–24.

21. $y'' - y = xe^x, \quad y(0) = 2, \quad y'(0) = 0$

22. $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}, \quad y(0) = 1, \quad y'(0) = 0$

23. $y'' + y = 2 \csc x \cot x, \quad y\left(\dfrac{\pi}{2}\right) = 1, \ y'\left(\dfrac{\pi}{2}\right) = 1$

24. $2y'' + y' - y = x + 1, \quad y(0) = 1, \quad y'(0) = 0$

In problems 25–30, obtain a general solution where the homogeneous solution is specified.

25. $(1 - x)y'' + xy' - y = 2(x - 1)^2 e^{-x}; \quad y_H = C_1 x + C_2 e^x$

26. $x^2 y'' - x(x + 2)y' + (x + 2)y = x^3; \quad y_H = C_1 x + C_2 x e^x$

27. $x(x - 2)y'' - (x^2 - 2)y' + 2(x - 1)y = 3x^2(x - 2)^2 e^x; \quad y_H = C_1 x^2 + C_2 e^x$

28. $xy'' - (1 + 2x^2)y' = x^5 e^{x^2}; \quad y_H = C_1 + C_2 e^{x^2}$

*29. $(1 - x^2)y'' - 2xy' = 2x;\quad y_H = C_1 + C_2 \log \dfrac{1 + x}{1 - x}$

*30. $x^2y'' + xy' + \left(x^2 - \dfrac{1}{4}\right)y = 3x^{3/2}\sin x, \quad x > 0;$
$y_H = C_1x^{-1/2}\cos x + C_2x^{-1/2}\sin x$

31. If $y = \phi(x)$ is a solution of

$$y'' + a_1(x)y' + a_o(x)y = f(x)$$

and $y = \psi(x)$ is a solution of

$$y'' + a_1(x)y' + a_0(x)y = g(x),$$

show that $y = \phi(x) + \psi(x)$ satisfies

$$y'' + a_1(x)y' + a_0(x)y = f(x) + g(x).$$

32. Use the result of problem 31 to solve

$$y'' + 8y = 2e^{-x} + 5x.$$

*33. Use the variation of parameter method to show that

$$y_P = \int_0^x \sin(x - s)f(s)\,ds$$

is a particular solution of the nonhomogeneous DE

$$y'' + y = f(x).$$

34. Use the result of problem 33 to solve the DE

$$y'' + y = 3e^{-x}.$$

*35. Use the variation of parameter method to show that

$$y_P = \int_0^x \sinh(x - s)f(s)\,ds$$

is a particular solution of the nonhomogeneous DE

$$y'' - y = f(x).$$

*36. Extend the variation of parameter technique to DEs of the third order,

$$y''' + a(x)y'' + b(x)y' + c(x)y = f(x),$$

by starting with

$$y_P = u(x)y_1(x) + v(x)y_2(x) + w(x)y_3(x),$$

where y_1, y_2, and y_3 are linearly independent solutions of the associated homogeneous DE.

*37. Referring to problem 36, solve the third-order DE

$$y''' - 2y'' - y' + 2y = e^{3x}.$$

*38. Find a general solution of the DE

$$(1 + x^2)y'' - 4xy' + 6y = 3(1 + x^2)^3$$

given that $y_1 = 1 - 3x^2$ is a solution of the associated homogeneous equation.

Thus far we have solved only constant-coefficient equations although most of the theory developed in the early sections of this chapter is applicable to general linear equations with variable coefficients. Unfortunately, we usually cannot solve these general linear equations as easily as constant-coefficient equations. That is, to solve variable-coefficient equations one must generally resort to some sort of power series method, as discussed in Chapter 9. An exception is the *Cauchy-Euler equation* (or *equidimensional equation*, as it is sometimes called)

$$ax^2y'' + bxy' + cy = F(x), \tag{48}$$

where a, b, and c are all constants. The significant feature here is that the power of x in each coefficient corresponds to the order of the derivative of y. Equations of this type can be transformed into constant-coefficient equations by means of a change of independent variable.

Let us make the change of variable $x = e^t$ (when $x > 0$) or $x = -e^t$ (when $x < 0$). By the chain rule, we have (when $x > 0$)

$$y' = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

and

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right),$$

which leads to (now using the notation $D = d/dt$)

$$xy' = Dy, \quad x^2y'' = D(D-1)y.$$

Under this transformation, our original DE (48) becomes the constant-coefficient equation

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = F(e^t). \tag{49}$$

Generalizations to higher-order DEs are treated in the exercises.

EXAMPLE 28 Solve $x^2y'' - 2xy' + 2y = 0$, $x > 0$.

$e^{2t} - 2e^{t}$

Solution Using the transformation $x = e^t$, we get

$m^2 - 3m + 2$

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0.$$

$(m-2)(m-$

The general solution of this equation is

$$y(t) = C_1e^t + C_2e^{2t},$$

and transforming back to the original variable x leads to

$$y(x) = C_1 x + C_2 x^2.$$

EXAMPLE 29 Solve $x^2y'' - xy' + 5y = 0$, $x > 0$.

Solution The related constant-coefficient equation is given by

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0,$$

with $1 \pm 2i$ as the roots of the auxiliary polynomial. Thus

$$y(t) = e^t(C_1 \cos 2t + C_2 \sin 2t)$$

or

$$y(x) = x[C_1 \cos(2\log x) + C_2 \sin(2 \log x)].$$

EXAMPLE 30 Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$, $x > 0$.

Solution This time the DE is nonhomogeneous, and so we must find a particular solution y_P (by variation of parameters) as well as the homogeneous solution y_H. The homogeneous DE $x^2y'' - 3xy' + 3y = 0$ transforms into

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 3y = 0$$

by making the change of variable $x = e^t$. Thus the roots of the auxiliary equation are $m = 1, 3$, giving us

$$y_H(t) = C_1e^t + C_2e^{3t}$$

or

$$y_H(x) = C_1x + C_2x^3.$$

Before using variation of parameters, we must put the DE into normal form by dividing both sides by x^2. This action leads to

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x,$$

from which we identify $f(x) = 2x^2e^x$.

Now it follows that

$$W(x, x^3) = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3$$

and thus

$$u'(x) = -\frac{x^3(2x^2e^x)}{2x^3} = -x^2e^x, \quad v'(x) = \frac{x(2x^2e^x)}{2x^3} = e^x.$$

Integration of these expressions leads to

$$u(x) = -x^2e^x + 2xe^x - 2e^x, \quad v(x) = e^x,$$

and hence

$$y_P = u(x)x + v(x)x^3 = 2x^2e^x - 2xe^x.$$

Finally, we obtain the general solution

$$y = y_P + y_H = 2x(x - 1)e^x + C_1x + C_2x^3.$$

EXERCISES 4.8

In problems 1–7, find the general solution of each homogeneous DE valid for $x > 0$.

1. $x^2y'' - 5xy' + 5y = 0$
2. $x^2y'' - 4xy' + 6y = 0$
3. $x^2y'' - xy' + y = 0$
4. $4x^2y'' + y = 0$
5. $3xy'' + 2y' = 0$
6. $x^2y'' + xy' + y = 0$
7. $x^2y'' - 5xy' + 25y = 0$

In problems 8–10, find the general solution of each nonhomogeneous DE by the method of undetermined coefficients. (Transform the entire DE by letting $x = e^t$, find the general solution of this transformed nonhomogeneous DE, and then transform back to the variable x.)

8. $x^2y'' - 2xy' + 2y = x^2$
9. $x^2y'' - 5xy' + 9y = 2x^3$
10. $x^2y'' + xy' + 4y = 2x\log x$

Solve problems 11–15 by any method (assume $x > 0$).

11. $x^2y'' - 4xy' + 6y = 4x - 6$
12. $x^2y'' - xy' + y = 4x\log x$
13. $x^2y'' + xy' - y = x$
14. $x^2y'' - 2xy' + 2y = x^3e^x$
15. $x^2y'' - 2xy' + 2y = x^3\log x^2$

Solve problems 16–20, subject to the prescribed initial conditions.

16. $x^2y'' - 2xy' - 10y = 0, \quad y(1) = 5, \quad y'(1) = 4$
17. $x^2y'' - 4xy' + 6y = 0, \quad y(2) = 0, \quad y'(2) = 4$
18. $4x^2y'' + 8xy' + y = 0, \quad y(1) = 1, \quad y'(1) = 0$

19. $x^2y'' - 3xy' + 13y = x^3, \quad y(1) = 1, \quad y'(1) = 0$

20. $x^2y'' + xy' + 4y = \sin(\log x), \quad y(1) = 1, \quad y'(1) = 0$

In problems 21–23, solve the equation by assuming a solution of the form $y = x^m$, where m must be determined.

21. $2x^2y'' + 3xy' - y = 0$

22. $x^2y'' - 2xy' + 2y = 0$

23. $x^2y'' + 7xy' + 5y = 0$

***24.** Show that under the transformtion $x = e^t$,

$$x^3y''' = D(D - 1)(D - 2)y,$$
$$x^4y^{(4)} = D(D - 1)(D - 2)(D - 3)y,$$

where $D = d/dt$, and deduce that in general

$$x^ny^{(n)} = D(D - 1)(D - 2) \cdots (D - n + 1)y.$$

In problems 25–30, use the result of problem 24 to find the general solution of each DE for $x > 0$.

25. $x^2y''' - xy'' + y' = 0$

26. $x^3y''' + x^2y'' - 2xy' + 2y = 0$

27. $x^3y''' + 2x^2y'' + xy' - y = 0$

***28.** $x^3y''' - 3x^2y'' + 6xy' - 6y = 3 + \log x^3$

***29.** $x^3y''' - x^2y'' + 2xy' - 2y = x^3$

***30.** $x^4y^{(4)} + 6x^3y''' + 15x^2y'' + 9xy' + 16y = 0$

Solve problems 31 and 32 by first making an appropriate change of independent variable to reduce the given DE to a Cauchy-Euler equation.

***31.** $(x + 5)^2y'' - (x + 5)y' - 3y = 0$

***32.** $(3x - 2)^2y'' - 6(3x - 2)y' + 12y = 0$

REFERENCES

Boyce, W. E., and DiPrima, R. C. 1977. *Elementary differential equations*. 3rd ed. New York: Wiley.

Coddington, E. A. 1961. *An introduction to ordinary differential equations*. Englewood Cliffs, N.J.: Prentice-Hall.

Ince, E. L. 1956. *Ordinary differential equations*. New York: Dover.

Rabenstein, A. L. 1972. *Introduction to ordinary differential equations*. 2nd ed. New York: Academic Press.

Rainville, E. D., and Bedient, P. E. 1974. *Elementary differential equations*. 5th ed. New York: Macmillan.

Ross, S. L. 1980. *Introduction to ordinary differential equations*. 3rd ed. New York: Wiley.

Zill, D. G. 1979. *A first course in differential equations with applications*. Boston, Mass.: Prindle, Weber & Schmidt.

Applications Involving Initial Value Problems

5

In the present chapter we consider applications of DEs to problems involving *mechanical vibrations* and *electric circuits*. Such applications naturally lead to *initial value problems*, since time is normally the independent variable and the auxiliary conditions are all prescribed at a particular instant of time.

Newton's law of motion is used in Section 5.2 to derive the DE governing the small vertical movements of a spring-mass system. *Free motions* are discussed here for cases of both *undamped* and *damped* systems. The latter type of motion results when frictional or other types of resistive forces in the system are taken into account. The three cases of *overdamped*, *underdamped*, and *critically damped* motions are carefully distinguished, and typical solution curves are illustrated for each case. In Section 5.3 we examine certain kinds of *forced motions* resulting from an external stimulus applied to the system. In particular, we consider the case of an impressed periodic force whose frequency is at or near that of the *natural frequency* of the system, leading to a state of *resonance*. Analogous systems involving simple electric circuits are briefly discussed in Section 5.4.

In Section 5.5 we introduce the notion of a *Green's function*, which permits us to develop general solution formulas for initial value problems similar to the formula derived in Section 2.5.1 in connection with first-order equations. The concept of an *impulse function*, which is so useful in circuit analysis, is presented in Section 5.6, and this in turn helps to describe a physical interpretation of the Green's function.

In Section 5.7 we look at equations with variable coefficients, which arise in applications where some of the system parameters vary over time. Since solution techniques for such DEs have not yet been developed (see Chapter 9), we limit our study to the *qualitative behavior* of the solutions. We are particularly interested in the oscillatory characteristics of the solutions, which are discussed in terms of the famous *Sturm separation* and *comparison theorems*.

For a shorter course, Sections 5.5, 5.6, and 5.7 can be omitted.

5.1 INTRODUCTION

Initial value problems arise in the study of particle motion, population dynamics, and electric circuits, as well as several other areas of application. The general problem is to solve the linear equation

$$M[y] = f(t), \quad t > t_0, \tag{1}$$

where M is the nth-order linear differential operator ($D = d/dt$)

$$M = D^n + a_{n-1}(t)D^{n-1} + \cdots + a_1(t)D + a_0(t), \tag{2}$$

subject to the n auxiliary conditions

$$y(t_0) = k_0, \quad y'(t_0) = k_1, \ldots, \quad y^{(n-1)}(t_0) = k_{n-1}, \tag{3}$$

all specified at a single point. Although the value t_0 is usually chosen as zero, the general theory does not require this choice. Equation (1) is said to be in *normal form*, and the operator M is said to be a *normal differential operator* on any interval for which the coefficients in (2) are continuous.

Remark. Since the independent variable in initial value problems is usually time, it is customary to use t rather than x.

An interesting and important property of initial value problems is that they *always* possess unique solutions when the coefficients and forcing function are continuous. Without proof, we state the following fundamental *existence-uniqueness theorem.**

Theorem 5.1 If $f(t)$ is continuous on the interval $t \geqslant t_0$ and M is a normal differential operator of order n on this same interval, then for any choice of the constants $k_0, k_1, \ldots, k_{n-1}$, there exists a unique solution of the initial value problem

$$M[y] = f(t), \quad t > t_0,$$
$$y(t_0) = k_0, \quad y'(t_0) = k_1, \ldots, \quad y^{(n-1)}(t_0) = k_{n-1}.$$

As an immediate consequence of Theorem 5.1, we have the following corollary.

Corollary 5.1 If M is a normal differential operator of order n on the interval $t \geqslant t_0$, then the initial value problem

$$M[y] = 0, \quad t > t_0,$$
$$y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0,$$

*For a proof of Theorem 5.1, see E. A. Coddington, *An Introduction to Ordinary Differential Equations* (Englewood Cliffs, N.J.: Prentice-Hall, 1961).

has only the trivial solution $y = 0$.

Proof: By inspection we see that $y = 0$ is a solution of the initial value problem, and by Theorem 5.1 it is the only solution. □

EXAMPLE 1 Show that $y = 0$ is the only solution of

$$y'' + y = 0, \quad y(0) = 0, \quad y'(0) = 0.$$

Solution The general solution of the DE is

$$y = C_1 \cos t + C_2 \sin t.$$

Imposing the prescribed initial conditions, we find

$$y(0) = C_1 + C_2 \cdot 0 = 0,$$
$$y'(0) = -C_1 \cdot 0 + C_2 = 0,$$

from which we deduce that $C_1 = C_2 = 0$. Clearly, $y = 0$ is the only solution.

5.2 SMALL MOTIONS OF A SPRING-MASS SYSTEM

When different weights are attached to an elastic spring suspended from a fixed support, the spring will stretch by an amount that varies with the weight. *Hooke's law** states that the spring will exert an upward restoring force proportional to the amount of stretch s (within reason); i.e., $F = ks$. The constant of proportionality, denoted by k, depends upon the "stiffness" of the spring and thus is different for each spring. For example, if a 10-pound weight stretches a spring 6 inches ($\frac{1}{2}$ foot), then $10 = k(\frac{1}{2})$, or $k = 20$ lb/ft, whereas if the weight only stretches the spring 2 inches, we find $k = 60$ lb/ft.

The two most common systems of units and their abbreviations are given in Table 5.1.

Suppose the natural length of a spring is b units and a weight $W = mg$ is attached to the spring. The weight, which is also referred to as the "mass," will then attain a position of equilibrium at $y = 0$, which is s units from the equilibrium position of the spring itself (see Figure 5.1). The upward restoring force is ks, which is offset by the weight mg. If the system is now subjected to an external force (downward) of magnitude $F(t)$, the weight will move in the vertical direction. Let us assume such motions are "small" so that Hooke's law will remain valid.

In addition to an external force, there frequently exists a retarding force caused by resistance of the medium in which the motion takes place or possibly by friction. For example, the weight could be suspended in a viscous medium, connected to a

*Named in honor of the English physicist ROBERT HOOKE (1635–1703).

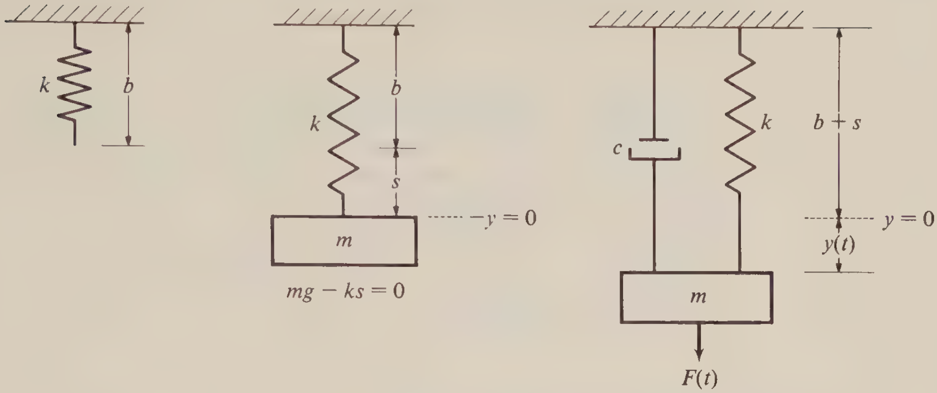


Figure 5.1 (a) Unstretched spring. (b) Equilibrium position with mass. (c) Spring-mass system with a forcing function and damping.

Table 5.1 Common Systems of Units and Their Abbreviations

System of Units	*Force*	*Length*	*Mass*	*Time*
International	newton (N)	meter (m)	kilogram (kg)	second (s)
English	pound (lb)	foot (ft)	slug	second (s)

Note: To convert from one system to another, we use the following relations:
$1 \text{ N} = 1 \text{ kg}\cdot\text{m/s}^2 = 0.2248 \text{ lb}$ $\quad 1 \text{ kg} = 0.0685 \text{ slug}$
$1 \text{ m} = 3.2808 \text{ ft}$ $\quad 1 \text{ lb} = 1 \text{ slug}\cdot\text{ft/s}^2 = 4.4482 \text{ N}$

dashpot damping device, etc. In practice, many such retarding forces are approximately proportional to the velocity y'. Hence, we will assume the resistive force is cy', where c is a positive constant, and this force acts in a direction opposing the motion. Now, taking into account all forces acting on the system, we deduce

$$\begin{aligned} my'' &= mg - k(y + s) - cy' + F(t) \\ &= \underbrace{mg - ks}_{\text{Zero}} - ky - cy' + F(t), \end{aligned}$$

which is a consequence of *Newton's second law of motion* ($F = ma$). Simplifying this equation, we have

$$my'' + cy' + ky = F(t). \tag{4}$$

Remark. Note that the terms on the left-hand side of (4) represent system forces such as restoring and damping forces, while the function $F(t)$ on the right-hand side represents an external force to the system. In this sense, the function $F(t)$ is referred to as a *forcing function* or *input function*, which gives rise to additional motions of the system superimposed onto the free motion that would result in the absence of $F(t)$.

Equation (4) describes the general motions of a spring-mass system. Observe that it is a nonhomogeneous, second-order, linear DE with constant coefficients. The motion is said to be *undamped* when $c = 0$ and *damped* when $c \neq 0$. The motion is further classified as *free* when $F(t)$ is absent and *forced* when $F(t)$ is present.

If the mass is initially displaced a distance y_0 from the equilibrium position and released from that point with velocity v_0, then we prescribe the initial conditions

$$y(0) = y_0, \quad y'(0) = v_0. \tag{5}$$

In order to investigate the solutions of (4) subject to the initial conditions (5), it is best to consider several special cases.

EXAMPLE 2 Derive the initial value problem for a 2-kg mass suspended by a spring with spring constant $k = 32$ N/m. A force of $0.5 \sin 3t$ is applied to the mass, and a dashpot damping mechanism is such that $c = 5$ kg/s. The mass is released from rest 3 cm below the equilibrium position.

Solution The relevant constants are $m = 2$, $c = 5$, and $k = 32$, leading to the DE

$$2y'' + 5y' + 32y = 0.5 \sin 3t.$$

The initial position is $y(0) = 3 \text{ cm} = 0.03$ m, and the initial velocity is $y'(0) = 0$ since the mass is released from rest.

5.2.1 Undamped Free Motion

For the case when c is sufficiently small compared with mk and the time span is short, it may be acceptable to neglect the damping term cy'. (All systems have a certain amount of damping, no matter how small the motions or how short the period of time.) If this is done and if no external force acts on the mass, then the initial value problem whose solution describes the motion is

$$my'' + ky = 0, \quad t > 0, \quad y(0) = y_0, \quad y'(0) = v_0. \tag{6}$$

The general solution of the DE above is

$$y = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t,$$

where we write $\omega_0 = \sqrt{k/m}$ for convenience. If we subject this solution function to the prescribed initial conditions, we see that

$$y(0) = C_1 \cos 0 + C_2 \sin 0 = y_0,$$

$$y'(0) = \omega_0(-C_1 \sin 0 + C_2 \cos 0) = v_0,$$

which identifies the constants $C_1 = y_0$ and $C_2 = v_0/\omega_0$. The motion of the mass is therefore described by

$$y = y_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t. \tag{7}$$

Making use of the trigonometric identity

$$\cos(a - b) = \cos a \cos b + \sin a \sin b,$$

we can rewrite (7) in the more compact form

$$y = A \cos(\omega_0 t - \phi). \tag{8}$$

The number A, defined by

$$A = \sqrt{y_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}, \tag{9}$$

is called the *amplitude* of the motion. It gives the maximum (positive) displacement of the mass from its equilibrium position. The angle ϕ is referred to as the *phase angle* and is chosen in such a way that

$$\left.\begin{aligned} \cos\phi &= \frac{y_0}{A} \\ \sin\phi &= \frac{v_0}{\omega_0 A} \end{aligned}\right\} \qquad \tan\phi = \frac{v_0}{y_0\omega_0}. \tag{10}$$

Any motion described by a single sinusoidal function as in (8) is called *simple harmonic motion*. Such motion is clearly periodic, since the mass will oscillate between $y = -A$ and $y = A$. The time between successive maxims, or the length of time required to complete one cycle of the motion, is the *period* of the motion and is given by

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}. \tag{11}$$

The reciprocal of the period, or the number of cycles per second*, is called the *natural frequency* of the system, which we denote by $f_0 = \omega_0/2\pi$. The value $\omega_0 = \sqrt{k/m}$ is known as the *angular frequency*.

Regardless of the values of the input parameters y_0 and v_0, the graph of (8) is simply that of a cosine curve (see Figure 5.2). By changing either y_0 or v_0, the

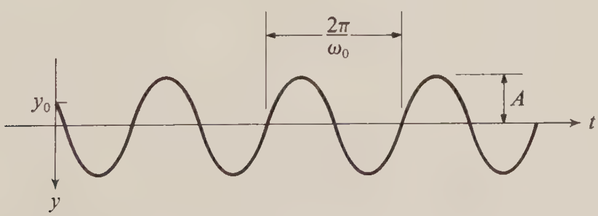


Figure 5.2 Simple harmonic motion.

*The unit cycles per second (cps) is now called hertz (Hz).

cosine curve will appear to shift a certain amount along the t-axis and possibly change amplitude.

EXAMPLE 3 Find the natural period of a spring-mass system for which the spring is stretched 4 in. by a 6-lb weight.

Solution Since 4 in. corresponds to $\frac{1}{3}$ ft, Hooke's law leads to

$$6 = k \cdot \frac{1}{3},$$

or $k = 18$ lb/ft. The mass $m = W/g = \frac{6}{32}$ slug, and hence

$$T = 2\pi\left(\frac{m}{k}\right)^{1/2} = 2\pi\left[\frac{6}{(32)(18)}\right]^{1/2} = \frac{\pi\sqrt{6}}{12}\text{ s.}$$

EXAMPLE 4 Suppose a 16-lb weight is attached to the spring in Example 3 and released 3 in. above the equilibrium point of the spring-mass system with an initial velocity of 2 ft/sec directed upward.* Describe the subsequent motion of the mass.

Solution In Example 3 we found that $k = 18$ lb/ft, and the 16-lb weight corresponds to a mass of $\frac{1}{2}$ slug. Thus the motion of the mass is described by the solution of the initial value problem

$$\frac{1}{2}y'' + 18y = 0, \quad y(0) = -\frac{1}{4}, \quad y'(0) = -2.$$

Solving, we get

$$y = -\frac{1}{4}\cos 6t - \frac{1}{3}\sin 6t.$$

Putting this solution in the form of Equation (8), we find the amplitude of motion given by

$$A = \sqrt{\frac{1}{16} + \frac{1}{9}} = \frac{5}{12},$$

whereas the phase angle satisfies $\cos\phi = -\frac{3}{5}$, $\sin\phi = -\frac{4}{5}$, or $\phi \cong 4.07$ rad (233°). Hence,

$$y = \frac{5}{12}\cos(6t - 4.07)$$

for which the period of motion is $T = 2\pi(m/k)^{1/2} = \pi/3$ (see Figure 5.3). The frequency is $f_0 = 3/\pi$ Hz.

---

*The *positive* y-axis has been chosen downward so that, for example, $y(0) = -y_0$ if the mass is pushed y_0 units above the equilibrium position. The same is true of the velocity.

Sometimes it is useful to know the values of time for which the graph of $y(t)$ crosses the positive t-axis. This corresponds physically to the mass passing through its equilibrium position. Writing the solution in the form of Equation (8) is very helpful for such calculations. For instance, using the solution in Example 4, we observe that $\cos(6t - 4.07) = 0$ when

$$6t - 4.07 = \frac{(2n-1)}{2}\pi,$$

where n is an integer. The first positive value of t that satisfies this relation is found to be $t_1 = 0.42$ rad ($n = 0$), whereas the next value is $t_2 = 0.94$ rad ($n = 1$), and so on (see Figure 5.3).

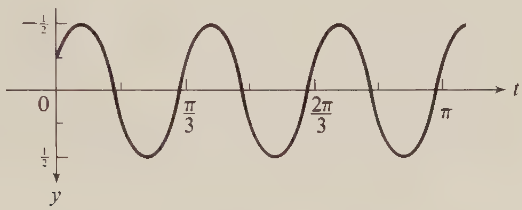


Figure 5.3

[O] 5.2.2 The Pendulum Problem

A mass m is suspended from the end of a rod of constant length b whose weight is negligible (see Figure 5.4). We wish to determine the equation of motion of the mass in terms of the angle of displacement θ.

Summing forces makes it clear that the weight component $mg \cos \theta$, acting in the normal direction to the path, is offset by the force of restraint in the rod. Therefore,

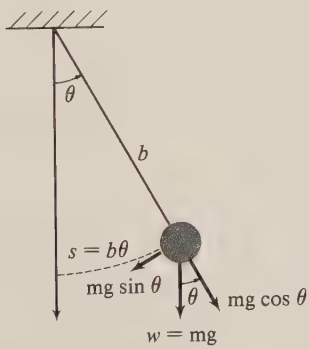


Figure 5.4

the only weight component contributing to the motion is $mg \sin \theta$, which acts in the direction of the tangent to the path. If we denote the arc length of the path by s, then Newton's second law of motion reads

$$m\frac{d^2s}{dt^2} = -mg \sin \theta,$$

where the minus sign illustrates that the tangential force component opposes the motion for s increasing. Now the arc length s of a circle of radius b is related to the central angle θ through the formula $s = b\theta$, and so the equation of motion (after simplification) transforms to

$$b\frac{d^2\theta}{dt^2} + g \sin \theta = 0. \tag{12}$$

The equation of motion (12) is *nonlinear* due to the term $\sin \theta$. To solve this equation exactly would necessitate introducing a special function called the *Jacobian elliptic function*,* since the equation has no solution that can be expressed in terms of elementary functions. However, if we restrict the motion so that θ is always a "small angle," then we might use the approximation $\sin \theta \cong \theta$ and replace (12) with the *linear* DE

$$b\frac{d^2\theta}{dt^2} + g\theta = 0. \tag{13}$$

In this form we recognize (13) as being equivalent in structure to the equation of motion of the undamped spring-mass system. Hence the pendulum problem for small motions is mathematically equivalent to the spring-mass problem, which once again illustrates the fact that the same DE can be used to describe contrasting physical phenomena.

5.2.3 Damped Free Motion

When damping effects are taken into account, the free motions of the spring-mass system are described by solutions of

$$my'' + cy' + ky = 0, \quad t > 0. \tag{14}$$

The auxiliary equation from which the solutions of (14) are determined is

$$m\lambda^2 + c\lambda + k = 0,$$

with roots

$$\lambda_1, \lambda_2 = \frac{-c \pm (c^2 - 4mk)^{1/2}}{2m}.\dagger \tag{15}$$

The solution obviously takes on three different forms, depending upon the magnitude of the damping term. The three cases are:

*See, for example, T. C. Bradbury, *Theoretical Mechanics* (New York: Wiley, 1968).

†We use the parameter λ here since m denotes mass.

Case I—Overdamping: $c^2 > 4mk$ (λ_1, λ_2 distinct and real)

Case II—Critical damping: $c^2 = 4mk$ $\left(\lambda_1 = \lambda_2 = -\dfrac{c}{2m}\right)$

Case III—Underdamping: $c^2 < 4mk$ (λ_1, λ_2 complex conjugates)

Let us discuss each case separately.

Case I—Overdamping: The damping is large compared with the spring constant. Both roots of the auxiliary equation are real and distinct, leading to the solution formula

$$y = e^{-ct/2m}\left\{C_1 \exp\left[(c^2 - 4mk)^{1/2}\frac{t}{2m}\right] + C_2 \exp\left[-(c^2 - 4mk)^{1/2}\frac{t}{2m}\right]\right\}. \quad (16)$$

This equation represents a smooth, nonoscillatory type of motion. Typical graphs of this motion are illustrated in Figure 5.5.

Case II—Critical damping: For this case the roots of the auxiliary equation are equal, so the solution takes the form

$$y = e^{-ct/2m}(C_1 + C_2 t). \quad (17)$$

The motions here are similar to those of the overdamped case, as shown in Figure 5.6.

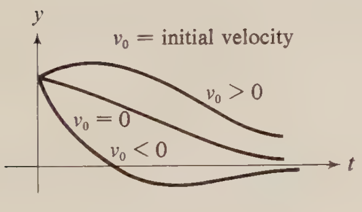


Figure 5.5 Overdamped motion.

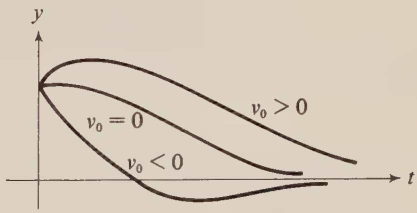


Figure 5.6 Critically damped motion.

Case III—Underdamping: This case is the most interesting of all three cases. The roots of the auxiliary equation are complex, and so the solution can be expressed as

$$y = e^{-ct/2m}(C_1 \cos \mu t + C_2 \sin \mu t), \quad (18)$$

where $\mu = (4mk - c^2)^{1/2}/2m$. Regardless of the initial conditions prescribed, the mass will oscillate back and forth across the equilibrium position with the amplitude of motion steadily decreasing in time (see Figure 5.7).

In all three cases, the solution of the homogeneous DE (14) contains the multiplicative factor $e^{-(c/2m)t}$, which tends to zero after a sufficiently long period of time.

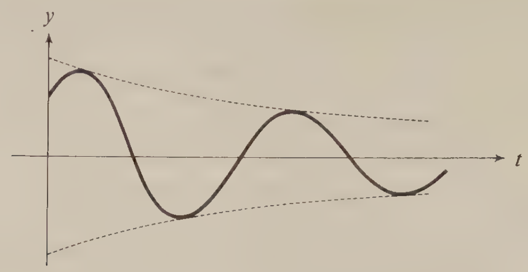


Figure 5.7 Underdamped motion.

Hence, regardless of the prescribed initial conditions, the solution of (14) must approach zero as time goes on. This situation confirms what we intuitively expect—without damping (friction), the motion of the system continues indefinitely; but with damping, the motion eventually dies out.

EXAMPLE 5 A spring-mass system involves a mass of 4 kg, a spring with $k = 64$ N/m, and a dashpot with $c = 32$ kg/s. The mass is lowered 1 m from its equilibrium position and released from rest. Determine the subsequent motion.

Solution The initial value problem describing the motion of the spring-mass system is

$$4y'' + 32y' + 64y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Dividing the equation by 4, we find the auxiliary equation $\lambda^2 + 8\lambda + 16 = 0$ with solutions $\lambda = -4, -4$. Thus the general solution of the DE is

$$y = (C_1 + C_2 t)e^{-4t},$$

and imposing the initial conditions leads to $C_1 = 1$ and $C_2 = 4$, resulting in

$$y = (1 + 4t)e^{-4t}.$$

The system is therefore critically damped. The mass will never pass through its equilibrium position $y = 0$, but will slowly approach it as time goes on.

Although damped motion is not truly periodic, it is sometimes convenient to introduce the notion of a *quasiperiod*, defined as the time between successive maxima of the displacement. That is, we write

$$\tilde{T} = \frac{2\pi}{\mu} = 2\pi\left(\frac{k}{m} - \frac{c^2}{4m^2}\right)^{-1/2}, \tag{19}$$

which can also be expressed in terms of T [given by (11)],

$$\tilde{T} = \frac{2\pi}{\omega_o}\left(1 - \frac{c^2}{4km}\right)^{-1/2} = T\left(1 - \frac{c^2}{4km}\right)^{-1/2}, \tag{20}$$

where $\omega_0 = (k/m)^{1/2}$. Here we see that when damping is small, i.e., when $c^2/4km << 1$, the quasiperiod $\bar{T}$ is approximately equal to T.

EXERCISES 5.2

1. Find the frequency of oscillation of a spring-mass system if the mass is 4 kg and the spring constant is 100 N/m. What is its natural period?

2. Calculate the time necessary for a 0.03-kg mass hanging from a spring with spring constant 0.5 N/m to undergo one full oscillation. What is the natural frequency of the system?

3. A spring-mass system is stretched 6 in. by a 12-lb weight. If the weight attached to the spring is pulled downward 4 in. below the equilibrium position and started upward with a velocity of 2 ft/s, show that the subsequent motion is described by the function $y = \frac{1}{3}\cos 8t - \frac{1}{4}\sin 8t$.

4. Show that the solution in problem 3 can be expressed in the form

$$y = A\cos(8t - \phi),$$

and find the first two positive values of time for which $y = 0$.

5. A 10-lb weight stretches a steel spring 2 in.

   (a) Determine the natural period of the spring-mass system.

   (b) If the spring is stretched an additional 2 in. and then released, describe the subsequent motion of the mass.

wrong answer

6. The period of free oscillations of a mass on a string is $\pi/2$ s. What is the numerical value of the length of the string in feet? *Hint*: See Equation (13).

7. A clock has a pendulum 1 m long. The clock ticks once for each time the pendulum makes a complete swing, returning to its original position. How many ticks will the clock make in 1 min?
Hint: Use Equation (13).

8. A 24-lb weight stretches a spring 4 in. Determine the equation of motion if the weight is released from a point 3 in. above the equilibrium position with a downward velocity of 2 ft/s.

9. The period of free, undamped oscillations of a spring-mass system is observed to be $\pi/4$ s. If the spring constant is given by 16 lb/ft, what is the numerical value of the weight in pounds?

10. Find the solution of the DE

$$my'' + ky = 0$$

in the form $A\cos(\omega_0 t - \phi)$ when the initial conditions are prescribed by

   (a) $y(0) = y_0, \quad y'(0) = 0.$

   (b) $y(0) = 0, \quad y'(0) = v_0.$

11. A mass of 1 slug is attached to a spring with $k = 9$ lb/ft. The mass initially starts moving from a point 1 ft above the equilibirum position with velocity $\sqrt{3}$ ft/s directed downward. Find the first positive value of time for which the mass is moving downward with a velocity of 3 ft/s.

12. Prove that the maximum value of the speed of a mass undergoing simple harmonic motion occurs when $y = 0$.

13. Determine the natural period of oscillation of the pendulum in problem 7 if the pendulum is 2 m long.

14. Solve Equation (13) of the pendulum problem when the weight is 8 lb and the rod is 1 ft long. Assume the weight is released from an angle of $\frac{1}{2}$ rad with a positive velocity of $\frac{3}{2}$ rad/s.

15. At what time does the pendulum in problem 14 first pass through the angle $\theta = 0$? What is its velocity at this time?

16. An 8-lb weight, attached to the end of a vertical spring, is pulled y_0 ft below its equilibrium position and released at time $t = 0$ with a downward velocity of 3 ft/s. Determine the spring constant k and the initial displacement y_0 if the amplitude of the resulting motion is known to be $\sqrt{5}$ and the period is $\pi/2$.

***17.** Show that *underdamped* free motion has the following characteristics:

(a) The characteristic angular frequency μ is independent of the initial conditions but decreases as c increases.

(b) The natural logarithm of the ratio of two consecutive maximum amplitudes is the constant $\delta = \pi c/m\mu$. The number δ is called the *logarithmic decrement* of the oscillation.

(c) Find δ in the case $y = e^{-t}\cos t$, and determine which values of t correspond to maximum and minimum displacements.

18. Show that *overdamped* free motion has the following characteristics:

(a) The mass cannot pass through $y = 0$ more than once.

(b) If the initial conditions are such that the constants C_1 and C_2 in the general solution have the same sign, the mass never passes through $y = 0$.

19. Under what conditions on y_0 and v_0, where $y_0 = y(0)$ and $v_0 = y'(0)$, will *critically damped* free motion have a maximum or a minimum for $t > 0$?

20. A spring is stretched 6 in. by a 3-lb weight, which is started from its equilibrium position with an upward velocity of 12 ft/s. If a retarding force equal in magnitude to 0.03v exists, find the resulting motion.

21. A certain straight line motion is described by the initial value problem

$$y'' + 2by' + 169y = 0, \quad y(0) = 0, \quad y'(0) = 8, \quad b > 0.$$

(a) Find the value of b that leads to critical damping.

(b) Find the solution for that value of b in (a).

(c) At what time does the motion momentarily stop, if at all?

22. A 4-lb weight is attached to a spring with spring constant 2 lb/ft. If the weight is released from 1 ft above the equilibrium position with a downward velocity of 8 ft/s, determine the time that the weight passes through the equilibrium position, assuming the retarding force is equal in magnitude to the instantaneous velocity. Find the time and position of the weight at its maximum displacement after passing through equilibrium.

23. Determine the maximum displacement of the free motion of a spring-mass system governed by the initial value problem

$$y'' + 5y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

Does the graph of y ever cross the t-axis?

24. Consider a spring-mass system experiencing a viscous damping term.*

(a) If the mass m is given an upward initial velocity of 50 m/s from the equilibrium position, find the motion given that $m = 4$ kg, $k = 64$ N/m, and $c = 40$ kg/s.

(b) Determine the time between consecutive maximum displacements of the mass when $m = 30$ kg, $k = 2000$ N/m, and $c = 300$ kg/s.

5.3 FORCED MOTIONS

If an external force $F(t)$ is also present, the initial value problem describing the possible forced motions of a spring-mass system reads

$$my'' + cy' + ky = F(t), \quad y(0) = y_0, \quad y'(0) = v_0. \tag{21}$$

To discuss such motions we will again take cases.

5.3.1 Undamped Forced Motion

For undamped motion we set $c = 0$ in (21) to get

$$my'' + ky = F(t). \tag{22}$$

In order to investigate undamped motions due entirely to the input function $F(t)$, it is convenient to assume the system is initially at rest. Thus we prescribe the conditions

$$y(0) = 0, \quad y'(0) = 0. \tag{23}$$

Let us suppose the external force is a constant described by $F(t) = P$. Assuming a particular solution of (22) of the form $y_p = A$, we find that $A = P/k$, giving the general solution

$$y = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + P/k,$$

where $\omega_0 = \sqrt{k/m}$ is the natural (angular) frequency of the system. Applying the initial conditions (23), we deduce

$$y = \frac{P}{k}(1 - \cos \omega_0 t). \tag{24}$$

In this case the mass oscillates at its natural frequency between the points $y = 0$ and $y = 2P/k$ with a period of $2\pi/\omega_0$ s.

If a sinusoidal force $F(t) = P \cos \omega t$ is applied, we assume a particular solution exists of the form (see Section 4.6)

*A viscous damping term arises if the spring-mass system is suspended in a fluid like oil or water, or if air resistance cannot reasonably be neglected.

$$y_p = A\cos\omega t + B\sin\omega t,$$

where the constants A and B are to be determined. The substitution of y_p into the DE leads to

$$(k - m\omega^2)(A\cos\omega t + B\sin\omega t) = P\cos\omega t.$$

Hence, equating like coefficients of $\cos\omega t$ and $\sin\omega t$, we obtain $A = P/(k - m\omega^2) = P/m(\omega_0^2 - \omega^2)$ and $B = 0$, so our general solution is

$$y = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{P}{m(\omega_0^2 - \omega^2)}\cos\omega t.$$

If we now impose the initial conditions (23), we see that $C_1 = -P/m(\omega_0^2 - \omega^2)$ and $C_2 = 0$, and therefore

$$y = \frac{P}{m(\omega_0^2 - \omega^2)}(\cos\omega t - \cos\omega_0 t) \tag{25}$$

provided $\omega \neq \omega_0$. This time the motion consists of two modes of vibration—the *natural mode* at frequency ω_0 and the *forced mode* at frequency ω.

An interesting phenomenon occurs when the forcing frequency in (25) is close to the natural frequency, i.e., when $|\omega - \omega_0|$ is small. By setting $\omega t = a + b$ and $\omega_0 t = a - b$, we can use the trigonometric identities

$$\cos(a \pm b) = \cos a\cos b \mp \sin a\sin b$$

to rewrite (25) in the form

$$y = \frac{2P}{m(\omega^2 - \omega_0^2)}\sin\left[(\omega - \omega_0)\frac{t}{2}\right]\sin\left[(\omega + \omega_0)\frac{t}{2}\right]. \tag{26}$$

Since $|\omega - \omega_0|$ is small, the period of the sine wave $\sin[(\omega - \omega_0)t/2]$ is large compared with the period of $\sin[(\omega + \omega_0)t/2]$. The motion described by (26) can then be visualized as a rapid oscillation with angular frequency $(\omega + \omega_0)/2$, but with a slowly varying sinusoidal amplitude (see Figure 5.8). Motion of this type, possessing a periodic variation of amplitude, exhibits what is called a *beat*. The

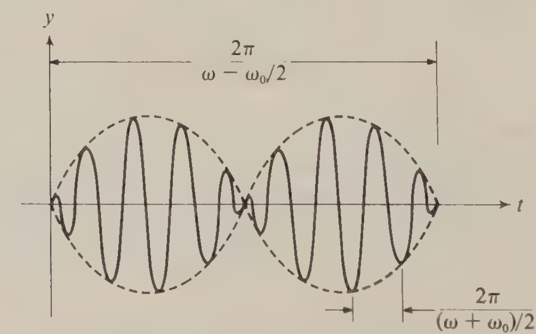


Figure 5.8 Phenomenon of beats.

phenomenon of beats can most easily be demonstrated with acoustic waves—for example, when two tuning forks of nearly the same frequency are sounded at the same time.

In the special case when the system is excited at its natural frequency ($\omega = \omega_0$), the response of the system becomes [from (25)]

$$y = \lim_{\omega \to \omega_0} \left[\frac{P}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t) \right],$$

which, evaluated through use of L'Hôpital's rule, leads to

$$y = \frac{P}{2m\omega_0} t \sin \omega_0 t. \tag{27}$$

It is clear that the amplitude of motion will become unbounded in (27) as $t \to \infty$, and thus we have the phenomenon of *resonance* (see Figure 5.9). Of course, we recognize that in a physical problem the amplitude cannot become unbounded. A certain amount of damping, however small, is always present, and this has the effect of limiting the amplitude. Also, if the amplitude should become large enough, the system is likely to fail. This situation has actually caused certain bridges to collapse, such as the Tacoma Narrows bridge at Puget Sound in the state of Washington. On November 7, 1940, only four months after its grand opening, a huge portion of the bridge collapsed into the water below. From the very beginning the bridge had experienced large undulations, which were later attributed to the wind blowing across the superstructure. Therefore, in designing such structures, it is very important to make the natural frequency of the structure different (if possible) from the frequency of any probable forcing function.

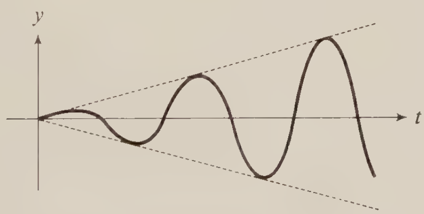


Figure 5.9 Phenomenon of resonance.

Remark. It should be noted that the reason soldiers do not march in step across bridges is to avoid any possibility of resonance occurring between the natural frequency of the bridge and the frequency of the uniformly stomping feet.

5.3.2 Damped Forced Motion

When damping effects are important, we can think of the solution of (21) as composed of two parts, i.e., $y = y_P + y_H$. Since it contains the multiplicative factor $e^{-(c/2m)t}$, the solution function y_H contributes only initial effects to the motion and it is called the *transient solution*. The function y_P dominates the response of the system after initial effects diminish and is therefore referred to as the *steady-state solution* of the system.*

Let us assume the forcing function is $F(t) = P\cos\omega t$. Then it seems natural to assume that the steady-state solution has the general form

$$y_p = A\cos\omega t + B\sin\omega t, \tag{28}$$

where A and B must be determined. Observe that we are assuming that the frequency of the steady-state motion is the same as the frequency of the forcing term producing the motion. When (28) is substituted into the DE, we obtain

$$[(k - m\omega^2)A + \omega cB]\cos\omega t + [(k - m\omega^2)B - \omega cA]\sin\omega t = P\cos\omega t.$$

Equating like coefficients of $\cos\omega t$ and $\sin\omega t$ results in the simultaneous solution of A and B

$$A = \frac{P(k - m\omega^2)}{(k - m\omega^2)^2 + \omega^2c^2}, \quad B = \frac{P\omega c}{(k - m\omega^2)^2 + \omega^2c^2}.$$

The particular solution, or steady-state solution, is then

$$y_P = \frac{P(k - m\omega^2)}{(k - m\omega^2)^2 + \omega^2c^2}\left[\cos\omega t + \frac{\omega c}{k - m\omega^2}\sin\omega t\right]. \tag{29}$$

Since the amplitude R of the oscillation is of some importance, let us rewrite (29) in the equivalent form

$$y_P = R\cos(\omega t - \phi), \tag{30}$$

where ($\omega_0 = \sqrt{k/m}$)

$$R = \frac{P}{[m^2(\omega_0^2 - \omega^2)^2 + \omega^2c^2]^{1/2}} \tag{31}$$

and the phase angle ϕ is such that

$$\tan\phi = \frac{\omega c}{k - m\omega^2} = \frac{\omega c}{m(\omega_0^2 - \omega^2)}. \tag{32}$$

Unlike the case of undamped motion, here we see that the maximum value of R does not occur when $\omega = \omega_0$. The maximum amplitude of the motion can be found by setting $dR/d\omega = 0$, which occurs when (see problem 19 in this section)

$$\omega^2 = \omega_0^2 - \frac{c^2}{2m^2}. \tag{33}$$

However, for sufficiently large damping such that $c^2 > 2m^2\omega_0^2$, there is no value of ω satisfying (33), and hence no maximum amplitude. For damping coefficients

*By steady state, we mean only that portion of y_P that does not go to 0 as $t \to \infty$.

satisying $c^2 < 2m^2\omega_0^2$, the maximum amplitude is found to be (see problem 20 in this section)

$$R_{\max} = \frac{2Pm}{c[4m^2\omega_0^2 - c^2]^{1/2}}. \tag{34}$$

The amplitude R given by (31) is plotted in Figure 5.10 as a function of the input frequency ω. It thus becomes clear that large amplitudes due to resonance can be avoided by a sufficient amount of damping.

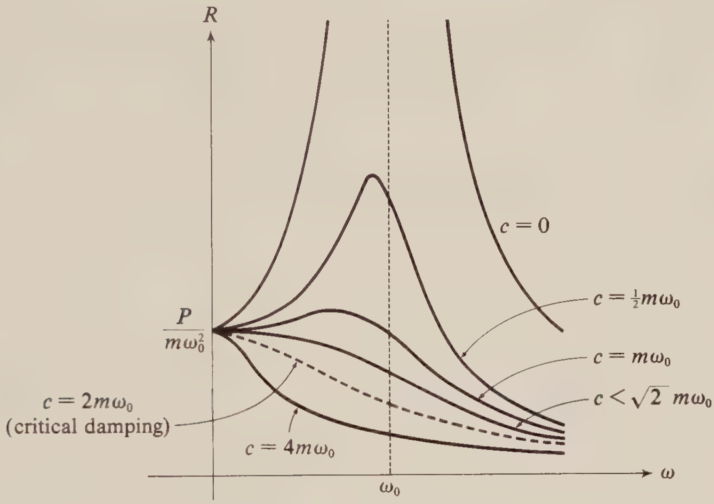


Figure 5.10 The amplitude R as a function of ω.

EXAMPLE 6 Interpret and solve the initial value problem

$$\frac{1}{2}y'' + 2y' + 10y = 5\cos 2t, \quad y(0) = \frac{1}{2}, \quad y'(0) = 0.$$

Solution We can interpret the problem as representing a spring-mass system consisting of a mass of 1/2 unit, spring constant equal to 10 units, a damping term equal to twice the instantaneous velocity, and a periodic forcing term equal to $5\cos 2t$. The mass is lowered 1/2 unit from the equilibrium position and released with zero velocity.

To solve this problem, we find it convenient to first rewrite the DE in the form

$$y'' + 4y' + 20y = 10\cos 2t.$$

The auxiliary equation of the associated homogeneous DE is $m^2 + 4m + 20 = 0$ with roots $m = -2 \pm 4i$. Hence,

$$y_H = e^{-2t}(C_1\cos 4t + C_2\sin 4t).$$

Using the method of undetermined coefficients, we set

$$y_P = A\cos 2t + B\sin 2t.$$

Upon differentiation and substitution into the nonhomogeneous DE, we find that the coefficients A and B must satisfy the equations

$$-8A + 16B = 0, \quad 16A + 8B = 10.$$

Calculating, we have $A = \frac{1}{2}$ and $B = \frac{1}{4}$, so that

$$y_P = \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t.$$

Combining y_H and y_P leads to the general solution

$$y = e^{-2t}(C_1\cos 4t + C_2\sin 4t) + \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t.$$

Applying the initial conditions, we see that

$$C_1 + \frac{1}{2} = \frac{1}{2},$$

$$2C_1 - 4C_2 - \frac{1}{2} = 0,$$

and hence it follows that $C_1 = 0$, $C_2 = -\frac{1}{8}$. Therefore,

$$y = \underbrace{-\frac{1}{8}e^{-2t}\sin 4t}_{\text{transient solution}} + \underbrace{\frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t}_{\text{steady-state solution}}.$$

After a short period of time ($t > 3$), the dominant part of the solution is the steady-state term, which represents simple harmonic motion with maximum amplitude

$$R = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2} \cong 0.56$$

and period π.

The undetermined-coefficient method of producing the steady-state solution is effective only when the forcing function is of a simple nature like Example 6. Of particular importance in many applications, however, is the case when the forcing function is a periodic function other than a simple sinusoid. When this situation occurs, the method of *Fourier series* becomes a useful device, and when the forcing function is of a more general nature, the method of *Green's function* (Section 5.5) can be quite effective.

EXERCISES 5.3

1. A 2-lb weight stretches a spring 6 in. An impressed force 16 sin $8t$ is acting upon the spring, and the weight is pulled down 3 in. below the equilibrium position and released. Determine the equation of motion.

2. If an impressed force $\frac{1}{8}\cos 4t$ is imposed upon the system in problem 3 of Exercises 5.2, determine the subsequent motion.

***3.** A spring with spring constant $k = 0.75$ lb/ft has a weight of 6 lb attached, which is at rest in the equilibrium position. A $1\frac{1}{2}$-lb force is applied to the weight in the downward direction for 4 s and then removed. Discuss the subsequent motion.

4. A spring stretches 6 in. when a 4-lb weight is attached. If the weight is started from the equilibrium position with an upward velocity of 4 ft/s and has an impressed force of $\frac{1}{2}\cos 8t$ acting on the weight, determine the position of the weight for all time. What is the position when $t = 2$ s?

5. A 2-kg mass is attached to a spring with $k = 32$ N/m. A force of $0.1 \sin 4t$ is applied to the mass, which is at rest. Neglecting damping, calculate the time required for failure to occur if the spring breaks when the amplitude of oscillation exceeds 0.5 m.

6. Show that the solution of

$$y'' + 25y = 10\cos 7t, \quad y(0) = 0, \quad y'(0) = 0,$$

is given by $y = \frac{5}{6}\sin t \sin 6t$. How many seconds are there between beats?

7. A 20-N weight is suspended by a frictionless spring for which $k = 98$ N/m. An external force of $2\cos 7t$ acts on the weight. Find the frequency of the beat, and determine the maximum amplitude of the motion that starts from rest.

8. How many seconds are there between the beats in problem 7?

9. Verify Equation (27) by solving directly the initial value problem

$$my'' + ky = P\cos\omega_0 t, \quad y(0) = y'(0) = 0, \quad \omega_0 = \sqrt{k/m}.$$

10. Assuming $c^2 - 4mk < 0$, determine the complete solution of the DE

$$my'' + cy' + ky = P\cos\omega t,$$

satisfying the following sets of initial conditions:

(a) $y(0) = y_0, \quad y'(0) = 0,$

(b) $y(0) = 0, \quad y'(0) = v_0,$

(c) $y(0) = y_0, \quad y'(0) = v_0.$

11. Show that the motion of a body rising with drag proportional to velocity is given by

$$my'' + cy' + mg = 0.$$

12. For problem 11, assume the initial velocity of the body is 100 m/s upward, $c = 0.4$ kg/s, and $m = 2$ kg. How high will the body rise?

13. For the body of problem 12, determine the time required for the body to reach its maximum height, and compare this with the time it takes for the body to fall back to its original position.

14. Verify Equation (31).

15. Determine both the transient and steady-state solutions of the initial value problem

$$y'' + 2y' + 2y = 4\cos t + 2\sin t, \quad y(0) = 0, \quad y'(0) = 3.$$

16. A mass of $\frac{1}{2}$ slug is attached to a spring with spring constant $k = 6$ lb/ft. A damping force numerically equal to twice the instantaneous velocity acts on the system.

(a) Find the steady-state response of the system due to an external driving force $F(t) = 40\sin 2t$.

(b) Will R_{max} occur?

(c) What is the amplitude in this case?

17. Find the steady-state response of the system in problem 16 if the driving force is constant; i.e., $F(t) = P$.

***18.** The ratio of successive maximum amplitudes of a particular underdamped spring-mass system is found to be 1.25 when the system undergoes free motion. If $k = 100$ N/m, $m = 4$ kg, and a driving force of $F(t) = 10\cos 4t$ is imposed on the system, determine the amplitude of the steady-state motion.

19. Show that $dR/d\omega = 0$, where R is defined by (31), occurs when the frequency ω satisfies $\omega^2 = \omega_0^2 - c^2/2m^2$.

20. Derive Equation (34).

5.4 SIMPLE ELECTRIC CIRCUITS

Let us consider an electric circuit composed of a resistance R, capacitance C, and inductance L, connected in series with a voltage source $E(t)$, as shown in Figure 5.11. When the switch is closed at time $t = 0$, a current $i = i(t)$ will flow in the loop. The network featured here is called an *RLC circuit*. The current is determined at each point in the network by solving appropriate differential equations that result from applying Kirchhoff's laws:

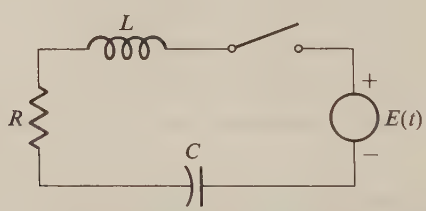


Figure 5.11 *RLC* circuit.

1. The sum of the currents into (or away from) any point is zero.

2. The sum of the instantaneous voltage drops in a specified direction is zero around any closed path.

The first Kirchhoff law indicates that the current is the same throughout the circuit. To apply the second law, we must know the voltage drop across each of the idealized elements in the *RLC* circuit. From experimental observations, we have

$$\text{voltage drop across a resistor} = Ri,$$

$$\text{voltage drop across an inductor} = L\frac{di}{dt},$$

$$\text{voltage drop across a capacitor} = \frac{q}{C},$$

where q denotes the electric charge on the capacitor and is related to the current i by $i = dq/dt$. The impressed electromotive force $E(t)$ contributes to a voltage gain. Applying the second Kirchhoff law to the circuit shown leads to the differential equation

$$L\frac{di}{dt} + Ri + C^{-1}q = E(t). \tag{35}$$

In terms of the charge q, we find

$$Lq'' + Rq' + C^{-1}q = E(t), \quad t > 0. \tag{36}$$

We recognize that (36) is the same equation in form as that which governs the damped motions of a spring-mass system. In fact, we notice the following analogies between mechanical and electrical systems.

1. *Charge q* corresponds to *position y*.
2. *Current i* corresponds to *velocity y'*.
3. *Inductance L* corresponds to *mass m*.
4. *Resistance R* corresponds to *damping constant c*.
5. *Inverse capacitance C^{-1}* corresponds to *spring constant k*.
6. *Electromotive force $E(t)$* corresponds to *input function $F(t)$*.

Such analogies between mechanical and electrical systems prove very useful in practice. For example, in studying a certain mechanical system that is either too complicated or too expensive to build, the electrical counterpart is often constructed instead for the purpose of analysis. Interestingly, the phenomenon of resonance also occurs in electrical systems, but it does not have the undesired side effects of mechanical resonance. Quite the contrary, it is primarily because of electrical resonance that we can tune a radio to the frequency of the transmitting radio station in order to obtain reception.

