COLLEGE ALGEBRA

JEROME E. KAUFMANN

8E

KAREN L. SCHWITTERS

BROOKS/COLE
CENGAGE Learning

***College Algebra*, Eighth Edition**
Jerome E. Kaufmann and
Karen L. Schwitters

Acquisitions Editor: Gary Whalen

Developmental Editor/Assistant Editor: Cynthia Ashton

Editorial Assistant: Sabrina Black

Media Editor: Lynh Pham

Senior Marketing Manager: Danae April

Marketing Communications Manager: Mary Anne Payumo

Content Project Manager: Jennifer Risden

Design Director: Rob Hugel

Art Director: Vernon Boes

Manufacturing Planner: Becky Cross

Rights Acquisitions Specialist: Roberta Broyer

Production Service: Graphic World Inc.

Text Designer: Diane Beasley

Photo Researcher: Bill Smith Group

Copy Editor: Graphic World Inc.

Illustrator: Network Graphics; Graphic World Inc.

Cover Designer: Irene Morris

Cover Image: Fotolia

Compositor: Graphic World Inc.

Library of Congress Control Number: 2011938660

ISBN-13: 978-1-111-99036-7
ISBN-10: 1-111-99036-0

Brooks/Cole
20 Davis Drive
Belmont, CA 94002-3098
USA

Cengage Learning is a leading provider of customized learning solutions with office locations around the globe, including Singapore, the United Kingdom, Australia, Mexico, Brazil, and Japan. Locate your local office at **www.cengage.com/global.**

Cengage Learning products are represented in Canada by Nelson Education, Ltd.

To learn more about Brooks/Cole, visit **www.cengage.com/brookscole**

Purchase any of our products at your local college store or at our preferred online store **www.CengageBrain.com**

Printed in the United States of America
1 2 3 4 5 6 7 16 15 14 13 12

0 Some Basic Concepts of Algebra: A Review

This statue of Fibonacci was constructed and erected in Pisa, Italy. Leonardo Fibonacci was a famous Italian middle-ages mathematician. He is known for spreading the Hindu-Arabic number system in the western world and the Fibonacci sequence of numbers.

© David Lyons/Alamy

The temperature in Big Lake, Alaska at 3 P.M. was $-4°$F. By 11 P.M. the temperature had dropped another 20°. We can use the *numerical expression* $-4 - 20$ to determine the temperature at 11 P.M.

Megan has p pennies, n nickels, d dimes, and q quarters. The *algebraic expression* $p + 5n + 10d + 25q$ can be used to represent the total amount of money in cents.

Algebra is often described as a generalized arithmetic. That description does not tell the whole story, but it does convey an important idea: A good understanding of arithmetic provides a sound basis for the study of algebra. In this chapter we will often use arithmetic examples to lead into a review of basic algebraic concepts. Then we will use the algebraic concepts in a wide variety of problem-solving situations. Your study of algebra should make you a better problem solver. Be sure that you can work effectively with the algebraic concepts reviewed in this first chapter.

0.1 Some Basic Ideas

OBJECTIVES

1. Recognize the vocabulary and symbolism associated with sets
2. Know the various subset classifications of the real number system
3. Find distance on a number line
4. Apply the definition of the absolute value of a number
5. Know the real number properties
6. Evaluate algebraic expressions
7. Review the Cartesian coordinate system

Let's begin by pulling together the basic tools we need for the study of algebra. In arithmetic, symbols such as 6, $\frac{2}{3}$, 0.27, and π are used to represent numbers. The operations of addition, subtraction, multiplication, and division are commonly indicated by the symbols $+$, $-$, $\times$, and $\div$, respectively. These symbols enable us to form specific **numerical expressions.** For example, the indicated sum of 6 and 8 can be written $6 + 8$.

In algebra, we use variables to generalize arithmetic ideas. For example, by using x and y to represent *any* two numbers, we can use the expression $x + y$ to represent the indicated sum of *any* two numbers. The x and y in such an expression are called **variables,** and the phrase $x + y$ is called an **algebraic expression.**

Many of the notational agreements we make in arithmetic can be extended to algebra, with a few modifications. The following chart summarizes those notational agreements regarding the four basic operations.

Operation	Arithmetic	Algebra	Vocabulary
Addition	$4 + 6$	$x + y$	The sum of x and y
Subtraction	$14 - 10$	$a - b$	The difference of a and b
Multiplication	7×5 or $7 \cdot 5$	$a \cdot b$, $a(b)$, $(a)b$, $(a)(b)$, or ab	The product of a and b
Division	$8 \div 4$, $\frac{8}{4}$, $8/4$ or $4\overline{)8}$	$x \div y$, $\frac{x}{y}$, x/y, or $y\overline{)x}$ $(y \neq 0)$	The quotient of x divided by y

Note the different ways of indicating a product, including the use of parentheses. The ab form is the simplest and probably the most widely used form. Expressions such as abc, $6xy$, and $14xyz$ all indicate multiplication. Notice the various forms used to indicate division. In algebra, the fraction forms $\frac{x}{y}$ and x/y are generally used, although the other forms do serve a purpose at times.

The Use of Sets

Some of the vocabulary and symbolism associated with the concept of sets can be effectively used in the study of algebra. A **set** is a collection of objects; the objects are called **elements** or **members of the set.** The use of capital letters to name sets and the use of set braces, { }, to enclose the elements or a description of the elements provide a convenient way to communicate about sets. For example, a set A that consists of the vowels of the English alphabet can be represented as follows:

	$A = \{\text{vowels of the English alphabet}\}$	Word description
or	$A = \{\text{a, e, i, o, u}\}$	List or roster description
or	$A = \{x \mid x \text{ is a vowel}\}$	Set-builder notation

A set consisting of no elements is called the **null set** or **empty set** and is written $\varnothing$.

Set-builder notation combines the use of braces and the concept of a variable. For example, $\{x \mid x \text{ is a vowel}\}$ is read "the set of all x such that x is a vowel." Note that the vertical line is read "such that."

Two sets are said to be **equal** if they contain exactly the same elements. For example, $\{1, 2, 3\} = \{2, 1, 3\}$ because both sets contain exactly the same elements; the order in which the elements are listed does not matter. A slash mark through an equality symbol denotes *not equal to*. Thus if $A = \{1, 2, 3\}$ and $B = \{3, 6\}$, we can write $A \neq B$, which is read "set A is not equal to set B."

Real Numbers

The following terminology is commonly used to classify different types of numbers:

$\{1, 2, 3, 4, \ldots\}$	Natural numbers, counting numbers, positive integers
$\{0, 1, 2, 3, \ldots\}$	Whole numbers, nonnegative integers
$\{\ldots, -3, -2, -1\}$	Negative integers
$\{\ldots, -3, -2, -1, 0\}$	Nonpositive integers
$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$	Integers

A **rational number** is defined as any number that can be expressed in the form a/b, where a and b are integers and b is not zero. The following are examples of rational numbers:

$$\frac{2}{3} \qquad -\frac{3}{4} \qquad \frac{-1}{7} \qquad \frac{9}{2}$$

$$6\frac{1}{2} \text{ because } 6\frac{1}{2} = \frac{13}{2} \qquad -4 \text{ because } -4 = \frac{-4}{1} = \frac{4}{-1}$$

$$0 \text{ because } 0 = \frac{0}{1} = \frac{0}{2} = \frac{0}{3}, \text{ etc.} \qquad 0.3 \text{ because } 0.3 = \frac{3}{10}$$

A rational number can also be defined in terms of a decimal representation. Before doing so, let's briefly review the different possibilities for decimal representations. Decimals can be classified as **terminating, repeating,** or **nonrepeating.** Here are some examples of each:

$$\left.\begin{array}{l} 0.3 \\ 0.46 \\ 0.789 \\ 0.2143 \end{array}\right] \quad \text{Terminating decimals}$$

$$\left.\begin{array}{l} 0.333\ldots \\ 0.1414\ldots \\ 0.7127127\ldots \\ 0.241717\ldots \end{array}\right] \quad \text{Repeating decimals}$$

$$\left.\begin{array}{l} 0.472195631\ldots \\ 0.21411711191111\ldots \\ 3.141592654\ldots \\ 1.414213562\ldots \end{array}\right] \quad \text{Nonrepeating decimals}$$

A **repeating decimal** has a block of digits that repeats indefinitely. This repeating block of digits may be of any size and may or may not begin immediately after the decimal point. A small horizontal bar is commonly used to indicate the repeating block. Thus 0.3333 . . . can be expressed as $0.\overline{3}$ and 0.24171717 . . . as $0.24\overline{17}$.

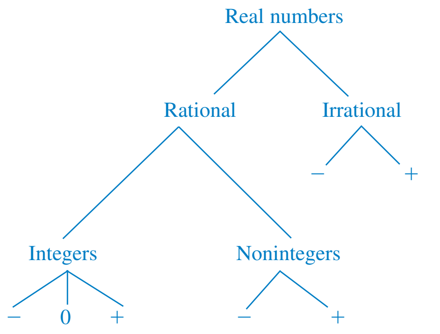
In terms of decimals, a rational number is defined as a number with either a terminating or a repeating decimal representation. The following examples illustrate some rational numbers written in $\frac{a}{b}$ form and in the equivalent decimal form:

$$\frac{3}{4} = 0.75 \qquad \frac{3}{11} = 0.\overline{27} \qquad \frac{1}{8} = 0.125 \qquad \frac{1}{7} = 0.\overline{142857} \qquad \frac{1}{3} = 0.\overline{3}$$

We define an **irrational number** as a number that cannot be expressed in $\frac{a}{b}$ form, form, where a and b are integers and b is not zero. Furthermore, an irrational number has a nonrepeating, nonterminating decimal representation. Following are some examples of irrational numbers and a partial decimal representation for each number. Note that the decimals do not terminate and do not repeat.

$$\sqrt{2} = 1.414213562373095\ldots$$
$$\sqrt{3} = 1.73205080756887\ldots$$
$$\pi = 3.14159265358979\ldots$$

The entire set of **real numbers** is composed of the rational numbers along with the irrationals. The following tree diagram can be used to summarize the various classifications of the real number system.



Any real number can be traced down through the tree. Here are some examples:

7 is real, rational, an integer, and positive.

$-\dfrac{2}{3}$ is real, rational, a noninteger, and negative.

$\sqrt{7}$ is real, irrational, and positive.

0.59 is real, rational, a noninteger, and positive.

The concept of a subset is convenient to use at this time. A set A is a **subset** of another set B if and only if every element of A is also an element of B. For example, if $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then A is a subset of B. This is written $A \subseteq B$ and is read "A is a subset of B." The slash mark can also be used here to denote negation. If $A = \{1, 2, 4, 6\}$ and $B = \{2, 3, 7\}$, we can say *A is not a subset of B* by writing $A \nsubseteq B$. The following statements use the subset vocabulary and symbolism; they are represented in Figure 0.1.

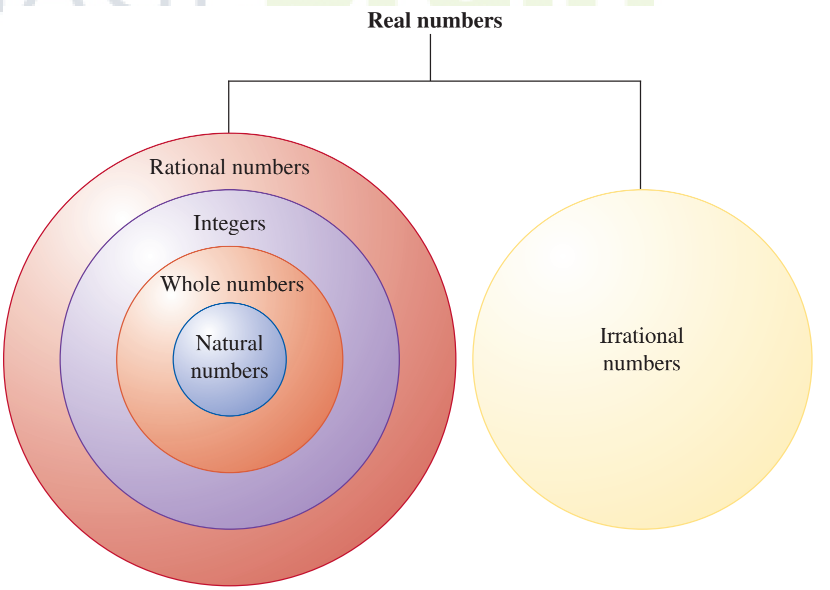


Figure 0.1

1. The set of whole numbers is a subset of the set of integers:

$$\{0, 1, 2, 3, \ldots\} \subseteq \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

2. The set of integers is a subset of the set of rational numbers:

$$\{\ldots, -2, -1, 0, 1, 2, \ldots\} \subseteq \{x \mid x \text{ is a rational number}\}$$

3. The set of rational numbers is a subset of the set of real numbers:

$$\{x \mid x \text{ is a rational number}\} \subseteq \{y \mid y \text{ is a real number}\}$$

Real Number Line and Absolute Value

It is often helpful to have a geometric representation of the set of real numbers in front of us, as indicated in Figure 0.2. Such a representation, called the **real number line,** indicates a one-to-one correspondence between the set of real numbers and the points on a line. In other words, to each real number there corresponds one and only one point on the line, and to each point on the line there corresponds one and only one real number. The number that corresponds to a particular point on the line is called the **coordinate** of that point.

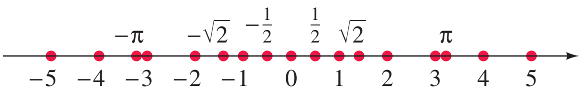


Figure 0.2

Many operations, relations, properties, and concepts pertaining to real numbers can be given a geometric interpretation on the number line. For example, the addition problem $(-1) + (-2)$ can be interpreted on the number line as shown in Figure 0.3.

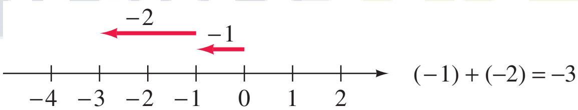


Figure 0.3

The inequality relations also have a geometric interpretation. The statement $a > b$ (read "a is greater than b") means that a is to the right of b, and the statement $c < d$ (read "c is less than d") means that c is to the left of d (see Figure 0.4).

Figure 0.4

The property $-(-x) = x$ can be pictured on the number line in a sequence of steps. See Figure 0.5.

1. Choose a point that has a coordinate of x.

2. Locate its opposite (written as $-x$) on the other side of zero.

3. Locate the opposite of $-x$ [written as $-(-x)$] on the other side of zero.

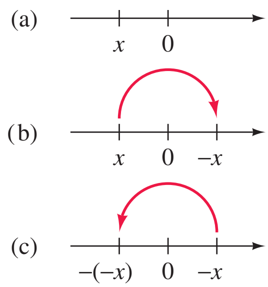


Figure 0.5

Therefore, we conclude that **the opposite of the opposite of any real number is the number itself,** and we express this symbolically by $-(-x) = x$.

Remark: The symbol -1 can be read "negative one," the "negative of one," the "opposite of one," or the "additive inverse of one." The opposite-of and additive-inverse-of terminology is especially meaningful when working with variables. For example, the symbol $-x$, read "the opposite of x or the additive inverse of x," emphasizes an important issue. Because x can be any real number, $-x$ (opposite of x) can be zero, positive, or negative. If x is positive, then $-x$ is negative. If x is negative, then $-x$ is positive. If x is zero, then $-x$ is zero. For example,

If $x = 4$, then $-x = -(4) = -4$.

If $x = -2$, then $-x = -(-2) = 2$.

If $x = 0$, then $-x = -(0) = 0$.

The concept of absolute value can be interpreted on the number line. Geometrically, the **absolute value** of any real number is the distance between that number and zero on the number line. For example, the absolute value of 2 is 2, the absolute value of -3 is 3, and the absolute value of zero is zero (see Figure 0.6).

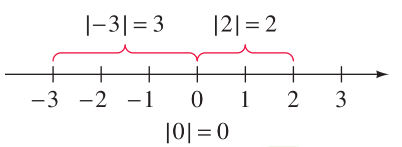


Figure 0.6

Symbolically, absolute value is denoted with vertical bars. Thus we write $|2| = 2$, $|-3| = 3$, and $|0| = 0$. More formally, the concept of absolute value is defined as follows.

Definition 0.1

For all real numbers a,

1. If $a \geq 0$, then $|a| = a$.

2. If $a < 0$, then $|a| = -a$.

According to Definition 0.1, we obtain

$\|6\| = 6$	by applying part 1
$\|0\| = 0$	by applying part 1
$\|-7\| = -(-7) = 7$	by applying part 2

Notice that the absolute value of a positive number is the number itself, but the absolute value of a negative number is its opposite. Thus the absolute value of any number except zero is positive, and the absolute value of zero is zero. Together, these facts

Unless otherwise noted, all art on this page is © Cengage Learning.

indicate that the absolute value of any real number is equal to the absolute value of its opposite. All of these ideas are summarized in the following properties.

Properties of Absolute Value

The variables a and b represent any real number.

1. $|a| \geq 0$ The absolute value of a real number is positive or zero.
2. $|a| = |-a|$ The absolute value of a real number is equal to the absolute value of its opposite
3. $|a - b| = |b - a|$ The expressions $a - b$ and $b - a$ are opposites of each other, hence their absolute values are equal.

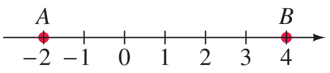


Figure 0.7

In Figure 0.7 the points A and B are located at -2 and 4, respectively. The distance *between* A and B is 6 units and can be calculated by using either $|-2 - 4|$ or $|4 - (-2)|$. In general, if two points on a number line have coordinates x_1 and x_2, then the distance between the two points is determined by using either

$$|x_2 - x_1| \quad \text{or} \quad |x_1 - x_2|$$

because, by property 3 above, they are the same quantity.

Properties of Real Numbers

As you work with the set of real numbers, the basic operations, and the relations of equality and inequality, the following properties will guide your study. Be sure that you understand these properties because they not only facilitate manipulations with real numbers but also serve as a basis for many algebraic computations. The variables a, b, and c represent real numbers.

Properties of Real Numbers

Closure properties	$a + b$ is a unique real number. ab is a unique real number.
Commutative properties	$a + b = b + a$ $ab = ba$
Associative properties	$(a + b) + c = a + (b + c)$ $(ab)c = a(bc)$

Unless otherwise noted, all art on this page is © Cengage Learning.

Identity properties	There exists a real number 0 such that $a + 0 = 0 + a = a$.
	There exists a real number 1 such that $a(1) = 1(a) = a$.
Inverse properties	For every real number a, there exists a unique real number $-a$ such that $a + (-a) = (-a) + a = 0$.
	For every nonzero real number a, there exists a unique real number $\frac{1}{a}$ such that $a\left(\frac{1}{a}\right) = \frac{1}{a}(a) = 1$.
Multiplication property of zero	$a(0) = (0)(a) = 0$
Multiplication property of negative one	$a(-1) = -1(a) = -a$
Distributive property	$a(b + c) = ab + ac$

Let's make a few comments about the properties of real numbers. The set of real numbers is said to be **closed** with respect to addition and multiplication. That is, the sum of two real numbers is a real number, and the product of two real numbers is a real number. **Closure** plays an important role when we are proving additional properties that pertain to real numbers.

Addition and multiplication are said to be **commutative operations.** This means that the order in which you add or multiply two real numbers does not affect the result. For example, $6 + (-8) = -8 + 6$ and $(-4)(-3) = (-3)(-4)$. It is important to realize that subtraction and division are *not* commutative operations; order does make a difference. For example, $3 - 4 = -1$, but $4 - 3 = 1$. Likewise, $2 \div 1 = 2$, but $1 \div 2 = \frac{1}{2}$.

Addition and multiplication are **associative operations.** The associative properties are grouping properties. For example, $(-8 + 9) + 6 = -8 + (9 + 6)$; changing the grouping of the numbers does not affect the final sum. Likewise, for multiplication, $[(-4)(-3)](2) = (-4)[(-3)(2)]$. Subtraction and division are *not* associative operations. For example, $(8 - 6) - 10 = -8$, but $8 - (6 - 10) = 12$. An example showing that division is not associative is $(8 \div 4) \div 2 = 1$, but $8 \div (4 \div 2) = 4$.

Zero is the **identity element for addition.** This means that the sum of any real number and zero is identically the same real number. For example, $-87 + 0 = 0 + (-87) = -87$. One is the **identity element for multiplication.** The product of any real number and 1 is identically the same real number. For example, $(-119)(1) = (1)(-119) = -119$.

The real number $-a$ is called the **additive inverse of *a*** or the **opposite of *a*.** The sum of a number and its additive inverse is the identity element for addition. For example, 16 and -16 are additive inverses, and their sum is zero. The additive inverse of zero is zero.

The real number $1/a$ is called the **multiplicative inverse** or **reciprocal of *a*.** The product of a number and its multiplicative inverse is the identity element for multiplication. For example, the reciprocal of 2 is $\frac{1}{2}$, and $2\left(\frac{1}{2}\right) = \frac{1}{2}(2) = 1$.

The product of any real number and zero is zero. For example, $(-17)(0) = (0)(-17) = 0$. The product of any real number and -1 is the opposite of the real number. For example, $(-1)(52) = (52)(-1) = -52$.

The **distributive property** ties together the operations of addition and multiplication. We say that *multiplication distributes over addition*. For example, $7(3 + 8) = 7(3) + 7(8)$. Furthermore, because $b - c = b + (-c)$, it follows that *multiplication also distributes over subtraction*. This can be expressed symbolically as $a(b - c) = ab - ac$. For example, $6(8 - 10) = 6(8) - 6(10)$.

Algebraic Expressions

Algebraic expressions such as

$$2x \qquad 8xy \qquad -3xy \qquad -4abc \qquad z$$

are called "terms." A **term** is an indicated product and may have any number of factors. The variables of a term are called "literal factors," and the numerical factor is called the "numerical coefficient." Thus in $8xy$, the x and y are **literal factors,** and 8 is the **numerical coefficient.** Because $1(z) = z$, the numerical coefficient of the term z is understood to be 1. Terms that have the same literal factors are called "similar terms" or "like terms." The distributive property in the form $ba + ca = (b + c)a$ provides the basis for simplifying algebraic expressions by *combining similar terms,* as illustrated in the following examples:

$$3x + 5x = (3 + 5)x = 8x$$
$$-6xy + 4xy = (-6 + 4)xy = -2xy$$
$$4x - x = 4x - 1x = (4 - 1)x = 3x$$

Sometimes we can simplify an algebraic expression by applying the distributive property to remove parentheses and combine similar terms, as the next examples illustrate:

$$\begin{aligned} 4(x + 2) + 3(x + 6) &= 4(x) + 4(2) + 3(x) + 3(6) \\ &= 4x + 8 + 3x + 18 \\ &= 7x + 26 \end{aligned}$$

$$\begin{aligned} -5(y + 3) - 2(y - 8) &= -5(y) - 5(3) - 2(y) - 2(-8) \\ &= -5y - 15 - 2y + 16 \\ &= -7y + 1 \end{aligned}$$

An algebraic expression takes on a numerical value whenever each variable in the expression is replaced by a real number. For example, when x is replaced by 5 and y by 9, the algebraic expression $x + y$ becomes the numerical expression $5 + 9$, which is equal to 14. We say that $x + y$ has a value of 14 when $x = 5$ and $y = 9$.

Consider the following examples, which illustrate the process of finding a value of an algebraic expression. The process is commonly referred to as **evaluating an algebraic expression.**

Classroom Example
Find the value of $-2a + 4bc$ when $a = -3$, $b = 5$, and $c = -1$.

EXAMPLE 1 Find the value of $3xy - 4z$ when $x = 2$, $y = -4$, and $z = -5$.

Solution

$$\begin{aligned} 3xy - 4z &= 3(2)(-4) - 4(-5) \qquad \text{when } x = 2,\ y = -4, \text{ and } z = -5 \\ &= -24 + 20 \\ &= -4 \end{aligned}$$

Classroom Example
Find the value of $3[2x - (5y - 4)]$ when $x = -1$ and $y = -6$.

EXAMPLE 2

Find the value of $a - [4b - (2c + 1)]$ when $a = -8$, $b = -7$, and $c = 14$.

Solution

$$\begin{aligned} a - [4b - (2c + 1)] &= -8 - [4(-7) - (2(14) + 1)] \\ &= -8 - [-28 - 29] \\ &= -8 - [-57] \\ &= 49 \end{aligned}$$

Classroom Example
Evaluate $\dfrac{-x - 3y}{x - y}$ when $x = -4$ and $y = 2$.

EXAMPLE 3

Evaluate $\dfrac{a - 2b}{3c + 5d}$ when $a = 14$, $b = -12$, $c = -3$, and $d = -2$.

Solution

$$\begin{aligned} \frac{a - 2b}{3c + 5d} &= \frac{14 - 2(-12)}{3(-3) + 5(-2)} \\ &= \frac{14 + 24}{-9 - 10} \\ &= \frac{38}{-19} = -2 \end{aligned}$$

Look back at Examples 1–3, and note that we use the following **order of operations** when simplifying numerical expressions.

1. Perform the operations inside the symbols of inclusion (parentheses, brackets, and braces) and above and below each fraction bar. Start with the innermost inclusion symbol.
2. Perform all multiplications and divisions in the order in which they appear, from left to right.
3. Perform all additions and subtractions in the order in which they appear, from left to right.

You should also realize that first simplifying by combining similar terms can sometimes aid in the process of evaluating algebraic expressions. The last example of this section illustrates this idea.

Classroom Example
Evaluate $2(-2y - 1) - 2(y + 4)$ when $y = -2$.

EXAMPLE 4 Evaluate $2(3x + 1) - 3(4x - 3)$ when $x = -5$.

Solution

$$\begin{aligned} 2(3x + 1) - 3(4x - 3) &= 2(3x) + 2(1) - 3(4x) - 3(-3) \\ &= 6x + 2 - 12x + 9 \\ &= -6x + 11 \end{aligned}$$

Now substituting -5 for x, we obtain

$$\begin{aligned} -6x + 11 &= -6(-5) + 11 \\ &= 30 + 11 \\ &= 41 \end{aligned}$$

Cartesian Coordinate System

Just as real numbers can be associated with points on a line, pairs of real numbers can be associated with points in a plane. To do this, we set up two number lines, one vertical and one horizontal, perpendicular to each other at the point associated with zero on both lines, as shown in Figure 0.8. We refer to these number lines as the **horizontal axis** and the **vertical axis** or together as the **coordinate axes.** They partition a plane into four regions called **quadrants.** The quadrants are numbered counterclockwise from I through IV as indicated in Figure 0.8. The point of intersection of the two axes is called the **origin.**

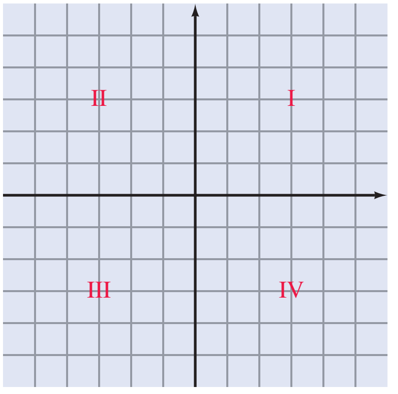


Figure 0.8

The positive direction on the horizontal axis is to the right, and the positive direction on the vertical axis is up. It is now possible to set up a one-to-one correspondence between **ordered pairs** of real numbers and the points in a plane. To each ordered pair of real numbers there corresponds a unique point in the plane, and to each point in the plane there corresponds a unique ordered pair of real numbers. A part of this correspondence is illustrated in Figure 0.9. For example, the ordered pair (3, 2) means that the point A is located 3 units to the right of and 2 units up from the origin. Likewise, the ordered pair $(-3, -5)$ means that the point D is located 3 units to the left of and 5 units down from the origin. The ordered pair (0, 0) is associated with the origin O.

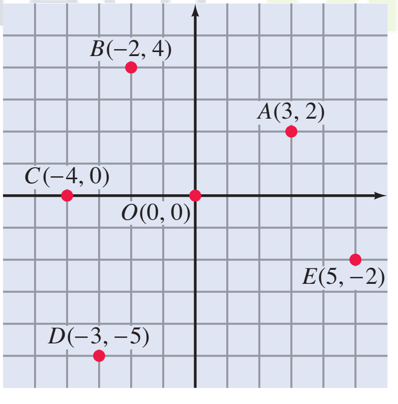


Figure 0.9

In general we refer to the real numbers a and b in an ordered pair (a, b) associated with a point as the **coordinates of the point.** The first number, a, called the **abscissa,** is the directed distance of the point from the vertical axis measured parallel to the horizontal axis. The second number, b, called the **ordinate,** is the directed distance of the point from the horizontal axis measured parallel to the vertical axis (Figure 0.10). Thus in the first quadrant, all points have a positive abscissa and a positive ordinate. In the second quadrant all points have a negative abscissa and a positive ordinate. We have indicated the sign situations for all four quadrants in Figure 0.11. This system of associating points in a plane with pairs of real numbers is called the **rectangular coordinate system** or the **Cartesian coordinate system.**

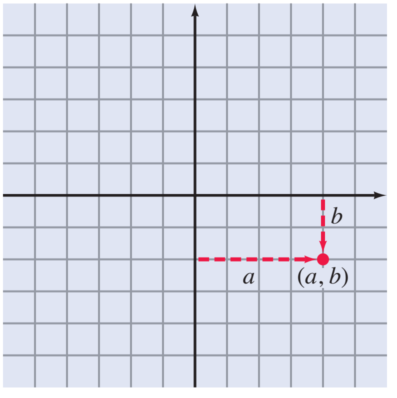


Figure 0.10

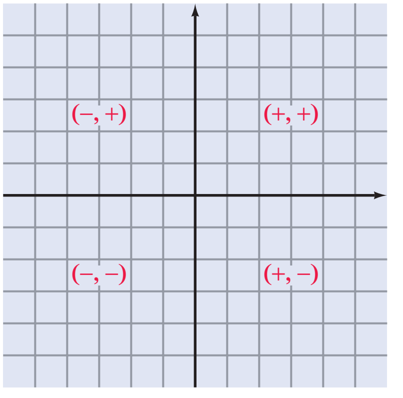


Figure 0.11

Historically, the rectangular coordinate system provided the basis for the development of the branch of mathematics called **analytic geometry,** or what we presently refer to as **coordinate geometry.** In this discipline, René Descartes, a French 17th-century mathematician, was able to transform geometric problems into an algebraic setting and then use the tools of algebra to solve the problems.

Basically, there are two kinds of problems to solve in coordinate geometry:

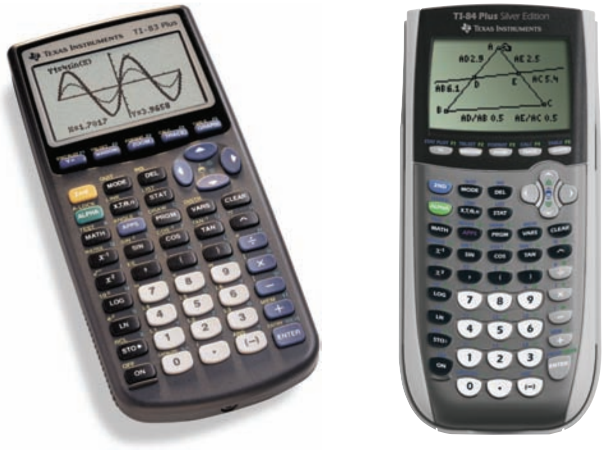
1. Given an algebraic equation, find its geometric graph.
2. Given a set of conditions pertaining to a geometric figure, find its algebraic equation.

Throughout this text we will consider a wide variety of situations dealing with both kinds of problems.

For most purposes in coordinate geometry, it is customary to label the horizontal axis the ***x*-axis** and the vertical axis the ***y*-axis.** Then ordered pairs of real numbers associated with points in the xy plane are of the form (x, y); that is, x is the first coordinate and y is the second coordinate.

Graphing Utilities

The term **graphing utility** is used in current literature to refer to either a graphing calculator (see Figure 0.12) or a computer with a graphing software package. (We will frequently use the phrase "use a graphing calculator" to mean either a graphing calculator or a computer with an appropriate software package.) We will introduce various features of graphing calculators as we need them in the text. Because so many different types of graphing utilities are available, we will use mostly generic terminology and let you consult a user's manual for specific key-punching instructions. We urge you to study the graphing calculator examples in this text even if you do not have access to a graphing utility. The examples are chosen to reinforce concepts under discussion. Furthermore, for those who do have access to a graphing utility, we provide "Graphing Calculator Activities" in many of the problem sets.



Courtesy of Texas Instruments

Figure 0.12

Graphing calculators have display windows large enough to show graphs. This window feature is also helpful when you're using a graphing calculator for computational purposes because it allows you to see the entries of the problem. Figure 0.13 shows a display window for an example of the distributive property. Note that we can check to see that the correct numbers and operational symbols have been entered. Also note that the answer is given below and to the right of the problem.

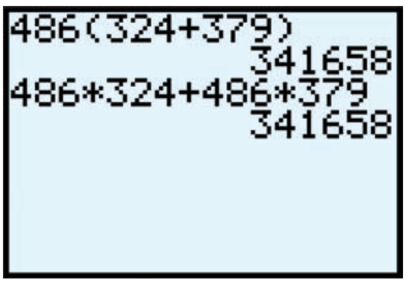


Figure 0.13

Most calculators, including graphing calculators, can be used to evaluate algebraic expressions. One calculator method for evaluating the algebraic expression in Example 1, $3xy - 4z$ for $x = 2$, $y = -4$, and $z = -5$, is to replace x with 2, y with -4, and z with -5, and then calculate the resulting numerical expression.