The units most commonly used are listed in Table 5.2

Table 5.2 Common System of Units and Their Abbreviations

Quantity	*Unit*
emf or voltage (E)	volt (V)
charge (q)	coulomb
current (i)	ampere (amp)
resistance (R)	ohm (Ω)
capacitance (C)	farad (F)
inductance (L)	henry (H)

EXAMPLE 7 The circuit shown in Figure 5.11 contains the components $L = 1\text{H}$, $R = 1000\ \Omega$, and $C = 4 \times 10^{-6}\,\text{F}$. At time $t = 0$, both current and charge are zero, and a battery supplying a constant voltage of 24 V is instantaneously switched on. Find the charge $q(t)$ on the capacitor and the current $i(t)$ for any later time.

Solution The DE to be solved is given by

$$q'' + 1000q' + \frac{1}{4} \times 10^6 q = 24.$$

The associated homogeneous equation has the general solution

$$q_H(t) = (C_1 + C_2 t)e^{-500t}.$$

To find a particular solution q_P, we substitute $q_P = A$ into the nonhomogeneous DE, getting

$$A = 9.6 \times 10^{-5}.$$

Therefore, since $q = q_P + q_H$, we find

$$q(t) = 9.6 \times 10^{-5} + (C_1 + C_2 t)e^{-500t},$$

and differentiating gives

$$q'(t) = i(t) = -500C_1 e^{-500t} + C_2(1 - 500t)e^{-500t}.$$

The prescribed initial conditions $q(0) = 0$ and $i(0) = 0$ lead to

$$0 = 9.6 \times 10^{-5} + C_1, \quad 0 = -500C_1 + C_2.$$

Hence, $C_1 = -9.6 \times 10^{-5}$ and $C_2 = -4.8 \times 10^{-2}$, from which we deduce

$$q(t) = 9.6 \times 10^{-5}(1 - e^{-500t}) - 4.8 \times 10^{-2} te^{-500t} \text{ coulombs}$$

and

$$i(t) = 24te^{-500t} \text{ amp}.$$

EXERCISES 5.4

1. If the resistance R is not included in the RLC circuit of Figure 5.11, we have what is called an LC circuit. Show that the current $i(t)$ of the LC circuit satisfies the relation

$$i'(0) = \frac{E(0)}{L} - \frac{q(0)}{LC}, \quad LC \neq 0,$$

where $q(0)$ is the charge in the capacitor at time $t = 0$.

2. Using problem 1, find the current in the LC circuit when $i(0) = 0$, $q(0) = 0$, and

(a) $L = 1$ H, $C = 0.25$ F, $E(t) = 30 \sin t$ V.

(b) $L = 10$ H, $C = 0.1$ F, $E(t) = 10t$ V.

3. A *steady-state current* in the RLC circuit results after a sufficient length of time ($t \to \infty$). Find the steady-state current where

(a) $R = 4\ \Omega$, $L = 1$ H, $C = 2 \times 10^{-4}$ F, $E(t) = 220$ V.

(b) $R = 10\ \Omega$, $L = 2$ H, $C = 0.5$ F, $E(t) = 10.9 \cos 2t$ V.

4. Show that the current $i(t)$ in the RLC circuit satisfies

$$i'(0) = \frac{E(0)}{L} - \frac{R}{L} i(0) - \frac{q(0)}{LC}, \quad LC \neq 0,$$

where $q(0)$ is the charge in the capacitor at time $t = 0$.

5. Find the current in the RLC circuit assuming $i(0) = q(0) = 0$, and

(a) $R = 6\ \Omega$, $L = 1$ H, $C = 0.04$ F, $E(t) = 24 \cos 5t$ V.

(b) $R = 80\ \Omega$, $L = 20$ H, $C = 0.01$ F, $E(t) = 100$ V.

6. What conditions on the circuit parameters R, C, and L must be satisfied for the charge variation to be (see Section 5.2.3)

(a) underdamped?

(b) overdamped?

(c) critically damped?

7. A series RLC circuit has components $L = \frac{1}{2}$ H, $R = 10\ \Omega$, $C = 10^{-2}$ F, and $E(t) = 150$ V. Determine the instantaneous charge on the capacitor for $t > 0$ if initially $q(0) = 1$ and $i(0) = 0$. What charge persists after a long period of time?

8. An electrical circuit has components $L = 10^{-3}$ H, $C = 2 \times 10^{-5}$ F, and a resistor R. Determine the critical resistance necessary to lead to an oscillatory current if the elements are connected in series.

***9.** The amplitudes of two successive maximum currents in a series circuit with $L = 10^{-4}$ H and $C = 10^{-6}$ F are measured to be 0.2 amp and 0.01 amp. Determine the resistance R.

10. A particular RLC circuit connected in series has an electromotive force given by $E(t) = E_0 \sin \omega t$.

(a) Show that the steady-state current (as $t \to \infty$) is

$$i(t) = \frac{E_0}{Z}\left(\frac{R}{Z}\sin \omega t - \frac{X}{Z}\cos \omega t\right),$$

where $X = L\omega - 1/C\omega$ and $Z = \sqrt{X^2 + R^2}$. The quantity X is called the *reactance* of the circuit, and Z is the *impedance* of the circuit.

(b) Show that when

$$\omega = \frac{1}{\sqrt{LC}},$$

the amplitude of the steady-state current is a maximum. *Electrical resonance* is said to occur for this value of ω.

[O] 5.5 THE METHOD OF GREEN'S FUNCTIONS

We now wish to develop a general solution technique applicable to initial value problems involving higher-order DEs. The method of attack is attributed to George Green* and is based upon the construction of a particular function known as the *Green's function*.

Since most of the DEs that commonly occur in practice are of the second order, we will develop the theory for these equations and generalize the results to order n in the exercises. Thus we will confine our attention to the initial value problem

$$M[y] = f(t), \quad t > t_0, \quad y(t_0) = k_0, \quad y'(t_0) = k_1, \tag{37}$$

where M is the normal second-order differential operator

$$M = D^2 + a_1(t)D + a_0(t). \tag{38}$$

For solution purposes it is convenient to separate (37) into two simpler problems:

$$M[y] = 0, \quad y(t_0) = k_0, \quad y'(t_0) = k_1, \tag{39}$$

and

$$M[y] = f(t), \quad y(t_0) = 0, \quad y'(t_0) = 0. \tag{40}$$

The solution of (39), which we denote by y_H, physically represents the free response of the system described by (37) entirely due to the initial conditions. Therefore, the solution of (40) can be interpreted as the response of the system which is at rest until time $t = t_0$, when it is subjected to the external disturbance $f(t)$. This latter solution function is denoted by y_P, and the sum $y = y_P + y_H$ then constitutes the solution of (37) (see problems 46 and 47 in this section).

Let the general solution of $M[y] = 0$ in (39) be denoted by

$$y_H = C_1y_1(t) + C_2y_2(t). \tag{41}$$

*George Green (1793–1841) gained recognition for his important works concerning the reflection and refraction of sound and light waves. He also extended the work of Poisson in the theory of electricity and magnetism.

Imposing the prescribed initial conditions on (41), we get

$$C_1y_1(t_0) + C_2y_2(t_0) = k_0,$$
$$C_1y_1'(t_0) + C_2y_2'(t_0) = k_1,$$

and using Cramer's rule leads to the unique determination

$$\begin{aligned} C_1 &= \frac{\begin{vmatrix} k_0 & y_2(t_0) \\ k_1 & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} = \frac{k_0y_2'(t_0) - k_1y_2(t_0)}{W(y_1, y_2)(t_0)}, \\ C_2 &= \frac{\begin{vmatrix} y_1(t_0) & k_0 \\ y_1'(t_0) & k_1 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} = \frac{k_1y_1(t_0) - k_0y_1'(t_0)}{W(y_1, y_2)(t_0)}. \end{aligned} \tag{42}$$

Remark. The explicit representation for the constants C_1 and C_2 as given by (42) is mostly of theoretical importance. In practice, it is usually just as convenient to solve for these constants directly, as illustrated in Example 8 below.

Notice that when $k_0 = k_1 = 0$, we get $C_1 = C_2 = 0$, so that necessarily $y_H = 0$. The physical implication of this result is that a system which is initially at rest and not subject to any external disturbance must remain at rest.

Theorem 5.2 The homogeneous initial value problem

$$M[y] = 0, \quad y(t_0) = 0, \quad y'(t_0) = 0,$$

has only the trivial solution $y_H = 0$.*

EXAMPLE 8 Solve $y'' + y = 0, \quad y(0) = 1, \quad y'(0) = -1.$

Solution The general solution of the DE is

$$y_H = C_1 \cos t + C_2 \sin t.$$

Imposing the initial conditions, we find

$$y(0) = C_1 + C_2 \cdot 0 = C_1,$$
$$y'(0) = -C_1 \cdot 0 + C_2 = C_2,$$

and deduce that $C_1 = 1$ and $C_2 = -1$. Thus, our solution is

$$y_H = \cos t - \sin t.$$

---

*Theorem 5.2 is a repeat of Corollary 5.1 for the case of second-order DEs.

5.5.1 The One-Sided Green's Function

In order to solve the problem described by (40), we will start by assuming y_P has the form

$$y_P = u(t)y_1(t) + v(t)y_2(t) \tag{43}$$

for some functions $u(t)$ and $v(t)$. Using the method of *variation of parameters* as discussed in Section 4.7, we find that

$$u(t) = -\int \frac{y_2(t)f(t)}{W(y_1, y_2)(t)}\,dt, \quad v(t) = \int \frac{y_1(t)f(t)}{W(y_1, y_2)(t)}\,dt,$$

where $W(y_1, y_2) = y_1y_2' - y_1'y_2$ is the Wronskian function. If we choose $u(t)$ and $v(t)$ as any indefinite integrals as indicated above, the resulting solution function y_P is unlikely to satisfy the prescribed homogeneous initial conditions. A proper choice turns out to be

$$u(t) = -\int_{t_0}^{t} \frac{y_2(\tau)f(\tau)}{W(y_1, y_2)(\tau)}\,d\tau, \quad v(t) = \int_{t_0}^{t} \frac{y_1(\tau)f(\tau)}{W(y_1, y_2)(\tau)}\,d\tau, \tag{44}$$

and substituting these expressions back into (43) leads to

$$y_P = -y_1(t)\int_{t_0}^{t} \frac{y_2(\tau)f(\tau)}{W(y_1, y_2)(\tau)}\,d\tau + y_2(t)\int_{t_0}^{t} \frac{y_1(\tau)f(\tau)}{W(y_1, y_2)(\tau)}\,d\tau,$$

which we choose to write as

$$y_P = \int_{t_0}^{t} g_1(t, \tau)f(\tau)\,d\tau, \tag{45}$$

where

$$g_1(t, \tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{W(y_1, y_2)(\tau)} = \frac{\begin{vmatrix} y_1(\tau) & y_2(\tau) \\ y_1(t) & y_2(t) \end{vmatrix}}{\begin{vmatrix} y_1(\tau) & y_2(\tau) \\ y_1'(\tau) & y_2'(\tau) \end{vmatrix}}. \tag{46}$$

To show that (45) satisfies the homogeneous initial conditions in (40), first observe that when $t = t_0$, we have

$$y_P(t_0) = \int_{t_0}^{t_0} g_1(t_0, \tau)f(\tau)\,d\tau = 0.$$

Next, using the Leibniz formula (from calculus)

$$\frac{d}{dt}\int_a^t F(x, t)\,dx = \int_a^t \frac{\partial F}{\partial t}(x, t)\,dx + F(t, t),$$

we find

$$y_P'(t_0) = \int_{t_0}^{t_0} \frac{\partial g_1}{\partial t}(t_0, \tau)f(\tau)\,d\tau + g_1(t_0, t_0)f(t_0),$$

which also is zero since $g_1(t_0, t_0) = 0$ by definition.

The function $g_1(t, \tau)$ is called the *one-sided Green's function* for the initial value problem described by (37). Its construction depends only upon knowledge of the homogeneous solutions $y_1(t)$ and $y_2(t)$; i.e., it is independent of t_0 and the prescribed initial conditions and is completely determined by the operator $M = D^2 + a_1(t)D + a_0(t)$.

Remark. It is important to observe that the particular solution (45) was derived under the assumption that the DE was in *normal form*. Strict adherence to this form is necessary for proper identification of the forcing function $f(t)$.

The original initial value problem

$$M[y] = f(t), \quad y(t_0) = k_0, \quad y'(t_0) = k_1,$$

has the solution $y = y_P + y_H$, which can now be expressed as

$$y = \int_{t_0}^{t} g_1(t, \tau) f(\tau)\, d\tau + C_1 y_1(t) + C_2 y_2(t), \tag{47}$$

where the constants C_1 and C_2 are defined by (42).

Remark. Although (47) represents a general solution formula for the given initial value problem, it does not always represent the simplest approach to finding the solution. The Green's function method is important for developing and understanding some of the general theory, and it can be useful in those situations where the same DE must be solved a number of times with various input functions. It is the responsibility of the practitioner to determine those occasions for which such general formulas are useful.

EXAMPLE 9 Determine the one-sided Green's function for the operator $M = D^2 + 1$.

Solution Linearly independent solutions of $M[y] = 0$ are simply $y_1(t) = \cos t$ and $y_2(t) = \sin t$. The Wronskian is

$$W(y_1, y_2)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1,$$

and hence, from (46), it follows that

$$g_1(t, \tau) = \begin{vmatrix} \cos \tau & \sin \tau \\ \cos t & \sin t \end{vmatrix} = \sin t \cos \tau - \cos t \sin \tau = \sin(t - \tau).$$

With $g_1(t, \tau)$ determined for the operator $M = D^2 + 1$ as given in Example 9, we are now in position to solve all DEs of the form $y'' + y = f(t)$, a fact that clearly illustrates the power and economy of using the Green's function.

EXAMPLE 10 Using the one-sided Green's function, solve the initial value problem

$$y'' + y = \sin t, \quad y(0) = 1, \quad y'(0) = -1.$$

Solution From Example 9, we have $g_1(t, \tau) = \sin(t - \tau)$. Thus,

$$\begin{aligned} y_P &= \int_0^t \sin(t - \tau) \sin \tau \, d\tau \\ &= \sin t \int_0^t \cos \tau \sin \tau \, d\tau - \cos t \int_0^t \sin^2 \tau \, d\tau \\ &= \frac{1}{2}(\sin t - t \cos t). \end{aligned}$$

From Example 8, it has been determined that $y_H = \cos t - \sin t$, and hence,

$$\begin{aligned} y &= \frac{1}{2}(\sin t - t \cos t) + \cos t - \sin t \\ &= \left(1 - \frac{t}{2}\right) \cos t - \frac{1}{2} \sin t. \end{aligned}$$

EXAMPLE 11 Solve the initial value problem

$$t^2y'' - 3ty' + 3y = 2t^4e^t, \quad y(1) = 0, \quad y'(1) = 2.$$

Solution This is a Cauchy-Euler equation. Using the method of Section 4.8, we find the general solution of the associated homogeneous DE to be

$$y_H = C_1t + C_2t^3.$$

Subjecting this solution to the initial conditions, we find

$$\begin{aligned} C_1 + C_2 &= 0, \\ C_1 + 3C_2 &= 2, \end{aligned}$$

from which it follows that $C_2 = -C_1 = 1$. Hence,

$$y_H = t^3 - t.$$

The Wronskian of $y_1 = t$ and $y_2 = t^3$ is

$$W(t, t^3) = \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} = 2t^3,$$

and therefore the one-sided Green's function is given by

$$g_1(t, \tau) = \frac{1}{2\tau^3} \begin{vmatrix} \tau & \tau^3 \\ t & t^3 \end{vmatrix} = \frac{\tau t^3 - t\tau^3}{2\tau^3}.$$

Dividing the DE by the leading coefficient t^2 puts the equation in normal form, from which we identify $f(t) = 2t^2e^t$. Therefore,

$$y_P = \int_1^t g_1(t, \tau)(2\tau^2e^\tau) \, d\tau^*$$

*Note that the lower limit of integration is 1 since the initial conditions are prescribed there.

$$= t^3 \int_1^t e^\tau \, d\tau - t \int_1^t \tau^2 e^\tau \, d\tau$$

$$= (t - t^3)e + 2t(t - 1)e^t.$$

Finally, combining solutions, we have

$$y = y_P + y_H = t(e - 1) + t^3(1 - e) + 2t(t - 1)e^t.$$

It should be observed that one of the distinct features of the Green's function technique is that the nonhomogeneous initial conditions are imposed only upon the solution of the associated homogeneous DE. This is in sharp contrast with the methods employed in Chapter 4, wherein it was first necessary to find a general solution of the nonhomogeneous DE before applying the prescribed initial conditions. The reason for this situation, of course, is that we are selecting a particular solution y_P that always satisfies homogeneous initial conditions.

5.5.2 A Table of Some One-Sided Green's Functions

A listing of one-sided Green's functions for some of the more common differential operators is provided in Table 5.3 for easy reference.

Table 5.3 Table of One-Sided Green's Functions

	$M(D)$	$g_1(t, \tau)$
1.	D^2	$t - \tau$
2.	$D^n,\ n = 1, 2, 3, \ldots$	$\dfrac{(t - \tau)^{n-1}}{(n - 1)!}$
3.	$D^2 + b^2$	$\dfrac{1}{b} \sin b(t - \tau)$
4.	$D^2 - b^2$	$\dfrac{1}{b} \sinh b(t - \tau)$
5.	$(D - a)(D - b),\ a \neq b$	$\dfrac{1}{a - b}\left[e^{a(t-\tau)} - e^{b(t-\tau)}\right]$
6.	$(D - a)^2$	$(t - \tau)e^{a(t-\tau)}$
7.	$(D - a)^n,\ n = 1, 2, 3, \ldots$	$\dfrac{(t - \tau)^{n-1}}{(n - 1)!} e^{a(t-\tau)}$
8.	$D^2 - 2aD + a^2 + b^2$	$\dfrac{1}{b} e^{a(t-\tau)} \sin b(t - \tau)$
9.	$D^2 - 2aD + a^2 - b^2$	$\dfrac{1}{b} e^{a(t-\tau)} \sinh b(t - \tau)$
10.	$t^2D^2 + tD - b^2$	$\dfrac{\tau}{2b}\left[\left(\dfrac{t}{\tau}\right)^b - \left(\dfrac{\tau}{t}\right)^b\right]$

EXERCISES 5.5

In problems 1–8, find the solution satisfying the prescribed initial conditions.

1. $y'' + y = 0, \quad y(0) = 0, \quad y'(0) = 1$
2. $y'' - y = 0, \quad y(0) = 0, \quad y'(0) = 1$
3. $y'' - 4y' + 3y = 0, \quad y(0) = -1, \quad y'(0) = 3$
4. $y'' + 2y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = -3$
5. $y''' + y'' = 0, \quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = -1$
6. $t^2y'' + 4ty' + 2y = 0, \quad y(1) = 1, \quad y'(1) = 2$
7. $t^2y'' - 3ty' + 4y = 0, \quad y(1) = 2, \quad y'(1) = 1$
8. $t^2y'' + ty' + 4y = 0, \quad y(1) = 2, \quad y'(1) = 8$

In problems 9–18, determine the one-sided Green's function for the given operator.

9. $M = D^2$
10. $M = D^2 + 5$
11. $M = D^2 - 5$
12. $M = D^2 + 4D + 4$
13. $M = 4D^2 - 8D + 5$
14. $M = D^2 - D - 2$
15. $M = t^2D^2 + tD - 16$
16. $M = t^2D^2 - tD^2 + 1$

*17. $M = D(1 - t^2)D$

*18. $M = tD^2 - (1 + 2t^2)D$

In problems 19–24, use the one-sided Green's function to solve the given initial value problem.

19. $y'' = 1, \quad y(0) = 0, \quad y'(0) = 0$
20. $y'' = e^t, \quad y(0) = 2, \quad y'(0) = -1$
21. $y'' - y = 1, \quad y(0) = 0, \quad y'(0) = 1$
22. $y'' + y = e^{t-1}, \quad y(1) = 0, \quad y'(1) = 0$
23. $y'' - 3y' - 4y = e^{-t}, \quad y(2) = 3, \quad y'(2) = 0$
24. $y'' + y = 2\csc t \cot t, \quad y\left(\frac{\pi}{2}\right) = 1, \quad y'\left(\frac{\pi}{2}\right) = 1$
25. Show that the one-sided Green's function associated with the undamped spring-mass system described by

$$my'' + ky = F(t), \quad y(0) = y_0, \quad y'(0) = v_0,$$

is

$$g_1(t, \tau) = \frac{1}{\omega_0}\sin[\omega_0(t - \tau)], \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

26. Using the one-sided Green's function given in problem 25, find the response of the spring-mass system described there when the system is initially at rest and subject to the forcing function

(a) $F(t) = P$ (constant).

(b) $F(t) = P \cos \omega t, \quad \omega \neq \omega_0$.

(c) $F(t) = P \cos \omega_0 t$.

*27. Show that the one-sided Green's function associated with the damped spring-mass system described by

$$my'' + cy' + ky = F(t), \quad y(0) = y_0, \quad y'(0) = v_0,$$

for each of the three cases of damping is as follows:

(a) *Underdamped*: $g_1(t, \tau) = \dfrac{1}{\mu} e^{-c(t-\tau)/2m} \sin [\mu(t - \tau)]$, where

$$\mu = \frac{(4mk - c^2)^{1/2}}{2m}.$$

(b) *Critically damped*: $g_1(t, \tau) = (t - \tau)e^{-c(t-\tau)/2m}$.

(c) *Overdamped*: $g_1(t, \tau) = \dfrac{1}{\alpha} e^{-c(t-\tau)/2m} \sinh [\alpha(t - \tau)]$, where

$$\alpha = \frac{(c^2 - 4mk)^{1/2}}{2m}.$$

*28. Using the one-sided Green's function given in problem 27(a), (b), and (c), find the steady-state response of the system in all three cases when the driving force is

(a) $F(t) = P$ (constant).

(b) $F(t) = P \cos \omega t$.

(c) $F(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1. \end{cases}$

(d) $F(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & t > \pi. \end{cases}$

In problems 29–32, solve the Cauchy-Euler initial value problems using the method of Green's function.

29. $t^2y'' + 7ty' + 5y = t, \quad y(1) = 0, \quad y'(1) = 0$

30. $t^2y'' - 5ty' + 8y = 2t^3, \quad y(-2) = 1, \quad y'(-2) = 7$

31. $t^2y'' - 6y = \log t, \quad y(1) = \frac{1}{6}, \quad y'(1) = -\frac{1}{6}$

32. $t^2y'' + ty' + 4y = \sin(\log t), \quad y(1) = 1, \quad y'(1) = 0$

33. Show that for all $\tau \geqslant t_0$,

(a) $g_1(\tau, \tau) = 0$.

(b) $g_1(t, \tau)$, along with its first and second derivatives with respect to t, is continuous.

34. Show that for a fixed value of τ, the function $\phi(t) = g_1(t, \tau)$ is a solution of the initial value problem

$$y'' + a_1(t)y' + a_0(t)y = 0, \quad \phi(\tau) = 0, \quad \phi'(\tau) = 1.$$

Hint: Use the result of problem 33.

*35. Consider the third-order initial value problem

$$y''' + a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t),$$
$$y(t_0) = k_0, \quad y'(t_0) = k_1, \quad y''(t_0) = k_2.$$

(a) Show that the one-sided Green's function is defined by

$$g_1(t, \tau) = \frac{\begin{vmatrix} y_1(\tau) & y_2(\tau) & y_3(\tau) \\ y_1'(\tau) & y_2'(\tau) & y_3'(\tau) \\ y_1(t) & y_2(t) & y_3(t) \end{vmatrix}}{W(y_1, y_2, y_3)(\tau)},$$

where y_1, y_2, and y_3 are linearly independent solutions of the associated homogeneous DE.

(b) Establish the solution formula

$$y = \int_{t_0}^{t} g_1(t, \tau) f(\tau)\, d\tau + C_1 y_1(t) + C_2 y_2(t) + C_3 y_3(t),$$

and derive determinant expressions for C_1, C_2, and C_3 similar to those in (42).

(c) Generalize the results in (a) and (b) to nth-order problems.

In problems 36–41, find the one-sided Green's function for the given operator using the result of problem 35.

36. $M = D^n, \quad n = 2, 3, 4, \ldots$

37. $M = D^2(D^2 - 1)$

38. $M = D(D^2 + 4)$

***39.** $M = D^3 + \frac{5}{2}D^2 - \frac{3}{2}$

***40.** $M = D^4 - 1$

***41.** $M = D^3 - 6D^2 + 11D - 6$

***42.** Show that the one-sided Green's function associated with an nth-order normal linear operator and its first $n - 2$ derivatives with respect to t vanish when $t = \tau$, but that the $n - 1$ derivative equals unity when $t = \tau$.

***43.** Show that for a fixed value of τ, the function $\phi(t) = g_1(t, \tau)$ is a solution of the initial value problem

$$M[\phi] = 0, \quad t > \tau$$
$$\phi(\tau) = 0, \quad \phi'(\tau) = 0, \quad \ldots, \quad \phi^{(n-2)}(\tau) = 0, \quad \phi^{(n-1)}(\tau) = 1,$$

where M is an nth-order normal linear differential operator.
Hint: Use the result of problem 42.

Solve the initial value problems 44 and 45 using the one-sided Green's function (see problem 35).

***44.** $y''' + y = te^t, \quad y(1) = 0, \quad y'(1) = 0, \quad y''(1) = 1$

***45.** $y''' - y'' + 4y' - 4y = 1, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = -1$

46. Prove that $M = D^2 + a_1(t)D + a_0(t)$ is a linear operator; i.e., that

$$M[C_1 y_1 + C_2 y_2] = C_1 M[y_1] + C_2 M[y_2]$$

for any constants C_1 and C_2.

47. If y_P satisfies the initial value problem

$$M[y] = f(t), \quad y(t_0) = 0, \quad y'(t_0) = 0$$

and y_H satisfies the initial value problem

$$M[y] = 0, \quad y(t_0) = k_0, \quad y'(t_0) = k_1,$$

show that the sum $y = y_P + y_H$ satisfies

$$M[y] = f(t), \quad y(t_0) = k_0, \quad y'(t_0) = k_1.$$

[O] 5.6 IMPULSE FUNCTIONS

In certain applications it is convenient to introduce the concept of an impulse function, which is the result of a sudden excitation administered to a system, such as a sharp blow or a voltage surge. Let us imagine that the sudden excitation, which we will denote by $d_a(t)$, has a nonzero value over the short interval of time $a - \epsilon < t < a + \epsilon$, but is otherwise zero. The total impulse (force times duration) imparted to the system is thus defined by

$$I = \int_{-\infty}^{\infty} d_a(t)\,dt = \int_{a-\epsilon}^{a+\epsilon} d_a(t)\,dt \quad (\epsilon > 0). \tag{48}$$

The value of I is a measure of the strength of the sudden excitation.

In order to provide a mathematical model of the function $d_a(t)$, it is convenient to think of it as having a constant value over the interval $a - \epsilon < t < a + \epsilon$ (see Figure 5.12). Furthermore, we wish to choose this constant value in such a way that the total impulse given by (48) is unity. Hence, we write

$$d_a(t) = \begin{cases} \dfrac{1}{2\epsilon}, & a - \epsilon < t < a + \epsilon \\ 0, & \text{otherwise}. \end{cases} \tag{49}$$

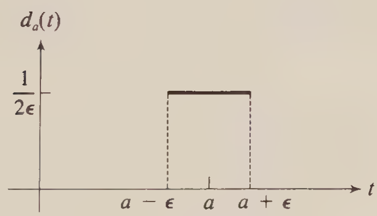


Figure 5.12 Impulse function.

Now let us idealize the function $d_a(t)$ by requiring it to act over shorter and shorter intervals of time by allowing $\epsilon \to 0$. Although the interval about $t = a$ is shrinking to zero, we still want $I = 1$; thus it follows that

$$\lim_{\epsilon \to 0} I = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} d_a(t)\,dt = 1. \tag{50}$$

We can use the results of this limit process to define an idealized *unit impulse function*, $\delta(t - a)$, which has the property of imparting a unit impulse to the system at time $t = a$ but being zero for all other values of t. The defining properties of this function are therefore

$$\delta(t-a) = 0, \quad t \neq a,$$

$$\int_{-\infty}^{\infty} \delta(t-a)\, dt = 1. \tag{51}$$

By a similar kind of limit process, it is possible to define the integral of a product of the unit impulse function and any continuous and bounded function f; i.e.,

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\, dt = f(a). \tag{52}$$

The verification of this result, which makes use of the mean value theorem for integrals, is left to the exercises (see problem 8).

Obviously the "function" $\delta(t - a)$, also known as the *Dirac delta function*,* is not a function in the usual sense of the word. It has significance only as part of an integrand. It is an example of what are commonly called *generalized functions*. In dealing with these functions, it is usually best to avoid the idea of assigning "functional values" and instead refer to its integral property (52), even though it has no meaning as an ordinary integral. Following more rigorous lines, generalized functions can be defined as a limit of an infinite sequence of well-behaved functions (see problems 14 and 15 in this section).

EXAMPLE 12 Solve the initial value problem

$$y'' + y = \delta(t - \pi), \quad t > 0, \quad y(0) = 0, \quad y'(0) = 0.$$

Solution Because of the homogeneous initial conditions, we only need consider the particular solution; i.e., $y = y_P$. The one-sided Green's function for the operator $M = D^2 + 1$ has previously been shown to be (see Example 9)

$$g_1(t, \tau) = \sin(t - \tau),$$

and thus it follows that

$$y = \int_0^t \sin(t-\tau)\, \delta(\tau - \pi)\, d\tau = \begin{cases} 0, & t < \pi \\ \sin(t-\pi), & t \geqslant \pi \end{cases}$$

where we have made use of (52).

We can interpret this solution as the response of some system that remains at rest until time $t = \pi$, when it is subjected to a unit impulse. After time $t = \pi$, the response of the system follows that of a simple sinusoid with frequency equal to that of the natural frequency of the system (see Figure 5.13).

*Named after PAUL A. M. DIRAC (1902–), who was awarded the Nobel Prize (with E. SCHRÖDINGER) in 1933 for his work in quantum mechanics.

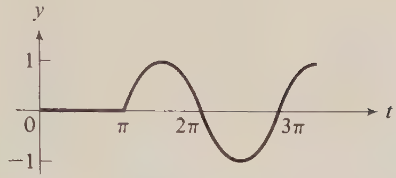


Figure 5.13

5.6.1 A More General Definition of the Green's Function

The unit impulse function is also useful for providing us with a physical interpretation of the one-sided Green's function. To see this, let us consider the initial value problem

$$M[y] = \delta(t - a), \quad y(t_0) = 0, \quad y'(t_0) = 0. \tag{53}$$

The solution of (53) is formally given by

$$y = \int_{t_0}^{t} g_1(t, \tau)\, \delta(\tau - a)\, d\tau = \begin{cases} 0, & t_0 < t < a \\ g_1(t, a), & t \geq a, \end{cases} \tag{54}$$

where $g_1(t, \tau)$ is the one-sided Green's function associated with the operator M. Thus the function $g_1(t, a)$ must represent the response of the system described by (53) for $t > a$, which was formerly at rest and then subjected to a unit disturbance (impulse) at time $t = a$.

Based on the above interpretation of $g_1(t, \tau)$, let us introduce the more general function

$$g(t, \tau) = \begin{cases} 0, & t_0 < t < \tau \\ g_1(t, \tau), & \tau \leq t < \infty, \end{cases}$$

or equivalently,

$$g(t, \tau) = \begin{cases} g_1(t, \tau), & t_0 \leq \tau \leq t \\ 0, & t < \tau < \infty, \end{cases} \tag{55}$$

so that we can express the response of the system to a general input function f as

$$y = \int_{t_0}^{\infty} g(t, \tau) f(\tau)\, d\tau, \tag{56}$$

where the integration now takes place over all $\tau \geq t_0$. The function $g(t, \tau)$ can then be interpreted as the response of the system for *all* time t due to a unit impulse delivered at time $t = \tau$. We will refer to $g(t, \tau)$ as simply the *Green's function*, or *influence function* as it is sometimes called.*

*The influence function in some textbooks is defined as a constant multiple of $g(t, \tau)$ rather than $g(t, \tau)$ itself.

For a fixed value of τ, we see that $g(t, \tau)$ must necessarily satisfy the differential equation $M[g] = \delta(t - \tau)$, where M is the differential operator associated with the construction of $g_1(t, \tau)$. The Green's function must also satisfy the homogeneous initial conditions $g(t_0, \tau) = (\partial g/\partial t)(t_0, \tau) = 0$ for any t_0 less than or equal to the fixed value of τ. It is a continuous function for all values of t since $g_1(\tau, \tau) = 0$ and $g_1(t, \tau)$ is itself a continuous function of t by definition. However, there is a jump discontinuity in the first derivative of $g(t, \tau)$ at $t = \tau$ of unit magnitude. That is, for $t < \tau$, the derivative is clearly zero, while

$$\left.\frac{\partial g}{\partial t}\right|_{t=\tau^+} = \frac{y_1(\tau)y_2'(\tau) - y_1'(\tau)y_2(\tau)}{W(y_1,y_2)(\tau)} = 1.$$

The jump discontinuity in the derivative turns out to be an essential feature of the Green's function.

In summary, we have the following definition.

Definition 5.1 *The **Green's function** $g(t, \tau)$ associated with the nonhomogeneous initial value problem*

$$M[y] = f(t), \quad t > t_0, \quad y(t_0) = k_0, \quad y'(t_0) = k_1,$$

is a function satisfying the following conditions:

(a) $M[g] = \delta(t - \tau)$ (τ *fixed*),

(b) $g(t_0, \tau) = \dfrac{\partial g}{\partial t}(t_0, \tau) = 0,$

(c) $g(\tau^+, \tau) = g(\tau^-, \tau)$ (*continuous function*),

(d) $\left.\dfrac{\partial g}{\partial t}\right|_{t=\tau^-}^{t=\tau^+} = 1.$

EXERCISES 5.6

In problems 1–5, find the solution of the given initial value problem.

1. $y'' + 2y' + 2y = \delta(t - \pi), \quad y(0) = 1, \quad y'(0) = 0$
2. $y'' + y = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 1$
3. $y'' + y = \delta(t - \pi) + A \cos \omega t, \quad y(0) = 0, \quad y'(0) = 0 \quad (\omega \neq 1)$
4. $y'' - y = 2\delta(t - 1), \quad y(0) = 1, \quad y'(0) = -1$
5. $y'' + y = A\delta\left(t - \frac{\pi}{2}\right) \sin t, \quad y(0) = 0, \quad y'(0) = 2$
6. A particular spring-mass system has spring constant $k = 72$ N/m, $m = 2$ kg, and $c = 40$ kg/s. If the system is at rest and at time $t = 0$ given a sharp blow of magnitude 100 N, what is the subsequent motion?
   Hint: Set $F(t) = 100\ \delta(t)$.

***7.** Solve the initial value problem

$$y''' + y = e^t + \delta(t-1), \quad y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 2.$$

8. Show that $\int_{-\infty}^{\infty} \delta(t-a)f(t)\,dt = f(a)$.

Hint: The mean value theorem of the integral calculus states that

$$\int_a^b f(x)\,dx = f(\xi)(b-a)$$

for some ξ in the interval $a < \xi < b$. Thus,

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} d_a(t)f(t)\,dt = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{a-\epsilon}^{a+\epsilon} f(t)\,dt = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \cdot f(\xi) \cdot 2\epsilon,$$

where $a - \epsilon < \xi < a + \epsilon$.

9. Show formally that $dh(t)/dt = \delta(t)$, where $h(t)$ is the *Heaviside unit function* defined by

$$h(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0. \end{cases}$$

Hint: Use integration by parts to show that

$$\int_{-\infty}^{\infty} \frac{dh}{dt}(t)f(t)\,dt = f(0).$$

10. Show that $\delta(t-x) = \delta(x-t)$.

***11.** Show formally that $f(t)\,\delta(t-a) = f(a)\,\delta(t-a)$, and use this result to deduce that $t\,\delta(t) = 0$.

Hint: Show that $\int_{-\infty}^{\infty} g(t)[f(t)\,\delta(t-a)]\,dt = \int_{-\infty}^{\infty} g(t)[f(a)\,\delta(t-a)]\,dt$.

12. If $a > 0$, show that $\int_{-\infty}^{\infty} \delta(at)f(t)\,dt = a^{-1}f(0)$.

13. Show formally (using integration by parts) that

(a) $\int_{-\infty}^{\infty} \delta'(t)f(t)\,dt = -f'(0)$.

(b) $\int_{-\infty}^{\infty} \delta^{(m)}(t)f(t)\,dt = (-1)^m f^{(m)}(0)$.

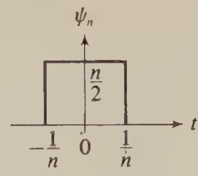
***14.** Consider the sequence of rectangle functions defined by

$$\psi_n(t) = \begin{cases} \dfrac{n}{2}, & |t| < \dfrac{1}{n} \\ 0, & |t| > \dfrac{1}{n} \end{cases}$$

for $n = 1, 2, 3, \ldots$.

(a) Show that for each n the area enclosed by the rectangle is unity, and deduce that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \psi_n(t)\,dt = 1.$$



Problem 14

(b) More generally, if $f(t)$ is any function continuous at $t = 0$ and bounded, show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \psi_n(t)f(t)\,dt = f(0).$$

Hint: See problem 8.

15. Any sequence of continuous and differentiable functions $\psi_1(t), \psi_2(t), \ldots, \psi_n(t), \ldots$ satisfying the condition of problem 14(a) is called a *delta sequence*. Show that the following sequences are delta sequences:

(a) $\psi_n(t) = \dfrac{n}{\pi(1 + n^2t^2)}, \quad n = 1, 2, 3, \ldots$

(b) $\psi_n(t) = \dfrac{n}{\sqrt{\pi}} e^{-n^2t^2}, \quad n = 1, 2, 3, \ldots$

Hint: $\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}$.

[O] 5.7 ELEMENTARY OSCILLATION THEORY

We now turn our attention to equations more general than the constant-coefficient DEs arising in most spring-mass systems or electric circuits. In particular, we want to discuss the DE

$$A_2(t)y'' + A_1(t)y' + A_o(t)y = 0, \tag{57}$$

in which certain parameters of the system vary over time. Although in general we cannot obtain explicit solutions of (57), we can study the qualitative behavior of these solutions by directly analyzing the equation itself. For example, information concerning the existence and relative positions of the zero-crossings can be obtained without formal solutions. In the theorems that follow, we will treat the coefficients of (57) as continuous functions on the interval of interest and assume $A_2(t) \neq 0$ on this interval.

Theorem 5.3 (*Sturm separation theorem*) *If* y_1 and y_2 are linearly independent solutions of the second-order equation

$$A_2(t)y'' + A_1(t)y' + A_0(t)y = 0$$

on some interval I, then y_1 has precisely one zero between any two consecutive zeros of y_2 on the interval I.

Proof: Let $t = a$ and $t = b$ be consecutive zeros of y_2 (Figure 5.14). The linear independence of y_1 and y_2 implies that the Wronskian $W(y_1, y_2)(t)$ does not vanish on the interval $a \le t \le b$. And since $y_2(a) = y_2(b) = 0$, it follows that

$$W(y_1, y_2)(a) = y_1(a)y_2'(a) \neq 0,$$
$$W(y_1, y_2)(b) = y_1(b)y_2'(b) \neq 0.$$

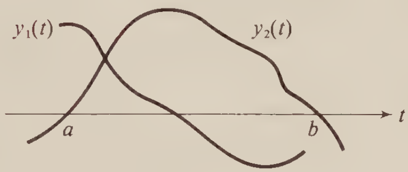


Figure 5.14

The Wronskian is clearly continuous on $a \leqslant t \leqslant b$, and since it does not vanish, it must have the same sign at all points in the interval, and in particular, the same sign at $t = a$ and $t = b$. On the other hand, since a and b are consecutive zeros of y_2, then y_2' must have opposite signs at $t = a$ and $t = b$. In order to prevent the Wronskian from changing signs, $y_1(a)$ and $y_1(b)$ must have opposite signs, and the continuity of y_1 implies that it must assume a zero value at least once between $t = a$ and $t = b$.

If we reverse the roles of y_1 and y_2 in the preceding argument, we conclude that y_2 must have at least one zero between consecutive zeros of y_1. Hence, y_1 cannot vanish more than once between $t = a$ and $t = b$. □

Put another way, Theorem 5.3 states that the zeros of y_1 and y_2 occur alternately, and that on any finite interval the number of zeros of y_1 and y_2 can differ by at most one. For example, the solutions $\cos t$ and $\sin t$ of $y'' + y = 0$ have zeros that alternate on the entire real axis. A somewhat less obvious consequence is that any two functions of the form

$$y_1 = C_1 \cos t + C_2 \sin t, \quad y_2 = C_3 \cos t + C_4 \sin t,$$

have alternating zeros whenever $C_1C_4 \neq C_2C_3$, as all such pairs of functions are also linearly independent.

The Sturm separation theorem does not imply the existence of any zeros of the solutions y_1 and y_2. Furthermore, it does not allow for a comparison of the number of zeros (rate of oscillation) of solutions of two distinct DEs. To pursue these matters it becomes convenient to put the differential equation in what we call the *standard form*.

If we divide (57) by the leading coefficient $A_2(t)$, the resulting equation is

$$y'' + a(t)y' + b(t)y = 0, \tag{58}$$

which is in normal form. Next we put $y = u(t)v(t)$ so that $y' = uv' + u'v$ and $y'' = uv'' + 2u'v' + u''v$; when these expressions are substituted into (58), we get

$$vu'' + (2v' + av)u' + (v'' + av' + bv)u = 0. \tag{59}$$

The coefficient of u' can be made to vanish by selecting

$$v(t) = \exp\left[-\frac{1}{2}\int a(t)dt\right], \tag{60}$$

and hence (59) reduces to

$$u'' + Q(t)u = 0, \tag{61}$$

where

$$Q(t) = b(t) - \frac{1}{4}[a(t)]^2 - \frac{1}{2}a'(t). \tag{62}$$

We will refer to (61) as the *standard form.*

Because the function v cannot vanish, the transformation $y = uv$ above can have no effect on the zeros of y, and hence leaves unaltered the oscillation phenomena we are investigating.

EXAMPLE 13 Find the standard form of *Bessel's equation*

$$t^2y'' + ty' + (t^2 - \nu^2)y = 0.$$

Solution We first rewrite the equation as

$$y'' + \frac{1}{t}y' + \left(1 - \frac{\nu^2}{t^2}\right)y = 0,$$

which identifies $a(t) = 1/t$ and $b(t) = 1 - \nu^2/t^2$. Hence,

$$Q(t) = 1 - \frac{\nu^2}{t^2} - \frac{1}{4t^2} + \frac{1}{2t^2} = 1 + \frac{1 - 4\nu^2}{4t^2},$$

and Bessel's equation reduces to

$$u'' + \left(1 + \frac{1 - 4\nu^2}{4t^2}\right)u = 0.*$$

We can now show that if $Q(t)$ is negative on a given interval, the solutions of (61) do not oscillate on that interval. To better understand this situation, consider the solutions of $y'' - y = 0$.

*Although we don't need it, we see that $v(t) = t^{-1/2}$ in this case.

Theorem 5.4 If $Q(t) < 0$ on the interval I and u is a nontrivial solution of $u'' + Q(t)u = 0$, then u has at most one zero on the interval I.

Proof. Let $t = a$ be a point on the interval such that $u(a) = 0$. Since u is a nontrivial solution, it follows from Theorem 5.2 that $u'(a) \neq 0$. Let us assume that $u'(a) > 0$ so that u is positive as t increases from $t = a$. Since $Q(t) < 0$, then $u'' = -Q(t)u$ must be positive as t increases. And since u'' is the rate at which the slope of the function u changes, we conclude that the slope is increasing, and hence u cannot have a zero for $t > a$. A similar argument can show that u has no zero for $t < a$, and the proof is also the same for $u'(a) < 0$. $\square$

Theorem 5.5 (*Sturm comparison theorem*) Let u_1 and u_2 represent nontrivial solutions of

$$u'' + Q_1(t)u = 0, \quad u'' + Q_2(t)u = 0,$$

respectively, and suppose that $Q_1(t) > Q_2(t)$ everywhere on the interval of interest. Then there exists at least one zero of u_1 between every two consecutive zeros of u_2.

Proof. Let $t = a$ and $t = b$ be consecutive zeros of u_2, and assume that u_1 does not vanish on the interval $a < t < b$. We will further assume that both u_1 and u_2 are positive on the interval $a < t < b$. Since the zeros of u_1 are the same as $-u_1$, this assumption is justified. Following the argument presented in the proof of Theorem 5.3, we have

$$W(u_1, u_2)(a) = u_1(a)u_2'(a), \quad W(u_1, u_2)(b) = u_1(b)u_2'(b).$$

However,

$$\begin{aligned}\frac{d}{dt}[W(u_1, u_2)(t)] &= \frac{d}{dt}(u_1u_2' - u_1'u_2)\\ &= u_1u_2'' - u_1''u_2\\ &= -u_1Q_2u_2 + u_2Q_1u_1\\ &= u_1u_2(Q_1 - Q_2),\end{aligned}$$

and since $Q_1(t) > Q_2(t)$, we conclude that the Wronskian is an increasing function on $a < t < b$. Also, since we have assumed that u_2 is positive on this interval, it follows that $u_2'(a) \geqslant 0$ and $u_2'(b) \leqslant 0$, and hence

$$W(u_1, u_2)(a) \geqslant 0, \quad W(u_1, u_2)(b) \leqslant 0.$$

But this condition implies that the Wronskian cannot be an increasing function. Thus, contrary to our assumption, we must conclude that u_1 vanishes at least once on the interval $a < t < b$. $\square$

An immediate consequence of the Sturm comparison theorem is that the larger the coefficient of u in $u'' + Q(t)u = 0$, the more rapidly the solution oscillates. For example, let us examine the solutions $\sin t$ and $\sin 2t$ of the two equations

$$y'' + y = 0, \quad y'' + 4y = 0.$$

The solution ($\sin 2t$) of the second equation oscillates more rapidly than the solution ($\sin t$) of the first equation. That is, the zeros of $\sin 2t$ are located at $t = 0, \pi/2, \pi, 3\pi/2, \ldots$, whereas the zeros of $\sin t$ are located at $t = 0, \pi, 2\pi, \ldots$ (see Figure 5.15).

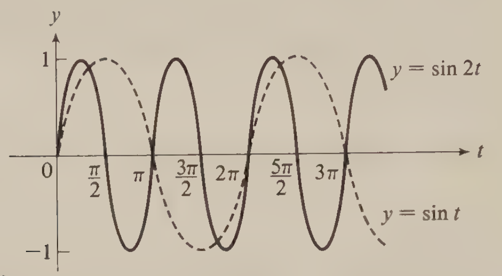


Figure 5.15 Graph of $y = \sin t$ and $y = \sin 2t$.

EXAMPLE 14 Show that every solution of $y'' + t^2 y = 0$ has infinitely many zeros on the interval $t > 1$.

Solution The equation $y'' + y = 0$ has one solution $\sin t$ with infinitely many zeros at $t = 0$ and $t = n\pi$, $n = 1, 2, 3, \ldots$. Since $t^2 > 1$, it follows from the *Sturm comparison theorem* that one nontrivial solution of $y'' + t^2 y = 0$ has at least one zero between $n\pi$ and $(n + 1)\pi$ for $n = 1, 2, 3, \ldots$, and hence has infinitely many zeros. By comparison with $\cos t$, it can be established that a second linearly independent solution of $y'' + t^2 y = 0$ has infinitely many zeros between the zeros of $\cos t$. Since all other solutions of this DE are linear combinations of these two linearly independent solutions, we have our intended result.

The zeros of *Bessel's equation* are of both practical and theoretical importance, and are found to satisfy the conditions of the following theorem by comparison with the zeros of solutions of $y'' + y = 0$.

Theorem 5.6 (*Zeros of Bessel's equation*) Every nontrivial solution of Bessel's equation, which has the standard form

$$u'' + \left(1 + \frac{1 - 4\nu^2}{4t^2}\right)u = 0,$$

has infinitely many zeros on the positive t-axis. Moreover, the distance between successive zeros is

(a) less than π for $0 \leq \nu < \frac{1}{2}$,
(b) equal to π for $\nu = \frac{1}{2}$, and
(c) greater than π for $\nu > \frac{1}{2}$.

Remark. Interestingly, the zeros of the solutions of Bessel's equation coincide with the zeros of the sinusoidal functions when $\nu = \frac{1}{2}$. The reason for this situation is that the solutions of Bessel's equation are directly related to the sinusoidal functions for this particular value of ν.

EXERCISES 5.7

1. Prove that the zeros of any solution of the second-order DE

$$A_2(t)y'' + A_1(t)y' + A_0(t)y = 0$$

are simple; i.e., if $y(t_0) = 0$, then $y'(t_0) \neq 0$.
Hint: Use Theorem 5.2.

2. Prove that if y_1 and y_2 are linearly independent solutions of

$$A_2(t)y'' + A_1(t)y' + A_0(t)y = 0,$$

they cannot vanish at the same point.
Hint: Assume $y_1(a) = y_2(a) = 0$ at some point $t = a$, and show that this leads to a contradiction that the Wronskian cannot vanish anywhere on the interval of interest.

3. Prove that between every pair of consecutive zeros of $\sin t$ there is one zero of $\sin t + \cos t$.

4. Show that the zeros of $y_1 = \sin(\log t)$ and $y_2 = \cos(\log t)$ alternate.
Hint: Find a DE for which y_1 and y_2 are both solutions.

5. Explain how the Sturm separation theorem applies to $y'' - y = 0$, if indeed it does.

6. Show that the substitution $y = t^{-1/2}u$ transforms Bessel's equation to standard form.

7. Show that every nontrivial solution of $y'' + (\sinh t)y = 0$ has at most one zero in the interval $t < 0$. What can be said about the interval $t > 0$?

8. How many zeros does every solution of Airy's equation $y'' + ty = 0$ have on the interval $t > 0$?

***9.** Show that every nontrivial solution of $y'' - 2ty' + ky = 0$ has

(a) at most one zero when $k \leq -1$.

(b) only finitely many zeros when $k > 0$.

10. Which of the equations

$$y'' + (1 + t^2)y = 0 \quad \text{and} \quad y'' + 2ty = 0$$

has the most rapidly oscillating solution in the interval $0 \leq t \leq 10$?

11. Show that the distance between consecutive zeros of a solution of $y'' + ty = 0$ is less than π for at least $t > \pi$.

12. Show that the distance between consecutive zeros of a solution of $y'' + t^2y = 0$ approaches zero as $t \to \infty$.

13. Put the DE $my'' + cy' + ky = 0$ in standard form, and show that it does not possess oscillatory solutions unless $c^2 - 4mk < 0$.

14. Prove Theorem 5.6.

REFERENCES

Birkhoff, G., and Rota, G.-C. 1978. *Ordinary differential equations*. 3rd ed. New York: Wiley.

Boyce, W. E., and DiPrima, R. C. 1977. *Elementary differential equations and boundary value problems*. 3rd ed. New York: Wiley.

Bradbury, T. C. 1968. *Theoretical mechanics*. New York: Wiley.

Coddington, E. A. 1961. *An introduction to ordinary differential equations*. Englewood Cliffs, N.J.: Prentice-Hall.

Finney, R. L., and Ostberg, D. R. 1976. *Elementary differential equations with linear algebra*, Reading, Mass.: Addison-Wesley.

Potter, M. C. 1978. *Mathematical methods in the physical sciences*. Englewood Cliffs, N.J.: Prentice-Hall.

Ross, S. L. 1980. *Introduction to ordinary differential equations*. 3rd ed. New York: Wiley.

The Laplace Transform

6

The Laplace transform is an efficient method for solving linear, constant-coefficient equations with prescribed auxiliary conditions, usually in the form of initial conditions. It offers the advantage of solving the problem directly without first producing the general solution or even having to solve separately for the homogeneous and particular solutions.

The transform method is formally equivalent to the operational calculus devised by Oliver Heaviside (1850–1925) for the solution of transient problems in physics and electrical engineering. It is an especially useful tool for solving problems in circuit analysis, where the nonhomogeneous terms are frequently of a discontinuous or impulse nature. More conventional methods tend to be clumsy when piecing together solutions valid in different intervals, whereas the Laplace transform has the effect of "smoothing" the problem in the transform domain and making it more tractable.

In Section 6.2 we calculate the transforms of some elementary functions directly from the integral definition. We follow this in Section 6.3 with some of the *operational properties* of the transform, which permit methods of calculating certain transforms other than straight evaluations of the defining integral.

We discuss methods of computing *inverse Laplace transforms* in Section 6.4, which rely mostly on operational properties and sometimes *partial fraction expansions*.

Solutions of initial value problems by the transform method are studied in Section 6.5 and the vibrating spring-mass system is revisited, illustrating some of the benefits of a transform analysis.

The use of the Laplace transform in dealing with *discontinuous* and *impulse functions* is discussed in Section 6.6, and in Section 6.7 the *convolution theorem* and its relation to the *one-sided Green's function* are examined. The last section contains a short *table of Laplace transforms*.

6.1 INTRODUCTION

The idea of transforming one function into another is commonplace in mathematics; for example, the operation of differentiation transforms the function f into the function f'. Another transformation that is prominent in the calculus is that of integration.

An *integral transform* is a relation of the form

$$F(s) = \int_{-\infty}^{\infty} K(s, t) f(t)\, dt \tag{1}$$

such that a given function $f(t)$ is transformed into another function $F(s)$ by means of an integral. The new function $F(s)$ is said to be the *transform* of $f(t)$, and $K(s, t)$ is called the *kernel* of the transformation.

In particular, if the kernel is defined by

$$K(s, t) = \begin{cases} 0, & t < 0 \\ e^{-st}, & t \geqslant 0, \end{cases}$$

the resulting integral transform

$$F(s) = \int_{0}^{\infty} e^{-st} f(t)\, dt \tag{2}$$

is called the *Laplace transform.** The symbol

$$\mathscr{L}\{f(t)\} = F(s) \tag{3}$$

is also used to denote this transform.

Because the defining integral (2) is improper, it must be evaluated through the limit process

$$\int_{0}^{\infty} e^{-st} f(t)\, dt = \lim_{b \to \infty} \int_{0}^{b} e^{-st} f(t)\, dt. \tag{4}$$

If the limit of the integral from 0 to b exists, we say that the integral *converges* to that limiting value; otherwise, it *diverges*.

6.2 THE LAPLACE TRANSFORM OF SOME ELEMENTARY FUNCTIONS

The Laplace transform of many elementary functions can be obtained through routine integration of the defining integral. Consider the following examples.

*Named in honor of the French mathematician PIERRE SIMON DE LAPLACE (1749–1827). Laplace made use of this particular integral transform in his work in probability theory, although it is believed that the integral was really discovered by Euler.

EXAMPLE 1 Let $f(t) = 1$, $t \geqslant 0$. Then

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s}$$

provided that $s > 0$ so that $e^{-st} \to 0$ as t tends to infinity.*

Although the variable s may be real or complex, in the computation of Laplace transforms we normally assume s is real and greater than some constant, as in Example 1. Such a restriction on s, however, has little effect in applications.

EXAMPLE 2 Let $f(t) = e^{at}$, $t \geqslant 0$. Then

$$\mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt.$$

In the case $s \leqslant a$, the integral diverges, while for $s > a$, we get

$$\mathscr{L}\{e^{at}\} = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty = \frac{1}{s-a}, \qquad s > a.$$

EXAMPLE 3 Let $f(t) = \sin t$, $t \geqslant 0$. Then integration by parts yields

$$\begin{aligned}\mathscr{L}\{f(t)\} &= \int_0^\infty e^{-st}\sin t\,dt \\ &= -e^{-st}\cos t\Big|_0^\infty - s\int_0^\infty e^{-st}\cos t\,dt \\ &= 1 - s\int_0^\infty e^{-st}\cos t\,dt, \qquad s > 0.\end{aligned}$$

A second integration by parts now gives

$$\begin{aligned}\mathscr{L}\{f(t)\} &= 1 - s^2\int_0^\infty e^{-st}\sin t\,dt \\ &= 1 - s^2\mathscr{L}\{f(t)\},\end{aligned}$$

and solving for $\mathscr{L}\{f(t)\}$, we find

$$\mathscr{L}\{f(t)\} = \frac{1}{s^2+1}, \qquad s > 0.$$

*For abbreviation, $\Big|_0^\infty$ will denote $\lim_{b\to\infty}(\ )\Big|_0^b$.

Not all functions of t have a Laplace transform, even if they are continuous, since the defining integral is improper. The basic requirement for the existence of the transform is that f be of *exponential order*.

Definition 6.1 *The function $f(t)$ is said to be of* **exponential order** *if there exists real constants c, M, and t_0 such that*

$$|f(t)| < Me^{ct}, \quad t > t_0.*$$

Saying a function is of exponential order means that its graph on the interval $t > t_0$ does not grow faster than the graph of Me^{ct} for appropriate values of M and c (see Figure 6.1). For instance, the functions 1, e^{at}, and $\sin t$ are all of exponential order, whereas the function $f(t) = e^{t^2}$ is not since its graph grows faster than any linear power of e for $t > t_0$ (see Figure 6.2).

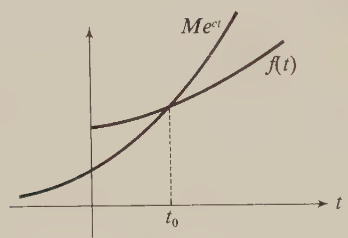


Figure 6.1 A function of exponential order.

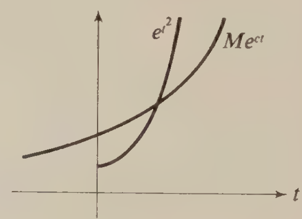


Figure 6.2 $f(t) = e^{t^2}$.

Although most of the functions we encounter in practice are continuous, they do not have to be in order to have a Laplace transform.

Definition 6.2 *A function $f(t)$ is said to be* **piecewise continuous** *in a given interval provided*

(*a*) *$f(t)$ is defined and continuous at all but a finite number of points in the interval, and*

(*b*) *the left-hand and right-hand limits exist at each point in the interval.*

Remark. The left-hand and right-hand limits are defined, respectively, by $\lim_{\epsilon\to 0^+} f(t - \epsilon) = f(t^-)$ and $\lim_{\epsilon\to 0^+} f(t+\epsilon) = f(t^+)$. Furthermore, when t is a point of continuity, $f(t^-) = f(t^+) = f(t)$.

A piecewise continuous function f need not be defined at every point in the interval of interest. In particular, it is often not defined at a point of discontinuity,

*This relation is also denoted by $f(t) = 0(e^{ct})$.

and even when it is, the functional value assigned at these points really doesn't matter. Also, the interval of interest may be open or closed, or open at one end and closed at the other (see Figure 6.3).

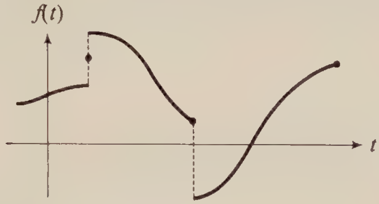


Figure 6.3 A piecewise continuous function.

Theorem 6.1 If $f(t)$ is piecewise continuous and of exponential order, then it has a Laplace transform.

Proof: Consider

$$\int_0^\infty e^{-st}f(t)\,dt = \int_0^{t_0} e^{-st}f(t)\,dt + \int_{t_0}^\infty e^{-st}f(t)\,dt.$$

The first integral on the right exists, since f is assumed piecewise continuous. Since f is also of exponential order, the second integral on the right satisfies the inequality

$$\left|\int_{t_0}^\infty e^{-st}f(t)\,dt\right| \leq \int_{t_0}^\infty e^{-st}|f(t)|\,dt \leq M\int_{t_0}^\infty e^{-(s-c)t}\,dt.$$

Hence, by direct integration, we get

$$\left|\int_0^\infty e^{-st}f(t)\,dt\right| \leq \frac{Me^{-(s-c)t_0}}{s-c}.$$

For $s > c$, this last expression vanishes in the limit as $t_0 \to \infty$ so that we say the integral is absolutely convergent. □

Most functions met in practice satisfy the conditions of Theorem 6.1. However, these conditions are sufficient rather than necessary to ensure that a function has a Laplace transform. For example, both t^{-1} and $t^{-1/2}$ have infinite discontinuities at $t = 0$, and while the integral

$$\mathcal{L}\{t^{-1}\} = \int_0^\infty e^{-st}t^{-1}\,dt$$

diverges, we can show that

$$\mathcal{L}\{t^{-1/2}\} = \int_0^\infty e^{-st}t^{-1/2}\,dt = \left(\frac{\pi}{s}\right)^{1/2}. \tag{5}$$

Hence $t^{-1/2}$ has a Laplace transform, but t^{-1} does not. Finally, we remark that if the transform of a function exists, it is *unique*, since the definite integral of a function is uniquely determined.

EXAMPLE 4 Find the Laplace transform of the piecewise continuous function $f(t)$ where

$$f(t) = \begin{cases} t, & 0 < t < 2 \\ 5, & t > 2. \end{cases}$$

Solution From the definition, we have

$$\mathscr{L}\{f(t)\} = \int_0^2 e^{-st} t\,dt + \int_2^\infty e^{-st} 5\,dt.$$

Using integration by parts on the first integral above, we deduce that

$$\begin{aligned} \mathscr{L}\{f(t)\} &= \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right] \Bigg|_0^2 + \left[-\frac{5}{s} e^{-st} \right] \Bigg|_2^\infty \\ &= -\frac{2}{s} e^{-2s} - \frac{1}{s^2} e^{-2s} + 0 + \frac{1}{s^2} - 0 + \frac{5}{s} e^{-2s}, \end{aligned}$$

which simplifies to

$$\mathscr{L}\{f(t)\} = \frac{1}{s^2} + \frac{3}{s} e^{-2s} - \frac{1}{s^2} e^{-2s}.$$

Remark. Observe that the original function $f(t)$ in Example 4 had a discontinuity at $t = 2$, whereas the transformed function $F(s)$ is continuous for all $s > 0$. This phenomenon is known as the "smoothing effect" of the transform.

Finding the transform of discontinuous functions by the method used in Example 4 is especially awkward when the function has several discontinuities, because of the many integrals to be evaluated. A more appealing method for transforming such functions will be discussed in Section 6.6.

EXERCISES 6.2

In problems 1–12, evaluate the Laplace transform of each given function directly from the defining integral.

1. $f(t) = t$

2. $f(t) = t^2$

3. $f(t) = t^n, \quad n = 1, 2, 3, \ldots$

4. $f(t) = \sin kt$

5. $f(t) = \cos kt$

6. $f(t) = 2e^{3t} - e^{-3t}$

7. $f(t) = e^{-at} - e^{-bt}$

8. $f(t) = \sinh kt$

9. $f(t) = \cosh kt$

10. $f(t) = \cos(at + b)$

11. $f(t) = t^2 - 3t + 5$

12. $f(t) = e^{-4t} + 3e^{-2t}$

In problems 13–16, evaluate the Laplace transform of each function, recalling the identities $\cosh x = (e^x + e^{-x})/2$ *and* $\sinh x = (e^x - e^{-x})/2$.

13. $f(t) = \cosh^2 kt$

14. $f(t) = \sinh^2 kt$

15. $f(t) = e^{at}\cosh kt$

16. $f(t) = e^{at}\sinh kt$

In problems 17–20, evaluate the Laplace transform of each function using integration by parts.

17. $f(t) = te^{at}$

18. $f(t) = t\sin kt$

19. $f(t) = t^2\cos kt$

20. $f(t) = te^{-t}\cos t$

21. Using the relation $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, find $\mathscr{L}\{\cos^2 kt\}$.

22. Using $\sin^2 x = 1 - \cos^2 x$ and the results of problem 21, find $\mathscr{L}\{\sin^2 kt\}$.

***23.** Show that if $\lim\limits_{t\to\infty} e^{-ct}f(t) = 0$, then $f(t)$ is of exponential order.

24. Using the criterion in problem 23, determine whether the following functions are of exponential order.

(a) $f(t) = t^{100}$ (b) $f(t) = e^{-t^2}$ (c) $f(t) = te^t$

(d) $f(t) = 3e^{t^2-t}$ (e) $f(t) = 5\sin(e^{t^2})$ (f) $f(t) = \dfrac{\sin t}{t}$

FXP UI

25. Sketch the graph of each of the following functions and evaluate its Laplace transform.

(a) $f(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ 1, & t > 1 \end{cases}$

(b) $f(t) = \begin{cases} \sin 2t, & 0 \leq t < \pi \\ 0, & t > \pi \end{cases}$ STUCK WITH IN

(c) $f(t) = \begin{cases} t, & 0 < t < 1 \\ 3 - t, & 1 < t < 2 \\ 1, & t > 2 \end{cases}$

***26.** The *gamma function*, denoted by $\Gamma(x)$, is defined by the integral

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt,$$

which converges for all $x > 0$.

(a) Show that $\Gamma(x + 1) = x\Gamma(x)$.

(b) Show that when $x = n$, $n = 0, 1, 2, \ldots$, then $\Gamma(n + 1) = n!$.

***27.** Using the results of problem 26 and the special value $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, verify the following identities.

(a) $\mathcal{L}\{t^{-1/2}\} = \left(\dfrac{\pi}{s}\right)^{1/2}, \quad s > 0.$

(b) $\mathcal{L}\{t^{1/2}\} = \dfrac{1}{2s}\left(\dfrac{\pi}{s}\right)^{1/2}, \quad s > 0.$

(c) $\mathcal{L}\{t^{5/2}\} = \dfrac{15}{s^3}\left(\dfrac{\pi}{s}\right)^{1/2}, \quad s > 0.$

(d) $\mathcal{L}\{t^x\} = \dfrac{\Gamma(x+1)}{s^{x+1}}, \quad s > 0.$

***28.** Referring to problem 26, show that

$$\mathcal{L}\{\log t\} = \frac{1}{s}\Gamma'(1) - \frac{\log s}{s},$$

where

$$\Gamma'(x) = \int_0^\infty e^{-t}(\log t)t^{x-1}\,dt, \quad x > 0.$$

***29.** Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Hint: Establish that $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-x^2}\,dx = 2\int_0^\infty e^{-y^2}\,dy$.

Then multiply these integrals and evaluate the resulting double integral by changing to polar coordinates.

6.3 OPERATIONAL PROPERTIES

Operational properties often offer an attractive alternative for evaluating the transform of a function to tedious and often cumbersome evaluations of the integral that defines the transform. For example, it becomes fairly easy to evaluate the transforms $\mathcal{L}\{e^{-3t}\cos 2t\}$, $\mathcal{L}\{t^3 \sin 5t\}$, and $\mathcal{L}\{t^6 e^{2t}\}$ with these operational properties once we know the transforms of $\mathcal{L}\{\cos 2t\}$, $\mathcal{L}\{\sin 5t\}$, and $\mathcal{L}\{t^6\}$, respectively. First among such properties is the *linearity property*, which is a simple consequence of integrals.

Theorem 6.2 (*Linearity*) If $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ are the Laplace transforms of $f(t)$ and $g(t)$, respectively, then

$$\mathcal{L}\{C_1 f(t) + C_2 g(t)\} = C_1\mathcal{L}\{f(t)\} + C_2\mathcal{L}\{g(t)\}$$

for any constants C_1 and C_2.

Proof: From the definition,

$$\begin{aligned}\mathcal{L}\{C_1 f(t) + C_2 g(t)\} &= \int_0^\infty e^{-st}[C_1 f(t) + C_2 g(t)]\,dt \\ &= C_1\int_0^\infty e^{-st} f(t)\,dt + C_2\int_0^\infty e^{-st} g(t)\,dt \\ &= C_1\mathcal{L}\{f(t)\} + C_2\mathcal{L}\{g(t)\}. \quad \square\end{aligned}$$

EXAMPLE 5 Evaluate $\mathscr{L}\{7 - 3e^{2t} + 5\sin t\}$.

Solution From the linearity property, we have

$$\mathscr{L}\{7 - 3e^{2t} + 5\sin t\} = 7\mathscr{L}\{1\} - 3\mathscr{L}\{e^{2t}\} + 5\mathscr{L}\{\sin t\}$$
$$= \frac{7}{s} - \frac{3}{s-2} + \frac{5}{s^2+1},$$

where we have used the results of Examples 1, 2, and 3.

EXAMPLE 6 Find $\mathscr{L}\{\sin kt\}$ and $\mathscr{L}\{\cos kt\}$.

Solution While both of these transforms can readily be found through routine integration methods, another approach using Theorem 6.2 is available to us. Setting $a = ik$ in $\mathscr{L}\{e^{at}\} = 1/(s-a)$, we get

$$\mathscr{L}\{e^{ikt}\} = \frac{1}{s-ik} = \frac{s}{s^2+k^2} + i\frac{k}{s^2+k^2}.$$

From Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, and Theorem 6.2, it follows that

$$\mathscr{L}\{e^{ikt}\} = \mathscr{L}\{\cos kt\} + i\mathscr{L}\{\sin kt\}.$$

Matching up real and imaginary parts, we deduce

$$\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad \mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}.$$

Functions multiplied by exponentials are easily handled because of the exponential function occurring in the defining integral of the Laplace transform.

Theorem 6.3 (*Shifting*) If $F(s) = \mathscr{L}\{f(t)\}$, then

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a).$$

Proof: By definition,

$$\mathscr{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}[e^{at}f(t)]\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt,$$

from which our result follows. $\square$

EXAMPLE 7 Given that $\mathscr{L}\{t^4\} = 4!/s^5$, evaluate $\mathscr{L}\{t^4e^{3t}\}$.

Solution Direct application of Theorem 6.3 leads to

$$\mathcal{L}\{t^4 e^{3t}\} = \left.\frac{4!}{s^5}\right|_{s \to s-3} = \frac{24}{(s-3)^5}.$$

EXAMPLE 8 Evaluate $\mathcal{L}\{e^{-2t}\cos 3t\}$.

Solution From Example 6, we know that $\mathcal{L}\{\cos 3t\} = s(s^2 + 9)^{-1}$, and hence, through the shifting property, it follows that

$$\mathcal{L}\{e^{-2t}\cos 3t\} = \frac{s+2}{(s+2)^2 + 9}.$$

6.3.1 The Laplace Transform of Derivatives and Integrals

The real merit of the Laplace transform is revealed by its effect on derivatives. Suppose that $f(t)$ is continuous with a piecewise continuous derivative $f'(t)$. Then by definition

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)\,dt \\ &= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt,\end{aligned}$$

where we have performed an integration by parts. If the function $f(t)$ has a Laplace transform $F(s)$, then $\lim_{t\to\infty} e^{-st} f(t) = 0$, so that

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0). \tag{6}$$

Remark. If $f(t)$ is continuous for $t \geqslant 0$ except for a finite jump discontinuity at $t = a$, then it can be shown that (6) takes the form

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) - [f(a^+) - f(a^-)]e^{-as}.$$

EXAMPLE 9 Use the result $\mathcal{L}\{\sin kt\} = k/(s^2 + k^2)$ and Equation (6) to evaluate $\mathcal{L}\{\cos kt\}$.

Solution Let $f(t) = \sin kt$ and then $f'(t) = k\cos kt$. From (6) it follows that

$$\begin{aligned}\mathcal{L}\{k\cos kt\} &= s\mathcal{L}\{\sin kt\} - 0 \\ &= \frac{ks}{s^2 + k^2},\end{aligned}$$

and hence we deduce that (as before)

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2 + k^2}.$$

Similarly, if $f(t)$ and $f'(t)$ are continuous and $f''(t)$ is piecewise continuous, and all three have transforms, it follows that

$$\begin{aligned}\mathscr{L}\{f''(t)\} &= s\mathscr{L}\{f'(t)\} - f'(0) \\ &= s^2F(s) - sf(0) - f'(0). \end{aligned} \tag{7}$$

Generalizing these results, we state the following theorem.

Theorem 6.4 (*Differentiation*) If $f(t)$, $f'(t)$, . . . , $f^{(n-1)}(t)$ are continuous for $t \geq 0$ and of exponential order, and if $f^{(n)}(t)$ is piecewise continuous and of exponential order, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

EXAMPLE 10 Evaluate $\mathscr{L}\{t^n\}$ for $n = 1, 2, 3, \ldots$.

Solution Let us first take the case when $n = 1$ and define $f(t) = t$. Then $f'(t) = 1$ and from previous results we know $\mathscr{L}\{1\} = 1/s$. Substituting these expressions into (6), we get

$$\frac{1}{s} = s\mathscr{L}\{t\} - f(0) = s\mathscr{L}\{t\}$$

or

$$\mathscr{L}\{t\} = \frac{1}{s^2}.$$

The function $f(t) = t^n$ and all its derivatives are continuous and of exponential order. Also we observe that

$$f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0, \quad f^{(n)}(t) = n!.$$

Substituting these results into Theorem 6.4 leads to

$$\frac{n!}{s} = s^n\mathscr{L}\{t^n\} - 0 - 0 - \cdots - 0,$$

and therefore

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots .$$

A different application of Theorem 6.4 involves the Laplace transform of integrals.

Theorem 6.5 If $f(t)$ is piecewise continuous and $F(s) = \mathscr{L}\{f(t)\}$, then

$$\mathscr{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}.$$

Proof: Let us define

$$g(t) = \int_0^t f(u)\,du,$$

which is continuous (why?). Furthermore, $g(0) = 0$ and $g'(t) = f(t)$. Hence,

$$\mathscr{L}\{f(t)\} = s\mathscr{L}\left\{\int_0^t f(u)\,du\right\} - 0$$

or

$$\mathscr{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}. \quad \Box$$

Remark. Theorem 6.5 is a special case of the *convolution theorem*, discussed in Section 6.7.

In these last properties, we found that differentiation and integration of a given function in the t-domain correspond roughly to multiplication and division, respectively, in the s-domain. In this fashion, the Laplace transform has the effect of relacing the operations of calculus in the t-domain with algebraic operations in the s-domain. It is primarily for this reason that the Laplace transform is so useful in applications.

6.3.2 Derivatives and Integrals of Laplace Transforms

Sometimes we need to evaluate the transform of a function that is expressed as $t^n f(t)$, when the transform of $f(t)$ is either known or readily obtained. In order to derive the needed property, let us start with the relation

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt \tag{8}$$

and formally differentiate both sides with respect to s. This action leads to

$$F'(s) = \int_0^\infty (-t) f(t) e^{-st}\,dt,$$

and thus it follows that

$$\mathscr{L}\{tf(t)\} = -F'(s) \tag{9}$$

EXAMPLE 11 Evaluate $\mathscr{L}\{t \sin t\}$.

Solution From the known relation $\mathscr{L}\{\sin t\} = (s^2 + 1)^{-1}$, simple differentiation with respect to s yields

$$\mathscr{L}\{t \sin t\} = -\frac{d}{ds}[(s^2 + 1)^{-1}] = \frac{2s}{(s^2 + 1)^2}.$$

EXAMPLE 12 Evaluate $\mathscr{L}\{te^{-2t} \cos t\}$.

Solution Starting with $\mathscr{L}\{\cos t\} = s(s^2 + 1)^{-1}$, we first evaluate

$$\mathscr{L}\{t \cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2 + 1}\right) = \frac{s^2 - 1}{(s^2 + 1)^2}$$

and then apply the shifting property to obtain

$$\mathscr{L}\{te^{-2t} \cos t\} = \frac{(s + 2)^2 - 1}{[(s + 2)^2 + 1]^2}.$$

If we differentiate (8) n times, we get

$$F^{(n)}(s) = \int_0^\infty (-t)^n f(t) e^{-st}\, dt,$$

from which we deduce the following theorem.

Theorem 6.6 If $F(s) = \mathscr{L}\{f(t)\}$, then for $n = 1, 2, 3, \ldots$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s).$$

The integration of both sides of (8) produces yet another interesting result.

omit

Theorem 6.7 If $F(s) = \mathscr{L}\{f(t)\}$, then

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\, du,$$

provided that $f(t)/t$ has a transform.

Proof. Integrating both sides of

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

with respect to s from s to ∞, and interchanging the order of integration on the right, we find

$$\int_s^\infty F(u)\,du = \int_0^\infty \left[\frac{e^{-st}}{-t}\right]_s^\infty f(t)\,dt$$
$$= \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right] dt,$$

which is our intended result. $\square$

EXAMPLE 13 Evaluate $\mathcal{L}\{(\sin t)/t\}$.

Solution Since $\mathcal{L}\{\sin t\} = (s^2 + 1)^{-1}$, we have

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{du}{u^2+1} = \text{Arctan}\frac{1}{s} = \text{Arccot}\, s.$$

EXAMPLE 14 Using properties of Laplace transforms, evaluate the integral

$$\int_0^\infty te^{-2t}\cos t\,dt.$$

Solution In Example 12, we have shown that

$$\mathcal{L}\{t\cos t\} = \int_0^\infty te^{-st}\cos t\,dt = \frac{s^2-1}{(s^2+1)^2}, \qquad s > 0.$$

Letting $s = 2$, we find

$$\int_0^\infty te^{-2t}\cos t\,dt = \frac{3}{25}.$$

EXERCISES 6.3

In problems 1–12, evaluate the Laplace transform of the given function using known results and any of the operational properties.

1. $f(t) = 3te^{2t}$
2. $f(t) = t^3e^{-2t}$
3. $f(t) = t^2\sin kt$
4. $f(t) = 2e^{-t}\sin 3t$
5. $f(t) = \cosh kt\cos kt$
6. $f(t) = e^{-t}(t^2 - 2t + 7)$
7. $f(t) = 3e^{-4t}(\cos 4t - t\sin 4t)$
8. $f(t) = \dfrac{\sinh t}{t}$
9. $f(t) = \dfrac{e^t - e^{-t}}{t}$
10. $f(t) = \int_0^t (u^2 - u + e^{-u})\,du$
11. $f(t) = e^{-t}\cos^2 t$
12. $f(t) = 5te^{3t}\sin^2 t$

13. Graph $f(t)$ and $f'(t)$, where $f(t) = t + 1$, $0 \leq t \leq 2$, and $f(t) = 3$, $t > 2$.

(a) Find $\mathcal{L}\{f(t)\}$.

(b) Find $\mathcal{L}\{f'(t)\}$ in two ways.

14. If $\mathcal{L}\{f(t)\} = F(s)$, show that $\mathcal{L}\{f(at)\} = (1/a)F(s/a)$, $a > 0$.

15. Using problem 14 and $\mathcal{L}\{\cos t\} = s(s^2 + 1)^{-1}$, find $\mathcal{L}\{\cos 4t\}$.

16. If n denotes a positive integer, obtain $\mathcal{L}\{t^n e^{at}\}$ from $\mathcal{L}\{e^{at}\} = 1/(s - a)$ using Theorem 6.6.

17. Using Theorems 6.5 and 6.7, determine the Laplace transform of the *sine integral*

$$Si(t) = \int_0^t \frac{\sin u}{u}\, du.$$

18. Using transforms, evaluate the following integrals.

(a) $\displaystyle\int_0^\infty \frac{\sin at}{t}\, dt, \quad a > 0.$

(b) $\displaystyle\int_0^\infty \frac{e^{-3t} - e^{-6t}}{t}\, dt.$

(c) $\displaystyle\int_0^\infty e^{-t} \frac{\sin t}{t}\, dt.$

(d) $\displaystyle\int_0^\infty t^2 e^{-3t} \sin t\, dt.$

Hint: Set $s = 0$ in $\displaystyle\int_0^\infty e^{-st} \frac{\sin at}{t}\, dt$ for part (a), etc.

***19.** Show that $\displaystyle\mathcal{L}\left\{\int_t^\infty \frac{f(u)}{u}\, du\right\} = \frac{1}{s}\int_0^\infty \frac{f(u)}{u}(1 - e^{-su})\, du.$

***20.** If $f(t)$ is a periodic function such that $f(t + T) = f(t)$,

(a) show that $\displaystyle\mathcal{L}\{f(t)\} = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)\, dt.$

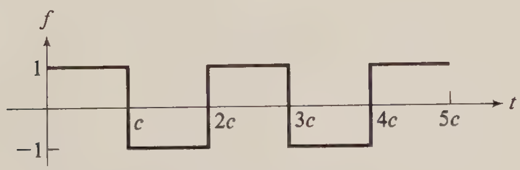
(b) Using the relation $\displaystyle\sum_{n=0}^{\infty} e^{-nst} = (1 - e^{-st})^{-1}$, verify that the Laplace transform of a periodic function with period T is obtained from

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}.$$

***21.** Use the result of problem 20 to establish the following Laplace transform relations:

(a) $\displaystyle\mathcal{L}\{f(t)\} = \frac{1}{s}\tanh\left(\frac{1}{2}cs\right),$ where $f(t + 2c) = f(t)$ and

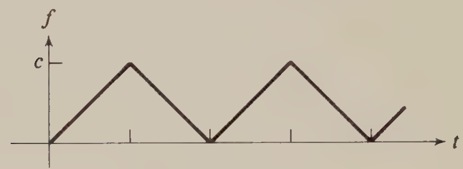
$$f(t) = \begin{cases} 1, & 0 \leq t \leq c \\ -1, & c < t < 2c. \end{cases}$$



Problem 21(a)

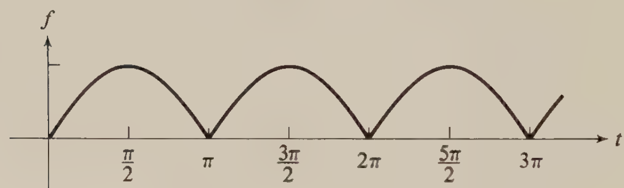
(b) $\mathscr{L}\{f(t)\} = \dfrac{1}{s^2}\tanh\left(\dfrac{1}{2}cs\right)$, where $f(t + 2c) = f(t)$ and

$$f(t) = \begin{cases} t, & 0 \leqslant t \leqslant c \\ 2c - t, & c < t < 2c. \end{cases}$$



Problem 21(b)

(c) $\mathscr{L}\{|\sin t|\} = (s^2 + 1)^{-1}\coth\left(\dfrac{1}{2}\pi s\right)$.



Problem 21(c)

***22.** The *Laguerre polynomials* are defined by $L_n(t) = \dfrac{e^t}{n!}\dfrac{d^n}{dt^n}(t^n e^{-t})$, $n = 0, 1, 2, \ldots$. Show that

$$\mathscr{L}\{L_n(t)\} = \frac{1}{s}\left(\frac{s-1}{s}\right)^n.$$

Hint: First find $\mathscr{L}\{e^{-t}L_n(t)\}$.

***23.** The *error function* (erf) and *complementary error function* (erfc) are defined, respectively, by

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}\,dx, \quad \operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-x^2}\,dx.$$

(a) Show that $\operatorname{erf}(\infty) = 1$.

Hint: Use the gamma function relation $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, given in problem 29 of Exercises 6.2.

(b) Show that $\operatorname{erfc}(t) = 1 - \operatorname{erf}(t)$.

***24.** Using problem 23, establish the following Laplace transform relations.

(a) $\mathscr{L}\{e^{-(1/4)t^2}\} = \sqrt{\pi}\, e^{s^2}\operatorname{erfc}(s)$.

Hint: Write $\frac{1}{4}t^2 + st = (\frac{1}{2}t + s)^2 - s^2$ and make the change of variable $y = \frac{1}{2}t + s$.

(b) $\mathcal{L}\{\mathrm{erf}(t)\} = s^{-1}e^{(1/4)s^2}\,\mathrm{erfc}\,(\frac{1}{2}s)$.

Hint: Change the order of integration.

(c) $\mathcal{L}\{\mathrm{erfc}\,(t^{-1/2})\} = \dfrac{1}{s}e^{-2\sqrt{s}}$.

Hint: $\displaystyle\int_0^\infty e^{-ax^2-b^2x^{-2}}\,dx = \left(\frac{\pi^{1/2}}{2a}\right)e^{-2ab}$.

(d) From (c), show that $\mathcal{L}\{\mathrm{erf}\,(t^{-1/2})\} = \dfrac{1}{s}(1 - e^{-2\sqrt{s}})$.

***25.** Expand $\sin t^{1/2}$ in an infinite series and

(a) show that $\mathcal{L}\{\sin t^{1/2}\} = (\pi^{1/2}/2s^{3/2})e^{-1/4s}$.

(b) From (a), determine $\mathcal{L}\{t^{-1}\sin t^{1/2}\}$. *Hint*: Use problem 23.

***26.** Starting with the integral relation for the *Bessel function*

$$J_0(t) = \frac{2}{\pi}\int_0^{\pi/2}\cos\,(t\sin\theta)\,d\theta,$$

(*a*) show that

$$\mathcal{L}\{J_0(t)\} = \frac{2s}{\pi}\int_0^{\pi/2}\frac{\csc^2\theta}{(s^2+1)+s^2\cot^2\theta}\,d\theta.$$

(b) Make the change of variables $x = \cot\theta$ in (a), evaluate the resulting integral, and establish the transform relation

$$\mathcal{L}\{J_0(t)\} = (s^2+1)^{-1/2}.$$

6.4 INVERSE LAPLACE TRANSFORMS

In applications the use of the Laplace transform is effective only if, given some function $F(s)$, we can find the *inverse transform* $f(t)$. In symbols, we write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}. \tag{10}$$

We might well wonder if an explicit representation for the inverse Laplace transform exists analogous to the integral representation of the transform itself. Such a representation does exist, but it requires integrations performed in the complex plane that involve techniques of complex variables. Therefore, we will rely on other methods for constructing the inverse transform.

In the process of finding the inverse transform $f(t)$ of a function $F(s)$, there are three basic questions of concern:

1. Does the inverse transform exist?
2. When the inverse transform exists, is it unique?
3. How do we find inverse transforms?

The answer to question 1 is not necessarily. In order for $F(s)$ to be the transform of some function, it must possess certain continuity requirements and behave suitably as $s\to\infty$, as stated in the following theorem.

Theorem 6.8 If $f(t)$ is piecewise continuous and of exponential order, then

$$\lim_{s\to\infty} F(s) = 0,$$

where $F(s) = \mathscr{L}\{f(t)\}$.

The proof of this theorem follows that of Theorem 6.1 and is left to the exercises. The real significançe of the theorem is that if $F(s)$ is any function such that $\lim_{s\to\infty} F(s) \neq 0$, then it does not represent the Laplace transform of any regular function $f(t)$. This condition immediately rules out many functions as possible Laplace transforms, such as polynomials in s, e^s, $\cos ks$, and so forth.

To respond to question 2, we observe that a discontinuous function can be transformed into a continuous function by use of the Laplace transform. If $f(t)$ and $g(t)$ are two identical functions except for a finite number of points, they will have the same transform, say $F(s)$. Hence, either $f(t)$ or $g(t)$ can be considered the inverse transform of $F(s)$. That is to say, the inverse transform of a given function is uniquely determined only up to an additive *null function* (*Lerche's theorem*). Since null functions are normally of little consequence in applications, the apparent difficulty of finding unique inverse transforms is of no practical concern.

Remark. A null function $n(t)$ is one satisfying $\int_0^t n(u)\,du = 0$ for all t.

As an illustration of the above remarks concerning uniqueness, recall from Section 6.2 that $\mathscr{L}\{1\} = 1/s$. Also, we note that the function defined by

$$g(t) = \begin{cases} 1, & 0 < t < 5 \\ 2, & t = 5 \\ 1, & t > 5 \end{cases}$$

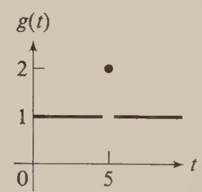


Figure 6.4

(Figure 6.4) has the transform

$$\begin{aligned} \mathscr{L}\{g(t)\} &= \int_0^\infty e^{-st} g(t)\,dt = \int_0^5 e^{-st}\,dt + \int_5^\infty e^{-st}\,dt \\ &= \frac{1}{s}, \quad s > 0. \end{aligned}$$

Hence, either 1 or $g(t)$ can be considered the inverse transform of $F(s) = 1/s$. When a choice can be made between a continuous function $f(t)$ and a discontinuous function $g(t)$ to represent the inverse Laplace transform of $F(s)$, we always choose the continuous function. There can only be one continuous inverse transform in any situation.

In answer to question 3, we find that for many routine problems the inverse Laplace transform can be obtained directly from existing tables of transforms. (A short table is provided in the last section of this chapter.) Also, many of the operational properties used in finding the transform itself can likewise be used in constructing the inverse transform. For instance, the linearity property for inverse transforms reads

$$\mathcal{L}^{-1}\{C_1F(s) + C_2G(s)\} = C_1\mathcal{L}^{-1}\{F(s)\} + C_2\mathcal{L}^{-1}\{G(s)\}. \tag{11}$$

Hence, optimum use of tables coupled with certain operational properties will generally produce the required inverse transform in most elementary problems.

EXAMPLE 15 Find $\mathcal{L}^{-1}\{(3s + 7)/(s^2 + 5)\}$.

Solution Using the linearity property (11), we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3s+7}{s^2+5}\right\} &= 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+5}\right\} + \frac{7}{\sqrt{5}}\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{s^2+5}\right\}\\ &= 3\cos\sqrt{5}\,t + \frac{7}{\sqrt{5}}\sin\sqrt{5}\,t.\end{aligned}$$

This last result is obtained from the tables.

EXAMPLE 16 Find $\mathcal{L}^{-1}\left\{\dfrac{s-5}{s^2+6s+13}\right\}$.

Solution Completing the square in the denominator, we get

$$\frac{s-5}{s^2+6s+13} = \frac{s-5}{(s+3)^2+4} = \frac{(s+3)-8}{(s+3)^2+4}.$$

Thus, using the shifting property (Theorem 6.3) in the form

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\},$$

we obtain

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s-5}{s^2+6s+13}\right\} &= e^{-3t}\mathcal{L}^{-1}\left\{\frac{s-8}{s^2+4}\right\}\\ &= e^{-3t}\left[\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - 4\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\right]\\ &= e^{-3t}(\cos 2t - 4\sin 2t).\end{aligned}$$

6.4.1 Partial Fractions

In many cases of practical importance we wish to find the inverse transform of a rational function, i.e., a function having the form

$$F(s) = \frac{P(s)}{Q(s)},$$

where $P(s)$ and $Q(s)$ are polynomials in s. The inverse transform in such a case can most easily be effected by representing $F(s)$ in partial fractions. The partial fraction representation is the same as that found in the calculus, for example, as a means of integrating certain rational functions. It is assumed that $P(s)$ and $Q(s)$ have no common factors and that the degree of $P(s)$ is lower than that of $Q(s)$. Let us illustrate the technique with some examples.

EXAMPLE 17 Find the inverse transform of

$$F(s) = \frac{2s^2 + 5s - 1}{s^3 - s}.$$

Solution Using partial fraction expansions, we have

$$\frac{2s^2 + 5s - 1}{s^3 - s} = \frac{2s^2 + 5s - 1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1},$$

or, upon clearing fractions,

$$2s^2 + 5s - 1 = A(s-1)(s+1) + Bs(s+1) + Cs(s-1).$$

By setting $s = 0$, $s = 1$, and $s = -1$, respectively, we deduce that $A = 1, B = 3$, and $C = -2$. Hence,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2s^2 + 5s - 1}{s^3 - s}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= 1 + 3e^{t} - 2e^{-t}.\end{aligned}$$

EXAMPLE 18 Find $\mathcal{L}^{-1}\{(s+1)^{-1}(s^2+1)^{-1}\}$.

Solution To determine the inverse transform, we first assume an expansion of the form

$$\frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs + C}{s^2 + 1}.$$

Clearing fractions gives

$$1 = A(s^2 + 1) + (Bs + C)(s + 1).$$

Setting $s = -1$, we find $A = \frac{1}{2}$, and equating the coefficients of s^2 and s^0 yields the equations

$$0 = A + B, \quad 1 = A + C,$$

from which we deduce $B = -\frac{1}{2}$ and $C = \frac{1}{2}$. Thus

$$\begin{aligned}\mathcal{L}^{-1}\{(s+1)^{-1}(s^2+1)^{-1}\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \frac{1}{2}(e^{-t} - \cos t + \sin t).\end{aligned}$$

EXAMPLE 19 Find $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2(s+2)^3}\right\}$.

Solution Let us write

$$\frac{s+1}{s^2(s+2)^3} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{(s+2)^2} + \frac{E}{(s+2)^3}.$$

Clearing fractions, we obtain

$$s + 1 = As(s+2)^3 + B(s+2)^2 + Cs^2(s+2)^2 + Ds^2(s+2) + Es^2.$$

Solving for the constants yields $C = -A = \frac{1}{16}$, $B = \frac{1}{8}$, $D = 0$, and $E = -\frac{1}{4}$. Hence

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s+2)^3}\right\} &= -\frac{1}{16}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{16}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} \\ &\quad - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\} \\ &= -\frac{1}{16} + \frac{1}{8}t + \frac{1}{16}e^{-2t} - \frac{1}{8}t^2e^{-2t}.\end{aligned}$$

The method of partial fractions can be greatly systematized by use of the Heaviside expansion theorems.* These theorems provide an approach to determining the unknown constants in partial fraction expansions that is more sophisticated than clearing fractions and matching like terms. The technique involves finding all linear factors (both real and complex) of the polynomial in the denominator, producing a partial fraction expansion in terms of these linear (and possibly repeated) factors, and then evaluating the unknown constants one by one. In other words, it is not necessary to solve simultaneous equations as we did here. Such an elaborate procedure, however, is not required for the elementary problems considered in this chapter.

*See, for example, E. Kreyszig, *Advanced Engineering Mathematics*, 4th ed. (New York: Wiley, 1979).

EXERCISES 6.4

In problems 1–12, determine the inverse Laplace transform of each function using Table 6.1 on p. 217 and various operational properties.

1. $F(s) = \dfrac{7}{s^3}$

2. $F(s) = \dfrac{3}{5s^2 + 25}$

3. $F(s) = \dfrac{2}{(s - 3)^5}$

4. $F(s) = \dfrac{15}{s^2 + 4s + 13}$

5. $F(s) = \dfrac{1}{s^2 - 6s + 10}$

6. $F(s) = \dfrac{s}{s^2 - 6s + 13}$

7. $F(s) = \dfrac{13}{s^2 + 8s + 16}$

8. $F(s) = \dfrac{s - 10}{s^2 + 6s + 13}$

9. $F(s) = \dfrac{7s - 3}{s^2 + 4s + 29}$

10. $F(s) = \dfrac{s^2}{(s + 2)^4}$

11. $F(s) = \dfrac{5s - 2}{3s^2 + 4s + 8}$

12. $F(s) = \dfrac{3s + 1}{(s + 1)^5}$

In problems 13–24, evaluate the inverse Laplace transform by the method of partial fractions.

13. $F(s) = (s^2 + s)^{-1}$

14. $F(s) = \dfrac{1}{(s - 1)(s + 2)(s + 4)}$

15. $F(s) = \dfrac{s^2}{(s + 2)^3}$

16. $F(s) = \dfrac{3s - 2}{s^3(s^2 + 4)}$

17. $F(s) = \dfrac{s + 1}{(s^2 - 4s)(s + 5)^2}$

18. $F(s) = (s^4 - 1)^{-1}$

19. $F(s) = \dfrac{s^2 + 1}{(s^2 - 1)(s^2 - 4)}$

20. $F(s) = \dfrac{4s^2 - 16}{s^3(s + 2)^2}$

21. $F(s) = \dfrac{s + 1}{s^3 + s^2 - 6s}$

***22.** $F(s) = \dfrac{3s^2 - 6s + 7}{(s^2 - 2s + 5)^2}$

***23.** $F(s) = \dfrac{s^2 - 3}{(s + 2)(s - 3)(s^2 + 2s + 5)}$

***24.** $F(s) = \dfrac{s^3 + 16s - 24}{s^4 + 20s^2 + 64}$

25. Given that $F(s) = \mathscr{L}\{f(t)\}$, show for constants a, b, and k, that

(a) $\mathscr{L}^{-1}\{F(ks)\} = \dfrac{1}{k} f\left(\dfrac{t}{k}\right), \quad k > 0.$

(b) $\mathscr{L}^{-1}\{F(as + b)\} = \dfrac{1}{a} e^{-bt/a} f\left(\dfrac{t}{a}\right), \quad a > 0.$

26. Using the results of problem 25, evaluate the inverse Laplace transform of each of the following functions.

(a) $F(s) = \left(\dfrac{3}{s}\right)^7$

(b) $F(s) = \dfrac{2s + 3}{4s^2 + 4s + 5}$

(c) $F(s) = \dfrac{1}{9s^2 + 12s + 3}$ (d) $F(s) = \dfrac{1}{2s - 1}$

27. If it is known that $\mathcal{L}^{-1}\{s^{-1/2}e^{-1/s}\} = (\pi t)^{-1/2}\cos(2t^{1/2})$, find

$$\mathcal{L}^{-1}\{s^{-1/2}e^{-a/s}\}, \quad a > 0.$$

Hint: Use problem 25.

28. Show that $\mathcal{L}^{-1}\{(s + a)^{-(n+1)}\} = \dfrac{t^n e^{-at}}{n!}$.

29. Show that $\mathcal{L}^{-1}\left\{\dfrac{s}{(s + a)^2 + b^2}\right\} = \dfrac{1}{b}e^{-at}(b\cos bt - a\sin bt)$.

***30.** Show that $\mathcal{L}^{-1}\left\{\log\left(\dfrac{s + 1}{s - 1}\right)\right\} = \dfrac{2\sinh t}{t}$.

Hint: Write

$$\log\left(\frac{s + 1}{s - 1}\right) = \log\left(\frac{1 + 1/s}{1 - 1/s}\right)$$

and use

$$\log\left(\frac{1 + x}{1 - x}\right) = 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n + 1}, \quad |x| < 1.$$

Find the inverse transform of the series and sum the result.

***31.** Show that (see problem 30)

(a) $\mathcal{L}^{-1}\left\{\log\left(1 + \dfrac{1}{s}\right)\right\} = \dfrac{1 - e^{-t}}{t}$.

(b) $\mathcal{L}^{-1}\left\{\log\left(\dfrac{s - a}{s - b}\right)\right\} = \dfrac{e^{bt} - e^{at}}{t}$.

***32.** Solve problem 30 by using Theorem 6.6 in the form $(n=1)$

$$f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}.$$

***33.** Solve problem 31 by using Theorem 6.6 (see problem 32).

***34.** Let $f(t) = \displaystyle\int_0^\infty \frac{\sin tx}{x}\,dx, \quad t > 0$.

(a) Show that by taking Laplace transforms of both sides, we formally obtain

$$F(s) = \int_0^\infty (x^2 + s^2)^{-1}\,dx.$$

(b) Evaluate the integral in (a), and by taking the inverse transform of the result, deduce the value of $f(t)$.

***35.** Using the technique of problem 34, show that

(a) $\displaystyle\int_0^\infty e^{-tx^2}\,dx = \frac{1}{2}\left(\frac{\pi}{t}\right)^{1/2}, \quad t > 0.$

(b) $\displaystyle\int_0^\infty \frac{\cos tx}{1 + x^2}\,dx = \frac{\pi}{2}e^{-t}, \quad t > 0.$

(c) $\int_0^{\infty} x^{-1/2} \sin tx \, dx = \left(\dfrac{\pi}{2t}\right)^{1/2}, \quad t > 0.$

***36.** Prove that $\lim_{s \to \infty} sF(s) = \lim_{t \to 0^+} f(t)$.

Hint: Set $f(t) = \sum_{n=0}^{\infty} c_n t^n$.

***37.** Prove Theorem 6.8.

6.5 SOLUTION OF INITIAL VALUE PROBLEMS

The Laplace transform is a powerful tool for solving linear DEs with constant coefficients—in particular, *initial value problems*. Let us illustrate the method with a simple example.

EXAMPLE 20 Solve $y' = 3e^t$, $y(0) = 7$.

Solution We first apply the Laplace transform to both sides of the DE:

$$\mathcal{L}\{y'(t)\} = \mathcal{L}\{3e^t\}.$$

From the tables we find that $\mathcal{L}\{3e^t\} = 3/(s - 1)$, and making use of Theorem 6.4, we get

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0) = sY(s) - 7.$$

Thus, the differential system is reduced to the algebraic equation

$$sY(s) - 7 = \frac{3}{s - 1},$$

and solving for $Y(s)$ yields

$$Y(s) = \frac{3}{s(s - 1)} + \frac{7}{s}.$$

Expressing $Y(s)$ in terms of its partial fractions, we get

$$Y(s) = \frac{3}{s - 1} + \frac{4}{s},$$

and hence the solution of the original problem is symbolically given by

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

or

$$y(t) = 3e^t + 4.$$

Although this example was rather trivial and could be solved more easily by another method, it does illustrate the basic features of the transform method. The usefulness of the Laplace transform rests primarily on the fact that the transform of the DE together with the prescribed initial conditions reduces the differential system to an algebraic equation in $Y(s)$. Such an algebraic equation is readily solved, and the inverse transform of its solution then yields the solution of the initial value problem; i.e., $y(t) = \mathcal{L}^{-1}\{Y(s)\}$. Furthermore, the solution of the DE satisfying certain initial conditions is found directly without first producing a general solution and then solving for the arbitrary constants.

EXAMPLE 21 Solve $y'' + 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = -4.$

Solution The transform of the DE leads to

$$[s^2Y(s) - 2s + 4] + 2[sY(s) - 2] + 5Y(s) = 0,$$

or

$$(s^2 + 2s + 5)Y(s) = 2s.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{2s}{s^2 + 2s + 5}.$$

Making use of the shifting property, we can write

$$\frac{2s}{s^2 + 2s + 5} = \frac{2s}{(s + 1)^2 + 4} = \frac{2(s + 1) - 2}{(s + 1)^2 + 4},$$

and thus

$$\begin{aligned} y(t) &= e^{-t}\mathcal{L}^{-1}\left\{\frac{2s - 2}{s^2 + 4}\right\} \\ &= e^{-t}\left(2\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\}\right) \\ &= e^{-t}(2\cos 2t - \sin 2t). \end{aligned}$$

EXAMPLE 22 Solve $y'' - 6y' + 9y = t^2e^{3t}, \quad y(0) = 2, \quad y'(0) = 6.$

Solution Taking the transform of the DE and simplifying gives us

$$(s^2 - 6s + 9)Y(s) = 2(s - 3) + \frac{2}{(s - 3)^3}.$$

Hence,

$$Y(s) = \frac{2}{s - 3} + \frac{2}{(s - 3)^5},$$

and taking inverse transforms, we find

$$\begin{aligned} y(t) &= 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{2}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{(s-3)^5}\right\} \\ &= 2e^{3t} + \frac{1}{12}t^4e^{3t}. \end{aligned}$$

It may be of interest to see exactly where each input parameter ends up in the transform domain by considering the general second-order DE

$$ay'' + by' + cy = f(t), \tag{12}$$

where a, b, and c are known constants.

If we apply the Laplace transform to each term in (12), we get

$$a[s^2Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s),$$

or

$$(as^2 + bs + c)Y(s) - (as + b)y(0) - ay'(0) = F(s),$$

where $F(s) = \mathcal{L}\{f(t)\}$. This algebraic equation can be rearranged in the form

$$Y(s) = \frac{(as + b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}. \tag{13}$$

Now, taking the inverse transform leads to

$$y(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{(as + b)y(0) + ay'(0)}{as^2 + bs + c}\right\}}_{y_H(t)} + \underbrace{\mathcal{L}^{-1}\left\{\frac{F(s)}{as^2 + bs + c}\right\}}_{y_P(t)}. \tag{14}$$

Here it is interesting to observe that the solution (14) has naturally split into two parts—the function y_H, which is a solution of the initial value problem

$$ay'' + by' + cy = 0, \quad y(0) = k_0, \quad y'(0) = k_1, \tag{15}$$

where k_0 and k_1 are prescribed values, and y_P, which satisfies

$$ay'' + by' + cy = f(t), \quad y(0) = 0, \quad y'(0) = 0. \tag{16}$$

We see, therefore, that the transform method splits the problem into two separate problems, much like we did in Section 5.5 because of physical significance. This makes the Laplace transform an effective tool for analyzing the basic characteristics of a system in response to each of the input parameters k_0 and k_1, or $f(t)$.

Finally, we should note that the coefficient $as^2 + bs + c$ of the function $Y(s)$ in the transform domain is precisely the polynomial that appears in the auxiliary equation of the DE. Thus, the use of the Laplace transform does not avoid the necessity of factoring this polynomial. Once again, this factorization may represent one of the most difficult aspects of the problem when the DE is of higher order.

As a last example, let us solve a third-order DE.

EXAMPLE 23 Solve $y''' + y'' - y' - y = 9e^{2t}$, $y(0) = 2$, $y'(0) = 4$, $y''(0) = 3$.

Solution Applying the transform, we have

$$[s^3Y(s) - 2s^2 - 4s - 3] + [s^2Y(s) - 2s - 4] - [sY(s) - 2] - Y(s) = \frac{9}{s-2},$$

and solving for $Y(s)$ gives

$$\begin{aligned} Y(s) &= \frac{2s^2 + 6s + 5}{(s-1)(s+1)^2} + \frac{9}{(s-2)(s-1)(s+1)^2} \\ &= \frac{(2s^2 + 6s + 5)(s-2) + 9}{(s-2)(s-1)(s+1)^2}. \end{aligned}$$

Using partial fractions, we write

$$Y(s) = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{(s+1)^2} + \frac{D}{s+1},$$

from which it is found that $A = B = C = 1$ and $D = 0$. Hence

$$y(t) = e^{2t} + e^t + te^{-t}.$$

6.5.1 Spring-Mass Systems

The equation of motion governing small movements of a damped spring-mass system is known to be (see Section 5.2)

$$my'' + cy' + ky = f(t), \quad t > 0, \tag{17}$$

where m denotes the mass, k is the spring constant, and $f(t)$* is an externally applied stimulus. The term cy', where c is a positive constant, represents a retarding force due to damping effects. Let us assume that the initial position and velocity of the mass are given by

$$y(0) = y_0, \quad y'(0) = v_0. \tag{18}$$

To discuss some of the possible motions of the system described by (17) and (18), let us first set $f(t) \equiv 0$. We will find it convenient to also introduce the new parameters

$$a = \frac{c}{2m}, \quad b^2 = \frac{k}{m},$$

so that the governing equation now takes the form

$$y'' + 2ay' + b^2y = 0, \quad y(0) = y_0, \quad y'(0) = v_0. \tag{19}$$

Because of the nature of the parameters involved, it is clear that $a \geq 0$†, and we may choose $b > 0$.

*We use a lowercase letter here for $f(t)$ in order to be consistent with our Laplace transform notation.

†$a = 0$ only in the absence of the damping term, i.e., when $c = 0$.

If we apply the Laplace transform to (19), we find

$$s^2Y(s) - sy_0 - v_0 + 2a[sY(s) - y_0] + b^2Y(s) = 0$$

with solution

$$Y(s) = \frac{sy_0 + v_0 + 2ay_0}{s^2 + 2as + b^2},$$

which we choose to write as

$$Y(s) = \frac{y_0(s + a) + (v_0 + ay_0)}{(s + a)^2 + b^2 - a^2}. \tag{20}$$

Hence,

$$y(t) = e^{-at}\mathcal{L}^{-1}\left\{\frac{y_0 s + (v_0 + ay_0)}{s^2 + b^2 - a^2}\right\}. \tag{21}$$

It is here where the benefits of using the Laplace transform become evident. In order to invert the expression in (21), we need to know whether the denominator $s^2 + b^2 - a^2$ has distinct real factors, has equal factors, or is the sum of two squares. These three cases correspond to the following:

Case I—Overdamping: $b < a$
Case II—Critical damping: $b = a$
Case III—Underdamping: $b > a$

Case I: When $b < a$, then also $b^2 - a^2 < 0$, and we write

$$b^2 - a^2 = -\alpha^2.$$

By taking the inverse transform in (21), we get the result

$$\begin{aligned} y(t) &= e^{-at}\mathcal{L}^{-1}\left\{\frac{y_0 s + (v_0 + ay_0)}{s^2 - \alpha^2}\right\} \\ &= e^{-at}\left(y_0 \cosh \alpha t + \frac{(v_0 + ay_0)}{\alpha}\sinh \alpha t\right), \end{aligned} \tag{22}$$

which describes *overdamped motion*.

Case II: *Critically damped motion* results when $b = a$, and under this condition

$$\begin{aligned} y(t) &= e^{-at}\mathcal{L}^{-1}\left\{\frac{y_0}{s} + \frac{(v_0 + ay_0)}{s^2}\right\} \\ &= e^{-at}[y_0 + (v_0 + ay_0)t]. \end{aligned} \tag{23}$$

Case III: Oscillatory motion can occur only when $b > a$, which means that the system is *underdamped*. Here we find it convenient to put

$$b^2 - a^2 = \mu^2,$$

and making the appropriate parameter change in (21), we obtain

$$\begin{aligned} y(t) &= e^{-at}\mathcal{L}^{-1}\left\{\frac{y_0 s + (v_0 + ay_0)}{s^2 + \mu^2}\right\} \\ &= e^{-at}\left(y_0 \cos \mu t + \frac{v_0 + ay_0}{\mu}\sin \mu t\right). \end{aligned} \tag{24}$$

The reader should compare all three solutions with those obtained in Section 5.2.3 for each of the three cases of damping.

When the forcing function $f(t)$ is not identically zero, the Laplace transform method leads to the solution form

$$y(t) = e^{-at}\mathcal{L}^{-1}\left\{\frac{y_0 s + (v_0 + ay_0)}{s^2 + b^2 - a^2}\right\} + \mathcal{L}^{-1}\left\{\frac{F(s)}{(s + a)^2 + b^2 - a^2}\right\}, \tag{25}$$

where $F(s) = \mathcal{L}\{f(t)\}$. Let us illustrate this case with an example.

EXAMPLE 24 Determine the response of a spring-mass system that is initially at rest and then at time $t = 0$ subjected to the sinusoidal forcing function $f(t) = P\cos \omega t$. Neglect damping effects.

Solution The spring-mass system is characterized by

$$y'' + \omega_0^2 y = \frac{P}{m}\cos \omega t, \quad y(0) = 0, \quad y'(0) = 0,$$

where $\omega_0 = \sqrt{k/m}$. The transform of this DE leads to

$$(s^2 + \omega_0^2)Y(s) = \frac{Ps/m}{s^2 + \omega^2}$$

with solution

$$Y(s) = \frac{Ps/m}{(s^2 + \omega_0^2)(s^2 + \omega^2)}.$$

No resonance: If $\omega \neq \omega_0$, a partial fraction expansion yields

$$\frac{s}{(s^2 + \omega_0^2)(s^2 + \omega^2)} = \frac{As + B}{s^2 + \omega^2} + \frac{Cs + D}{s^2 + \omega_0^2},$$

from which we find $A = -C = 1/(\omega_0^2 - \omega^2)$ and $B = D = 0$. Hence

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \frac{P}{m(\omega_0^2 - \omega^2)}(\cos \omega t - \cos \omega_0 t). \end{aligned}$$

This solution represents the superposition of two harmonic motions whose frequencies correspond to the angular frequency ω of the forcing term and the natural angular frequency ω_0 of the system.

Resonance: When $\omega = \omega_0$, we get

$$y(t) = \frac{P}{m}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+\omega_0^2)^2}\right\}$$

$$= \frac{P}{2m\omega_0} t \sin \omega_0 t,$$

this last result being obtained from the tables. The solution this time corresponds to resonance in the system, since $y \to \infty$ as $t \to \infty$ (see also Section 5.3.2).

EXERCISES 6.5

In problems 1–20, solve the initial value problem using the Laplace transform method.

1. $y' = e^{2t}, \quad y(0) = -1$
2. $y' - y = -e^{-t}, \quad y(0) = 1$
3. $y' + y = e^{t}, \quad y(0) = 0$
4. $y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$
5. $y'' + y = 0, \quad y(0) = 0, \quad y'(0) = 1$
6. $y'' + y = 1, \quad y(0) = 0, \quad y'(0) = 0$
7. $y'' + y = 2e^{t}, \quad y(0) = 0, \quad y'(0) = 0$
8. $y'' + y' - 2y = -4, \quad y(0) = 2, \quad y'(0) = 3$
9. $y'' + 2y' + 2y = \sin 2t - 2\cos 2t, \quad y(0) = 0, \quad y'(0) = 0$
10. $y'' + 2y' + y = 3te^{-t}, \quad y(0) = 4, \quad y'(0) = 2$
11. $y'' + 4y' + 6y = 1 + e^{-t}, \quad y(0) = 1, \quad y'(0) = -4$
12. $y'' + 16y = \cos 4t, \quad y(0) = 0, \quad y'(0) = 1$
13. $y'' - 4y' + 4y = t, \quad y(0) = 1, \quad y'(0) = 0$
14. $y'' - y' = e^{t}\cos t, \quad y(0) = 0, \quad y'(0) = 0$
15. $y''' + 6y'' + 11y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = -1$
16. $y''' - y'' + 4y' - 4y = t, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$
17. $2y''' + 3y'' - 3y' - 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$
18. $y^{(4)} - y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 0$
19. $y^{(4)} - y = t, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$

*20. $y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0,$ $y'''(0) = 1$

In problems 21–31, use the Laplace transform to solve the problem specified.

21. Problem 3, Exercises 5.2
22. Problem 5, Exercises 5.2

23. Problem 8, Exercises 5.2

24. Problem 20, Exercises 5.2

25. Problem 21, Exercises 5.2

26. Example 6, Section 5.3

27. Problem 4, Exercises 5.3

28. Problem 15, Exercises 5.3

29. Problem 5, Exercises 5.4

30. Problem 7, Exercises 5.4

31. Problem 8, Exercises 5.4

***32.** Show that the Laplace transform of the differential system

$$ty'' + 2(t - 1)y' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 4,$$

leads to the first-order DE in the transform domain

$$(s^2 + 2s)\frac{dY}{ds} + (4s + 4)Y = 0.$$

Solve this first-order DE to obtain $Y(s)$, and invert it to find the solution $y(t)$. (This problem is one of the few variable-coefficient DEs for which the Laplace transform method proves fruitful.)

***33.** Apply the method of problem 32 to *Bessel's equation* of order zero,

$$ty'' + y' + ty = 0.$$

(a) Show that $Y(s)$ satisfies the first-order DE

$$(1 + s^2)\frac{dY}{ds} + sY(s) = 0.$$

(b) Show that the general solution of the DE in (a) is

$$Y(s) = C(1 + s^2)^{-1/2},$$

where C is an arbitrary constant.

(c) Express the term $(1 + s^2)^{-1/2}$ in a binomial series valid for $s > 1$. Assuming it is permissible to take the inverse transform termwise, deduce that

$$y = C\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2} = CJ_0(t),$$

where J_0 is the *Bessel function* of the first kind of order zero. Although a second solution of Bessel's equation can be found by other methods, it is not bounded at $t = 0$ and so does not arise in the transform method. Because it is not bounded at $t = 0$, it is discarded anyway in most applications.

6.6 DISCONTINUOUS FUNCTIONS

The Laplace transform has the effect of "smoothing" discontinuous functions in the transform domain. Thus, one of the most interesting and useful applications of the transform method is in solving linear DEs with discontinuous or impulse forcing functions. Equations of this type are commonplace in circuit analysis problems as well as in some problems involving mechanical systems.

In order to effectively deal with functions having finite jump discontinuities, it is helpful to introduce the *Heaviside unit function*, or *unit step function* as it is often called. We denote this function by the symbol $h(t - a)$ and define it by

$$h(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a, \end{cases} \tag{26}$$

where $a \geqslant 0$ (see Figure 6.5).

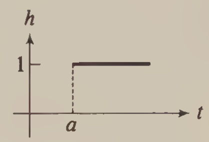


Figure 6.5 Heaviside unit function.

EXAMPLE 25 Sketch the graph of $f(t) = h(t-a) - h(t-b)$, $a < b$.

Solution From the definition of h, we find

$$f(t) = \begin{cases} 0 - 0 = 0, & 0 < t < a \\ 1 - 0 = 1, & a < t < b \\ 1 - 1 = 0, & t > b. \end{cases}$$

The graph of this function is shown in Figure 6.6.