Another method is shown in Figure 0.14, in which the values for x, y, and z are stored and then the algebraic expression $3xy - 4z$ is evaluated.

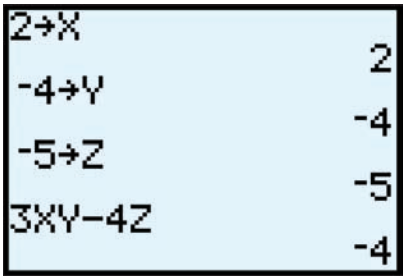


Figure 0.14

Concept Quiz 0.1

For Problems 1–10, answer true or false.

1. The null set is written as $\{\varnothing\}$.
2. The sets {a, b, c, d} and {a, d, c, b} are equal sets.
3. Decimal numbers that are classified as repeating or terminating decimals represent rational numbers.
4. The absolute value of x is equal to x.
5. The axes of the rectangular coordinate system intersect in a point called the center.
6. Subtraction is a commutative operation.
7. Every real number has a multiplicative inverse.
8. The associative properties are grouping properties.
9. On the rectangular coordinate system, the point of intersection of the two axes is called the origin.
10. The horizontal axis is customarily referred to as the y-axis.

Remark: You can find answers to the Concept Quiz questions at the end of the next Problem Set.

Problem Set 0.1

For Problems 1–10, identify each statement as *true* or *false*. **(Objective 2)**

1. Every rational number is a real number.
2. Every irrational number is a real number.
3. Every real number is a rational number.
4. If a number is real, then it is irrational.
5. Some irrational numbers are also rational numbers.
6. All integers are rational numbers.
7. The number zero is a rational number.
8. Zero is a positive integer.
9. Zero is a negative number.
10. All whole numbers are integers.

For Problems 11–18, list those elements of the set of numbers

$$\left\{0, \sqrt{5}, -\sqrt{2}, \frac{7}{8}, -\frac{10}{13}, 7\frac{1}{8}, 0.279, 0.4\overline{67}, -\pi, -14, 46, 6.75\right\}$$

that belong to each of the following sets. **(Objective 2)**

11. The natural numbers

12. The whole numbers

13. The integers

14. The rational numbers

15. The irrational numbers

16. The nonnegative integers

17. The nonpositive integers

18. The real numbers

For Problems 19–32, use the following set designations:

$N = \{x|x \text{ is a natural number}\}$
$W = \{x|x \text{ is a whole number}\}$
$I = \{x|x \text{ is an integer}\}$
$Q = \{x|x \text{ is a rational number}\}$
$H = \{x|x \text{ is an irrational number}\}$
$R = \{x|x \text{ is a real number}\}$

Place $\subseteq$ or $\nsubseteq$ in each blank to make a true statement. **(Objectives 1 and 2)**

19. N ________ R **20.** R ________ N

21. N ________ I **22.** I ________ Q

23. H ________ Q **24.** Q ________ H

25. W ________ I **26.** N ________ W

27. I ________ W **28.** I ________ N

29. $\{0, 2, 4, \ldots\}$ ________ W

30. $\{1, 3, 5, 7, \ldots\}$ ________ I

31. $\{-2, -1, 0, 1, 2\}$ ________ W

32. $\{0, 3, 6, 9, \ldots\}$ ________ N

For Problems 33–42, list the elements of each set. For example, the elements of $\{x|x$ is a natural number less than 4$\}$ can be listed $\{1, 2, 3\}$. **(Objectives 1 and 2)**

33. $\{x|x$ is a natural number less than 2$\}$

34. $\{x|x$ is a natural number greater than 5$\}$

35. $\{n|n$ is a whole number less than 4$\}$

36. $\{y|y$ is an integer greater than $-3\}$

37. $\{y|y$ is an integer less than 2$\}$

38. $\{n|n$ is a positive integer greater than $-4\}$

39. $\{x|x$ is a whole number less than 0$\}$

40. $\{x|x$ is a negative integer greater than $-5\}$

41. $\{n|n$ is a nonnegative integer less than 3$\}$

42. $\{n|n$ is a nonpositive integer greater than 1$\}$

43. Find the distance on the real number line between two points whose coordinates are the following. **(Objective 3)**

(a) 17 and 35 **(b)** -14 and 12

(c) 18 and -21 **(d)** -17 and -42

(e) -56 and -21 **(f)** 0 and -37

44. Evaluate each of the following if x is a nonzero real number. **(Objective 4)**

(a) $\dfrac{|x|}{x}$ **(b)** $\dfrac{x}{|x|}$

(c) $\dfrac{|-x|}{-x}$ **(d)** $|x| - |-x|$

In Problems 45–58, state the property that justifies each of the statements. For example, $3 + (-4) = (-4) + 3$ because of the commutative property of addition. **(Objective 5)**

45. $x(2) = 2(x)$

46. $(7 + 4) + 6 = 7 + (4 + 6)$

47. $1(x) = x$

48. $43 + (-18) = (-18) + 43$

49. $(-1)(93) = -93$

50. $109 + (-109) = 0$

51. $5(4 + 7) = 5(4) + 5(7)$

52. $-1(x + y) = -(x + y)$

53. $7yx = 7xy$

54. $(x + 2) + (-2) = x + [2 + (-2)]$

55. $6(4) + 7(4) = (6 + 7)(4)$

56. $\left(\frac{2}{3}\right)\left(\frac{3}{2}\right) = 1$

57. $4(5x) = (4 \cdot 5)x$

58. $[(17)(8)](25) = (17)[(8)(25)]$

For Problems 59–80, evaluate each of the algebraic expressions for the given values of the variables. **(Objective 6)**

59. $5x + 3y$; $x = -2$ and $y = -4$

60. $7x - 4y$; $x = -1$ and $y = 6$

61. $-3ab - 2c$; $a = -4$, $b = 7$, and $c = -8$

62. $x - (2y + 3z)$; $x = -3$, $y = -4$, and $z = 9$

63. $(a - 2b) + (3c - 4)$; $a = 6$, $b = -5$, and $c = -11$

64. $3a - [2b - (4c + 1)]$; $a = 4$, $b = 6$, and $c = -8$

65. $\frac{-2x + 7y}{x - y}$; $x = -3$ and $y = -2$

66. $\frac{x - 3y + 2z}{2x - y}$; $x = 4$, $y = 9$, $z = -12$

67. $(5x - 2y)(-3x + 4y)$; $x = -3$ and $y = -7$

68. $(2a - 7b)(4a + 3b)$; $a = 6$ and $b = -3$

69. $5x + 4y - 9y - 2y$; $x = 2$ and $y = -8$

70. $5a + 7b - 9a - 6b$; $a = -7$ and $b = 8$

71. $-5x + 8y + 7y + 8x$; $x = 5$ and $y = -6$

72. $|x - y| - |x + y|$; $x = -4$ and $y = -7$

73. $|3x + y| + |2x - 4y|$; $x = 5$ and $y = -3$

74. $\left|\frac{x - y}{y - x}\right|$; $x = -6$ and $y = 13$

75. $\left|\frac{2a - 3b}{3b - 2a}\right|$; $a = -4$ and $b = -8$

76. $5(x - 1) + 7(x + 4)$; $x = 3$

77. $2(3x + 4) - 3(2x - 1)$; $x = -2$

78. $-4(2x - 1) - 5(3x + 7)$; $x = -1$

79. $5(a - 3) - 4(2a + 1) - 2(a - 4)$; $a = -3$

80. $-3(2y - 7) - (y + 10) + 8y + 5$; $y = 10$

For Problems 81–86, plot the following points on a rectangular coordinate system. **(Objective 7)**

81. $(-4, 1)$

82. $(3, -2)$

83. $(0, -3)$

84. $(-2, -2)$

85. $(5, -1)$

86. $(1, 4)$

For Problems 87–92, state the quadrant that contains the point. **(Objective 7)**

87. $(4, -2)$

88. $(-3, 1)$

89. $(-6, -2)$

90. $(5, 2)$

91. $(1, 8)$

92. $(-7, -7)$

Thoughts Into Words

93. Do you think $3\sqrt{2}$ is a rational or an irrational number? Defend your answer.

94. Explain why $\frac{0}{8} = 0$ but $\frac{8}{0}$ is undefined.

95. The solution of the following simplification problem is incorrect. The answer should be -11. Find and correct the error.

$$\begin{aligned} 8 \div (-4)(2) - 3(4) \div 2 + (-1) &= (-2)(2) - 12 \div 1 \\ &= -4 - 12 \\ &= -16 \end{aligned}$$

96. Explain the difference between "simplifying a numerical expression" and "evaluating an algebraic expression."

Graphing Calculator Activities

97. Different graphing calculators use different sequences of key strokes to evaluate algebraic expressions. Be sure that you can do Problems 59–80 with your calculator.

Answers to the Concept Quiz

1. False **2.** True **3.** True **4.** False **5.** False **6.** False **7.** False **8.** True **9.** True **10.** False

0.2 Exponents

OBJECTIVES

1. Evaluate numerical expressions that have integer exponents
2. Apply the properties of exponents to simplify algebraic expressions
3. Write numbers in scientific notation
4. Convert numbers from scientific notation to ordinary decimal notation
5. Perform calculations with numbers in scientific form

Positive integers are used as *exponents* to indicate repeated multiplication. For example, $4 \cdot 4 \cdot 4$ can be written 4^3, where the raised 3 indicates that 4 is to be used as a factor three times. The following general definition is helpful.

Definition 0.2

If n is a positive integer, and b is any real number, then

$$b^n = \underbrace{bbb \cdots b}_{n \text{ factors of } b}$$

The number b is referred to as the **base**, and n is called the **exponent**. The expression b^n can be read "b to the nth power." The terms **squared** and **cubed** are commonly associated with exponents of 2 and 3, respectively. For example, b^2 is read "b squared" and b^3 as "b cubed." An exponent of 1 is usually not written, so b^1 is simply written b. The following examples illustrate Definition 0.2:

$$2^3 = 2 \cdot 2 \cdot 2 = 8 \qquad \left(\frac{1}{2}\right)^5 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$$

$$3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81 \qquad (0.7)^2 = (0.7)(0.7) = 0.49$$

$$(-5)^2 = (-5)(-5) = 25 \qquad -5^2 = -(5 \cdot 5) = -25$$

We especially want to call your attention to the last example in each column. Note that $(-5)^2$ means that -5 is the base used as a factor twice. However, -5^2 means that 5 is the base, and after it is squared, we take the opposite of the result.

Properties of Exponents

In a previous algebra course, you may have seen some properties pertaining to the use of positive integers as exponents. Those properties can be summarized as follows.

Property 0.1 Properties of Exponents

If a and b are real numbers, and m and n are positive integers, then

1. $b^n \cdot b^m = b^{n+m}$

2. $(b^n)^m = b^{mn}$

3. $(ab)^n = a^n b^n$

4. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ $\qquad b \neq 0$

5. $\dfrac{b^n}{b^m} = b^{n-m}$ $\qquad$ when $n > m$, $b \neq 0$

$\dfrac{b^n}{b^m} = 1$ $\qquad$ when $n = m$, $b \neq 0$

$\dfrac{b^n}{b^m} = \dfrac{1}{b^{m-n}}$ $\qquad$ when $n < m$, $b \neq 0$

Each part of Property 0.1 can be justified by using Definition 0.2. For example, to justify part 1, we can reason as follows:

$$\begin{aligned} b^n \cdot b^m &= \underbrace{(bbb \cdots b)}_{\substack{n \text{ factors} \\ \text{of } b}} \cdot \underbrace{(bbb \cdots b)}_{\substack{m \text{ factors} \\ \text{of } b}} \\ &= \underbrace{bbb \cdots b}_{(n+m) \text{ factors of } b} \\ &= b^{n+m} \end{aligned}$$

Similar reasoning can be used to verify the other parts of Property 0.1. The following examples illustrate the use of Property 0.1 along with the commutative and associative properties of the real numbers. We have chosen to show all of the steps; however many of the steps can be performed mentally.

Classroom Example
Find the indicated product, $(5a^3b^2)(-2ab^4)$.

EXAMPLE 1 Find the indicated product, $(3x^2y)(4x^3y^2)$.

Solution

$$\begin{aligned}(3x^2y)(4x^3y^2) &= 3 \cdot 4 \cdot x^2 \cdot x^3 \cdot y \cdot y^2 \\ &= 12x^{2+3}y^{1+2} && b^n \cdot b^m = b^{n+m} \\ &= 12x^5y^3\end{aligned}$$

Classroom Example
Find the indicated product, $(-3x^2)^4$.

EXAMPLE 2 Find the indicated product, $(-2y^3)^5$.

Solution

$$\begin{aligned}(-2y^3)^5 &= (-2)^5(y^3)^5 && (ab)^n = a^n b^n \\ &= -32y^{15} && (b^n)^m = b^{mn}\end{aligned}$$

Classroom Example
Find the indicated quotient, $\left(\frac{x^5}{y}\right)^3$.

EXAMPLE 3 Find the indicated quotient, $\left(\frac{a^2}{b^4}\right)^7$.

Solution

$$\begin{aligned}\left(\frac{a^2}{b^4}\right)^7 &= \frac{(a^2)^7}{(b^4)^7} && \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \\ &= \frac{a^{14}}{b^{28}} && (b^n)^m = b^{mn}\end{aligned}$$

Classroom Example
Find the indicated quotient, $\frac{21a^8}{-3a^3}$.

EXAMPLE 4 Find the indicated quotient, $\frac{-56x^9}{7x^4}$.

Solution

$$\begin{aligned}\frac{-56x^9}{7x^4} &= -8x^{9-4} && \frac{b^n}{b^m} = b^{n-m} \quad \text{when } n > m \\ &= -8x^5\end{aligned}$$

Zero and Negative Integers As Exponents

Now we can extend the concept of an exponent to include the use of zero and negative integers. First, let's consider the use of zero as an exponent. We want to use zero in a way that Property 0.1 will continue to hold. For example, if $b^n \cdot b^m = b^{n+m}$ is to hold, then

$x^4 \cdot x^0$ should equal x^{4+0}, which equals x^4. In other words, x^0 *acts like* 1 because $x^4 \cdot x^0 = x^4$. Look at the following definition.

Definition 0.3

If b is a nonzero real number, then

$$b^0 = 1$$

Therefore, according to Definition 0.3, the following statements are all true:

$$5^0 = 1 \qquad (-413)^0 = 1$$

$$\left(\frac{3}{11}\right)^0 = 1 \qquad (x^3y^4)^0 = 1 \quad \text{if } x \neq 0 \text{ and } y \neq 0$$

A similar line of reasoning can be used to motivate a definition for the use of negative integers as exponents. Consider the example $x^4 \cdot x^{-4}$. If $b^n \cdot b^m = b^{n+m}$ is to hold, then $x^4 \cdot x^{-4}$ should equal $x^{4+(-4)}$, which equals $x^0 = 1$. Therefore, x^{-4} must be the reciprocal of x^4 because their product is 1. That is, $x^{-4} = 1/x^4$. This suggests the following definition.

Definition 0.4

If n is a positive integer, and b is a nonzero real number, then

$$b^{-n} = \frac{1}{b^n}$$

According to Definition 0.4, the following statements are true:

$$x^{-5} = \frac{1}{x^5} \qquad 2^{-4} = \frac{1}{2^4} = \frac{1}{16}$$

$$\left(\frac{3}{4}\right)^{-2} = \frac{1}{\left(\frac{3}{4}\right)^2} = \frac{1}{\frac{9}{16}} = \frac{16}{9} \qquad \frac{2}{x^{-3}} = \frac{2}{\frac{1}{x^3}} = 2x^3$$

The first four parts of Property 0.1 hold true *for all integers*. Furthermore, we do not need all three equations in part 5 of Property 0.1. The first equation,

$$\frac{b^n}{b^m} = b^{n-m}$$

can be used for *all integral exponents*. Let's restate Property 0.1 as it pertains to integers. We will include name tags for easy reference.

Property 0.2

If m and n are integers, and a and b are real numbers, with $b \neq 0$ whenever it appears in a denominator, then

1. $b^n \cdot b^m = b^{n+m}$ — Product of two powers
2. $(b^n)^m = b^{mn}$ — Power of a power
3. $(ab)^n = a^n b^n$ — Power of a product
4. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ — Power of a quotient
5. $\frac{b^n}{b^m} = b^{n-m}$ — Quotient of two powers

Having the use of all integers as exponents allows us to work with a large variety of numerical and algebraic expressions. Let's consider some examples that illustrate the various parts of Property 0.2.

Classroom Example
Evaluate each of the numerical expressions.
(a) $(3^{-2} \cdot 4)^{-1}$ **(b)** $\left(\frac{5^{-2}}{2^{-3}}\right)^{-2}$

EXAMPLE 5 Evaluate each of the following numerical expressions.

(a) $(2^{-1} \cdot 3^2)^{-1}$ **(b)** $\left(\frac{2^{-3}}{3^{-2}}\right)^{-2}$

Solution

(a)
$$\begin{aligned}(2^{-1} \cdot 3^2)^{-1} &= (2^{-1})^{-1}(3^2)^{-1} && \text{Power of a product}\\ &= (2^1)(3^{-2}) && \text{Power of a power}\\ &= (2)\left(\frac{1}{3^2}\right)\\ &= 2\left(\frac{1}{9}\right) = \frac{2}{9}\end{aligned}$$

(b)
$$\begin{aligned}\left(\frac{2^{-3}}{3^{-2}}\right)^{-2} &= \frac{(2^{-3})^{-2}}{(3^{-2})^{-2}} && \text{Power of a quotient}\\ &= \frac{2^6}{3^4} && \text{Power of a power}\\ &= \frac{64}{81}\end{aligned}$$

Classroom Example

Find the indicated products and quotients, and express the final results with positive integral exponents only.

(a) $(5a^{-3}b^{-1}c^{-3})(3a^{-2}bc^{7})$

(b) $\dfrac{-28x^2y^3}{7x^4y^{-2}}$

(c) $\left(\dfrac{12a^3b^{-2}}{3a^4b^{-3}}\right)^{-1}$

EXAMPLE 6

Find the indicated products and quotients, and express the final results with positive integral exponents only.

(a) $(3x^2y^{-4})(4x^{-3}y)$ **(b)** $\dfrac{12a^3b^2}{-3a^{-1}b^5}$ **(c)** $\left(\dfrac{15x^{-1}y^2}{5xy^{-4}}\right)^{-1}$

Solution

(a)
$$\begin{aligned}(3x^2y^{-4})(4x^{-3}y) &= 12x^{2+(-3)}y^{-4+1} && \text{Product of powers}\\ &= 12x^{-1}y^{-3}\\ &= \frac{12}{xy^3}\end{aligned}$$

(b)
$$\begin{aligned}\frac{12a^3b^2}{-3a^{-1}b^5} &= -4a^{3-(-1)}b^{2-5} && \text{Quotient of powers}\\ &= -4a^4b^{-3}\\ &= -\frac{4a^4}{b^3}\end{aligned}$$

(c)
$$\begin{aligned}\left(\frac{15x^{-1}y^2}{5xy^{-4}}\right)^{-1} &= (3x^{-1-1}y^{2-(-4)})^{-1} && \text{First simplify inside parentheses}\\ &= (3x^{-2}y^6)^{-1}\\ &= 3^{-1}x^2y^{-6} && \text{Power of a product}\\ &= \frac{x^2}{3y^6}\end{aligned}$$

The next two examples illustrate the simplification of numerical and algebraic expressions involving sums and differences. In such cases, Definition 0.4 can be used to change from negative to positive exponents so that we can proceed in the usual ways.

Classroom Example

Simplify $4^{-1} + 2^{-1}$.

EXAMPLE 7

Simplify $2^{-3} + 3^{-1}$.

Solution

$$\begin{aligned}2^{-3} + 3^{-1} &= \frac{1}{2^3} + \frac{1}{3^1}\\ &= \frac{1}{8} + \frac{1}{3}\\ &= \frac{3}{24} + \frac{8}{24}\\ &= \frac{11}{24}\end{aligned}$$

Classroom Example
Simplify $(2^{-2} - 3^{-1})^{-1}$.

EXAMPLE 8 Simplify $(4^{-1} - 3^{-2})^{-1}$.

Solution

$$\begin{aligned}(4^{-1} - 3^{-2})^{-1} &= \left(\frac{1}{4^1} - \frac{1}{3^2}\right)^{-1}\\ &= \left(\frac{1}{4} - \frac{1}{9}\right)^{-1}\\ &= \left(\frac{9}{36} - \frac{4}{36}\right)^{-1}\\ &= \left(\frac{5}{36}\right)^{-1}\\ &= \frac{1}{\left(\frac{5}{36}\right)^1} = \frac{36}{5}\end{aligned}$$

Figure 0.15 shows calculator windows for Examples 7 and 8. Note that the answers are given in decimal form. If your calculator also handles common fractions, then the display window may appear as in Figure 0.16.

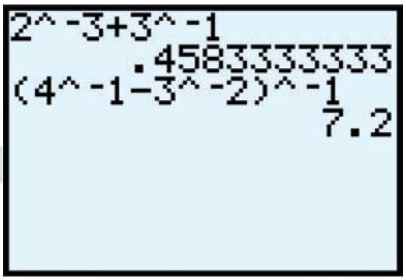


Figure 0.15

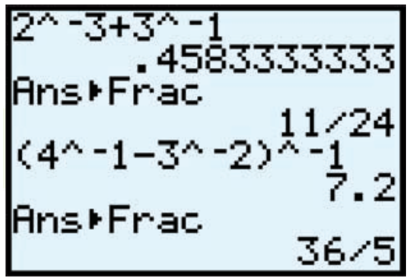


Figure 0.16

Classroom Example
Express $x^{-3} + y^{-2}$ as a single fraction with positive exponent.

EXAMPLE 9

Express $a^{-1} + b^{-2}$ as a single fraction involving positive exponents only.

Solution

$$\begin{aligned}a^{-1} + b^{-2} &= \frac{1}{a^1} + \frac{1}{b^2}\\ &= \left(\frac{1}{a}\right)\left(\frac{b^2}{b^2}\right) + \left(\frac{1}{b^2}\right)\left(\frac{a}{a}\right)\\ &= \frac{b^2}{ab^2} + \frac{a}{ab^2}\\ &= \frac{b^2 + a}{ab^2}\end{aligned}$$

Scientific Notation

The expression $(n)(10^k)$ (where n is a number greater than or equal to 1 and less than 10, written in decimal form, and k is any integer) is commonly called **scientific notation** or the **scientific form** of a number. The following are examples of numbers expressed in scientific form:

$$(4.23)(10^4) \qquad (8.176)(10^{12}) \qquad (5.02)(10^{-3}) \qquad (1)(10^{-5})$$

Very large and very small numbers can be conveniently expressed in scientific notation. For example, a light year (the distance that a ray of light travels in one year) is approximately 5,900,000,000,000 miles, and this can be written as $(5.9)(10^{12})$. The weight of an oxygen molecule is approximately 0.000000000000000000000053 of a gram, and this can be expressed as $(5.3)(10^{-23})$.

To change from ordinary decimal notation to scientific notation, the following procedure can be used.

> Write the given number as the product of a number greater than or equal to 1 and less than 10, and a power of 10. The exponent of 10 is determined by counting the number of places that the decimal point was moved when going from the original number to the number greater than or equal to 1 and less than 10. This exponent is (a) negative if the original number is less than 1, (b) positive if the original number is greater than 10, and (c) 0 if the original number itself is between 1 and 10.

Thus we can write

$$0.00092 = (9.2)(10^{-4})$$
$$872{,}000{,}000 = (8.72)(10^{8})$$
$$5.1217 = (5.1217)(10^{0})$$

To change from scientific notation to ordinary decimal notation, the following procedure can be used.

> Move the decimal point the number of places indicated by the exponent of 10. Move the decimal point to the right if the exponent is positive. Move it to the left if the exponent is negative.

Thus we can write

$$(3.14)(10^{7}) = 31{,}400{,}000$$
$$(7.8)(10^{-6}) = 0.0000078$$

Scientific notation can be used to simplify numerical calculations. We merely change the numbers to scientific notation and use the appropriate properties of exponents. Consider the following examples.

Classroom Example
Use scientific notation to perform the indicated operations.

(a) $\dfrac{(0.0048)(20{,}000)}{(0.0000016)(400)}$

(b) $\sqrt{4{,}000{,}000}$

EXAMPLE 10 Use scientific notation to perform the indicated operations.

(a) $\dfrac{(0.00063)(960{,}000)}{(3200)(0.0000021)}$ **(b)** $\sqrt{90{,}000}$

Solution

(a)
$$\begin{aligned}\frac{(0.00063)(960{,}000)}{(3200)(0.0000021)} &= \frac{(6.3)(10^{-4})(9.6)(10^{5})}{(3.2)(10^{3})(2.1)(10^{-6})}\\ &= \frac{(6.3)(9.6)(10^{1})}{(3.2)(2.1)(10^{-3})}\\ &= (9)(10^{4})\\ &= 90{,}000\end{aligned}$$

(b)
$$\begin{aligned}\sqrt{90{,}000} &= \sqrt{(9)(10^{4})}\\ &= \sqrt{9}\sqrt{10^{4}}\\ &= (3)(10^{2})\\ &= 3(100)\\ &= 300\end{aligned}$$

Many calculators are equipped to display numbers in scientific notation. The display panel shows the number between 1 and 10 and the appropriate exponent of 10. For example, evaluating $(3{,}800{,}000)^2$ yields

$\boxed{\text{1.444E13}}$

Thus $(3{,}800{,}000)^2 = (1.444)(10^{13}) = 14{,}440{,}000{,}000{,}000$. Similarly, the answer for $(0.000168)^2$ is displayed as

$\boxed{\text{2.8224E}-8}$

Thus $(0.000168)^2 = (2.8224)(10^{-8}) = 0.000000028224$.

Calculators vary in the number of digits they display between 1 and 10 when they represent a number in scientific notation. For example, we used two different calculators to estimate $(6729)^6$ and obtained the following results:

$\boxed{\text{9.283316768E22}}$

$\boxed{\text{9.28331676776E22}}$

Obviously, you need to know the capabilities of your calculator when working with problems in scientific notation.

Many calculators also allow you to enter a number in scientific notation. Such calculators are equipped with an enter-the-exponent key often labeled $\boxed{\text{EE}}$. Thus a number such as $(3.14)(10^8)$ might be entered as follows:

Enter	Press	Display
3.14	$\boxed{\text{EE}}$	3.14E
8		3.14E8

A MODE key is often used on calculators to let you choose normal decimal notation, scientific notation, or engineering notation. (The abbreviations Norm, Sci, and Eng are commonly used.) If the calculator is in scientific mode, then a number can be entered and changed to scientific form with the ENTER key. For example, when we enter 589 and press the ENTER key, the display will show 5.89E2. Likewise, when the calculator is in scientific mode, the answers to computational problems are given in scientific form. For example, the answer for (76)(533) is given as 4.0508E4.

It should be evident from this brief discussion that even when you are using a calculator, you need to have a thorough understanding of scientific notation.

Concept Quiz 0.2

For Problems 1–10, answer true or false.

1. Exponents are used to indicate repeated multiplications.
2. An exponent cannot be zero.
3. $2^{-2} = -4$
4. $(-1)^{-2} = 2$
5. In the expression 6^3, the number 6 is referred to as the baseline number.
6. $(2 + 5)^2 = 4 + 25$
7. $(3^{-2} + 3^{-4}) = 3^{-6}$
8. When writing a number in scientific notation, $(n)(10^k)$, the number n must be greater than 1 and less than or equal to 10.
9. Single-digit numbers can be expressed in scientific notation.
10. The number 357,000 is written as $(35.7)(10^4)$ in scientific notation.

Problem Set 0.2

For Problems 1–42, evaluate each numerical expression. **(Objective 1)**

1. 2^{-3}
2. 3^{-2}
3. -10^{-3}
4. 10^{-4}
5. $\dfrac{1}{3^{-3}}$
6. $\dfrac{1}{2^{-5}}$
7. $\left(\dfrac{1}{2}\right)^{-2}$
8. $-\left(\dfrac{1}{3}\right)^{-2}$
9. $\left(-\dfrac{2}{3}\right)^{-3}$
10. $\left(\dfrac{5}{6}\right)^{-2}$
11. $\left(-\dfrac{1}{5}\right)^{0}$
12. $\dfrac{1}{\left(\dfrac{3}{5}\right)^{-2}}$
13. $\dfrac{1}{\left(\dfrac{4}{5}\right)^{-2}}$
14. $\left(\dfrac{4}{5}\right)^{0}$
15. $2^5 \cdot 2^{-3}$
16. $3^{-2} \cdot 3^5$
17. $10^{-6} \cdot 10^4$
18. $10^6 \cdot 10^{-9}$
19. $10^{-2} \cdot 10^{-3}$
20. $10^{-1} \cdot 10^{-5}$
21. $(3^{-2})^{-2}$
22. $((-2)^{-1})^{-3}$

23. $(4^2)^{-1}$

24. $(3^{-1})^3$

25. $(3^{-1} \cdot 2^2)^{-1}$

26. $(2^3 \cdot 3^{-2})^{-2}$

27. $(4^2 \cdot 5^{-1})^2$

28. $(2^{-2} \cdot 4^{-1})^3$

29. $\left(\dfrac{2^{-2}}{5^{-1}}\right)^{-2}$

30. $\left(\dfrac{3^{-1}}{2^{-3}}\right)^{-2}$

31. $\left(\dfrac{3^{-2}}{8^{-1}}\right)^{2}$

32. $\left(\dfrac{4^{2}}{5^{-1}}\right)^{-1}$

33. $\dfrac{2^3}{2^{-3}}$

34. $\dfrac{2^{-3}}{2^3}$

35. $\dfrac{10^{-1}}{10^4}$

36. $\dfrac{10^{-3}}{10^{-7}}$

37. $3^{-2} + 2^{-3}$

38. $2^{-3} + 5^{-1}$

39. $\left(\dfrac{2}{3}\right)^{-1} - \left(\dfrac{3}{4}\right)^{-1}$

40. $3^{-2} - 2^3$

41. $(2^{-4} + 3^{-1})^{-1}$

42. $(3^{-2} - 5^{-1})^{-1}$

Simplify Problems 43–62; express final results without using zero or negative integers as exponents. **(Objective 2)**

43. $x^3 \cdot x^{-7}$

44. $x^{-2} \cdot x^{-3}$

45. $a^2 \cdot a^{-3} \cdot a^{-1}$

46. $b^{-3} \cdot b^5 \cdot b^{-4}$

47. $(a^{-3})^2$

48. $(b^5)^{-2}$

49. $(x^3y^{-4})^{-1}$

50. $(x^4y^{-2})^{-2}$

51. $(ab^2c^{-1})^{-3}$

52. $(a^2b^{-1}c^{-2})^{-4}$

53. $(2x^2y^{-1})^{-2}$

54. $(3x^4y^{-2})^{-1}$

55. $\left(\dfrac{x^{-2}}{y^{-3}}\right)^{-2}$

56. $\left(\dfrac{y^{4}}{x^{-1}}\right)^{-3}$

57. $\left(\dfrac{2a^{-1}}{3b^{-2}}\right)^{-2}$

58. $\left(\dfrac{3x^{2}y}{4a^{-1}b^{-3}}\right)^{-1}$

59. $\dfrac{x^{-5}}{x^{-2}}$

60. $\dfrac{a^{-3}}{a^5}$

61. $\dfrac{a^2b^{-3}}{a^{-1}b^{-2}}$

62. $\dfrac{x^{-1}y^{-2}}{x^3y^{-1}}$

For Problems 63–70, find the indicated products, quotients, and powers; express answers without using zero or negative integers as exponents. **(Objective 2)**

63. $(4x^3y^2)(-5xy^3)$

64. $(-6xy)(3x^2y^4)$

65. $(-3xy^3)^3$

66. $(-2x^2y^4)^4$

67. $\left(\dfrac{2x^2}{3y^3}\right)^3$

68. $\left(\dfrac{4x}{5y^2}\right)^3$

69. $\dfrac{72x^8}{-9x^2}$

70. $\dfrac{108x^6}{-12x^2}$

For Problems 71–80, find the indicated products and quotients; express results using positive integral exponents only. **(Objective 2)**

71. $(2x^{-1}y^2)(3x^{-2}y^{-3})$

72. $(4x^{-2}y^3)(-5x^3y^{-4})$

73. $(-6a^5y^{-4})(-a^{-7}y)$

74. $(-8a^{-4}b^{-5})(-6a^{-1}b^8)$

75. $\dfrac{24x^{-1}y^{-2}}{6x^{-4}y^3}$

76. $\dfrac{56xy^{-3}}{8x^2y^2}$

77. $\dfrac{-35a^3b^{-2}}{7a^5b^{-1}}$

78. $\dfrac{27a^{-4}b^{-5}}{-3a^{-2}b^{-4}}$

79. $\left(\dfrac{14x^{-2}y^{-4}}{7x^{-3}y^{-6}}\right)^{-2}$

80. $\left(\dfrac{24x^{5}y^{-3}}{-8x^{6}y^{-1}}\right)^{-3}$

For Problems 81–88, express each as a single fraction involving positive exponents only. **(Objective 2)**

81. $x^{-1} + x^{-2}$

82. $x^{-2} + x^{-4}$

83. $x^{-2} - y^{-1}$

84. $2x^{-1} - 3y^{-3}$

85. $3a^{-2} + 2b^{-3}$

86. $a^{-2} + a^{-1}b^{-2}$

87. $x^{-1}y - xy^{-1}$

88. $x^2y^{-1} - x^{-3}y^2$

For Problems 89–98, find the following products and quotients. Assume that all variables appearing as exponents represent integers. **(Objective 2)** For example,

$$(x^{2b})(x^{-b+1}) = x^{2b+(-b+1)} = x^{b+1}$$

89. $(3x^{a})(4x^{2a+1})$

90. $(5x^{-a})(-6x^{3a-1})$

91. $(x^{a})(x^{-a})$

92. $(-2y^{3b})(-4y^{b+1})$

93. $\dfrac{x^{3a}}{x^{a}}$

94. $\dfrac{4x^{2a+1}}{2x^{a-2}}$

95. $\dfrac{-24y^{5b+1}}{6y^{-b-1}}$

96. $(x^{a})^{2b}(x^{b})^{a}$

97. $\dfrac{(xy)^{b}}{y^{b}}$

98. $\dfrac{(2x^{2b})(-4x^{b+1})}{8x^{-b+2}}$

For Problems 99–102, express each number in scientific notation. **(Objective 3)**

99. 62,000,000

100. 17,000,000,000

101. 0.000412

102. 0.000000078

For Problems 103–106, change each number from scientific notation to ordinary decimal form. **(Objective 4)**

103. $(1.8)(10^{5})$

104. $(5.41)(10^{7})$

105. $(2.3)(10^{-6})$

106. $(4.13)(10^{-9})$

For Problems 107–112, use scientific notation and the properties of exponents to help perform the indicated operations. **(Objective 5)**

107. $\dfrac{0.00052}{0.013}$

108. $\dfrac{(0.000075)(4{,}800{,}000)}{(15{,}000)(0.0012)}$

109. $\sqrt{900{,}000{,}000}$

110. $\sqrt{0.000004}$

111. $\sqrt{0.0009}$

112. $\dfrac{(0.00069)(0.0034)}{(0.0000017)(0.023)}$

Thoughts Into Words

113. Explain how you would simplify $(3^{-1} \cdot 2^{-2})^{-1}$ and also how you would simplify $(3^{-1} + 2^{-2})^{-1}$.

114. How would you explain why the product of x^2 and x^4 is x^6 and not x^8?

Graphing Calculator Activities

115. Use your calculator to check your answers for Problems 107–112.

116. Use your calculator to evaluate each of the following. Express final answers in ordinary notation.

(a) $(27{,}000)^2$ **(b)** $(450{,}000)^2$

(c) $(14{,}800)^2$ **(d)** $(1700)^3$

(e) $(900)^4$ **(f)** $(60)^5$

(g) $(0.0213)^2$ **(h)** $(0.000213)^2$

(i) $(0.000198)^2$ **(j)** $(0.000009)^3$

117. Use your calculator to estimate each of the following. Express final answers in scientific notation with the number between 1 and 10 rounded to the nearest one-thousandth.

(a) $(4576)^4$ **(b)** $(719)^{10}$

(c) $(28)^{12}$ **(d)** $(8619)^6$

(e) $(314)^5$ **(f)** $(145{,}723)^2$

118. Use your calculator to estimate each of the following. Express final answers in ordinary notation rounded to the nearest one-thousandth.

(a) $(1.09)^5$ **(b)** $(1.08)^{10}$
(c) $(1.14)^7$ **(d)** $(1.12)^{20}$
(e) $(0.785)^4$ **(f)** $(0.492)^5$

Answers to the Concept Quiz

1. True **2.** False **3.** False **4.** False **5.** False **6.** False **7.** False **8.** False **9.** True **10.** False

0.3 Polynomials

OBJECTIVES

1. Add and subtract polynomials
2. Multiply polynomials
3. Perform binomial expansions
4. Divide a polynomial by a monomial

Recall that algebraic expressions such as $5x$, $-6y^2$, $2x^{-1}y^{-2}$, $14a^2b$, $5x^{-4}$, and $-17ab^2c^3$ are called **terms**. Terms that contain variables with only nonnegative integers as exponents are called **monomials**. Of the previously listed terms, $5x$, $-6y^2$, $14a^2b$, and $-17ab^2c^3$ are monomials. The **degree** of a monomial is the sum of the exponents of the literal factors. For example, $7xy$ is of degree 2, whereas $14a^2b$ is of degree 3, and $-17ab^2c^3$ is of degree 6. If the monomial contains only one variable, then the exponent of that variable is the degree of the monomial. For example, $5x^3$ is of degree 3, and $-8y^4$ is of degree 4. Any nonzero constant term, such as 8, is of degree zero.

A **polynomial** is a monomial or a finite sum of monomials. Thus all of the following are examples of polynomials:

$4x^2$	$3x^2 - 2x - 4$	$7x^4 - 6x^3 + 5x^2 - 2x - 1$
$3x^2y + 2y$	$\frac{1}{5}a^2 - \frac{2}{3}b^2$	14

In addition to calling a polynomial with one term a monomial, we classify polynomials with two terms as **binomials** and those with three terms as **trinomials**. The **degree of a polynomial** is the degree of the term with the highest degree in the polynomial. The following examples illustrate some of this terminology:

The polynomial $4x^3y^4$ is a monomial in two variables of degree 7.

The polynomial $4x^2y - 2xy$ is a binomial in two variables of degree 3.

The polynomial $9x^2 - 7x - 1$ is a trinomial in one variable of degree 2.

Addition and Subtraction of Polynomials

Both adding polynomials and subtracting them rely on the same basic ideas. The commutative, associative, and distributive properties provide the basis for rearranging, regrouping, and combining similar terms. Consider the following addition problems:

$$\begin{aligned}(4x^2 + 5x + 1) + (7x^2 - 9x + 4) &= (4x^2 + 7x^2) + (5x - 9x) + (1 + 4)\\ &= 11x^2 - 4x + 5\end{aligned}$$

$$\begin{aligned}(5x - 3) + (3x + 2) + (8x + 6) &= (5x + 3x + 8x) + (-3 + 2 + 6)\\ &= 16x + 5\end{aligned}$$

The definition of subtraction as *adding the opposite* $[a - b = a + (-b)]$ extends to polynomials in general. The opposite of a polynomial can be formed by taking the opposite of each term. For example, the opposite of $3x^2 - 7x + 1$ is $-3x^2 + 7x - 1$. Symbolically, this is expressed as

$$-(3x^2 - 7x + 1) = -3x^2 + 7x - 1$$

You can also think in terms of the property $-x = -1(x)$ and the distributive property. Therefore,

$$-(3x^2 - 7x + 1) = -1(3x^2 - 7x + 1) = -3x^2 + 7x - 1$$

Now consider the following subtraction problems:

$$\begin{aligned}(7x^2 - 2x - 4) - (3x^2 + 7x - 1) &= (7x^2 - 2x - 4) + (-3x^2 - 7x + 1)\\ &= (7x^2 - 3x^2) + (-2x - 7x) + (-4 + 1)\\ &= 4x^2 - 9x - 3\end{aligned}$$

$$\begin{aligned}(4y^2 + 7) - (-3y^2 + y - 2) &= (4y^2 + 7) + (3y^2 - y + 2)\\ &= (4y^2 + 3y^2) + (-y) + (7 + 2)\\ &= 7y^2 - y + 9\end{aligned}$$

Multiplying Polynomials

The distributive property is usually stated as $a(b+c) = ab+ac$, but it can be extended as follows:

$$a(b + c + d) = ab + ac + ad$$
$$a(b + c + d + e) = ab + ac + ad + ae \qquad \text{etc.}$$

The commutative and associative properties, the properties of exponents, and the distributive property work together to form the basis for finding the product of a monomial and a polynomial with more than one term. The following example illustrates this idea:

$$\begin{aligned}3x^2(2x^2 + 5x + 3) &= 3x^2(2x^2) + 3x^2(5x) + 3x^2(3)\\ &= 6x^4 + 15x^3 + 9x^2\end{aligned}$$

Extending the method of finding the product of a monomial and a polynomial to finding the product of two polynomials, each of which has more than one term, is again based on the distributive property:

$$\begin{aligned}(x + 2)(y + 5) &= x(y + 5) + 2(y + 5)\\ &= x(y) + x(5) + 2(y) + 2(5)\\ &= xy + 5x + 2y + 10\end{aligned}$$

In the next example, notice that each term of the first polynomial multiplies each term of the second polynomial:

$$\begin{aligned}(x - 3)(y + z + 3) &= x(y + z + 3) - 3(y + z + 3)\\ &= xy + xz + 3x - 3y - 3z - 9\end{aligned}$$

Frequently, multiplying polynomials produces similar terms that can be combined, which simplifies the resulting polynomial:

$$\begin{aligned}(x + 5)(x + 7) &= x(x + 7) + 5(x + 7)\\ &= x^2 + 7x + 5x + 35\\ &= x^2 + 12x + 35\end{aligned}$$

In a previous algebra course, you may have developed a shortcut for multiplying binomials, as illustrated by Figure 0.17.

$$(2x + 5)(3x - 2) = 6x^2 + 11x - 10$$

Figure 0.17

STEP 1 Multiply $(2x)(3x)$.

STEP 2 Multiply $(5)(3x)$ and $(2x)(-2)$ and combine.

STEP 3 Multiply $(5)(-2)$.

Remark: Shortcuts can be very helpful for certain manipulations in mathematics. But a word of caution: Do not lose the understanding of what you are doing. Make sure that you are able to do the manipulation without the shortcut.

Keep in mind that the shortcut illustrated in Figure 0.17 applies only to multiplying two binomials. The next example applies the distributive property to find the product of a binomial and a trinomial:

$$\begin{aligned}(x - 2)(x^2 - 3x + 4) &= x(x^2 - 3x + 4) - 2(x^2 - 3x + 4)\\ &= x^3 - 3x^2 + 4x - 2x^2 + 6x - 8\\ &= x^3 - 5x^2 + 10x - 8\end{aligned}$$

In this example we are claiming that

$$(x - 2)(x^2 - 3x + 4) = x^3 - 5x^2 + 10x - 8$$

for all real numbers. In addition to going back over our work, how can we verify such a claim? Obviously, we cannot try all real numbers, but trying at least one number gives us a partial check. Let's try the number 4:

$$\begin{aligned}(x - 2)(x^2 - 3x + 4) &= (4 - 2)(4^2 - 3(4) + 4)\\ &= 2(16 - 12 + 4)\\ &= 2(8)\\ &= 16\end{aligned}$$

$$\begin{aligned}x^3 - 5x^2 + 10x - 8 &= 4^3 - 5(4)^2 + 10(4) - 8\\ &= 64 - 80 + 40 - 8\\ &= 16\end{aligned}$$

We can also use a graphical approach as a partial check for such a problem. In Figure 0.18, we let $Y_1 = (x - 2)(x^2 - 3x + 4)$ and $Y_2 = x^3 - 5x^2 + 10x - 8$ and graphed them on the same set of axes. Note that the graphs appear to be identical.

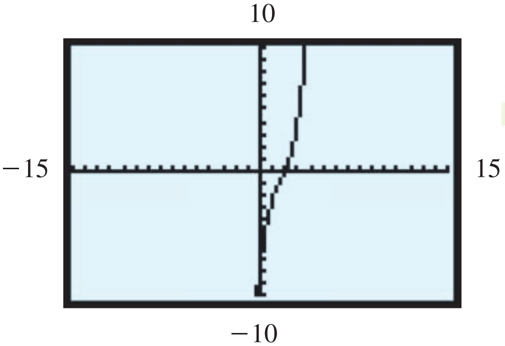


Figure 0.18

Remark: Graphing on the Cartesian coordinate system is not formally reviewed in this text until Chapter 2. However, we feel confident that your knowledge of this topic from previous mathematics courses is sufficient for what we are doing at this time.

Exponents can also be used to indicate repeated multiplication of polynomials. For example, $(3x - 4y)^2$ means $(3x - 4y)(3x - 4y)$, and $(x + 4)^3$ means $(x + 4)(x + 4)(x + 4)$. Therefore, raising a polynomial to a power is merely another multiplication problem.

$$\begin{aligned}(3x - 4y)^2 &= (3x - 4y)(3x - 4y)\\ &= 9x^2 - 24xy + 16y^2\end{aligned}$$

[*Hint*: When squaring a binomial, be careful not to forget the middle term. That is, $(x+5)^2 \neq x^2 + 25$; instead, $(x+5)^2 = x^2 + 10x + 25$.]

$$\begin{aligned}(x+4)^3 &= (x+4)(x+4)(x+4)\\ &= (x+4)(x^2+8x+16)\\ &= x(x^2+8x+16) + 4(x^2+8x+16)\\ &= x^3 + 8x^2 + 16x + 4x^2 + 32x + 64\\ &= x^3 + 12x^2 + 48x + 64\end{aligned}$$

Special Patterns

In multiplying binomials, you should learn to recognize some special patterns. These patterns can be used to find products, and some of them will be helpful later when you are factoring polynomials.

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2\\ (a-b)^2 &= a^2 - 2ab + b^2\\ (a+b)(a-b) &= a^2 - b^2\\ (a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3\\ (a-b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3\end{aligned}$$

The three following examples illustrate the first three patterns, respectively:

$$\begin{aligned}(2x+3)^2 &= (2x)^2 + 2(2x)(3) + (3)^2\\ &= 4x^2 + 12x + 9\\ (5x-2)^2 &= (5x)^2 - 2(5x)(2) + (2)^2\\ &= 25x^2 - 20x + 4\end{aligned}$$

$$(3x+2y)(3x-2y) = (3x)^2 - (2y)^2 = 9x^2 - 4y^2$$

In the first two examples, the resulting trinomial is called a **perfect-square trinomial**; it is the result of squaring a binomial. In the third example, the resulting binomial is called the **difference of two squares**. Later, we will use both of these patterns extensively when factoring polynomials.

The cubing-of-a-binomial patterns are helpful primarily when you are multiplying. These patterns can shorten the work of cubing a binomial, as the next two examples illustrate:

$$\begin{aligned}(3x+2)^3 &= (3x)^3 + 3(3x)^2(2) + 3(3x)(2)^2 + (2)^3\\ &= 27x^3 + 54x^2 + 36x + 8\\ (5x-2y)^3 &= (5x)^3 - 3(5x)^2(2y) + 3(5x)(2y)^2 - (2y)^3\\ &= 125x^3 - 150x^2y + 60xy^2 - 8y^3\end{aligned}$$

Keep in mind that these multiplying patterns are useful shortcuts, but if you forget them, simply revert to applying the distributive property.

Binomial Expansion Pattern

It is possible to write the expansion of $(a + b)^n$, where n is *any* positive integer, without showing all of the intermediate steps of multiplying and combining similar terms. To do this, let's observe some patterns in the following examples; each one can be verified by direct multiplication:

$$
\begin{aligned}
(a+b)^1 &= a + b\\
(a+b)^2 &= a^2 + 2ab + b^2\\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3\\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{aligned}
$$

First, note the patterns of the exponents for a and b on a term-by-term basis. The exponents of a begin with the exponent of the binomial and decrease by 1, term by term, until the last term, which has $a^0 = 1$. The exponents of b begin with zero ($b^0 = 1$) and increase by 1, term by term, until the last term, which contains b to the power of the original binomial. In other words, the variables in the expansion of $(a + b)^n$ have the pattern

$$a^n, \quad a^{n-1}b, \quad a^{n-2}b^2, \quad \ldots, \quad ab^{n-1}, \quad b^n$$

where, for each term, the *sum* of the exponents of a and b is n.

Next, let's arrange the *coefficients* in a triangular formation; this yields an easy-to-remember pattern.

$$
\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Row number n in the formation contains the coefficients of the expansion of $(a + b)^n$. For example, the fifth row contains 1 5 10 10 5 1, and these numbers are the coefficients of the terms in the expansion of $(a + b)^5$. Furthermore, each can be formed from the previous row as follows:

1. Start and end each row with 1.
2. All other entries result from adding the two numbers in the row immediately above, one number to the left and one number to the right.

Thus from row 5, we can form row 6.

Row 5: 1 5 10 10 5 1

Add Add Add Add Add

Row 6: 1 6 15 20 15 6 1

Now we can use these seven coefficients and our discussion about the exponents to write out the expansion for $(a + b)^6$.

$$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Remark: The triangular formation of numbers that we have been discussing is often referred to as *Pascal's triangle*. This is in honor of Blaise Pascal, a 17th-century mathematician, to whom the discovery of this pattern is attributed.

Let's consider two more examples using Pascal's triangle and the exponent relationships.

Classroom Example
Expand $(x - y)^5$.

EXAMPLE 1 Expand $(a - b)^4$.

Solution

We can treat $a - b$ as $a + (-b)$ and use the fourth row of Pascal's triangle (1, 4, 6, 4, 1) to obtain the coefficients:

$$\begin{aligned}[a + (-b)]^4 &= a^4 + 4a^3(-b) + 6a^2(-b)^2 + 4a(-b)^3 + (-b)^4 \\ &= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4\end{aligned}$$

Classroom Example
Expand $(3a + 2b)^4$.

EXAMPLE 2 Expand $(2x + 3y)^5$.

Solution

Let $2x = a$ and $3y = b$. The coefficients (1, 5, 10, 10, 5, 1) come from the fifth row of Pascal's triangle:

$$\begin{aligned}(2x + 3y)^5 &= (2x)^5 + 5(2x)^4(3y) + 10(2x)^3(3y)^2 + 10(2x)^2(3y)^3 + 5(2x)(3y)^4 + (3y)^5 \\ &= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5\end{aligned}$$

Dividing Polynomials by Monomials

In Section 0.5 we will review the addition and subtraction of rational expressions using the properties

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} \quad \text{and} \quad \frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$$

These properties can also be viewed as

$$\frac{a + c}{b} = \frac{a}{b} + \frac{c}{b} \quad \text{and} \quad \frac{a - c}{b} = \frac{a}{b} - \frac{c}{b}$$

Together with our knowledge of dividing monomials, these properties provide the basis for dividing polynomials by monomials. Consider the following examples:

$$\frac{18x^3 + 24x^2}{6x} = \frac{18x^3}{6x} + \frac{24x^2}{6x} = 3x^2 + 4x$$

$$\frac{35x^2y^3 - 55x^3y^4}{5xy^2} = \frac{35x^2y^3}{5xy^2} - \frac{55x^3y^4}{5xy^2} = 7xy - 11x^2y^2$$

Therefore, to divide a polynomial by a monomial, we divide each term of the polynomial by the monomial. As with many skills, once you feel comfortable with the process, you may then choose to perform some of the steps mentally. Your work could take the following format:

$$\frac{40x^4y^5 + 72x^5y^7}{8x^2y} = 5x^2y^4 + 9x^3y^6$$

$$\frac{36a^3b^4 - 48a^3b^3 + 64a^2b^5}{-4a^2b^2} = -9ab^2 + 12ab - 16b^3$$

Concept Quiz 0.3

For Problems 1–8, answer true or false.

1. The variables of a monomial term have exponents that are either positive integers or zero.
2. The term, 3^2xy^2, is of degree 5.
3. Any nonzero constant term is of degree zero.
4. A polynomial is a monomial or a finite sum of monomials.
5. A polynomial with three terms is classified as a binomial.
6. $(x - 6)^2 = x^2 + 36$
7. A perfect-square trinomial is the result when a trinomial is squared.
8. Row number 4 in Pascal's triangle contains the coefficients of the expansion of $(a + b)^3$.

Problem Set 0.3

For Problems 1–10, perform the indicated operations. **(Objective 1)**

1. $(5x^2 - 7x - 2) + (9x^2 + 8x - 4)$
2. $(-9x^2 + 8x + 4) + (7x^2 - 5x - 3)$
3. $(14x^2 - x - 1) - (15x^2 + 3x + 8)$
4. $(-3x^2 + 2x + 4) - (4x^2 + 6x - 5)$
5. $(3x - 4) - (6x + 3) + (9x - 4)$
6. $(7a - 2) - (8a - 1) - (10a - 2)$
7. $(8x^2 - 6x - 2) + (x^2 - x - 1) - (3x^2 - 2x + 4)$
8. $(12x^2 + 7x - 2) - (3x^2 + 4x + 5) + (-4x^2 - 7x - 2)$
9. $5(x - 2) - 4(x + 3) - 2(x + 6)$
10. $3(2x - 1) - 2(3x + 4) - 4(5x - 1)$

For Problems 11–54, find the indicated products. Remember the special patterns that we discussed in this section. **(Objective 2)**

11. $3xy(4x^2y + 5xy^2)$

12. $-2ab^2(3a^2b - 4ab^3)$

13. $6a^3b^2(5ab - 4a^2b + 3ab^2)$

14. $-xy^4(5x^2y - 4xy^2 + 3x^2y^2)$

15. $(x + 8)(x + 12)$

16. $(x - 9)(x + 6)$

17. $(n - 4)(n - 12)$

18. $(n + 6)(n - 10)$

19. $(s - t)(x + y)$

20. $(a + b)(c + d)$

21. $(3x - 1)(2x + 3)$

22. $(5x + 2)(3x + 4)$

23. $(4x - 3)(3x - 7)$

24. $(4n + 3)(6n - 1)$

25. $(x + 4)^2$

26. $(x - 6)^2$

27. $(2n + 3)^2$

28. $(3n - 5)^2$

29. $(x + 2)(x - 4)(x + 3)$

30. $(x - 1)(x + 6)(x - 5)$

31. $(x - 1)(2x + 3)(3x - 2)$

32. $(2x + 5)(x - 4)(3x + 1)$

33. $(x - 1)(x^2 + 3x - 4)$

34. $(t + 1)(t^2 - 2t - 4)$

35. $(t - 1)(t^2 + t + 1)$

36. $(2x - 1)(x^2 + 4x + 3)$

37. $(3x + 2)(2x^2 - x - 1)$

38. $(3x - 2)(2x^2 + 3x + 4)$

39. $(x^2 + 2x - 1)(x^2 + 6x + 4)$

40. $(x^2 - x + 4)(2x^2 - 3x - 1)$

41. $(5x - 2)(5x + 2)$

42. $(3x - 4)(3x + 4)$

43. $(x^2 - 5x - 2)^2$

44. $(-x^2 + x - 1)^2$

45. $(2x + 3y)(2x - 3y)$

46. $(9x + y)(9x - y)$

47. $(x + 5)^3$

48. $(x - 6)^3$

49. $(2x + 1)^3$

50. $(3x + 4)^3$

51. $(4x - 3)^3$

52. $(2x - 5)^3$

53. $(5x - 2y)^3$

54. $(x + 3y)^3$

For Problems 55–66, use Pascal's triangle to help expand each expression. **(Objective 3)**

55. $(a + b)^7$

56. $(a + b)^8$

57. $(x - y)^5$

58. $(x - y)^6$

59. $(x + 2y)^4$

60. $(2x + y)^5$

61. $(2a - b)^6$

62. $(3a - b)^4$

63. $(x^2 + y)^7$

64. $(x + 2y^2)^7$

65. $(2a - 3b)^5$

66. $(4a - 3b)^3$

For Problems 67–72, perform the indicated divisions.

67. $\dfrac{15x^4 - 25x^3}{5x^2}$

68. $\dfrac{-48x^8 - 72x^6}{-8x^4}$

69. $\dfrac{30a^5 - 24a^3 + 54a^2}{-6a}$

70. $\dfrac{18x^3y^2 + 27x^2y^3}{3xy}$

71. $\dfrac{-20a^3b^2 - 44a^4b^5}{-4a^2b}$

72. $\dfrac{21x^5y^6 + 28x^4y^3 - 35x^5y^4}{7x^2y^3}$

For Problems 73–82, find the indicated products. Assume all variables that appear as exponents represent integers. **(Objectives 2 and 3)**

73. $(x^a + y^b)(x^a - y^b)$

74. $(x^{2a} + 1)(x^{2a} - 3)$

75. $(x^b + 4)(x^b - 7)$

76. $(3x^a - 2)(x^a + 5)$

77. $(2x^b - 1)(3x^b + 2)$

78. $(2x^a - 3)(2x^a + 3)$

79. $(x^{2a} - 1)^2$

80. $(x^{3b} + 2)^2$

81. $(x^a - 2)^3$

82. $(x^b + 3)^3$

Thoughts Into Words

83. Describe how to multiply two binomials.

84. Describe how to multiply a binomial and a trinomial.

85. Determine the number of terms in the product of $(x + y)$ and $(a + b + c + d)$ without doing the multiplication. Explain how you arrived at your answer.

Graphing Calculator Activities

86. Use the computing feature of your graphing calculator to check at least one real number for your answers for Problems 29–40.

87. Use the graphing feature of your graphing calculator to give visual support for your answers for Problems 47–52.

88. Some of the product patterns can be used to do arithmetic computations mentally. For example, let's use the pattern $(a + b)^2 = a^2 + 2ab + b^2$ to compute 31^2 mentally. Your thought process should be "$31^2 = (30 + 1)^2 = 30^2 + 2(30)(1) + 1^2 = 961$." Compute each of the following numbers mentally, and then check your answers with your calculator.

(a) 21^2 **(b)** 41^2
(c) 71^2 **(d)** 32^2
(e) 52^2 **(f)** 82^2

89. Use the pattern $(a - b)^2 = a^2 - 2ab + b^2$ to compute each of the following numbers mentally, and then check your answers with your calculator.

(a) 19^2 **(b)** 29^2
(c) 49^2 **(d)** 79^2
(e) 38^2 **(f)** 58^2

90. Every whole number with a units digit of 5 can be represented by the expression $10x + 5$, where x is a whole number. For example, $35 = 10(3) + 5$ and $145 = 10(14) + 5$. Now let's observe the following pattern when squaring such a number:

$$\begin{aligned}(10x + 5)^2 &= 100x^2 + 100x + 25\\ &\boxed{= 100x(x + 1) + 25}\end{aligned}$$

The pattern inside the dashed box can be stated as "add 25 to the product of x, $x + 1$, and 100." Thus to compute 35^2 mentally, we can think "$35^2 = 3(4)(100) + 25 = 1225$." Compute each of the following numbers mentally, and then check your answers with your calculator.

(a) 15^2 **(b)** 25^2
(c) 45^2 **(d)** 55^2
(e) 65^2 **(f)** 75^2
(g) 85^2 **(h)** 95^2
(i) 105^2

Answers to the Concept Quiz

1. True **2.** False **3.** True **4.** True **5.** False **6.** False **7.** False **8.** False

0.4 Factoring Polynomials

OBJECTIVES

1. Factor out a common factor
2. Factor by grouping
3. Factor the difference of two squares
4. Factor trinomials
5. Factor the sum or difference of two cubes
6. Apply more than one factoring technique

If a polynomial is equal to the product of other polynomials, then each polynomial in the product is called a **factor** of the original polynomial. For example, because $x^2 - 4$ can be expressed as $(x + 2)(x - 2)$, we say that $x + 2$ and $x - 2$ are factors of $x^2 - 4$. The process of expressing a polynomial as a product of polynomials is called **factoring.** In this section we will consider methods of factoring polynomials with integer coefficients.

In general, factoring is the reverse of multiplication, so we can use our knowledge of multiplication to help develop factoring techniques. For example, we previously used the distributive property to find the product of a monomial and a polynomial, as the next examples illustrate.

$$3(x + 2) = 3(x) + 3(2) = 3x + 6$$
$$3x(x + 4) = 3x(x) + 3x(4) = 3x^2 + 12x$$

For factoring purposes, the distributive property [now in the form $ab + ac = a(b + c)$] can be used to reverse the process.

$$3x + 6 = 3(x) + 3(2) = 3(x + 2)$$
$$3x^2 + 12x = 3x(x) + 3x(4) = 3x(x + 4)$$

Polynomials can be factored in a variety of ways. Consider some factorizations of $3x^2 + 12x$:

$$3x^2 + 12x = 3x(x + 4) \quad \text{or} \quad 3x^2 + 12x = 3(x^2 + 4x) \quad \text{or}$$
$$3x^2 + 12x = x(3x + 12) \quad \text{or} \quad 3x^2 + 12x = \frac{1}{2}(6x^2 + 24x)$$

We are, however, primarily interested in the first of these factorization forms; we refer to it as the **completely factored form.** A polynomial with integral coefficients is in completely factored form if:

1. it is expressed as a product of polynomials with *integral coefficients,* and
2. no polynomial, other than a monomial, within the factored form can be further factored into polynomials with integral coefficients.

Do you see why only the first of the factored forms of $3x^2 + 12x$ is said to be in completely factored form? In each of the other three forms, the polynomial inside the parentheses can be factored further. Moreover, in the last form, $\frac{1}{2}(6x^2 + 24x)$, the condition of using only integers is violated.

This application of the distributive property is often referred to as **factoring out the highest common monomial factor.** The following examples illustrate the process:

$$12x^3 + 16x^2 = 4x^2(3x + 4)$$
$$8ab - 18b = 2b(4a - 9)$$
$$6x^2y^3 + 27xy^4 = 3xy^3(2x + 9y)$$
$$30x^3 + 42x^4 - 24x^5 = 6x^3(5 + 7x - 4x^2)$$

Sometimes there may be a common *binomial* factor rather than a common monomial factor. For example, each of the two terms in the expression $x(y + 2) + z(y + 2)$ has a binomial factor of $y + 2$. Thus we can factor $y + 2$ from each term and obtain the following result:

$$x(y + 2) + z(y + 2) = (y + 2)(x + z)$$

Consider a few more examples involving a common binomial factor:

$$a^2(b + 1) + 2(b + 1) = (b + 1)(a^2 + 2)$$
$$x(2y - 1) - y(2y - 1) = (2y - 1)(x - y)$$
$$x(x + 2) + 3(x + 2) = (x + 2)(x + 3)$$

Factoring by Grouping

It may seem that a given polynomial exhibits no apparent common monomial or binomial factor. Such is the case with $ab + 3c + bc + 3a$. However, by using the commutative property to rearrange the terms, we can factor it as follows:

$$\begin{aligned} ab + 3c + bc + 3a &= ab + 3a + bc + 3c \\ &= a(b + 3) + c(b + 3) && \text{Factor } a \text{ from the first two terms and } c \text{ from the last two terms} \\ &= (b + 3)(a + c) && \text{Factor } b + 3 \text{ from both terms} \end{aligned}$$

This factoring process is referred to as **factoring by grouping.** Let's consider another example of this type.

$$\begin{aligned} ab^2 - 4b^2 + 3a - 12 &= b^2(a - 4) + 3(a - 4) && \text{Factor } b^2 \text{ from the first two terms, 3 from the last two} \\ &= (a - 4)(b^2 + 3) && \text{Factor the common binomial from both terms} \end{aligned}$$

Difference of Two Squares

In Section 0.3 we called your attention to some special multiplication patterns. One of these patterns was

$$(a + b)(a - b) = a^2 - b^2$$

This same pattern, viewed as a factoring pattern,

$$a^2 - b^2 = (a + b)(a - b)$$

is referred to as the **difference of two squares.** Applying the pattern is a fairly simple process, as these next examples illustrate.

$$x^2 - 16 = (x)^2 - (4)^2 = (x + 4)(x - 4)$$
$$4x^2 - 25 = (2x)^2 - (5)^2 = (2x + 5)(2x - 5)$$

Because multiplication is commutative, the order in which we write the factors is not important. For example, $(x + 4)(x - 4)$ can also be written $(x - 4)(x + 4)$.

You must be careful not to assume an analogous factoring pattern for the *sum* of two squares; *it does not exist*. For example, $x^2 + 4 \neq (x + 2)(x + 2)$ because $(x + 2)(x + 2) = x^2 + 4x + 4$. We say that a polynomial such as $x^2 + 4$ is **not factorable using integers.**

Sometimes the difference-of-two-squares pattern can be applied more than once, as the next example illustrates:

$$16x^4 - 81y^4 = (4x^2 + 9y^2)(4x^2 - 9y^2) = (4x^2 + 9y^2)(2x + 3y)(2x - 3y)$$

It may also happen that the squares are not just simple monomial squares. These next three examples illustrate such polynomials.

$$(x + 3)^2 - y^2 = [(x + 3) + y][(x + 3) - y] = (x + 3 + y)(x + 3 - y)$$

$$\begin{aligned} 4x^2 - (2y + 1)^2 &= [2x + (2y + 1)][2x - (2y + 1)] \\ &= (2x + 2y + 1)(2x - 2y - 1) \end{aligned}$$

$$\begin{aligned} (x - 1)^2 - (x + 4)^2 &= [(x - 1) + (x + 4)][(x - 1) - (x + 4)] \\ &= (x - 1 + x + 4)(x - 1 - x - 4) \\ &= (2x + 3)(-5) \end{aligned}$$

It is possible that both the technique of factoring out a common monomial factor and the pattern of the difference of two squares can be applied to the same problem. *In general, it is best to look first for a common monomial factor*. Consider the following examples.

$$\begin{aligned} 2x^2 - 50 &= 2(x^2 - 25) \\ &= 2(x + 5)(x - 5) \end{aligned}$$

$$\begin{aligned} 48y^3 - 27y &= 3y(16y^2 - 9) \\ &= 3y(4y + 3)(4y - 3) \end{aligned}$$

$$\begin{aligned} 9x^2 - 36 &= 9(x^2 - 4) \\ &= 9(x + 2)(x - 2) \end{aligned}$$

Factoring Trinomials

Expressing a trinomial as the product of two binomials is one of the most common factoring techniques used in algebra. As before, to develop a factoring technique we first look at some multiplication ideas. Let's consider the product $(x + a)(x + b)$, using the distributive property to show how each term of the resulting trinomial is formed:

$$\begin{aligned}(x + a)(x + b) &= x(x + b) + a(x + b)\\ &= x(x) + x(b) + a(x) + a(b)\\ &= x^2 + (a + b)x + ab\end{aligned}$$

Notice that the coefficient of the middle term is the *sum* of a and b and that the last term is the *product* of a and b. These two relationships can be used to factor trinomials. Let's consider some examples.