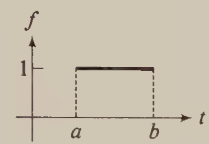


Figure 6.6

Suppose we have a function f that has nonzero values only on the interval $a < t < b$ (Figure 6.7). That is,

$$f(t) = \begin{cases} f_1(t), & a < t < b \\ 0, & \text{otherwise}. \end{cases} \tag{27}$$

In terms of the Heaviside unit function, we can write

$$f(t) = f_1(t)[h(t-a) - h(t-b)] \tag{28}$$

rather than expressing f piecewise as in (27). More generally, if

$$f(t) = \begin{cases} f_1(t), & 0 < t < a \\ f_2(t), & a < t < b \\ f_3(t), & t > b \end{cases} \tag{29}$$

(Figure 6.8), then we can write

$$\begin{aligned} f(t) &= f_1(t)[h(t) - h(t-a)] + f_2(t)[h(t-a) - h(t-b)] + f_3(t)h(t-b) \\ &= f_1(t) + [f_2(t) - f_1(t)]h(t-a) + [f_3(t) - f_2(t)]h(t-b) \end{aligned} \tag{30}$$

for $t \geqslant 0$.

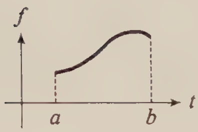


Figure 6.7

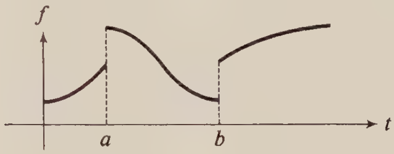


Figure 6.8

EXAMPLE 26 Express the function

$$f(t) = \begin{cases} t^2, & 0 < t < 2 \\ 3, & t > 2 \end{cases}$$

(Figure 6.9) in terms of the Heaviside unit function.

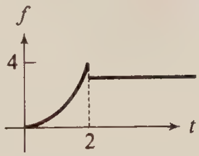


Figure 6.9

Solution Following (30), we write

$$\begin{aligned} f(t) &= t^2[h(t) - h(t-2)] + 3h(t-2) \\ &= t^2 + (3 - t^2)h(t-2), \quad t \geqslant 0. \end{aligned}$$

The Heaviside unit function is also useful in translating a function to the right a distance of a units. For example, the function $y = g(t)$ is shown in Figure 6.10, while the function $y = g(t-a)h(t-a)$ shown in Figure 6.11 represents a translation of g by a distance a in the positive t direction.

Calculating the Laplace transform of $h(t-a)$, we find directly

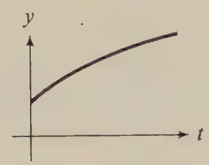


Figure 6.10 $y = g(t)$.

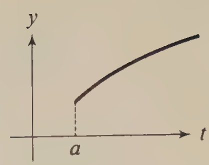


Figure 6.11 $y = g(t - a)h(t - a)$.

$$\mathcal{L}\{h(t-a)\} = \int_0^\infty e^{-st}h(t-a)\,dt$$
$$= \int_a^\infty e^{-st}\,dt$$

or

$$\mathcal{L}\{h(t-a)\} = e^{-as}/s, \quad s > 0. \tag{31}$$

Observe that when $a = 0$, $\mathcal{L}\{h(t)\} = \mathcal{L}\{1\} = 1/s$.

The next theorem, which relates the Laplace transform of the function $g(t-a)h(t-a)$ to that of $g(t)$, is most important in our present discussion.

Theorem 6.9 (*Translation*) If $\mathcal{L}\{g(t)\} = G(s)$, then

$$\mathcal{L}\{g(t-a)h(t-a)\} = e^{-as}G(s).$$

Proof: From definition,

$$\mathcal{L}\{g(t-a)h(t-a)\} = \int_0^\infty e^{-st}g(t-a)h(t-a)\,dt$$
$$= \int_a^\infty e^{-st}g(t-a)\,dt.$$

$t-a=x$

Introducing the new variable $x = t - a$, we have

$dt=dx$

$$\mathcal{L}\{g(t-a)h(t-a)\} = \int_0^\infty e^{-s(x+a)}g(x)\,dx$$
$$= e^{-as}\int_0^\infty e^{-sx}g(x)\,dx$$
$$= e^{-as}G(s),$$

and the theorem is proved. □

EXAMPLE 27 Find the Laplace transform of

$$f(t) = \begin{cases} t^2, & 0 < t < 2 \\ 3, & t > 2. \end{cases}$$

Solution From Example 26, we have

$$f(t) = t^2 + (3 - t^2)h(t - 2).$$

In order to use Theorem 6.9, we must rewrite $3 - t^2$ as a function of $t - 2$. That is,

$$\begin{aligned} 3 - t^2 &= 3 - (t - 2)^2 - 4t + 4 \\ &= 3 - (t - 2)^2 - 4(t - 2) + 4 - 8 \\ &= -1 - 4(t - 2) - (t - 2)^2. \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{t^2\} + e^{-2s}\mathcal{L}\{-1 - 4t - t^2\} \\ &= \frac{2}{s^3} - e^{-2s}\left(\frac{1}{s} + \frac{4}{s^2} + \frac{2}{s^3}\right). \end{aligned}$$

Another way of solving Example 27 is to express the result of Theorem 6.9 in the form

$$\mathcal{L}\{g(t)h(t - a)\} = e^{-as}\mathcal{L}\{g(t + a)\},$$

and then it follows that $f(t) = t^2 + (3 - t^2)h(t - 2),$

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{t^2\} + e^{-2s}\mathcal{L}\{3 - (t + 2)^2\} \\ &= \mathcal{L}\{t^2\} + e^{-2s}\mathcal{L}\{-1 - 4t - t^2\} \\ &= \frac{2}{s^3} - e^{-2s}\left(\frac{1}{s} + \frac{4}{s^2} + \frac{2}{s^3}\right). \end{aligned}$$

Using this latter technique, we avoid the necessity of first writing $3 - t^2$ as a function of $t - 2$.

EXAMPLE 28 Find the inverse transform of

$$F(s) = \frac{1 - 3e^{-5s}}{s^2}.$$

Solution Writing Theorem 6.9 as

$$\mathcal{L}^{-1}\{e^{-as}G(s)\} = g(t - a)h(t - a),$$

we find

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1 - 3e^{-5s}}{s^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s^2}\right\} \\ &= t - 3(t - 5)h(t - 5), \end{aligned}$$

or

$$f(t) = \begin{cases} t, & 0 \leq t < 5 \\ 15 - 2t, & t > 5. \end{cases}$$

EXAMPLE 29 Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(t) = \begin{cases} \sin t, & 0 \leq t < \dfrac{\pi}{2} \\ 0, & t > \dfrac{\pi}{2}. \end{cases}$$

Solution We can interpret this problem as a spring-mass system (no damping) that is at rest until time $t = 0$ and then subject to the sinusoidal forcing function $\sin t$ until time $t = \pi/2$, after which the forcing function is removed.

Writing $f(t) = \sin[1 - h(t = \pi/2)]$ and finding

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2 + 1} - e^{-\pi s/2}\mathcal{L}\left\{\sin\left(t + \frac{\pi}{2}\right)\right\}$$

$$= \frac{1}{s^2 + 1} - e^{-\pi s/2}\mathcal{L}\{\cos t\}$$

$$= \frac{1}{s^2 + 1} - \frac{se^{-\pi s/2}}{s^2 + 1},$$

it now follows that

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{f(t)\},$$

or

$$s^2Y(s) + Y(s) = \frac{1}{s^2 + 1} - \frac{se^{-\pi s/2}}{s^2 + 1}$$

with solution

$$Y(s) = \frac{1}{(s^2 + 1)^2} - \frac{se^{-\pi s/2}}{(s^2 + 1)^2}.$$

Thus,

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s^2 + 1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{se^{-\pi s/2}}{(s^2 + 1)^2}\right\}$$

$$= \frac{1}{2}(\sin t - t\cos t) - \frac{1}{2}h\left(t - \frac{\pi}{2}\right)\left(t - \frac{\pi}{2}\right)\sin\left(t - \frac{\pi}{2}\right),$$

or, equivalently,

$$y(t) = \begin{cases} \dfrac{1}{2}(\sin t - t\cos t), & 0 \le t < \dfrac{\pi}{2} \\ \dfrac{1}{2}\sin t - \dfrac{\pi}{4}\cos t, & t > \dfrac{\pi}{2}. \end{cases}$$

Observe that the amplitude of vibration becomes steady as soon as the external force is removed. Otherwise resonance would take place, since the forcing function has the natural frequency of the system.

[O] 6.6.1 Impulse Functions

Closely associated with the Heaviside unit function is the *impulse function* $\delta(t - a)$, also called the *Dirac delta function* (see Section 5.6). For $t \neq a$, the derivative of $h(t - a)$ is clearly zero, and at $t = a$ this function doesn't have a derivative in the usual sense because of the discontinuity. However, in a generalized sense the derivative can be defined in terms of the impulse function. Consider the following argument.

Since $\mathscr{L}\{h(t - a)\} = e^{-as}/s$, it follows from Theorem 6.4 that

$$\mathscr{L}\left\{\frac{dh}{dt}(t - a)\right\} = s\mathscr{L}\{h(t - a)\} = e^{-as}. \tag{32}$$

Also, using integral properties of the impulse function, we have

$$\mathscr{L}\{\delta(t - a)\} = \int_0^\infty e^{-st}\delta(t - a)\,dt = e^{-as}. \tag{33}$$

The two results (32) and (33) suggest that

$$\frac{dh}{dt}(t - a) = \delta(t - a). \tag{34}$$

The interpretation of (34) is that the unit step function has a jump discontinuity of unit magnitude at $t = a$. This is not too surprising since we defined it that way! The significance of the result, however, is that it allows us to generalize the concept of differentiation to include functions with finite discontinuities. Wherever a discontinuity occurs in the function, a delta function occurs in the derivative of this function, and moreover, the magnitude of the jump discontinuity appears as a multiplicative constant of the delta function.

Remark. Observe that $\mathscr{L}\{\delta(t)\} = 1$, which follows from (33) by setting $a = 0$. Although $\lim_{s\to\infty} F(s) \neq 0$ for $F(s) = 1$, we are not in violation of Theorem 6.8, since $\delta(t)$ is *not* a regular function.

EXAMPLE 30 Solve the initial value problem

$$y'' + y = \delta(t - \pi), \quad y(0) = 0, \quad y'(0) = 0.$$

Solution We previously solved this problem in Section 5.6 (Example 12) by using the method of Green's function. Now, taking the Laplace transform of each term in the equation, we get

$$s^2Y(s) + Y(s) = e^{-\pi s},$$

with solution

$$Y(s) = \frac{e^{-\pi s}}{s^2 + 1}.$$

Using the result $\mathcal{L}^{-1}\{1/(s^2 + 1)\} = \sin t$, we have

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2 + 1}\right\} = \sin(t - \pi)h(t - \pi),$$

or (see Figure 6.12)

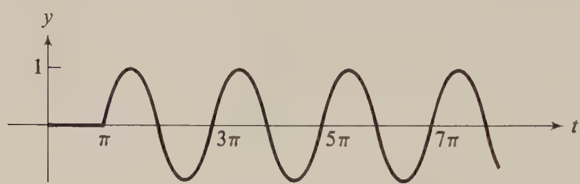


Figure 6.12

$$y(t) = \begin{cases} 0, & 0 \leq t < \pi \\ \sin(t - \pi), & t > \pi. \end{cases}$$

We might interpret this problem as representing an electric circuit to which an instantaneous unit voltage impulse is applied at time $t = \pi$. Hence, there is no response until time $t = \pi$ (due to the homogeneous initial conditions), but the impulse voltage produces a response that lasts indefinitely (there is no resistor for damping effects). Interestingly, the solution is a continuous function for all $t \geq 0$ in spite of the singular nature of the impulse function. There is, however, a jump discontinuity in y' at the point $t = \pi$. (See also the discussion in Section 5.6.1 concerning the Green's function.)

EXERCISES 6.6

In problems 1–6, sketch graphs for $t \geq 0$.

1. $f(t) = t^2h(t) + (5 - t^2)h(t - 3)$
2. $f(t) = h(t - 1) + 2h(t - 3) - 6h(t - 4)$
3. $f(t) = h(t) - h(t - \pi) + \sin t\, h(t - 2\pi)$

4. $f(t) = g(t-2)h(t-2)$, where $g(t) = t^2$

5. $f(t) = t^2 - (t-2)^2 h(t-2)$

6. $f(t) = t^2 - t^2 h(t-2)$

In problems 7–14, express each function in terms of the Heaviside unit function and find F(s).

7. $f(t) = \begin{cases} 2, & 0 \le t \le 1 \\ t, & t > 1 \end{cases}$

8. $f(t) = \begin{cases} t^2, & 0 \le t < 3 \\ 4, & t > 3 \end{cases}$

9. $f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t > \pi \end{cases}$

10. $f(t) = \begin{cases} \sin 3t, & 0 \le t < \dfrac{\pi}{2} \\ 0, & t > \dfrac{\pi}{2} \end{cases}$

11. $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ e^{-t}, & t > 3 \end{cases}$

12. $f(t) = \begin{cases} e^{-t}, & 0 \le t < 3 \\ 0, & t > 3 \end{cases}$

13. $f(t) = \begin{cases} t^2, & 0 \le t < 1 \\ 3, & 1 < t < 4 \\ 0, & t > 4 \end{cases}$

14. $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 < t < 2 \\ 1, & 2 < t < 3 \\ 0, & t > 3 \end{cases}$

In problems 15–22, find the inverse Laplace transform.

15. $F(s) = \dfrac{5e^{-3s} - e^{-s}}{s}$

16. $F(s) = \dfrac{e^{-s}}{s^2}$

17. $F(s) = \dfrac{3e^{-2s} - 1}{s^2}$

18. $F(s) = \dfrac{e^{-3s}}{s+2}$

19. $F(s) = \dfrac{e^{-3s}}{(s+2)^3}$

20. $F(s) = \dfrac{1 - e^{-\pi s}}{s^2 + 4}$

21. $F(s) = \dfrac{s(1 + e^{-\pi s})}{s^2 + 4}$

22. $F(s) = \dfrac{(s-2)e^{-s}}{s^2 - 4s + 3}$

Solve the initial value problems 23–34.

23. $y' + y = f(t)$, $y(0) = 1$, where $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & t > 1. \end{cases}$

24. $y' + y = f(t)$, $y(0) = 0$, where $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t > 1. \end{cases}$

25. $y'' + 4y = f(t)$, $y(0) = 1$, $y'(0) = 0$, where $f(t) = \begin{cases} 4t, & 0 \le t < 1 \\ 4, & t > 1 \end{cases}$

26. $y'' + y = f(t)$, $y(0) = 0$, $y'(0) = 0$, where $f(t) = \begin{cases} 4, & 0 \le t < 2 \\ t + 2, & t > 2 \end{cases}$

27. $y'' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = 1, \quad \text{where } f(t) = \begin{cases} \cos 4t, & 0 \le t < \pi \\ 0, & t > \pi \end{cases}$

28. $y'' + 4y = \sin t - h(t - 2\pi)\sin(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0$

29. $y'' - 5y' + 6y = h(t - 1), \quad y(0) = 0, \quad y'(0) = 1$

30. $y'' + 2y' + 2y = \delta(t - \pi), \quad y(0) = 0, \quad y'(0) = 0$

31. $y'' + 2y' + 2y = \delta(t - \pi), \quad y(0) = 1, \quad y'(0) = 0$

32. $y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0$

33. $y'' + 2y' + y = \delta(t) + h(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 1$

34. $y'' + y = \delta(t - \pi)\cos t, \quad y(0) = 0, \quad y'(0) = 1$

35. A spring-mass system at rest until time $t = 0$ is then subjected to the sinusoidal force $f(t) = P\sin \omega t$. At time $t = \pi$ seconds, the mass is given a sharp blow from below that instantaneously imparts an upward impulse of 5 units to the system. Neglecting damping effects, describe the motion of the system.

6.7 THE CONVOLUTION THEOREM

In applications we must often find the inverse transform of a function that is the simple product of two other transforms. Unfortunately, it happens that

$$\mathcal{L}^{-1}\{F(s)G(s)\} \neq \mathcal{L}^{-1}\{F(s)\}\mathcal{L}^{-1}\{G(s)\}.$$

We will presently show that the inverse transform of this product is equal instead to the convolution of two functions $f(t)$ and $g(t)$.

The Laplace *convolution* of two functions $f(t)$ and $g(t)$ is defined by

$$(f*g)(t) = \int_0^t f(t - u)g(u)\,du. \tag{35}$$

There are several immediate consequences of this definition. First, we note that making the change of variables $v = t - u$ leads to

$$(f*g)(t) = -\int_t^0 f(v)g(t - v)\,dv = \int_0^t f(v)g(t - v)\,dv,$$

from which we conclude

$$(f*g)(t) = (g*f)(t) \quad \text{(commutative law)}. \tag{36}$$

EXAMPLE 31 Calculate both $(f*g)(t)$and $(g*f)(t)$, where $f(t) = t$ and $g(t) = \sin kt$.

Solution From (35) we have

$$(f*g)(t) = \int_0^t (t - u)\sin ku\,du$$

$$= t\int_0^t \sin ku\,du - \int_0^t u\sin ku\,du$$

$$= \frac{t}{k}(1 - \cos kt) - \frac{1}{k^2}\sin kt + \frac{t}{k}\cos kt$$

$$= \frac{1}{k}\left(t - \frac{1}{k}\sin kt\right),$$

whereas

$$(g*f)(t) = \int_0^t u \sin[k(t - u)]\, du$$

$$= \sin kt \int_0^t u \cos ku\, du - \cos kt \int_0^t u \sin ku\, du$$

$$= \frac{1}{k^2}\sin kt \cos kt + \frac{t}{k}\sin^2 kt - \frac{1}{k^2}\sin kt - \frac{1}{k^2}\cos kt \sin kt + \frac{t}{k}\cos^2 kt$$

$$= \frac{1}{k}\left(t - \frac{1}{k}\sin kt\right).^*$$

Other properties that readily follow from the definition are the following, the proofs of which are left to the exercises:

$$f*(kg) = (kf)*g = k(f*g), \quad k \text{ constant} \tag{37}$$

$$f*(g + h) = f*g + f*h \quad \text{(distributive law)} \tag{38}$$

$$f*(g * h) = (f*g) * h \quad \text{(associative law)} \tag{39}$$

The important result we need is that concerning the Laplace transform of the convolution. Let us consider

$$\mathcal{L}\{(f*g)(t)\} = \int_0^\infty e^{-st} \int_0^t f(t - u)g(u)\, du\, dt,$$

which we can write as a double integral,

$$\mathcal{L}\{(f*g)(t)\} = \int_o^\infty \int_0^t e^{-st} f(t - u)g(u)\, du\, dt.$$

We can interpret the integrals on the right as an iterated integral over the region $0 \leq u \leq t$, $0 \leq t < \infty$, as shown in Figure 6.13. If we interchange the order of integration, we find that the region is characterized by $u \leq t < \infty$, $0 \leq u < \infty$, and thus

$$\mathcal{L}\{(f*g)(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(t - u)g(u)\, dt\, du.$$

The change of variables $x = t - u$ leads to the expression

*Although $f*g = g*f$, one of the integrals is often easier to evaluate than the other, which should be taken into account when using the convolution integral.

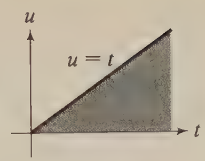


Figure 6.13

$$\mathcal{L}\{(f*g)(t)\} = \int_0^\infty \int_0^\infty e^{-(x+u)s} f(x)g(u)\,dx\,du$$

$$= \int_0^\infty e^{-sx} f(x)\,dx \cdot \int_0^\infty e^{-su} g(u)\,du$$

or

$$\mathcal{L}\{(f*g)(t)\} = F(s)G(s). \tag{40}$$

In summary, we have the following important theorem.

Theorem 6.10 (*Convolution*) If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-u)g(u)\,du = (f*g)(t).$$

EXAMPLE 32 Find $\mathcal{L}^{-1}\{s^{-2}(s^2+k^2)^{-1}\}$.

Solution Let us select $F(s) = 1/s^2$ and $G(s) = (s^2+k^2)^{-1}$. From the tables, we find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = f(t) = t.$$

$$\mathcal{L}^{-1}\{(s^2+k^2)^{-1} = g(t) = \frac{1}{k}\sin kt.$$

Thus we write

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+k^2)}\right\} = (f*g)(t)$$

$$= \int_0^t (t-u)\frac{1}{k}\sin ku\,du$$

$$= \frac{1}{k^2}\left(t - \frac{1}{k}\sin kt\right).$$

[O] 6.7.1 The One-Sided Green's Function

The Laplace transform also turns out to be a useful tool for constructing the one-sided Green's function (see Section 5.5) for constant-coefficient DEs.

Consider the initial value problem

$$y'' + ay' + by = f(t), \quad y(0) = 0, \quad y'(0) = 0, \tag{41}$$

where a and b are constants. The Laplace transform of this DE leads to the algebraic solution

$$Y(s) = \frac{F(s)}{s^2 + as + b}.$$

Thus, if we define the function

$$k(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + as + b}\right\}, \tag{42}$$

the convolution formula yields the solution

$$y(t) = \int_0^t k(t - \tau) f(\tau)\, d\tau. \tag{43}$$

Comparison of (43) with the solution obtained through use of the one-sided Green's function,

$$y(t) = \int_0^t g_1(t, \tau) f(\tau)\, d\tau, \tag{44}$$

identifies the function $k(t - \tau)$ appearing in (43) as the one-sided Green's function; i.e.,

$$k(t - \tau) = g_1(t, \tau). \tag{45}$$

EXAMPLE 33 Find the one-sided Green's function associated with the differential operator $M = D^2 - 2D + 5$.

Solution We first determine

$$k(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 - 2s + 5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s - 1)^2 + 4}\right\},$$

which leads to

$$k(t) = \frac{1}{2} e^t \sin 2t.$$

Hence,

$$k(t - \tau) \equiv g_1(t, \tau) = \frac{1}{2} e^{t-\tau} \sin 2(t - \tau).$$

Comparison of the convolution theorem of the Laplace transform and the Green's function method show that they are equivalent for constant-coefficient equations. In fact, it now follows that the one-sided Green's function for a constant-coefficient DE can always be expressed as a function of the variable $t - \tau$. The method of Green's function is more general than that of Laplace transforms, however, since it can also be applied to variable-coefficient equations (at least in theory) and is more readily adapted to problems when the initial data is prescribed at a point other than $t = 0$.

EXERCISES 6.7

In problems 1–4, find the Laplace transform of each convolution integral.

1. $\int_0^t (t-u)\sin 2u\,du$

2. $\int_0^t (t-u)^2 e^{-2u}\,du$

3. $e^{-t}\int_0^t e^u \cos u\,du$

4. $\int_0^t \cos(t-u)\sin u\,du$

In problems 5–10, find the inverse transform of each function using the convolution theorem.

5. $Y(s) = \dfrac{1}{s^2(s+1)}$

6. $Y(s) = \dfrac{1}{(s^2+1)^2}$

7. $Y(s) = \dfrac{1}{s^4(s^2+1)}$

8. $Y(s) = \dfrac{1}{(s+1)(s^2+4)}$

9. $Y(s) = \dfrac{1}{(s+1)^2(s^2+4)}$

10. $Y(s) = \dfrac{1}{(s^2+a^2)(s^2+b^2)}, \quad a \neq b$

***11.** Establish the properties (37), (38), and (39) of the convolution integral.

12. Show that $(f*1)(t) \neq f(t)$ in general.
Hint: Find a counterexample.

13. Show that $(f*f)(t)$ is not necessarily nonnegative by letting $f(t) = \sin t$.

14. Given $Y(s) = (s+1)/(s^2+4)$, explain why we cannot set $F(s) = s+1$ and $G(s) = 1/(s^2+4)$ in the convolution theorem to evaluate $y(t)$; i.e., explain why

$$y(t) \neq \int_0^t f(t-u)g(u)\,du,$$

where $f(t) = \mathscr{L}^{-1}\{F(s)\}$ and $g(t) = \mathscr{L}^{-1}\{G(s)\}$.

15. Given that $\mathscr{L}\{f(t)\} = (s^2 - a^2)^{-1/2}$, evaluate $\int_0^t f(t-u)f(u)\,du$.

16. Show that $\int_0^t J_0(t-u)J_0(u)\,du = \sin t$, where J_0 is the *Bessel function* of the first kind of order zero.
Hint: $\mathscr{L}\{J_0(t)\} = (s^2+1)^{-1/2}$.

17. Solve the integral equation* for $y(t)$,

*The equation receives its name from the fact that the unknown function y appears under an integral.

$$y(t) = 4t - 3\int_0^t y(u)\sin(t-u)\,du.$$

18. Solve the integral equation for $y(t)$,

$$\int_0^t (t-u)^{-1/2} y(u)\,du = t^{1/2}.$$

19. Solve the integrodifferential equation* for $y(t)$,

$$y'(t) = \int_0^t y(u)\cos(t-u)\,du, \quad y(0) = 1.$$

***20.** Starting with $f(t) = \int_0^t u^{x-1}(t-u)^{y-1}\,du, x, y > 0,$

(a) use the convolution theorem to show

$$F(s) = \frac{\Gamma(x)\Gamma(y)}{s^{x+y}},$$

where Γ denotes the *gamma function* (see problem 26, Exercises 6.2).

(b) Take the inverse transform of $F(s)$ and establish the formula

$$\int_0^1 u^{x-1}(1-u)^{y-1}\,du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

21. Show that

(a) $t * t * t = \dfrac{t^5}{5!}$

(b) $t^{m-1} * t^n = \dfrac{(m-1)!n!t^{m+n}}{(m+n)!}$

22. Using the convolution theorem, show that the solution of the spring-mass system

$$my'' + ky = f(t), \quad y(0) = 0, \quad y'(0) = 0,$$

can be expressed in the form ($\omega_0 = \sqrt{k/m}$)

$$y = (m\omega_0)^{-1}\int_0^t \sin[\omega_0(t-u)]f(u)\,du.$$

23. Using the result of problem 22, determine the response of the spring-mass system when the forcing function is given by

(a) $f(t) = P$ (constant).

(b) $f(t) = P\cos\omega t, \quad \omega \neq \omega_0.$

(c) $f(t) = P\cos\omega_o t.$

24. Determine the current $i(t)$ in a single-loop RLC circuit when $L = 0.1\ H$, $R = 20\ \Omega$, $C = 10^{-3}\ F$, $i(0) = 0$, and the impressed voltage is

$$E(t) = \begin{cases} 120t, & 0 \le t < 1 \\ 0, & t > 1. \end{cases}$$

The system is characterized by

*The equation receives its name from the fact that the unknown function y appears under both an integral and a derivative.

$$0.1\frac{di}{dt} + 20i + 10^3\int_0^t i(u)\,du = E(t), \quad i(0) = 0.$$

***25.** Abel (1802–1829) studied a particular Volterra integral equation that has many important applications. In particular, suppose a particle of mass m is constrained to move (without friction) along a certain path in a vertical plane under the influence of gravity alone. Given the time T required for the particle to descend the curve, we wish to determine the equation of the curve. This problem reduces to finding the solution $f(y)$ of the *Volterra integral equation of the first kind*

$$T = k(y) = \int_0^y \frac{f(u)\,du}{\sqrt{2g(y-u)}},$$

where g is the gravitational constant and $f(y)$ is the length of the path.

(a) Take the Laplace transform of the integral equation and deduce that $K(s) = (\pi/2gs)^{1/2}F(s)$.

(b) Solve for $F(s)$, and by taking inverse Laplace transforms, show that

$$f(y) = \frac{(2g)^{1/2}}{\pi}\int_0^y (y-u)^{-1/2}k'(u)\,du.$$

Hint: Write $F(s) = (2g)^{1/2}(\pi s)^{-1/2}sK(s)$ and use Theorem 6.4.

26. Determine the one-sided Green's function for the following differential operators using the Laplace transform method.

(a) $M = D^2$

(b) $M = (D - a)^2$

(c) $M = D^2 + 2D + 5$

(d) $M = (D - a)(D - b), \quad a \neq b$

***27.** Show that the one-sided Green's function for an nth-order linear DE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = f(t)$$

is given by $k(t - \tau) = \mathcal{L}^{-1}\{1/p(s)\}$, where

$$p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0$$

is the auxiliary polynomial in the transform domain.

28. Use the result of problem 27 to determine the one-sided Green's function associated with the following differential operators:

(a) $M = D^n, \quad n = 2, 3, 4, \ldots$

(b) $M = D^2(D^2 - 1)$

(c) $M = D^4 - 1$

(d) $M = D^3 - 6D^2 + 11D - 6$

6.8 TABLE OF SOME LAPLACE TRANSFORMS

Listed on pages 217–218 is a short table of Laplace transforms and their inverses that commonly occur in applications.

Table 6.1 Table of Laplace Transforms

	$F(s) = \mathscr{L}\{f(t)\}$	$f(t) = \mathscr{L}^{-1}\{F(s)\}$
1.	$\dfrac{1}{s}$	1
2.	$\dfrac{1}{s^2}$	t
3.	$\dfrac{1}{s^n}$ $(n = 1, 2, 3, \ldots)$	$\dfrac{t^{n-1}}{(n-1)!}$
4.	$\dfrac{1}{s^{1/2}}$	$\dfrac{1}{(\pi t)^{1/2}}$
5.	$\dfrac{1}{s^{3/2}}$	$2\left(\dfrac{t}{\pi}\right)^{1/2}$
6.	$\dfrac{1}{s^x}$ $(x > 0)$	$\dfrac{t^{x-1}}{\Gamma(x)}$
7.	$\dfrac{1}{s-a}$	e^{at}
8.	$\dfrac{1}{(s-a)^2}$	te^{at}
9.	$\dfrac{1}{(s-a)^n}$ $(n = 1, 2, 3, \ldots)$	$\dfrac{t^{n-1}e^{at}}{(n-1)!}$
10.	$\dfrac{1}{(s-a)^x}$ $(x > 0)$	$\dfrac{t^{x-1}e^{at}}{\Gamma(x)}$
11.	$\dfrac{1}{(s-a)(s-b)}$ $(a \neq b)$	$\dfrac{e^{at} - e^{bt}}{a-b}$
12.	$\dfrac{s}{(s-a)(s-b)}$ $(a \neq b)$	$\dfrac{ae^{at} - be^{bt}}{a-b}$
13.	$\dfrac{1}{s^2+k^2}$	$\dfrac{1}{k}\sin kt$
14.	$\dfrac{s}{s^2+k^2}$	$\cos kt$
15.	$\dfrac{1}{s^2-k^2}$	$\dfrac{1}{k}\sinh kt$
16.	$\dfrac{s}{s^2-k^2}$	$\cosh kt$
17.	$\dfrac{1}{(s-a)^2+b^2}$	$\dfrac{1}{b}e^{at}\sin bt$
18.	$\dfrac{s-a}{(s-a)^2+b^2}$	$e^{at}\cos bt$

Table 6.1 Table of Laplace Transforms (*cont.*)

	$F(s) = \mathscr{L}\{f(t)\}$	$f(t) = \mathscr{L}^{-1}\{F(s)\}$
19.	$\dfrac{1}{s(s^2 + k^2)}$	$\dfrac{1}{k^2}(1 - \cos kt)$
20.	$\dfrac{1}{s^2(s^2 + k^2)}$	$\dfrac{1}{k^3}(kt - \sin kt)$
21.	$\dfrac{1}{(s^2 + k^2)^2}$	$\dfrac{1}{2k^3}(\sin kt - kt \cos kt)$
22.	$\dfrac{s}{(s^2 + k^2)^2}$ $\quad \dfrac{2ks}{(\quad)^2} \longleftrightarrow t \sin kt$	$\dfrac{t}{2k} \sin kt$
23.	$\dfrac{s^2}{(s^2 + k^2)^2}$	$\dfrac{1}{2k}(\sin kt + kt \cos kt)$
24.	$\dfrac{s}{(s^2 + a^2)(s^2 + b^2)} \quad (a^2 \neq b^2)$	$\dfrac{1}{b^2 - a^2}(\cos at - \cos bt)$
25.	$\dfrac{1}{s^4 + 4k^4}$	$\dfrac{1}{4k^3}(\sin kt \cosh kt - \cos kt \sinh kt)$
26.	$\dfrac{s}{s^4 + 4k^4}$	$\dfrac{1}{2k^2} \sin kt \sinh kt$
27.	$\dfrac{1}{s^4 - k^4}$	$\dfrac{1}{2k^3}(\sinh kt - \sin kt)$
28.	$\dfrac{s}{s^4 - k^4}$	$\dfrac{1}{2k^2}(\cosh kt - \cos kt)$
29.	$\dfrac{e^{-as}}{s} \quad (a > 0)$	$h(t - a)$
30.	$e^{-as}G(s) \quad (a > 0)$	$g(t - a)h(t - a)$
31.	$F(s - a)$	$e^{at}f(t)$
32.	$\dfrac{1}{a}F\left(\dfrac{s}{a}\right) \quad (a > 0)$	$f(at)$
33.	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	$f^{(n)}(t)$
34.	$F^{(n)}(s)$	$(-t)^n f(t)$
35.	$e^{-as} \quad (a > 0)$	$\delta(t - a)$
36.	$F(s)G(s)$	$\int_0^t f(t - u)g(u)\,du$

REFERENCES

Churchill, R. V. 1958. *Operational mathematics*. 2nd ed. New York: McGraw-Hill.

Finney, R. C., and Ostberg, D. R. 1976. *Elementary differential equations with linear algebra*. Reading, Mass.: Addison-Wesley.

Kreyszig, E. R. 1979. *Advanced engineering mathematics*. 4th ed. New York: Wiley.

LePage, W. R. 1980. *Complex variables and the Laplace transform for engineers*. New York: Dover.

Miles, J. W. 1971. *Integral transforms in applied mathematics*. London: Cambridge University Press.

Sneddon, I. H. 1972. *The use of integral transforms*. New York: McGraw-Hill.

Spiegel, M. R. 1965. *Laplace transforms*. New York: Schaum.

7 Systems of Equations

The first five sections of this chapter are devoted to solving *linear systems of differential equations with constant coefficients*. Several methods can be employed to solve such equations.

In Section 7.2 we introduce a method that changes the system of equations to a single DE by successive elimination of the unknowns. The technique can be greatly systematized with the aid of *Cramer's rule*. We use the *Laplace transform* in Section 7.3 to reduce the differential system to an algebraic system solvable by elementary techniques and then apply the inverse transform to produce the desired solution set. Another method, presented in Section 7.4, consists of assuming a particular trial solution form that, when substituted into the system, reduces it to a linear system of algebraic equations. Solving the algebraic system identifies the parameters occurring in the trial solution set. Most of the general theory concerning linear differential systems is also discussed in this section. In Section 7.5 we briefly discuss the use of matrix techniques that permit the system to be expressed as a single-matrix DE in a form similar to that of a first-order linear equation.

The *qualitative* aspects of certain kinds of *nonlinear systems* are discussed in Section 7.6 without any explicit, formal solution functions. The general questions of concern are mainly associated with the idea of *stability* of a solution. In particular, it is important to know whether small changes in the input data (initial conditions) produce only small changes (stability) or large changes (instability) in the output (solution). When a nonlinear system is approximated by a linear system, it is equally important to know whether such an approximation is "reasonable." That is to say, some nonlinear systems do not lend themselves to reasonable linear approximations.

7.1 INTRODUCTION

In applied mathematics, many applications involve not one but several dependent variables, each of which is a function of a single independent variable, usually time. The formulation of such problems leads to a *system* of differential equations rather than a single equation, as we have studied thus far.

For example, the motion of a single particle in space governed by Newton's second law of motion leads to the vector DE

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}, \tag{1}$$

which is equivalent to the system of scalar DEs

$$\begin{aligned} mx'' &= F_1, \\ my'' &= F_2, \\ mz'' &= F_3, \end{aligned} \tag{2}$$

where m is the mass of the particle, $\vec{r} = (x, y, z)$ is the position vector of the particle, and $\vec{F} = (F_1, F_2, F_3)$ is the external force acting on the particle. Systems of equations are also used to describe the motions of coupled spring-mass systems and *RLC* circuits connected in parallel. Examples of such systems will be given later.

Systems of equations arise in the classical ecological problem of the prey and the predator, which involves the struggle for survival among different species of animals living in the same environment. One kind of animal eats the other as a means of survival, while the other develops methods of evasion in order to avoid being eaten. For example, suppose $x(t)$ denotes the population of rabbits at any time on an isolated island, and $y(t)$ the number of foxes at any time on this same island. The foxes eat the rabbits, and the rabbits eat clover, which is in ample supply. When the rabbits are plentiful, so too are the foxes, and their population grows. When the foxes become too numerous and eat too many rabbits, they enter a period of famine and their population begins to diminish. The rabbits eventually become relatively safe again, and their population increases, which also initiates new increases in the number of foxes. Under the proper ecological balance, these cycles of population increases and decreases can be repeated incessantly. On the other hand, if the balance of nature is disturbed in the right way, both species could die out.

Problems of this nature were independently modeled by A. J. Lotka (1880–1949) in 1925 and by V. Volterra (1860–1940) in 1926. They suggested that the instantaneous populations $x(t)$ and $y(t)$ of both species are solutions of the system of equations

$$\begin{aligned} x' &= ax - bxy, \\ y' &= -cy + dxy, \end{aligned} \tag{3}$$

where a, b, c, and d are all positive constants. The constants a and c represent the growth rate of the prey (rabbits) and death rate of the predator (foxes), respectively, whereas b and d are measures of the effect of the interaction between the species. These equations are now widely known as the *Lotka-Volterra equations*.

Finally, another way in which a system of equations can arise is to convert an nth-order DE to a system of first-order equations. That is, given the nth-order DE (not necessarily linear)

$$y^{(n)} = F[t, y, y', \dots, y^{(n-1)}], \tag{4}$$

let us introduce the variables $x_1, x_2, \dots, x_n$ defined by

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad \dots, \quad x_n = y^{(n-1)}.$$

Hence (4) is equivalent to the first-order system of equations

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_3, \\ &\cdots\cdots\cdots \\ x_{n-1}' &= x_n, \\ x_n' &= F(t, x_1, x_2, \dots, x_n). \end{aligned} \tag{5}$$

7.2 THE OPERATOR METHOD

Before proceeding with this section, it may be helpful to review the operator notation introduced in Section 4.4.1.

Let us consider the simple system of equations

$$\begin{aligned} x' - y &= 0, \\ y' + 4x &= 0, \end{aligned} \tag{6}$$

or, equivalently (using operator notation $D = d/dt$),

$$\begin{aligned} Dx - y &= 0, \\ 4x + Dy &= 0. \end{aligned} \tag{7}$$

Suppose we operate on the first equation by D to get

$$\begin{aligned} D^2x - Dy &= 0, \\ 4x + Dy &= 0. \end{aligned}$$

Adding the two equations now eliminates y from the system. Hence,

$$(D^2 + 4)x = 0$$

with general solution

$$x(t) = C_1 \cos 2t + C_2 \sin 2t. \tag{8}$$

In a similar manner we multiply the first DE in (7) by 4, operate on the second DE by D, and then subtract the results to eliminate x from the system. This action leads to

$$(D^2 + 4)y = 0,$$

from which we get

$$y(t) = C_3 \cos 2t + C_4 \sin 2t. \tag{9}$$

Although (8) and (9) represent solutions of the system (7), they are not solutions for every choice of the constants C_1, C_2, C_3, and C_4. Therefore, it is necessary to find a proper relationship between these constants. To do this, we substitute (8) and (9) into the first equation in (7) to obtain

$$-2C_1 \sin 2t + 2C_2 \text{cost}\ 2t - C_3 \cos 2t - C_4 \sin 2t = 0,$$

or

$$(2C_1 - C_3)\cos 2t - (2C_1 + C_4)\sin 2t = 0,$$

from which we deduce $C_3 = 2C_2$ and $C_4 = -2C_1$. Hence, the solution of (7) is given by

$$\begin{aligned} x(t) &= C_1 \cos 2t + C_2 \sin 2t, \\ y(t) &= 2C_2 \cos 2t - 2C_1 \sin 2t. \end{aligned} \tag{10}$$

In order to systematize the procedure we just used, let us now consider the general system of equations

$$\begin{aligned} L_1[x] + L_2[y] &= f_1(t), \\ L_3[x] + L_4[y] &= f_2(t), \end{aligned} \tag{11}$$

where L_1, L_2, L_3, and L_4 are any *linear differential operators with constant coefficients*. Operating on the first equation by L_4 and the second equation by L_2, and subtracting the results, we eliminate y from the system and are left with

$$(L_1L_4 - L_2L_3)\, x = g_1(t), \tag{12}$$

where $g_1(t) = L_4[f_1(t)] - L_2[f_2(t)]$. In a similar fashion, the elimination of x gives

$$(L_1L_4 - L_2L_3)\, y = g_2(t), \tag{13}$$

where $g_2(t) = L_1[f_2(t)] - L_3[f_1(t)]$. Equations (12) and (13) can now be solved independently.

The above procedure can be systematized even further by formulating it in determinant notation. Observe that if we write

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} x = \begin{vmatrix} f_1(t) & L_2 \\ f_2(t) & L_4 \end{vmatrix}, \tag{14}$$

which is precisely what Cramer's rule would yield for an algebraic system, then the expansion of the determinants on each side reduces (14) to (12). However, it is important to keep in mind that the determinant on the right-hand side has the proper meaning only if the operators L_2 and L_4 operate on $f_2(t)$ and $f_1(t)$, respectively. That is, the expansion of this determinant must produce the function $g_1(t)$ defined above. Likewise, we can express (13) as

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} y = \begin{vmatrix} L_1 & f_1(t) \\ L_3 & f_2(t) \end{vmatrix}. \tag{15}$$

The use of Cramer's rule to solve systems of equations can be applied to systems of any number n of equations, regardless of the order of the operators $L_1, L_2, \ldots, L_n, \ldots, L_{2n}$. From a practical point of view, it works best on systems of two or three equations with either first- or second-order operators, since the amount of computation becomes unwieldy for larger systems. Also, we might note that the auxiliary polynomials for x and y are always the same, since the determinants on the left-hand sides are identical [see (14) and (15)]. Therefore, with the exception of the arbitrary constants, the homogeneous solutions (complementary functions) of x and y are identical. The number of independent arbitrary constants in the final solutions should be equal to the order of operator determinant on the left-hand sides. For a 2×2 system, for example, this is the order of $L_1L_4 - L_2L_3$.

Remark. If $L_1L_4 - L_2L_3 = 0$, the system either has infinitely many solutions or no solutions, depending upon whether the determinants on the right-hand sides of (14) and (15) vanish. Thus the situation here is similar to that of a system of algebraic equations.

EXAMPLE 1 Solve the system of equations

$$x' = x + 2y,$$
$$y' = 3x + 2y.$$

Solution We first express the system in operator notation

$$(D - 1)x - 2y = 0,$$
$$-3x + (D - 2)y = 0,$$

and then formulate as two determinant equations

$$\begin{vmatrix} D-1 & -2 \\ -3 & D-2 \end{vmatrix} x = \begin{vmatrix} 0 & -2 \\ 0 & D-2 \end{vmatrix},$$

$$\begin{vmatrix} D-1 & -2 \\ -3 & D-2 \end{vmatrix} y = \begin{vmatrix} D-1 & 0 \\ -3 & 0 \end{vmatrix}.$$

Hence, we obtain the two homogeneous DEs

$$(D^2 - 3D - 4)x = 0,$$
$$(D^2 - 3D - 4)y = 0,$$

with general solutions

$$x(t) = C_1e^{-t} + C_2e^{4t},$$
$$y(t) = C_3e^{-t} + C_4e^{4t}.$$

Substituting these solutions into the first DE of the system, we have

$$-C_1e^{-t} + 4C_2e^{4t} = (C_1 + 2C_3)e^{-t} + (C_2 + 2C_4)e^{4t}.$$

The constants must therefore satisfy the equations

$$C_1 + C_3 = 0,$$
$$3C_2 - 2C_4 = 0,$$

or $C_3 = -C_1$ and $C_4 = \frac{3}{2}C_2$. Thus

$$x(t) = C_1e^{-t} + C_2e^{4t},$$
$$y(t) = -C_1e^{-t} + \frac{3}{2}C_2e^{4t}.$$

We might point out here that instead of solving both homogeneous DEs for x and y, we could have solved only for, say, x, and then used the first DE of the system in the form $y = \frac{1}{2}(x' - x)$ to find y. Such observations can be helpful in some problems.

EXAMPLE 2 Solve the system of equations

$$x' = 2x + y + t,$$
$$y' = x + 2y + t^2.$$

Solution In operator notation, the system becomes

$$(D - 2)x - y = t,$$
$$-x + (D - 2)y = t^2,$$

which leads to the determinantal formulation

$$\begin{vmatrix} D-2 & -1 \\ -1 & D-2 \end{vmatrix} x = \begin{vmatrix} t & -1 \\ t^2 & D-2 \end{vmatrix},$$

$$\begin{vmatrix} D-2 & -1 \\ -1 & D-2 \end{vmatrix} y = \begin{vmatrix} D-2 & t \\ -1 & t^2 \end{vmatrix}.$$

After expansion of the determinants, we find

$$(D^2 - 4D + 3)x = t^2 - 2t + 1,$$
$$(D^2 - 4D + 3)y = -2t^2 + 3t.$$

The homogeneous solution for x is

$$x_H(t) = C_1e^t + C_2e^{3t},$$

while a particular solution is found to be

$$x_P(t) = \frac{1}{3}t^2 + \frac{2}{9}t + \frac{11}{27}.$$

Hence,

$$\begin{aligned} x(t) &= x_H(t) + x_P(t) \\ &= C_1e^t + C_2e^{3t} + \frac{1}{3}t^2 + \frac{2}{9}t + \frac{11}{27}. \end{aligned}$$

Rather than solve the nonhomogeneous DE above for y, we merely observe that $y = x' - 2x - t$, obtained from the first DE of the system. Therefore,

$$y(t) = -C_1e^t + C_2e^{3t} - \frac{2}{3}t^2 - \frac{7}{9}t - \frac{16}{27}.$$

EXERCISES 7.2

In problems 1–10, find the general solution of the system.

1. $x' = 2x - y$
$y' = 3x - 2y$

2. $x' = 4x - 3y$
$y' = 5x - 4y$

3. $x' = 4x - 3y$
$y' = 8x - 6y$

4. $x' = 3x - 18y$
$y' = 2x - 9y$

5. $x' = x - 4y$
$y' = x + y$

6. $x' = 3x - 2y$
$y' = 2x + 3y$

7. $x' = 6x - 5y$
$y' = x + 2y$

8. $x' = 3x + 2y$
$y' = -5x + y$

***9.** $x' = x - y + 4z$
$y' = 3x + 2y - z$
$z' = 2x + y - z$

***10.** $x' = -x + z$
$y' = -y + z$
$z' = -x + y$

11. Find the general solution of

$$x' = ax + by,$$
$$y' = bx + ay,$$

given that $b \neq 0$.

In problems 12–28, solve the given system of equations. Make sure the proper number of constants appears in the general solution.

12. $x' - 2x + 5y = -\sin 2t, \quad x(0) = 0$
$y' - x + 2y = t, \quad y(0) = 1$

13. $x' - 2x + y = t$
$y' - 3x + 2y = 2t$

14. $x' - 2x - y = e^t, \quad x(0) = 1$
$y' - 4x + y = -e^t, \quad y(0) = -1$

15. $x' + y' + 2y = 0$
$x' - 3x - 2y = 0$

16. $x'' + 5x - 2y = 0$
$-2x + y'' + 2y = 0$

17. $x'' + x' - x + y'' - 3y' + 2y = 0$
$x' + 2x + 2y' - 4y = 0$

18. $(D^2 - 3D + 2)x + (D - 1)y = 0$
$(D - 2)x + (D + 1)y = 0$

19. $(D^2 - 4D + 4)x + (D^2 + 2D)y = 0$
$(D^2 - 2D)x + (D^2 + 4D + 4)y = 0$

20. $x' + x + y' - y = 2$
$3x + y' + 2y = -1$

21. $2x' - 3x - 2y' = t$
$2x' + 3x + 2y' + 8y = 2$

22. $x' - 2x + y' - 4y = e^t$
$x' + y' - y = e^{4t}$

23. $x' + y' + 2y = \sin t$
$x' - x + y' - y = 0$

24. $(2D - 1)x + (D - 1)y = 1$
$(D + 2)x + (D - 1)y = t$

25. $(D^2 - 4D + 4)x + 3Dy = 1$
$(D - 2)x + (D + 2)y = 0$

***26.** $2x' + y' - y + z' + 2z = 0$
$x' + 2x + 2y' - 3y - z' + 6z = 0$
$2x' - y' - 3y - z' = 0$

***27.** $x' - 6y = 0$
$x - y' + z = 0$
$x + y - z' = 0$

***28.** $x' + z = e^t$
$x' - x + y' + z' = 0$
$x + 2y + z' = e^t$

In problems 29–31, show that each system is degenerate in that it either has no solution or infinitely many solutions.

29. $x' + y' + y = e^t$
$x' + y' + y = e^t + 3$

30. $x'' - x + y' - y = \sin t$
$x' + x + y = 2e^t$

31. $x'' + 3x' + 2x + y'' + 2y' = 0$
$x' + x + y' = 0$

7.3 THE METHOD OF LAPLACE TRANSFORMS

If initial conditions are prescribed along with a system of differential equations, the Laplace transform method is generally more convenient to use than the operator method of the last section.

The Laplace transform applied to a differential system reduces it to an algebraic system in the transformed functions. Standard solution techniques can then be applied to solve the algebraic system, and the inverse transform of the algebraic solution functions will produce the solution functions of the original differential system.

EXAMPLE 3 Solve the differential system

$$x'' - x + 5y' = t,$$

$$y'' - 4y - 2x' = -2,$$

subject to the initial conditions

$$x(0) = 0, \quad x'(0) = 0, \quad y(0) = 0, \quad y'(0) = 0.$$

Solution If we let $X(s) = \mathscr{L}\{x(t)\}$ and $Y(s) = \mathscr{L}\{y(t)\}$, then application of the Laplace transform to each equation results in the pair of algebraic equations

$$s^2X(s) - X(s) + 5sY(s) = \frac{1}{s^2},$$

$$s^2Y(s) - 4Y(s) - 2sX(s) = -\frac{2}{s}.$$

Upon simplification, we have

$$(s^2 - 1)X(s) + 5sY(s) = \frac{1}{s^2},$$

$$-2sX(s) + (s^2 - 4)Y(s) = -\frac{2}{s},$$

the simultaneous solution of which is given by

$$X(s) = \frac{11s^2 - 4}{s^2(s^2 + 1)(s^2 + 4)},$$

$$Y(s) = \frac{-2s^2 + 4}{s(s^2 + 1)(s^2 + 4)}.$$

In order to invert these expressions, we first expand the right-hand members in terms of their partial fractions

$$X(s) = -\frac{1}{s^2} + \frac{5}{s^2 + 1} - \frac{4}{s^2 + 4},$$

$$Y(s) = \frac{1}{s} - \frac{2s}{s^2 + 1} + \frac{s}{s^2 + 4}.$$

Now,

$$\begin{aligned} x(t) &= -\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} + 5\mathscr{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} - 2\mathscr{L}^{-1}\left\{\frac{2}{s^2 + 4}\right\} \\ &= -t + 5 \sin t - 2 \sin 2t, \end{aligned}$$

and, similarly, we find

$$y(t) = 1 - 2 \cos t + \cos 2t.$$

7.3.1 Coupled Spring-Mass Systems

One elementary application leading to a system of DEs involves the coupling of spring-mass systems. Suppose two masses m_1 and m_2 are connected to two springs

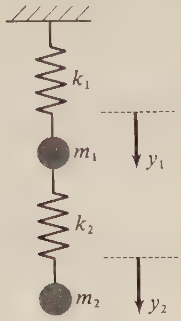


Figure 7.1 Coupled spring-mass system.

having spring constants k_1 and k_2, respectively (see Figure 7.1). Let y_1 and y_2 denote the vertical displacements of the masses from their equilibrium positions. When the system is in motion, the stretching of the lower spring is $y_2 - y_1$, exerting a force of $k_2(y_2 - y_1)$ on the upper mass m_1 (Hooke's law). Also, the upper spring exerts the force $-k_1y_1$ on this mass, and so, invoking Newton's second law of motion, we have

$$m_1y_1'' = -k_1y_1 + k_2(y_2 - y_1) \tag{16}$$

describing the small motions of the upper mass m_1. In a similar manner, it follows that the equation of motion for the lower mass m_2 is

$$m_2y_2'' = -k_2(y_2 - y_1). \tag{17}$$

Hence, the coupled spring-mass system in the absence of damping effects and external forces is governed by the system of second-order DEs

$$\begin{aligned} m_1y_1'' + (k_1 + k_2)y_1 - k_2y_2 &= 0, \\ m_2y_2'' - k_2y_1 + k_2y_2 &= 0. \end{aligned} \tag{18}$$

EXAMPLE 4 Determine the free motions of a double spring-mass system composed of two unit masses and springs with constants $k_1 = 6$ and $k_2 = 4$. Assume that both masses start from their equilibrium positions, but that m_1 has a downward unit initial velocity and m_2 an upward unit initial velocity. Neglect damping.

Solution The system here is described by Equations (18) above, and when the proper parameter values are written in, we obtain

$$\begin{aligned} y_1'' + 10y_1 - 4y_2 &= 0, \\ y_2'' - 4y_1 + 4y_2 &= 0, \end{aligned}$$

with initial conditions

$$y_1(0) = 0, \quad y_1'(0) = 1, \quad y_2(0) = 0, \quad y_2'(0) = -1.$$

Using the notation $Y_1(s) = \mathcal{L}\{y_1(t)\}$ and $Y_2(s) = \mathcal{L}\{y_2(t)\}$, the transform of the differential system reduces to

$$s^2Y_1(s) - 1 + 10Y_1(s) - 4Y_2(s) = 0,$$
$$s^2Y_2(s) + 1 - 4Y_1(s) + 4Y_2(s) = 0,$$

or

$$(s^2 + 10)Y_1(s) - 4Y_2(s) = 1,$$
$$-4Y_1(s) + (s^2 + 4)Y_2(s) = -1.$$

Solving these equations simultaneously, we find

$$Y_1(s) = \frac{s^2}{(s^2 + 2)(s^2 + 12)},$$

$$Y_2(s) = -\frac{s^2 + 6}{(s^2 + 2)(s^2 + 12)}.$$

The partial fraction expansion of these functions yields

$$Y_1(s) = -\frac{1/5}{s^2 + 2} + \frac{6/5}{s^2 + 12},$$

$$Y_2(s) = -\frac{2/5}{s^2 + 2} - \frac{3/5}{s^2 + 12},$$

and taking the inverse transform of each expression gives

$$y_1(t) = -\frac{\sqrt{2}}{10} \sin \sqrt{2}\,t + \frac{\sqrt{3}}{5} \sin 2\sqrt{3}\,t,$$

$$y_2(t) = -\frac{\sqrt{2}}{5} \sin \sqrt{2}\,t - \frac{\sqrt{3}}{10} \sin 2\sqrt{3}\,t.$$

Hence, each mass is governed by the superposition of two harmonic motions.

7.3.2 Electrical Networks

An electrical network composed of components connected in parallel also gives rise to simultaneous DEs. For example, the *RLC* circuit in Figure 7.2 has two loops. By Kirchhoff's voltage law (Section 5.4), we obtain

$$Li_1' + Ri_2 = E(t),$$
$$C^{-1}i_3 - Ri_2' = 0, \tag{19}$$

where i_1, i_2, and i_3 denote the current in each part of the circuit. However, by Kirchhoff's current law,

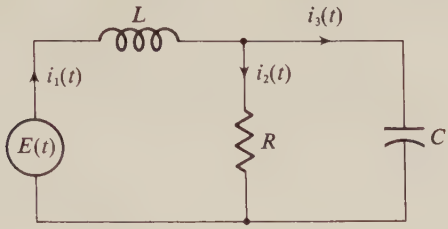


Figure 7.2 *RLC* circuit.

$$i_1 = i_2 + i_3,$$

and thus we can eliminate i_3 by writing $i_3 = i_1 - i_2$, and then (19) becomes

$$\begin{aligned} &Li_1' + Ri_2 = E(t), \\ &i_2' + \frac{1}{RC}(i_2 - i_1) = 0. \end{aligned} \tag{20}$$

EXAMPLE 5 Let $R = 100\ \Omega$, $L = 4$ H, $C = 10^{-4}$ F, and $E(t) = 100$ V in the network of Figure 7.2. If initially the currents i_1 and i_2 are both zero, find the currents at all later times.

Solution Substituting the appropriate values of the parameters into (20), we have

$$\begin{aligned} &4i_1' + 100i_2 = 100, \\ &i_2' + 100(i_2 - i_1) = 0, \end{aligned}$$

with $i_1(0) = i_2(0) = 0$. The Laplace transform applied to each equation gives (after simplification)

$$\begin{aligned} sI_1(s) + 25I_2(s) &= \frac{25}{s}, \\ -100I_1(s) + (s + 100)I_2(s) &= 0, \end{aligned}$$

where $I_1(s) = \mathcal{L}\{i_1(t)\}$ and $I_2(s) = \mathcal{L}\{i_2(t)\}$, and solving these equations, we find

$$I_1(s) = \frac{25(s + 100)}{s(s + 50)^2} = \frac{1}{s} - \frac{25}{(s + 50)^2} - \frac{1}{s + 50},$$

$$I_2(s) = \frac{2500}{s(s + 50)^2} = \frac{1}{s} - \frac{50}{(s + 50)^2} - \frac{1}{s + 50}.$$

Hence,

$$\begin{aligned} i_1(t) &= 1 - 25te^{-50t} - e^{-50t}, \\ i_2(t) &= 1 - 50te^{-50t} - e^{-50t}, \end{aligned}$$

and

$$i_3(t) = i_1(t) - i_2(t) = 25te^{-50t}.$$

Observe that both i_1 and i_2 approach a constant unit value as $t \to \infty$, but that i_3 tends to zero.

EXERCISES 7.3

In problem 1–14, solve the system of equations by use of the Laplace transform.