Classroom Example
Factor $a^2 + 12a + 32$.

EXAMPLE 1 Factor $x^2 + 12x + 20$.

Solution

We need two integers whose sum is 12 and whose product is 20. The numbers are 2 and 10, and we can complete the factoring as follows:

$$x^2 + 12x + 20 = (x + 2)(x + 10)$$

Classroom Example
Factor $y^2 - 10y - 24$.

EXAMPLE 2 Factor $x^2 - 3x - 54$.

Solution

We need two integers whose sum is -3 and whose product is -54. The integers are -9 and 6, and we can factor as follows:

$$x^2 - 3x - 54 = (x - 9)(x + 6)$$

Classroom Example
Factor $x^2 + 2x + 12$.

EXAMPLE 3 Factor $x^2 + 7x + 16$.

Solution

We need two integers whose sum is 7 and whose product is 16. The only possible pairs of factors of 16 are $1 \cdot 16$, $2 \cdot 8$, and $4 \cdot 4$. A sum of 7 is not produced by any of these pairs, so the polynomial $x^2 + 7x + 16$ is *not factorable using integers*.

Trinomials of the Form $ax^2 + bx + c$

Now let's consider factoring trinomials where the coefficient of the squared term is not one. First, let's illustrate an informal trial-and-error technique that works well for certain types of trinomials. This technique is based on our knowledge of multiplication of binomials.

Classroom Example
Factor $5a^2 + 8a + 3$.

EXAMPLE 4 Factor $3x^2 + 5x + 2$.

Solution

By looking at the first term, $3x^2$, and the positive signs of the other two terms, we know that the binomials are of the form

$$(x + \_\_)(3x + \_\_)$$

Because the factors of the last term, 2, are 1 and 2, we have only the following two possibilities to try.

$$(x + 2)(3x + 1) \qquad \text{or} \qquad (x + 1)(3x + 2)$$

By checking the middle term formed in each of these products, we find that the second possibility yields the desired middle term of $5x$. Therefore

$$3x^2 + 5x + 2 = (x + 1)(3x + 2)$$

Classroom Example
Factor $6x^2 + 17xy + 5y^2$.

EXAMPLE 5 Factor $8x^2 - 30xy + 7y^2$.

Solution

First, observe that the first term, $8x^2$, can be written as $2x \cdot 4x$ or $x \cdot 8x$. Second, because the middle term is negative and the last term is positive, we know that the binomials are of the form

$$(2x - \_\_)(4x - \_\_) \qquad \text{or} \qquad (x - \_\_)(8x - \_\_)$$

Third, because the factors of the last term, $7y^2$, are $1y$ and $7y$, the following possibilities exist.

$$(2x - 1y)(4x - 7y) \qquad (2x - 7y)(4x - 1y)$$
$$(x - 1y)(8x - 7y) \qquad (x - 7y)(8x - 1y)$$

By checking the middle term formed in each of these products, we find that $(2x - 7y)(4x - 1y)$ produces the desired middle term of $-30xy$. Therefore

$$8x^2 - 30xy + 7y^2 = (2x - 7y)(4x - y)$$

Classroom Example
Factor $6a^2 + 46a + 28$.

EXAMPLE 6 Factor $10x^2 - 36x - 16$.

Solution

First, note that there is a common factor of 2. By using the distributive property we obtain $10x^2 - 36x - 16 = 2(5x^2 - 18x - 8)$. Now, let's determine if $5x^2 - 18x - 8$ can be factored. The first term, $5x^2$, can be written as $x \cdot 5x$. The last term, -8, can be written as $(-2)(4)$, $(2)(-4)$, $(-1)(8)$, or $(1)(-8)$. Therefore we have the following possibilities to try:

$$(x - 2)(5x + 4) \qquad (x + 4)(5x - 2) \qquad (x - 1)(5x + 8) \qquad (x + 8)(5x - 1)$$
$$(x + 2)(5x - 4) \qquad (x - 4)(5x + 2) \qquad (x + 1)(5x - 8) \qquad (x - 8)(5x + 1)$$

By checking the middle terms, we find that $(x - 4)(5x + 2)$ yields the desired middle term of $-18x$. Thus

$$10x^2 - 36x - 16 = 2(5x^2 - 18x - 8) = 2(x - 4)(5x + 2)$$

Classroom Example
Factor $2y^2 + 11y + 6$.

EXAMPLE 7 Factor $4x^2 + 6x + 9$.

Solution

The first term, $4x^2$, and the positive signs of the middle and last terms indicate that the binomials are of the form

$$(x + \_\_)(4x + \_\_) \qquad \text{or} \qquad (2x + \_\_)(2x + \_\_)$$

Because the factors of the last term, 9, are 1 and 9 or 3 and 3, we have the following possibilities to try:

$$(x + 1)(4x + 9)$$
$$(x + 9)(4x + 1)$$
$$(x + 3)(4x + 3)$$
$$(2x + 1)(2x + 9)$$
$$(2x + 3)(2x + 3)$$

None of these possibilities yields a middle term of $6x$. Therefore $4x^2 + 6x + 9$ is *not factorable using integers*.

Certainly, as the number of possibilities increases, this trial-and-error technique for factoring becomes more tedious. The key idea is to organize your work so that all possibilities are considered. We have suggested one possible format in the previous examples. However, as you practice such problems, you may devise a format that works better for you. Whatever works best for you is the right approach.

There is another, more systematic technique that you may wish to use with some trinomials. It is an extension of the technique we used earlier with trinomials where the coefficient of the squared term was one. To see the basis of this technique, consider the following general product:

$$\begin{aligned}(px + r)(qx + s) &= px(qx) + px(s) + r(qx) + r(s)\\ &= (pq)x^2 + ps(x) + rq(x) + rs\\ &= (pq)x^2 + (ps + rq)x + rs\end{aligned}$$

Notice that the product of the coefficient of x^2 and the constant term is $pqrs$. Likewise, the product of the two coefficients of x (ps and rq) is also $pqrs$. Therefore, the coefficient of x must be a sum of the form $ps + rq$, such that the product of the coefficient of x^2 and the constant term is $pqrs$. Now let's see how this works in some specific examples.

Classroom Example
Factor $8a^2 + 14a + 3$.

EXAMPLE 8 Factor $6x^2 + 17x + 5$.

Solution

$$6x^2 + 17x + 5 \qquad \text{Sum of } 17$$

Product of $6 \cdot 5 = 30$

We need two integers whose sum is 17 and whose product is 30. The integers 2 and 15 satisfy these conditions. Therefore the middle term, $17x$, of the given trinomial can be expressed as $2x + 15x$, and we can proceed as follows:

$$\begin{aligned} 6x^2 + 17x + 5 &= 6x^2 + 2x + 15x + 5 \\ &= 2x(3x + 1) + 5(3x + 1) \qquad \text{Factor by grouping} \\ &= (3x + 1)(2x + 5) \end{aligned}$$

Classroom Example
Factor $3y^2 + 16y - 12$.

EXAMPLE 9 Factor $5x^2 - 18x - 8$.

Solution

$5x^2 - 18x - 8$ Sum of -18

Product of $5(-8) = -40$

We need two integers whose sum is -18 and whose product is -40. The integers -20 and 2 satisfy these conditions. Therefore the middle term, $-18x$, of the trinomial can be written $-20x + 2x$, and we can factor as follows:

$$\begin{aligned} 5x^2 - 18x - 8 &= 5x^2 - 20x + 2x - 8 \\ &= 5x(x - 4) + 2(x - 4) \\ &= (x - 4)(5x + 2) \end{aligned}$$

Classroom Example
Factor $8a^2 + 22a - 21$.

EXAMPLE 10 Factor $24x^2 + 2x - 15$.

Solution

$24x^2 + 2x - 15$ Sum of 2

Product of $24(-15) = -360$

We need two integers whose sum is 2 and whose product is -360. To help find these integers, let's factor 360 into primes:

$$360 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$$

Now by grouping these factors in various ways, we find that $2 \cdot 2 \cdot 5 = 20$ and $2 \cdot 3 \cdot 3 = 18$, so we can use the integers 20 and -18 to produce a sum of 2 and a product of -360. Therefore, the middle term, $2x$, of the trinomial can be expressed as $20x - 18x$, and we can proceed as follows:

$$\begin{aligned} 24x^2 + 2x - 15 &= 24x^2 + 20x - 18x - 15 \\ &= 4x(6x + 5) - 3(6x + 5) \\ &= (6x + 5)(4x - 3) \end{aligned}$$

Probably the best way to check a factoring problem is to make sure the conditions for a polynomial to be completely factored are satisfied, and the product of the factors equals the given polynomial. We can also give some visual support to a factoring problem by graphing the given polynomial and its completely factored form on the same set of axes, as shown for Example 10 in Figure 0.19. Note that the graphs for $Y_1 = 24x^2 + 2x - 15$ and $Y_2 = (6x + 5)(4x - 3)$ appear to be identical.

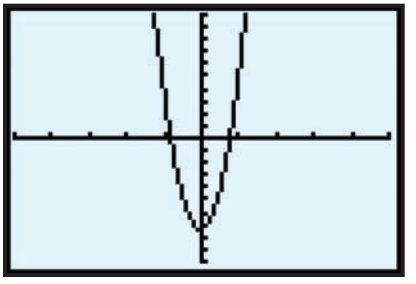


Figure 0.19

Sum and Difference of Two Cubes

Earlier in this section we discussed the difference-of-squares factoring pattern. We pointed out that no analogous sum-of-squares pattern exists; that is, a polynomial such as $x^2 + 9$ is not factorable using integers. However, there do exist patterns for both the *sum* and the *difference of two cubes*. These patterns come from the following special products:

$$\begin{aligned}(x + y)(x^2 - xy + y^2) &= x(x^2 - xy + y^2) + y(x^2 - xy + y^2)\\ &= x^3 - x^2y + xy^2 + x^2y - xy^2 + y^3\\ &= x^3 + y^3\end{aligned}$$

$$\begin{aligned}(x - y)(x^2 + xy + y^2) &= x(x^2 + xy + y^2) - y(x^2 + xy + y^2)\\ &= x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3\\ &= x^3 - y^3\end{aligned}$$

Thus we can state the following factoring patterns:

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Note how these patterns are used in the next three examples:

$$x^3 + 8 = x^3 + 2^3 = (x + 2)(x^2 - 2x + 4)$$
$$8x^3 - 27y^3 = (2x)^3 - (3y)^3 = (2x - 3y)(4x^2 + 6xy + 9y^2)$$
$$8a^6 + 125b^3 = (2a^2)^3 + (5b)^3 = (2a^2 + 5b)(4a^4 - 10a^2b + 25b^2)$$

Applying More Than One Factoring Technique

We do want to leave you with one final word of caution. **Be sure to factor completely.** Sometimes more than one technique needs to be applied, or perhaps the same technique can be applied more than once. Study the following examples very carefully:

$$2x^2 - 8 = 2(x^2 - 4) = 2(x + 2)(x - 2)$$
$$3x^2 + 18x + 24 = 3(x^2 + 6x + 8) = 3(x + 4)(x + 2)$$
$$3x^3 - 3y^3 = 3(x^3 - y^3) = 3(x - y)(x^2 + xy + y^2)$$
$$a^4 - b^4 = (a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b)$$
$$x^4 - 6x^2 - 27 = (x^2 - 9)(x^2 + 3) = (x + 3)(x - 3)(x^2 + 3)$$
$$\begin{aligned} 3x^4y + 9x^2y - 84y &= 3y(x^4 + 3x^2 - 28) \\ &= 3y(x^2 + 7)(x^2 - 4) \\ &= 3y(x^2 + 7)(x + 2)(x - 2) \end{aligned}$$
$$\begin{aligned} x^2 - y^2 + 8y - 16 &= x^2 - (y^2 - 8y + 16) \\ &= x^2 - (y - 4)^2 \\ &= (x - (y - 4))(x + (y - 4)) \\ &= (x - y + 4)(x + y - 4) \end{aligned}$$

Concept Quiz 0.4

For Problems 1–8, answer true or false.

1. The process of expressing a polynomial as a product of polynomials is called factoring.
2. $x^2(5x - 10)$ is the completely factored form of $5x^2 - 10x^2$.
3. The polynomial, $3a^3b - 4c^2d + 5bd$, does not have a common factor.
4. The sum of two squares is not factorable using integers.
5. The sum of two cubes is not factorable using integers.
6. A factoring problem can be partially checked by making sure the product of the factors equals the polynomial.
7. All trinomials are factorable using integers.
8. All common factors are monomial factors.

Problem Set 0.4

For Problems 1–6, factor completely by factoring out the common factor. **(Objective 1)**

1. $6xy - 8xy^2$
2. $4a^2b^2 + 12ab^3$
3. $12x^2y^3z^4 - 6x^4y^3z^3 + 6x^2y^3z^2$
4. $3m^2n - 6m^4n^3 - 9m^5n^4$
5. $x(z + 3) + y(z + 3)$
6. $5(x + y) + a(x + y)$

For Problems 7–10, factor completely by using grouping. **(Objective 2)**

7. $3x + 3y + ax + ay$
8. $ac + bc + a + b$
9. $ax - ay - bx + by$
10. $2a^2 - 3bc - 2ab + 3ac$

For Problems 11–18, factor by applying the difference-of-squares pattern. **(Objective 3)**

11. $9x^2 - 25$
12. $36x^2 - 121$
13. $1 - 81n^2$
14. $9x^2y^2 - 64$
15. $(x + 4)^2 - y^2$
16. $x^2 - (y - 1)^2$
17. $9s^2 - (2t - 1)^2$
18. $4a^2 - (3b + 1)^2$

For Problems 19–36, factor each trinomial. Indicate any that are not factorable using integers. **(Objective 4)**

19. $x^2 - 5x - 14$
20. $a^2 + 5a - 24$
21. $15 - 2x - x^2$
22. $40 - 6x - x^2$
23. $x^2 + 7x - 36$
24. $x^2 - 4xy - 5y^2$
25. $3x^2 - 11x + 10$
26. $2x^2 - 7x - 30$
27. $10x^2 + 17x + 7$
28. $8y^2 + 22y - 21$
29. $10x^2 + 39x - 27$
30. $3x^2 + x - 5$
31. $36a^2 - 12a + 1$
32. $18n^3 + 39n^2 - 15n$
33. $8x^2 + 2xy - y^2$
34. $12x^2 + 7xy - 10y^2$
35. $2n^2 - n - 5$
36. $6x^2 - x - 12$

For Problems 37–40, factor the sum or difference of two cubes. **(Objective 5)**

37. $x^3 - 8$
38. $x^3 + 64$
39. $64x^3 + 27y^3$
40. $27x^3 - 8y^3$

For Problems 41–66, factor each polynomial completely. Indicate any that are not factorable using integers. **(Objective 6)**

41. $4x^4 + 16$
42. $n^3 - 49n$
43. $x^3 - 9x$
44. $12n^2 + 59n + 72$
45. $9a^2 - 42a + 49$
46. $1 - 16x^4$
47. $2n^3 + 6n^2 + 10n$
48. $25t^2 - 100$
49. $2n^3 + 14n^2 - 20n$
50. $25n^2 + 64$
51. $4x^3 + 32$
52. $2x^3 - 54$
53. $x^4 - 4x^2 - 45$
54. $x^4 - x^2 - 12$
55. $2x^4y - 26x^2y - 96y$
56. $3x^4y - 15x^2y - 108y$
57. $(a + b)^2 - (c + d)^2$
58. $(a - b)^2 - (c - d)^2$
59. $x^2 + 8x + 16 - y^2$
60. $4x^2 + 12x + 9 - y^2$
61. $x^2 - y^2 - 10y - 25z$
62. $y^2 - x^2 + 16x - 64$
63. $60x^2 - 32x - 15$
64. $40x^2 + 37x - 63$
65. $84x^3 + 57x^2 - 60x$
66. $210x^3 - 102x^2 - 180x$

For Problems 67–76, factor each of the following, and assume that all variables appearing as exponents represent integers.

67. $x^{2a} - 16$
68. $x^{4n} - 9$
69. $x^{3n} - y^{3n}$
70. $x^{3a} + y^{6a}$
71. $x^{2a} - 3x^a - 28$
72. $x^{2a} + 10x^a + 21$
73. $2x^{2n} + 7x^n - 30$
74. $3x^{2n} - 16x^n - 12$
75. $x^{4n} - y^{4n}$
76. $16x^{2a} + 24x^a + 9$

77. Suppose that we want to factor $x^2 + 34x + 288$. We need to complete the following with two numbers whose sum is 34 and whose product is 288.

$$x^2 + 34x + 288 = (x + \_\_)(x + \_\_)$$

These numbers can be found as follows: Because we need a product of 288, let's consider the prime factorization of 288.

$$288 = 2^5 \cdot 3^2$$

Now we need to use five 2s and two 3s in the statement

$$(\quad) + (\quad) = 34$$

Because 34 is divisible by 2 but not by 4, four factors of 2 must be in one number and one factor of 2 in the other number. Also, because 34 is not divisible by 3, both factors of 3 must be in the same number. These facts aid us in determining that

$$(2 \cdot 2 \cdot 2 \cdot 2) + (2 \cdot 3 \cdot 3) = 34$$

or

$$16 \quad + \quad 18 \quad = 34$$

Thus we can complete the original factoring problem:

$$x^2 + 34x + 288 = (x + 16)(x + 18)$$

Use this approach to factor each of the following expressions.

a. $x^2 + 35x + 96$
b. $x^2 + 27x + 176$
c. $x^2 - 45x + 504$
d. $x^2 - 26x + 168$
e. $x^2 + 60x + 896$
f. $x^2 - 84x + 1728$

Thoughts Into Words

78. Describe, in words, the pattern for factoring the sum of two cubes.

79. What does it mean to say that the polynomial $x^2 + 5x + 7$ is not factorable using integers?

80. What role does the distributive property play in the factoring of polynomials?

81. Explain your thought process when factoring $30x^2 + 13x - 56$.

82. Consider the following approach to factoring $12x^2 + 54x + 60$:

$$\begin{aligned} 12x^2 + 54x + 60 &= (3x + 6)(4x + 10) \\ &= 3(x + 2)(2)(2x + 5) \\ &= 6(x + 2)(2x + 5) \end{aligned}$$

Is this factoring process correct? What can you suggest to the person who used this approach?

Answers to the Concept Quiz

1. True **2.** False **3.** True **4.** True **5.** False **6.** True **7.** False **8.** False

0.5 Rational Expressions

OBJECTIVES

1. Simplify rational expressions
2. Multiply and divide rational expressions
3. Add and subtract rational expressions
4. Simplify complex fractions

Indicated quotients of algebraic expressions are called **algebraic fractions** or **fractional expressions**. The indicated quotient of two polynomials is called a **rational expression**. (This is analogous to defining a rational number as the indicated quotient of two integers.) The following are examples of rational expressions:

$$\frac{3x^2}{5} \qquad \frac{x-2}{x+3} \qquad \frac{x^2+5x-1}{x^2-9} \qquad \frac{xy^2+x^2y}{xy} \qquad \frac{a^3-3a^2-5a-1}{a^4+a^3+6}$$

Because division by zero must be avoided, no values can be assigned to variables that will create a denominator of zero. Thus the rational expression $\frac{x-2}{x+3}$ is meaningful for all real number values of x except $x = -3$. Rather than making restrictions for each individual expression, we will merely assume that **all denominators represent nonzero real numbers**.

The basic properties of the real numbers can be used for working with rational expressions. For example, the property

$$\frac{a \cdot k}{b \cdot k} = \frac{a}{b}$$

which is used to reduce rational numbers, is also used to *simplify* rational expressions. Consider the following examples:

$$\frac{15xy}{25y} = \frac{3 \cdot \cancel{5} \cdot x \cdot \cancel{y}}{\cancel{5} \cdot 5 \cdot \cancel{y}} = \frac{3x}{5}$$

$$\frac{-9}{18x^2y} = -\frac{\overset{1}{\cancel{9}}}{\underset{2}{\cancel{18}}x^2y} = -\frac{1}{2x^2y}$$

Note that slightly different formats were used in these two examples. In the first one, we factored the coefficients into primes and then proceeded to simplify; however, in the second problem we simply divided a common factor of 9 out of both the numerator and denominator. This is basically a format issue and depends on your personal preference. Also notice that in the second example, we applied the property $\frac{-a}{b} = -\frac{a}{b}$. This is part of the general property that states

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$$

The properties $(b^n)^m = b^{mn}$ and $(ab)^n = a^nb^n$ may also play a role when simplifying a rational expression, as the next example demonstrates.

$$\frac{(4x^3y)^2}{6x(y^2)^2} = \frac{4^2 \cdot (x^3)^2 \cdot y^2}{6 \cdot x \cdot y^4} = \frac{\overset{8}{\cancel{16}}\overset{x^5}{\cancel{x^6}}\cancel{y^2}}{\underset{3}{\cancel{6}}\cancel{x}\underset{y^2}{\cancel{y^4}}} = \frac{8x^5}{3y^2}$$

The factoring techniques discussed in the previous section can be used to factor numerators and denominators so that the property $(a \cdot k)/(b \cdot k) = a/b$ can be applied. Consider the following examples:

$$\frac{x^2 + 4x}{x^2 - 16} = \frac{x\cancel{(x + 4)}}{(x - 4)\cancel{(x + 4)}} = \frac{x}{x - 4}$$

$$\frac{5n^2 + 6n - 8}{10n^2 - 3n - 4} = \frac{\cancel{(5n - 4)}(n + 2)}{\cancel{(5n - 4)}(2n + 1)} = \frac{n + 2}{2n + 1}$$

$$\frac{x^3 + y^3}{x^2 + xy + 2x + 2y} = \frac{(x + y)(x^2 - xy + y^2)}{x(x + y) + 2(x + y)}$$

$$= \frac{\cancel{(x + y)}(x^2 - xy + y^2)}{\cancel{(x + y)}(x + 2)} = \frac{x^2 - xy + y^2}{x + 2}$$

$$\frac{6x^3y - 6xy}{x^3 + 5x^2 + 4x} = \frac{6xy(x^2 - 1)}{x(x^2 + 5x + 4)} = \frac{6\cancel{x}y\cancel{(x + 1)}(x - 1)}{\cancel{x}\cancel{(x + 1)}(x + 4)} = \frac{6y(x - 1)}{x + 4}$$

Note that in the last example we left the numerator of the final fraction in factored form. This is often done if expressions other than monomials are involved. Either

$$\frac{6y(x-1)}{x+4} \quad \text{or} \quad \frac{6xy-6y}{x+4}$$

is an acceptable answer.

Remember that the quotient of any nonzero real number and its opposite is -1. For example, $6/-6=-1$ and $-8/8=-1$. Likewise, the indicated quotient of any polynomial and its opposite is equal to -1. For example,

$$\frac{a}{-a}=-1 \qquad \text{because } a \text{ and } -a \text{ are opposites}$$

$$\frac{a-b}{b-a}=-1 \qquad \text{because } a-b \text{ and } b-a \text{ are opposites}$$

$$\frac{x^2-4}{4-x^2}=-1 \qquad \text{because } x^2-4 \text{ and } 4-x^2 \text{ are opposites}$$

The next example illustrates how we use this idea when simplifying rational expressions.

$$\begin{aligned}\frac{4-x^2}{x^2+x-6} &= \frac{(2+x)}{(x+3)}\boxed{\frac{(2-x)}{(x-2)}}\\ &= (-1)\left(\frac{x+2}{x+3}\right) \qquad \frac{2-x}{x-2}=-1\\ &= -\frac{x+2}{x+3} \quad \text{or} \quad \frac{-x-2}{x+3}\end{aligned}$$

Multiplying and Dividing Rational Expressions

Multiplication of rational expressions is based on the following property:

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

In other words, we multiply numerators and we multiply denominators and express the final product in simplified form. Study the following examples carefully and pay special attention to the formats used to organize the computational work.

$$\frac{3x}{4y}\cdot\frac{8y^2}{9x}=\frac{\cancel{3}\cdot\overset{2}{\cancel{8}}\cdot\cancel{x}\cdot\overset{y}{\cancel{y^2}}}{\cancel{4}\cdot\underset{3}{\cancel{9}}\cdot\cancel{x}\cdot\cancel{y}}=\frac{2y}{3}$$

$$\frac{12x^2y}{-18xy}\cdot\frac{-24xy^2}{56y^3}=\frac{\overset{2}{\cancel{12}}\cdot\overset{\cancel{3}}{\cancel{24}}\cdot\overset{x^2}{\cancel{x^3}}\cdot\cancel{y^3}}{\underset{\cancel{3}}{\cancel{18}}\cdot\underset{7}{\cancel{56}}\cdot\cancel{x}\cdot\underset{y}{\cancel{y^4}}}=\frac{2x^2}{7y} \qquad \frac{12x^2y}{-18xy}=-\frac{12x^2y}{18xy} \text{ and } \frac{-24xy^2}{56y^3}=-\frac{24xy^2}{56y^3}$$

so the product is positive.

$$\frac{y}{x^2-4}\cdot\frac{x+2}{y^2}=\frac{\cancel{y}\cancel{(x+2)}}{\underset{y}{\cancel{y^2}}\cancel{(x+2)}(x-2)}=\frac{1}{y(x-2)}$$

$$\frac{x^2-x}{x+5}\cdot\frac{x^2+5x+4}{x^4-x^2}=\frac{\cancel{x}\cancel{(x-1)}\cancel{(x+1)}(x+4)}{(x+5)\underset{x}{\cancel{(x^2)}}\cancel{(x+1)}\cancel{(x-1)}}=\frac{x+4}{x(x+5)}$$

To divide rational expressions, we merely apply the following property:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

That is, the quotient of two rational expressions is the product of the first expression times the reciprocal of the second. Consider the following examples:

$$\frac{16x^2y}{24xy^3} \div \frac{9xy}{8x^2y^2} = \frac{16x^2y}{24xy^3} \cdot \frac{8x^2y^2}{9xy} = \frac{16 \cdot \cancel{8} \cdot \overset{x^2}{\cancel{x^4}} \cdot \cancel{y^3}}{\underset{3}{\cancel{24}} \cdot 9 \cdot \cancel{x^2} \cdot \underset{y}{\cancel{y^4}}} = \frac{16x^2}{27y}$$

$$\frac{3a^2 + 12}{3a^2 - 15a} \div \frac{a^4 - 16}{a^2 - 3a - 10} = \frac{3a^2 + 12}{3a^2 - 15a} \cdot \frac{a^2 - 3a - 10}{a^4 - 16}$$

$$= \frac{\cancel{3}\cancel{(a^2 + 4)}\cancel{(a - 5)}\cancel{(a + 2)}}{\cancel{3}a\cancel{(a - 5)}\cancel{(a^2 + 4)}\cancel{(a + 2)}(a - 2)}$$

$$= \frac{1}{a(a - 2)}$$

Adding and Subtracting Rational Expressions

The following two properties provide the basis for adding and subtracting rational expressions:

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$$

$$\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$$

These properties state that rational expressions with a common denominator can be added (or subtracted) by adding (or subtracting) the numerators and placing the result over the common denominator. Let's illustrate this idea.

$$\frac{8}{x - 2} + \frac{3}{x - 2} = \frac{8 + 3}{x - 2} = \frac{11}{x - 2}$$

$$\frac{9}{4y} - \frac{7}{4y} = \frac{9 - 7}{4y} = \frac{2}{4y} = \frac{1}{2y}$$

Don't forget to simplify the final result.

$$\frac{n^2}{n - 1} - \frac{1}{n - 1} = \frac{n^2 - 1}{n - 1} = \frac{(n + 1)\cancel{(n - 1)}}{\cancel{n - 1}} = n + 1$$

If we need to add or subtract rational expressions that do not have a common denominator, then we apply the property $a/b = (a \cdot k)/(b \cdot k)$ to obtain equivalent fractions with a common denominator. Study the next examples and again pay special attention to the format we used to organize our work.

Remark: Remember that the **least common multiple** of a set of whole numbers is the smallest nonzero whole number divisible by each of the numbers in the set. When we add or subtract rational numbers, the least common multiple of the denominators of those numbers is the **least common denominator (LCD).** This concept of a least common denominator can be extended to include polynomials.

Classroom Example
Add $\frac{3x+2}{5} + \frac{x+6}{4}$.

EXAMPLE 1 Add $\frac{x+2}{4} + \frac{3x+1}{3}$.

Solution

By inspection we see that the LCD is 12.

$$\begin{aligned}\frac{x+2}{4} + \frac{3x+1}{3} &= \left(\frac{x+2}{4}\right)\left(\frac{3}{3}\right) + \left(\frac{3x+1}{3}\right)\left(\frac{4}{4}\right)\\ &= \frac{3(x+2)}{12} + \frac{4(3x+1)}{12}\\ &= \frac{3x+6+12x+4}{12}\\ &= \frac{15x+10}{12}\end{aligned}$$

Classroom Example
Perform the indicated operations.
$\frac{x+7}{20} + \frac{3x-4}{12} - \frac{x-3}{18}$.

EXAMPLE 2 Perform the indicated operations.

$$\frac{x+3}{10} + \frac{2x+1}{15} - \frac{x-2}{18}$$

Solution

If you cannot determine the LCD by inspection, then use the prime-factored forms of the denominators:

$$10 = 2 \cdot 5 \qquad 15 = 3 \cdot 5 \qquad 18 = 2 \cdot 3 \cdot 3$$

The LCD must contain one factor of 2, two factors of 3, and one factor of 5. Thus the LCD is $2 \cdot 3 \cdot 3 \cdot 5 = 90$.

$$\begin{aligned}\frac{x+3}{10} + \frac{2x+1}{15} - \frac{x-2}{18} &= \left(\frac{x+3}{10}\right)\left(\frac{9}{9}\right) + \left(\frac{2x+1}{15}\right)\left(\frac{6}{6}\right) - \left(\frac{x-2}{18}\right)\left(\frac{5}{5}\right)\\ &= \frac{9(x+3)}{90} + \frac{6(2x+1)}{90} - \frac{5(x-2)}{90}\\ &= \frac{9x+27+12x+6-5x+10}{90}\\ &= \frac{16x+43}{90}\end{aligned}$$

The presence of variables in the denominators does not create any serious difficulty; our approach remains the same. Study the following examples very carefully. For each problem we use the same basic procedure: (1) Find the LCD. (2) Change each fraction to an equivalent fraction having the LCD as its denominator. (3) Add or subtract numerators and place this result over the LCD. (4) Look for possibilities to simplify the resulting fraction.

Classroom Example
Add $\frac{2}{3a} + \frac{4}{5b}$.

EXAMPLE 3 Add $\frac{3}{2x} + \frac{5}{3y}$.

Solution

Using an LCD of $6xy$, we can proceed as follows:

$$\frac{3}{2x} + \frac{5}{3y} = \left(\frac{3}{2x}\right)\left(\frac{3y}{3y}\right) + \left(\frac{5}{3y}\right)\left(\frac{2x}{2x}\right)$$
$$= \frac{9y}{6xy} + \frac{10x}{6xy}$$
$$= \frac{9y + 10x}{6xy}$$

Classroom Example
Subtract $\frac{1}{6x^2} - \frac{5}{9xy}$.

EXAMPLE 4 Subtract $\frac{7}{12ab} - \frac{11}{15a^2}$.

Solution

We can factor the numerical coefficients of the denominators into primes to help find the LCD.

$$\left.\begin{aligned} 12ab &= 2 \cdot 2 \cdot 3 \cdot a \cdot b \\ 15a^2 &= 3 \cdot 5 \cdot a^2 \end{aligned}\right\} \quad \text{LCD} = 2 \cdot 2 \cdot 3 \cdot 5 \cdot a^2 \cdot b = 60a^2b$$

$$\frac{7}{12ab} - \frac{11}{15a^2} = \left(\frac{7}{12ab}\right)\left(\frac{5a}{5a}\right) - \left(\frac{11}{15a^2}\right)\left(\frac{4b}{4b}\right)$$
$$= \frac{35a}{60a^2b} - \frac{44b}{60a^2b}$$
$$= \frac{35a - 44b}{60a^2b}$$

Classroom Example
Add $\frac{3}{y} + \frac{6}{y^2 - 2y}$.

EXAMPLE 5 Add $\frac{8}{x^2 - 4x} + \frac{2}{x}$.

Solution

$$\left.\begin{aligned} x^2 - 4x &= x(x - 4) \\ x &= x \end{aligned}\right\} \quad \text{LCD} = x(x - 4)$$

$$\begin{aligned}\frac{8}{x(x-4)}+\frac{2}{x} &= \frac{8}{x(x-4)}+\left(\frac{2}{x}\right)\left(\frac{x-4}{x-4}\right)\\ &= \frac{8}{x(x-4)}+\frac{2(x-4)}{x(x-4)}\\ &= \frac{8+2x-8}{x(x-4)}\\ &= \frac{2\cancel{x}}{\cancel{x}(x-4)}\\ &= \frac{2}{x-4}\end{aligned}$$

In Figure 0.20 we give some visual support for our answer in Example 5 by graphing $Y_1 = \frac{8}{x^2-4x}+\frac{2}{x}$ and $Y_2 = \frac{2}{x-4}$. Certainly their graphs appear to be identical, but a word of caution is needed here. Actually, the graph of $Y_1 = \frac{8}{x^2-4x}+\frac{2}{x}$ has a hole at $\left(0, -\frac{1}{2}\right)$ because x cannot equal zero. When you use a graphing calculator, this hole may not be detected. Except for the hole, the graphs are identical, and we are claiming that $\frac{8}{x^2-4x}+\frac{2}{x}=\frac{2}{x-4}$ for all values of x except 0 and 4.

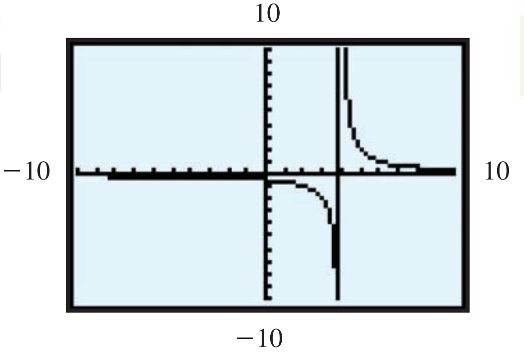


Figure 0.20

Classroom Example
Add $\frac{4m}{m^2+3m-4}+\frac{5}{m^2+m-2}$.

EXAMPLE 6 Add $\frac{3n}{n^2+6n+5}+\frac{4}{n^2-7n-8}$.

Solution

$$\left.\begin{aligned}n^2+6n+5 &= (n+5)(n+1)\\ n^2-7n-8 &= (n-8)(n+1)\end{aligned}\right\}\quad \text{LCD} = (n+1)(n+5)(n-8)$$

Unless otherwise noted, all art on this page is © Cengage Learning.

$$\frac{3n}{n^2+6n+5}+\frac{4}{n^2-7n-8}=\left[\frac{3n}{(n+5)(n+1)}\right]\left(\frac{n-8}{n-8}\right)+\left[\frac{4}{(n-8)(n+1)}\right]\left(\frac{n+5}{n+5}\right)$$

$$=\frac{3n(n-8)}{(n+5)(n+1)(n-8)}+\frac{4(n+5)}{(n+5)(n+1)(n-8)}$$

$$=\frac{3n^2-24n+4n+20}{(n+5)(n+1)(n-8)}$$

$$=\frac{3n^2-20n+20}{(n+5)(n+1)(n-8)}$$

Simplifying Complex Fractions

Fractional forms that contain rational expressions in the numerator and/or the denominator are called **complex fractions.** The following examples illustrate some approaches to simplifying complex fractions.

Classroom Example

Simplify $\dfrac{\frac{4}{a}-\frac{5}{b}}{\frac{3}{a}+\frac{6}{b^2}}$.

EXAMPLE 7

Simplify $\dfrac{\frac{3}{x}+\frac{2}{y}}{\frac{5}{x}-\frac{6}{y^2}}$.

Solution A

Treating the numerator as the sum of two rational expressions and the denominator as the difference of two rational expressions, we can proceed as follows.

$$\frac{\frac{3}{x}+\frac{2}{y}}{\frac{5}{x}-\frac{6}{y^2}}=\frac{\left(\frac{3}{x}\right)\left(\frac{y}{y}\right)+\left(\frac{2}{y}\right)\left(\frac{x}{x}\right)}{\left(\frac{5}{x}\right)\left(\frac{y^2}{y^2}\right)-\left(\frac{6}{y^2}\right)\left(\frac{x}{x}\right)}$$

$$=\frac{\frac{3y}{xy}+\frac{2x}{xy}}{\frac{5y^2}{xy^2}-\frac{6x}{xy^2}}=\frac{\frac{3y+2x}{xy}}{\frac{5y^2-6x}{xy^2}}$$

$$=\frac{3y+2x}{\cancel{xy}}\cdot\frac{\overset{y}{\cancel{xy^2}}}{5y^2-6x}$$

$$=\frac{y(3y+2x)}{5y^2-6x}$$

Solution B

The LCD of all four denominators (x, y, x, and y^2) is xy^2. Let's multiply the entire complex fraction by a form of 1—namely, $(xy^2)/(xy^2)$:

$$\frac{\frac{3}{x}+\frac{2}{y}}{\frac{5}{x}-\frac{6}{y^2}} = \left(\frac{\frac{3}{x}+\frac{2}{y}}{\frac{5}{x}-\frac{6}{y^2}}\right)\left(\frac{xy^2}{xy^2}\right)$$

$$= \frac{(xy^2)\left(\frac{3}{x}\right) + (xy^2)\left(\frac{2}{y}\right)}{(xy^2)\left(\frac{5}{x}\right) - (xy^2)\left(\frac{6}{y^2}\right)}$$

$$= \frac{3y^2 + 2xy}{5y^2 - 6x} \quad \text{or} \quad \frac{y(3y + 2x)}{5y^2 - 6x}$$

Certainly either approach (Solution A or Solution B) will work with a problem such as Example 7. We suggest that you study Solution B very carefully. This approach works effectively with complex fractions when the LCD of all the denominators is easy to find. Let's look at a type of complex fraction used in certain calculus problems.

Classroom Example

Simplify $\dfrac{\frac{3}{x+h}-\frac{3}{x}}{h}$.

EXAMPLE 8

Simplify $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$.

Solution

$$\frac{\frac{1}{x+h}-\frac{1}{x}}{\frac{h}{1}} = \left[\frac{x(x+h)}{x(x+h)}\right]\left[\frac{\frac{1}{x+h}-\frac{1}{x}}{\frac{h}{1}}\right]$$

$$= \frac{x(x+h)\left(\frac{1}{x+h}\right) - x(x+h)\left(\frac{1}{x}\right)}{x(x+h)(h)}$$

$$= \frac{x-(x+h)}{hx(x+h)} = \frac{x-x-h}{hx(x+h)}$$

$$= \frac{-h}{hx(x+h)} = -\frac{1}{x(x+h)}$$

Example 9 illustrates another way to simplify complex fractions.

Classroom Example

Simplify $4 - \dfrac{y}{1 - \dfrac{4}{y}}$.

EXAMPLE 9 Simplify $1 - \dfrac{n}{1 - \dfrac{1}{n}}$.

Solution

We first simplify the complex fraction by multiplying by n/n:

$$\left(\frac{n}{1 - \frac{1}{n}}\right)\left(\frac{n}{n}\right) = \frac{n^2}{n - 1}$$

Now we can perform the subtraction:

$$\begin{aligned} 1 - \frac{n^2}{n - 1} &= \left(\frac{n - 1}{n - 1}\right)\left(\frac{1}{1}\right) - \frac{n^2}{n - 1} \\ &= \frac{n - 1}{n - 1} - \frac{n^2}{n - 1} \\ &= \frac{n - 1 - n^2}{n - 1} \quad \text{or} \quad \frac{-n^2 + n - 1}{n - 1} \end{aligned}$$

Finally, we need to recognize that complex fractions are sometimes the result of applying the definition $b^{-n} = \dfrac{1}{b^n}$. Our final example illustrates this idea.

Classroom Example

Simplify $\dfrac{5a^{-2} + b^{-1}}{a + 2b^{-1}}$.

EXAMPLE 10 Simplify $\dfrac{2x^{-1} + y^{-1}}{x - 3y^{-2}}$.

Solution

First, let's apply $b^{-n} = \dfrac{1}{b^n}$.

$$\frac{2x^{-1} + y^{-1}}{x - 3y^{-2}} = \frac{\frac{2}{x} + \frac{1}{y}}{x - \frac{3}{y^2}}$$

Now we can proceed as in the previous examples:

$$\begin{aligned} \left(\frac{\frac{2}{x} + \frac{1}{y}}{x - \frac{3}{y^2}}\right)\left(\frac{xy^2}{xy^2}\right) &= \frac{\frac{2}{x}(xy^2) + \frac{1}{y}(xy^2)}{x(xy^2) - \frac{3}{y^2}(xy^2)} \\ &= \frac{2y^2 + xy}{x^2y^2 - 3x} \end{aligned}$$

Concept Quiz 0.5

For Problems 1–7, answer true or false.

1. The indicated quotient of two polynomials is called a rational expression.
2. The rational expression $\frac{3x - 4}{x + 2}$ is defined for all values of x.
3. The rational expressions $\frac{a^2 - 4}{b - 2}$ and $-\frac{4 - a^2}{2 - b}$ are equivalent.
4. The quotient of any nonzero polynomial and its opposite is -1.
5. To multiply rational expressions that do not have a common denominator, we need to obtain equivalent fractions with a common denominator.
6. Complex fractions are fractional forms that contain rational expressions in the numerator and/or the denominator.
7. The difference of $\frac{3x - 4}{7x + 8}$ and $\frac{5x - 1}{7x + 8}$ would equal zero if $3x - 4 = 5x - 1$.
8. Under what conditions would the product of $\frac{x + 2}{x}$ and $\frac{x - 2}{x}$ be equal to zero?

Problem Set 0.5

For Problems 1–18, simplify each rational expression. **(Objective 1)**

1. $\frac{14x^2y}{21xy}$
2. $\frac{-26xy^2}{65y}$
3. $\frac{-63xy^4}{-81x^2y}$
4. $\frac{x^2 - y^2}{x^2 + xy}$
5. $\frac{(2x^2y^2)^3}{(3xy)^2}$
6. $\frac{(3a^3b)^2}{6a^2(b^2)^2}$
7. $\frac{a^2 + 7a + 12}{a^2 - 6a - 27}$
8. $\frac{6x^2 + x - 15}{8x^2 - 10x - 3}$
9. $\frac{2x^3 + 3x^2 - 14x}{x^2y + 7xy - 18y}$
10. $\frac{3x - x^2}{x^2 - 9}$
11. $\frac{x^3 - y^3}{x^2 + xy - 2y^2}$
12. $\frac{ax - 3x + 2ay - 6y}{2ax - 6x + ay - 3y}$
13. $\frac{2y - 2xy}{x^2y - y}$
14. $\frac{16x^3y + 24x^2y^2 - 16xy^3}{24x^2y + 12xy^2 - 12y^3}$
15. $\frac{8x^2 + 4xy - 2x - y}{4x^2 - 4xy - x + y}$
16. $\frac{2x^3 + 2y^3}{2x^2 + 6x + 2xy + 6y}$
17. $\frac{27x^3 + 8y^3}{3x^2 - 15x + 2xy - 10y}$
18. $\frac{x^3 + 64}{3x^2 + 11x - 4}$

For Problems 19–68, perform the indicated operations involving rational expressions. Express final answers in simplest form. **(Objectives 2 and 3)**

19. $\frac{4x^2}{5y^2} \cdot \frac{15xy}{24x^2y^2}$
20. $\frac{5xy}{8y^2} \cdot \frac{18x^2y}{15}$
21. $\frac{-14xy^4}{18y^2} \cdot \frac{24x^2y^3}{35y^2}$
22. $\frac{6xy}{9y^4} \cdot \frac{30x^3y}{-48x}$
23. $\frac{7a^2b}{9ab^3} \div \frac{3a^4}{2a^2b^2}$
24. $\frac{9a^2c}{12bc^2} \div \frac{21ab}{14c^3}$

25. $\frac{5xy}{x+6} \cdot \frac{x^2-36}{x^2-6x}$

26. $\frac{2a^2+6}{a^2-a} \cdot \frac{a^3-a^2}{8a-4}$

27. $\frac{5a^2+20a}{a^3-2a^2} \cdot \frac{a^2-a-12}{a^2-16}$

28. $\frac{t^4-81}{t^2-6t+9} \cdot \frac{6t^2-11t-21}{5t^2+8t-21}$

29. $\frac{x^2+5xy-6y^2}{xy^2-y^3} \cdot \frac{2x^2+15xy+18y^2}{xy+4y^2}$

30. $\frac{10n^2+21n-10}{5n^2+33n-14} \cdot \frac{2n^2+6n-56}{2n^2-3n-20}$

31. $\frac{9y^2}{x^2+12x+36} \div \frac{12y}{x^2+6x}$

32. $\frac{x^2-4xy+4y^2}{7xy^2} \div \frac{4x^2-3xy-10y^2}{20x^2y+25xy^2}$

33. $\frac{2x^2+3x}{2x^3-10x^2} \cdot \frac{x^2-8x+15}{3x^3-27x} \div \frac{14x+21}{x^2-6x-27}$

34. $\frac{a^2-4ab+4b^2}{6a^2-4ab} \cdot \frac{3a^2+5ab-2b^2}{6a^2+ab-b^2} \div \frac{a^2-4b^2}{8a+4b}$

35. $\frac{x+4}{6} + \frac{2x-1}{4}$

36. $\frac{3n-1}{9} - \frac{n+2}{12}$

37. $\frac{x+1}{4} + \frac{x-3}{6} - \frac{x-2}{8}$

38. $\frac{x-2}{5} - \frac{x+3}{6} + \frac{x+1}{15}$

39. $\frac{7}{16a^2b} + \frac{3a}{20b^2}$

40. $\frac{5b}{24a^2} - \frac{11a}{32b}$

41. $\frac{1}{n^2} + \frac{3}{4n} - \frac{5}{6}$

42. $\frac{3}{n^2} - \frac{2}{5n} + \frac{4}{3}$

43. $\frac{3}{4x} + \frac{2}{3y} - 1$

44. $\frac{5}{6x} - \frac{3}{4y} + 2$

45. $\frac{3}{2x+1} + \frac{2}{3x+4}$

46. $\frac{5}{x-1} - \frac{3}{2x-3}$

47. $\frac{4x}{x^2+7x} + \frac{3}{x}$

48. $\frac{6}{x^2+8x} - \frac{3}{x}$

49. $\frac{4a-4}{a^2-4} - \frac{3}{a+2}$

50. $\frac{6a+4}{a^2-1} - \frac{5}{a-1}$

51. $\frac{3}{x-1} - \frac{2}{4x-4}$

52. $\frac{3x+2}{4x-12} + \frac{2x}{6x-18}$

53. $\frac{4}{n^2-1} + \frac{2}{3n+3}$

54. $\frac{5}{n^2-4} - \frac{7}{3n-6}$

55. $\frac{3}{x+1} + \frac{x+5}{x^2-1} - \frac{3}{x-1}$

56. $\frac{5}{x} - \frac{5x-30}{x^2+6x} + \frac{x}{x+6}$

57. $\frac{5}{x^2+10x+21} + \frac{4}{x^2+12x+27}$

58. $\frac{8}{a^2-3a-18} - \frac{10}{a^2-7a-30}$

59. $\frac{5}{x^2-1} - \frac{2}{x^2+6x-16}$

60. $\frac{4}{x^2+2} - \frac{7}{x^2+x-12}$

61. $\frac{3x}{x^2-6x+9} - \frac{2}{x-3}$

62. $\frac{6}{x^2-9} - \frac{9}{x^2-6x+9}$

63. $x - \frac{x^2}{x-1} + \frac{1}{x^2-1}$

64. $x - \frac{x^2}{x+7} - \frac{x}{x^2-49}$

65. $\frac{2n^2}{n^4-16} - \frac{n}{n^2-4} + \frac{1}{n+2}$

66. $\frac{n}{n^2+1} + \frac{n^2+3n}{n^4-1} - \frac{1}{n-1}$

67. $\frac{2x+1}{x^2-3x-4} + \frac{3x-2}{x^2+3x-28}$

68. $\frac{3x-4}{2x^2-9x-5} - \frac{2x-1}{3x^2-11x-20}$

69. Consider the addition problem $\frac{8}{x-2} + \frac{5}{2-x}$. Note that the denominators are opposites of each other. If the property $\frac{a}{-b} = -\frac{a}{b}$ is applied to the second fraction, we obtain $\frac{5}{2-x} = -\frac{5}{x-2}$. Thus we can proceed as follows:

$$\frac{8}{x-2} + \frac{5}{2-x} = \frac{8}{x-2} - \frac{5}{x-2}$$
$$= \frac{8-5}{x-2} = \frac{3}{x-2}$$

Use this approach to do the following problems.

a. $\frac{7}{x-1} + \frac{2}{1-x}$

b. $\frac{5}{2x-1} + \frac{8}{1-2x}$

c. $\frac{4}{a-3} - \frac{1}{3-a}$

d. $\frac{10}{a-9} - \frac{5}{9-a}$

e. $\frac{x^2}{x-1} - \frac{2x-3}{1-x}$

f. $\frac{x^2}{x-4} - \frac{3x-28}{4-x}$

For Problems 70–92, simplify each complex fraction. **(Objective 4)**

70. $\dfrac{\frac{2}{x} + \frac{7}{y}}{\frac{3}{x} - \frac{10}{y}}$

71. $\dfrac{\frac{5}{x^2} - \frac{3}{x}}{\frac{1}{y} + \frac{2}{y^2}}$

72. $\dfrac{\frac{1}{x} + 3}{\frac{2}{y} + 4}$

73. $\dfrac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$

74. $\dfrac{3 - \frac{2}{n-4}}{5 + \frac{4}{n-4}}$

75. $\dfrac{1 - \frac{1}{n+1}}{1 + \frac{1}{n-1}}$

76. $\dfrac{\frac{2}{x-3} - \frac{3}{x+3}}{\frac{5}{x^2-9} - \frac{2}{x-3}}$

77. $\dfrac{\frac{-2}{x} - \frac{4}{x+2}}{\frac{3}{x^2+2x} + \frac{3}{x}}$

78. $\dfrac{\frac{-1}{y-2} + \frac{5}{x}}{\frac{3}{x} - \frac{4}{xy-2x}}$

79. $1 + \dfrac{x}{1 + \frac{1}{x}}$

80. $2 - \dfrac{x}{3 - \frac{2}{x}}$

81. $\dfrac{a}{\frac{1}{a} + 4} + 1$

82. $\dfrac{3a}{2 - \frac{1}{a}} - 1$

83. $\dfrac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$

84. $\dfrac{\frac{1}{(x+h)^3} - \frac{1}{x^3}}{h}$

85. $\dfrac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h}$

86. $\dfrac{\frac{3}{x+h} - \frac{3}{x}}{h}$

87. $\dfrac{\frac{2}{2x+2h-1} - \frac{2}{2x-1}}{h}$

88. $\dfrac{\frac{3}{4x+4h+5} - \frac{3}{4x+5}}{h}$

89. $\dfrac{x^{-1} + 2y^{-1}}{x - y}$

90. $\dfrac{x + y}{x^{-1} + y^{-1}}$

91. $\dfrac{x + 2x^{-1}y^{-2}}{4x^{-1} - 3y^{-2}}$

92. $\dfrac{x^{-2} - 2y^{-1}}{3x^{-1} + y^{-2}}$

Thoughts Into Words

93. What role does factoring play in the simplifying of rational expressions?

94. Explain in your own words how to multiply two rational expressions.

95. Give a step-by-step description of how to add $\frac{2x-1}{4} + \frac{3x+5}{14}$.

96. Look back at the two approaches shown in Example 7. Which approach would you use to simplify $\dfrac{\frac{1}{4}+\frac{1}{6}}{\frac{1}{2}-\frac{3}{4}}$? Which approach would you use to simplify $\dfrac{\frac{5}{8}+\frac{4}{9}}{\frac{5}{14}-\frac{2}{21}}$? Explain the reason for your choice of approach for each problem.

Graphing Calculator Activities

97. Use the graphing feature of your graphing calculator to give visual support for your answers for Problems 60–68.

98. For each of the following, use your graphing calculator to help you decide whether the two given expressions are equivalent for all defined values of x.

(a) $\dfrac{6x^2-7x+2}{8x^2+6x-5}$ and $\dfrac{3x-2}{4x+5}$

(b) $\dfrac{4x^2-15x-54}{4x^2+13x+9}$ and $\dfrac{x-6}{x+1}$

(c) $\dfrac{2x^2+3x-2}{12x^2+19x+5}$ and $\dfrac{2x-1}{4x+5}$

(d) $\dfrac{x^3+2x^2-3x}{x^3+6x^2+5x-12}$ and $\dfrac{x}{x+4}$

(e) $\dfrac{-5x^2-11x+2}{3x^2-13x+14}$ and $\dfrac{-5x-1}{3x-7}$

Answers to the Concept Quiz

1. True **2.** False **3.** False **4.** True **5.** False **6.** True **7.** True **8.** If $x = 2$ or $x = -2$

0.6 Radicals

OBJECTIVES

1. Evaluate roots of numbers
2. Write radical expressions in simplest radical form
3. Simplify an indicated sum of radical expressions
4. Multiply radical expressions
5. Rationalize radical expressions

Recall from our work with exponents that to **square a number** means to raise it to the second power—that is, to use the number as a factor twice. For example, $4^2 = 4 \cdot 4 = 16$, and $(-4)^2 = (-4)(-4) = 16$. A **square root of a number** is one of its two equal factors.

Thus 4 and -4 are both square roots of 16. In general, a is a square root of b if $a^2 = b$. The following statements generalize these ideas:

1. Every positive real number has two square roots; one is positive and the other is negative. They are opposites of each other.
2. Negative real numbers have no real-number square roots because the square of any nonzero real number is positive.
3. The square root of zero is zero.

The symbol $\sqrt{\ }$, called a **radical sign,** is used to designate the *nonnegative* square root, which is called the **principal square root.** The number under the radical sign is called the **radicand,** and the entire expression, such as $\sqrt{16}$, is referred to as a **radical.**

The following examples demonstrate the use of the square root notation:

$\sqrt{16} = 4$ — $\sqrt{16}$ indicates the *nonnegative* or *principal square root* of 16.

$-\sqrt{16} = -4$ — $-\sqrt{16}$ indicates the negative square root of 16.

$\sqrt{0} = 0$ — Zero has only one square root. Technically, we could also write $-\sqrt{0} = -0 = 0$.

$\sqrt{-4}$ — Not a real number

$-\sqrt{-4}$ — Not a real number

To **cube a number** means to raise it to the third power—that is, to use the number as a factor three times. For example, $2^3 = 2 \cdot 2 \cdot 2 = 8$ and $(-2)^3 = (-2)(-2)(-2) = -8$. A **cube root of a number** is one of its three equal factors. Thus 2 is a cube root of 8, and as we will discuss later, it is the only real number that is a cube root of 8. Furthermore, -2 is the only real number that is a cube root of -8. In general, a is a cube root of b if $a^3 = b$. The following statements generalize these ideas:

1. Every positive real number has one positive real-number cube root.
2. Every negative real number has one negative real-number cube root.
3. The cube root of zero is zero.

Remark: Every nonzero real number has three cube roots, but only one of them is a real number. The other roots are complex numbers, which we will discuss in Section 0.8.

The symbol $\sqrt[3]{\ }$ is used to designate the cube root of a number. Thus we can write

$$\sqrt[3]{8} = 2 \qquad \sqrt[3]{-8} = -2 \qquad \sqrt[3]{\frac{1}{27}} = \frac{1}{3} \qquad \sqrt[3]{-\frac{1}{27}} = -\frac{1}{3}$$

The concept of root can be extended to fourth roots, fifth roots, sixth roots, and, in general, nth roots. If n is an *even positive integer,* then the following statements are true:

1. Every positive real number has exactly two real nth roots, one positive and one negative. For example, the real fourth roots of 16 are 2 and -2.
2. Negative real numbers do not have real nth roots. For example, there are no real fourth roots of -16.

If n is an *odd positive integer* greater than 1, then the following statements are true.

1. Every real number has exactly one real nth root.
2. The real nth root of a positive number is positive. For example, the fifth root of 32 is 2.
3. The real nth root of a negative number is negative. For example, the fifth root of -32 is -2.

In general, the following definition is useful.

Definition 0.5

$\sqrt[n]{b} = a \quad$ if and only if $a^n = b$

In Definition 0.5, if n is an even positive integer, then a and b are both nonnegative. If n is an odd positive integer greater than 1, then a and b are both nonnegative or both negative. The symbol $\sqrt[n]{}$ designates the principal root.

The following examples are applications of Definition 0.5:

$$\sqrt[4]{81} = 3 \qquad \text{because } 3^4 = 81$$
$$\sqrt[5]{32} = 2 \qquad \text{because } 2^5 = 32$$
$$\sqrt[5]{-32} = -2 \qquad \text{because } (-2)^5 = -32$$

To complete our terminology, the n in the radical $\sqrt[n]{b}$ is called the **index** of the radical. If $n = 2$, we commonly write $\sqrt{b}$ instead of $\sqrt[2]{b}$. In this text, when we use symbols such as $\sqrt[n]{b}$, $\sqrt[m]{y}$, and $\sqrt[r]{x}$, we will assume the previous agreements relative to the existence of real roots without listing the various restrictions, unless a special restriction is needed.

From Definition 0.5 we see that if n is any positive integer greater than 1 and $\sqrt[n]{b}$ exists, then

$$\left(\sqrt[n]{b}\right)^n = b$$

For example, $\left(\sqrt{4}\right)^2 = 4$, $\left(\sqrt[3]{-8}\right)^3 = -8$, and $\left(\sqrt[4]{81}\right)^4 = 81$. Furthermore, if $b \geq 0$ and n is any positive integer greater than 1, or if $b < 0$ and n is an odd positive integer greater than 1, then

$$\sqrt[n]{b^n} = b$$

For example, $\sqrt{4^2} = 4$, $\sqrt[3]{(-2)^3} = -2$, and $\sqrt[5]{6^5} = 6$. But we must be careful because

$$\sqrt{(-2)^2} \neq -2 \qquad \text{and} \qquad \sqrt[4]{(-2)^4} \neq -2$$

Simplest Radical Form

Let's use some examples to motivate another useful property of radicals.

$$\sqrt{16 \cdot 25} = \sqrt{400} = 20 \quad \text{and} \quad \sqrt{16} \cdot \sqrt{25} = 4 \cdot 5 = 20$$
$$\sqrt[3]{8 \cdot 27} = \sqrt[3]{216} = 6 \quad \text{and} \quad \sqrt[3]{8} \cdot \sqrt[3]{27} = 2 \cdot 3 = 6$$
$$\sqrt[3]{-8 \cdot 64} = \sqrt[3]{-512} = -8 \quad \text{and} \quad \sqrt[3]{-8} \cdot \sqrt[3]{64} = -2 \cdot 4 = -8$$

In general, the following property can be stated.

Property 0.3

$\sqrt[n]{bc} = \sqrt[n]{b}\sqrt[n]{c}$ if $\sqrt[n]{b}$ and $\sqrt[n]{c}$ are real numbers.

Property 0.3 states that **the *n*th root of a product is equal to the product of the *n*th roots.**

The definition of *n*th root, along with Property 0.3, provides the basis for changing radicals to simplest radical form. The concept of **simplest radical form** takes on additional meaning as we encounter more complicated expressions, but for now it simply means that the radicand does not contain any perfect powers of the index. Consider the following examples of reductions to simplest radical form:

$$\sqrt{45} = \sqrt{9 \cdot 5} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$$
$$\sqrt{52} = \sqrt{4 \cdot 13} = \sqrt{4}\sqrt{13} = 2\sqrt{13}$$
$$\sqrt[3]{24} = \sqrt[3]{8 \cdot 3} = \sqrt[3]{8}\sqrt[3]{3} = 2\sqrt[3]{3}$$

A variation of the technique for changing radicals with index n to simplest form is to factor the radicand into primes and then to look for the perfect nth powers in exponential form, as in the following examples:

$$\sqrt{80} = \sqrt{2^4 \cdot 5} = \sqrt{2^4}\sqrt{5} = 2^2\sqrt{5} = 4\sqrt{5}$$
$$\sqrt[3]{108} = \sqrt[3]{2^2 \cdot 3^3} = \sqrt[3]{3^3}\sqrt[3]{2^2} = 3\sqrt[3]{4}$$

The distributive property can be used to combine radicals that have the same index and the same radicand:

$$3\sqrt{2} + 5\sqrt{2} = (3 + 5)\sqrt{2} = 8\sqrt{2}$$
$$7\sqrt[3]{5} - 3\sqrt[3]{5} = (7 - 3)\sqrt[3]{5} = 4\sqrt[3]{5}$$

Sometimes it is necessary to simplify the radicals first and then to combine them by applying the distributive property:

$$\begin{aligned} 3\sqrt{8} + 2\sqrt{18} - 4\sqrt{2} &= 3\sqrt{4}\sqrt{2} + 2\sqrt{9}\sqrt{2} - 4\sqrt{2} \\ &= 6\sqrt{2} + 6\sqrt{2} - 4\sqrt{2} \\ &= (6 + 6 - 4)\sqrt{2} \\ &= 8\sqrt{2} \end{aligned}$$

Multiplying Radicals

Property 0.3 can also be viewed as $\sqrt[n]{b}\sqrt[n]{c} = \sqrt[n]{bc}$. Then, along with the commutative and associative properties of the real numbers, it provides the basis for multiplying radicals that have the same index. Consider the following two examples:

$$\begin{aligned} (7\sqrt{6})(3\sqrt{8}) &= 7 \cdot 3 \cdot \sqrt{6} \cdot \sqrt{8} \\ &= 21\sqrt{48} \\ &= 21\sqrt{16}\sqrt{3} \end{aligned}$$

$$\begin{aligned} &= 21 \cdot 4 \cdot \sqrt{3} \\ &= 84\sqrt{3} \end{aligned}$$

$$\begin{aligned} (2\sqrt[3]{6})(5\sqrt[3]{4}) &= 2 \cdot 5 \cdot \sqrt[3]{6} \cdot \sqrt[3]{4} \\ &= 10\sqrt[3]{24} \\ &= 10\sqrt[3]{8}\sqrt[3]{3} \\ &= 10 \cdot 2 \cdot \sqrt[3]{3} \\ &= 20\sqrt[3]{3} \end{aligned}$$

The distributive property, along with Property 0.3, provides a way of handling special products involving radicals, as the next examples illustrate:

$$\begin{aligned} 2\sqrt{2}(4\sqrt{3} - 5\sqrt{6}) &= (2\sqrt{2})(4\sqrt{3}) - (2\sqrt{2})(5\sqrt{6}) \\ &= 8\sqrt{6} - 10\sqrt{12} \\ &= 8\sqrt{6} - 10\sqrt{4}\sqrt{3} \\ &= 8\sqrt{6} - 20\sqrt{3} \end{aligned}$$

$$\begin{aligned} (2\sqrt{2} - \sqrt{7})(3\sqrt{2} + 5\sqrt{7}) &= 2\sqrt{2}(3\sqrt{2} + 5\sqrt{7}) - \sqrt{7}(3\sqrt{2} + 5\sqrt{7}) \\ &= (2\sqrt{2})(3\sqrt{2}) + (2\sqrt{2})(5\sqrt{7}) - (\sqrt{7})(3\sqrt{2}) - (\sqrt{7})(5\sqrt{7}) \\ &= 6 \cdot 2 + 10\sqrt{14} - 3\sqrt{14} - 5 \cdot 7 \\ &= -23 + 7\sqrt{14} \end{aligned}$$

$$\begin{aligned} (\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2}) &= \sqrt{5}(\sqrt{5} - \sqrt{2}) + \sqrt{2}(\sqrt{5} - \sqrt{2}) \\ &= (\sqrt{5})(\sqrt{5}) - (\sqrt{5})(\sqrt{2}) + (\sqrt{2})(\sqrt{5}) - (\sqrt{2})(\sqrt{2}) \\ &= 5 - \sqrt{10} + \sqrt{10} - 2 \\ &= 3 \end{aligned}$$

Pay special attention to the last example. It fits the special-product pattern $(a + b)(a - b) = a^2 - b^2$. We will use that idea in a moment.

More About Simplest Radical Form

Another property of nth roots is motivated by the following examples:

$$\sqrt{\frac{36}{9}} = \sqrt{4} = 2 \quad \text{and} \quad \frac{\sqrt{36}}{\sqrt{9}} = \frac{6}{3} = 2$$

$$\sqrt[3]{\frac{64}{8}} = \sqrt[3]{8} = 2 \quad \text{and} \quad \frac{\sqrt[3]{64}}{\sqrt[3]{8}} = \frac{4}{2} = 2$$

In general, the following property can be stated.

Property 0.4

$$\sqrt[n]{\frac{b}{c}} = \frac{\sqrt[n]{b}}{\sqrt[n]{c}} \quad \text{if } \sqrt[n]{b} \text{ and } \sqrt[n]{c} \text{ are real numbers, and } c \neq 0$$

Property 0.4 states that **the nth root of a quotient is equal to the quotient of the nth roots.**

To evaluate radicals such as $\sqrt{\frac{4}{25}}$ and $\sqrt[3]{\frac{27}{8}}$, where the numerator and the denominator of the fractional radicands are perfect nth powers, we can either use Property 0.4 or rely on the definition of nth root.

$$\sqrt{\frac{4}{25}} = \frac{\sqrt{4}}{\sqrt{25}} = \frac{2}{5} \quad \text{or} \quad \sqrt{\frac{4}{25}} = \frac{2}{5} \quad \text{because } \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}$$

$$\sqrt[3]{\frac{27}{8}} = \frac{\sqrt[3]{27}}{\sqrt[3]{8}} = \frac{3}{2} \quad \text{or} \quad \sqrt[3]{\frac{27}{8}} = \frac{3}{2} \quad \text{because } \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} = \frac{27}{8}$$

Radicals such as $\sqrt{\frac{28}{9}}$ and $\sqrt[3]{\frac{24}{27}}$, where only the denominators of the radicand are perfect nth powers, can be simplified as follows:

$$\sqrt{\frac{28}{9}} = \frac{\sqrt{28}}{\sqrt{9}} = \frac{\sqrt{4}\sqrt{7}}{3} = \frac{2\sqrt{7}}{3}$$

$$\sqrt[3]{\frac{24}{27}} = \frac{\sqrt[3]{24}}{\sqrt[3]{27}} = \frac{\sqrt[3]{8}\sqrt[3]{3}}{3} = \frac{2\sqrt[3]{3}}{3}$$

Before we consider more examples, let's summarize some ideas about simplifying radicals. A radical is said to be in **simplest radical form** if the following conditions are satisfied.

1. No fraction appears within a radical sign. — Thus $\sqrt{\frac{3}{4}}$ violates this condition.
2. No radical appears in the denominator. — Thus $\frac{\sqrt{2}}{\sqrt{3}}$ violates this condition.
3. No radicand contains a perfect power of the index. — Thus $\sqrt{7^2 \cdot 5}$ violates this condition.