1. $x' = -x + y, \quad x(0) = 0$
$y' = 2x, \quad y(0) = 1$

2. $x' = y, \quad x(0) = 1$
$y' = x, \quad y(0) = 0$

3. $x' = x - 2y, \quad x(0) = -1$
$y' = 5x - y, \quad y(0) = 2$

4. $x' = 4x - 2y, \quad x(0) = 2$
$y' = 5x + 2y, \quad y(0) = -2$

5. $x' = x - y + e^t, \quad x(0) = 1$
$y' = 2x + 3y + e^{-t}, \quad y(0) = 0$

6. $x' - 4x + 2y = e^t, \quad x(0) = 1$
$y' - 5x - 2y = -t, \quad y(0) = 0$

7. $2x' + y' - y = t, \quad x(0) = 1$
$x' + y' = t^2, \quad y(0) = 0$

8. $2x' + y' - 2x = 1, \quad x(0) = 0$
$x' + y' - 3x - 3y = 2, \quad y(0) = 0$

9. $x'' + x - y = 0, \quad x(0) = 0, \quad x'(0) = -2$
$y'' + y - x = 0, \quad y(0) = 0, \quad y'(0) = 1$

10. $x' - 4x + y''' = 6 \sin t, \quad x(0) = 0$
$x' - 2y''' + 2x = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0$

11. $x'' + 2x - y' = 2t + 5, \quad x(0) = 3, \quad x'(0) = 0$
$x' - x + y' + y = -2t - 1, \quad y(0) = -3$

***12.** $x'' - x + 5y' = f(t), \quad x(0) = 0, \quad x'(0) = 0$
$y'' - 4y - 2x' = 0, \quad y(0) = 0, \quad y'(0) = 0$

where $f(t) = \begin{cases} 6t, & 0 \leq t < 2 \\ 12, & t > 2. \end{cases}$

13. Solve the coupled spring-mass system described by (18) when $m_1 = m_2 = 1$, $k_1 = 3$, $k_2 = 2$, and the initial conditions are prescribed by $y_1(0) = 0$, $y_1'(0) = 1$, $y_2(0) = 1$, $y_2'(0) = 0$.

14. Solve the coupled spring-mass system described by (18) when $m_1 = 4$, $m_2 = 2$, $k_1 = 8$, $k_2 = 4$, and the initial conditions are $y_1(0) = 0$, $y_1'(0) = 0$, $y_2(0) = 0$, $y_2'(0) = -2$. What are the natural frequencies of the system?

15. Solve problem 14 if a force $f(t) = 40 \sin 3t$ is suddenly applied to m_1 while the system is in equilibrium.

16. Solve problem 15 if the force is applied to m_2 instead of m_1.

17. Show that the spring-mass system in the accompanying figure is governed by the system of equations

$$y_1'' + 2ky_1 - ky_2 = 0,$$
$$y_2'' + 2ky_2 - ky_1 = 0.$$

Problem 17

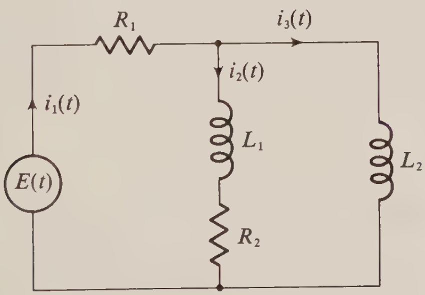
18. Solve the system of equations in problem 17 if $k = 3$ and the initial conditions are $y_1(0) = 1$, $y_1'(0) = 3$, $y_2(0) = 1$, $y_2'(0) = -3$.

19. Find the currents i_1, i_2, and i_3 in the network in Figure 7.2 when $R = 100\ \Omega$, $L = 1$ H, $C = 10^{-4}$ F, and $E(t) = 100$ V. Assume that i_1 and i_2 are initially zero.

20. Find the currents i_1, i_2, and i_3 in the network in Figure 7.2 when $R = 100\ \Omega$, $L = 4$ H, $C = 10^{-4}$ F, $i_1(0) = i_2(0) = 0$, and

(a) $E(t) = 100e^{-1000t}$ V. (b) $E(t) = 100 \sin 60\pi t$ V.

21. Show that the network illustrated in the accompanying figure is governed by the system of equations



Problem 21

$$R_1 i_1 + L i_2' + R_2 i_2 = E(t),$$

$$R_1 i_1 + L_2 i_3' = E(t).$$

Eliminate i_3 and rewrite the system of equations in terms of i_1 and i_2 alone.

22. Solve the system of equations in problem 21 when $R_1 = 6\ \Omega$, $R_2 = 5\ \Omega$, $L_1 = L_2 = 1$ H, $E(t) = 50 \sin t$ V, and i_1 and i_2 are initially zero.

***23.** A double pendulum oscillates in a vertical plane under the influence of gravity alone. For small displacements, it can be shown that the equations of motion are given by

$$(m_1 + m_2)b_1^2\theta_1'' + m_2 b_1 b_2 \theta_2'' + (m_1 + m_2) b_1 g \theta_1 = 0,$$

$$m_2 b_2^2 \theta_2'' + m_2 b_1 b_2 \theta_1'' + m_2 b_2 g \theta_2 = 0.$$

Find the natural frequencies of the system when (assume $g = 32$)

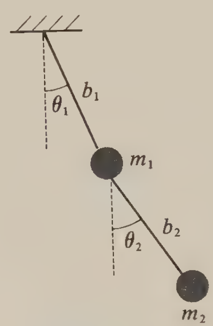
(a) $m_1 = m_2 = 1, \quad b_1 = b_2 = 1.$

(b) $m_1 = 2, \quad m_2 = 1, \quad b_1 = b_2 = 1.$

(c) $m_1 = 1, \quad m_2 = 2, \quad b_1 = b_2 = 1.$

(d) $m_1 = m_2 = 1, \quad b_1 = 1, \quad b_2 = 2.$

(e) $m_1 = m_2 = 1, \quad b_1 = 2, \quad b_2 = 1.$



Problem 23

***24.** Solve the system of equations in problem 23 when the initial conditions are

$$\theta_1(0) = 0, \quad \theta_1'(0) = 0, \quad \theta_2(0) = \frac{1}{2}, \quad \theta_2'(0) = 0,$$

and $m_1 = 3$, $m_2 = 1$, and $b_1 = b_2 = 16$.

7.4 FIRST-ORDER LINEAR SYSTEMS

In this section we wish to discuss some of the general theory associated with linear systems involving first-order DEs. Although the results can be generalized to systems of n equations, we will find it convenient to develop the theory for systems of two equations. Therefore we will be interested in the general *nonhomogeneous system*

$$\begin{aligned} x' &= a_{11}(t)x + a_{12}(t)y + f_1(t), \\ y' &= a_{21}(t)x + a_{22}(t)y + f_2(t). \end{aligned} \tag{21}$$

By the *associated homogeneous system* we mean

$$\begin{aligned} x' &= a_{11}(t)x + a_{12}(t)y, \\ y' &= a_{21}(t)x + a_{22}(t)y, \end{aligned} \tag{22}$$

which is obtained from (21) by setting $f_1(t) \equiv f_2(t) \equiv 0$. The point of view presented here will emphasize the similarities between such systems and the linear second-order DEs discussed in Chapter 4. That such similarities exist is a consequence of the fact that all second-order linear DEs can be expressed as a system like (21).

EXAMPLE 6 Reduce the second-order DE $my'' + cy' + ky = f(t)$ to a system of first-order linear DEs.

Solution Write the DE as

$$y'' = -\frac{c}{m}y' - \frac{k}{m}y + \frac{f(t)}{m},$$

and then let $y = x_1$, $y' = x_2$. Thus, since $x_1' = y' = x_2$, we have

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -\frac{c}{m}x_2 - \frac{k}{m}x_1 + \frac{f(t)}{m}. \end{aligned}$$

Most linear systems that are not first-order systems can also be rewritten as first-order systems but will generally involve more than two equations. To accomplish this it is usually best to solve for the highest-order derivative in each of the unknowns before introducing new variables. Let us illustrate with an example.

EXAMPLE 7 Reduce the system

$$\begin{aligned} x'' - 2x' - y' &= -e^{2t}, \\ -6x + y' &= t, \end{aligned}$$

to a system of first-order DEs.

Solution Let us rewrite the system as

$$x'' = 2x' + y' - e^{2t},$$
$$y' = 6x + t,$$

and introduce $u = x$, $v = y$, $w = x'$. Thus, $u' = w$ and

$$w' = 2w + v' - e^{2t},$$
$$v' = 6u + t,$$

or

$$u' = w,$$
$$v' = 6u + t,$$
$$w' = 6u + 2w + t - e^{2t}.$$

One of the reasons we are so interested in first-order systems is that most computational algorithms in numerical techniques are established for first-order equations, and in most practical applications the problem ultimately requires some numerical calculations. A clear understanding of the general theory is therefore desired to help facilitate the use of these numerical methods.

By a *solution* of the system (21), we mean simply a set of functions $\{x(t), y(t)\}$ with continuous derivatives that simultaneously satisfy both equations of the system identically on some interval $a \leq t \leq b$.

Theorem 7.1 If the functions $a_{11}(t)$, $a_{12}(t)$, $a_{21}(t)$, $a_{22}(t)$, $f_1(t)$, and $f_2(t)$ are continuous in some interval $a \leq t \leq b$ containing the point t_0, then there exists exactly one solution pair $\{x(t), y(t)\}$ of the system (21) such that $x(t_0) = x_0$, $y(t_0) = y_0$.

Theorem 7.1 is the basic *existence-uniqueness theorem* for the system (21), which is comparable to Theorem 5.1 for a single DE.

The *superposition principle*, which is so important in the study of linear homogeneous DEs, also applies to systems of DEs as stated in the next theorem. The proof is left as an exercise (problem 33).

Theorem 7.2 *If* $\{x_1(t), y_1(t)\}$ and $\{x_2(t), y_2(t)\}$ are both solution sets of the homogeneous system (22), then the pair

$$x(t) = C_1x_1(t) + C_2x_2(t),$$
$$y(t) = C_1y_1(t) + C_2y_2(t),$$

is also a solution for any constants C_1 and C_2.

EXAMPLE 8 The function sets $\{e^{-t}, -e^{-t}\}$ and $\{e^{4t}, \frac{3}{2}e^{4t}\}$ are each solutions of

$$x' = x + 2y,$$

$$y' = 3x + 2y.$$

Verify that $x(t) = C_1e^{-t} + C_2e^{4t}$ and $y(t) = -C_1e^{-t} + \frac{3}{2}C_2e^{4t}$ are also solutions.

Solution Direct substitution of $x(t)$ and $y(t)$ into the first DE gives

$$\begin{aligned} x' - x - 2y &= -C_1e^{-t} + 4C_2e^{4t} - C_1e^{-t} - C_2e^{4t} + 2C_1e^{-t} - 3C_2e^{4t} \\ &= 0. \end{aligned}$$

and, similarly, for the second DE,

$$\begin{aligned} y' - 3x - 2y &= C_1e^{-t} + 6C_2e^{4t} - 3C_1e^{-t} - 3C_2e^{4t} + 2C_1e^{-t} - 3C_2e^{4t} \\ &= 0. \end{aligned}$$

Definition 7.1 *The solution sets $\{x_1(t), y_1(t)\}$ and $\{x_2(t), y_2(t)\}$ are said to be* ***linearly dependent*** *on some interval $a \leqslant t \leqslant b$ if and only if there exist constants C_1 and C_2, not both zero, such that*

$$C_1x_1(t) + C_2x_2(t) = 0,$$

$$C_1y_1(t) + C_2y_2(t) = 0,$$

for every t in the interval. If these relations are true only for $C_1 = C_2 = 0$, we say the solution sets are* ***linearly independent.***

EXAMPLE 9 Show that the sets $\{e^{3t}, 2e^{t}\}$ and $\{-4e^{3t}, -8e^{t}\}$ are linearly dependent on every interval, while $\{e^{3t}, 2e^{t}\}$ and $\{-4e^{3t}, e^{t}\}$ are linearly independent on every interval.

Solution In the first case, we find

$$C_1e^{3t} - 4C_2e^{3t} = 0,$$

$$2C_1e^{t} - 8C_2e^{t} = 0,$$

which is satisfied for every choice of C_1 and C_2 such that $C_1 = 4C_2$. Hence the first pairs are linearly dependent. For the second set of functions, the conditions are

$$C_1e^{3t} - 4C_2e^{3t} = 0,$$

$$2C_1e^{t} + C_2e^{t} = 0.$$

These conditions imply that $C_1 = C_2 = 0$, and thus the sets are linearly independent.

*In the case of only two solution sets, we can also say they are linearly dependent if one set is a constant multiple of the other.

As a criterion for linear independence concerning a single DE, we introduced the concept of a *Wronskian*. Here once again we have a similar criterion for systems of equations, which we state in the following theorem without proof.

Theorem 7.3 If $\{x_1(t), y_1(t)\}$ and $\{x_2(t), y_2(t)\}$ are solution sets of the homogeneous system (22) on some interval $a \leq t \leq b$, a necessary and sufficient condition that these solution sets be linearly independent is that the Wronskian

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{vmatrix}$$

never vanishes on $a \leq t \leq b$.

Remark. As was the case in Chapter 4, the Wronskian defined in Theorem 7.3 can be shown to be either identically zero on the interval $a \leq t \leq b$ or never vanishing there. Also notice that our present definition of the Wronskian does not involve differentiation.

Based on the above results, if $\{x_1(t), y_1(t)\}$ and $\{x_2(t), y_2(t)\}$ are linearly independent solution pairs of the homogeneous system (22) on some interval $a \leq t \leq b$, then we define a *general solution* of this system to be the solution pair $\{x_H(t), y_H(t)\}$, where

$$\begin{aligned} x_H(t) &= C_1x_1(t) + C_2x_2(t), \\ y_H(t) &= C_1y_1(t) + C_2y_2(t), \end{aligned} \tag{23}$$

and where C_1 and C_2 are arbitrary constants.

EXAMPLE 10 Verify by the Wronskian test that the solution sets $\{e^{-t}, -e^{-t}\}$ and $\{e^{4t}, \frac{3}{2}e^{4t}\}$ given in Example 8 are linearly independent, and write the general solution of the system

$$\begin{aligned} x' &= x + 2y, \\ y' &= 3x + 2y. \end{aligned}$$

Solution The Wronskian of $\{e^{-t}, -e^{-t}\}$ and $\{e^{4t}, \frac{3}{2}e^{4t}\}$ is

$$W(t) = \begin{vmatrix} e^{-t} & e^{4t} \\ -e^{-t} & \frac{3}{2}e^{4t} \end{vmatrix} = \frac{5}{2}e^{3t},$$

which is never zero. Hence, the solution sets are linearly independent, and from Equation (23) the general solution is

$$\begin{aligned} x_H(t) &= C_1e^{-t} + C_2e^{4t}, \\ y_H(t) &= -C_1e^{-t} + \frac{3}{2}C_2e^{4t}. \end{aligned}$$

Finally, with respect to the nonhomogeneous system (21), we state the following theorem, which is analogous to Theorem 4.9. The proof also parallels that of Theorem 4.9 and is left to the exercises (problem 34).

Theorem 7.4 If $\{x_P(t), y_P(t)\}$ is any solution pair of the nonhomogeneous system (21) and $\{x_H(t), y_H(t)\}$ is the general solution of the associated homogeneous system (22), then the general solution of (21) is given by $\{x(t), y(t)\}$, where

$$x(t) = x_H(t) + x_P(t) = C_1x_1(t) + C_2x_2(t) + x_P(t),$$
$$y(t) = y_H(t) + y_P(t) = C_1y_1(t) + C_2y_2(t) + y_P(t).$$

7.4.1 Homogeneous Linear Systems with Constant Coefficients

In this section we wish to confine our attention to the homogeneous system

$$\begin{aligned} x' &= a_{11}x + a_{12}y, \\ y' &= a_{21}x + a_{22}y, \end{aligned} \tag{24}$$

where the coefficients are *constant*. In order to solve this system, we will steal an idea from the solution technique used in solving a single DE with constant coefficients. That is, let us assume that a solution of (24) exists of the form $x(t) = Ae^{\lambda t}$, $y(t) = Be^{\lambda t}$, where A, B, and λ are constants yet to be determined. The direct substitution of these trial solutions into (24) yields

$$A\lambda e^{\lambda t} = a_{11}Ae^{\lambda t} + a_{12}Be^{\lambda t},$$
$$B\lambda e^{\lambda t} = a_{21}Ae^{\lambda t} + a_{22}Be^{\lambda t}.$$

After dividing by the nonzero factor $e^{\lambda t}$ and rearranging the remaining terms, we are left with the algebraic system

$$\begin{aligned} (a_{11} - \lambda)A + a_{12}B &= 0, \\ a_{21}A + (a_{22} - \lambda)B &= 0. \end{aligned} \tag{25}$$

We wish to find values of λ such that the system (25) has a nontrivial solution for A and B, i.e., where A and B are not both zero. This situation is possible if and only if the coefficient determinant

$$\Delta(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} \tag{26}$$

vanishes. Setting $\Delta = 0$, we are led at once to the quadratic equation

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0. \tag{27}$$

We refer to (27) as the *auxiliary equation* of the system (24), or the *characteristic equation*, as it is sometimes called.

The roots λ_1 and λ_2 of (27) are called the *characteristic roots*. If $x(t) = Ae^{\lambda t}$, $y(t) = Be^{\lambda t}$ are to be solutions of the system (24), then λ must be one of these roots.

For $\lambda = \lambda_1$, we substitute this value back into the algebraic system (25) to obtain the nontrivial solutions A_1 and B_1. Hence, with these values of λ, A, and B, we have one solution set $\{x_1(t), y_1(t)\}$, where

$$\begin{aligned} x_1(t) &= A_1 e^{\lambda_1 t}, \\ y_1(t) &= B_1 e^{\lambda_1 t}. \end{aligned} \tag{28}$$

Obtaining a second linearly independent solution set corresponding to $\lambda = \lambda_2$ will vary depending upon whether the characteristic roots λ_1 and λ_2 are real and distinct, real and equal, or complex conjugates.

Case I—Distinct real roots: When the roots λ_1 and λ_2 of the auxiliary equation (27) are real and distinct, there are two linearly independent solution sets corresponding to $\{A_1e^{\lambda_1 t}, B_1e^{\lambda_1 t}\}$ and $\{A_2e^{\lambda_2 t}, B_2e^{\lambda_2 t}\}$ (see problem 25). The constants A_2 and B_2 are found in a similar way as A_1 and B_1, but this time the value $\lambda = \lambda_2$ is substituted into (25). The general solution in this case is therefore

$$\begin{aligned} x(t) &= C_1A_1e^{\lambda_1 t} + C_2A_2e^{\lambda_2 t}, \\ y(t) &= C_1B_1e^{\lambda_1 t} + C_2B_2e^{\lambda_2 t}. \end{aligned} \tag{29}$$

EXAMPLE 11 Solve the system

$$\begin{aligned} x' &= x + 2y, \\ y' &= 3x + 2y. \end{aligned}$$

Solution Evaluating Δ, we have

$$\Delta(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 6 = 0,$$

or

$$\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0.$$

Hence, $\lambda_1 = -1$ and $\lambda_2 = 4$. For $\lambda_1 = -1$, the system of equations (25) yields

$$\begin{aligned} 2A_1 + 2B_1 &= 0, \\ 3A_1 + 3B_1 &= 0, \end{aligned}$$

which shows that $B_1 = -A_1$. For $A_1 = 1$, a possible solution pair is $\{e^{-t}, -e^{-t}\}$. Likewise, with $\lambda_2 = 4$, we obtain from (25)

$$\begin{aligned} -3A_2 + 2B_2 &= 0, \\ 3A_2 - 2B_2 &= 0, \end{aligned}$$

or $B_2 = \frac{3}{2}A_2$. Thus, $\{e^{4t}, \frac{3}{2}e^{4t}\}$ is another solution pair, and our general solution is

$$x(t) = C_1e^{-t} + C_2e^{4t},$$

$$y(t) = -C_1e^{-t} + \frac{3}{2}C_2e^{4t}.$$

Case II—Equal roots: If the two roots λ_1 and λ_2 are real and equal, then setting $\lambda = \lambda_1 = \lambda_2$ produces only one solution set of the system. By analogy with second-order DEs, we might expect to find a second linearly independent solution set of the form

$$x_2(t) = A_2te^{\lambda t},$$
$$y_2(t) = B_2te^{\lambda t}.$$

However, this is not the case. Without providing the details, it turns out that we must seek a second solution set of the slightly more general form

$$\begin{aligned} x_2(t) &= (A_2t + A_3)e^{\lambda t}, \\ y_2(t) &= (B_2t + B_3)e^{\lambda t}. \end{aligned} \tag{30}$$

This being the case, our general solution this time becomes

$$\begin{aligned} x(t) &= C_1A_1e^{\lambda t} + C_2(A_2t + A_3)e^{\lambda t}, \\ y(t) &= C_1B_1e^{\lambda t} + C_2(B_2t + B_3)e^{\lambda t}. \end{aligned} \tag{31}$$

Let us illustrate this situation with an example.

EXAMPLE 12 Solve the system

$$x' = -4x - y,$$
$$y' = x - 2y.$$

Solution Here we have

$$\Delta(\lambda) = \begin{vmatrix} -4-\lambda & -1 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 9 = 0,$$

which has the double root $\lambda_1 = \lambda_2 = -3$. Setting $\lambda = -3$ in (25), we see that

$$-A_1 - B_1 = 0,$$
$$A_1 + B_1 = 0.$$

A possible nontrivial solution is $A_1 = -B_1 = 1$, from which we get the solution pair $\{e^{-3t}, -e^{-3t}\}$. We now substitute the trial solution set

$$x_2(t) = (A_2t + A_3)e^{-3t},$$
$$y_2(t) = (B_2t + B_3)e^{-3t},$$

into the original system of DEs. Such action leads to

$$e^{-3t}(A_2 - 3A_3 - 3A_2t) = -4(A_2t + A_3)e^{-3t} - (B_2t + B_3)e^{-3t},$$

$$e^{-3t}(B_2 - 3B_3 - 3B_2t) = (A_2t + A_3)e^{-3t} - 2(B_2t + B_3)e^{-3t}.$$

Equating the coefficients of like terms, we obtain

$$\begin{aligned} A_2 - 3A_3 &= -4A_3 - B_3, \\ -3A_2 &= -4A_2 - B_2, \\ B_2 - 3B_3 &= A_3 - 2B_3, \\ -3B_2 &= A_2 - 2B_2. \end{aligned}$$

One possible solution of these equations is $A_2 = A_3 = 1$, $B_2 = -1$, and $B_3 = -2$. Thus a second solution pair is $\{(t + 1)e^{-3t}, -(t + 2)e^{-3t}\}$. It is very easy to verify that the solution pairs given here are linearly independent since $|W(t)| = 1$, and therefore a general solution set is

$$\begin{aligned} x(t) &= C_1e^{-3t} + C_2(t + 1)e^{-3t}, \\ y(t) &= -C_1e^{-3t} - C_2(t + 2)e^{-3t}. \end{aligned}$$

Remark. It should be observed that the special case when $a_{11} = a_{22}$ and $a_{12} = a_{21} = 0$ leads to the uncoupled system $x' = a_{11}x$, $y' = a_{11}y$. Here $\{A_1e^{\lambda t}, 0\}$ and $\{0, B_2e^{\lambda t}\}$ are linearly independent solution pairs. However, our general procedure outlined above is for coupled systems.

Case III—Complex conjugate roots: When the characteristic roots are complex conjugates, $\lambda_1 = p + iq$ and $\lambda_2 = p - iq$, we obtain two distinct solution pairs

$$\begin{aligned} x_1(t) &= A_1e^{(p+iq)t}, \quad x_2(t) = \overline{A}_1e^{(p-iq)t}, \\ y_1(t) &= B_1e^{(p+iq)t}, \quad y_2(t) = \overline{B}_1e^{(p-iq)t}, \end{aligned} \tag{32}$$

where $\overline{A}_1$ and $\overline{B}_1$ denote the complex conjugates of A_1 and B_1. (The verification that the coefficients are indeed complex conjugates is left to the exercises.) Moreover, these solution pairs are linearly independent, although they are *complex*. To obtain *real* solutions, we proceed in a fashion similar to that in Section 4.4, where we previously obtained real solutions from complex solutions. Let $A_1 = a_1 + ia_2$ and $B_1 = b_1 + ib_2$ so that $\overline{A}_1 = a_1 - ia_2$ and $\overline{B}_1 = b_1 - ib_2$. Using Euler's formulas

$$e^{i\theta} = \cos\theta + i\sin\theta, \; e^{-i\theta} = \cos\theta - i\sin\theta,$$

and the superposition principle, we can write

$$\begin{aligned} \frac{1}{2}[x_1(t) + x_2(t)] &= \frac{1}{2}(a_1 + ia_2)e^{pt}(\cos qt + i\sin qt) \\ &\quad + \frac{1}{2}(a_1 - ia_2)e^{pt}(\cos qt - i\sin qt) \\ &= e^{pt}(a_1\cos qt - a_2\sin qt) \end{aligned}$$

and

$$\frac{1}{2}[y_1(t) + y_2(t)] = e^{pt}(b_1 \cos qt - b_2 \sin qt).$$

These sums represent another solution pair of the system involving only real functions. In a similar fashion, we see that

$$\begin{aligned} -\frac{i}{2}[x_1(t) - x_2(t)] &= -\frac{i}{2}(a_1 + ia_2)(\cos qt + i \sin qt) \\ &\quad + \frac{i}{2}(a_1 - ia_2)(\cos qt - i \sin qt) \\ &= e^{pt}(a_2 \cos qt - a_1 \sin qt) \end{aligned}$$

and

$$-\frac{i}{2}[y_1(t) - y_2(t)] = e^{pt}(b_2 \cos qt + b_1 \sin qt),$$

which is also a solution pair. The Wronskian of these solution pairs is found to be

$$W(t) = e^{2pt}(a_1 b_2 - a_2 b_1).$$

For complex λ, the constant B_1 cannot be a real multiple of A_1 and thus $a_1 b_2 - a_2 b_1 \neq 0$. These last solution pairs are therefore linearly independent, and the general solution for this case can now be written as

$$\begin{aligned} x(t) &= e^{pt}[C_1(a_1 \cos qt - a_2 \sin qt) + C_2(a_2 \cos qt + a_1 \sin qt)], \\ y(t) &= e^{pt}[C_1(b_1 \cos qt - b_2 \sin qt) + C_2(b_2 \cos qt + b_1 \sin qt)]. \end{aligned} \tag{33}$$

EXAMPLE 13 Solve the system

$$\begin{aligned} x' &= 6x - y, \\ y' &= 5x + 4y. \end{aligned}$$

Solution The auxiliary equation is

$$\begin{vmatrix} 6 - \lambda & -1 \\ 5 & 4 - \lambda \end{vmatrix} = \lambda^2 - 10\lambda + 29 = 0,$$

with complex roots $\lambda_1 = 5 + 2i$ and $\lambda_2 = 5 - 2i$. For $\lambda_1 = 5 + 2i$, we must solve

$$\begin{aligned} (1 - 2i)A_1 - B_1 &= 0, \\ 5A_1 - (1 + 2i)B_1 &= 0. \end{aligned}$$

Therefore, if we choose $A_1 = a_1 + ia_2 = 1$ and $B_1 = b_1 + ib_2 = 1 - 2i$, then $a_1 = 1$, $a_2 = 0$, $b_1 = 1$, and $b_2 = -2$. The general solution then becomes

$$\begin{aligned} x(t) &= e^{5t}(C_1 \cos 2t + C_2 \sin 2t), \\ y(t) &= e^{5t}[(C_1 - 2C_2) \cos 2t + (2C_1 + C_2) \sin 2t]. \end{aligned}$$

Notice that it was not necessary to substitute λ_2 into (25) for this case. (Can you explain why?)

The results of this section can be generalized to systems of equations larger than two. Let us illustrate with an example.

EXAMPLE 14 Solve the system

$$x' = x + z,$$

$$y' = y + 2z,$$

$$z' = x + 2y + 5z.$$

Solution To obtain the characteristic equation, we can expand the determinant

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 2 \\ 1 & 2 & 5-\lambda \end{vmatrix} = \lambda^3 - 7\lambda^2 + 6\lambda = \lambda(\lambda - 1)(\lambda - 6).$$

Hence, we find $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 6$. The substitution of $\lambda_1 = 0$ into the algebraic system

$$(1 - \lambda)A + C = 0,$$

$$(1 - \lambda)B + 2C = 0,$$

$$A + 2B + (5 - \lambda)C = 0,$$

yields

$$A_1 + C_1 = 0,$$

$$B_1 + 2C_1 = 0,$$

$$A_1 + 2B_1 + 5C_1 = 0,$$

from which we obtain $A_1 = 1$, $B_1 = 2$, and $C_1 = -1$. Similarly, the substitution of $\lambda_2 = 1$ and $\lambda_3 = 6$, respectively, into the algebraic system leads to $A_2 = 2$, $B_2 = -1$, $C_2 = 0$, $A_3 = 1$, $B_3 = 2$, and $C_3 = 5$. The general solution of this system is therefore

$$x(t) = C_1 + 2C_2e^t + C_3e^{6t},$$

$$y(t) = 2C_1 - C_2e^t + 2C_3e^{6t},$$

$$z(t) = -C_1 + 5C_3e^{6t}.$$

7.4.2 Nonhomogeneous Linear Systems

We now turn our attention to the nonhomogeneous system of DEs

$$\begin{aligned} x' &= a_{11}x + a_{12}y + f_1(t), \\ y' &= a_{21}x + a_{22}y + f_2(t). \end{aligned} \tag{34}$$

For the solution technique to be discussed, the coefficients need not be constants, although that is the only case we will solve. The general solution of this system (Theorem 7.4) is the pair $\{x_H(t) + x_P(t), y_H(t) + y_P(t)\}$, where $\{x_H(t), y_H(t)\}$ denotes the solution set of the associated homogeneous system and $\{x_P(t), y_P(t)\}$ is a particular solution set yet to be determined.

A general method for finding a particular solution set is the *variation of parameter method*, which is similar to the technique previously given this name in Section 4.7. We seek a particular solution of the form

$$\begin{aligned} x_P(t) &= u_1(t)x_1(t) + u_2(t)x_2(t), \\ y_P(t) &= u_1(t)y_1(t) + u_2(t)y_2(t), \end{aligned} \tag{35}$$

where we have replaced the arbitrary constants in the homogeneous solution by unknown functions. The substitution of (35) into (34) gives

$$\begin{aligned} u_1x_1' + u_1'x_1 + u_2x_2' + u_2'x_2 &= a_{11}u_1x_1 + a_{11}u_2x_2 + a_{12}u_1y_1 + a_{12}u_2y_2 + f_1, \\ u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2 &= a_{21}u_1x_1 + a_{21}u_2x_2 + a_{22}u_1y_1 + a_{22}u_2y_2 + f_2. \end{aligned}$$

Rearranging terms, we find

$$\begin{aligned} u_1x_1' + u_1'x_1 + u_2x_2' + u_2'x_2 &= u_1\overbrace{(a_{11}x_1 + a_{12}y_1)}^{x_1'} + u_2\overbrace{(a_{11}x_2 + a_{12}y_2)}^{x_2'} + f_1, \\ u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2 &= u_1\underbrace{(a_{21}x_1 + a_{22}y_1)}_{y_1'} + u_2\underbrace{(a_{21}x_2 + a_{22}y_2)}_{y_2'} + f_2, \end{aligned}$$

or

$$\begin{aligned} x_1u_1' + x_2u_2' &= f_1, \\ y_1u_1' + y_2u_2' &= f_2. \end{aligned}$$

Solving for u_1' and u_2', we obtain

$$u_1'(t) = \frac{\begin{vmatrix} f_1(t) & x_2(t) \\ f_2(t) & y_2(t) \end{vmatrix}}{W(t)}, \qquad u_2'(t) = \frac{\begin{vmatrix} x_1(t) & f_1(t) \\ y_1(t) & f_2(t) \end{vmatrix}}{W(t)}, \tag{36}$$

where $W(t) = x_1(t)y_2(t) - x_2(t)y_1(t)$ is the nonvanishing Wronskian defined in Theorem 7.3. Hence, a single integration yields $u_1(t)$ and $u_2(t)$, from which a particular solution set can be constructed.

EXAMPLE 15 Solve the system

$$x' = -4x + 2y + \frac{1}{t},$$

$$y' = 2x - y + 4 + \frac{2}{t}.$$

Solution We first solve the associated homogeneous system. Thus we set

$$\begin{vmatrix} -4-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 + 5\lambda = 0,$$

which gives $\lambda_1 = 0$ and $\lambda_2 = -5$. Setting $\lambda_1 = 0$ in the algebraic system (25) yields

$$-4A_1 + 2B_1 = 0,$$
$$2A_1 - B_1 = 0,$$

or $A_1 = 1$ and $B_1 = 2$. Also, the value $\lambda_2 = -5$ leads to $A_2 = 2$ and $B_2 = -1$. Therefore,

$$x_H(t) = C_1 + 2C_2e^{-5t},$$
$$y_H(t) = 2C_1 - C_2e^{-5t}.$$

Assuming a particular solution of the form

$$x_P(t) = u_1(t) + 2u_2(t)e^{-5t},$$
$$y_P(t) = 2u_1(t) - u_2(t)e^{-5t},$$

we find

$$u_1'(t) = \frac{\begin{vmatrix} \frac{1}{t} & 2e^{-5t} \\ 4+\frac{2}{t} & -e^{-5t} \end{vmatrix}}{\begin{vmatrix} 1 & 2e^{-5t} \\ 2 & -e^{-5t} \end{vmatrix}} = \frac{8}{5} + \frac{1}{t},$$

$$u_2'(t) = \frac{\begin{vmatrix} 1 & \frac{1}{t} \\ 2 & 4+\frac{2}{t} \end{vmatrix}}{\begin{vmatrix} 1 & 2e^{-5t} \\ 2 & -e^{-5t} \end{vmatrix}} = -\frac{4}{5}e^{5t}.$$

Therefore,

$$u_1(t) = \frac{8}{5}t + \log t,$$

$$u_2(t) = -\frac{4}{25}e^{5t},$$

giving us the particular solution set

$$x_P(t) = \frac{8}{5}t + \log t - \frac{8}{25},$$

$$y_P(t) = \frac{16}{5}t + 2\log t + \frac{4}{25}.$$

Finally,

$$x(t) = C_1 + 2C_2e^{-5t} + \frac{8}{5}t + \log t - \frac{8}{25},$$

$$y(t) = 2C_1 - C_2e^{-5t} + \frac{16}{5}t + 2\log t + \frac{4}{25}.$$

EXERCISES 7.4

In problems 1–7, rewrite the DEs as first-order systems of DEs.

1. $y'' + k^2y = P\sin\omega t$

2. $4y''' + y' = e^t$

3. $y^{(4)} - k^4y = f(t)$

4. $t^2y'' + ty' - y = t^2\log t$

5. $x'' - x - y' = e^t$
$x + y' = t - 10$

6. $2x'' + x - 2y' = 7$
$x' - y' = e^{3t}$

7. $m_1y_1'' + (k_1 + k_2)y_1 - k_2y_2 = 0$
$m_2y_2'' - k_1y_1 + k_2y_2 = 0$

In problems 8–20, solve by the method of this section.

8. $x' = 2x - y$
$y' = 3x - 2y$

9. $x' = 4x - 3y$
$y' = 5x - 4y$

10. $x' = 4x - 3y$
$y' = 8x - 6y$

11. $x' = 3x + 2y$
$y' = 6x - y$

12. $x' = x - 4y$
$y' = x + y$

13. $x' = 3x - 2y$
$y' = 2x + 3y$

14. $x' = 6x - 5y$
$y' = x + 2y$

15. $x' = 3x + 2y$
$y' = -5x + y$

16. $x' = 3x - 18y$
$y' = 2x - 9y$

17. $x' = -2x - 3y$
$y' = 3x + 4y$

18. $x' = x + y + z$
$y' = 2x + y - z$
$z' = -3x + 2y + 4z$

19. $x' = x + y + z$
$y' = 2x + y - z$
$z' = -y + z$

20. $x' = x + z$
$y' = x + y$
$z' = -2x - z$

In problems 21–24, determine whether the given set of functions is linearly dependent or independent using the Wronskian test.

21. $\{e^{5t}, 2e^{3t}\}$ and $\{-3e^{5t}, -e^{3t}\}$

22. $\{e^{5t}, 2e^{3t}\}$ and $\{-3e^{5t}, -6e^{3t}\}$

23. $\{e^t, 2e^t + 8te^t\}$ and $\{-e^t, 6e^t - 8te^t\}$

*24. $\{1 + t, -2 + 2t, 4 + 2t\}$, $\{1, -2, 4\}$, and $\{3 + 2t, -6 + 4t, 12 + 4t\}$

25. Show that the solution (29) is composed of linearly independent solution sets.

*26. Show that two of the characteristic roots of

$$x' = x - y + z,$$
$$y' = -x + y + z,$$
$$z' = -x - y + 3z,$$

are the same but that three linearly independent solution sets can be found which are all of the form $\{Ae^{\lambda t}, Be^{\lambda t}, Ce^{\lambda t}\}$.

In problems 27–32, solve the nonhomogeneous system of equations.

27. $x' = 2x - 5y - \sin 2t$, $y' = x - 2y + t$

28. $x' = 2x - y + t$, $y' = 3x - 2y + 2t$

29. $x' = 2x + y + e^t$, $y' = 4x - y - e^t$

30. $x' = 4x - 2y + e^t$, $y' = 5x + 2y - t$

31. $x' = x + 2y$, $y' = -\frac{1}{2}x + y + e^t \tan t$

32. $x' = 2x - 5y + \csc t$, $y' = x - 2y + \sec t$

*33. Prove Theorem 7.2.

*34. Prove Theorem 7.4.

*35. Show that the complex solutions (32) do indeed have coefficients A_1 and B_1, and $\overline{A}_1$ and $\overline{B}_1$, all of which are complex conjugates.

*36. Show that the Wronskian associated with the homogeneous system

$$x' = a_{11}(t)x + a_{12}(t)y,$$
$$y' = a_{21}(t)x + a_{22}(t)y,$$

is given by *Abel's formula*

$$W(t) = C\exp\left\{\int [a_{11}(t) + a_{22}(t)]\,dt\right\},$$

where C is a constant (see also Lemma 4.1).

[O] 7.5 MATRIX METHODS

In this section we assume that the reader is somewhat familiar with matrices and their basic operations, as taught in a beginning course in matrix methods or in linear algebra. Those readers lacking this background can skip this section without loss of continuity.

The theory and solution techniques discussed in the last section can all be formulated in terms of standard matrix operations. For example, the system itself

$$\begin{aligned} x' &= a_{11}(t)x + a_{12}(t)y + f_1(t), \\ y' &= a_{21}(t)x + a_{22}(t)y + f_2(t), \end{aligned} \tag{37}$$

can be expressed as a single matrix equation

$$\mathbf{Y}' = \mathbf{A}(t)\mathbf{Y} + \mathbf{F}(t), \tag{38}$$

where

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \tag{39}$$

is the coefficient matrix and

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \text{and} \quad \mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \tag{40}$$

are both vectors.* [The form of Equation (38) suggests that the notation for a matrix equation for a system of n equations is as simple as that for a system of two equations.] The use of vectors and matrices not only is notationally expedient but also facilitates calculations and emphasizes the similarity between systems of equations and first-order linear (scalar) equations. We say that the vector $\mathbf{Y}$ is a *solution* of (38) provided it is differentiable and its components satisfy the system of equations (37).

7.5.1 Homogeneous Equations

When $\mathbf{F}(t) \equiv \mathbf{O}$ in Equation (38), we obtain the *associated homogeneous equation*

$$\mathbf{Y}' = \mathbf{A}(t)\mathbf{Y}. \tag{41}$$

Suppose we know that

$$\mathbf{Y}^{(1)}(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \quad \text{and} \quad \mathbf{Y}^{(2)}(t) = \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} \tag{42}$$

are both solutions of (41). Then, from the superposition principle, it follows that the linear combination

$$\mathbf{Y} = C_1\mathbf{Y}^{(1)}(t) + C_2\mathbf{Y}^{(2)}(t) \tag{43}$$

is also a solution of (41) for any constants C_1 and C_2.

Let us now introduce the matrix

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix}, \tag{44}$$

whose columns are the solution vectors $\mathbf{Y}^{(1)}$ and $\mathbf{Y}^{(2)}$. We then say that $\mathbf{Y}^{(1)}$ and $\mathbf{Y}^{(2)}$ are *linearly independent if* $\det(\mathbf{X}) \neq 0$† (cf. Definition 7.1 in Section 7.4). However, this determinant is also the *Wronskian* defined in Theorem 7.3, i.e.,

$$\det(\mathbf{X}) = W(t). \tag{45}$$

*The terms *vector* and *column matrix* are used interchangeably in the literature on linear algebra.

†det (X) means the determinant of $\mathbf{X}$.

Thus, if we have that $\mathbf{Y}^{(1)}$ and $\mathbf{Y}^{(2)}$ are indeed linearly independent vectors on some interval I and are also solutions of (41), then a *general solution* of the homogeneous equation (41) on I is given by (43).

7.5.2 Linear Equations with Constant Coefficients

We now restrict our attention to the linear system

$$\mathbf{Y}' = \mathbf{A}\mathbf{Y}, \tag{46}$$

where $\mathbf{A}$ is a 2×2 matrix with constant elements. By analogy with our solution treatment of second-order constant-coefficient equations in Section 4.4, we seek solutions of (46) that are of the form

$$\mathbf{Y} = \mathbf{E}e^{\lambda t}, \tag{47}$$

where λ and the constant vector $\mathbf{E}$ are to be determined. The substitution of (47) into (46) leads to

$$\lambda \mathbf{E}e^{\lambda t} = \mathbf{A}\mathbf{E}e^{\lambda t},$$

which can be rearranged in the form (after canceling the common factor $e^{\lambda t}$)

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{E} = \mathbf{0}, \tag{48}$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{49}$$

is the identity matrix. Now from the general theory of matrix equations, we know that (48) can have a nontrivial solution if and only if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \Delta(\lambda) = 0. \tag{50}$$

Thus we are led once again to Equation (27) in Section 7.4.1 for finding the characteristic roots λ_1 and λ_2.

In the matrix notation given here, we recognize that finding the characteristic roots of (50) is simply finding the *eigenvalues* of the constant-coefficient matrix $\mathbf{A}$, and the *eigenvectors* of $\mathbf{A}$ are then used to construct the nontrivial solutions obtained from (47). Let us formalize these statements in the following theorem.

Theorem 7.5 For each eigenvalue λ_j of the constant-coefficient matrix $\mathbf{A}$ and each eigenvector $\mathbf{E}^{(j)}$ belonging to λ_j, the function

$$\mathbf{Y} = \mathbf{E}^{(j)}e^{\lambda_j t}$$

is a solution of the homogeneous matrix equation

$$\mathbf{Y}' = \mathbf{A}\mathbf{Y}.$$

EXAMPLE 16 Solve the system

$$\mathbf{Y}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \mathbf{Y}$$

by matrix methods.

Solution (We previously solved this system in Example 11 of Section 7.4.1 by conventional methods.) Here we assume $\mathbf{Y} = \mathbf{E}e^{\lambda t}$, which leads to the system of equations

$$\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Nontrivial solutions of this equation are possible only if

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = 0.$$

This equation has roots $\lambda_1 = -1$ and $\lambda_2 = 4$, which are the eigenvalues of **A**. For $\lambda_1 = -1$, the above system of equations reduces to the single equation (after simplification)

$$e_1 + e_2 = 0.$$

Thus $e_2 = -e_1$, and the eigenvector corresponding to the eigenvalue λ_1 can be represented by (any constant multiple will also suffice)

$$\mathbf{E}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

In the same way, corresponding to $\lambda_2 = 4$ we find the eigenvector

$$\mathbf{E}^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Based upon these results, the corresponding solutions of the system of equations are

$$\mathbf{Y}^{(1)} = \mathbf{E}^{(1)}e^{-t}, \quad \mathbf{Y}^{(2)} = \mathbf{E}^{(2)}e^{4t},$$

which lead to the general solution

$$\mathbf{Y} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t},$$

where C_1 and C_2 are any constants.

Remark. If any $n \times n$ matrix **A** is real and *symmetrical*, i.e., if it is equal to its transpose, then the eigenvalues and eigenvectors of **A** have important and useful properties. In particular, all eigenvalues and eigenvectors will be real. Moreover, a full set of linearly independent eigenvectors always exists, even in the case of repeated eigenvalues. There is no guarantee of these properties when $\mathbf{A}$ is not symmetrical. For example, complex eigenvalues and eigenvectors can arise as in Case III, discussed in Section 7.4.1.

The matrix formulation permits us to generalize the solution formula for a single first-order (scalar) DE with a constant coefficient. For instance, the equation

$$y' = ay \tag{51}$$

has the general solution (see Section 2.4.1)

$$y = Ce^{at}, \tag{52}$$

where C is any constant. Likewise, the matrix equation

$$\mathbf{Y}' = \mathbf{A}\mathbf{Y}, \tag{53}$$

where $\mathbf{A}$ is a constant matrix, has a solution that can be represented by (where $\mathbf{C}$ is any constant vector)

$$\mathbf{Y} = e^{\mathbf{A}t}\mathbf{C}. \tag{54}$$

Of course, we must now define the exponential matrix function $e^{\mathbf{A}t}$ so that (54) is meaningful (see problem 11). This can be done, but the use of this matrix function as a practical means of solving systems of DEs would necessitate a fairly thorough knowledge of matrix theory. Such knowledge is considered beyond the scope of this text.

7.5.3 Fundamental Matrices

The theory of systems of equations can be further enhanced by introducing the notion of a fundamental matrix. Suppose that $\mathbf{Y}^{(1)}$ and $\mathbf{Y}^{(2)}$ are linearly independent solutions of

$$\mathbf{Y}' = \mathbf{A}(t)\mathbf{Y}, \tag{55}$$

where $\mathbf{A}(t)$ is not necessarily a constant matrix. We then define the *fundamental matrix*

$$\mathbf{\Psi}(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix}, \tag{56}$$

whose columns are the vectors $\mathbf{Y}^{(1)}$ and $\mathbf{Y}^{(2)}$. Observe that $\mathbf{\Psi}$ is nonsingular, i.e., $\det(\mathbf{\Psi}) \neq 0$, since this determinant is also the nonvanishing Wronskian $W(t)$. In terms of the fundamental matrix, the solution of (55) is simply

$$\mathbf{Y} = \mathbf{\Psi}(t)\mathbf{C}, \tag{57}$$

where $\mathbf{C}$ is an arbitrary column vector.

In the case of an initial value problem, we seek a solution of (55) such that

$$\mathbf{Y}(0) = \mathbf{Y}_0, \tag{58}$$

where $\mathbf{Y}_0$ is a prescribed vector. Imposing this condition on the solution (57), we find

$$\mathbf{\Psi}(0)\mathbf{C} = \mathbf{Y}_0, \tag{59}$$

which has the formal solution

$$\mathbf{C} = \mathbf{\Psi}^{-1}(0)\mathbf{Y}_0, \tag{60}$$

where $\mathbf{\Psi}^{-1}$ denotes a matrix inverse.* Hence the solution of the initial value problem described by (55) and (58) is formally given by

$$\mathbf{Y} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(0)\mathbf{Y}_0. \tag{61}$$

EXAMPLE 17 Solve the initial value problem

$$\mathbf{Y}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$

Solution Referring to Example 16, we know that

$$\mathbf{Y}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}, \quad \mathbf{Y}^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t},$$

are linearly independent solutions of the equation. Hence, the fundamental matrix is

$$\mathbf{\Psi}(t) = \begin{pmatrix} e^{-t} & 2e^{4t} \\ -e^{-t} & 3e^{4t} \end{pmatrix}.$$

Next, we set $t = 0$ to get

$$\mathbf{\Psi}(0) = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, \quad \det(\mathbf{\Psi}(0)) = 5,$$

from which we compute

$$\mathbf{\Psi}^{-1}(0) = \frac{1}{5}\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}.$$

The solution we seek is therefore

*Recall from matrix theory that the inverse of a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is given by

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \quad \det(\mathbf{A}) \neq 0.$$

$$\mathbf{Y} = \begin{pmatrix} e^{-t} & 2e^{4t} \\ -e^{-t} & 3e^{4t} \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} e^{-t} & 2e^{4t} \\ -e^{-t} & 3e^{4t} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3e^{-t} + 2e^{4t} \\ -3e^{-t} + 3e^{4t} \end{pmatrix}.$$

7.5.4 Nonhomogeneous Equations

When the system of equations is *nonhomogeneous*, i.e., when

$$\mathbf{Y}' = \mathbf{A}(t)\mathbf{Y} + \mathbf{F}(t), \tag{62}$$

then a *general solution* has the form

$$\mathbf{Y} = \mathbf{Y}_H + \mathbf{Y}_P, \tag{63}$$

where $\mathbf{Y}_H$ is a general solution of the associated homogeneous equation and $\mathbf{Y}_P$ is any particular solution of (62).

To find $\mathbf{Y}_P$, we again call upon the method of *variation of parameters*. Thus we assume

$$\mathbf{Y}_P = \mathbf{\Psi}(t)\mathbf{u}(t), \tag{64}$$

where $\mathbf{u}(t)$ is a vector to be found. The substitution of (64) into (62) leads to

$$\mathbf{\Psi}'(t)\mathbf{u}(t) + \mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{A}(t)\mathbf{\Psi}(t)\mathbf{u}(t) + \mathbf{F}(t). \tag{65}$$

However, since $\mathbf{\Psi}$ is a fundamental matrix, it follows that

$$\mathbf{\Psi}'(t) = \mathbf{A}(t)\mathbf{\Psi}(t),$$

and therefore (65) reduces to simply

$$\mathbf{\Psi}(t)\mathbf{u}'(t) = \mathbf{F}(t). \tag{66}$$

Hence,

$$\mathbf{u}'(t) = \mathbf{\Psi}^{-1}(t)\mathbf{F}(t), \tag{67}$$

and, upon integration, we deduce that

$$\mathbf{u}(t) = \int \mathbf{\Psi}^{-1}(t)\mathbf{F}(t)\,dt. \tag{68}$$

Putting this result back into (64) gives

$$\mathbf{Y}_P = \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{F}(t)\,dt, \tag{69}$$

and so a general solution of (62) takes the form

$$\mathbf{Y} = \mathbf{\Psi}(t)\mathbf{C} + \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{F}(t)\,dt, \tag{70}$$

where **C** is any constant vector.

Finally, if an initial condition

$$\mathbf{Y}(0) = \mathbf{Y}_0 \tag{71}$$

is also prescribed, the complete solution of the initial value problem becomes (see problem 21)

$$\mathbf{Y} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(0)\mathbf{Y}_0 + \mathbf{\Psi}(t)\int_0^t \mathbf{\Psi}^{-1}(\tau)F(\tau)\,d\tau. \tag{72}$$

EXAMPLE 18 Use matrix methods to solve the nonhomogeneous system

$$\mathbf{Y}' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\mathbf{Y} + \begin{pmatrix} \dfrac{1}{t} \\ 4 + \dfrac{2}{t} \end{pmatrix}.$$

Solution (We previously solved this system in Example 15 of Section 7.4.2 by conventional methods.) The coefficient matrix **A** has eigenvalues given by solutions of

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 + 5\lambda = 0.$$

Hence $\lambda_1 = 0$ and $\lambda_2 = -5$, and the corresponding eigenvectors are found to be

$$\mathbf{E}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{E}^{(2)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

From these results we construct the fundamental matrix

$$\mathbf{\Psi}(t) = \begin{pmatrix} 1 & 2e^{-5t} \\ 2 & -e^{-5t} \end{pmatrix}.$$

The matrix inverse of $\mathbf{\Psi}$ is

$$\mathbf{\Psi}^{-1}(t) = \begin{pmatrix} \dfrac{1}{5} & \dfrac{2}{5} \\ \dfrac{2}{5}e^{5t} & -\dfrac{1}{5}e^{5t} \end{pmatrix},$$

so from (69) we obtain

$$\begin{aligned}\mathbf{Y}_P &= \mathbf{\Psi}(t)\int \mathbf{\Psi}^{-1}(t)\mathbf{F}(t)\,dt \\ &= \begin{pmatrix} 1 & 2e^{-5t} \\ 2 & -e^{-5t} \end{pmatrix}\int \begin{pmatrix} \dfrac{1}{5} & \dfrac{2}{5} \\ \dfrac{2}{5}e^{5t} & -\dfrac{1}{5}e^{5t} \end{pmatrix}\begin{pmatrix} \dfrac{1}{t} \\ 4 + \dfrac{2}{t} \end{pmatrix} dt\end{aligned}$$

$$= \begin{pmatrix} 1 & 2e^{-5t} \\ 2 & -e^{-5t} \end{pmatrix} \int \begin{pmatrix} \frac{8}{5} + \frac{1}{t} \\ -\frac{4}{5}e^{5t} \end{pmatrix} dt$$

$$= \begin{pmatrix} 1 & 2e^{-5t} \\ 2 & -e^{-5t} \end{pmatrix} \begin{pmatrix} \frac{8}{5}t + \log t \\ -\frac{4}{25}e^{5t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8}{5}t + \log t - \frac{8}{25} \\ \frac{16}{5}t + 2\log t + \frac{4}{25} \end{pmatrix}.$$

Therefore, from (70) the general solution is

$$\mathbf{Y} = \begin{pmatrix} 1 & 2e^{-5t} \\ 2 & -e^{-5t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \begin{pmatrix} \frac{8}{5}t + \log t - \frac{8}{25} \\ \frac{16}{5}t + 2\log t + \frac{4}{25} \end{pmatrix},$$

or, equivalently,

$$\mathbf{Y} = C_1 \begin{pmatrix} 1 \\ 2e^{-5t} \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ -e^{-5t} \end{pmatrix} + \begin{pmatrix} \frac{8}{5} \\ \frac{16}{5} \end{pmatrix} t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \log t + \begin{pmatrix} -\frac{8}{25} \\ \frac{4}{25} \end{pmatrix}$$

EXERCISES 7.5

In problems 1–7, use matrix methods to solve the linear system.

1. $\mathbf{Y}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{Y}$ **2.** $\mathbf{Y}' = \begin{pmatrix} 4 & -3 \\ 5 & -4 \end{pmatrix} \mathbf{Y}$ **3.** $\mathbf{Y}' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} \mathbf{Y}$

4. $\mathbf{Y}' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} \mathbf{Y}$ **5.** $\mathbf{Y}' = \begin{pmatrix} 3 & 2 \\ -5 & 1 \end{pmatrix} \mathbf{Y}$ **6.** $\mathbf{Y}' = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \mathbf{Y}$

7. $\mathbf{Y}' = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \mathbf{Y}$

In problems 8–10, find a unique solution of the system of equations satisfying the prescribed initial condition.

8. $\mathbf{Y}' = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$ **9.** $\mathbf{Y}' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

***10.** $\mathbf{Y}' = \begin{pmatrix} 5 & 2 & -2 \\ 7 & 0 & -2 \\ 11 & 1 & -3 \end{pmatrix}\mathbf{Y}, \quad \mathbf{Y}(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

***11.** By analogy with the Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

we define the matrix exponential function

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \mathbf{A}^3\frac{t^3}{3!} + \cdots,$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{A}^2 = \mathbf{AA}$, $\mathbf{A}^3 = \mathbf{A}(\mathbf{A}^2)$, and so on.

(a) Use the approximation

$$e^{\mathbf{A}t} \cong \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!}$$

to compute the matrix exponential function when $t = 1$ and

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(b) Compute $e^{\mathbf{A}t}$ exactly for the matrix $\mathbf{A}$ given in part (a) when $t = 1$ and compare the result with that for (a).

(c) Compute $e^{\mathbf{A}t}$ exactly for any t when

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

***12.** Use the results of problem 11 to solve the linear systems

(a) $\mathbf{Y}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{Y}.$ (b) $\mathbf{Y}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{Y}.$

In problems 13–18, use matrix methods to solve the nonhomogeneous linear systems.

13. $\mathbf{Y}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{Y} + \begin{pmatrix} -\sin 2t \\ t \end{pmatrix}$ **14.** $\mathbf{Y}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{Y} + \begin{pmatrix} t \\ 2t \end{pmatrix}$

15. $\mathbf{Y}' = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}\mathbf{Y} + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$ **16.** $\mathbf{Y}' = \begin{pmatrix} 4 & -2 \\ 5 & 2 \end{pmatrix}\mathbf{Y} + \begin{pmatrix} e^t \\ -t \end{pmatrix}$

17. $\mathbf{Y}' = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix}\mathbf{Y} + \begin{pmatrix} 0 \\ e^t \tan t \end{pmatrix}$ **18.** $\mathbf{Y}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}\mathbf{Y} + \begin{pmatrix} \csc t \\ \sec t \end{pmatrix}$

19. Find a unique solution of the nonhomogeneous initial value problem

$$\mathbf{Y}' = \begin{pmatrix} -10 & 6 \\ -12 & 7 \end{pmatrix}\mathbf{Y} + \begin{pmatrix} 10e^{-3t} \\ 18e^{-3t} \end{pmatrix}, \quad \mathbf{Y}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

20. Find a unique solution of the nonhomogeneous initial value problem

$$\mathbf{Y}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{Y} + \begin{pmatrix} 3e^{t} \\ 3e^{t} \end{pmatrix}, \quad \mathbf{Y}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

***21.** Verify that Equation (72) satisfies the initial value problem described by Equations (62) and (71).

[O] 7.6 NONLINEAR SYSTEMS AND STABILITY

A DE provides the basic model from which a mathematical analyst pursues investigations into the existence, characterization, and construction of solutions to physical systems. For a large class of problems, either the *oscillatory nature* of the system under study or its *stability* is of greatest interest. But because finding the solution of many DEs in a convenient and useful explicit form is either difficult or impossible, we often resort to *qualitative methods* for much of the analysis, as we did in Section 5.7 in studying the oscillatory characteristics of certain *linear systems*. We now wish to use qualitative methods once again in studying certain *nonlinear systems*, but with respect primarily to their stability.