Rationalizing Radical Expressions

Now let's consider an example in which neither the numerator nor the denominator of the radicand is a perfect nth power:

$$\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}} \cdot \underbrace{\frac{\sqrt{3}}{\sqrt{3}}}_{\text{Form of 1}} = \frac{\sqrt{6}}{3}$$

The process used to simplify the radical in this example is referred to as **rationalizing the denominator.** There is more than one way to rationalize the denominator, as illustrated by the next example.

Classroom Example
Simplify $\frac{\sqrt{7}}{\sqrt{18}}$.

EXAMPLE 1 Simplify $\frac{\sqrt{5}}{\sqrt{8}}$.

Solution A

$$\frac{\sqrt{5}}{\sqrt{8}} = \frac{\sqrt{5}}{\sqrt{8}} \cdot \frac{\sqrt{8}}{\sqrt{8}} = \frac{\sqrt{40}}{8} = \frac{\sqrt{4}\sqrt{10}}{8} = \frac{2\sqrt{10}}{8} = \frac{\sqrt{10}}{4}$$

Solution B

$$\frac{\sqrt{5}}{\sqrt{8}} = \frac{\sqrt{5}}{\sqrt{8}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{10}}{\sqrt{16}} = \frac{\sqrt{10}}{4}$$

Solution C

$$\frac{\sqrt{5}}{\sqrt{8}} = \frac{\sqrt{5}}{\sqrt{4}\sqrt{2}} = \frac{\sqrt{5}}{2\sqrt{2}} = \frac{\sqrt{5}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{10}}{4}$$

The three approaches in Example 1 again illustrate the need to think first and then push the pencil. You may find one approach easier than another.

Classroom Example
Simplify $\frac{\sqrt{6}}{\sqrt{27}}$.

EXAMPLE 2 Simplify $\frac{\sqrt{6}}{\sqrt{8}}$.

Solution

$$\frac{\sqrt{6}}{\sqrt{8}} = \sqrt{\frac{6}{8}}$$ Remember that $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

$$= \sqrt{\frac{3}{4}}$$ Reduce the fraction

$$= \frac{\sqrt{3}}{\sqrt{4}}$$

$$= \frac{\sqrt{3}}{2}$$

Classroom Example
Simplify $\frac{\sqrt[3]{7}}{\sqrt[3]{2}}$.

EXAMPLE 3 Simplify $\frac{\sqrt[3]{5}}{\sqrt[3]{9}}$.

Solution

$$\frac{\sqrt[3]{5}}{\sqrt[3]{9}} = \frac{\sqrt[3]{5}}{\sqrt[3]{9}} \cdot \frac{\sqrt[3]{3}}{\sqrt[3]{3}}$$

$$= \frac{\sqrt[3]{15}}{\sqrt[3]{27}}$$

$$= \frac{\sqrt[3]{15}}{3}$$

Now let's consider an example in which the denominator is of binomial form.

Classroom Example

Simplify $\dfrac{7}{\sqrt{5}-\sqrt{3}}$ by rationalizing the denominator.

EXAMPLE 4 Simplify $\dfrac{4}{\sqrt{5}+\sqrt{2}}$ by rationalizing the denominator.

Solution

Remember that a moment ago we found that $(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2}) = 3$. Let's use that idea here:

$$\frac{4}{\sqrt{5}+\sqrt{2}} = \left(\frac{4}{\sqrt{5}+\sqrt{2}}\right)\left(\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}\right)$$
$$= \frac{4(\sqrt{5}-\sqrt{2})}{(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})} = \frac{4(\sqrt{5}-\sqrt{2})}{3}$$

The process of rationalizing the denominator does agree with the previously listed conditions. However, for certain problems in calculus, it is necessary to **rationalize the numerator.** Again, the fact that $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a - b$ can be used.

Classroom Example

Change the form of $\dfrac{\sqrt{3x+3h}-\sqrt{3x}}{h}$ by rationalizing the numerator.

EXAMPLE 5 Change the form of $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ by rationalizing the *numerator.*

Solution

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$
$$= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$$
$$= \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$$
$$= \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

Radicals Containing Variables

Before we illustrate how to simplify radicals that contain variables, there is one important point we should call to your attention. Let's look at some examples to illustrate the idea.

Consider the radical $\sqrt{x^2}$ for different values of x.

Let $x = 3$; then $\sqrt{x^2} = \sqrt{3^2} = \sqrt{9} = 3$.

Let $x = -3$; then $\sqrt{x^2} = \sqrt{(-3)^2} = \sqrt{9} = 3$.

Thus if $x \geq 0$, then $\sqrt{x^2} = x$, but if $x < 0$, then $\sqrt{x^2} = -x$. Using the concept of absolute value, we can state that **for all real numbers,** $\boldsymbol{\sqrt{x^2} = |x|}$**.**

Now consider the radical $\sqrt{x^3}$. Because x^3 is negative when x is negative, we need to restrict x to the nonnegative real numbers when working with $\sqrt{x^3}$. Thus we can write

$$\text{if } x \geq 0, \quad \text{then } \sqrt{x^3} = \sqrt{x^2}\sqrt{x} = x\sqrt{x}$$

and no absolute value sign is needed.

Finally, let's consider the radical $\sqrt[3]{x^3}$.

Let $x = 2$; then $\sqrt[3]{x^3} = \sqrt[3]{2^3} = \sqrt[3]{8} = 2$.

Let $x = -2$; then $\sqrt[3]{x^3} = \sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$.

Thus it is correct to write

$$\sqrt[3]{x^3} = x \quad \text{for all real numbers}$$

and again, no absolute value sign is needed.

The previous discussion indicates that, technically, every radical expression with variables in the radicand needs to be analyzed individually to determine the necessary restrictions on the variables. However, to avoid having to do this on a problem-by-problem basis, we shall merely **assume that all variables represent positive real numbers.**

Let's conclude this section by simplifying some radical expressions that contain variables.

$$\sqrt{72x^2y^7} = \sqrt{36x^2y^6}\sqrt{2xy} = 6xy^3\sqrt{2xy}$$

$$\sqrt[3]{40x^4y^8} = \sqrt[3]{8x^3y^6}\,\sqrt[3]{5xy^2} = 2xy^2\,\sqrt[3]{5xy^2}$$

$$\frac{\sqrt{5}}{\sqrt{12a^3}} = \frac{\sqrt{5}}{\sqrt{12a^3}} \cdot \frac{\sqrt{3a}}{\sqrt{3a}} = \frac{\sqrt{15a}}{\sqrt{36a^4}} = \frac{\sqrt{15a}}{6a^2}$$

$$\frac{3}{\sqrt[3]{4x}} = \frac{3}{\sqrt[3]{4x}} \cdot \frac{\sqrt[3]{2x^2}}{\sqrt[3]{2x^2}} = \frac{3\sqrt[3]{2x^2}}{\sqrt[3]{8x^3}} = \frac{3\sqrt[3]{2x^2}}{2x}$$

Concept Quiz 0.6

For Problems 1–8, answer true or false.

1. The symbol $\sqrt{}$ is used to designate the principal square root.
2. Every positive real number has two principal square roots.
3. The square root of zero does not exist in the real number system.
4. Every real number has one real number cube root.
5. The $\sqrt{25}$ could be 5 or -5.

6. $\sqrt{(-3)^2} = -3$
7. If $x < 0$, then $\sqrt{x^2} = -x$.
8. For real numbers, the process of rationalizing the denominator changes the denominator from an irrational number to a rational number.

Problem Set 0.6

For Problems 1–8, evaluate. **(Objective 1)**

1. $\sqrt{81}$
2. $-\sqrt{49}$
3. $\sqrt[3]{125}$
4. $\sqrt[4]{81}$
5. $\sqrt{\frac{36}{49}}$
6. $\sqrt{\frac{256}{64}}$
7. $\sqrt[3]{-\frac{27}{8}}$
8. $\sqrt[3]{\frac{64}{27}}$

For Problems 9–30, express each in simplest radical form. All variables represent positive real numbers. **(Objective 2)**

9. $\sqrt{24}$
10. $\sqrt{54}$
11. $\sqrt{112}$
12. $6\sqrt{28}$
13. $-3\sqrt{44}$
14. $-5\sqrt{68}$
15. $\frac{3}{4}\sqrt{20}$
16. $\frac{3}{8}\sqrt{72}$
17. $\sqrt{12x^2}$
18. $\sqrt{45xy^2}$
19. $\sqrt{64x^4y^7}$
20. $3\sqrt{32a^3}$
21. $\frac{3}{7}\sqrt{45xy^6}$
22. $\sqrt[3]{32}$
23. $\sqrt[3]{128}$
24. $\sqrt[3]{54x^3}$
25. $\sqrt[3]{16x^4}$
26. $\sqrt[3]{81x^5y^6}$
27. $\sqrt[4]{48x^5}$
28. $\sqrt[4]{162x^6y^7}$
29. $\sqrt{\frac{12}{25}}$
30. $\sqrt{\frac{75}{81}}$

For Problems 31–44, rationalize the denominator and express the result in simplest radical form. **(Objective 5)**

31. $\sqrt{\frac{7}{8}}$
32. $\frac{\sqrt{35}}{\sqrt{7}}$
33. $\frac{4\sqrt{6}}{\sqrt{10}}$
34. $\frac{\sqrt{27}}{\sqrt{18}}$
35. $\frac{6\sqrt{3}}{7\sqrt{6}}$
36. $\sqrt{\frac{3x}{2y}}$
37. $\frac{\sqrt{5}}{\sqrt{12x^4}}$
38. $\frac{\sqrt{5y}}{\sqrt{18x^3}}$
39. $\frac{\sqrt{12a^2b}}{\sqrt{5a^3b^3}}$
40. $\frac{5}{\sqrt[3]{3}}$
41. $\frac{\sqrt[3]{27}}{\sqrt[3]{4}}$
42. $\sqrt[3]{\frac{5}{2x}}$
43. $\frac{\sqrt[3]{2y}}{\sqrt[3]{3x}}$
44. $\frac{\sqrt[3]{12xy}}{\sqrt[3]{3x^2y^5}}$

For Problems 45–52, use the distributive property to help simplify each. **(Objective 3)** For example,

$$\begin{aligned} 3\sqrt{8} + 5\sqrt{2} &= 3\sqrt{4}\sqrt{2} + 5\sqrt{2} \\ &= 6\sqrt{2} + 5\sqrt{2} \\ &= (6 + 5)\sqrt{2} \\ &= 11\sqrt{2} \end{aligned}$$

45. $5\sqrt{12} + 2\sqrt{3}$
46. $4\sqrt{50} - 9\sqrt{32}$
47. $2\sqrt{28} - 3\sqrt{63} + 8\sqrt{7}$
48. $4\sqrt[3]{2} + 2\sqrt[3]{16} - \sqrt[3]{54}$
49. $\frac{5}{6}\sqrt{48} - \frac{3}{4}\sqrt{12}$
50. $\frac{2}{5}\sqrt{40} + \frac{1}{6}\sqrt{90}$
51. $\frac{2\sqrt{8}}{3} - \frac{3\sqrt{18}}{5} - \frac{\sqrt{50}}{2}$
52. $\frac{3\sqrt[3]{54}}{2} + \frac{5\sqrt[3]{16}}{3}$

For Problems 53–68, multiply and express the results in simplest radical form. All variables represent nonnegative real numbers. **(Objective 4)**

53. $(4\sqrt{3})(6\sqrt{8})$ **54.** $(5\sqrt{8})(3\sqrt{7})$

55. $2\sqrt{3}(5\sqrt{2} + 4\sqrt{10})$

56. $3\sqrt{6}(2\sqrt{8} - 3\sqrt{12})$

57. $3\sqrt{x}(\sqrt{6xy} - \sqrt{8y})$

58. $\sqrt{6y}(\sqrt{8x} + \sqrt{10y^2})$

59. $(\sqrt{3} + 2)(\sqrt{3} + 5)$ **60.** $(\sqrt{2} - 3)(\sqrt{2} + 4)$

61. $(4\sqrt{2} + \sqrt{3})(3\sqrt{2} + 2\sqrt{3})$

62. $(2\sqrt{6} + 3\sqrt{5})(3\sqrt{6} + 4\sqrt{5})$

63. $(6 + 2\sqrt{5})(6-2\sqrt{5})$

64. $(7 - 3\sqrt{2})(7 + 3\sqrt{2})$ **65.** $(\sqrt{x} + \sqrt{y})^2$

66. $(2\sqrt{x} - 3\sqrt{y})^2$

67. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$

68. $(3\sqrt{x} + 5\sqrt{y})(3\sqrt{x} - 5\sqrt{y})$

For Problems 69–80, rationalize the denominator and simplify. All variables represent positive real numbers. **(Objective 5)**

69. $\dfrac{3}{\sqrt{5} + 2}$ **70.** $\dfrac{7}{\sqrt{10} - 3}$

71. $\dfrac{4}{\sqrt{7} - \sqrt{3}}$ **72.** $\dfrac{2}{\sqrt{5} + \sqrt{3}}$

73. $\dfrac{\sqrt{2}}{2\sqrt{5} + 3\sqrt{7}}$ **74.** $\dfrac{5}{5\sqrt{2} - 3\sqrt{5}}$

75. $\dfrac{\sqrt{x}}{\sqrt{x} - 1}$ **76.** $\dfrac{\sqrt{x}}{\sqrt{x} + 2}$

77. $\dfrac{\sqrt{x}}{\sqrt{x} + \sqrt{y}}$ **78.** $\dfrac{2\sqrt{x}}{\sqrt{x} - \sqrt{y}}$

79. $\dfrac{2\sqrt{x} + \sqrt{y}}{3\sqrt{x} - 2\sqrt{y}}$ **80.** $\dfrac{3\sqrt{x} - 2\sqrt{y}}{2\sqrt{x} + 5\sqrt{y}}$

For Problems 81–84, *rationalize the numerator*. All variables represent positive real numbers. **(Objective 5)**

81. $\dfrac{\sqrt{2x + 2h} - \sqrt{2x}}{h}$

82. $\dfrac{\sqrt{x + h + 1} - \sqrt{x + 1}}{h}$

83. $\dfrac{\sqrt{x + h - 3} - \sqrt{x - 3}}{h}$

84. $\dfrac{2\sqrt{x + h} - 2\sqrt{x}}{h}$

Thoughts Into Words

85. Is the equation $\sqrt{x^2y} = x\sqrt{y}$ true for all real-number values for x and y? Defend your answer.

86. Is the equation $\sqrt{x^2y^2} = xy$ true for all real-number values for x and y? Defend your answer.

87. Give a step-by-step description of how you would change $\sqrt{252}$ to simplest radical form.

88. Why is $\sqrt{-9}$ not a real number?

89. How could you find a whole-number approximation for $\sqrt{2750}$ if you did not have a calculator or table available?

Further Investigations

Do the following problems, where the variable could be any real number as long as the radical represents a real number. Use absolute-value signs in the answers as necessary.

90. $\sqrt{125x^2}$ **91.** $\sqrt{16x^4}$

92. $\sqrt{8b^3}$ **93.** $\sqrt{3y^5}$

94. $\sqrt{288x^6}$ **95.** $\sqrt{28m^8}$

96. $\sqrt{128c^{10}}$ **97.** $\sqrt{18d^7}$

98. $\sqrt{49x^2}$ **99.** $\sqrt{80n^{20}}$

100. $\sqrt{81h^3}$

Graphing Calculator Activities

101. Sometimes it is more convenient to express a large or very small number as a product of a power of 10 and a number that is not between 1 and 10. For example, suppose that we want to calculate $\sqrt{640{,}000}$. We can proceed as follows:

$$\begin{aligned}\sqrt{640{,}000} &= \sqrt{(64)(10)^4}\\ &= ((64)(10)^4)^{1/2}\\ &= (64)^{1/2}(10^4)^{1/2}\\ &= (8)(10)^2\\ &= 8(100) = 800\end{aligned}$$

Compute each of the following without a calculator, and then use a calculator to check your answers.

(a) $\sqrt{49{,}000{,}000}$ **(b)** $\sqrt{0.0025}$

(c) $\sqrt{14{,}400}$ **(d)** $\sqrt{0.000121}$

(e) $\sqrt[3]{27{,}000}$ **(f)** $\sqrt[3]{0.000064}$

102. There are several methods of approximating square roots without using a calculator. One such method works on a "clamping between values" principle. For example, to find a whole-number approximation for $\sqrt{128}$, we can proceed as follows: $11^2 = 121$ and $12^2 = 144$. Therefore $11 < \sqrt{128} < 12$. Because 128 is closer to 121 than to 144, we say that 11 is a whole-number approximation for $\sqrt{128}$. If a more precise approximation is needed, we can do more clamping. We would find that $(11.3)^2 = 127.69$ and $(11.4)^2 = 129.96$. Because 128 is closer to 127.69 than to 129.96, we conclude that $\sqrt{128} = 11.3$, to the nearest tenth.

For each of the following, use the clamping idea to find a whole-number approximation. Then check your answers using a calculator and the square root key.

(a) $\sqrt{52}$ **(b)** $\sqrt{93}$ **(c)** $\sqrt{174}$

(d) $\sqrt{200}$ **(e)** $\sqrt{275}$ **(f)** $\sqrt{350}$

103. The clamping process discussed in Problem 102 works for any whole-number root greater than or equal to 2. For example, a whole-number approximation for $\sqrt[3]{80}$ is 4 because $4^3 = 64$ and $5^3 = 125$, and 80 is closer to 64 than to 125.

For each of the following, use the clamping idea to find a whole-number approximation. Then use your calculator and the appropriate root keys to check your answers.

(a) $\sqrt[3]{24}$ **(b)** $\sqrt[3]{32}$ **(c)** $\sqrt[3]{150}$

(d) $\sqrt[3]{200}$ **(e)** $\sqrt[4]{50}$ **(f)** $\sqrt[4]{250}$

Answers to the Concept Quiz

1. True **2.** False **3.** False **4.** True **5.** False **6.** False **7.** True **8.** True

0.7 Relationship Between Exponents and Roots

OBJECTIVES

1. Evaluate a number raised to a rational exponent
2. Simplify expressions with rational exponents
3. Apply rational exponents to simplify radical expressions

Recall that we used the basic properties of positive integral exponents to motivate a definition of negative integers as exponents. In this section, we shall use the properties of integral exponents to motivate definitions for rational numbers as exponents. These definitions will tie together the concepts of *exponent* and *root*. Let's consider the following comparisons:

From our study of radicals we know that	If $(b^m)^n = b^{nm}$ is to hold when m is a rational number of the form $1/p$, where p is a positive integer greater than 1 and $n = p$, then
$(\sqrt{5})^2 = 5$	$(5^{1/2})^2 = 5^{2(1/2)} = 5^1 = 5$
$(\sqrt[3]{8})^3 = 8$	$(8^{1/3})^3 = 8^{3(1/3)} = 8^1 = 8$
$(\sqrt[4]{21})^4 = 21$	$(21^{1/4})^4 = 21^{4(1/4)} = 21^1 = 21$

Such examples motivate the following definition.

Definition 0.6

If b is a real number, n is a positive integer greater than 1, and $\sqrt[n]{b}$ exists, then

$$b^{1/n} = \sqrt[n]{b}$$

Definition 0.6 states that $b^{1/n}$ means the nth root of b. We shall assume that b and n are chosen so that $\sqrt[n]{b}$ exists in the real number system. For example, $(-25)^{1/2}$ is not meaningful at this time because $\sqrt{-25}$ is not a real number. The following examples illustrate the use of Definition 0.6:

$$25^{1/2} = \sqrt{25} = 5 \qquad 16^{1/4} = \sqrt[4]{16} = 2$$
$$8^{1/3} = \sqrt[3]{8} = 2 \qquad (-27)^{1/3} = \sqrt[3]{-27} = -3$$

Now the following definition provides the basis for the use of *all* rational numbers as exponents.

Definition 0.7

If m/n is a rational number expressed in lowest terms, where n is a positive integer greater than 1, and m is any integer, and if b is a real number such that $\sqrt[n]{b}$ exists, then

$$b^{m/n} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$$

In Definition 0.7, whether we use the form $\sqrt[n]{b^m}$ or $\left(\sqrt[n]{b}\right)^m$ for computational purposes depends somewhat on the magnitude of the problem. Let's use both forms on the following two problems:

$$8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4 \quad \text{or} \quad 8^{2/3} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$$
$$27^{2/3} = \sqrt[3]{27^2} = \sqrt[3]{729} = 9 \quad \text{or} \quad 27^{2/3} = \left(\sqrt[3]{27}\right)^2 = (3)^2 = 9$$

To compute $8^{2/3}$, both forms work equally well. However, to compute $27^{2/3}$, the form $(\sqrt[3]{27})^2$ is much easier to handle. The following examples further illustrate Definition 0.7:

$$25^{3/2} = (\sqrt{25})^3 = 5^3 = 125$$

$$(32)^{-2/5} = \frac{1}{(32)^{2/5}} = \frac{1}{(\sqrt[5]{32})^2} = \frac{1}{2^2} = \frac{1}{4}$$

$$(-64)^{2/3} = (\sqrt[3]{-64})^2 = (-4)^2 = 16$$

$$-8^{4/3} = -(\sqrt[3]{8})^4 = -(2)^4 = -16$$

It can be shown that all of the results pertaining to integral exponents listed in Property 0.2 (on page 23) also hold for all rational exponents. Let's consider some examples to illustrate each of those results.

$$\begin{aligned} x^{1/2} \cdot x^{2/3} &= x^{1/2+2/3} && b^n \cdot b^m = b^{n+m} \\ &= x^{3/6+4/6} \\ &= x^{7/6} \end{aligned}$$

$$\begin{aligned} (a^{2/3})^{3/2} &= a^{(3/2)(2/3)} && (b^n)^m = b^{nm} \\ &= a^1 = a \end{aligned}$$

$$\begin{aligned} (16y^{2/3})^{1/2} &= (16)^{1/2}(y^{2/3})^{1/2} && (ab)^n = a^n b^n \\ &= 4y^{1/3} \end{aligned}$$

$$\begin{aligned} \frac{y^{3/4}}{y^{1/2}} &= y^{3/4-1/2} && \frac{b^n}{b^m} = b^{n-m} \\ &= y^{3/4-2/4} \\ &= y^{1/4} \end{aligned}$$

$$\begin{aligned} \left(\frac{x^{1/2}}{y^{1/3}}\right)^6 &= \frac{(x^{1/2})^6}{(y^{1/3})^6} && \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \\ &= \frac{x^3}{y^2} \end{aligned}$$

The link between exponents and roots provides a basis for multiplying and dividing some radicals even if they have different indexes. The general procedure is to change from radical to exponential form, apply the properties of exponents, and then change back to radical form. Let's apply these procedures in the next three examples:

$$\sqrt{2}\sqrt[3]{2} = 2^{1/2} \cdot 2^{1/3} = 2^{1/2+1/3} = 2^{5/6} = \sqrt[6]{2^5} = \sqrt[6]{32}$$

$$\begin{aligned} \sqrt{xy}\sqrt[5]{x^2y} &= (xy)^{1/2}(x^2y)^{1/5} \\ &= x^{1/2}y^{1/2}x^{2/5}y^{1/5} \\ &= x^{1/2+2/5}y^{1/2+1/5} \\ &= x^{9/10}y^{7/10} \\ &= (x^9y^7)^{1/10} = \sqrt[10]{x^9y^7} \end{aligned}$$

$$\frac{\sqrt{5}}{\sqrt[3]{5}} = \frac{5^{1/2}}{5^{1/3}} = 5^{1/2-1/3} = 5^{1/6} = \sqrt[6]{5}$$

Earlier we agreed that a radical such as $\sqrt[3]{x^4}$ is not in simplest form because the radicand contains a perfect power of the index. Thus we simplified $\sqrt[3]{x^4}$ by expressing it as $\sqrt[3]{x^3}\sqrt[3]{x}$, which in turn can be written $x\sqrt[3]{x}$. Such simplification can also be done in exponential form, as follows:

$$\sqrt[3]{x^4} = x^{4/3} = x^{3/3} \cdot x^{1/3} = x \cdot x^{1/3} = x\sqrt[3]{x}$$

Note the use of this type of simplification in the following examples.

Classroom Example
Perform the indicated operations and express the answers in simplest radical form.

(a) $\sqrt[3]{x^2}\sqrt{x}$

(b) $\sqrt[4]{3}\sqrt[3]{9}$

(c) $\dfrac{\sqrt[3]{4}}{\sqrt{2}}$

EXAMPLE 1

Perform the indicated operations and express the answers in simplest radical form.

(a) $\sqrt[3]{x^2}\sqrt[4]{x^3}$ **(b)** $\sqrt{2}\sqrt[3]{4}$ **(c)** $\dfrac{\sqrt{27}}{\sqrt[3]{3}}$

Solutions

(a) $\sqrt[3]{x^2}\sqrt[4]{x^3} = x^{2/3} \cdot x^{3/4} = x^{2/3+3/4} = x^{17/12} = x^{12/12} \cdot x^{5/12} = x\sqrt[12]{x^5}$

(b) $\sqrt{2}\sqrt[3]{4} = 2^{1/2} \cdot 4^{1/3} = 2^{1/2}(2^2)^{1/3} = 2^{1/2} \cdot 2^{2/3}$

$$= 2^{1/2+2/3} = 2^{7/6} = 2^{6/6} \cdot 2^{1/6} = 2\sqrt[6]{2}$$

(c) $\dfrac{\sqrt{27}}{\sqrt[3]{3}} = \dfrac{27^{1/2}}{3^{1/3}} = \dfrac{(3^3)^{1/2}}{3^{1/3}} = \dfrac{3^{3/2}}{3^{1/3}} = 3^{3/2-1/3} = 3^{7/6}$

$$= 3^{6/6} \cdot 3^{1/6} = 3\sqrt[6]{3}$$

The process of rationalizing the denominator can sometimes be handled more easily in exponential form. Consider the following examples, which illustrate this procedure.

Classroom Example
Rationalize the denominator and express the answer in simplest radical form.

(a) $\dfrac{3}{\sqrt[3]{y^2}}$ **(b)** $\dfrac{\sqrt{m}}{\sqrt[5]{n^2}}$

EXAMPLE 2

Rationalize the denominator and express the answer in simplest radical form.

(a) $\dfrac{2}{\sqrt[3]{x}}$ **(b)** $\dfrac{\sqrt[3]{x}}{\sqrt{y}}$

Solutions

(a) $\dfrac{2}{\sqrt[3]{x}} = \dfrac{2}{x^{1/3}} = \dfrac{2}{x^{1/3}} \cdot \dfrac{x^{2/3}}{x^{2/3}} = \dfrac{2x^{2/3}}{x} = \dfrac{2\sqrt[3]{x^2}}{x}$

(b) $\dfrac{\sqrt[3]{x}}{\sqrt{y}} = \dfrac{x^{1/3}}{y^{1/2}} = \dfrac{x^{1/3}}{y^{1/2}} \cdot \dfrac{y^{1/2}}{y^{1/2}} = \dfrac{x^{1/3} \cdot y^{1/2}}{y} = \dfrac{x^{2/6} \cdot y^{3/6}}{y} = \dfrac{\sqrt[6]{x^2y^3}}{y}$

Note in part b that if we had changed back to radical form at the step $\frac{x^{1/3}y^{1/2}}{y}$, we would have obtained the product of two radicals, $\sqrt[3]{x}\sqrt{y}$, in the numerator. Instead we used the exponential form to find this product and express the final result with a single radical in the numerator. Finally, let's consider an example involving *the root of a root*.

Classroom Example
Simplify $\sqrt[4]{\sqrt[3]{5}}$.

EXAMPLE 3 Simplify $\sqrt[3]{\sqrt{2}}$.

Solution

$$\sqrt[3]{\sqrt{2}} = (2^{1/2})^{1/3} = 2^{1/6} = \sqrt[6]{2}$$

Concept Quiz 0.7

For Problems 1–4, select the equivalent radical form.

1. $x^{\frac{3}{5}}$ A. $\sqrt[3]{x^5}$ B. $x\sqrt[3]{x^2}$ C. $\sqrt[5]{x^3}$
2. $y^{-\frac{1}{3}}$ A. $\frac{1}{\sqrt{y}}$ B. $\frac{\sqrt[3]{y^2}}{y}$ C. $-\sqrt[3]{y}$
3. $-w^{-\frac{1}{2}}$ A. $-\frac{\sqrt{w}}{w}$ B. $\sqrt{w}$ C. $-\sqrt{w}$
4. $\sqrt[n]{x}\sqrt[m]{x}$ A. $\sqrt[mn]{x}$ B. $x^{\frac{m+n}{mn}}$ C. $x^{\frac{1}{m+n}}$

For Problems 5–8, answer true or false.

5. Assuming the nth root of x exists, $\sqrt[n]{x}$ can be expressed as $x^{\frac{1}{n}}$.
6. The expression $\sqrt[n]{x^m}$ is $(\sqrt[n]{x})^m$.
7. The process of rationalizing the denominator can be done with rational exponents.
8. An exponent of $\frac{1}{3}$ indicates the cube root.

Problem Set 0.7

For Problems 1–16, evaluate. **(Objective 1)**

1. $49^{1/2}$
2. $64^{1/3}$
3. $32^{3/5}$
4. $(-8)^{1/3}$
5. $-8^{2/3}$
6. $64^{-1/2}$
7. $\left(\frac{1}{4}\right)^{-1/2}$
8. $\left(-\frac{27}{8}\right)^{-1/3}$
9. $16^{3/2}$
10. $(0.008)^{1/3}$
11. $(0.01)^{3/2}$
12. $\left(\frac{1}{27}\right)^{-2/3}$
13. $64^{-5/6}$
14. $-16^{5/4}$
15. $\left(\frac{1}{8}\right)^{-1/3}$
16. $\left(-\frac{1}{8}\right)^{2/3}$

For Problems 17–32, perform the indicated operations and simplify. Express final answers using positive exponents only. **(Objective 2)**

17. $(3x^{1/4})(5x^{1/3})$

18. $(2x^{2/5})(6x^{1/4})$

19. $(y^{2/3})(y^{-1/4})$

20. $(2x^{1/3})(x^{-1/2})$

21. $(4x^{1/4}y^{1/2})^3$

22. $(5x^{1/2}y)^2$

23. $\dfrac{24x^{3/5}}{6x^{1/3}}$

24. $\dfrac{18x^{1/2}}{9x^{1/3}}$

25. $\dfrac{56a^{1/6}}{8a^{1/4}}$

26. $\dfrac{48b^{1/3}}{12b^{3/4}}$

27. $\left(\dfrac{2x^{1/3}}{3y^{1/4}}\right)^4$

28. $\left(\dfrac{6x^{2/5}}{7y^{2/3}}\right)^2$

29. $\left(\dfrac{x^2}{y^3}\right)^{-1/2}$

30. $\left(\dfrac{a^3}{b^{-2}}\right)^{-1/3}$

31. $\left(\dfrac{4a^2x}{2a^{1/2}x^{1/3}}\right)^3$

32. $\left(\dfrac{3ax^{-1}}{a^{1/2}x^{-2}}\right)^2$

For Problems 33–48, perform the indicated operations and express the answer in simplest radical form. **(Objective 3)**

33. $\sqrt{2}\sqrt[4]{2}$

34. $\sqrt[3]{3}\sqrt{3}$

35. $\sqrt[3]{x}\sqrt[4]{x}$

36. $\sqrt[3]{x^2}\sqrt[5]{x^3}$

37. $\sqrt{xy}\sqrt[4]{x^3y^5}$

38. $\sqrt[3]{x^2y^4}\sqrt[4]{x^3y}$

39. $\sqrt[3]{a^2b^2}\sqrt[4]{a^3b}$

40. $\sqrt{ab}\sqrt[3]{a^4b^5}$

41. $\sqrt[3]{4}\sqrt{8}$

42. $\sqrt[3]{9}\sqrt{27}$

43. $\dfrac{\sqrt{2}}{\sqrt[3]{2}}$

44. $\dfrac{\sqrt{9}}{\sqrt[3]{3}}$

45. $\dfrac{\sqrt[3]{8}}{\sqrt[4]{4}}$

46. $\dfrac{\sqrt[3]{16}}{\sqrt[6]{4}}$

47. $\dfrac{\sqrt[4]{x^9}}{\sqrt[3]{x^2}}$

48. $\dfrac{\sqrt[5]{x^7}}{\sqrt[3]{x}}$

For Problems 49–56, rationalize the denominator and express the final answer in simplest radical form. **(Objective 3)**

49. $\dfrac{5}{\sqrt[3]{x}}$

50. $\dfrac{3}{\sqrt[3]{x^2}}$

51. $\dfrac{\sqrt{x}}{\sqrt[3]{y}}$

52. $\dfrac{\sqrt[4]{x}}{\sqrt{y}}$

53. $\dfrac{\sqrt[4]{x^3}}{\sqrt[5]{y^3}}$

54. $\dfrac{2\sqrt{x}}{3\sqrt[3]{y}}$

55. $\dfrac{5\sqrt[3]{y^2}}{4\sqrt[4]{x}}$

56. $\dfrac{\sqrt{xy}}{\sqrt[3]{a^2b}}$

57. Simplify each of the following, expressing the final result as one radical. For example,

$$\sqrt{\sqrt{3}} = (3^{1/2})^{1/2} = 3^{1/4} = \sqrt[4]{3}$$

(a) $\sqrt[3]{\sqrt{2}}$

(b) $\sqrt[3]{\sqrt[4]{3}}$

(c) $\sqrt[3]{\sqrt{x^3}}$

(d) $\sqrt{\sqrt[3]{x^4}}$

Thoughts Into Words

58. Your friend keeps getting an error message when evaluating $-4^{5/2}$ on his calculator. What error is he probably making?