7.6.1 The Phase Plane and Critical Points

Suppose we consider the initial value problem (primes denote derivatives with respect to t)

$$x'' = F(x, x'), \quad x(0) = \alpha, \quad x'(0) = \beta, \tag{73}$$

where F is assumed to have continuous first partial derivatives with respect to x and x'. If we put $y = x'$, then we can replace (73) with the equivalent system of equations

$$\begin{aligned} x' &= y, \quad x(0) = \alpha, \\ y' &= F(x, y), \quad y(0) = \beta. \end{aligned} \tag{74}$$

By a solution of (74), we mean a set of differentiable functions $\{x(t), y(t)\}$, which on some interval containing $t = 0$ reduces (74) to a set of identities.

We can think of the functions $x = x(t)$ and $y = y(t)$ as parametric equations of an arc in the xy-plane that passes through the point (α, β). From this point of view, it is customary to refer to the xy-plane as the *phase plane* of the system (74) and the arc defined by its solution as a *trajectory*, *path*, or *orbit*. The direction of increasing t is considered the positive direction along a given trajectory.

The trajectories of a system can be found by eliminating the parameter t between the equations $x = x(t)$ and $y = y(t)$. In other cases where a solution cannot be found, the trajectories can be determined by forming the ratio y'/x', which leads to

$$\frac{dy}{dx} = \frac{F(x, y)}{y}, \tag{75}$$

and then solving this first-order DE.

EXAMPLE 19 Determine the trajectories associated with the simple harmonic oscillator whose governing equation is

$$mx'' + kx = 0.$$

Solution We first reexpress the DE as the first-order system

$$x' = y,$$

$$y' = -\frac{kx}{m},$$

and then form the ratio y'/x' to get

$$\frac{dy}{dx} = -\frac{kx}{my}.$$

The general solution of this DE can be found by separating the variables, which leads to $kx\,dx + my\,dy = 0$, or

$$kx^2 + my^2 = C,$$

where C is an arbitrary constant. By allowing C to assume various values, we find the trajectories to be the one-parameter family of ellipses shown in Figure 7.3. Each trajectory represents a possible motion of the system (depending upon the prescribed initial conditions), and each point on a given trajectory represents an instantaneous state of the system. The direction of the arrows in Figure 7.3 suggests that the representative point moves clockwise along a given trajectory. This is so since $y = x'$, and $y > 0$ implies that $x(t)$ is increasing, and $y < 0$ implies that $x(t)$ is decreasing.

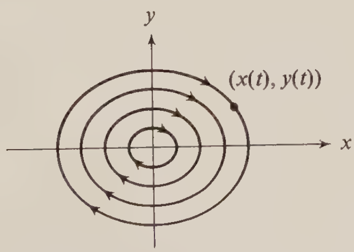


Figure 7.3 Trajectories of a harmonic oscillator.

EXAMPLE 20 Find the trajectories of the nonlinear system

$$x' = x + 2y + x\cos y,$$
$$y' = -y - \sin y.$$

Solution Dividing the second equation by the first, we obtain

$$\frac{dy}{dx} = \frac{-y - \sin y}{x + 2y + x\cos y},$$

or

$$(y + \sin y)\,dx + (x + 2y + x\cos y)\,dy = 0.$$

This DE happens to be exact and can be easily solved by the method of Section 2.3.1. Its solution is

$$xy + y^2 + x\sin y = C,$$

where C is any constant. Typical configurations of the trajectories are shown in Figure 7.4 for various values of C.

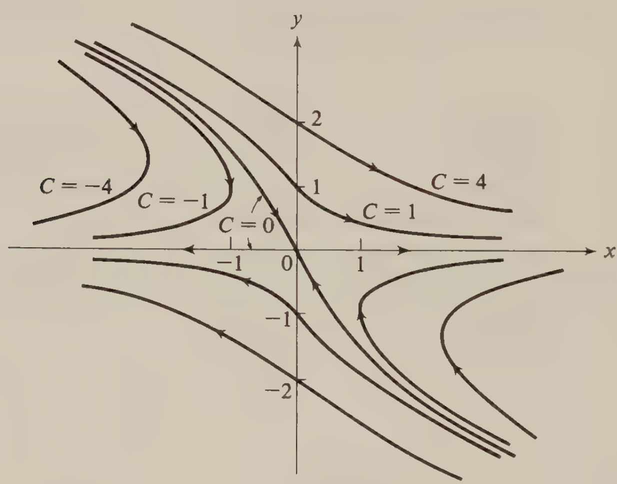


Figure 7.4

Remark. Since the parameterization of a curve is not unique, the terms *solution* and *trajectory* are not synonymous. For example, $x = e^t$, $y = e^{2t}$ and $x = t$, $y = t^2$ are both parameterizations of the parabola $y = x^2$, but in the first case only the right half $(x > 0)$ of the parabola is defined by the parametric equations.

Example 20 is characteristic of nonlinear systems described by

$$\begin{aligned} x' &= P(x, y), \\ y' &= Q(x, y). \end{aligned} \tag{76}$$

Since the independent variable t does not explicitly appear in (76), the system is said to be *autonomous*. This means that the physical parameters of the system are not time dependent, which frequently is the case in practice and is the only kind of system we will discuss in the remainder of this chapter. Forming the quotient of the two equations in (76), we obtain

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}, \tag{77}$$

the integration of which provides the integral curves or trajectories except at points for which $P(x, y)$ vanishes. (If $P = 0$, we simply consider $dx/dy = P/Q$.) If both P and Q vanish at a point (x_0, y_0) of the phase plane, that point is said to be a *critical point*, for then a unique slope is not defined. However, in reference to the original system (76), the critical point (x_0, y_0) is best described as an *equilibrium point* because both x and y are stationary there. That is to say, once we reach this point, we can never leave it. In mechanics, a critical point is one for which both velocity and acceleration vanish, and hence the motion stops.

EXAMPLE 21 Determine the critical points of the nonlinear system

$$\begin{aligned} x' &= -x^2 + y, \\ y' &= x - y^2. \end{aligned}$$

Solution Solving simultaneously,

$$\begin{aligned} -x^2 + y &= 0, \\ x - y^2 &= 0, \end{aligned}$$

we see that there are two critical points, $(0, 0)$ and $(1, 1)$.

It is usually of interest to know how the solution of a DE is altered if either the initial conditions or the input function of the equation is slightly changed. If small changes in these input parameters produce only small changes in the solution function, we say the solution is *stable*. More precisely, we have the following definition.

Definition 7.2 *Let $\{x(t), y(t)\}$ denote the solution set of the system*

$$\begin{aligned} x' &= P(x,y), \quad x(0) = \alpha, \\ y' &= Q(x,y), \quad y(0) = \beta, \end{aligned}$$

which has a critical point at (x_0, y_0). The critical point is said to be (*a*) ***stable*** *if for every* $\epsilon > 0$ *there exists some number* $\delta > 0$ *such that*

$$[(x_0 - x)^2 + (y_0 - y)^2]^{1/2} < \epsilon$$

for all $t \geq 0$ *whenever*

$$[(x_0 - \alpha)^2 + (y_0 - \beta)^2]^{1/2} < \delta;$$

(*b*) ***asymptotically stable*** *if it is stable and there exists a positive number A such that*

$$\lim_{t\to\infty} [(x_0 - x)^2 + (y_0 - y)^2]^{1/2} = 0$$

whenever

$$[(x_0 - \alpha)^2 + (y_0 - \beta)^2]^{1/2} < A; \textit{ and}$$

(*c*) ***unstable*** *otherwise.*

EXAMPLE 22 Discuss the stability of the critical point (0, 0) of the linear system

$$x' = -x, \quad x(0) = 2,$$
$$y' = -2y, \quad y(0) = 1.$$

Solution Because the system is linear with constant coefficients, we readily find the solution $\{2e^{-t}, e^{-2t}\}$. Thus, according to Definition 7.2, given $\epsilon > 0$, we wish to find a δ such that whenever

$$(4 + 1)^{1/2} = \sqrt{5} < \delta,$$

we have

$$(4e^{-2t} + e^{-4t})^{1/2} < \epsilon$$

for all $t \geq 0$. Here we see that by choosing $\delta = \epsilon$, it follows that

$$(4e^{-2t} + e^{-4t})^{1/2} \leq \sqrt{5} < \delta, \quad t \geq 0.$$

Hence the critical point (0, 0) is stable, and since $x(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$, the critical point is also asymptotically stable.

The following statements can be made about the trajectories of autonomous systems, although we will not present any proofs.

1. There exists at most one trajectory passing through any point of the phase plane that is not a critical point.
2. A particle starting at a point other than a critical point cannot reach the critical point (if indeed it reaches it at all) in a finite amount of time. If the solution is asymptotically stable, it will approach the critical point as $t \to \infty$ (see Figure

7.5). If the solution is unstable, no matter how close to the critical point the particle starts, there are solutions that move away from it.

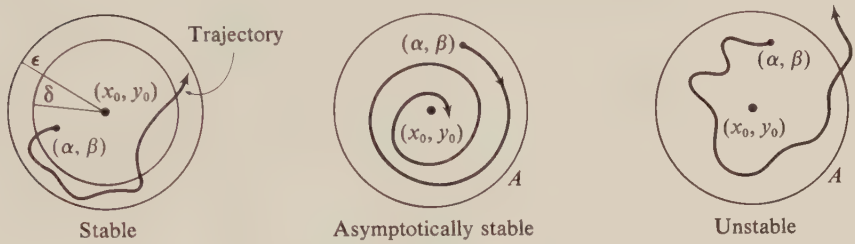


Figure 7.5 Stability trajectories.

3. If a trajectory crosses itself at a point of the phase plane that is not a critical point, that trajectory is a closed path and corresponds to a periodic solution of the system (see Figure 7.6).

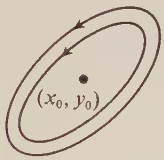


Figure 7.6 Periodic motion.

These statements suggest that a particle (solution) either approaches a critical point as $t \to \infty$, moves along a closed path, approaches a closed path, or goes off to infinity for increasing time. Thus the study of critical points is most important in the analysis of autonomous systems.

In order to test for stability by using Definition 7.2, the solution set of the system must be known. Since nonlinear systems often do not have a known solution set, it is desirable to be able to test for stability without first explicitly finding the solution of the system. To see how this is done, let us begin by examining more closely *linear* autonomous systems and then extend our ideas to similar nonlinear systems.

7.6.2 Stability of Linear Systems

Consider the linear system

$$x' = ax + by,$$
$$y' = cx + dy, \tag{78}$$

where a, b, c, and d, are constants such that $ad - bc \neq 0$. This system has one critical point at the origin of the phase plane.

Remark. The requirement that $ad - bc \neq 0$ is to ensure that only isolated critical points occur. If $ad = bc$, all points on the line $ax + by = 0$ (or the line $cx + dy = 0$) are nonisolated critical points. Also note that the most general linear system is $x' = ax + by + c_1$, $y' = cx + dy + c_2$, where c_1 and c_2 are constants. However, if the critical point is (x_0, y_0), then the simple translation $X = x - x_0$, $Y = y - y_0$ reduces this linear system to the form (78) with (0, 0) as the critical point. Hence, there is no loss of generality in treating the case (78).

The auxiliary equation associated with (78) is (see Section 7.4)

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0, \tag{79}$$

with roots λ_1 and λ_2. Observe that $\lambda = 0$ cannot be a root since $ad - bc \neq 0$. From our definition of stability, it is clear that if either or both λ_1 and λ_2 have a positive real part, the critical point is unstable, since $x(t)$ and $y(t)$ would both become infinite as $t \to \infty$. However, if both λ_1 and λ_2 have negative real parts, then the trajectory described by every nontrivial solution of (78) approaches (0, 0) as $t \to \infty$. In summary, we state the following theorem.

Theorem 7.6 The critical point (0, 0) of the linear system

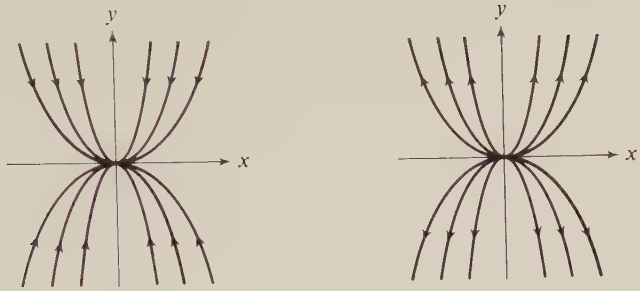
$$x' = ax + by, \quad ad - bc \neq 0$$
$$y' = cx + dy,$$

is stable if and only if both roots of the auxiliary equation

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

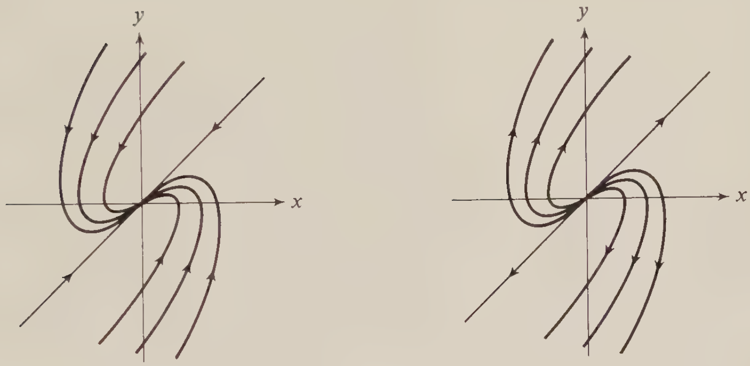
have nonpositive real parts. The critical point has asymptotic stability if both roots have negative real parts.

Some of the possible trajectories for the linear system (78) are shown in Figures 7.7 through 7.11. The critical point (0, 0) is referred to as a *node* in Figures 7.7 and 7.8, a *focus* in Figure 7.9, a *center* in Figure 7.10, and a *saddle point* in Figure 7.11. These figures make clear that small changes in the initial conditions produce only small changes in the solution for stable systems but can lead to large changes in the solution for unstable systems.



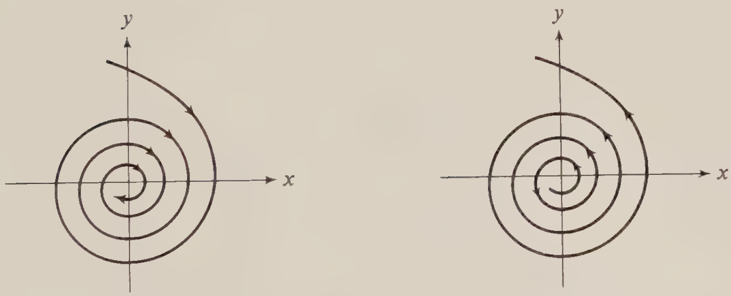
$\lambda_2 < \lambda_1 < 0$ (stable) $\lambda_1 > \lambda_2 > 0$ (unstable)

Figure 7.7 Real distinct roots leading to nodes.



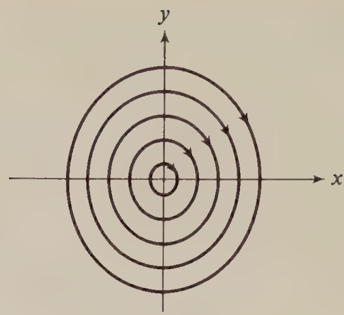
$\lambda_1 = \lambda_2 < 0$ (stable) $\lambda_1 = \lambda_2 > 0$ (unstable)

Figure 7.8 Equal roots leading to nodes.



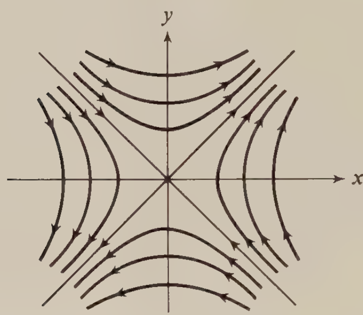
$p < 0$ (stable) $p > 0$ (unstable)

Figure 7.9 Complex roots $\lambda = p \pm iq$ leading to foci.



(stable)

Figure 7.10 Pure imaginary roots $\lambda = \pm iq$ leading to a center.



$\lambda_1 > 0 > \lambda_2$ (unstable)

Figure 7.11 Real distinct roots leading to a saddle point.

EXAMPLE 23 Discuss the stability of the system

$$x' = x - 3y,$$
$$y' = x - y.$$

Solution The auxiliary equation is $\lambda^2 + 2 = 0$ with roots $\lambda = \pm\sqrt{2}\,i$. Thus the origin is a *center* and the solution is stable.

EXAMPLE 24 Discuss the stability of the system

$$x' = -4x - y,$$
$$y' = x - 2y.$$

Solution The auxiliary equation is $\lambda^2 + 6\lambda + 9 = 0$ with double root $\lambda = -3$. Hence the origin is a *node*, which is asymptotically stable.

EXAMPLE 25 Discuss the stability of the system

$$x' = 4x - y,$$
$$y' = 6x - 3y.$$

Solution Here we find $\lambda^2 - \lambda - 6 = 0$ with roots $\lambda_1 = 3$ and $\lambda_2 = -2$. The origin is thus a *saddle point*, which is unstable.

Since periodic solutions are of special interest in many problems, it should be noted that they arise in the case of a linear system when and only when the roots of the auxiliary equation (79) are pure imaginary. In such a case, the trajectories are closed curves (ellipses), and the critical point is a center. In other words, closed trajectories imply periodic motion.

7.6.3 Local Stability of Nonlinear Systems

Here now we wish to examine certain types of nonlinear systems

$$\begin{aligned} x' &= P(x, y), \\ y' &= Q(x, y), \end{aligned} \tag{80}$$

where for simplicity we assume $P(0, 0) = Q(0, 0) = 0$ so that the origin is a critical point. We further assume that $P(x, y)$ and $Q(x, y)$ possess at least third partial derivatives so that they can be expanded in Taylor series,

$$P(x, y) = P(0, 0) + P_x(0, 0)x + P_y(0, 0)y + P_{xx}(0, 0)\frac{x^2}{2}$$
$$+ P_{xy}(0, 0)xy + P_{yy}(0, 0)\frac{y^2}{2} + \cdots,$$
$$Q(x, y) = Q(0, 0) + Q_x(0, 0)x + Q_y(0, 0)y + Q_{xx}(0, 0)\frac{x^2}{2}$$
$$+ Q_{xy}(0, 0)xy + Q_{yy}(0, 0)\frac{y^2}{2} + \cdots.$$

Then, using the fact that $P(0, 0) = Q(0, 0) = 0$, we can write

$$P(x, y) = ax + by + p(x, y),$$
$$Q(x, y) = cx + dy + q(x, y),$$

where $p(x, y)$ and $q(x, y)$ have continuous first partial derivatives and are small enough near the origin in the sense that

$$\lim_{x,y\to 0} \frac{p(x, y)}{(x^2 + y^2)^{1/2}} = \lim_{x,y\to 0} \frac{q(x, y)}{(x^2 + y^2)^{1/2}} = 0.$$

Our nonlinear system (80) now has the form of a *perturbed linear system* (nearly linear system)

$$\begin{aligned} x' &= ax + by + p(x, y), \\ y' &= cx + dy + q(x, y). \end{aligned} \tag{81}$$

In most cases, stability behavior of the perturbed linear system in the neighborhood of a critical point (0, 0) is closely related to the stability behavior of the corresponding linear system arising when $p(x, y) = q(x, y) = 0$. In particular, we have the following theorem, which we state without proof.

Theorem 7.7 Let (0, 0) be a critical point of the perturbed linear system

$$x' = ax + by + p(x, y),$$
$$y' = cx + dy + q(x, y).$$

If the critical point (0, 0) of the associated linear system

$$x' = ax + by, \quad ad - bc \neq 0,$$
$$y' = cx + dy,$$

is

(a) asymptotically stable, then the nonlinear system is asymptotically stable near this point;
(b) unstable, then the same critical point of the nonlinear system is unstable; and
(c) stable, but not asymptotically stable, then the same critical point of the nonlinear system may be asymptotically stable, stable, or unstable (i.e., no conclusion).

Even though the stability of a perturbed nonlinear system may be the same as that of the associated linear system, the trajectories of the nonlinear system may differ greatly from those of the linear system. Also, if we move sufficiently far from a stable critical point of the nonlinear system, it may no longer be a point of stability of the system. This is in sharp contrast with linear systems, where stability is not localized.

EXAMPLE 26 Discuss the stability of the simple pendulum whose equation of motion is

$$\theta'' + k^2 \sin\theta = 0.$$

Solution Setting $x = \theta$ and $y = \theta'$, we obtain the nonlinear system

$$x' = y,$$
$$y' = -k^2 \sin x.$$

This system has an infinite number of critical points (0, 0) and $(n\pi, 0)$, where $n = \pm 1, \pm 2, \ldots$. The associated linear system near the critical point (0, 0) is ($\sin x \cong x$ as $x \to 0$)

$$x' = y,$$
$$y' = -k^2 x.$$

Here we find that the roots of the auxiliary equation are pure imaginary, which means that (0, 0) is a center of the linear system. Based on Theorem 7.7, however, we cannot conclude anything about the stability of the nonlinear system near (0, 0).

At the critical point $(\pi, 0)$, the situation is different. Expanding $\sin x$ about the point $x = \pi$ yields

$$\sin x = (\pi - x) - \frac{1}{3!}(\pi - x)^3 + \ldots,$$

so that the associated linear system this time becomes

$$x' = y,$$
$$y' = k^2(x - \pi).$$

Letting $z = x - \pi$, we see that the roots of the auxiliary equation $\lambda^2 - k^2 = 0$ are $\lambda_1 = k$ and $\lambda_2 = -k$, so that $(\pi, 0)$ is an unstable critical point of the linear system. Therefore, it is an unstable critical point of the nonlinear system.

At the point $x = \pi$, the pendulum is pointing vertically upward. Clearly, such a position is unstable, since a small displacement will cause large motions of the system. Also, the point $x = 0$ corresponds to the pendulum hanging vertically downward. It is equally clear that such a position is stable based on physical considerations in spite of the fact that we could not derive this conclusion from Theorem 7.7. In fact, it is evident that all critical points of the form $(n\pi, 0)$ will be of the same type as (0, 0) for even values of n and of the type $(\pi, 0)$ for odd values of n. Thus the critical points are alternately stable (but not asymptotically stable) centers and unstable saddle points.

The trajectories of this system are shown in Figure 7.12. Close to the stable

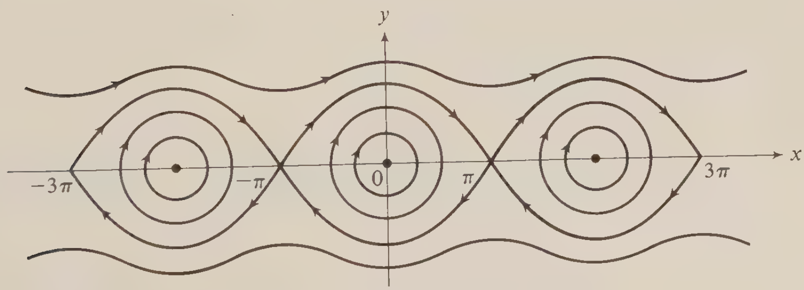


Figure 7.12 Trajectories of a pendulum.

critical points are the closed paths suggesting periodic motion. The trajectories that cross at the unstable critical points are called *separatrices*. Finally, the wavy paths outside the separatrices correspond to whirling motions of the pendulum.

For purposes of contrast with Example 26, it may be interesting to consider the effect of frictional forces acting on the pendulum. Assuming such a force is proportional to the angular velocity θ', the equation of motion becomes

$$\theta'' + c\theta' + k^2 \sin\theta = 0, \quad c > 0, \tag{82}$$

or, equivalently,

$$\begin{aligned} x' &= y, \\ y' &= -k^2 \sin x - cy, \end{aligned} \tag{83}$$

where $x = \theta$. Again we observe that the origin is a critical point, but it now becomes asymptotically stable. To see this, let us examine the associated linear system

$$\begin{aligned} x' &= y, \\ y' &= -k^2 x - cy, \end{aligned} \tag{84}$$

with auxiliary equation $\lambda^2 + c\lambda + k^2 = 0$. The roots are

$$\lambda_1, \lambda_2 = \frac{-c \pm (c^2 - 4k^2)^{1/2}}{2},$$

and hence the origin is a stable node if $c \geq 2k$, or a stable focus if $c < 2k$. According to Theorem 7.7, in all cases the origin of the nonlinear system is also asymptotically stable.

A more direct method than presented here is also available for analyzing the stability of nonlinear systems. This method, which rests upon the construction of a suitable auxiliary function, is a more powerful method in that it provides more global type of information. This method is a generalization of the physical principles associated with a conservative system and is due to A. M. Liapunov (1857–1918). The Liapunov theory, however, would take us too far afield for our purposes, so we refer the interested reader to the references.

EXERCISES 7.6

In problems 1–4, sketch the trajectory corresponding to the solution satisfying the prescribed initial conditions, and indicate the positive direction of motion.

1. $x' = -y, \quad x(0) = 3$
$y' = x, \quad y(0) = 0$

2. $x' = y, \quad x(0) = 1$
$y' = x, \quad y(0) = 0$

3. $x' = y, \quad x(0) = -1$
$y' = x, \quad y(0) = 0$

4. $x' = 2x + 4y, \quad x(0) = 4$
$y' = -2x + 6y, \quad y(0) = 0$

In problems 5–10, determine the critical points of the system.

5. $x' = y$
$y' = -x$

6. $x' = x$
$y' = x + y$

7. $x' = 3x + y$
$y' = x + 3y$

8. $x' = x + 2y - 3$
$y' = 3x + 2y + 1$

9. $x' = x - xy$
$y' = y + 2xy$

10. $x' = x - x^2 - xy$
$y' = \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy$

In problems 11–17, find the one-parameter family of trajectories by solving Equation (77).

11. $x' = y$
$y' = x + 1$

12. $x' = 2x - 2y$
$y' = 4x - 2y + 5$

13. $x' = 3x + y$
$y' = x + 3y$

14. $x' = -2x^3y$
$y' = x^4 + y^4$

15. $x' = x - y^2$
$y' = x^2 - y$

16. $x' = 2xy - \sin y$
$y' = 1 - y^2$

17. $x' = 3x^2y^2 - 3x + 4y$
$y' = y(3 - 2xy^2)$

18. Consider the linear autonomous system

$$x' = x,$$
$$y' = x + y.$$

(a) Find the solution of this system that satisfies the initial conditions $x(0) = 1$, $y(0) = 3$.

(b) Repeat (a) for $x(4) = e$, $y(4) = 4e$.

(c) Show that the two different solutions in (a) and (b) both represent the same trajectory.

In problems 19–25, determine the nature of the critical point (0, 0) and tell whether it is stable, asymptotically stable, or unstable.

19. $x' = 2x - 7y$
$y' = 3x - 8y$

20. $x' = 2x + 4y$
$y' = -2x + 6y$

21. $x' = x + 3y$
$y' = 3x + y$

22. $x' = 2x + 5y$
$y' = x - 2y$

23. $x' = x - 2y$
$y' = 4x + 5y$

24. $x' = x - y$
$y' = x + 5y$

25. $x' = x + 7y$
$y' = 3x + 5y$

***26.** Given the system

$$x' = ax - y,$$
$$y' = x + ay,$$

show that the trajectories are spirals when $a \neq 0$. Discuss the stability of the critical point $(0, 0)$ for both cases $a > 0$ and $a < 0$. What are the trajectories when $a = 0$? *Hint:* Transform to polar coordinates to show spiral trajectories.

In problems 27–32, determine the nature of the critical point (0, 0) by analyzing the related linear system.

27. $x' = x + 2y + x\cos y,$
$y' = -y - \sin y$

28. $x' = y$
$y' = -x - y^2$

29. $x' = y$
$y' = -x - y^3$

30. $x' = 3x + 4y + x^2$
$y' = 4x - 3y - 2xy$

31. $x' = x + 2y + 2\sin y$
$y' = -3y - xe^x$

32. $x' = e^{-x+y} - \cos x$
$y' = \sin(x - 3y)$

Determine the critical points in problems 33–35, and discuss their nature and stability.

33. $x' = 8x - y^2$
$y' = -6y + 6x^2$

34. $x' = 2y + x^2$
$y' = -2x - 4y$

35. $x' = 1 - xy$
$y' = x - y^3$

***36.** The *Lotka-Volterra equations* for the predator-prey problem are given by (see Section 7.1)

$$x' = ax - bxy,$$
$$y' = -cy + dxy.$$

(a) Show that the change of variables $x = cX/d$, $y = aY/b$ leads to the system
$$X' = a(X - XY),$$
$$Y' = -c(Y - XY).$$

(b) Show that the trajectories are given by
$$(e^X/X)^c = K(Y/e^Y)^a \quad (K \text{ constant}).$$

(c) The point $(0, 0)$ is obviously a critical point of the system in part (a). Find a second critical point, and discuss the stability of the system at each critical point when $a = 2$ and $c = 1$.

(d) If x and y represent two competing biological species and they start initially with "small" populations, do you expect the populations to become extinct, or can they continue to exist without the threat of extinction? Explain your answers.

REFERENCES

Birkhoff, G., and Rota, G.-C. 1978. *Ordinary differential equations.* 3rd ed. New York: Wiley.

Boyce, W. E. and DiPrima, R. C. 1977. *Elementary differential equations and boundary value problems.* 3rd ed. New York: Wiley.

Campbell, H. G. 1971. *Linear algebra with applications.* New York: Appleton-Century-Crofts.

Cullen, C. G. 1979. *Linear algebra and differential equations.* Boston, Mass.: Prindle, Weber & Schmidt.

Finney, R. L., and Ostberg, D. R. 1976. *Elementary differential equations with linear algebra.* Reading, Mass.: Addison-Wesley.

Ross, S. L. 1980. *Introduction to ordinary differential equations.* 3rd ed. New York: Wiley.

Numerical Methods 8

Numerical methods are becoming more and more important in applications, partly because of the difficulties encountered in obtaining exact analytical solutions but also, more recently, because of the ease with which numerical techniques can be used in conjunction with today's high-speed automatic computers.

Several numerical procedures for solving initial value problems involving first-order DEs are discussed briefly in Section 8.2, all of which are based upon Taylor series approximations. They include *Euler's method,* the *improved Euler's method,* and the *Runge-Kutta method.* Of these, the Runge-Kutta method is the most widely used because it is far more accurate.

These same numerical techniques are generalized in Section 8.3 to include systems of first-order equations and higher-order equations, which are solved by reducing them to a system of first-order DEs.

Although error analysis is an important part of any numerical procedure, we have limited our discussion primarily to the use of the procedure itself. The theory of errors is sometimes fairly complex and goes beyond the intended scope of this chapter. The interested reader should consult a text on numerical analysis.

8.1 INTRODUCTION

In applications we must often solve an initial value or boundary value problem that is either difficult or impossible to solve exactly by analytical methods. For this reason it becomes either convenient or necessary to employ some method that yields accurate numerical estimates of the true behavior of the system. However, the method itself can involve a considerable amount of analysis concerning the errors involved in using the method as well as errors due to rounding off in the computations. We do not intend to discuss these matters deeply, as is done in courses on numerical analysis; instead, we will simply present some techniques that provide quantitative information about certain systems we wish to study.

8.2 NUMERICAL METHODS FOR FIRST-ORDER EQUATIONS

Let us consider the initial value problem

$$y' = F(t, y), \quad y(t_0) = y_0, \tag{1}$$

where the function F is suitably "well behaved." We will assume the problem has a unique solution in some interval containing the point t_0 (see Theorem 2.1).

The methods to be discussed here are *step-by-step procedures*, wherein each calculated value makes use of the previously calculated value. We start at the initial point on the solution curve (t_0, y_0) and increase t_0 by the fixed (positive) number h to get $t_1 = t_0 + h$, and then compute a value y_1 that approximates the solution value $y(t_1)$. In the second step of the procedure, we find an approximate value y_2 for $y(t_2)$, where $t_2 = t_1 + h = t_0 + 2h$, and continue in this fashion.

The calculations are all done by the same formula at each step of the process. These formulas are suggested by the Taylor series

$$y(t + h) = y(t) + hy'(t) + \frac{1}{2}h^2y''(t) + \cdots, \tag{2}$$

where from (1) we find $y'(t) = F(t, y)$ and

$$y''(t) = \frac{d}{dt}F(t, y) = F_t(t, y) + F_y(t, y)\frac{dy}{dt},$$

and so forth. The subscript variables indicate partial differentiation with respect to the designated variable. In the first step of the procedure, we set $t = t_0$ and $y(t_o) = y_0$ to calculate

$$y_1 = y_0 + hF(t_0, y_0) + \frac{1}{2}h^2\frac{d}{dt}F(t_0, y_0) + \cdots,$$

and in the second step we calculate

$$y_2 = y_1 + hF(t_1, y_1) + \frac{1}{2}h^2\frac{dF}{dt}(t_1, y_1) + \cdots,$$

whereas, in general,

$$y_{n+1} = y_n + hF(t_n, y_n) + \frac{1}{2}h^2\frac{dF}{dt}(t_n, y_n) + \cdots. \tag{3}$$

For computational purposes, the series in (3) must be truncated after a certain number of terms. This leads to an error, which is appropriately referred to as a *truncation error*. If h is picked to be sufficiently small, terms involving h^2, h^3, and higher powers of h can often be neglected in (3), making a *first-order approximation*. The truncation error per step is then of the order h^2. When higher-order terms of (3) are required for greater accuracy, we must compute the derivatives of F. Since these derivatives must be done by hand, using (3) directly is often not feasible when F is of a complicated nature. In such instances it is usually preferable to replace the derivatives by certain numerical equivalences so that the formulas lend themselves to computer calculations. That is precisely the approach used in the *improved Euler method* and *Runge-Kutta methods* to be discussed.

8.2.1 The Euler Method

In the *Euler method* we make our calculations using the formula

$$y_{n+1} = y_n + hF(t_n, y_n), \tag{4}$$

obtained from (3) by truncating all terms involving powers of h higher than 1. Geometrically, this technique approximates the true solution curve with a polygon whose first side is tangent to the solution curve at (t_0, y_0) (see Figure 8.1). Hence it is also called the *method of tangent lines*.

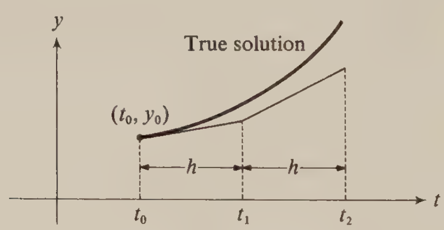


Figure 8.1

EXAMPLE 1 Use Euler's method to approximate the solution of the initial value problem

$$y' = -2ty^2, \quad y(0) = 1,$$

on the interval $0 \leqslant t \leqslant 0.5$, using $h = 0.1$ and also $h = 0.05$.

Solution We identify $F(t_n, y_n) = -2t_n y_n^2$ so that (4) becomes

$$y_{n+1} = y_n - 2ht_n y_n^2, \quad n = 0, 1, 2, \ldots.$$

For $h = 0.1$, we set $n = 0$ and calculate

$$
\begin{aligned}
n = 0: \qquad y_1 &= y_0 - 2(0.1)t_0 y_0^2 \\
&= 1 - 2(0.1)(0)(1)^2 \\
&= 1,
\end{aligned}
$$

which is our estimate to the value $y(0.1)$. Continuing, we find

$$
\begin{aligned}
n = 1: \qquad y_2 &= y_1 - 2(0.1)t_1 y_1^2 \\
&= 1 - 2(0.1)(0.1)(1)^2 \\
&= 0.98, \\
n = 2: \qquad y_3 &= y_2 - 2(0.1)t_2 y_2^2 \\
&= 0.98 - 2(0.1)(0.2)(0.98)^2 \\
&= 0.9416,
\end{aligned}
$$

and so forth. These, plus a few additional calculations as well as the exact values, are summarized in Tables 8.1 and 8.2. [By separating the variables, the exact solution is found to be $y = 1/(t^2 + 1)$.]

Table 8.1 Euler's method for $y' = -2ty^2$, $y(0) = 1$, with $h = 0.1$

t	y_n	*Exact Value*	*Error*
0.00	1.0000	1.0000	0.0000
0.10	1.0000	0.9901	0.0099
0.20	0.9800	0.9615	0.0185
0.30	0.9416	0.9174	0.0242
0.40	0.8884	0.8621	0.0263
0.50	0.8253	0.8000	0.0253

Table 8.2 Euler's method for $y' = -2ty^2$, $y(0) = 1$, with $h = 0.05$

t	y_n	*Exact Value*	*Error*
0.00	1.0000	1.0000	0.0000
0.05	1.0000	0.9975	0.0025
0.10	0.9950	0.9901	0.0049
0.15	0.9851	0.9780	0.0071
0.20	0.9705	0.9615	0.0090
0.25	0.9517	0.9412	0.0105
0.30	0.9291	0.9174	0.0117
0.35	0.9032	0.8909	0.0123
0.40	0.8746	0.8621	0.0125
0.45	0.8440	0.8316	0.0124
0.50	0.8119	0.8000	0.0119

Since we had the exact solution in Example 1, we could calculate the amount of error incurred in the method at each step of the procedure. For instance, over the interval in which we made the calculations, the *maximum percentage error* in the first case is found to be

$$\frac{|\text{error}|}{\text{exact value}} \times 100 = \frac{0.0263}{0.8621} \times 100 = 3.05\%,$$

while in the second case we have

$$\frac{0.0125}{0.8621} \times 100 = 1.45\%.$$

We might conclude that the Euler method works quite well, since the errors are acceptable in many applications. However, this kind of accuracy is usually not realized by Euler's method in practice (see Example 2).

EXAMPLE 2 Use Euler's method to solve the initial value problem

$$y' = 1 - t + 4y, \quad y(0) = 1,$$

on the interval $0 \leqslant t \leqslant 0.5$ with $h = 0.1$.

Solution Here we find $F(t_n, y_n) = 1 - t_n + 4y_n$ and thus

$$y_{n+1} = y_n + (0.1)(1 - t_n + 4y_n).$$

Our first calculations yield

$$\begin{aligned} n = 0: \quad y_1 &= y_0 + (0.1)(1 - t_0 + 4y_0) \\ &= 1 + (0.1)(1 - 0 + 4) \\ &= 1.5, \\ n = 1: \quad y_2 &= y_1 + (0.1)(1 - t_1 + 4y_1) \end{aligned}$$

Table 8.3 Euler's method for $y' = 1 - t + 4y$, $y(0) = 1$, with $h = 0.1$

t	y_n	*Exact Value*	*Error*
0.00	1.0000	1.0000	0.0000
0.10	1.5000	1.6090	0.1090
0.20	2.1900	2.5053	0.3153
0.30	3.1460	3.8301	0.6841
0.40	4.4744	5.7942	1.3192
0.50	6.3242	8.7120	2.3878

$$= 1.5 + (0.1)(1 - 0.1 + 6)$$
$$= 2.19.$$

Additional values are provided in Table 8.3 along with the exact values for comparison.

The maximum percentage error in this last example is 27.4%, significantly greater than in our previous example. We recognize, of course, that the error is controlled to some extent by the size of the increment h. By reducing the size of h we can reduce the error, but the number of calculations then increases. Also, the error normally builds up as we move farther away from the initial point on the solution curve so that it eventually exceeds what is deemed acceptable, no matter how small we choose the value of h. Therefore Euler's method is seldom used in practice even with the high-speed digital computers available today. It is introduced here primarily because it is simple and may be helpful in understanding the basic procedure used in other numerical techniques.

8.2.2 The Improved Euler Method

Euler's method can be made more accurate for a fixed value of h by first computing the auxiliary value

$$y^*_{n+1} = y_n + hF(t_n, y_n) \tag{5}$$

and then the new value

$$y_{n+1} = y_n + \frac{1}{2}h[F(t_n, y_n) + F(t_{n+1}, y^*_{n+1})]. \tag{6}$$

This technique, called the *improved Euler method* (or *Heun's method*), is an example of what is called a *predictor-corrector method*. That is, at each step we predict a value by (5) and then correct it by (6).

The geometric interpretation of this new method is that we approximate the true solution y in the interval from t_n to $t_n + \frac{1}{2}h$ by the straight line through (t_n, y_n) with slope $F(t_n, y_n)$, and then along a new line with slope $F(t_{n+1}, y^*_{n+1})$ up to t_{n+1} (see Figure 8.2). Therefore, we might interpret the sum $\frac{1}{2}[F(t_n, y_n) + F(t_{n+1}, y^*_{n+1})]$ as

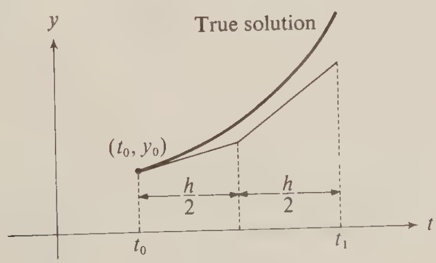


Figure 8.2

some average slope over the interval $t_n \leq t \leq t_{n+1}$. It can be shown that (6) is equivalent to a Taylor series through the term containing h^2, but it has the advantage that dF/dt does not have to be calculated.

EXAMPLE 3 Use the improved Euler formula to obtain an approximate solution of

$$y' = -2ty^2, \quad y(0) = 1.$$

Solution We will illustrate the technique for $h = 0.1$ on the interval $0 \leq t \leq 0.5$. We first calculate

$$n = 0: \qquad \begin{aligned} y_1^* &= y_0 - 2ht_0y_0^2 \\ &= 1 - 2(0.1)(0)(1)^2 \\ &= 1, \end{aligned}$$

and then

$$\begin{aligned} y_1 &= y_0 + \frac{1}{2}h[-2t_0y_0^2 - 2t_1(y_1^*)^2] \\ &= 1 - (0.1)[(0)(1)^2 + (0.1)(1)^2] \\ &= 0.99. \end{aligned}$$

The remaining calculations are provided in Table 8.4 along with those obtained by Euler's method and the exact values.

Table 8.4 Approximate solutions of $y' = -2ty^2$, $y(0) = 1$, using the Euler method and the improved Euler method with $h = 0.1$.

t	*Euler* y_n	*Improved Euler* y_n	*Exact Value*
0.00	1.0000	1.0000	1.0000
0.10	1.0000	0.9900	0.9901
0.20	0.9800	0.9614	0.9615
0.30	0.9416	0.9173	0.9174
0.40	0.8884	0.8620	0.8621
0.50	0.8253	0.8001	0.8000

Example 3 shows that the results using the improved Euler method with $h = 0.1$ are better than those obtained with the Euler method even for $h = 0.05$. In general this is the case, and although a few more calculations are required at each step using the improved Euler's method, the greater accuracy is worth the extra effort.

8.2.3 The Runge-Kutta Method

Perhaps the most commonly used as well as most accurate technique is the *Runge-Kutta method*.* At each step we must first compute four auxiliary quantities

$$\begin{aligned} k_1 &= F(t_n, y_n), \\ k_2 &= F\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right), \\ k_3 &= F\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2\right), \\ k_4 &= F(t_{n+1}, y_n + hk_3), \end{aligned} \tag{7}$$

and then calculate the new value

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \tag{8}$$

The purpose of the Runge-Kutta method is to achieve the accuracy of a Taylor series expansion without having to calculate higher-order derivatives. For example, the algorithm given in (7) and (8) was derived by obtaining appropriate constants A, B, C, and D so that

$$y_{n+1} = y_n + Ak_1 + Bk_2 + Ck_3 + Dk_4$$

agrees with the Taylor expansion out to h^4, or the fifth term of the series (see problem 26). Thus the technique is often called the *fourth-order Runge-Kutta method*. The term $(k_1 + 2k_2 + 2k_3 + k_4)/6$ appearing in (8) can be interpreted as an average slope over the interval $t_n \leqslant t \leqslant t_{n+1}$.

Although the Runge-Kutta formula is more complex than our previous formulas, it yields results that are many times more accurate than even the improved Euler formula. And it generally achieves this accuracy with larger comparative increments of h. Furthermore, with the use of a modern computer, the calculations are routine and can be performed in a few seconds for most problems.

Finally, we note that if the function F does not explicitly depend on y, then

$$k_1 = F(t_n), \quad k_2 = k_3 = F\left(t_n + \frac{1}{2}h\right), \quad k_4 = F(t_n + h),$$

and (8) can be written in the form

$$y_{n+1} - y_n = \frac{h}{6}\left[F(t_n) + 4F\left(t_n + \frac{1}{2}h\right) + F(t_n + h)\right],$$

which is recognized as Simpson's rule of integration† on the interval $t_n \leqslant t \leqslant t_n + h$.

*Named after the German mathematicians CARL RUNGE (1856–1927) and WILHELM KUTTA (1867–1944).

†Named after THOMAS SIMPSON (1710–1761).

EXAMPLE 4 Find an approximate solution to the initial value problem

$$y' = -2ty^2, \quad y(0) = 1,$$

on the interval $0 \leqslant t \leqslant 1$ using the Runge-Kutta method with $h = 0.2$.

Solution With $F(t, y) = -2ty^2$, we first set up the expressions

$$k_1 = -2t_n y_n^2,$$
$$k_2 = -2(t_n + 0.1)(y_n + 0.1k_1)^2,$$
$$k_3 = -2(t_n + 0.1)(y_n + 0.1k_2)^2,$$
$$k_4 = -2(t_{n+1})(y_n + 0.2k_3)^2.$$

For $n = 0$, our calculations yield

$$n = 0: \quad k_1 = 0, \quad k_2 = -0.2, \quad k_3 = -0.192, \quad k_4 = -0.37,$$

and

$$y_1 = 1 - \frac{0.2}{6}[2(0.2) + 2(0.192) + 0.37] = 0.9615.$$

The remaining calculations are given in Table 8.5 along with similar results obtained by the Euler methods. The improved Euler method with $h = 0.1$ is also included for comparison.

Table 8.5 Approximate solutions of $y' = -2ty^2$, $y(0) = 1$, using the Euler methods and the Runge-Kutta method with $h = 0.2$.

t	*Euler* y_n	*Improved Euler* y_n	*Runge-Kutta* y_n	*Improved Euler* $y_n(h = 0.1)$	*Exact Values*
0.00	1.0000	1.0000	1.0000	1.0000	1.0000
0.20	1.0000	0.9600	0.9615	0.9614	0.9615
0.40	0.9200	0.8603	0.8621	0.8620	0.8621
0.60	0.7846	0.7350	0.7353	0.7356	0.7353
0.80	0.6369	0.6115	0.6098	0.6104	0.6098
1.00	0.5071	0.5033	0.5000,	0.5009	0.5000

EXAMPLE 5 Find an approximate solution to the initial value problem

$$y' = 1 - t + 4y, \quad y(0) = 1,$$

on the interval $0 \leqslant t \leqslant 1$ using the Runge-Kutta method with $h = 0.1$.

Solution Here $F(t, y) = 1 - t + 4y$, which leads to $(n = 0)$

$$k_1 = F(0,1) = 5,$$

$$k_2 = F(0 + 0.05,\ 1 + 0.25) = 5.95,$$

$$k_3 = F(0 + 0.05,\ 1 + 0.2975) = 6.14,$$

$$k_4 = F(0.1,\ 1 + 0.614) = 7.356.$$

Thus,

$$y_1 = 1 + \frac{0.1}{6}[5 + 2(5.95) + 2(6.14) + 7.356]$$

$$= 1.6089.$$

A tabulation of the remaining values is given in Table 8.6 as well as the results obtained from the Euler methods. The superiority of the Runge-Kutta method is clearly demonstrated by this example.

Table 8.6 Approximate values of $y' = 1 - t + 4y$, $y(0) = 1$, using the Euler methods and the Runge-Kutta method with $h = 0.1$.

t	Euler y_n	Improved Euler y_n	Runge-Kutta y_n	Exact Values
0.00	1.0000	1.0000	1.0000	1.0000
0.10	1.5000	1.5950	1.6089	1.6090
0.20	2.1900	2.4636	2.5050	2.5053
0.30	3.1460	3.7371	3.8294	3.8301
0.40	4.4774	5.6099	5.7928	5.7942
0.50	6.3242	8.3697	8.7093	8.7120
0.60	8.9038	12.442	13.048	13.053
0.70	12.505	18.457	19.507	19.516
0.80	17.537	27.348	29.131	29.145
0.90	24.572	40.494	43.474	43.498
1.00	34.411	59.938	64.858	64.898

Remark. The Runge-Kutta method has certain drawbacks in that it requires time-consuming calculations of the function $F(t, y)$ at successive steps of the procedure. In some applications this technique may be too expensive. These calculations can be greatly reduced by use of certain *predictor-corrector methods*, such as the *Adams-Moulton method*. However, these methods create some inconveniences of their own in many cases by requiring a change in step size as the calculations proceed. The proper numerical method to choose will greatly depend upon the application and the budget allowed for making the necessary computer calculations.

EXERCISES 8.2

In problems 1–10, use the Euler formula to obtain an approximation to four places after the decimal, to the indicated value of y with $h = 0.1$.

1. $y' = 2ty,\quad y(1) = 1,\quad y(1.5) = ?$

2. $y' = 1 + y^2, y(0) = 0,\quad y(0.5) = ?$

3. $y' = (t + y - 1)^2,\quad y(0) = 2,\quad y(0.5) = ?$

4. $y' = t + y^2,\quad y(0) = 1,\quad y(0.5) = ?$

5. $y' = t^2 + y^2,\quad y(0) = 1,\quad y(0.5) = ?$

6. $y' = ty + \sqrt{y},\quad y(0) = 1,\quad y(0.5) = ?$

7. $y' = ty^2 - \dfrac{y}{t},\quad y(1) = 1,\quad y(1.5) = ?$

8. $y' = \sin(t + y),\quad y(0) = 0,\quad y(0.5) = ?$

9. $y' = e^{-y},\quad y(0) = 0,\quad y(0.5) = ?$

10. $y' = y - y^2,\quad y(0) = \dfrac{1}{2},\quad y(0.5) = ?$

11. Repeat the calculations in problems 1–10 using the improved Euler formula.

12. Repeat the calculations in problems 1–10 using the Runge-Kutta method.

13. Using Euler's formula with $h = 0.2$, find an approximate value for $y(1)$ where y is the particular solution of the initial value problem

$$y' = y,\quad y(0) = 1.$$

(Your answer approximates the value of e. Can you explain why?)

14. Solve problem 13 using the improved Euler's formula.

15. Solve problem 13 using the Runge-Kutta method.

16. Using Euler's formula with $h = 0.2$, find an approximate value for $y(1)$ where y is the particular solution of the initial value problem

$$y' = (t^2 + 1)^{-1},\quad y(0) = 0.$$

Use your answer to approximate the value of π.
Hint: The exact solution is $y = \operatorname{Arctan}(t)$.

17. Solve problem 16 using the improved Euler's formula.

18. Solve problem 16 using the Runge-Kutta method.

***19.** Derive Euler's formula (4) by integrating both sides of the DE $y' = F(t, y)$ from x_n to x_{n+1} and then approximating the integral by

$$\int_a^b f(x)\,dx \cong (b - a)f(a).$$

Give some justification for using the above approximation for the integral.

20. Using the step size $h = 0.2$, approximate the value $y(1.4)$, where y is a particular solution of the initial value problem

$$y' = t^2 + y^3,\quad y(1) = 1,$$

(a) by Euler's formula.

(b) by the improved Euler's formula.

(c) by the Runge-Kutta method.

*21. Show that the Runge-Kutta method reduces to Simpson's rule of integration on the interval $x_n \leqslant x \leqslant x_{n+1}$ when the DE is of the form $y' = F(x)$.

*22. Using the first three terms of the Taylor series (2), show that it leads to the approximation formula

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n,$$

where $y' = F(t, y)$ and $y'' = F_t(t, y) + F_y(t, y)y'$. Such a formula is called a *three-term Taylor formula*.

23. Use the three-term Taylor formula in problem 22 to approximate the value $y(0.5)$ for the solution of the initial value problem

$$y' = -2ty^2, \quad y(0) = 1,$$

using $h = 0.1$, and compare your answer with those listed in Table 8.4.

*24. Show that when $F(t, y)$ is linear in t and y, the three-term Taylor formula in problem 22 reduces to the improved Euler's formula.

*25. Derive a *four-term Taylor formula* similar to the result given in problem 22, and use it to show that $y(0.4) \cong 1.5152$, where y is a particular solution of

$$y' = t^2 + y, \quad y(0) = 1.$$

Use the step size $h = 0.1$.

*26. Prove that the fourth-order Runge-Kutta formula (8) agrees with the Taylor expansion out to h^4; i.e., show that (8) and (2) are equivalent through the fifth term of (2).

8.3 SYSTEMS OF EQUATIONS

The methods discussed in the preceding section for solving initial value problems featuring first-order DEs can be extended to a system of first-order equations. For instance, let us consider a system of two equations

$$\begin{aligned} x' &= F(t, x, y), \quad x(t_0) = x_0, \\ y' &= G(t, x, y), \quad y(t_0) = y_0. \end{aligned} \tag{9}$$

As before, we will assume that F and G are suitably behaved so that (9) has a unique solution (for linear systems, see Theorem 7.1).

For illustration purposes we will generalize the Euler method, since it is the simplest to apply. Thus, we first calculate

$$x_1 = x_0 + hF(t_0, x_0, y_0),$$

$$y_1 = y_0 + hG(t_0, x_0, y_0),$$

for some prechosen positive increment h, whereas in general our calculations involve

$$\begin{aligned} x_{n+1} &= x_n + hF(t_n, x_n, y_n), \\ y_{n+1} &= y_n + hG(t_n, x_n, y_n), \end{aligned} \tag{10}$$

where $t_n = t_0 + nh$.

EXAMPLE 6 Approximate the solutions of the system of equations

$$\begin{aligned} x' &= x - y + e^t, \quad x(0) = 1, \\ y' &= 2x + 3y + e^{-t}, \quad y(0) = 0, \end{aligned}$$

at the points $t = 0.1$ and $t = 0.2$ using the Euler method.

Solution Choosing $h = 0.1$, we first calculate

$$\begin{aligned} x_1 &= x_0 + h(x_0 - y_0 + e^{t_0}) \\ &= 1 + (0.1)(1 - 0 + 1) \\ &= 1.2 \end{aligned}$$

and

$$\begin{aligned} y_1 &= y_0 + h(2x_0 + 3y_0 + e^{-t_0}) \\ &= 0 + (0.1)[2(1) + 3(0) + 1] \\ &= 0.3. \end{aligned}$$

Similarly,

$$\begin{aligned} x_2 &= x_1 + h(x_1 - y_1 + e^{t_1}) \\ &= 1.2 + (0.1)(1.2 - 0.3 + e^{0.1}) \\ &= 1.4005 \end{aligned}$$

and

$$\begin{aligned} y_2 &= y_1 + h(2x_1 + 3y_1 + e^{-t_1}) \\ &= 0.3 + (0.1)[2(1.2) + 3(0.3) + e^{-0.1}] \\ &= 0.7205. \end{aligned}$$

For comparison purposes, the exact solution is

$$\begin{aligned} x(t) &= \frac{1}{10}e^{2t}(21\cos t - 13\sin t) - e^t - \frac{1}{10}e^{-t}, \\ y(t) &= \frac{1}{5}e^{2t}(-4\cos t + 17\sin t) + e^t - \frac{1}{5}e^{-t}, \end{aligned}$$

from which we calculate

$$x(0.1) = 1.1980, \quad y(0.1) = 0.3665,$$

$$x(0.2) = 1.3818, \quad y(0.2) = 0.8957.$$

8.3.1 Higher-Order Equations

In Chapter 7 we found that higher-order DEs can always be reduced to a system of first-order DEs. For example, the initial value problem

$$x'' = F(t, x, x'), \quad x(0) = x_0, \quad x'(0) = y_0, \tag{11}$$

is equivalent to the system of equations

$$\begin{aligned} x' &= y, \quad x(0) = x_0, \\ y' &= F(t, x, y), \quad y(0) = y_0, \end{aligned} \tag{12}$$

and thus can be solved by methods already discussed.*

EXAMPLE 7 Approximate the solution of the initial value problem

$$x'' + t^2x' + 3x = t, \quad x(0) = 1, \quad x'(0) = 2,$$

at the points $t = 0.1$ and $t = 0.2$ using the Euler method with $h = 0.1$.

Solution The equivalent system of equations is

$$\begin{aligned} x' &= y, \quad x(0) = 1, \\ y' &= t - 3x - t^2y, \quad y(0) = 2. \end{aligned}$$

From (10) we have

$$\begin{aligned} x_1 &= x_0 + hy_0 \\ &= 1 + (0.1)(2) \\ &= 1.2, \\ y_1 &= y_0 + h(t_0 - 3x_0 - t_0^2y_0) \\ &= 2 + (0.1)0 - 3(1) - 0 \\ &= 1.7, \end{aligned}$$

and

*Some numerical analysts believe that greater accuracy is achieved by applying a numerical procedure directly to the higher-order DE rather than reducing the DE to a system of first-order equations and then applying a numerical procedure. See Peter Henrici, *Discrete Variable Methods in Ordinary Differential Equations* (New York: Wiley, 1962).

$$
\begin{aligned}
x_2 &= x_1 + hy_1 \\
&= 1.2 + (0.1)(1.7) \\
&= 1.37, \\
y_2 &= y_1 + h(t_1 - 3x_1 - t_1^2 y_1) \\
&= 1.7 + (0.1)(0.1) - 3(1.2) - (0.1)^2(1.7) \\
&= 1.3483.
\end{aligned}
$$

Hence, $x(0.1) \cong 1.2$ and $x(0.2) \cong 1.37$.

EXERCISES 8.3

In problems 1–5, use the Euler method with $h = 0.1$ to determine approximate values of the solution at $t = 0.1$ and $t = 0.2$.

1. $x' = x - 4y, \quad x(0) = 1$
$y' = -x + y, \quad y(0) = 0$

2. $x' = x + y, \quad x(0) = 1$
$y' = x - y, \quad y(0) = 2$

3. $x' = 2x + ty, \quad x(0) = 1$
$y' = xy, \quad y(0) = 1$

4. $x' = x + y + t, \quad x(0) = 1$
$y' = 4x - 2y, \quad y(0) = 0$

5. $x' = -tx - y - 1, \quad x(0) = 1$
$y' = x, \quad y(0) = 1$

6. The generalizations of the Runge-Kutta method to the system of equations

$$
\begin{aligned}
x' &= F(t, x, y), \quad x(t_0) = x_0, \\
y' &= G(t, x, y), \quad y(t_0) = y_0,
\end{aligned}
$$

leads to the formulas

$$
\begin{aligned}
x_{n+1} &= x_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4), \\
y_{n+1} &= y_n + \frac{h}{6}(L_1 + 2L_2 + 2L_3 + L_4),
\end{aligned}
$$

where

$$
\begin{aligned}
K_1 &= F(t_n, x_n, y_n), \\
L_1 &= G(t_n, x_n, y_n), \\
K_2 &= F(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hK_1, y_n + \frac{1}{2}hL_1), \\
L_2 &= G(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hK_1, y_n + \frac{1}{2}hL_1), \\
K_3 &= F(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hK_2, y_n + \frac{1}{2}hL_2), \\
L_3 &= G(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hK_2, y_n + \frac{1}{2}hL_2),
\end{aligned}
$$

$$K_4 = F(t_n + h, x_n + hK_3, y_n + hL_3),$$
$$L_4 = G(t_n + h, x_n + hK_3, y_n + hL_3).$$

Use these formulas to solve problem 1 at $t = 0.1$.