59. Explain how you would evaluate $27^{2/3}$ without a calculator.

Further Investigations

Sometimes we meet the following type of simplification problem in calculus:

$$\frac{(x-1)^{1/2} - x(x-1)^{-(1/2)}}{[(x-1)^{1/2}]^2}$$

$$= \left(\frac{(x-1)^{1/2} - x(x-1)^{-(1/2)}}{(x-1)^{2/2}}\right) \cdot \left(\frac{(x-1)^{1/2}}{(x-1)^{1/2}}\right)$$

$$= \frac{x-1-x(x-1)^0}{(x-1)^{3/2}}$$

$$= \frac{x-1-x}{(x-1)^{3/2}}$$

$$= \frac{-1}{(x-1)^{3/2}} \quad \text{or} \quad -\frac{1}{(x-1)^{3/2}}$$

For Problems 60–65, simplify each expression as we did in the previous example.

60. $\dfrac{2(x+1)^{1/2} - x(x+1)^{-(1/2)}}{[(x+1)^{1/2}]^2}$

61. $\dfrac{2(2x-1)^{1/2} - 2x(2x-1)^{-(1/2)}}{[(2x-1)^{1/2}]^2}$

62. $\dfrac{2x(4x+1)^{1/2} - 2x^2(4x+1)^{-(1/2)}}{[(4x+1)^{1/2}]^2}$

63. $\dfrac{(x^2+2x)^{1/2} - x(x+1)(x^2+2x)^{-(1/2)}}{[(x^2+2x)^{1/2}]^2}$

64. $\dfrac{(3x)^{1/3} - x(3x)^{-(2/3)}}{[(3x)^{1/3}]^2}$

65. $\dfrac{3(2x)^{1/3} - 2x(2x)^{-(2/3)}}{[(2x)^{1/3}]^2}$

Graphing Calculator Activities

66. Use your calculator to evaluate each of the following.

(a) $\sqrt[3]{1728}$ **(b)** $\sqrt[3]{5832}$

(c) $\sqrt[4]{2401}$ **(d)** $\sqrt[4]{65{,}536}$

(e) $\sqrt[5]{161{,}051}$ **(f)** $\sqrt[5]{6{,}436{,}343}$

67. In Definition 0.7 we stated that $b^{m/n} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$. Use your calculator to verify each of the following.

(a) $\sqrt[3]{27^2} = \left(\sqrt[3]{27}\right)^2$ **(b)** $\sqrt[3]{8^5} = \left(\sqrt[3]{8}\right)^5$

(c) $\sqrt[4]{16^3} = \left(\sqrt[4]{16}\right)^3$ **(d)** $\sqrt[3]{16^2} = \left(\sqrt[3]{16}\right)^2$

(e) $\sqrt[5]{9^4} = \left(\sqrt[5]{9}\right)^4$ **(f)** $\sqrt[3]{12^4} = \left(\sqrt[3]{12}\right)^4$

68. Use your calculator to evaluate each of the following.

(a) $16^{5/2}$ **(b)** $25^{7/2}$

(c) $16^{9/4}$ **(d)** $27^{5/3}$

(e) $343^{2/3}$ **(f)** $512^{4/3}$

69. Use your calculator to estimate each of the following to the nearest thousandth.

(a) $7^{4/3}$ **(b)** $10^{4/5}$

(c) $12^{2/5}$ **(d)** $19^{2/5}$

(e) $7^{3/4}$ **(f)** $10^{5/4}$

Answers to the Concept Quiz

1. C **2.** B **3.** A **4.** B **5.** True **6.** True **7.** True **8.** True

0.8 Complex Numbers

OBJECTIVES

1. Express the square root of a negative number in terms of i
2. Add and subtract complex numbers
3. Multiply and divide complex numbers

So far we have dealt only with real numbers. However, as we get ready to solve equations in the next chapter, there is a need for *more numbers*. There are some very simple equations that do not have solutions if we restrict ourselves to the set of real numbers. For example, the equation $x^2 + 1 = 0$ has no solutions among the real numbers. To solve such equations, we need to extend the real number system. In this section we will introduce a set of numbers that contains some numbers with squares that are negative real numbers. Then in the next chapter and in Chapter 4 we will see that this set of numbers, called the **complex numbers,** provides solutions not only for equations such as $x^2 + 1 = 0$ but also for *any* polynomial equation in general.

Let's begin by defining a number i such that

$$i^2 = -1$$

The number i is not a real number and is often called the **imaginary unit,** but the number i^2 is the real number -1. The imaginary unit i is used to define a complex number as follows.

Definition 0.8

A **complex number** is any number that can be expressed in the form

$$a + bi$$

where a and b are real numbers, and i is the imaginary unit.

The form $a + bi$ is called the **standard form** of a complex number. The real number a is called the **real part** of the complex number, and b is called the **imaginary part.** (Note that b is a real number even though it is called the imaginary part.) Each of the following represents a complex number:

$6 + 2i$ is already expressed in the form $a + bi$. Traditionally, complex numbers for which $a \neq 0$ and $b \neq 0$ have been called imaginary numbers.

$5 - 3i$ can be written $5 + (-3i)$ even though the form $5 - 3i$ is often used.

$-8 + i\sqrt{2}$ can be written $-8 + \sqrt{2}i$. It is easy to mistake $\sqrt{2}i$ for $\sqrt{2i}$. Thus we commonly write $i\sqrt{2}$ instead of $\sqrt{2}i$ to avoid any difficulties with the radical sign.

$-9i$ can be written $0 + (-9i)$. Complex numbers such as $-9i$, for which $a = 0$ and $b \neq 0$, traditionally have been called **pure imaginary numbers.**

5 can be written $5 + 0i$.

The set of real numbers is a subset of the set of complex numbers. The following diagram indicates the organizational format of the complex number system:

Complex numbers

$a + bi$, where a and b are real numbers

Real numbers

$a + bi$, where $b = 0$

Imaginary numbers

$a + bi$, where $b \neq 0$

Pure imaginary numbers

$a + bi$, where $a = 0$ and $b \neq 0$

Two complex numbers $a + bi$ and $c + di$ are said to be *equal* if and only if $a = c$ and $b = d$. In other words, two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

Adding and Subtracting Complex Numbers

The following definition provides the basis for adding complex numbers:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

We can use this definition to find the sum of two complex numbers.

$$(4 + 3i) + (5 + 9i) = (4 + 5) + (3 + 9)i = 9 + 12i$$

$$(-6 + 4i) + (8 - 7i) = (-6 + 8) + (4 - 7)i = 2 - 3i$$

$$\left(\frac{1}{2} + \frac{3}{4}i\right) + \left(\frac{2}{3} + \frac{1}{5}i\right) = \left(\frac{1}{2} + \frac{2}{3}\right) + \left(\frac{3}{4} + \frac{1}{5}\right)i$$

$$= \left(\frac{3}{6} + \frac{4}{6}\right) + \left(\frac{15}{20} + \frac{4}{20}\right)i = \frac{7}{6} + \frac{19}{20}i$$

$$\left(3 + i\sqrt{2}\right) + \left(-4 + i\sqrt{2}\right) = [3 + (-4)] + \left(\sqrt{2} + \sqrt{2}\right)i = -1 + 2i\sqrt{2}$$

Note the form for writing $2\sqrt{2}i$.

The set of complex numbers is **closed with respect to addition**; that is, the sum of two complex numbers is a complex number. Furthermore, the commutative and associative properties of addition hold for all complex numbers. The additive identity element is $0 + 0i$, or simply the real number 0. The additive inverse of $a + bi$ is $-a - bi$ because

$$(a + bi) + (-a - bi) = [a + (-a)] + [b + (-b)]i = 0$$

Therefore, to *subtract* $c + di$ from $a + bi$, we add the additive inverse of $c + di$:

$$\begin{aligned}(a + bi) - (c + di) &= (a + bi) + (-c - di)\\ &= (a - c) + (b - d)i\end{aligned}$$

The following examples illustrate the subtraction of complex numbers:

$$(9 + 8i) - (5 + 3i) = (9 - 5) + (8 - 3)i = 4 + 5i$$

$$(3 - 2i) - (4 - 10i) = (3 - 4) + [-2 - (-10)]i = -1 + 8i$$

$$\left(-\frac{1}{2} + \frac{1}{3}i\right) - \left(\frac{3}{4} + \frac{1}{2}i\right) = \left(-\frac{1}{2} - \frac{3}{4}\right) + \left(\frac{1}{3} - \frac{1}{2}\right)i = -\frac{5}{4} - \frac{1}{6}i$$

Multiplying and Dividing Complex Numbers

Because $i^2 = -1$, the number i is a square root of -1, so we write $i = \sqrt{-1}$. It should also be evident that $-i$ is a square root of -1 because

$$(-i)^2 = (-i)(-i) = i^2 = -1$$

Therefore, in the set of complex numbers, -1 has two square roots—namely, i and $-i$. This is expressed symbolically as

$$i = \sqrt{-1} \quad \text{and} \quad -i = -\sqrt{-1}$$

Let's extend the definition so that in the set of complex numbers, every negative real number has two square roots. For any positive real number b,

$$\left(i\sqrt{b}\right)^2 = i^2(b) = -1(b) = -b$$

Therefore, let's denote the **principal square root of $-b$** by $\sqrt{-b}$ and define it to be

$$\sqrt{-b} = i\sqrt{b}$$

where b is any positive real number. In other words, the principal square root of any negative real number can be represented as the product of a real number and the imaginary unit i. Consider the following examples:

$$\sqrt{-4} = i\sqrt{4} = 2i$$

$$\sqrt{-17} = i\sqrt{17}$$

$$\sqrt{-24} = i\sqrt{24} = i\sqrt{4}\sqrt{6} = 2i\sqrt{6}$$

Note that we simplified the radical $\sqrt{24}$ to $2\sqrt{6}$

We should also observe that $-\sqrt{-b}$, where $b > 0$, is a square root of $-b$ because

$$\left(-\sqrt{-b}\right)^2 = \left(-i\sqrt{b}\right)^2 = i^2(b) = (-1)b = -b$$

Thus in the set of complex numbers, $-b$ (where $b > 0$) has two square roots: $i\sqrt{b}$ and $-i\sqrt{b}$. These are expressed as

$$\sqrt{-b} = i\sqrt{b} \quad \text{and} \quad -\sqrt{-b} = -i\sqrt{b}$$

We must be careful with the use of the symbol $\sqrt{-b}$, where $b > 0$. Some properties that are true in the set of real numbers involving the square root symbol do not hold if the square root symbol does not represent a real number. For example, $\sqrt{a}\sqrt{b} = \sqrt{ab}$ *does not hold if a and b are both negative numbers*.

Correct $\quad \sqrt{-4}\sqrt{-9} = (2i)(3i) = 6i^2 = 6(-1) = -6$

Incorrect $\quad \sqrt{-4}\sqrt{-9} = \sqrt{(-4)(-9)} = \sqrt{36} = 6$

To avoid difficulty with this idea, you should rewrite all expressions of the form $\sqrt{-b}$, where $b > 0$, in the form $i\sqrt{b}$ *before* doing any computations. The following examples further illustrate this point:

$$\sqrt{-5}\sqrt{-7} = (i\sqrt{5})(i\sqrt{7}) = i^2\sqrt{35} = (-1)\sqrt{35} = -\sqrt{35}$$
$$\sqrt{-2}\sqrt{-8} = (i\sqrt{2})(i\sqrt{8}) = i^2\sqrt{16} = (-1)(4) = -4$$
$$\sqrt{-2}\sqrt{8} = (i\sqrt{2})(\sqrt{8}) = i\sqrt{16} = 4i$$
$$\sqrt{-6}\sqrt{-8} = (i\sqrt{6})(i\sqrt{8}) = i^2\sqrt{48} = i^2\sqrt{16}\sqrt{3} = 4i^2\sqrt{3} = -4\sqrt{3}$$
$$\frac{\sqrt{-2}}{\sqrt{3}} = \frac{i\sqrt{2}}{\sqrt{3}} = \frac{i\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{i\sqrt{6}}{3}$$
$$\frac{\sqrt{-48}}{\sqrt{12}} = \frac{i\sqrt{48}}{\sqrt{12}} = i\sqrt{\frac{48}{12}} = i\sqrt{4} = 2i$$

Because complex numbers have a *binomial form,* we can find the product of two complex numbers in the same way that we find the product of two binomials. Then, by replacing i^2 with -1 we can simplify and express the final product in the standard form of a complex number. Consider the following examples:

$$\begin{aligned}(2+3i)(4+5i) &= 2(4+5i) + 3i(4+5i)\\ &= 8 + 10i + 12i + 15i^2\\ &= 8 + 22i + 15(-1)\\ &= 8 + 22i - 15\\ &= -7 + 22i\end{aligned}$$

$$\begin{aligned}(1-7i)^2 &= (1-7i)(1-7i)\\ &= 1(1-7i) - 7i(1-7i)\\ &= 1 - 7i - 7i + 49i^2\\ &= 1 - 14i + 49(-1)\\ &= 1 - 14i - 49\\ &= -48 - 14i\end{aligned}$$

$$\begin{aligned}(2+3i)(2-3i) &= 2(2-3i) + 3i(2-3i)\\ &= 4 - 6i + 6i - 9i^2\\ &= 4 - 9(-1)\\ &= 4 + 9\\ &= 13\end{aligned}$$

Remark: Don't forget that when multiplying complex numbers, we can also use the multiplication patterns

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)(a-b) = a^2 - b^2$$

The last example illustrates an important idea. The complex numbers $2 + 3i$ and $2 - 3i$ are called *conjugates* of each other. In general, the two complex numbers $a + bi$ and $a - bi$ are called **conjugates** of each other, and **the product of a complex number and its conjugate is a real number.** This can be shown as follows:

$$\begin{aligned}(a + bi)(a - bi) &= a(a - bi) + bi(a - bi)\\ &= a^2 - abi + abi - b^2i^2\\ &= a^2 - b^2(-1)\\ &= a^2 + b^2\end{aligned}$$

Conjugates are used to simplify an expression such as $3i/(5 + 2i)$, which *indicates the quotient of two complex numbers*. To eliminate i in the denominator and to change the indicated quotient to the standard form of a complex number, we can multiply both the numerator and denominator by the conjugate of the denominator.

$$\begin{aligned}\frac{3i}{5 + 2i} &= \frac{3i}{5 + 2i} \cdot \frac{5 - 2i}{5 - 2i}\\ &= \frac{3i(5 - 2i)}{(5 + 2i)(5 - 2i)}\\ &= \frac{15i - 6i^2}{25 - 4i^2}\\ &= \frac{15i - 6(-1)}{25 - 4(-1)}\\ &= \frac{6 + 15i}{29}\\ &= \frac{6}{29} + \frac{15}{29}i\end{aligned}$$

The following examples further illustrate the process of dividing complex numbers:

$$\begin{aligned}\frac{2 - 3i}{4 - 7i} &= \frac{2 - 3i}{4 - 7i} \cdot \frac{4 + 7i}{4 + 7i}\\ &= \frac{(2 - 3i)(4 + 7i)}{(4 - 7i)(4 + 7i)}\\ &= \frac{8 + 14i - 12i - 21i^2}{16 - 49i^2}\\ &= \frac{8 + 2i - 21(-1)}{16 - 49(-1)}\\ &= \frac{29 + 2i}{65} = \frac{29}{65} + \frac{2}{65}i\end{aligned}$$

$$\begin{aligned}
\frac{4-5i}{2i} &= \frac{4-5i}{2i}\cdot\frac{-2i}{-2i} \\
&= \frac{(4-5i)(-2i)}{(2i)(-2i)} \\
&= \frac{-8i+10i^2}{-4i^2} \\
&= \frac{-8i+10(-1)}{-4(-1)} \\
&= \frac{-10-8i}{4} = -\frac{5}{2}-2i
\end{aligned}$$

For a problem such as the last one, in which the denominator is a pure imaginary number, we can change to standard form by choosing a multiplier other than the conjugate of the denominator. Consider the following alternative approach:

$$\begin{aligned}
\frac{4-5i}{2i} &= \frac{4-5i}{2i}\cdot\frac{i}{i} \\
&= \frac{(4-5i)(i)}{(2i)(i)} \\
&= \frac{4i-5i^2}{2i^2} \\
&= \frac{4i-5(-1)}{2(-1)} \\
&= \frac{5+4i}{-2} \\
&= -\frac{5}{2}-2i
\end{aligned}$$

Concept Quiz 0.8

For Problems 1–8, answer true or false.

1. The number i is not a real number.
2. The number i^2 is a real number.
3. The form $ai + b$ is called the standard form of a complex number.
4. Every real number is a member of the set of complex numbers.
5. The principal square root of any negative real number can be represented as the product of a real number and the imaginary unit i.
6. $6 - 4i$ and $-6 + 4i$ are additive inverses.
7. The conjugate of the number $-2 - 3i$ is $2 + 3i$.
8. The product of a complex number and its conjugate is a real number.

Problem Set 0.8

For Problems 1–14, add or subtract as indicated. **(Objective 2)**

1. $(5 + 2i) + (8 + 6i)$

2. $(-9 + 3i) + (4 + 5i)$

3. $(8 + 6i) - (5 + 2i)$

4. $(-6 + 4i) - (4 + 6i)$

5. $(-7 - 3i) + (-4 + 4i)$ **6.** $(6 - 7i) - (7 - 6i)$

7. $(-2 - 3i) - (-1 - i)$

8. $\left(\frac{1}{3} + \frac{2}{5}i\right) + \left(\frac{1}{2} + \frac{1}{4}i\right)$

9. $\left(-\frac{3}{4} - \frac{1}{4}i\right) + \left(\frac{3}{5} + \frac{2}{3}i\right)$

10. $\left(\frac{5}{8} + \frac{1}{2}i\right) - \left(\frac{7}{8} + \frac{1}{5}i\right)$

11. $\left(\frac{3}{10} - \frac{3}{4}i\right) - \left(-\frac{2}{5} + \frac{1}{6}i\right)$

12. $\left(4 + i\sqrt{3}\right) + \left(-6 - 2i\sqrt{3}\right)$

13. $(5 + 3i) + (7 - 2i) + (-8 - i)$

14. $(5 - 7i) - (6 - 2i) - (-1 - 2i)$

For Problems 15–30, write each in terms of i and simplify. **(Objective 1)** For example,

$$\sqrt{-20} = i\sqrt{20} = i\sqrt{4}\sqrt{5} = 2i\sqrt{5}$$

15. $\sqrt{-9}$ **16.** $\sqrt{-49}$

17. $\sqrt{-19}$ **18.** $\sqrt{-31}$

19. $\sqrt{-\frac{4}{9}}$ **20.** $\sqrt{-\frac{25}{36}}$

21. $\sqrt{-8}$ **22.** $\sqrt{-18}$

23. $\sqrt{-27}$ **24.** $\sqrt{-32}$

25. $\sqrt{-54}$ **26.** $\sqrt{-40}$

27. $3\sqrt{-36}$ **28.** $5\sqrt{-64}$

29. $4\sqrt{-18}$ **30.** $6\sqrt{-8}$

Some of the solution sets for quadratic equations in the next chapter will contain complex numbers such as $\frac{-4 + \sqrt{-12}}{2}$ and $\frac{-4 - \sqrt{-12}}{2}$. We can simplify the first number as follows.

$$\frac{-4 + \sqrt{-12}}{2} = \frac{-4 + i\sqrt{12}}{2} =$$

$$\frac{-4 + 2i\sqrt{3}}{2} = \frac{\cancel{2}(-2 + i\sqrt{3})}{\cancel{2}} = -2 + i\sqrt{3}$$

For Problems 31–36, simplify each of the following complex numbers.

31. $\frac{-4 - \sqrt{-12}}{2}$ **32.** $\frac{6 + \sqrt{-24}}{4}$

33. $\frac{-3 - \sqrt{-18}}{3}$ **34.** $\frac{-6 + \sqrt{-27}}{3}$

35. $\frac{12 + \sqrt{-45}}{6}$ **36.** $\frac{4 - \sqrt{-48}}{2}$

For Problems 37–50, write each in terms of i, perform the indicated operations, and simplify. **(Objective 1)** For example,

$$\sqrt{-9}\sqrt{-16} = \left(i\sqrt{9}\right)\left(i\sqrt{16}\right) = (3i)(4i)$$
$$= 12i^2 = 12(-1) = -12$$

37. $\sqrt{-4}\sqrt{-16}$ **38.** $\sqrt{-25}\sqrt{-9}$

39. $\sqrt{-2}\sqrt{-3}$ **40.** $\sqrt{-3}\sqrt{-7}$

41. $\sqrt{-5}\sqrt{-4}$ **42.** $\sqrt{-7}\sqrt{-9}$

43. $\sqrt{-6}\sqrt{-10}$ **44.** $\sqrt{-2}\sqrt{-12}$

45. $\sqrt{-8}\sqrt{-7}$ **46.** $\sqrt{-12}\sqrt{-5}$

47. $\frac{\sqrt{-36}}{\sqrt{-4}}$ **48.** $\frac{\sqrt{-64}}{\sqrt{-16}}$

49. $\frac{\sqrt{-54}}{\sqrt{-9}}$ **50.** $\frac{\sqrt{-18}}{\sqrt{-3}}$

For Problems 51–70, find each product and express the answers in standard form. **(Objective 3)**

51. $(3i)(7i)$

52. $(-5i)(8i)$

53. $(4i)(3 - 2i)$

54. $(5i)(2 + 6i)$

55. $(3 + 2i)(4 + 6i)$

56. $(7 + 3i)(8 + 4i)$

57. $(4 + 5i)(2 - 9i)$

58. $(1 + i)(2 - i)$

59. $(-2 - 3i)(4 + 6i)$

60. $(-3 - 7i)(2 + 10i)$

61. $(6 - 4i)(-1 - 2i)$

62. $(7 - 3i)(-2 - 8i)$

63. $(3 + 4i)^2$

64. $(4 - 2i)^2$

65. $(-1 - 2i)^2$

66. $(-2 + 5i)^2$

67. $(8 - 7i)(8 + 7i)$

68. $(5 + 3i)(5 - 3i)$

69. $(-2 + 3i)(-2 - 3i)$

70. $(-6 - 7i)(-6 + 7i)$

For Problems 71–84, find each quotient and express the answers in standard form. **(Objective 3)**

71. $\dfrac{4i}{3 - 2i}$

72. $\dfrac{3i}{6 + 2i}$

73. $\dfrac{2 + 3i}{3i}$

74. $\dfrac{3 - 5i}{4i}$

75. $\dfrac{3}{2i}$

76. $\dfrac{7}{4i}$

77. $\dfrac{3 + 2i}{4 + 5i}$

78. $\dfrac{2 + 5i}{3 + 7i}$

79. $\dfrac{4 + 7i}{2 - 3i}$

80. $\dfrac{3 + 9i}{4 - i}$

81. $\dfrac{3 - 7i}{-2 + 4i}$

82. $\dfrac{4 - 10i}{-3 + 7i}$

83. $\dfrac{-1 - i}{-2 - 3i}$

84. $\dfrac{-4 + 9i}{-3 - 6i}$

85. Using $a + bi$ and $c + di$ to represent two complex numbers, verify the following properties.

(a) The conjugate of the sum of two complex numbers is equal to the sum of the conjugates of the two numbers.

(b) The conjugate of the product of two complex numbers is equal to the product of the conjugates of the numbers.

Thoughts Into Words

86. Is every real number also a complex number? Explain your answer.

87. Can the product of two nonreal complex numbers be a real number? Explain your answer.

Further Investigations

88. Observe the following powers of i:

$$i = \sqrt{-1}$$
$$i^2 = -1$$
$$i^3 = i^2 \cdot i = -1(i) = -i$$
$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$$

Any power of i greater than 4 can be simplified to i, -1, $-i$, or 1 as follows:

$$i^9 = (i^4)^2(i) = (1)(i) = i$$
$$i^{14} = (i^4)^3(i^2) = (1)(-1) = -1$$
$$i^{19} = (i^4)^4(i^3) = (1)(-i) = -i$$
$$i^{28} = (i^4)^7 = (1)^7 = 1$$

Express each of the following as $i, -1, -i$, or 1.

(a) i^5 **(b)** i^6 **(c)** i^{11}

(d) i^{12} **(e)** i^{16} **(f)** i^{22}

(g) i^{33} **(h)** i^{63}

89. We can use the information from Problem 88 and the binomial expansion patterns to find powers of complex numbers as follows:

$$\begin{aligned}(3+2i)^3 &= (3)^3 + 3(3)^2(2i) + 3(3)(2i)^2 + (2i)^3\\ &= 27 + 54i + 36i^2 + 8i^3\\ &= 27 + 54i + 36(-1) + 8(-i)\\ &= -9 + 46i\end{aligned}$$

Find the indicated power of each expression:

(a) $(2+i)^3$ **(b)** $(1-i)^3$

(c) $(1-2i)^3$ **(d)** $(1+i)^4$

(e) $(2-i)^4$ **(f)** $(-1+i)^5$

Answers to the Concept Quiz

1. True **2.** True **3.** False **4.** True **5.** True **6.** True **7.** False **8.** True

Chapter 0 SUMMARY

OBJECTIVE	SUMMARY	EXAMPLE
Recognize the vocabulary and symbolism associated with sets. **(Section 0.1/Objective 1)**	Be sure of the following key concepts about sets: Elements, null set, equal sets, subsets, and set builder notation.	Answer True or False. **(a)** $\{a, b, c\} = \{b, a, c\}$ **(b)** $\{1, 3, 5\} \subset \{0, 1, 2, 3, 4, 5\}$ **Solution** **(a)** True **(b)** True
Know the various subset classifications of the real number system. **(Section 0.1/Objective 2)**	The sets of natural numbers, whole numbers, integers, rational numbers, and irrational numbers are all subsets of the real number system.	Name each of the following sets. **(a)** $\{0, 1, 2, 3, \ldots\}$ **(b)** $\{\ldots -3, -2, -1, 0\}$ **(c)** $\{1, 2, 3, \ldots\}$ **Solution** **(a)** Whole numbers **(b)** Nonpositive integers **(c)** Natural numbers
Apply the definition of the absolute value of a number. **(Section 0.1/Objective 4)**	For all real numbers a, If $a \geq 0$, then $\|a\| = a$. If $a < 0$, then $\|a\| = -a$. The following properties of absolute value are useful: **1.** $\|a\| \geq 0$ **2.** $\|a\| = \|-a\|$ **3.** $\|a - b\| = \|b - a\|$	Evaluate $\frac{2\|x - y\|}{\|y - x\|}$. **Solution** By the absolute value properties, $\|x - y\| = \|y - x\|$. Therefore $\frac{2\|x - y\|}{\|y - x\|} = 2(1) = 2$.
Know the real number properties. **(Section 0.1/Objective 5)**	As you study the operations on the set of real numbers, the following properties will serve as the bases for many algebraic operations. • Commutative properties for addition and multiplication • Associative properties for addition and multiplication • Identity properties for addition and multiplication • Inverse properties for addition and multiplication • Distributive property	State the property that justifies the statement. **(a)** $(a + b) + c = (b + a) + c$ **(b)** $(x + y) + z = x + (y + z)$ **(c)** $4m + 4n = 4(m + n)$ **Solution** **(a)** Commutative property of addition **(b)** Associative property of addition **(c)** Distributive property

(continued)

OBJECTIVE	SUMMARY	EXAMPLE
Evaluate algebraic expressions. **(Section 0.1/Objective 6)**	An algebraic expression takes on a numerical value whenever each variable in the expression is replaced by a real number. It is good practice to use parentheses when replacing the variable with a number.	Evaluate $\frac{a-4b}{(a-b)^2}$ when $a = 4$ and $b = -1$. **Solution** $\frac{a-4b}{(a-b)^2} = \frac{4-4(-1)}{(4-(-1))^2} = \frac{4+4}{(5)^2} = \frac{8}{25}$ when $a = 4$ and $b = -1$.
Apply the properties of exponents to simplify algebraic expressions. **(Section 0.2/Objective 2)**	Read Property 0.2 on page 23. A quick summary of some of that information is as follows: **1.** When multiplying like bases, add the exponents. **2.** When dividing like bases, subtract the exponents. **3.** When a power is raised to another power, multiply the exponents.	Simplify $\left(\frac{6x^3y^{-4}}{3x^{-2}y^{-1}}\right)^2$. **Solution** $\left(\frac{6x^3y^{-4}}{3x^{-2}y^{-1}}\right)^2 = (2x^{3-(-2)}y^{-4-(-1)})^2$ $= (2x^5y^{-3})^2$ $= 2^2x^{10}y^{-6}$ $= \frac{4x^{10}}{y^6}$
Write numbers in scientific notation. **(Section 0.2/Objective 3)**	Scientific notation is often used to write numbers that are very small or very large in magnitude. The scientific form of a number is expressed as $(N)(10^k)$, where N is a number greater than or equal to 1 and less than 10, written in decimal form, and k is an integer.	Write each of the following in scientific notation. **(a)** 0.00000342 **(b)** 678,000,000,000 **Solution** **(a)** $0.00000342 = 3.42(10^{-6})$ **(b)** $678{,}000{,}000{,}000 = 6.78(10^{11})$
Convert numbers from scientific notation to ordinary decimal notation. **(Section 0.2/Objective 4)**	To switch from scientific notation to ordinary decimal notation, move the decimal point the number of places indicated by the exponent of the 10. The decimal point is moved to the right if the exponent is positive and to the left if the exponent is negative.	Write each of the following in ordinary decimal notation. **(a)** $(8.5)(10^{-5})$ **(b)** $(3.4)(10^6)$ **Solution** **(a)** $(8.5)(10^{-5}) = 0.000085$ **(b)** $(3.4)(10^6) = 3{,}400{,}000$

OBJECTIVE	SUMMARY	EXAMPLE
Perform calculations with numbers in scientific form. **(Section 0.2/Objective 5)**	Scientific notation can be used to simplify numerical operations by changing the numbers to scientific notation and using the appropriate properties of exponents.	Simplify $\dfrac{0.0000068}{0.04}$. **Solution** $\dfrac{0.0000068}{0.04} = \dfrac{(6.8)(10^{-6})}{(4)(10^{-2})}$ $= (1.7)(10^{-4}) = 0.00017$
Add and subtract polynomials. **(Section 0.3/Objective 1)**	Similar or like terms have the same literal factors. The commutative, associative, and distributive properties provide the basis for rearranging, regrouping, and combining similar terms.	Simplify $5x - [3x^2 - 4(6x - 2x^2)]$. **Solution** $5x - [3x^2 - 4(6x - 2x^2)]$ $= 5x - [3x^2 - 24x + 8x^2]$ $= 5x - [11x^2 - 24x]$ $= 5x - 11x^2 + 24x$ $= -11x^2 + 29x$
Multiply polynomials. **(Section 0.3/Objective 2)**	To multiply two polynomials, every term of the first polynomial is multiplied by each term of the second polynomial. Multiplying polynomials often produces similar terms that can be combined to simplify the resulting polynomial.	Find the indicated product. $(3x + 5)(x^2 - 2x + 7)$ **Solution** $(3x + 5)(x^2 - 2x + 7)$ $= 3x(x^2 - 2x + 7) + 5(x^2 - 2x + 7)$ $= 3x^3 - 6x^2 + 21x + 5x^2 - 10x + 35$ $= 3x^3 - x^2 + 11x + 35$
Perform binomial expansions. **(Section 0.3/Objective 3)**	It is possible to write the expansion of $(a + b)^n$, where n is a natural number, without doing all the intermediate steps. This can be done by realizing the pattern of the exponents for each term of the expansion and using Pascal's triangle to determine the coefficient for each term.	Expand $(2x + y)^4$. **Solution** $(2x + y)^4$ $= (2x)^4 + 4(2x)^3y + 6(2x)^2y^2 + 4(2x)y^3 + y^4$ $= 16x^4 + 32x^3y + 24x^2y^2 + 8xy^3 + y^4$

(continued)

OBJECTIVE	SUMMARY	EXAMPLE
Divide a polynomial by a monomial. **(Section 0.3/Objective 4)**	To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.	Perform the indicated division. $\frac{15a^3b^4 - 30a^5b^7 + 5a^2b^3}{5a^2b^3}$ **Solution** Rewrite the problem as separate fractions obtained by each term in the numerator divided by the denominator. Then simplify each fraction. $\frac{15a^3b^4 - 30a^5b^7 + 5a^2b^3}{5a^2b^3}$ $= \frac{15a^3b^4}{5a^2b^3} - \frac{30a^5b^7}{5a^2b^3} + \frac{5a^2b^3}{5a^2b^3}$ $= 3ab - 6a^3b^4 + 1$
Factor out a common factor. **(Section 0.4/Objective 1)**	The distributive property in the form $ab + ac = a(b + c)$ is the basis for factoring out a common factor. The common factor can be a binomial factor, as when performing factoring by grouping.	Factor $-6x^5y^4 - 3x^6y^3 - 24x^7y^2$. **Solution** The common factor is $-3x^5y^2$. $-6x^5y^4 - 3x^6y^3 - 24x^7y^2$ $= -3x^5y^2(2y^2 + xy + 8x^2)$
Factor by grouping. **(Section 0.4/Objective 2)**	It may be that the polynomial exhibits no common monomial or binomial factor. However, by factoring common factors from groups of terms, a common factor may be evident.	Factor $2xz + 6x + yz + 3y$. **Solution** $2xz + 6x + yz + 3y$ $= 2x(z + 3) + y(z + 3)$ $= (z + 3)(2x + y)$
Factor the difference of two squares. **(Section 0.4/Objective 3)**	The factoring pattern $a^2 - b^2 = (a + b)(a - b)$ is called the difference of two squares.	Factor $36a^2 - 25b^2$. **Solution** $36a^2 - 25b^2 = (6a - 5b)(6a + 5b)$