7. Use the Runge-Kutta formulas in problem 6 to solve problem 2 at $t = 0.1$.

8. Use the Runge-Kutta formulas in problem 6 to solve problem 3 at $t = 0.1$.

9. Use the Runge-Kutta formulas in problem 6 to solve problem 4 at $t = 0.1$.

10. Use the Runge-Kutta formulas in problem 6 to solve problem 5 at $t = 0.1$.

Change the initial value problems 11 and 12 to a system of first-order DEs, and use the Euler method with $h = 0.1$ to determine approximate values of the exact solution at $t = 0.1$ and $t = 0.2$.

11. $x'' + tx' + x = 0, \quad x(0) = 1, \quad x'(0) = 2$

12. $x'' + t^2x' + 3x = t, \quad x(0) = 1, \quad x'(0) = 1$

13. Use the Runge-Kutta formulas in problem 6 to solve problem 11 at $t = 0.1$.

14. Use the Runge-Kutta formulas in problem 6 to solve problem 12 at $t = 0.1$.

REFERENCES

Acton, F. S. 1970. *Numerical methods that work*. New York: Harper & Row.

Burden, R. L., Faires, J. D., and Reynolds, A. C. 1978. *Numerical analysis*. Boston, Mass.: Prindle, Weber & Schmidt.

Henrici, P. 1964. *Elements of numerical analysis*. New York: Wiley.

Hildebrand, F. B. 1974. *Introduction to numerical analysis*. 2nd ed. New York: McGraw-Hill.

The Power Series Method

9

Up to this point we have been able to solve homogeneous linear DEs of order greater than 1 entirely by algebraic methods when the coefficients were constant. In fact, we were even successful in extending the technique to a special variable-coefficient DE called the Cauchy-Euler equation. Unfortunately, this method cannot (in general) be extended to other variable-coefficient DEs because their solutions typically involve nonelementary functions. Variable-coefficient equations like *Bessel's*, *Legendre's*, and the *hypergeometric equation*, along with many others, arise in numerous important engineering applications, and so other solution techniques must be found. The method to be discussed produces solutions in the form of power series, and for that reason the procedure is referred to as the *power series method*. Because of computers, this method is no longer as useful as it once was. Nevertheless, the theory is still very important since it can be used to determine regions where the solutions are analytic, and this information is essential even in the application of numerical techniques.

The general method is discussed in Section 9.2 for the case of power series expansions about *ordinary points*. For this case we always find two linearly independent solutions of second-order equations.

In Section 9.3 we first distinguish between *regular singular points* and *irregular singular points*. The *Frobenius method* for finding a solution of a DE about a regular singular point is discussed at length in terms of three separate cases depending upon the nature of the roots of the *indicial equation*. The three cases correspond to the *roots differing by a noninteger*, *equal roots*, and *roots differing by a nonzero integer*. Finding solutions about irregular singular points is not considered at all, since there is no general theory for this case.

9.1 INTRODUCTION

A power series is an infinite series of the form

$$c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x - x_0)^n, \tag{1}$$

where $c_0, c_1, \ldots, c_n, \ldots$ are called the coefficients of the series and x_0 is the center of the series. The series has the sum c_0 when $x = x_0$, but generally we are interested in whether the series also has a sum for other values of x.

Definition 9.1 *A power series* $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ *is said to* ***converge*** *for a particular value of* x *if*

$$\lim_{N\to\infty} \sum_{n=0}^{N} c_n(x - x_0)^n$$

exists. Otherwise the series is said to ***diverge***.

The values of x for which the series (1) converges is called the *interval of convergence*. That is, to each series corresponds a number R, called the *radius of convergence*, with the property that the series converges if $|x - x_0| < R$ and diverges if $|x - x_0| > R$. The radius of convergence of many power series can be found by means of the *ratio test*.

Ratio Test If

$$L = \lim_{n\to\infty} \left| \frac{c_{n+1}(x - x_0)^{n+1}}{c_n(x - x_0)^n} \right| = |x - x_0| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|,$$

then the series $\sum_{n=0}^{\infty} c_n(x - x_0)^n$ converges when $L < 1$ and diverges when $L > 1$.

The ratio test administered to the series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots,$$

for example, shows that it converges for $|x| < 1$ and diverges for $|x| > 1$.

A power series in $x - x_0$ with a positive radius of convergence R defines a function f in the interval of convergence by the rule

$$f(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n, \tag{2}$$

where $c_n = f^{(n)}(x_0)/n!$. This function is necessarily continuous and possesses derivatives of all orders everywhere in the original interval of convergence. Moreover, these derivatives can be found by termwise differentiation of (2). Familiar examples of functions with power series expansions are

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1, \tag{3}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty, \tag{4}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty, \tag{5}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty. \tag{6}$$

Many of the series of interest have centers at $x = 0$ as in (3) through (6), but not all power series do; for example,

$$\log x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n, \quad 0 < x < 2. \tag{7}$$

Observe, however, that by making the change of variable $X = x - 1$ in (7), we obtain a power series with center at $X = 0$,

$$\log(1 + X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} X^n, \quad -1 < X < 1. \tag{8}$$

For this reason much of the discussion in subsequent sections will be confined to power series for which $x_0 = 0$ without loss of generality.

9.2 THE GENERAL METHOD

From the calculus we know that a differentiable function can be expanded in a power series. Since the solution of a DE must satisfy certain differentiability requirements, we assume it too can be expressed in the form

$$y = \sum_{n=0}^{\infty} c_n x^n. \tag{9}$$

It follows that power series for y', y'', . . . can be obtained from (9) by termwise differentiation. That is,

$$y' = \sum_{n=0}^{\infty} nc_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}, \quad \ldots . \tag{10}$$

The method now is somewhat like the method of undetermined coefficients in that all that remains is to determine the constants c_n appearing in (9) in order to have the solution.

Let us illustrate the basic procedure by applying it to the simple equation

$$y' - y = 0. \tag{11}$$

The general solution of this DE is known to be $y = Ce^x$, but now we wish to find it by the method of power series. Substituting the series for y and y' into (11), we have

$$y' - y = \sum_{n=0}^{\infty} nc_n x^{n-1} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{n \to n-1} = 0.$$

To combine these two series, we must have the same exponent for x in each corresponding term, and both summation indices must start at the same value. Let us begin by replacing the index n by $(n - 1)$ in the second sum to get

$$\sum_{n=0}^{\infty} nc_n x^{n-1} - \sum_{n=1}^{\infty} c_{n-1} x^{n-1} = 0,$$

and then we can write

$$0 \cdot c_0 x^{-1} + \sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=1}^{\infty} c_{n-1} x^{n-1} = 0,$$

or, adding the two series termwise,

$$0 \cdot c_0 x^{-1} + \sum_{n=1}^{\infty} (nc_n - c_{n-1}) x^{n-1} = 0. \tag{12}$$

Now, since x can take on various values, the coefficient of x^{n-1} must vanish in order to have (12) identically zero; i.e.,

$$nc_n - c_{n-1} = 0, \quad n = 1, 2, 3, \ldots, \tag{13}$$

while c_0 can remain arbitrary since the term in which it appears is already zero. The relationship (13) is called a *recurrence formula* for the unknown coefficients. Since $n \neq 0$, we can rewrite (13) as

$$c_n = \frac{c_{n-1}}{n}, \quad n = 1, 2, 3, \ldots. \tag{14}$$

Successively substituting $n = 1, 2, 3, \ldots$ into (14) yields

$$c_1 = c_0,$$

$$c_2 = \frac{1}{2}c_1 = \frac{1}{2}c_0,$$

$$c_3 = \frac{1}{3}c_2 = \frac{c_0}{3 \cdot 2} = \frac{c_0}{3!},$$

$$c_4 = \frac{1}{4}c_3 = \frac{c_0}{4 \cdot 3!} = \frac{c_0}{4!},$$

$$\cdots\cdots\cdots\cdots\cdots$$

Hence, from (9) we are able to write

$$\begin{aligned} y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \\ &= c_0 + c_0 x + \frac{c_0}{2!}x^2 + \frac{c_0}{3!}x^3 + \frac{c_0}{4!}x^4 + \cdots \\ &= c_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right), \end{aligned}$$

or

$$y = c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

In this case we recognize the infinite series as that of e^x and can therefore write the general solution as

$$y = c_0 e^x, \tag{15}$$

in agreement with our previous result.

Although the above example was quite elementary, it illustrates the basic manipulations required to solve a DE by the power series method. Even when the coefficients of the DE are polynomials in x rather than constants, the calculations required differ little from those used in this simple example.

The power series method does not provide a general solution to all variable-coefficient DEs, even for the special case when the coefficients are polynomials in x. Before discussing the method any further, however, let us first clarify the two rules of manipulation that were used in the above example and will be used in all future examples.

Rule 1: When making an index change, always make all exponents on x equal to the smallest one occurring in the various series.

Rule 2: When combining series under one summation sign, start all series with the largest of all the beginning values. Terms preceding the new first value of the summations must then be added outside the summation sign.

9.2.1 Ordinary and Singular Points

Consider the second-order DE

$$A_2(x)y'' + A_1(x)y' + A_0(x)y = 0, \tag{16}$$

where the coefficients $A_0(x)$, $A_1(x)$, and $A_2(x)$ are any polynomials with no common factors. The method of solution and the behavior of the solutions of (16) at a point $x = x_0$ depends upon whether this point is an *ordinary point* or a *singular point* of the equation.

Definition 9.2 *A point $x = x_0$ is called an* ***ordinary point*** *of the DE*

$$A_2(x)y'' + A_1(x)y' + A_0(x)y = 0$$

provided $A_2(x_0) \neq 0$. If $A_2(x_0) = 0$, we say that $x = x_0$ is a ***singular point*** *of the equation.*

Remark. When classifying points as singular or ordinary, complex values must be considered as well. Also, infinite values can be included, although we will limit our discussion to finite values.

EXAMPLE 1 Identify all singular points of the DEs

$$\text{(a)}\quad (1 - x^2)y'' - 2xy' + 2y = 0,$$

$$\text{(b)}\quad xy'' + y' + xy = 0, \text{ and}$$

$$\text{(c)}\quad (x^2 + 4)y'' + xy' - 2y = 0.$$

Solution In (a), we set $1 - x^2 = 0$ and find $x = \pm 1$ as the singular points. All other points, real and complex, are ordinary points.

The only singular point in (b) is $x = 0$, and thus all others are ordinary points.

The DE in (c) has singular points at the solutions of $x^2 + 4 = 0$, or $x = \pm 2i$. Again, all other points are ordinary.

More generally, if the coefficients in (16) are not polynomials, we first rewrite the equation in normal form

$$y'' + a(x)y' + b(x)y = 0, \tag{17}$$

and then define *ordinary point* as one for which both $a(x)$ and $b(x)$ are *analytic*. If one or both of these functions fails to be analytic at the point $x = x_0$, we say that point is a *singular point*.

Remark. A function analytic at $x = x_0$ has a power series expansion about $x = x_0$ with a positive radius of convergence.

EXAMPLE 2 Determine whether the DE $2xy'' + (\sin x)y = 0$ has any singular points.

Solution Writing the equation in normal form

$$y'' + \left(\frac{\sin x}{2x}\right)y = 0,$$

we see that all points are ordinary points, including $x = 0$, since $(\sin x)/2x$ has the power series expansion

$$\frac{\sin x}{2x} = \frac{1}{2}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right),$$

which converges for all x and therefore must be an analytic function.

In the remaining sections of this chapter, we will confine our discussion to only those DEs that have polynomial coefficients. Furthermore, we will present the theory only for second-order DEs and make no attempt at generalizing to higher-order equations.

9.2.2 Solutions Near an Ordinary Point

For most problems we are mainly concerned with whether the point $x = 0$ is an ordinary point or a singular point of the DE, since this point is the easiest point at which to apply the power series method. If $x = 0$ is an ordinary point the following theorem is applicable. We state it without proof.

Theorem 9.1 If $x = 0$ is an ordinary point of the equation

$$A_2(x)y'' + A_1(x)y' + A_0(x)y = 0,$$

then there exists two linearly independent solutions, each of the form

$$y = \sum_{n=0}^{\infty} c_n x^n.$$

Moreover, these series will converge at least in the interval $|x| < R$, where R is the distance from 0 to the nearest singular point.

Remark. If the nearest singular point is complex, then we define the modulus of this point as the distance from the origin. For example, the modulus of $1 - 2i$ is $|1 - 2i| = \sqrt{5}$.

Theorem 9.1 is easily generalized to any ordinary point $x = x_0$. The series solution in such a case will then be of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

EXAMPLE 3 Solve $(1 - x^2)y'' - 2xy' + 2y = 0$.*

Solution Since $x = 0$ is an ordinary point, we assume

$$y = \sum_{n=0}^{\infty} c_n x^n,$$

from which we formally obtain

$$y' = \sum_{n=0}^{\infty} nc_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}.$$

Replacing y, y', and y'' in the DE with their series, we have

$$(1 - x^2)\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=0}^{\infty} nc_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0,$$

*This DE is a special case of *Legendre's equation*.

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} - \underbrace{\sum_{n=0}^{\infty} n(n-1)c_n x^n}_{n \to n-2} - \underbrace{\sum_{n=0}^{\infty} 2nc_n x^n}_{n \to n-2} + \underbrace{\sum_{n=0}^{\infty} 2c_n x^n}_{n \to n-2} = 0.$$

Now let us replace n by $(n - 2)$ in the last three sums to get

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} (n-2)(n-3)c_{n-2} x^{n-2} - \sum_{n=2}^{\infty} 2(n-2)c_{n-2} x^{n-2} + \sum_{n=2}^{\infty} 2c_{n-2} x^{n-2} = 0$$

or

$$0 \cdot c_0 x^{-2} + 0 \cdot c_1 x^{-1} + \sum_{n=2}^{\infty} [n(n-1)c_n - (n-2)(n-3)c_{n-2} - 2(n-2)c_{n-2} + 2c_{n-2}]x^{n-2} = 0.$$

Clearly, c_0 and c_1 are arbitrary, but the remaining constants must satisfy

$$n(n-1)c_n - [(n-2)(n-3) + 2(n-2) - 2]c_{n-2} = 0.$$

This last expression can be simplified to

$$c_n = \left(\frac{n-3}{n-1}\right) c_{n-2}, \quad n = 2, 3, 4, \ldots,$$

which is our recurrence formula.

Setting $n = 2, 3, 4, \ldots$ into the recurrence formula yields

$$\begin{aligned}
c_2 &= -c_0, \\
c_3 &= 0, \\
c_4 &= \frac{1}{3}c_2 = -\frac{1}{3}c_0, \\
c_5 &= \frac{2}{4}c_3 = 0, \\
c_6 &= \frac{3}{5}c_4 = -\frac{1}{5}c_0, \\
c_7 &= \frac{4}{6}c_5 = 0, \\
c_8 &= \frac{5}{7}c_6 = -\frac{1}{7}c_0, \\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

Thus we get

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$

$$= c_0 + c_1 x - c_0 x^2 - \frac{c_0}{3}x^4 - \frac{c_0}{5}x^5 - \frac{c_0}{7}x^7 - \cdots$$

$$= c_0\left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \frac{1}{7}x^8 - \cdots\right) + c_1 x,$$

which has the form

$$y = c_0 y_1(x) + c_1 y_2(x).$$

If the pattern in each series is clear, it is sometimes useful to write the solutions in summation notation. For instance, in the above example we can write

$$y_1(x) = 1 - \sum_{n=1}^{\infty} \frac{x^{2n}}{2n - 1},$$

whereas $y_2(x) = x$ is already in this form. The ratio test can readily be applied to solutions written in this form to determine the radius of convergence. In the present example it is easy to verify that the series converges for $|x| < 1$.

Remark. In some cases the interval of convergence extends beyond the closest singularity.

EXAMPLE 4 Solve $y'' + xy' + y = 0, \quad y(0) = 3, \quad y'(0) = -7.$

Solution Again we see that $x = 0$ is an ordinary point. Thus we have

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=0}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0,$$

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \underbrace{\sum_{n=0}^{\infty} nc_n x^n}_{n \to n-2} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{n \to n-2} = 0.$$

Reindexing to obtain equal powers of x, we get

$$\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} (n-2)c_{n-2} x^{n-2} + \sum_{n=2}^{\infty} c_{n-2} x^{n-2} = 0$$

or

$$0 \cdot c_0 x^{-2} + 0 \cdot c_1 x^{-1} + \sum_{n=2}^{\infty} [n(n-1)c_n + (n-2)c_{n-2} + c_{n-2}]x^{n-2} = 0.$$

We now see that c_0 and c_1 are arbitrary, and the recurrence formula is

$$n(n-1)c_n + (n-2)c_{n-2} + c_{n-2} = 0,$$

or, upon simplification,

$$c_n = -\frac{1}{n}c_{n-2}, \quad n = 2, 3, 4, \ldots .$$

Therefore,

$$c_2 = -\frac{1}{2}c_0,$$

$$c_3 = -\frac{1}{3}c_1,$$

$$c_4 = -\frac{1}{4}c_2 = \frac{c_0}{2 \cdot 4},$$

$$c_5 = -\frac{1}{5}c_3 = \frac{c_1}{3 \cdot 5},$$

$$c_6 = -\frac{1}{6}c_4 = -\frac{c_0}{2 \cdot 4 \cdot 6},$$

$$c_7 = -\frac{1}{7}c_5 = -\frac{c_1}{3 \cdot 5 \cdot 7},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Hence the power series for y is

$$\begin{aligned} y &= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \cdots \\ &= c_0\left(1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \cdots\right) + c_1\left(x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} - \frac{x^7}{3 \cdot 5 \cdot 7} + \cdots\right), \end{aligned}$$

or

$$y = c_0y_1(x) + c_1y_2(x).$$

Although it is not obvious, the first series can be written in terms of an elementary function. That is,

$$\begin{aligned} y_1(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} - \frac{x^6}{2 \cdot 4 \cdot 6} + \cdots \\ &= 1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2!}\left(-\frac{x^2}{2}\right)^2 + \frac{1}{3!}\left(-\frac{x^2}{2}\right)^3 + \cdots, \end{aligned}$$

or

$$y_1(x) = e^{-x^2/2}.$$

This is not the case for the second series, so it must be left as an infinite series. Hence we write

$$y = c_0e^{-x^2/2} + c_1\left(x - \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} - \frac{x^7}{3 \cdot 5 \cdot 7} + \cdots\right).$$

Imposing the first initial condition $y(0) = 3$ on this general solution, we immediately find that

$$c_0 = 3.$$

In order to impose the second initial condition $y'(0) = -7$, we first calculate

$$y' = -c_0 x e^{-x^2/2} + c_1\left(1 - x + \frac{x^4}{3} - \frac{x^6}{3\cdot 5} + \cdots\right),$$

from which it follows that

$$c_1 = -7.$$

Thus the solution of the initial value problem is (for all x)

$$y = 3e^{-x^2/2} - 7\left(x - \frac{x^3}{3} + \frac{x^5}{3\cdot 5} - \frac{x^7}{3\cdot 5\cdot 7} + \cdots\right).$$

If the initial values of y and y' in Example 4 were prescribed at some point other than $x = 0$, then the solution technique would be altered. For example, suppose we wish to solve the initial value problem

$$y'' + xy' + y = 0, \quad y(1) = 3, \quad y'(1) = -7. \tag{18}$$

Since the initial conditions are prescribed at $x = 1$, it would be best to seek solutions in powers of $(x - 1)$ rather than in powers of x; that is, seek solutions in the form

$$y = \sum_{n=0}^{\infty} c_n(x - 1)^n. \tag{19}$$

To do this, we set $X = x - 1$ in (18), which leads to the equivalent initial value problem

$$y'' + (X + 1)y' + y = 0, \quad y(0) = 3, \quad y'(0) = -7, \tag{20}$$

where the primes now denote differentiation with respect to X. Setting $X = x - 1$ also in (19), we can now proceed as before.

Also, in some cases the substitution $X = x - x_0$ is used to simplify the algebraic structure of the DE. Consider the next example.

EXAMPLE 5 Find a power series solution of

$$y'' + (x - 1)^2 y' - 4(x - 1)y = 0$$

about the point $x = 1$.

Solution We are seeking a power series solution expressed in powers of $(x - 1)$. To simplify matters, let us make the translation of axes $X = x - 1$, which changes the DE to

$$y'' + X^2 y - 4Xy = 0,$$

where the primes now denote differentiation with respect to X.

Assuming

$$y = \sum_{n=0}^{\infty} c_n X^n,$$

we find that

$$\sum_{n=0}^{\infty} n(n-1)c_n X^{n-2} + \underbrace{\sum_{n=0}^{\infty} nc_n X^{n+1}}_{n \to n-3} - \underbrace{\sum_{n=0}^{\infty} 4c_n X^{n+1}}_{n \to n-3} = 0,$$

and with a shift in index, this expression becomes

$$0 \cdot c_0 X^{-2} + 0 \cdot c_1 X^{-1} + 2c_2 X^0 + \sum_{n=3}^{\infty} [n(n-1)c_n + (n-3)c_{n-3} - 4c_{n-3}]X^{n-2} = 0.$$

Once more c_0 and c_1 are arbitrary, but we must set $c_2 = 0$. The remaining constants are then determined from the recurrence formula

$$n(n-1)c_n + (n-7)c_{n-3} = 0$$

or

$$c_n = -\frac{(n-7)}{n(n-1)} c_{n-3}, \quad n = 3, 4, 5, \ldots .$$

Calculating the first few constants, we see that $C_2 =$

$$c_3 = -\frac{(-4)}{3 \cdot 2} c_0 = \frac{4}{3 \cdot 2} c_0,$$

$$c_4 = -\frac{(-3)}{4 \cdot 3} c_1 = \frac{3}{4 \cdot 3} c_1,$$

$$c_5 = -\frac{(-2)}{5 \cdot 4} c_2 = 0,$$

$$c_6 = -\frac{(-1)}{6 \cdot 5} c_3, = \frac{4}{6 \cdot 5 \cdot 3 \cdot 2} c_0,$$

$$c_7 = 0,$$

$$c_8 = 0,$$

$$c_9 = -\frac{2}{9 \cdot 8} c_6 = -\frac{2 \cdot 4}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} c_0,$$

$$c_{10} = -\frac{3}{10 \cdot 9} c_7 = 0,$$

$$c_{11} = 0,$$

and so on. Thus,

$$y = c_0 + c_1X + c_2X^2 + c_3X^3 + c_4X^4 + \cdots$$
$$= c_0\left(1 + \frac{4}{3\cdot 2}X^3 + \frac{4}{6\cdot 5\cdot 3\cdot 2}X^6 - \frac{2\cdot 4}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}X^9 + \cdots\right) + c_1\left(X + \frac{1}{4}X^4\right),$$

or, since $X = x - 1$,

$$y = c_0\left[1 + \frac{4}{3\cdot 2}(x-1)^3 + \frac{4}{6\cdot 5\cdot 3\cdot 2}(x-1)^6 - \frac{2\cdot 4}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}(x-1)^9 + \cdots\right]$$
$$+ c_1\left[(x-1) + \frac{1}{4}(x-1)^4\right].$$

Because the original DE has no singular point in the finite plane, this solution is valid for all finite values of x.

In some cases the recurrence formula involves more than two terms in the unknown coefficients. Consider the next example.

EXAMPLE 6 Solve $y'' + (1 + x)y = 0$.

Solution Making the appropriate substitutions for y, y', and y'', we find after simplification

$$\sum_{n=0}^{\infty} n(n-1)c_nx^{n-2} + \underbrace{\sum_{n=0}^{\infty} c_nx^n}_{n\to n-2} + \underbrace{\sum_{n=0}^{\infty} c_nx^{n+1}}_{n\to n-3} = 0,$$

$$\sum_{n=0}^{\infty} n(n-1)c_nx^{n-2} + \sum_{n=2}^{\infty} c_{n-2}x^{n-2} + \sum_{n=3}^{\infty} c_{n-3}x^{n-2} = 0,$$

$$0\cdot c_0x^{-2} + 0\cdot c_1x^{-1} + (2c_2 + c_0)x^0 + \sum_{n=3}^{\infty}[n(n-1)c_n + c_{n-2} + c_{n-3}]x^{n-2} = 0.$$

Thus, c_0 and c_1 are arbitrary, and also

$$c_2 = -\frac{c_0}{2},$$

$$c_n = -\frac{c_{n-2} + c_{n-3}}{n(n-1)}, \quad n = 3, 4, 5, \ldots.$$

To simplify matters here and still obtain two linearly independent solutions, let us first set $c_1 = 0$ and let c_0 remain arbitrary. Therefore

$$c_2 = -\frac{1}{2}c_0,$$

$$c_3 = -\frac{c_1 + c_0}{3\cdot 2} = -\frac{c_0}{3!},$$

$$c_4 = -\frac{c_2 + c_1}{4\cdot 3} = -\frac{c_0}{4!},$$

$$c_5 = -\frac{c_3 + c_2}{5\cdot 4} = \frac{4c_0}{5!},$$

$$\cdots\cdots\cdots\cdots\cdots,$$

and so one solution is

$$y = c_0\left(1 - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{4x^5}{5!} + \cdots\right) = c_0 y_1(x).$$

Similarly, if we set $c_0 = 0$ and leave c_1 arbitrary, we find

$$c_2 = 0,$$

$$c_3 = -\frac{c_1 + c_0}{3\cdot 2} = -\frac{c_1}{3!},$$

$$c_4 = -\frac{c_2 + c_1}{4\cdot 3} = -\frac{2c_1}{4!},$$

$$c_5 = -\frac{c_3 + c_2}{5\cdot 4} = \frac{c_1}{5!},$$

$$\cdots\cdots\cdots\cdots\cdots,$$

giving the second solution

$$y = c_1\left(x - \frac{x^3}{3!} - \frac{2x^4}{4!} + \frac{x^5}{5!} + \cdots\right) = c_1 y_2(x).$$

The general solution is therefore (for all x)

$$y = c_0 y_1(x) + c_1 y_2(x).$$

EXERCISES 9.2

In problems 1–4, make an appropriate index change to rewrite the series as directed.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{n(n+3)}{n!} x^{n-2}$ as a series in x^n.

2. $\displaystyle\sum_{n=0}^{\infty} \frac{3(n+1)}{n!} x^{n+1}$ as a series in x^n.

3. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ as a series in x^{2n-1}.

4. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2)(4)\cdots(2n)} x^{n-1}$ as a series in x^{n+1}.

In problems 5–8, list all of the singular points in the finite plane.

5. $(x^2+1)y'' - 6xy' + y = 0$

6. $x(3+x)y'' - (3+x)y' + 2xy = 0$

7. $5y'' + 6xy' - x^2y = 0$

8. $y'' - \frac{4}{x(1+4x^2)}y' + \frac{1}{x^2(1+4x^2)}y = 0$

In problems 9–12, find a power series solution about $x = 0$ for the given first-order DE.

9. $(1+x)y' + y = 0$

10. $y' + 2xy = 0, \quad y(0) = 3$

***11.** $y' + (\sin x)y = 0$

Hint: Expand $\sin x$ in a power series about $x = 0$ and find only the first *three* nonzero terms in the general solution.

12. $y' + y = x$

In problems 13–22, find two linearly independent power series solutions around the origin and state the region of validity as predicted by Theorem 9.1.

13. $y'' + 4y = 0$

14. $y'' - 4y = 0$

15. $(1+4x^2)y'' - 8y = 0$

16. $y'' + 2xy' + 5y = 0$

17. $(1-x^2)y'' - 2xy' + 12y = 0$

18. $(x^2+4)y'' + 2xy - 12y = 0$

19. $y'' + xy' + (x^2+2)y = 0$

20. $y'' + x^2y' + 3xy = 0$

21. $(x-1)y'' + y' = 0$

22. $(x+3)y'' + (x+2)y' + y = 0$

In the initial value problems 23–26, find the first four *nonzero terms in the power series expansion about* x = 0 *of the solution.*

23. $y'' - xy' - y = 0, \quad y(0) = 1, \quad y'(0) = 0$

24. $y'' + xy' - 2y = 0, \quad y(0) = 0, \quad y'(0) = 1$

25. $(x^2-1)y'' + 3xy' + xy = 0, \quad y(0) = 4, \quad y'(0) = 6$

***26.** $3y'' - y' + (x+1)y = 1, \quad y(0) = 0, \quad y'(0) = 0$

In problems 27–29, find the power series expansions about x = 1 *of the DE.*

27. $y'' + (x-1)y = 0$

***28.** $xy'' + y' + xy = 0, \quad y(1) = 0, \quad y'(1) = -1$

29. $(x^2-2x+2)y'' - 4(x-1)y' + 6y = 0$

***30.** Find the first four nonzero terms in the power series solution of

$$y'' - 4xy' - 4y = e^x.$$

Hint: Expand e^x in a power series about $x = 0$.

In problems 31 and 32, find the power series expansion about $x = 0$ of the general solution.

***31.** $y''' - 3xy' - y = 0$

***32.** $y''' + x^2y'' + 5xy' + 3y = 0$

***33.** The DE

$$y'' - 2xy' + 2ny = 0, \quad n \geq 0,$$

is called *Hermite's equation*.* Obtain two linearly independent solutions for the cases when

(a) $n = 1$.

(b) $n = 4$.

(c) Show that when n is any nonnegative integer, one solution of Hermite's DE is always a polynomial of degree n.

***34.** The DE

$$(1 - x^2)y'' - xy' + n^2y = 0, \quad n \geq 0,$$

is called *Chebyshëv's equation*.† Obtain two linearly independent solutions for the cases when

(a) $n = 1$.

(b) $n = 4$.

(c) Show that when n is any nonnegative integer, one solution of Chebyshëv's DE is always a polynomial of degree n.

9.3 SOLUTIONS NEAR A REGULAR SINGULAR POINT

The solution of a DE in the neighborhood of a singular point usually exhibits some type of peculiar behavior, and for this reason the behavior of the physical system governed by such an equation is frequently most interesting around this singular point. Therefore, rather than avoiding the singular points in our solution technique, we need to investigate precisely these points in many situations.

Unfortunately, when $x = 0$ is a singular point of the DE, it may not be possible to find a power series solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n.$$

*Named after the French mathematician CHARLES HERMITE (1822–1901). Hermite showed in 1858 that the general fifth-degree equation can be solved by elliptic functions and proved in 1873 that e is a transcendental number (i.e., a type of irrational number). He made many contributions to the fields of number theory and analysis.

†Named after the Russian mathematician PAFNUTI LVOVICH CHEBYSHËV (1821–1894). Much of his work involved prime numbers, but he is also known for his work in probability, the theory of numbers, and the approximation of functions. Other spellings of his name include *Tchebysheff* and *Tchebycheff*,

Under proper conditions, however, we may be able to find a solution of the more general form

$$y = x^s \sum_{n=0}^{\infty} c_n x^n, \tag{21}$$

where s is an unknown parameter to be determined.

In general, a singular point of the equation

$$y'' + a(x)y' + b(x)y = 0 \tag{22}$$

has been defined as one for which either (or both) $a(x)$ or $b(x)$ fails to be analytic. Singular points are further classified as *regular* or *irregular* according to the following definition.

Definition 9.3 *If $x = x_0$ is a singular point of* (22), *it is classified as a* ***regular singular point*** *(R.S.P.) if both*

$$(x - x_0)a(x) \quad \text{and} \quad (x - x_0)^2 b(x)$$

are analytic functions at $x = x_0$; otherwise, we say that $x = x_0$ is an ***irregular singular point*** *(I.S.P.).*

Remark. In the special case when $a(x)$ and $b(x)$ are rational functions reduced to lowest terms, an R.S.P. is a singular point for which the factor $(x - x_0)$ appears at most to the first power in the denominator of $a(x)$ and at most to the second power in the denominator of $b(x)$.

EXAMPLE 7 Classify the singular points of

$$xy'' + y' + xy = 0.$$

Solution Clearly, $x = 0$ is the only singular point, and dividing the DE by x shows that

$$a(x) = \frac{1}{x} \quad \text{and} \quad b(x) = 1.$$

Thus, since both $xa(x) = 1$ and $x^2 b(x) = x^2$ are analytic at $x = 0$, we conclude that $x = 0$ is an R.S.P.

EXAMPLE 8 Classify the singular points of

$$x(x - 1)^2(x + 3)y'' + 5x^2y' + (2x^3 + 1)y = 0.$$

Solution The singular points are $x = 0, 1, -3$. Putting the DE in normal form identifies

$$a(x) = \frac{5x}{(x - 1)^2(x + 3)}$$

and

$$b(x) = \frac{2x^3 + 1}{x(x - 1)^2(x + 3)}.$$

Therefore, since x and $(x + 3)$ appear at most to the first power in the denominators of $a(x)$ and $b(x)$, it is clear that $x = 0$ and $x = -3$ are both R.S.P.'s. Likewise, since the factor $(x - 1)$ appears to the second power in the denominator of $a(x)$, $x = 1$ is an I.S.P.

9.3.1 The Method of Frobenius

In 1873, a method was published by the German mathematician G. Frobenius* for finding a solution of a DE about a regular singular point that is based upon the following theorem.

Theorem 9.2 If $x = 0$ is an R.S.P. of the DE

$$y'' + a(x)y' + b(x)y = 0,$$

then at least one solution of the form

$$y = x^s \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+s}, \quad x > 0,$$

always exists with a positive radius of convergence.

In finding a solution of the form suggested in Theorem 9.2, it is necessary to determine values of the parameter s in addition to the coefficients $c_0, c_1, c_2, \ldots$. The restriction $x > 0$ is necessary to prevent complex solutions that might arise for certain values of s. If we need a solution that is valid for $x < 0$, we can make the simple change of variable $X = -x$ and solve the resulting DE for $X > 0$.

If we formally differentiate the series

$$y = \sum_{n=0}^{\infty} c_n x^{n+s},$$

we find

$$y' = \sum_{n=0}^{\infty} (n + s)c_n x^{n+s-1}$$

*GEORG F. FROBENIUS (1849–1917) is known for his research in algebra and analysis. In addition to his infinite series solution of a DE at a regular singular point, he made great contributions to the theory of groups.

and

$$y'' = \sum_{n=0}^{\infty} (n + s)(n + s - 1)c_n x^{n+s-2}.$$

Using these expressions for y, y' and y'', we can rewrite

$$y'' + a(x)y' + b(x)y = 0$$

as

$$\sum_{n=0}^{\infty} (n + s)(n + s - 1)c_n x^{n+s-2} + a(x)\sum_{n=0}^{\infty} (n + s)c_n x^{n+s-1} + b(x)\sum_{n=0}^{\infty} c_n x^{n+s} = 0$$

or, more compactly, as

$$\sum_{n=0}^{\infty} [(n + s)(n + s - 1) + xa(x)(n + s) + x^2 b(x)]c_n x^{n+s-2} = 0. \tag{23}$$

Assuming $x = 0$ is an R.S.P., both $xa(x)$ and $x^2b(x)$ have power series expansions; i.e.,

$$xa(x) = a_0 + a_1x + a_2x^2 + \cdots,$$

$$x^2b(x) = b_0 + b_1x + b_2x^2 + \cdots.$$

If we substitute these expressions into (23), the smallest exponent of x occurring in (23) is $(s - 2)$, corresponding to $n = 0$. Let us separate this term from the rest and write

$$[s(s - 1) + a_0s + b_0]c_0x^{s-2} + \sum_{n=1}^{\infty} [(n + s)(n + s - 1)$$

$$+ (a_1x + a_2x^2 + \cdots)(n + s) + (b_1x + b_2x^2 + \cdots)]c_n x^{n+s-2} = 0.$$

As before, this equation can be satisfied only if the coefficients of all powers of x vanish independently. This yields a system of equations in the unknown constants c_n. In particular, we must set

$$[s(s - 1) + a_0s + b_0]c_0 = 0.$$

But in order to obtain at least one solution of the DE, we must be assured of at least one arbitrary constant. We select this arbitrary constant to be c_0 and therefore set its coefficient to zero, giving us

$$s^2 + (a_0 - 1)s + b_0 = 0. \tag{24}$$

This important quadratic equation is called the *indicial equation* of the DE (21). There are two roots of the indicial equation, s_1 and s_2, leading to three different procedures for generating a second linearly independent solution, depending on the nature of the roots:

Case I—Roots differing by a noninteger
Case II—Equal roots
Case III—Roots differing by a nonzero integer

We will consider the three cases separately under the assumption that s_1 and s_2 are real solutions of the indicial equation (24). Although these roots can be complex, we will not discuss this case.

9.3.2 Roots Differing by a Noninteger

This case always leads to two linearly independent solutions of the form

$$y_1 = x^{s_1}\sum_{n=0}^{\infty} c_n(s_1)x^n, \quad c_0(s_1) = 1, \tag{25}$$

and

$$y_2 = x^{s_2}\sum_{n=0}^{\infty} c_n(s_2)x^n, \quad c_0(s_2) = 1. \tag{26}$$

These solutions are clearly linearly independent, since y_1/y_2 cannot be constant for our choice of s_1 and s_2. The two sets of coefficients $c_n(s_1)$ and $c_n(s_2)$ are obtained independently by replacing s by s_1 and s by s_2, respectively, in the recurrence formula.

EXAMPLE 9 Solve $2xy'' + (1 - 2x)y' - y = 0$.

Solution We first observe that $x = 0$ is an R.S.P. Next, substituting the infinite series expressions for y, y', and y'', we find

$$\sum_{n=0} 2(n + s)(n + s - 1)c_n x^{n+s-1} + \sum_{n=0}^{\infty} (n+s)c_n x^{n+s-1}$$
$$- \underbrace{\sum_{n=0}^{\infty} (n + s)c_n x^{n+s}}_{n\to n-1} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+s}}_{n\to n-1} = 0.$$

Making the appropriate change of indices and combining all terms under one summation, we are led to

$$[2s(s - 1) + s]c_0 x^{s-1} + \sum_{n=1}^{\infty} [2(n + s)(n + s - 1)c_n$$
$$+ (n + s)c_n - 2(n + s - 1)c_{n-1} - c_{n-1}]x^{n+s-1} = 0.$$

The coefficient of c_0 equated to zero gives the indicial equation

$$2s^2 - s = 0^*$$

with roots $s_1 = 0$ and $s_2 = \frac{1}{2}$. The recurrence formula is

$$(n + s)[2(n + s) - 1]c_n = [2(n + s) - 1]c_{n-1},$$

or, since $2(n + s) - 1 \neq 0$ for either choice of s,

$$(n + s)c_n = c_{n-1}, \quad n = 1, 2, 3, \ldots .$$

Now putting $s = 0$ in this recurrence formula and setting $c_0 = 1$ for convenience, we find $c_n = c_{n-1}/n$, which gives

$$\begin{aligned} c_0 &= 1, \\ c_1 &= c_0 = 1, \\ c_2 &= \frac{c_1}{2} = \frac{1}{2}, \\ c_3 &= \frac{c_2}{3} = \frac{1}{3!}, \\ c_4 &= \frac{c_3}{4} = \frac{1}{4!}, \\ &\ldots\ldots\ldots\ldots, \end{aligned}$$

leading to the first solution

$$y_1(x) = x^0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = e^x.$$

Similarly, for $s = \frac{1}{2}$ we get $c_n = 2c_{n-1}/(2n + 1)$, i.e.,

$$\begin{aligned} c_0 &= 1, \\ c_1 &= \frac{2c_0}{3} = \frac{2}{3}, \\ c_2 &= \frac{2c_1}{5} = \frac{2^2}{3 \cdot 5}, \\ c_3 &= \frac{2c_2}{7} = \frac{2^3}{3 \cdot 5 \cdot 7}, \\ &\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

and thus

$$y_2(x) = x^{1/2}\left(1 + \frac{2x}{3} + \frac{2^2x^2}{3 \cdot 5} + \frac{2^3x^3}{3 \cdot 5 \cdot 7} + \cdots\right).$$

The general solution is therefore

*Observe that we could find the indicial equation directly from (24) by noting that $xa(x) = \frac{1}{2} - x$ and $x^2b(x) = -x/2$, and hence $a_0 = \frac{1}{2}$ and $b_0 = 0$.

$$
\begin{aligned}
y &= Ay_1(x) + By_2(x) \\
&= Ae^x + Bx^{1/2}\left(1 + \frac{2x}{3} + \frac{2^2x^2}{3\cdot 5} + \frac{2^3x^3}{3\cdot 5\cdot 7} + \cdots\right),
\end{aligned}
$$

where A and B are arbitrary constants.

9.3.3 Equal Roots

When $s_1 = s_2$, the procedure used in the last case will yield only one solution. Nonetheless, two linearly independent solutions can be shown to exist in this case corresponding to

$$
y_1(x) = y(x, s_1) = x^{s_1}\sum_{n=0}^{\infty} c_n(s_1)x^n, \quad c_0(s_1) = 1, \tag{27}
$$

and

$$
\begin{aligned}
y_2(x) &= \left.\frac{\partial y(x, s)}{\partial s}\right|_{s=s_1} \\
&= y_1(x)\log x + \sum_{n=1}^{\infty} c_n'(s_1)x^{n+s_1},
\end{aligned} \tag{28}
$$

where

$$
y(x, s) = x^s + \sum_{n=1}^{\infty} c_n(s)x^{n+s}. \tag{29}
$$

Remark. By (29), we simply mean the general function obtained for arbitrary s, which becomes y_1 upon setting $s = s_1$.

Another way of obtaining a second linearly independent solution in this case is to use the method of Section 4.3. That is, a "second" linearly independent solution of $y'' + a(x)y' + b(x)y = 0$ is given by the expression

$$
y_2(x) = y_1(x)\int\frac{\exp[-\int a(x)\,dx]}{y_1^2(x)}\,dx, \tag{30}
$$

where $y_1(x) = y(x, s_1)$.

EXAMPLE 10 Solve $xy'' + y' + xy = 0$.

Solution The assumption $y = \sum_{n=0}^{\infty} c_nx^{n+s}$ leads to

$$
[s(s-1) + s]c_0x^{s-1} + [(s+1)s + s + 1]c_1x^s + \sum_{n=2}^{\infty}[(n+s)(n+s-1)c_n + (n+s)c_n + c_{n-2}]x^{n+s-1} = 0.
$$

Setting the coefficient of c_0 to zero, we see that the indicial equation reduces to $s^2 = 0$ with double root $s_1 = s_2 = 0$. For this choice of s, the coefficient of c_1 does not vanish and so we are forced to set $c_1 = 0$. For values of n greater than 1, we have the recurrence formula

$$(n + s)^2 c_n + c_{n-2} = 0, \quad n = 2, 3, 4, \ldots,$$

or

$$c_n(s) = -\frac{c_{n-2}(s)}{(s + n)^2}, \quad n = 2, 3, 4, \ldots.$$

In this last form of the recurrence formula we are emphasizing the dependence of these constants on the parameter s. Since the solution formula (28) requires derivatives of the c's with respect to s, we will not substitute the value $s = 0$ into the recurrence formula until later. From the recurrence formula, there follows (with $c_0 = 1$ once again for convenience)

$$c_2(s) = -\frac{c_0}{(s + 2)^2} = -\frac{1}{(s + 2)^2},$$

$$c_3(s) = -\frac{c_1}{(s + 3)^2} = 0,$$

$$c_4(s) = -\frac{c_2}{(s + 4)^2} = \frac{1}{(s + 2)^2(s + 4)^2},$$

$$c_5(s) = 0,$$

$$c_6(s) = -\frac{c_4}{(s + 6)^2} = -\frac{1}{(s + 2)^2(s + 4)^2(s + 6)^2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

while in general we deduce that $c_n = 0$ for $n = 1, 3, 5, \ldots$, and for even values of n we write

$$c_n(s) \equiv c_{2m}(s) = \frac{(-1)^m}{(s + 2)^2(s + 4)^2 \cdots (s + m)^2},$$

where $m = 1, 2, 3, \ldots$. To obtain the first solution, we now set $s = 0$, finding

$$c_{2m}(0) = \frac{(-1)^m}{(2^2)(4^2) \cdots (m^2)} = \frac{(-1)^m}{2^{2m}(m!)^2}, \quad m = 1, 2, 3, \ldots,$$

and therefore

$$y_1(x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2} x^{2m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}.$$

Next, using logarithmic differentiation, we deduce that

$$c'_{2m}(s) = \left(-\frac{2}{s + 2} - \frac{2}{s + 4} - \cdots - \frac{2}{s + m}\right) c_{2m}(s),$$

which for $s = 0$ leads to

$$c'_{2m}(0) = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right).$$

It is common to introduce the notation

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$

for the partial sum of the harmonic series. Thus, from (28) we obtain the second solution

$$y_2(x) = y_1(x)\log x + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}H_m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}.$$

The general solution, valid for $x > 0$, is

$$\begin{aligned} y &= Ay_1(x) + By_2(x) \\ &= (A + B\log x)\sum_{m=0}^{\infty}\frac{(-1)^m(x/2)^{2m}}{(m!)^2} + B\sum_{m=1}^{\infty}\frac{(-1)^{m-1}H_m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}. \end{aligned}$$

An alternate solution technique for finding $y_2(x)$ in Example 10 rests upon the use of (30), which leads to

$$y_2(x) = y_1(x)\int \frac{dx}{xy_1^2(x)}.$$

Performing some lengthy calculations, we have

$$\begin{aligned} \frac{1}{y_1^2(x)} &= \frac{1}{[1 - (x^2/4) + (x^4/64) - (x^6/2304) + \cdots]^2} \\ &= \frac{1}{1 - (x^2/2) + (3x^4/32) - (5x^6/576) + \cdots} \\ &= 1 + \frac{x^2}{2} + \frac{5x^4}{32} + \frac{23x^6}{576} + \cdots, \end{aligned}$$

this last step resulting from long division. Thus,

$$\begin{aligned} y_2(x) &= y_1(x)\int\left(\frac{1}{x} + \frac{x}{2} + \frac{5x^3}{32} + \frac{23x^5}{576} + \cdots\right)dx \\ &= y_1(x)\left(\log x + \frac{x^2}{4} + \frac{5x^4}{128} + \frac{23x^6}{3456} + \cdots\right) \\ &= y_1(x)\log x + \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots\right)\left(\frac{x^2}{4} + \frac{5x^4}{128} + \frac{23x^6}{3456} + \cdots\right), \end{aligned}$$

or

$$y_2(x) = y_1(x)\log x + \frac{x^2}{4} - \frac{3x^4}{128} + \frac{11x^6}{13824} + \cdots. \tag{31}$$

We have therefore obtained the first few terms of the second solution by this alternate technique. Although it is not immediately obvious, this is the same solution found in Example 10 for y_2, the verification of which is left to the reader.

The solution function y_1 in Example 10 defines a particular function called the *Bessel function of the first kind* of order zero and is denoted by the special symbol

$$y_1 = J_0(x). \tag{32}$$

In most applications involving the Bessel function, it is customary to choose a certain linear combination of J_0 and y_2 and to take this combination as the "second" solution of the DE rather than simply to take y_2. This special combination is called the *Bessel function of the second kind* of order zero and is defined by

$$Y_0(x) = \frac{2}{\pi}[y_2(x) + (\gamma - \log 2)J_0(x)], \tag{33}$$

where γ is called *Euler's constant* given by

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) \cong 0.5772.$$

Using the Bessel functions, we can then express the general solution of

$$xy'' + y' + xy = 0, \tag{34}$$

known as *Bessel's equation* of order zero, in the form

$$y = AJ_0(x) + BY_0(x), \tag{35}$$

where A and B are any constants.

9.3.4 Roots Differing by a Nonzero Integer

Let us suppose $s_1 - s_2 = N$, where N is a positive integer. Thus $s_1 > s_2$, and in this case there exist two linearly independent solutions of the form

$$y_1(x) = y(x, s_1) = \sum_{n=0}^{\infty} c_n(s_1)x^{n+s_1}, \quad c_0(s_1) = 1, \tag{36}$$

and

$$\begin{aligned} y_2(x) &= \left.\frac{\partial y(x, s)}{\partial s}\right|_{s = s_2} \\ &= Ky_1(x)\log x + \sum_{n=0}^{\infty} c_n'(s_2)x^{n+s_2}, \end{aligned} \tag{37}$$

where K is a constant that in some instances is zero, and

$$y(x, s) = x^s\sum_{n=0}^{\infty} c_n(s)x^n, \quad c_0(s) = s - s_2. \tag{38}$$

Therefore, we are always assured of one solution of the form (36) corresponding to the larger root s_1 of the indicial equation. Using the smaller root, however, we can always produce two solutions of the DE, although there are two subcases to consider here. In most situations we can produce the second linearly independent solution by a procedure similar to that used in Case II in Section 9.3.3 which leads to a logarithmic type of solution as suggested by (37). An essential point of dissimilarity between the two cases is the choice $c_0(s) = s - s_2$ in constructing $y(x, s)$, rather than setting $c_0(s) = 1$ as before. By constructing the function $y(x, s)$ in this manner, it can be shown that

$$y_1(x) = y(x, s_2) \tag{39}$$

is equivalent to calculating y_1 by use of the larger root of the indicial equation. Hence, by using (39) instead of (36), fewer calculations are necessary. In some special cases the use of the smaller root of the indicial equation will lead to two linearly independent solutions, neither of which involves a logarithmic term. Typically this will happen when s_1 is a positive integer and both c_0 and c_{s_1} turn out to be arbitrary. Otherwise c_{s_1} will be impossible to calculate (with the assumption $c_0 \neq 0$), and the general solution will then contain a logarithm. It is usually best to first try the smaller root s_2 of the indicial equation in the hopes that it will produce two solutions of the DE not involving logarithms. Let us illustrate these cases with some examples.

EXAMPLE 11 (*Nonlog case*) Solve $xy'' - (4 + x)y' + 2y = 0$, $x > 0$.

Solution The substitution of $y = \sum_{n=0}^{\infty} c_n x^{n+s}$ into the DE leads to

$$\sum_{n=0}^{\infty} (n + s)(n + s - 1)c_n x^{n+s-1} - \sum_{n=0}^{\infty} 4(n + s)c_n x^{n+s-1}$$
$$- \underbrace{\sum_{n=0}^{\infty} (n + s)c_n x^{n+s}}_{n \to n-1} + \underbrace{\sum_{n=0}^{\infty} 2c_n x^{n+s}}_{n \to n-1} = 0,$$

which simplifies to

$$[s(s - 1) - 4s]c_0 x^{s-1} + \sum_{n=1}^{\infty} [(n + s)(n + s - 1)c_n - 4(n + s)c_n$$
$$- (n + s - 1)c_{n-1} + 2c_{n-1}]x^{n+s-1} = 0.$$

Therefore, the indicial equation $s^2 - 5s = 0$ gives $s_1 = 5$ and $s_2 = 0$ as roots. If we try the smaller root $s_2 = 0$, the recurrence formula reduces to

$$n(n - 5)c_n = (n - 3)c_{n-1}, \quad n = 1, 2, 3, \ldots.$$

Since division by $(n - 5)$ is not permitted when $n = 5$, it is best to write out separate relations for the c's through $n = 5$. Thus,

$$
\begin{aligned}
n = 1: &\quad -4c_1 = -2c_0,\\
n = 2: &\quad -6c_2 = -c_1,\\
n = 3: &\quad -6c_3 = -0 \cdot c_2,\\
n = 4: &\quad -4c_4 = c_3,\\
n = 5: &\quad 0 \cdot c_5 = 2c_4,
\end{aligned}
$$

from which we deduce

$$c_1 = \frac{1}{2}c_0, \quad c_2 = \frac{1}{6}c_1 = \frac{1}{12}c_0, \quad c_3 = c_4 = 0,$$

and since $0 \cdot c_5 = 0$ is satisfied for any value c_5, we see that c_5 is arbitrary. For $n > 5$, it follows that

$$c_n = \frac{(n-3)c_{n-1}}{n(n-5)}, \quad n = 6, 7, 8, \ldots .$$

Proceeding now as usual, we find

$$
\begin{aligned}
c_6 &= \frac{3}{6 \cdot 1}c_5,\\
c_7 &= \frac{4}{7 \cdot 2}c_6 = \frac{3 \cdot 4}{7 \cdot 6 \cdot 2 \cdot 1}c_5,\\
c_8 &= \frac{5}{8 \cdot 3}c_7 = \frac{3 \cdot 4 \cdot 5}{8 \cdot 7 \cdot 6 \cdot 3 \cdot 2 \cdot 1}c_5,\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

Collecting terms, we have the general solution

$$y = x^0(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \cdots),$$

or finally

$$
\begin{aligned}
y = c_0&\left(1 + \frac{1}{2}x + \frac{1}{12}x^2\right)\\
&+ c_5x^5\left(1 + \frac{3}{6 \cdot 1}x + \frac{3 \cdot 4}{7 \cdot 6 \cdot 2 \cdot 1}x^2 + \frac{3 \cdot 4 \cdot 5}{8 \cdot 7 \cdot 6 \cdot 3 \cdot 2 \cdot 1}x^3 + \cdots\right).
\end{aligned}
$$

EXAMPLE 12 (*Nonlog case*) Solve $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$, $x > 0$.

Solution Proceeding as before, we have

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)c_nx^{n+s} + \sum_{n=0}^{\infty}(n+s)c_nx^{n+s}$$

$$+\underbrace{\sum_{n=0}^{\infty} c_n x^{n+s+2}}_{n\to n-2} - \sum_{n=0}^{\infty} \frac{1}{4} c_n x^{n+s} = 0,$$

or

$$\left(s^2 - \frac{1}{4}\right)c_0 x^s + \left(s + \frac{3}{2}\right)\left(s + \frac{1}{2}\right)c_1 x^{s+1}$$
$$+ \sum_{n=2}^{\infty}\left[(n + s)(n + s - 1)c_n + (n + s)c_n + c_{n-2} - \frac{1}{4}c_n\right]x^{n+s} = 0.$$

We find that $s_1 = \frac{1}{2}$ and $s_2 = -\frac{1}{2}$ are roots of the indicial equation $s^2 - \frac{1}{4} = 0$. Using the smaller root $s_2 = -\frac{1}{2}$ leads to both c_0 and c_1 arbitrary in this instance, and the remaining constants are then determined by

$$c_n = -\frac{c_{n-2}}{n(n - 1)}, \qquad n = 2, 3, 4, \ldots .$$

We obtain successively

$$c_2 = -\frac{c_0}{2\cdot 1} = -\frac{c_0}{2!},$$
$$c_3 = -\frac{c_1}{3\cdot 2} = -\frac{c_1}{3!},$$
$$c_4 = -\frac{c_2}{4\cdot 3} = -\frac{c_0}{4!},$$
$$c_5 = -\frac{c_3}{5\cdot 4} = -\frac{c_1}{5!},$$
$$\ldots\ldots\ldots\ldots,$$

and thus

$$y = x^{-1/2}(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots)$$
$$= x^{-1/2}c_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + x^{-1/2}c_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right),$$

which can also be expressed in the form

$$y = c_0\frac{\cos x}{\sqrt{x}} + c_1\frac{\sin x}{\sqrt{x}}, \qquad x > 0.$$

EXAMPLE 13 (*Log case*) Solve $xy'' + 3y' - y = 0$, $x > 0$.

Solution If $y = \sum_{n=0}^{\infty} c_n x^{n+s}$, then

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)c_n x^{n+s-1} + \underbrace{\sum_{n=0}^{\infty} 3(n+s)x^{n+s}}_{n \to n-1} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+s}}_{n \to n-1} = 0,$$

$$[s(s-1) + 3s]c_0 x^{s-1} + \sum_{n=1}^{\infty}[(n+s)(n+s-1)c_n$$

$$+ 3(n+s)c_n - c_{n-1}]x^{n+s-1} = 0.$$

Here we find that the coefficient of c_0 set to zero gives $s^2 + 2s = 0$, or $s_1 = 0$ and $s_2 = -2$. Making the substitution $s = -2$ in the recurrence formula

$$(n+s)(n+s+2)c_n = c_{n-1}$$

yields

$$n(n-2)c_n = c_{n-1}, \quad n = 1, 2, 3, \ldots .$$

Again we do not divide by the factor $(n-2)$ until $n > 2$. Thus,

$$n = 1: \qquad -c_1 = c_0,$$

$$n = 2: \qquad 0 \cdot c_2 = c_1,$$

and

$$c_n = \frac{c_{n-1}}{n(n-2)}, \quad n = 3, 4, 5, \ldots .$$

These relations can be satisfied only for $c_1 = c_0 = 0$, which contradicts our assumption that $c_0 \neq 0$. Therefore we have failed to produce two solutions this time that do not involve logarithmic terms. (It may at first seem that the way to proceed is to set $c_0 = c_1 = 0$ and leave c_2 arbitrary. Doing so will only produce one solution, which will be reobtained automatically when solving the DE by the method discussed here.)

Let us write the recurrence formula above in the form

$$c_n(s) = \frac{c_{n-1}(s)}{(s+n)(s+n+2)}, \quad n = 1, 2, 3, \ldots,$$

where we now choose $c_0(s) = s + 2$. Therefore,

$$c_1(s) = \frac{c_0}{(s+1)(s+3)} = \frac{s+2}{(s+1)(s+3)},$$

$$c_2(s) = \frac{c_1}{(s+2)(s+4)} = \frac{1}{(s+1)(s+3)(s+4)},$$

$$c_3(s) = \frac{c_2}{(s+3)(s+5)} = \frac{1}{(s+1)(s+3)^2(s+4)(s+5)},$$

$$c_4(s) = \frac{c_3}{(s+4)(s+6)}$$

$$= \frac{1}{(s+1)(s+3)^2(s+4)^2(s+5)(s+6)},$$

and so on. We have

$$\begin{aligned} y(x, s) &= x^s \sum_{n=0}^{\infty} c_n(s) x^n \\ &= (s+2)x^s + \frac{(s+2)x^{s+1}}{(s+1)(s+3)} + \frac{x^{s+2}}{(s+1)(s+3)(s+4)} \\ &\quad + \frac{x^{s+3}}{(s+1)(s+3)^2(s+4)(s+5)} \\ &\quad + \frac{x^{s+4}}{(s+1)(s+3)^2(s+4)^2(s+5)(s+6)} + \cdots, \end{aligned}$$

from which we can generate y_1 by setting $s = -2$; i.e.,

$$\begin{aligned} y_1(x) &= y(x, -2) \\ &= 0 \cdot x^{-2} + 0 \cdot x^{-1} - \frac{1}{2!} - \frac{x}{3!} - \frac{x^2}{2!4!} - \frac{x^3}{3!5!} - \cdots \end{aligned}$$

or

$$y_1(x) = -\sum_{n=0}^{\infty} \frac{x^n}{n!(n+2)!}.$$

Also,

$$\begin{aligned} \frac{\partial y(x,s)}{\partial s} &= y(x,s)\log x + x^s + \frac{(s+2)\,x^{s+1}}{(s+1)(s+2)}\left(\frac{1}{s+2} - \frac{1}{s+1} - \frac{1}{s+3}\right) \\ &\quad + \frac{x^{s+2}}{(s+1)(s+3)(s+4)}\left(-\frac{1}{s+1} - \frac{1}{s+3} - \frac{1}{s+4}\right) \\ &\quad + \frac{x^{s+3}}{(s+1)(s+3)^2(s+4)(s+5)} \\ &\quad \times \left(-\frac{1}{s+1} - \frac{2}{s+3} - \frac{1}{s+4} - \frac{1}{s+5}\right) \\ &\quad + \frac{x^{s+4}}{(s+1)(s+3)^2(s+4)^2(s+5)(s+6)} \\ &\quad \times \left(-\frac{1}{s+1} - \frac{2}{s+3} - \frac{2}{s+4} - \frac{1}{s+5} - \frac{1}{s+6}\right) \\ &\quad + \cdots, \end{aligned}$$

and therefore,

$$y_2(x) = \left.\frac{\partial y(x, s)}{\partial s}\right|_{s=-2}$$

$$= y_1(x)\log x + \frac{1}{x^2} - \frac{1}{x} + \frac{1}{4} + \frac{11}{36}x + \frac{31}{576}x^2 + \cdots,$$

leading to the general solution

$$y = Ay_1(x) + By_2(x), \quad x > 0,$$

for any constants A and B. The solution is valid for all $x > 0$, since $x = 0$ is the only singular point of the DE.

EXERCISES 9.3

In problems 1–8, locate and classify the singular points of the DE.

1. $x^3(x^2 - 1)y'' - x(x + 1)y' + (x - 1)y = 0$
2. $(x^4 - 1)y'' + xy' + y = 0$
3. $x^4(x^2 + 1)(x - 1)^2y'' + 4x^3(x - 1)y' + (x + 1)y = 0$
4. $y'' + xy = 0$
5. $x^2(x - 4)^2y'' + 3xy' - (x - 4)y = 0$
6. $x^2(x + 1)^2y'' + (x^2 - 1)y' + 2y = 0$
7. $xy'' + (x - 3)^{-2}y = 0$
8. $(x^5 + x^4 - 6x^3)y'' + 3x^2y' + (x - 2)y = 0$

In problems 9–18, show that the roots of the indicial equation do not differ by an integer and obtain two linearly independent solutions by the method of Frobenius about the point x = 0.

9. $2xy'' + (1 + x)y' - 2y = 0$
10. $4xy'' + 3y' + 3y = 0$
11. $2xy'' + 5(1 + 2x)y' + 5y = 0$
12. $x^2y'' + x\left(x - \frac{1}{2}\right)y' + \frac{1}{2}y = 0$
13. $2x^2y'' + 3xy' - y = 0$
14. $2x^2y'' + xy' - y = 0$
15. $2x^2y'' - xy' + (1 + x)y = 0$
16. $2x^2y'' - xy' + (x - 5)y = 0$
17. $2x^2y'' + xy' + (x^2 - 3)y = 0$
18. $2xy'' + (1 + 2x)y' - 5y = 0$

In problems 19–26, show that the roots of the indicial equation are equal and obtain two linearly independent solutions by the method of Frobenius about the point x = 0.

19. $xy'' + y' - 4y = 0$
20. $x^2y'' + 3xy' + (1 - 2x)y = 0$
21. $x^2y'' - x(1 + x)y' + y = 0$
22. $4x^2y'' + (1 - 2x)y = 0$
23. $x^2y'' + 5xy' + 4y = 0$
24. $x^2y'' + 3xy' + (1 + 4x^2)y = 0$

25. $x^2y'' + x(x - 1)y' + (1 - x)y = 0$

26. $xy'' + (1 - x)y' - y = 0$

In problems 27–36, show that the roots of the indicial equation differ by a nonzero integer and obtain two linearly independent solutions by the method of Frobenius about the point $x = 0$.

Nonlog cases

27. $x^2y'' + 2x(x - 2)y' + 2(2 - 3x)y = 0$

28. $xy'' + (x - 6)y' - 3y = 0$

29. $xy'' - (x + 3)y' + 2y = 0$

30. $x^2y'' + x^2y' - 2y = 0$

31. $x^2y'' + x^2y' + (x - 2)y = 0$

Log cases

32. $xy'' + y = 0$

33. $xy'' + (3 - 2x)y' + 8y = 0$

34. $x^2y'' + (x^2 - 3x)y' + 3y = 0$

35. $x^2y'' + x(1 - x)y' - (1 + 3x)y = 0$

36. $x^2y'' + xy' + (x^2 - 1)y = 0$

***37.** The DE

$$x(1 - x)y'' + [c - (a + b + 1)x]y' - aby = 0,$$

where a, b, and c are all constants, is called the *hypergeometric equation*, and its solutions are called *hypergeometric functions*.

(a) Show that $x = 0$ and $x = 1$ are R.S.P.'s of the DE.

(b) Assuming $c \neq 0, -1, -2, \ldots$, show that one solution of this DE is

$$y_1(x) \equiv F(a, b; c; x)^* = 1 + \frac{ab}{c \cdot 1!}x + \frac{a(a + 1)b(b + 1)}{c(c + 1)2!}x^2 + \cdots.$$

(c) When $a = 1$ and $c = b$, show that the series in (b) reduces to

$$y_1(x) = F(1, b; b; x) = \frac{1}{(1 - x)}.$$

(d) Assuming $1 - c$ is not an integer or zero, show that

$$y_2(x) = x^{1-c}F(1 - c + a, 1 - c + b; 2 - c; x)$$

is a second linearly independent solution of the DE.

***38.** The DE

$$xy'' + (c - x)y' - ay = 0,$$

where a and c are constants, is called the *confluent hypergeometric equation*, and its solutions are likewise called *confluent hypergeometric functions*.

(a) Show that $x = 0$ is an R.S.P. of the DE.

(b) Assuming $c \neq 0, -1, -2, \ldots$, show that one solution of this DE is

$$y_1(x) \equiv M(a; c; x)^* = 1 + \frac{a}{c \cdot 1!}x + \frac{a(a + 1)}{c(c + 1)2!}x^2 + \cdots.$$

(c) When $c = a$, show that the solution in (b) reduces to

*The semicolons used in this term separate numerator parameters, denominator parameters, and the argument x.

$$y_1(x) = M(a; a; x) = e^x.$$

(d) Assuming $1 - c$ is not an integer or zero, show that

$$y_2(x) = x^{1-c}M(a + 1 - c; 2 - c; x)$$

is a second linearly independent solution of the DE.

REFERENCES

Boyce, W. E., and DiPrima, R. C. 1977. *Elementary differential equations and boundary value problems*. 3rd ed. New York: Wiley.

Coddington, E. A. 1961. *An introduction to ordinary differential equations*. Englewood Cliffs, N.J.: Prentice-Hall.

Kaplan, W. 1958. *Ordinary differential equations*. Reading, Mass.: Addison-Wesley.

Rainville, E. D., and Bedient, P. E. 1974. *Elementary differential equations*. 5th ed. New York: Macmillan.

Zill, D. G. 1979. *A first course in differential equations with applications*. Boston, Mass.: Prindle, Weber & Schmidt.

Answers to Odd-Numbered Exercises

CHAPTER 1

Section 1.2

1. linear, second-order
3. nonlinear, first-order
5. linear, third-order
7. linear, fourth-order
9. linear, first-order
11. linear, second-order
13. linear in x, first-order
15. nonlinear, third-order
17. nonlinear, first-order
19. nonlinear, first-order

Section 1.3

15. $m = 2$
17. $m = 2, -3$
19. $m = 0, 1, -4$
21. $y = Ae^{x}$
23. $y = \dfrac{Ax + B}{x + C}$
25. $y = A\sin x + B\sin 3x$
29. $xy' - 3y = 12$
31. $y' - 2y = 4x$
33. $(y')^2 - 9y^2 = 1$
35. $2yy' = 2x + y'$
37. $y = 0$ (singular solution)
39. no

Section 1.4

1. $y = e^{2x}$
3. $y = \sinh x$
5. $y = \frac{1}{2}x^2$
7. $y = x(x - 1)$
9. $y = e^{x}(\cos x - \sin x)$
11. $y = x$
13. $y = C_1 \cos x$ (C_1 arbitrary)
15. $y = 0$
17. $k = 0, \pm 1, \pm 2 \ldots$

CHAPTER 2

Section 2.2

1. $y^3 = Cx^2$
3. $x^2 + y^2 = C$
5. $\dfrac{1 + y}{1 - y} = C\left(\dfrac{1 + x}{1 - x}\right)$
7. $\dfrac{1}{2}x + \dfrac{1}{4}\sin 2x - \cos y = C$
9. $P^{-1} = \dfrac{b}{a} + Ce^{-at}$
11. $\log N = (t - 1)e^{t+2} - t + C$
13. $(x^2 + 1)^{1/2} = C(y + 1)^2 e^{1/2y^2 - y}$
15. $x^x y^y = C$
17. $y^2 = x^2 - x + C$
19. $x + y + \log \dfrac{y}{x^x} = C$
21. $y = e^{C(x^2 - 1)}$
23. $(y - x)^2 - 2C(y + x) + C^2 = 0, \quad y = 0$

25. $4y^3 = 9(e^x + C)^2$

27. $y = y_0 e^{-x^2}$

29. $\sin x - x\cos x = e^y(y - 1)$

31. (c) $y = \pm 2$

33. (a) $y = \dfrac{1 - Cx^2}{1 + Cx^2}$
(b) $y = -1$ (singular solution)
(c) no solution

35. $y = \dfrac{a}{1 + C^2}[(1 - C^2)\sin x + 2C\cos x]$, $y = \pm a$

Section 2.3

1. $x^3 - 3x^2y - y^2 = C$

3. $x^2(\cancel{1} - t) = C$

5. $\sin^2\theta + r^2(1 - \theta^2) = C$

7. $u^2 - ue^{3v} + \sin 3v = C$

9. $\cos x \sin y - \log(\cos x) = C$

11. $x^2y(y - 1) = C$

13. $(xy + y^2)e^x = C$

15. $2x - \log(x^2 + y^2) = C$

17. $\dfrac{x}{y} = \dfrac{1}{2}\cancel{x^2} + C$

19. $(x^3 + y^3) + \dfrac{3y}{x} = C$

21. $x^4y^3(3 + x^5y^3) = C$

23. $y = C_1e^{-2x}$

25. $y = \cancel{C_1}e^{-x^4}$

27. $y = e^{-x^2}\left(C_1 + \dfrac{1}{2}x^2\right)$

31. $x = \dfrac{1}{\sqrt{1 + y^2}}[C_1 + \log(y + \sqrt{1 + y^2})]$

33. $2\log x + \dfrac{y}{x} - \dfrac{y^2}{3x^2} = C$

35. $\dfrac{1}{2}\log(u^2 + v^2) + \tan^{-1}\left(\dfrac{u}{v}\right) = C$

37. $\log y + \dfrac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\dfrac{x}{y}\right) = C$

39. $(x - y)\log x + y\log y = Cx + y$

41. $2(x + 2y) + (x + y)\log(x + y) = 0$

43. $x^2y^3 - 3xy + 2y^2 = C$

45. $y = \dfrac{1}{4}x - \dfrac{2}{3} + \dfrac{C_1}{x^3}$

47. $\dfrac{1}{2}x^2(y^2 + 1) - 2xy + 3x - y^2 = 1$

49. $\dfrac{1}{2}\log(x^2 - y^2) = \dfrac{1}{3}y^3 + C$

51. $4x^3y^2 = x^4y^4 + C$

53. $ax^2 - dy^2 + 2bxy = 2c$

Section 2.4

3. $y = C_1e^{-bx/a}$

5. $y = C_1e^{x^2/2}$

7. $y = C_1e^{1/x}$

9. $y = C_1x^{-x}e^x$

11. $y = \dfrac{C_1}{t + 1}$

13. $y = 2e^{-\sin x}$

15. $w = -7e^{x^5}$

17. $y = -e^{1-e^x}$

19. $y = e^{3-x^2-2x^3}$

21. $y = x$

23. $y = \dfrac{2}{1 + x}$

25. prob. 21, continuous at $x = 0$
prob. 22, discontinuous at $x = 0$
prob. 23, discontinuous at $x = -1$
prob. 24, continuous at $x = n\pi$, $n = 0, 1, 2, \ldots$

Section 2.5

1. $y = \dfrac{1}{5} + C_1e^{-5x}$

3. $y = \dfrac{1}{x}(\sin x + C_1) - \cos x$

5. $y = x^4[(x - 1)e^x + C_1]$

7. $z = \frac{1}{4} + C_1e^{-2x^2}$

9. $y = (x + C_1)\csc x$

11. $y = 1 + C_1(x^2 + 9)^{-1/2}$

13. $y = (x + C_1)e^{mx}$

15. $y = \left(\dfrac{1}{2}x^2 + C_1\right)e^{-x^2}$

17. $y = \dfrac{b}{k - a}x^k + C_1x^a$

19. $y = 5 + C_1 e^{1/x}$

21. $i(t) = \dfrac{E}{R}(1 - e^{-Rt/L})$

23. $y = \dfrac{1}{2}(2x + 3)^{1/2}\log(2x + 3)$

25. $y = \dfrac{\sin x - x\cos x - 1 + \pi^2/4}{x^2}$

27. $y = \begin{cases} 1 - e^{-x}, & 0 \le x < 1 \\ (e - 1)e^{-x}, & x > 1 \end{cases}$

29. $y = \begin{cases} \dfrac{\frac{1}{2}x^2}{1 + x^2}, & 0 \le x < 1 \\ \dfrac{1 - \frac{1}{2}x^2}{1 + x^2}, & x > 1 \end{cases}$

35. $y = x + 1 + C_1 e^{-x}$

Section 2.6

1. $y^2 = \dfrac{1}{2}x^2 + \dfrac{C_1}{x^2}$

3. $y^{-2} = \dfrac{2}{5x} + C_1 x^4$

5. $x^{-3}y = 2 + C_1 y^{1/2}$

CHAPTER 3

Section 3.2

1. $x^2 - y^2 = k$

3. $y = ke^{-2x}$

5. $xy = k$

7. $\log y + \log\left(\dfrac{x^2}{y^2} + 1\right) = k$

9. $2\log y = \log\left(3 - \dfrac{2x^2}{y^2}\right) + k$

11. $y^{5/3} = x^{5/3} + k$

15. $r = b(1 + \sin\theta)$

17. $r^2 = b\sin\theta$

23. $v = \sqrt{\dfrac{mg}{c}}\left(\dfrac{1 - e^{-2\sqrt{cg/m}\,t}}{1 + e^{-2\sqrt{cg/m}\,t}}\right), \quad v_\infty = \sqrt{\dfrac{mg}{c}}$

25. (a) $v = 320(1 - e^{-t/10}), \quad 0 \le t < 5$

(b) $v = 16\left(\dfrac{1 + 0.775\,e^{-4(t-5)}}{1 - 0.775\,e^{-4(t-5)}}\right), \quad t > 5$

(c) $v_\infty = 320$ ft/sec if chute never opens
$v_\infty = 16$ ft/sec if chute opens

27. v_0

29. 3.1 mi/sec

31. $v = 493$ ft/sec

33. approx. 16 hr, 16 min

35. approx. 20 min, 47 s

37. $y = \dfrac{1}{4}a\left[\left(\dfrac{x}{a}\right)^2 - 1\right] - \dfrac{1}{2}a\log\dfrac{x}{a}$, no

39. $y = \dfrac{1}{2}a\left[\dfrac{(x/a)^{k+1} - 1}{k + 1} + \dfrac{(a/x)^{k-1} - 1}{k - 1}\right], \quad k = \dfrac{v}{w}$

$D = x\left\{1 + \dfrac{1}{2}\left[\left(\dfrac{x}{a}\right)^k - \left(\dfrac{a}{k}\right)^k\right]^2\right\}^{1/2}$

41. 24 units, $\dfrac{27}{2}$ units

43. $y = \dfrac{1}{2}a\left(1 - \dfrac{x^2}{a^2}\right)$; no

45. $y^2 = 2Cx + C^2$ (parabolic shape)

Section 3.3

1. 9.33 g

3. 55,820 yr

5. 1.142×10^{32}

7. $P(t) = (100b + P_0 - \frac{1000}{3})e^{-3t} + \underbrace{\tfrac{1000}{3} + 100b(3\sin t - \cos t)}_{\text{steady-state solution}}$

9. (b) $k = 1$: $M(10) = 1.790M_0$
$k = 4$: $M(10) = 1.814M_0$
$k = 365$: $M(10) = 1.821M_0$

(c) $\dfrac{dM}{dt} = 0.06M, \quad M(0) = M_0$
$M(t) = M_0 e^{0.06t}$
$M(10) = 1.822M_0$

11. $v(t) = 40(1 - e^{-4t/5})$

13. (a) $t = \dfrac{m}{c}\log\left(1 + \dfrac{v_0 c}{mg}\right)$

(b) 0.884

15. $i(t) = \dfrac{E_0 C\omega}{1 - R^2C^2\omega^2}\cos\omega t - \dfrac{E_0 RC^2\omega^2}{1 - R^2C^2\omega^2}\sin\omega t$

17. (a) $i(t) = \dfrac{E_0}{R}(1 - e^{-Rt/L}) + i_0 e^{-Rt/L}$

(b) $i(t) = \dfrac{E_0 R}{R^2 - L^2\omega^2}\sin\omega t + \dfrac{E_0 L\omega}{R^2 - L^2\omega^2}(e^{-Rt/L} - \cos\omega t) + i_0 e^{-Rt/L}$

(c) $i(t) = \begin{cases} 6, & 0 \leq t < 10 \\ 6e^{1-t/10}, & t > 10 \end{cases}$

19. $58.5°$

21. $x(t) = 600 - 550e^{-t/150}$
$x(60) = 231.3$ lb
$x_\infty = 600$ lb

23. (a) $x(t) = 1200 - 1150e^{-t/300}$
$x(60) = 258.5$ lb
$x_\infty = 1200$ lb

(b) $x(t) = 400 - 350e^{-t/100}$
$x(60) = 207.9$ lb
$x_\infty = 400$ lb

CHAPTER 4

Section 4.2

1. independent
3. dependent
5. independent
9. Ce^{3x}
11. C/x
15. $y = (C_1 + C_2 x)e^{3x}$

Section 4.3

1. $y_2 = e^{-2x}$
3. $y_2 = xe^{3x}$
5. $y_2 = \cos x$
7. $y_2 = e^x \sin 2x$
9. $y_2 = x^{-2}$
11. $y_2 = x^{1/2}$
13. $y_2 = \log\dfrac{1 + x}{1 - x}$
15. $y_2 = x^{-1/2}\cos x$

Section 4.4

1. $y = C_1 + C_2 e^{x/3}$
3. $y = C_1 e^x + C_2 e^{-3x}$
5. $y = e^{5x}(C_1 \cosh 2\sqrt{2}x + C_2 \sinh 2\sqrt{2}x)$
7. $y = (C_1 + C_2 x)e^x$
9. $y = (C_1 + C_2 x)e^{-x/3}$
11. $y = (C_1 + C_2 x)e^{2x/3}$
13. $y = e^{-x/2}\left(C_1 \cos\dfrac{\sqrt{3}}{2}x + C_2 \sin\dfrac{\sqrt{3}}{2}x\right)$
15. $y = e^{3x}(C_1 \cos 4x + C_2 \sin 4x)$
17. $y = e^{x/4}\left(C_1 \cos\dfrac{\sqrt{7}}{4}x + C_2 \sin\dfrac{\sqrt{7}}{4}x\right)$
23. $4D^2 - 17D - 15$
25. $D^3 - 1$
27. $D^2 + 1 - x^2$
29. $x^2 D^2 + 2xD - 6$
31. $y = C_1 e^{5x} + C_2 e^{-3x/4}$
33. $y = C_1 e^{-x/3} + C_2 e^{5x}$
35. $y = C_1 e^{-x} + C_2 e^{3x/5} + C_3 e^{-4x}$
37. $y = e^{-x} - e^{3x}$
39. $y = \dfrac{1}{e^2(e - 1)}(e^{2x-1} - e^x)$

Section 4.5

1. dependent
3. independent
5. dependent
9. $y = (C_1 + C_2 x + C_3 x^2)e^x$
11. $y = (C_1 + C_2 x + C_3 x^2)e^{4x} + (C_5 + C_6 x)e^{-2x}$
13. $y = (C_1 + C_2 x + C_3 x^2)e^{-x}$
15. $y = C_1 e^{-x} + e^{x/2}\left(C_3 \cos\dfrac{\sqrt{3}}{2}x + C_4 \sin\dfrac{\sqrt{3}}{2}x\right)$
17. $y = C_1 + C_2 e^{-4x} + C_3 e^x$
19. $y = C_1 + (C_2 + C_3 x)e^{-x/2}$
21. $y = C_1 e^x + C_2 e^{2x} + C_3 e^{10x/3}$
23. $y = (C_1 + C_2 x)e^{2x} + (C_3 + C_4 x)e^{-3x/2}$

25. $y = C_1e^{-x} + C_2e^{2x} + C_3e^{-x/2} + C_4e^{3x/2}$

27. $y = C_1 + C_2x + C_3x^2 + (C_4 + C_5x)\cos 3x + (C_6 + C_7x)\sin 3x + (C_8 + C_9x + C_{10}x^2 + C_{11}x^3)e^x$

29. $y = C_1 + (C_2 + C_3x)\cos x + (C_4 + C_5x)\sin x$

31. $y = (1 - \frac{1}{2}x)e^{-x}$

33. $y = 2 + 3x + 2x^2 + \frac{5}{6}x^3$

35. $y = (C_1 + C_2x + C_3x^2 + C_4x^3)e^{3x} + e^x(C_5 + C_6x)\cos x + (C_7 + C_8x)\sin x + C_9e^{5x} + C_{10}e^{-x}$

37. $(D^3 - 7D^2 + 17D - 15)y = 0$

39. $y = e^{-x}[(C_1 + C_2x)\cos 2x + (C_3 + C_4x)\sin 2x]$

41. (a) $y = 0$ (b) $y = C\sin n\pi x$, where C is arbitrary and $n = 1, 2, \ldots$

Section 4.6

1. $y = C_1 + C_2e^{-x} + \frac{1}{2}(\cos x - \sin x)$

3. $y = (C_1 + C_2x)e^{3x} + \frac{1}{4}e^x$

5. $y = C_1\cos(2\sqrt{2}x) + C_2\sin(2\sqrt{2}x) + \frac{2}{9}e^{-x} + \frac{5}{8}x$

7. $y = C_1\cos 3x + C_2\sin 3x + \frac{1}{36}x\sin 3x - \frac{1}{12}x^2\cos 3x$

9. $y = C_1\cos 2x + C_2\sin 2x - \frac{1}{4}x\cos 2x - \frac{7}{4}$

11. $y = C_1e^x + C_2e^{-x} + 2x(x-1)e^x$

13. $y = C_1 + C_2e^{-x} + \frac{1}{4}x^4 + \frac{3}{2}x^2 - 2x$

15. $y = e^{-x/2}\left(C_1\cos\frac{\sqrt{3}}{2}x + C_2\sin\frac{\sqrt{3}}{2}x\right) + 1 + \frac{1}{13}(\cos 2x - 2\sin 2x)$

17. $y = C_1 + (C_2 + C_3x)e^{-2x} + (1 - x)e^{-x}$

19. $y = C_1e^{x/2} + C_2e^{-x/2} + C_3\cos\frac{x}{2} + C_4\sin\frac{x}{2} + \frac{3}{4}xe^{x/2}$

21. $y_P = Ax^2 + Bx + C + \boxed{Dx^4e^x}$

23. $y_P = (Ax^3 + Bx^2 + Cx + D)e^{2x} + Ex^2 + Fx$

25. $y = \frac{1}{2}(3 - x)\sin x - \cos x$

27. $y = -\pi\cos x - \frac{11}{3}\sin x - \frac{8}{3}\cos 2x + 2x\cos x$

29. $y = 4(x-1)e^{-x} + (1 - 2x)$

31. $y = (C + x)\sin x$, where C is arbitrary

33. $y = -\frac{6}{5}\cos 2x + \left[\frac{3 + \frac{1}{5}e^{-2} - \frac{12}{5}\sin(4)}{2\cos(4)}\right]\sin 2x + \frac{1}{5}e^{-x}$

Section 4.7

1. $y = C_1e^{-x} + C_2e^x + \frac{1}{4}(2x - 1)e^x$

3. $y = C_1\cos 3x + C_2\sin 3x - \frac{1}{6}\cos 3x(x - \frac{1}{6}\sin 6x) + \frac{1}{18}\sin^3 3x$

5. $y = (C_1 - \log|\cos x|)\cos x + (C_2 + x)\sin x$

7. $y = C_1\cos x + C_2\sin x + \frac{1}{6}\sec^2 x - \frac{1}{2} + \frac{1}{2}\sin x\log|\sec x + \tan x|$

9. $y = C_1\cos x + (C_2 - \log|\csc x + \cot x|)\sin x$

11. $y = (C_1 - \log|\sec x + \tan x|)\cos x + (C_2 - \log|\csc x + \cot x|)\sin x$

13. $y = (C_1 + C_2x + \frac{3}{2}x^2)e^{2x}$

15. $y = (C_1 + C_2x + \frac{3}{2}x^2\log|x| - \frac{3}{4}x^2)e^{-x}$

17. $y = C_1e^x + C_2e^{-x} - \sin(e^{-x}) - e^x\cos(e^{-x})$

19. $y = (C_1 + \frac{1}{2}\cot^2 x)\cos x + (C_2 - \frac{1}{3}\cot^3 x)\sin x$

21. $y = (\frac{1}{4}x^2 - \frac{1}{4}x + \frac{9}{8})e^x + \frac{7}{8}e^{-x}$

23. $y = (1 + \pi - 2x)\sin x - (3 + 2\log|\sin x|)\cos x$

25. $y = C_1x + C_2e^x + (\frac{1}{2} - x)e^{-x}$

27. $y = C_1x^2 + C_2e^x + x^2(x - 3)e^x$

29. $y = C_1 + C_2\log\frac{1 + x}{1 - x} - x$

37. $y = C_1e^{-x} + C_2e^x + C_2e^{2x} + \frac{1}{8}e^{3x}$

Section 4.8

1. $y = C_1x + C_2x^5$

3. $y = (C_1 + C_2\log x)x$

5. $y = C_1 + C_2x^{1/3}$

7. $y = x^3[C_1\cos(4\log x) + C_2\sin(4\log x)]$

9. $y = (C_1 + C_2\log x)x^3 + x^3(\log x)^2$

11. $y = C_1x^2 + C_2x^3 + 2x - 1$

13. $y = C_1x^{-1} + C_2x + \frac{1}{2}x\log x$

15. $y = C_1x + C_2x^2 + x^3(\log x - \frac{3}{2})$

17. $y = x^3 - 2x^2$

19. $y = \frac{1}{10}x^2[9\cos(3\log x) - 7\sin(3\log x)] + \frac{1}{10}x^3$

21. $y = C_1x^{-1} + C_2x^{1/2}$

23. $y = C_1x^{-5} + C_2x^{-1}$

25. $y = C_1 + (C_2 + C_3\log x)x^2$

27. $y = C_1x + C_2\cos(\log x) + C_3\sin(\log x)$

29. $y = (C_1 + C_2\log x)x + C_3x^2 + \frac{1}{4}x^3$

31. $y = C_1(x+5)^{-1} + C_2(x+5)^3$

CHAPTER 5

Section 5.2

1. $\frac{5}{2\pi}$ Hz, $\frac{2\pi}{5}$ s

5. (a) $\frac{\pi}{4\sqrt{3}}$ s (b) $y = \frac{1}{\cancel{12}}\cos(8\sqrt{3}t)$ 6

7. 29 ticks

9. 8 lb

11. $\frac{\pi}{6}$ s

13. 2.84 s

15. 0.364 s, -3.20 rad/s

17. (c) $\delta = 2\pi$, max at $t = 0, 2\pi, 4\pi, \ldots$ min at $t = \pi, 3\pi, 5\pi, \ldots$

19. $v_0 + cy_0/2m \neq 0$

21. (a) 13 (b) $y = 8te^{-13t}$ (c) $\frac{1}{13}$ s

23. 1.069; no

Section 5.3

1. $\frac{1}{16}y'' + 4y = 16\sin 8t$, $y(0) = \frac{1}{4}$, $y'(0) = 0$

3. $y(t) = \begin{cases} 2(1 - \cos 2t), & 0 < t < 4 \\ 2\cos 2(t-4) - 2\cos 2t, & t > 4 \end{cases}$

5. 80 s

7. 0.0056 Hz, 2 m

13. 5.56 s, 8.8 s

15. $e^{-t}\sin t$ (transient), $2\sin t$ (steady-state)

17. $\frac{P}{6}$

Section 5.4

3. (a) 0 (b) $i_P(t) = \cos 2t + \frac{3}{10}\sin 2t$

5. (a) $i(t) = e^{-3t}(3\sin 4t - 4\cos 4t) + 4\cos 5t$
(b) $i(t) = 5e^{-2t}\sin t$

7. $q(t) = \frac{3}{2} - \frac{1}{2}e^{-10t}(\cos 10t + \sin 10t)$, $\frac{3}{2}$ coulombs

9. $R = 9.41 \times 10^{-4}$ ohms

Section 5.5

1. $y = \sin t$

3. $y = 2e^{3t} - 3e^t$

5. $y = 3 - e^{-t}$

7. $y = (2 - 3\log t)t^2$

9. $g_1(t, \tau) = t - \tau$

11. $g_1(t, \tau) = \frac{1}{\sqrt{5}}\sinh\sqrt{5}(t - \tau)$

13. $g_1(t, \tau) = \frac{1}{3}e^{5(t-\tau)/2} - e^{-(t-\tau)/2}$

15. $g_1(t, \tau) = \frac{\tau}{8}\left[\left(\frac{t}{\tau}\right)^4 - \left(\frac{\tau}{t}\right)^4\right]$

17. $g_1(t, \tau) = \frac{1}{2}(1 - \tau^2)\log\frac{(1+t)(1-\tau)}{(1-t)(1+\tau)}$

19. $y = \frac{1}{2}t^2$

21. $y = e^t - 1$

23. $y = \frac{1}{25}[(60e^2 + 7 - 5t)e^{-t} + (15e^{-8} + e^{-10})e^{4t}]$

29. $y = \frac{1}{24}t^{-5} - \frac{1}{8}t^{-1} + \frac{1}{12}t$

31. $y = \frac{1}{12}t^{-2} + \frac{1}{18}t^3 + \frac{1}{36} - \frac{1}{6}\log t$

37. $g_1(t, \tau) = \frac{1}{2}(e^{t-\tau} - e^{\tau - t}) - (t - \tau)$

39. $g_1(t, \tau) = \frac{(a+1)e^{b(t-\tau)} - (b+1)e^{a(t-\tau)} + (b-a)e^{t-\tau}}{(a-b)(a+1)(b+1)}$ where $a = \frac{-3 - \sqrt{33}}{4}$, $b = \frac{-3 + \sqrt{33}}{4}$

41. $g_1(t, \tau) = \frac{1}{2}e^{t-\tau} - e^{2(t-\tau)} + \frac{1}{2}e^{3(t-\tau)}$

45. $y = \frac{1}{4}\cos 2t + \frac{1}{2}\sin 2t - \frac{1}{4}$

Section 5.6

1. $y(t) = \begin{cases} e^{-t}(\cos t + 2\sin t), & 0 < t < \pi \\ e^{-t}(\cos t + 2\sin t) - e^{-(t-\pi)}\sin t, & t > \pi \end{cases}$

3. $y(t) = \begin{cases} \dfrac{A}{\omega^2 - 1}(\cos t - \cos \omega t), & 0 < t < \pi \\ \dfrac{A}{\omega^2 - 1}(\cos t - \cos \omega t) - \sin t, & t > \pi \end{cases}$

5. $y(t) = \begin{cases} 2\sin t, & 0 < t < \dfrac{\pi}{2} \\ 2\sin t - A\cos t, & t > \dfrac{\pi}{2} \end{cases}$

7. $y(t) = \begin{cases} (t+3)e^t - 2e^{t/2}\cos\dfrac{\sqrt{3}}{2}t, & 0 < t < 1 \\ (t+3)e^t - 2e^{t/2}\cos\dfrac{\sqrt{3}}{2}t + e^{t-1} \\ \quad - e^{(t-1)/2}\left[\cos\dfrac{\sqrt{3}}{2}(t-1) + \dfrac{1}{\sqrt{3}}\sin\dfrac{\sqrt{3}}{2}(t-1)\right], & t > 1 \end{cases}$

CHAPTER 6

Section 6.2

1. $\dfrac{1}{s^2}$
3. $\dfrac{n!}{s^{n+1}}$
5. $\dfrac{s}{s^2 + k^2}$
7. $\dfrac{b - a}{(s + a)(s + b)}$
9. $\dfrac{s}{s^2 - k^2}$
11. $\dfrac{2}{s^3} - \dfrac{3}{s^2} + \dfrac{5}{s}$
13. $\dfrac{s^2 - 2k^2}{s(s^2 - 4k^2)}$
15. $\dfrac{s - a}{(s - a)^2 - k^2}$
17. $\dfrac{1}{(s - a)^2}$
19. $\dfrac{2s(s^2 - 3k^2)}{(s^2 + k^2)^3}$
21. $\dfrac{s^2 + 2k^2}{s(s^2 + 4k^2)}$
25. (a) $\dfrac{e^{-s}}{s}$ (b) $\dfrac{2(1 - e^{-\pi s})}{s^2 + 4}$ (c) $\dfrac{(1 + e^{-2s})}{s^2} + \dfrac{(s - 2)}{s^2}e^{-s}$

Section 6.3

1. $\dfrac{3}{(s - 2)^2}$
3. $\dfrac{2k(3s^2 - k^2)}{(s^2 + k^2)^3}$
5. $\dfrac{s^3}{s^4 + 4k^4}$
7. $\dfrac{3(s + 4)}{(s + 4)^2 + 16} - \dfrac{12(s + 4)}{[(s + 4)^2 + 16]^2}$
9. $\log\dfrac{s + 1}{s - 1}$
11. $\dfrac{(s + 1)^2 + 2}{(s + 1)[(s + 1)^2 + 4]}$
13. (a) $\dfrac{1}{s} + \dfrac{1}{s^2}(1 - e^{-2s})$ (b) $\dfrac{1}{s}(1 - e^{-2s})$
15. $\dfrac{s}{s^2 + 16}$
17. $\dfrac{1}{s}\arctan\dfrac{1}{s}$
25. (b) $\pi\,\mathrm{erf}\left(\dfrac{1}{2\sqrt{s}}\right)$

Section 6.4

1. $\dfrac{7t^2}{2}$
3. $\dfrac{1}{12}t^4e^{-3t}$
5. $e^{3t}\sin t$
7. $13te^{-4t}$
9. $e^{-2t}\left(7\cos 5t - \dfrac{17}{5}\sin 5t\right)$
11. $\dfrac{1}{3}e^{-2t/3}\left(5\cos\dfrac{2}{3}\sqrt{5}t - 4\sqrt{5}\sin\dfrac{2}{3}\sqrt{5}t\right)$
13. $1 - e^{-t}$
15. $e^{-2t}(1 - 4t + 2t^2)$
17. $-\dfrac{1}{20} + \dfrac{5}{324}e^{4t} + e^{-5t}\left(\dfrac{35}{162} - \dfrac{4}{45}t\right)$
19. $\dfrac{1}{3}(e^{-t} - e^t) + \dfrac{5}{12}(e^{2t} - e^{-2t})$

21. $\frac{3}{10}e^{2t} - \frac{1}{6} - \frac{2}{15}e^{-3t}$ 23. $\frac{3}{50}e^{3t} - \frac{1}{25}e^{-2t} + \frac{1}{50}e^{-t}(9\sin 2t - \cos 2t)$ 25. $(\pi t)^{-1/2}\cos 2(at)^{1/2}$

Section 6.5

1. $y(t) = \frac{1}{2}(e^{2t} - 3)$ 3. $y(t) = \sinh t$ 5. $y(t) = \sin t$ 7. $y(t) = e^t - (\cos t + \sin t)$

9. $y(t) = (e^{-t} - 1)\sin t$ 11. $y(t) = \frac{1}{6} + \frac{1}{3}e^{-t} + e^{-2t}\left(\frac{1}{2}\cos\sqrt{2}t - \frac{8}{3\sqrt{2}}\sin\sqrt{2}t\right)$

13. $y(t) = \frac{1}{8}(1 + 2t) + \frac{7}{8}e^{2t}(1 - 2t)$ 15. $y(t) = 8e^{-t} - 9e^{-2t} + 3e^{-3t}$

17. $y(t) = \frac{5}{18}e^t - \frac{8}{9}e^{-t/2} + \frac{1}{9}e^{-2t} + \frac{1}{2}e^{-t}$ 19. $y(t) = \frac{1}{2}(\sinh t + \sin t) - t$

Section 6.6

7. $\frac{2}{s} + \left(\frac{1}{s^2} - \frac{1}{s}\right)e^{-s}$ 9. $\frac{1 + se^{-\pi s}}{s^2 + 1}$ 11. $\frac{e^{-3(s-1)}}{s + 1}$

13. $\frac{2}{s^3} + \left(\frac{2}{s} - \frac{2}{s^2} - \frac{2}{s^3}\right)e^{-s} - \frac{3}{s}e^{-4s}$ 15. $5h(t - 3) - h(t - 1)$

17. $3(t - 2)h(t - 2) - t$ 19. $e^{-2(t-3)}h(t - 3)$ 21. $\cos 2t[1 + h(t - \pi)]$

23. $y(t) = e^{-t} + [1 - e^{-(t-1)}]h(t - 1)$ 25. $y(t) = \cos 2t + t - \frac{1}{2}\sin t - [t - 1 - \frac{1}{2}\sin 2(t - 1)]h(t - 1)$

27. $y(t) = \frac{1}{12}(\cos 2t - \cos 4t)[1 + h(t - \pi)] + \frac{1}{2}\sin 2t$ 29. $y(t) = e^{3t} - e^{2t} + \frac{1}{6}[1 + 2e^{3(t-1)} - 3e^{2(t-1)}]h(t - 1)$

31. $y(t) = e^{-t}(\cos t + \sin t) - e^{-(t-\pi)}\sin t\, h(t - \pi)$

33. $y(t) = 2te^{-t} + [1 - e^{-(t-2\pi)} - (t - 2\pi)e^{-(t-2\pi)}]h(t - 2\pi)$

35. $y(t) = \frac{P}{m(\omega^2 - \omega_0^2)}\left(\frac{1}{\omega_0}\sin\omega_0 t - \frac{1}{\omega}\sin\omega t\right) - \frac{5}{m\omega_0}\sin\omega_0(t - \pi)$, where $\omega_0^2 = \frac{k}{m}$

Section 6.7

1. $\frac{2}{s(s^2 + 4)}$ 3. $\frac{s}{(s + 1)(s^2 + 1)}$ 5. $e^{-t} + t - 1$ 7. $\frac{1}{6}t^3 - t + \sin t$

9. $\frac{1}{25}(2 + 5t)e^{-t} - \frac{1}{50}(4\cos 2t + 3\sin 2t)$ 15. $\frac{1}{a}\sinh at$ 17. $y(t) = t + \frac{3}{2}\sin 2t$

19. $y(t) = 1 + \frac{t^2}{2}$ 23. (a) $y(t) = \frac{P}{k}(1 - \cos\omega_0 t)$

(b) $y(t) = \frac{P}{m(\omega_0^2 - \omega^2)}(\cos\omega t - \cos\omega_0 t)$ (c) $y(t) = \frac{P}{2m\omega_0}t\sin\omega_0 t$

CHAPTER 7

Section 7.2

1. $x(t) = C_1e^t + C_2e^{-t}$
$y(t) = C_1e^t + 3C_2e^{-t}$

3. $x(t) = C_1 + C_2e^{-2t}$
$y(t) = \frac{4}{3}C_1 + 2C_2e^{-2t}$

5. $x(t) = e^t(C_1\cos 2t + C_2\sin 2t)$
$y(t) = \frac{1}{2}e^t(C_1\sin 2t - C_2\cos 2t)$

7. $x(t) = e^{4t}(C_1\cos t + C_2\sin t)$
$y(t) = e^{4t}[(2C_1 - C_2)\cos t + (2C_2 + C_1)\sin t]$

9. $x(t) = C_1e^t + C_2e^{-2t} + C_3e^{3t}$
$y(t) = -4C_1e^t - C_2e^{-2t} + 2C_3e^{3t}$
$z(t) = -C_1e^t - C_2e^{-2t} + C_3e^{3t}$

11. $x(t) = C_1e^{(a+b)t} + C_2e^{(a-b)t}$
$y(t) = C_1e^{(a+b)t} - C_2e^{(a-b)t}$

13. $x(t) = C_1e^t + C_2e^{-t} - 1$
$y(t) = C_1e^t + 3C_2e^{-t} + t - 2$

15. $x(t) = C_1e^{-3t} + C_2e^{2t}$
$y(t) = -3C_1e^{-3t} - \frac{1}{2}C_2e^{2t}$

17. $x(t) = 2C_1 + 6C_2e^{-t}$
$y(t) = C_1 + C_2e^{-t} + C_3e^{2t}$

19. $x(t) = C_1e^{2t}$
$y(t) = C_2e^{-2t}$

21. $x(t) = C_1e^t + C_2e^{-3t} - \frac{1}{3}t - \frac{11}{36}$
$y(t) = -\frac{1}{2}C_1e^t + \frac{3}{2}C_2e^{-3t} + \frac{1}{8}t + \frac{5}{12}$

23. $x(t) = C_1e^t - \frac{1}{2}\sin t$
$y(t) = -\frac{1}{3}C_1e^t + \frac{1}{2}\sin t$

25. $x(t) = C_1e^{-t} + C_2e^{2t} + 3C_3e^{4t} + \frac{1}{4}$
$y(t) = 3C_1e^{-t} - C_3e^{4t} + \frac{1}{4}$

27. $x(t) = -6C_1e^{-t} - 3C_2e^{-2t} + 2C_3e^{3t}$
$y(t) = C_1e^{-t} + C_2e^{-2t} + C_3e^{3t}$
$z(t) = 5C_1e^{-t} + C_2e^{-2t} + C_3e^{3t}$

29. no solutions

31. infinitely many solutions

Section 7.3

1. $x(t) = \frac{1}{3}(e^t - e^{-2t})$
$y(t) = \frac{1}{3}(2e^t + e^{-2t})$

3. $x(t) = -\cos 3t - \frac{5}{3}\sin 3t$
$y(t) = 2\cos 3t - \frac{7}{3}\sin 3t$

5. $x(t) = -e^t - \frac{1}{10}e^{-t} + \frac{1}{10}e^{2t}(21\cos t - 13\sin t)$
$y(t) = e^t - \frac{1}{5}e^{-t} + \frac{1}{5}e^{2t}(-4\cos t + 7\sin t)$

7. $x(t) = 5e^{-t} - 4 + 5t - 2t^2 + \frac{1}{3}t^3$
$y(t) = -5e^{-t} + 5 - 5t + 2t^2$

9. $x(t) = -\frac{1}{2}t - \frac{3}{4}\sqrt{2}\sin\sqrt{2}t$
$y(t) = -\frac{1}{2}t + \frac{3}{4}\sqrt{2}\sin\sqrt{2}t$

11. $x(t) = t + 2 + e^{-2t} + \sin t$
$y(t) = 1 - t - 3e^{-2t} - \cos t$

13. $y_1(t) = \dfrac{3}{5}\sin t + \dfrac{2}{5}\cos t - \dfrac{2}{5}\cos\sqrt{6}t + \dfrac{\sqrt{6}}{15}\sin\sqrt{6}t$
$y_2(t) = \dfrac{6}{5}\sin t + \dfrac{4}{5}\cos t + \dfrac{1}{5}\cos\sqrt{6}t - \dfrac{\sqrt{6}}{30}\sin\sqrt{6}t$

15. $y_1(t) = \dfrac{13}{42}\sin t + \dfrac{145}{42}\sin 2t - \dfrac{25}{14}\sin 3t$
$y_2(t) = \dfrac{13}{21}\sin t - \dfrac{145}{42}\sin 2t + \dfrac{10}{7}\sin 3t$

19. $i_1(t) = 1 + e^{-50t}\left(\dfrac{1}{\sqrt{3}}\sin 50\sqrt{3}t - \cos 50\sqrt{3}t\right)$
$i_2(t) = 1 - e^{-50t}\left(\cos 50\sqrt{3}t + \dfrac{1}{\sqrt{3}}\sin 50\sqrt{3}t\right)$

23. (a) 0.69, 1.66 (b) 0.72, 1.38
(c) 0.67, 2.10 (d) 0.56, 1.46
(e) 0.56, 1.46

Section 7.4

1. $x_1' = x_2$
$x_2' = -kx_1 + P\sin\omega t$

3. $x_1' = x_2$
$x_2' = x_3$
$x_3' = x_4$
$x_4' = k^4x_1 + f(t)$

5. $u' = w$
$v' = -u + t - 10$
$w' = t - 10 + e^t$

7. $x_1' = x_2$
$x_2' = -\dfrac{1}{m}(k_1 + k_2)x_1 + \dfrac{k_2}{m}x_3$
$x_3' = x_4$
$x_4' = \dfrac{k_1}{m}x_1 - \dfrac{k_2}{m}x_3$

9. $x(t) = C_1e^t + 3C_2e^{-t}$
$y(t) = C_1e^t + 5C_2e^{-t}$

11. $x(t) = C_1e^{-3t} + C_2e^{5t}$
$y(t) = -3C_1e^{-3t} + C_2e^{5t}$

13. $x(t) = e^{3t}(C_1\cos 2t + C_2\sin 2t)$
$y(t) = e^{3t}(C_1\sin 2t - C_2\cos 2t)$

15. $x(t) = 2e^{2t}(C_1\cos 3t + C_2\sin 3t)$
$y(t) = e^{2t}[(3C_2 - C_1)\cos 3t - (3C_1 + C_2)\sin 3t]$

17. $x(t) = C_1e^t + C_2te^t$
$y(t) = -C_1e^t - C_2(t + \frac{1}{2})e^t$

19. $x(t) = -3C_1e^{-t} + C_3e^{2t}$
$y(t) = 4C_1e^{-t} + C_2e^{2t} + C_3te^{2t}$
$z(t) = 2C_1e^{-t} - C_2e^{2t} + C_3(1 - t)e^{2t}$

21. independent

23. independent

27. $x(t) = C_1\cos t + C_2\sin t + \frac{2}{3}(\cos 2t + \sin 2t) - 5t$
$y(t) = \frac{1}{5}(2C_1 - C_2)\cos t + \frac{1}{5}(2C_2 + C_1)\sin t + \frac{1}{3}\sin 2t - 2t + 1$

29. $x(t) = C_1e^{-2t} + C_2e^{3t} - \frac{1}{6}e^t$
$y(t) = -4C_1e^{-2t} + C_2e^{3t} - \frac{5}{6}e^t$

31. $x(t) = e^t[C_1\sin t - 2C_2\cos t - 2\cos t\log|\sec t + \tan t|]$
$y(t) = e^t[\frac{1}{2}C_1\cos t + C_2\sin t - 1 + \sin t\log|\sec t + \tan t|]$

Section 7.5

1. $\mathbf{Y}(t) = C_1\begin{pmatrix}1\\1\end{pmatrix}e^t + C_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t}$

3. $\mathbf{Y}(t) = C_1\begin{pmatrix}1\\\frac{4}{3}\end{pmatrix} + C_2\begin{pmatrix}1\\2\end{pmatrix}e^{-t}$

5. $\mathbf{Y}(t) = C_1\begin{pmatrix}2\cos 3t\\-\cos 3t - 3\sin 3t\end{pmatrix}e^t + C_2\begin{pmatrix}2\sin 3t\\3\cos 3t - \sin 3t\end{pmatrix}e^{2t}$

7. $\mathbf{Y}(t) = C_1\begin{pmatrix}1\\6\\-13\end{pmatrix} + C_2\begin{pmatrix}1\\-2\\-1\end{pmatrix}e^{-4t} + C_3\begin{pmatrix}1\\\frac{3}{2}\\-1\end{pmatrix}e^{3t}$

9. $\mathbf{Y}(t) = \frac{5}{3}\begin{pmatrix}1\\1\end{pmatrix}e^{-2t} + \frac{1}{3}\begin{pmatrix}1\\-2\end{pmatrix}e^{-5t}$

11. (a) $\begin{pmatrix}1.5 & 1\\1 & 1.5\end{pmatrix}$ (b) $\begin{pmatrix}1.543 & 1.175\\1.175 & 1.543\end{pmatrix}$ (c) $\begin{pmatrix}\cosh t & \sinh t\\\sinh t & \cosh t\end{pmatrix}$

13. $\mathbf{Y}(t) = C_1\left[\begin{pmatrix}1\\\frac{2}{5}\end{pmatrix}\cos t + \begin{pmatrix}0\\\frac{1}{5}\end{pmatrix}\sin t\right] + C_2\left[\begin{pmatrix}0\\\frac{1}{5}\end{pmatrix}\cos t + \begin{pmatrix}1\\\frac{2}{5}\end{pmatrix}\sin t\right] + \frac{1}{3}\begin{pmatrix}2\cos 2t + 2\sin 2t\\\sin 2t\end{pmatrix} + \begin{pmatrix}-5t\\-2t + 1\end{pmatrix}$

15. $\mathbf{Y}(t) = C_1\begin{pmatrix}1\\-4\end{pmatrix}e^{-2t} + C_2\begin{pmatrix}1\\1\end{pmatrix}e^{3t} - \frac{1}{6}\begin{pmatrix}1\\5\end{pmatrix}e^t$

17. $\mathbf{Y}(t) = C_1\begin{pmatrix}\sin t\\\frac{1}{2}\cos t\end{pmatrix}e^t + C_2\begin{pmatrix}-2\cos t\\\sin t\end{pmatrix}e^t + \begin{pmatrix}-2\cos t\log|\sec t + \tan t|\\-1 + \sin t\log|\sec t + \tan t|\end{pmatrix}e^t$

19. $\mathbf{Y}(t) = \begin{pmatrix}2\\3\end{pmatrix}e^{-t} - \begin{pmatrix}6\\8\end{pmatrix}e^{-2t} + \begin{pmatrix}4\\3\end{pmatrix}e^{-3t}$

Section 7.6

5. (0, 0)

7. (0, 0)

9. (0, 0), $(-\frac{1}{2}, 1)$

11. $y^2 - (x + 1)^2 = C$

13. $(x - y)^2 = C(x + y)$

15. $x^3 - 3xy + y^3 = C$

17. $3xy - x^2y^3 - 2y^2 = C$

19. node, asymptotically stable

21. saddle point, unstable

23. foci, unstable

25. unstable

27. unstable saddle point

29. no conclusion

31. asymptotically stable

33. (0, 0) unstable
(2, 4) unstable

35. (1, 1) unstable
(−1, −1) asymptotically stable

CHAPTER 8

Section 8.2

1. 2.9278 (Euler),
3.4509 (Improved Euler)

3. 3.2261 (Euler)
3.8254 (Improved Euler)

5. 1.8370 (Euler)
2.0486 (Improved Euler)

7. 1.2194 (Euler)
1.3260 (Improved Euler)

9. 0.4198 (Euler)
0.4053 (Improved Euler)

11. answers given above in 1–9

13. $y(1) = 2.4883$,
$e = 2.71828\ldots$ (actual value)

15. $y(1) = 2.7182$,
$e = 2.71828\ldots$ (actual value)

17. $y(1) = 0.7839$, $\pi \cong 3.1355$

23. $y(0.5) = 0.7971$

Section 8.3

1. $x(0.1) = 1.1$, $y(0.1) = -0.1$
$x(0.2) = 1.25$, $y(0.2) = -0.22$

3. $x(0.1) = 1.2$, $y(0.1) = 1.1$
$x(0.2) = 1.451$, $y(0.2) = 1.32$

5. $x(0.1) = 0.8$, $y(0.1) = 1.1$
$x(0.2) = 0.582$, $y(0.2) = 1.18$

7. $x(0.1) = 1.3110$, $y(0.1) = 1.9197$

9. $x(0.1) = 1.1305$, $y(0.1) = 0.3851$

11. $x(0.1) = 1.2$, $x(0.2) = 1.39$

13. $x(0.1) = 1.1947$

CHAPTER 9

Section 9.2

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(n+2)(n+5)}{(n+2)!} x^n$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$

5. $x = \pm i$

7. $x = 0, -3$

9. $y = c_0 \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n$

11. $y = c_0(1 - \frac{1}{2}x^2 + \frac{1}{6}x^4 - \cdots)$

13. $y = c_0 \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$, $|x| < \infty$

15. $y = c_0(1 + 4x^2) + c_1(x + \frac{4}{3}x^3 - \frac{16}{15}x^5 + \cdots)$, $|x| < \frac{1}{2}$

17. $y = c_0(1 - 6x^2 + 3x^4 - \cdots) + c_1(x - \frac{5}{3}x^3)$, $|x| < 1$

19. $y = c_0(1 - x^2 + \frac{1}{4}x^4 - \cdots) + c_1(x - \frac{1}{2}x^3 + \frac{3}{40}x^5 - \cdots)$, $|x| < \infty$

21. $y = c_0 + c_1(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots)$, $|x| < 1$

23. $y = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{48}x^6 + \cdots$

25. $y = 4 + 6x + \frac{11}{3}x^3 + \frac{1}{2}x^4 + \cdots$

27. $y = c_0[1 - \frac{1}{6}(x-1)^3 + \frac{1}{180}(x-1)^6 - \cdots] + c_1[(x-1) - \frac{1}{12}(x-1)^4 + \frac{1}{504}(x-1)^7 - \cdots]$

29. $y = c_0[1 - 3(x-1)^2] + c_1[(x-1) - \frac{1}{3}(x-1)^3]$

31. $y = c_0(1 + \frac{7}{3!}x^3 + \frac{102}{6!}x^6 + \cdots) + c_1(x + \frac{10}{4!}x^4 + \frac{190}{7!}x^7 + \cdots) + c_2(x^2 + \frac{26}{5!}x^5 + \frac{572}{8!}x^8 + \cdots)$

33. (a) $y = c_0(1 - x^2 - \frac{1}{6}x^4 - \frac{1}{30}x^6 - \cdots) + c_1 x$
(b) $y = c_0(1 - 4x^2 + \frac{4}{3}x^4) + c_1(x - x^3 + \frac{1}{10}x^5 - \cdots)$

Section 9.3

1. I.S.P. at $x = 0$, R.S.P. at $x = \pm 1$

3. R.S.P. at $x = \pm i, \pm 1$, I.S.P. at $x = 0$

5. R.S.P. at $x = 0$, I.S.P. at $x = 4$

7. R.S.P. at $x = 0, 3$

9. $y = A(1 + 2x + \frac{1}{3}x^2) + Bx^{1/2}(1 + \frac{1}{2}x + \frac{1}{40}x^2 + \cdots)$

11. $y = A(1 - x + \frac{15}{14}x^2 - \cdots) + Bx^{-3/2}(1 - 10x)$

13. $y = Ax^{-1} + Bx^{1/2}$

15. $y = Ax^{1/2}(1 - x + \frac{1}{6}x^2 - \cdots) + Bx(1 - \frac{1}{3}x + \frac{1}{30}x^2 - \cdots)$

17. $y = Ax^{-1}(1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \cdots) + Bx^{3/2}(1 - \frac{1}{18}x^2 + \frac{1}{936}x^4 - \cdots)$

19. $y = (A + B\log x)(1 + 4x + 4x^2 + \cdots) - B(8x + 12x^2 + \frac{176}{27}x^3 + \cdots)$

21. $y = (A + B\log x)xe^x - B(x^2 + \frac{3}{4}x^3 + \frac{11}{36}x^4 + \cdots)$

23. $y = (A + B\log x)x^{-2}$

25. $y = (A + B\log x)x - B(x^2 - \frac{1}{4}x^3 + \frac{1}{18}x^4 - \cdots)$

27. $y = c_0(x - 2x^2 + 2x^3) + c_3(x^4 - \frac{1}{2}x^5 + \frac{1}{3}x^6 - \cdots)$

29. $y = c_0(1 + \frac{2}{3}x + \frac{1}{6}x^2) + c_4(x^4 + \frac{2}{3}x^5 + \frac{1}{10}x^6 + \cdots)$

31. $y = c_0x^{-1} + c_3(x^2 + \frac{3}{4}x^3 + \frac{3}{10}x^4 + \cdots)$

33. $y = (A + B\log x)(-12 + 32x - 40x^2 + \cdots) + B\left(\dfrac{1}{x^2} + \dfrac{4}{x} - 4 - 16x + \cdots\right)$

35. $y = (A + B\log x)\left(-3x - 4x - \dfrac{5}{2}x^2 - \cdots\right) + B\left(\dfrac{1}{x} - 2 - x - 3x^2 + \cdots\right)$

Index

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15800