OBJECTIVE	SUMMARY	EXAMPLE
Factor trinomials of the form $x^2 + bx + c$ and trinomials of the form $ax^2 + bx + c$. (Section 0.4/Objective 4)	Expressing a trinomial (for which the coefficient of the squared term is 1) as a product of two binomials is based on the relationship $(x + a)(x + b) = x^2 + (a + b)x + ab$ The coefficient of the middle term is the sum of a and b, and the last term is the product of a and b. Two methods were presented for factoring trinomials of the form $ax^2 + bx + c$. One technique is to try the various possibilities of factors and check by multiplying. This method is referred to as trial-and-error. The other method is structured technique and is shown in Section 0.4 Examples 8 and 9.	Factor $x^2 - 2x - 35$. **Solution** $x^2 - 2x - 35 = (x - 7)(x + 5)$ Factor $4x^2 + 16x + 15$. **Solution** Multiply 4 times 15 to get 60. The factors of 60 that add to 16 are 6 and 10. Rewrite the problem and factor by grouping. $4x^2 + 16x + 15$ $= 4x^2 + 10x + 6x + 15$ $= 2x(2x + 5) + 3(2x + 5)$ $= (2x + 5)(2x + 3)$
Factor the sum or difference of two cubes. (Section 0.4/Objective 5)	The factoring patterns $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ are called the sum of two cubes or the difference of two cubes.	Factor $8x^3 + 27y^3$. **Solution** $8x^3 + 27y^3 = (2x + 3y)(4x^2 - 6xy + 9y^2)$
Apply more than one factoring technique. (Section 0.4/Objective 6)	Be sure to factor completely. Some problems require that more than one factoring technique may be necessary or perhaps the same technique used twice.	Factor $81a^4 - 16b^4$. **Solution** $81a^4 - 16b^4$ $= (9a^2 + 4b^2)(9a^2 - 4b^2)$ $= (9a^2 + 4b^2)(3a + 2b)(3a - 2b)$
Simplify rational expressions. (Section 0.5/Objective 1)	A rational expression is defined as the indicated quotient of two polynomials. The Fundamental Principle of Fractions, $\frac{a \cdot k}{b \cdot k} = \frac{a}{b}$, is used when reducing rational numbers or rational expressions.	Simplify $\frac{x^2 - 2x - 15}{x^2 + x - 6}$. **Solution** $\frac{x^2 - 2x - 15}{x^2 + x - 6}$ $= \frac{(x + 3)(x - 5)}{(x + 3)(x - 2)} = \frac{x - 5}{x - 2}$

(continued)

OBJECTIVE	SUMMARY	EXAMPLE
Multiply and divide rational expressions. **(Section 0.5/Objective 2)**	Multiplication of rational expressions is based on the following definition: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ Division of rational expressions is based on the following definition: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$	Find the quotient $\frac{6xy}{x^2 - 6x + 9} \div \frac{18x}{x^2 - 9}$. **Solution** $\frac{6xy}{x^2 - 6x + 9} \div \frac{18x}{x^2 - 9}$ $= \frac{6xy}{x^2 - 6x + 9} \cdot \frac{x^2 - 9}{18x}$ $= \frac{6xy}{(x-3)(x-3)} \cdot \frac{(x+3)(x-3)}{18x}$ $\frac{\cancel{6x}y}{(x-3)\cancel{(x-3)}} \cdot \frac{(x+3)\cancel{(x-3)}}{\underset{3}{\cancel{18x}}}$ $= \frac{y(x+3)}{3(x-3)}$
Add and subtract rational expressions. **(Section 0.5/Objective 3)**	Addition and subtraction of rational expressions are based on the following definitions: $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$ Addition $\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$ Subtraction The following basic procedure is used to add or subtract rational expressions: **1.** Factor the denominators. **2.** Find the LCD. **3.** Change each fraction to an equivalent fraction that has the LCD as the denominator. **4.** Combine the numerators and place over the LCD. **5.** Simplify by performing the addition or subtraction in the numerator. **6.** If possible, reduce the resulting fraction.	Subtract $\frac{2}{x^2 - 2x - 3} - \frac{5}{x^2 + 5x + 4}$. **Solution** $\frac{2}{x^2 - 2x - 3} - \frac{5}{x^2 + 5x + 4}$ $= \frac{2}{(x-3)(x+1)} - \frac{5}{(x+1)(x+4)}$ The LCD is $(x-3)(x+1)(x+4)$. $= \frac{2(x+4)}{(x-3)(x+1)(x+4)} - \frac{5(x-3)}{(x+1)(x+4)(x-3)}$ $= \frac{2(x+4) - 5(x-3)}{(x-3)(x+1)(x+4)}$ $= \frac{2x + 8 - 5x + 15}{(x-3)(x+1)(x+4)}$ $= \frac{-3x + 23}{(x-3)(x+1)(x+4)}$

OBJECTIVE	SUMMARY	EXAMPLE
Simplify complex fractions. **(Section 0.5/Objective 4)**	Fractions that contain rational expressions in the numerators or denominators are called complex fractions. In Section 0.5 two methods were shown for simplifying complex fractions.	Simplify $\dfrac{\frac{2}{x} - \frac{3}{y}}{\frac{4}{x^2} + \frac{5}{y}}$. **Solution** $\dfrac{\frac{2}{x} - \frac{3}{y}}{\frac{4}{x^2} + \frac{5}{y}}$ Multiply the numerator and denominator by x^2y. $\dfrac{x^2y\left(\frac{2}{x} - \frac{3}{y}\right)}{x^2y\left(\frac{4}{x^2} + \frac{5}{y}\right)} = \dfrac{x^2y\left(\frac{2}{x}\right) + x^2y\left(-\frac{3}{y}\right)}{x^2y\left(\frac{4}{x^2}\right) + x^2y\left(\frac{5}{y}\right)}$ $= \dfrac{2xy - 3x^2}{4y + 5x^2}$
Write radical expressions in simplest radical form. **(Section 0.6/Objective 2)**	A radical expression is in simplest form if **1.** No fraction appears within a radical sign. **2.** No radical appears in the denominator. **3.** No radicand contains a perfect power of the index. The following properties are used to express radicals in simplest form: $\sqrt[n]{bc} = \sqrt[n]{b}\sqrt[n]{c}$ $\sqrt[n]{\dfrac{b}{c}} = \dfrac{\sqrt[n]{b}}{\sqrt[n]{c}}$	Simplify $\sqrt{150a^3b^2}$. Assume all variables represent nonnegative values. **Solution** $\sqrt{150a^3b^2} = \sqrt{25a^2b^2}\sqrt{6a}$ $= 5ab\sqrt{6a}$
Simplify an indicated sum of radical expressions. **(Section 0.6/Objective 3)**	The distributive property can be used to combine radicals that have the same index and the same radicand. Sometimes the problem requires that the given radicals be expressed in simplest form.	Simplify $\sqrt{24} - \sqrt{54} + 8\sqrt{6}$. **Solution** $\sqrt{24} - \sqrt{54} + 8\sqrt{6}$ $= \sqrt{4}\sqrt{6} - \sqrt{9}\sqrt{6} + 8\sqrt{6}$ $= 2\sqrt{6} - 3\sqrt{6} + 8\sqrt{6}$ $= 7\sqrt{6}$

(continued)

OBJECTIVE	SUMMARY	EXAMPLE
Multiply radical expressions. **(Section 0.6/Objective 4)**	Property 0.3 can be viewed as $\sqrt[n]{b}\sqrt[n]{c} = \sqrt[n]{bc}$. This property, along with commutative, associative, and distributive properties of real numbers, provides a basis for multiplying radicals that have the same index.	Multiply $\sqrt{2x}(\sqrt{6x} + \sqrt{18xy})$ and simplify where possible. **Solution** $\sqrt{2x}(\sqrt{6x} + \sqrt{18xy})$ $= \sqrt{12x^2} + \sqrt{36x^2y}$ $= \sqrt{4x^2}\sqrt{3} + \sqrt{36x^2}\sqrt{y}$ $= 2x\sqrt{3} + 6x\sqrt{y}$
Rationalize radical expressions. **(Section 0.6/Objective 5)**	If a radical appears in the denominator, then it will be necessary to rationalize the denominator for the expression to be in simplest form. To rationalize a binomial denominator, multiply the numerator and denominator by the conjugate of the denominator. The factors $a - b$ and $a + b$ are called conjugates.	Simplify $\dfrac{3}{\sqrt{7} - \sqrt{5}}$. **Solution** $\dfrac{3}{\sqrt{7} - \sqrt{5}}$ $= \dfrac{3}{(\sqrt{7} - \sqrt{5})} \cdot \dfrac{(\sqrt{7} + \sqrt{5})}{(\sqrt{7} + \sqrt{5})}$ $= \dfrac{3(\sqrt{7} + \sqrt{5})}{\sqrt{49} - \sqrt{25}} = \dfrac{3(\sqrt{7} + \sqrt{5})}{7 - 5}$ $= \dfrac{3(\sqrt{7} + \sqrt{5})}{2}$
Evaluate a number raised to a rational exponent. **(Section 0.7/Objective 1)**	If b is a real number, n is a positive integer greater than 1, and $\sqrt[n]{b}$ exists, then $b^{1/n} = \sqrt[n]{b}$. Thus $b^{1/n}$ means the nth root of b.	Simplify $16^{3/2}$. **Solution** $16^{3/2} = (16^{1/2})^3 = 4^3 = 64$
Simplify expressions with rational exponents. **(Section 0.7/Objective 2)**	Properties of exponents are used to simplify products and quotients involving rational exponents.	Simplify $(4x^{1/3})(-3x^{3/4})$and express the result with positive exponents only. **Solution** $(4x^{1/3})(-3x^{-3/4}) = -12x^{1/3-3/4}$ $= -12x^{-5/12}$ $= \dfrac{-12}{x^{5/12}}$

OBJECTIVE	SUMMARY	EXAMPLE
Apply rational exponents to simplify radical expressions. (Section 0.7/Objective 3)	To multiply or divide radical expressions with different indexes, change from radical to exponential form. Then apply the properties of exponents. Finally change back to radical form.	Perform the indicated operation and express the answers in simplest radical form. $\sqrt[4]{xy^3}\sqrt[5]{x^2y}$ **Solution** $\sqrt[4]{xy^3}\sqrt[5]{x^2y} = (x^{1/4}y^{3/4})(x^{2/5}y^{1/5})$ $= x^{1/4+2/5}y^{3/4+1/5}$ $= x^{13/20}y^{19/20} = \sqrt[20]{x^{13}y^{19}}$
Express the square root of a negative number in terms of i. (Section 0.8/Objective 1)	We can represent a square root of any negative real number as the product of a real number and the imaginary unit i. That is, $\sqrt{-b} = i\sqrt{b}$, where b is a positive real number.	Write $\sqrt{-48}$ in terms of i and simplify. **Solution** $\sqrt{-48} = \sqrt{-1}\sqrt{48}$ $= i\sqrt{16}\sqrt{3}$ $= 4i\sqrt{3}$
Add and subtract complex numbers. (Section 0.8/Objective 2)	We describe the addition and subtraction of complex numbers as follows: $(a + bi) + (c + di)$ $= (a + c) + (b + d)i$ $(a + bi) - (c + di)$ $= (a - c) + (b - d)i$	Add the complex numbers $(3 - 6i) + (-7 - 3i)$. **Solution** $(3 - 6i) + (-7 - 3i)$ $= (3 - 7) + (-6 - 3)i$ $= -4 - 9i$
Multiply and divide complex numbers. (Section 0.8/Objective 3)	The product of two complex numbers follows the same pattern as the product of two binomials. When simplifying replace any i^2 with -1. To simplify expressions that indicate the quotient of complex numbers, like $\frac{4 + 3i}{5 - 2i}$, multiply the numerator and denominator by the conjugate of the denominator. The **conjugate** of $a + bi$ is $a - bi$. The product of a complex number and its conjugate is a real number.	Find the quotient $\frac{2 + 3i}{4 - i}$ and express the answer in standard form of a complex number. **Solution** Multiply the numerator and denominator by $4 + i$, the conjugate of the denominator. $\frac{2 + 3i}{4 - i} = \frac{(2 + 3i)}{(4 - i)} \cdot \frac{(4 + i)}{(4 + i)}$ $= \frac{8 + 14i + 3i^2}{16 - i^2}$ $= \frac{8 + 14i + 3(-1)}{16 - (-1)}$ $= \frac{5 + 14i}{17} = \frac{5}{17} + \frac{14}{17}i$

Chapter 0 Review Problem Set

For Problems 1–10, evaluate.

1. 5^{-3}

2. -3^{-4}

3. $\left(\frac{3}{4}\right)^{-2}$

4. $\dfrac{1}{\left(\frac{1}{3}\right)^{-2}}$

5. $-\sqrt{64}$

6. $\sqrt[3]{\dfrac{27}{8}}$

7. $\sqrt[5]{-\dfrac{1}{32}}$

8. $36^{-1/2}$

9. $\left(\frac{1}{8}\right)^{-2/3}$

10. $-32^{3/5}$

For Problems 11–18, perform the indicated operations and simplify. Express the final answers using positive exponents only.

11. $(3x^{-2}y^{-1})(4x^4y^2)$

12. $(5x^{2/3})(-6x^{1/2})$

13. $(-8a^{-1/2})(-6a^{1/3})$

14. $(3x^{-2/3}y^{1/5})^3$

15. $\dfrac{64x^{-2}y^3}{16x^3y^{-2}}$

16. $\dfrac{56x^{-1/3}y^{2/5}}{7x^{1/4}y^{-3/5}}$

17. $\left(\dfrac{-8x^2y^{-1}}{2x^{-1}y^2}\right)^2$

18. $\left(\dfrac{36a^{-1}b^4}{-12a^2b^5}\right)^{-1}$

For Problems 19–34, perform the indicated operations.

19. $(-7x - 3) + (5x - 2) + (6x + 4)$

20. $(12x + 5) - (7x - 4) - (8x + 1)$

21. $3(a - 2) - 2(3a + 5) + 3(5a - 1)$

22. $(4x - 7)(5x + 6)$

23. $(-3x + 2)(4x - 3)$

24. $(7x - 3)(-5x + 1)$

25. $(x + 4)(x^2 - 3x - 7)$

26. $(2x + 1)(3x^2 - 2x + 6)$

27. $(5x - 3)^2$

28. $(3x + 7)^2$

29. $(2x - 1)^3$

30. $(3x + 5)^3$

31. $(x^2 - 2x - 3)(x^2 + 4x + 5)$

32. $(2x^2 - x - 2)(x^2 + 6x - 4)$

33. $\dfrac{24x^3y^4 - 48x^2y^3}{-6xy}$

34. $\dfrac{-56x^2y + 72x^3y^2}{8x^2}$

For Problems 35–46, factor each polynomial completely. Indicate any that are not factorable using integers.

35. $9x^2 - 4y^2$

36. $3x^3 - 9x^2 - 120x$

37. $4x^2 + 20x + 25$

38. $(x - y)^2 - 9$

39. $x^2 - 2x - xy + 2y$

40. $64x^3 - 27y^3$

41. $15x^2 - 14x - 8$

42. $3x^3 + 36$

43. $2x^2 - x - 8$

44. $3x^3 + 24$

45. $x^4 - 13x^2 + 36$

46. $4x^2 - 4x + 1 - y^2$

For Problems 47–56, perform the indicated operations involving rational expressions. Express final answers in simplest form.

47. $\dfrac{8xy}{18x^2y} \cdot \dfrac{24xy^2}{16y^3}$

48. $\dfrac{-14a^2b^2}{6b^3} \div \dfrac{21a}{15ab}$

49. $\dfrac{x^2 + 3x - 4}{x^2 - 1} \cdot \dfrac{3x^2 + 8x + 5}{x^2 + 4x}$

50. $\dfrac{9x^2 - 6x + 1}{2x^2 + 8} \cdot \dfrac{8x + 20}{6x^2 + 13x - 5}$

51. $\dfrac{3x - 2}{4} + \dfrac{5x - 1}{3}$

52. $\dfrac{2x - 6}{5} - \dfrac{x + 4}{3}$

53. $\dfrac{3}{n^2} + \dfrac{4}{5n} - \dfrac{2}{n}$

54. $\dfrac{5}{x^2 + 7x} - \dfrac{3}{x}$

55. $\dfrac{3x}{x^2 - 6x - 40} + \dfrac{4}{x^2 - 16}$

56. $\dfrac{2}{x-2} - \dfrac{2}{x+2} - \dfrac{4}{x^3-4x}$

For Problems 57– 59, simplify each complex fraction.

57. $\dfrac{\dfrac{3}{x} - \dfrac{2}{y}}{\dfrac{5}{x^2} + \dfrac{7}{y}}$

58. $\dfrac{3 - \dfrac{2}{x}}{4 + \dfrac{3}{x}}$

59. $\dfrac{\dfrac{3}{(x+h)^2} - \dfrac{3}{x^2}}{h}$

60. Simplify the expression

$$\frac{6(x^2+2)^{1/2} - 6x^2(x^2+2)^{-1/2}}{[(x^2+2)^{1/2}]^2}$$

For Problems 61–68, express each in simplest radical form. All variables represent positive real numbers.

61. $5\sqrt{48}$

62. $3\sqrt{24x^3}$

63. $\sqrt[3]{32x^4y^5}$

64. $\dfrac{3\sqrt{8}}{2\sqrt{6}}$

65. $\sqrt{\dfrac{5x}{2y^2}}$

66. $\dfrac{3}{\sqrt{2}+5}$

67. $\dfrac{4\sqrt{2}}{3\sqrt{2}+\sqrt{3}}$

68. $\dfrac{3\sqrt{x}}{\sqrt{x}-2\sqrt{y}}$

For Problems 69–74, perform the indicated operations and express the answers in simplest radical form.

69. $\sqrt{5}\sqrt[3]{5}$

70. $\sqrt[3]{x^2}\sqrt[4]{x}$

71. $\sqrt{x^3}\sqrt[3]{x^4}$

72. $\sqrt{xy}\sqrt[5]{x^3y^2}$

73. $\dfrac{\sqrt{5}}{\sqrt[3]{5}}$

74. $\dfrac{\sqrt[3]{x^2}}{\sqrt[4]{x^3}}$

For Problems 75–86, perform the indicated operations and express the resulting complex number in standard form.

75. $(-7+3i) + (-4-9i)$

76. $(2-10i) - (3-8i)$

77. $(-1+4i) - (-2+6i)$

78. $(3i)(-7i)$

79. $(2-5i)(3+4i)$

80. $(-3-i)(6-7i)$

81. $(4+2i)(-4-i)$

82. $(5-2i)(5+2i)$

83. $\dfrac{5}{3i}$

84. $\dfrac{2+3i}{3-4i}$

85. $\dfrac{-1-2i}{-2+i}$

86. $\dfrac{-6i}{5+2i}$

For Problems 87–92, write each in terms of i and simplify.

87. $\sqrt{-100}$

88. $\sqrt{-40}$

89. $4\sqrt{-80}$

90. $(\sqrt{-9})(\sqrt{-16})$

91. $(\sqrt{-6})(\sqrt{-8})$

92. $\dfrac{\sqrt{-24}}{\sqrt{-3}}$

For Problems 93 and 94, use scientific notation and the properties of exponents to help with the computations.

93. $\dfrac{(0.0064)(420{,}000)}{(0.00014)(0.032)}$

94. $\dfrac{(8600)(0.0000064)}{(0.0016)(0.000043)}$

Chapter 0 Test

1. Evaluate each of the following.

   a. -7^{-2} b. $\left(\frac{3}{2}\right)^{-3}$

   c. $\left(\frac{4}{9}\right)^{3/2}$ d. $\sqrt[3]{\frac{27}{64}}$

2. Find the product $(-3x^{-1}y^2)(5x^{-3}y^{-4})$ and express the result using positive exponents only.

For Problems 3–7, perform the indicated operations.

3. $(-3x - 4) - (7x - 5) + (-2x - 9)$
4. $(5x - 2)(-6x + 4)$
5. $(x + 2)(3x^2 - 2x - 7)$
6. $(4x - 1)^3$
7. $\dfrac{-18x^4y^3 - 24x^5y^4}{-2xy^2}$

For Problems 8–11, factor each polynomial completely.

8. $18x^3 - 15x^2 - 12x$
9. $30x^2 - 13x - 10$
10. $8x^3 + 64$
11. $x^2 + xy - 2y - 2x$

For Problems 12–16, perform the indicated operations involving rational expressions. Express final answers in simplest form.

12. $\dfrac{6x^3y^2}{5xy} \div \dfrac{8y}{7x^3}$
13. $\dfrac{x^2 - 4}{2x^2 + 5x + 2} \cdot \dfrac{2x^2 + 7x + 3}{x^3 - 8}$
14. $\dfrac{3n - 2}{4} - \dfrac{4n + 1}{6}$
15. $\dfrac{5}{2x^2 - 6x} + \dfrac{4}{3x^2 + 6x}$
16. $\dfrac{4}{n^2} - \dfrac{3}{2n} - \dfrac{5}{n}$
17. Simplify the complex fraction $\dfrac{\frac{2}{x} - \frac{5}{y}}{\frac{3}{x} + \frac{4}{y^2}}$.

For Problems 18–21, express each radical expression in simplest radical form. All variables represent positive real numbers.

18. $6\sqrt{28x^5}$
19. $\dfrac{5\sqrt{6}}{3\sqrt{12}}$
20. $\dfrac{\sqrt{6}}{2\sqrt{2} - \sqrt{3}}$
21. $\sqrt[3]{48x^4y^5}$

For Problems 22–25, perform the indicated operations and express the resulting complex numbers in standard form.

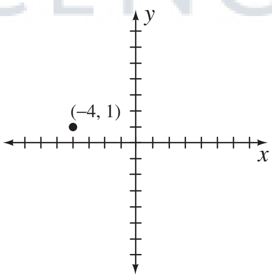
22. $(-2 - 4i) - (-1 + 6i) + (-3 + 7i)$
23. $(5 - 7i)(4 + 2i)$
24. $(7 - 6i)(7 + 6i)$
25. $\dfrac{1 + 2i}{3 - i}$

Answers to Odd-Numbered Problems and All Chapter Review, Chapter Test, and Cumulative Review Problems

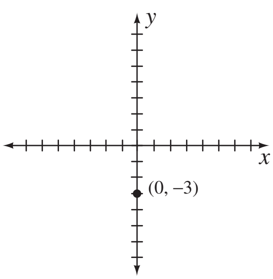
Chapter 0

Problem Set 0.1 (page 16)

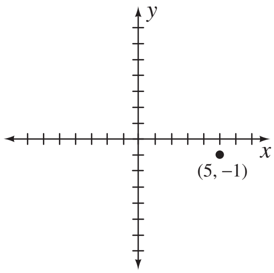
1. True **3.** False **5.** False **7.** True
9. False **11.** $\{46\}$ **13.** $\{0, -14, 46\}$
15. $\{\sqrt{5}, -\sqrt{2}, -\pi\}$ **17.** $\{0, -14\}$ **19.** $\subseteq$
21. $\subseteq$ **23.** $\not\subseteq$ **25.** $\subseteq$ **27.** $\not\subseteq$
29. $\subseteq$ **31.** $\not\subseteq$ **33.** $\{1\}$ **35.** $\{0, 1, 2, 3\}$
37. $\{\ldots, -2, -1, 0, 1\}$ **39.** $\varnothing$ **41.** $\{0, 1, 2\}$
43. **a.** 18 **c.** 39 **e.** 35
45. Commutative property of multiplication
47. Identity property of multiplication
49. Multiplication property of negative one
51. Distributive property
53. Commutative property of multiplication
55. Distributive property
57. Associative property of multiplication
59. -22 **61.** 100 **63.** -21 **65.** 8
67. 19 **69.** 66 **71.** -75 **73.** 34
75. 1 **77.** 11 **79.** 4
81.



83.



85.



87. Quadrant IV **89.** Quadrant III **91.** Quadrant I

Problem Set 0.2 (page 28)

1. $\frac{1}{8}$ **3.** $-\frac{1}{1000}$ **5.** 27 **7.** 4

9. $-\frac{27}{8}$ **11.** 1 **13.** $\frac{16}{25}$ **15.** 4

17. $\frac{1}{100}$ or 0.01 **19.** $\frac{1}{100{,}000}$ or 0.00001 **21.** 81

23. $\frac{1}{16}$ **25.** $\frac{3}{4}$ **27.** $\frac{256}{25}$ **29.** $\frac{16}{25}$

31. $\frac{64}{81}$ **33.** 64 **35.** $\frac{1}{100{,}000}$ or 0.00001

37. $\frac{17}{72}$ **39.** $\frac{1}{6}$ **41.** $\frac{48}{19}$ **43.** $\frac{1}{x^4}$

45. $\frac{1}{a^2}$ **47.** $\frac{1}{a^6}$ **49.** $\frac{y^4}{x^3}$ **51.** $\frac{c^3}{a^3b^6}$

53. $\frac{y^2}{4x^4}$ **55.** $\frac{x^4}{y^6}$ **57.** $\frac{9a^2}{4b^4}$ **59.** $\frac{1}{x^3}$

61. $\frac{a^3}{b}$ **63.** $-20x^4y^5$ **65.** $-27x^3y^9$

67. $\frac{8x^6}{27y^9}$ **69.** $-8x^6$ **71.** $\frac{6}{x^3y}$ **73.** $\frac{6}{a^2y^3}$

75. $\frac{4x^3}{y^5}$ **77.** $-\frac{5}{a^2b}$ **79.** $\frac{1}{4x^2y^4}$ **81.** $\frac{x+1}{x^2}$

83. $\dfrac{y - x^2}{x^2 y}$ **85.** $\dfrac{3b^3 + 2a^2}{a^2b^3}$ **87.** $\dfrac{y^2 - x^2}{xy}$
89. $12x^{3a+1}$ **91.** 1 **93.** x^{2a} **95.** $-4y^{6b+2}$
97. x^b **99.** $(6.2)(10)^7$ **101.** $(4.12)(10)^{-4}$
103. 180,000 **105.** 0.0000023 **107.** 0.04
109. 30,000 **111.** 0.03
117. a. $(4.385)(10^{14})$ **c.** $(2.322)(10^{17})$ **e.** $(3.052)(10^{12})$

Problem Set 0.3 (page 38)

1. $14x^2 + x - 6$ **3.** $-x^2 - 4x - 9$
5. $6x - 11$ **7.** $6x^2 - 5x - 7$ **9.** $-x - 34$
11. $12x^3y^2 + 15x^2y^3$ **13.** $30a^4b^3 - 24a^5b^3 + 18a^4b^4$
15. $x^2 + 20x + 96$ **17.** $n^2 - 16n + 48$
19. $sx + sy - tx - ty$ **21.** $6x^2 + 7x - 3$
23. $12x^2 - 37x + 21$ **25.** $x^2 + 8x + 16$
27. $4n^2 + 12n + 9$ **29.** $x^3 + x^2 - 14x - 24$
31. $6x^3 - x^2 - 11x + 6$ **33.** $x^3 + 2x^2 - 7x + 4$
35. $t^3 - 1$ **37.** $6x^3 + x^2 - 5x - 2$
39. $x^4 + 8x^3 + 15x^2 + 2x - 4$ **41.** $25x^2 - 4$
43. $x^4 - 10x^3 + 21x^2 + 20x + 4$ **45.** $4x^2 - 9y^2$
47. $x^3 + 15x^2 + 75x + 125$ **49.** $8x^3 + 12x^2 + 6x + 1$
51. $64x^3 - 144x^2 + 108x - 27$
53. $125x^3 - 150x^2y + 60xy^2 - 8y^3$
55. $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$
57. $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$
59. $x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$
61. $64a^6 - 192a^5b + 240a^4b^2 - 160a^3b^3 + 60a^2b^4 - 12ab^5 + b^6$
63. $x^{14} + 7x^{12}y + 21x^{10}y^2 + 35x^8y^3 + 35x^6y^4 + 21x^4y^5 + 7x^2y^6 + y^7$
65. $32a^5 - 240a^4b + 720a^3b^2 - 1080a^2b^3 + 810ab^4 - 243b^5$ **67.** $3x^2 - 5x$ **69.** $-5a^4 + 4a^2 - 9a$
71. $5ab + 11a^2b^4$ **73.** $x^{2a} - y^{2b}$
75. $x^{2b} - 3x^b - 28$ **77.** $6x^{2b} + x^b - 2$
79. $x^{4a} - 2x^{2a} + 1$ **81.** $x^{3a} - 6x^{2a} + 12x^a - 8$

Problem Set 0.4 (page 49)

1. $2xy(3 - 4y)$ **3.** $6x^2y^3z^2(2z^2 - x^2z + 1)$
5. $(z + 3)(x + y)$ **7.** $(x + y)(3 + a)$
9. $(x - y)(a - b)$ **11.** $(x + 5)(x - 5)$
13. $(1 + 9n)(1 - 9n)$ **15.** $(x + 4 + y)(x + 4 - y)$
17. $(3s + 2t - 1)(3s - 2t + 1)$ **19.** $(x - 7)(x + 2)$
21. $(5 + x)(3 - x)$ **23.** Not factorable
25. $(3x - 5)(x - 2)$ **27.** $(10x + 7)(x + 1)$
29. $(5x - 3)(2x + 9)$ **31.** $(6a - 1)^2$
33. $(4x - y)(2x + y)$ **35.** Not factorable
37. $(x - 2)(x^2 + 2x + 4)$
39. $(4x + 3y)(16x^2 - 12xy + 9y^2)$
41. $4(x^2 + 4)$ **43.** $x(x + 3)(x - 3)$
45. $(3a - 7)^2$ **47.** $2n(n^2 + 3n + 5)$
49. $2n(n^2 + 7n - 10)$ **51.** $4(x + 2)(x^2 - 2x + 4)$
53. $(x + 3)(x - 3)(x^2 + 5)$
55. $2y(x + 4)(x - 4)(x^2 + 3)$
57. $(a + b + c + d)(a + b - c - d)$
59. $(x + 4 + y)(x + 4 - y)$
61. $(x + y + 5)(x - y - 5)$ **63.** $(10x + 3)(6x - 5)$
65. $3x(7x - 4)(4x + 5)$ **67.** $(x^a + 4)(x^a - 4)$
69. $(x^n - y^n)(x^{2n} + x^ny^n + y^{2n})$
71. $(x^a + 4)(x^a - 7)$ **73.** $(2x^n - 5)(x^n + 6)$
75. $(x^{2n} + y^{2n})(x^n + y^n)(x^n - y^n)$
77. a. $(x + 32)(x + 3)$ **c.** $(x - 21)(x - 24)$
e. $(x + 28)(x + 32)$

Problem Set 0.5 (page 61)

1. $\dfrac{2x}{3}$ **3.** $\dfrac{7y^3}{9x}$ **5.** $\dfrac{8x^4y^4}{9}$ **7.** $\dfrac{a + 4}{a - 9}$
9. $\dfrac{x(2x + 7)}{y(x + 9)}$ **11.** $\dfrac{x^2 + xy + y^2}{x + 2y}$ **13.** $-\dfrac{2}{x + 1}$
15. $\dfrac{2x + y}{x - y}$ **17.** $\dfrac{9x^2 - 6xy + 4y^2}{x - 5}$ **19.** $\dfrac{x}{2y^3}$
21. $-\dfrac{8x^3y^3}{15}$ **23.** $\dfrac{14}{27a}$ **25.** $5y$
27. $\dfrac{5(a + 3)}{a(a - 2)}$ **29.** $\dfrac{(x + 6y)^2(2x + 3y)}{y^3(x + 4y)}$
31. $\dfrac{3xy}{4(x + 6)}$ **33.** $\dfrac{x - 9}{42x^2}$ **35.** $\dfrac{8x + 5}{12}$
37. $\dfrac{7x}{24}$ **39.** $\dfrac{35b + 12a^3}{80a^2b^2}$ **41.** $\dfrac{12 + 9n - 10n^2}{12n^2}$
43. $\dfrac{9y + 8x - 12xy}{12xy}$ **45.** $\dfrac{13x + 14}{(2x + 1)(3x + 4)}$
47. $\dfrac{7x + 21}{x(x + 7)}$ **49.** $\dfrac{1}{a - 2}$ **51.** $\dfrac{5}{2(x - 1)}$
53. $\dfrac{2n + 10}{3(n + 1)(n - 1)}$ **55.** $\dfrac{1}{x + 1}$
57. $\dfrac{9x + 73}{(x + 3)(x + 7)(x + 9)}$
59. $\dfrac{3x^2 + 30x - 78}{(x + 1)(x - 1)(x + 8)(x - 2)}$ **61.** $\dfrac{x + 6}{(x - 3)^2}$
63. $\dfrac{-x^2 - x + 1}{(x + 1)(x - 1)}$ **65.** $\dfrac{-8}{(n^2 + 4)(n + 2)(n - 2)}$
67. $\dfrac{5x^2 + 16x + 5}{(x + 1)(x - 4)(x + 7)}$
69. a. $\dfrac{5}{x - 1}$ **c.** $\dfrac{5}{a - 3}$ **e.** $x + 3$ **71.** $\dfrac{5y^2 - 3xy^2}{x^2y + 2x^2}$
73. $\dfrac{x + 1}{x - 1}$ **75.** $\dfrac{n - 1}{n + 1}$ **77.** $\dfrac{-6x - 4}{3x + 9}$

79. $\frac{x^2+x+1}{x+1}$ **81.** $\frac{a^2+4a+1}{4a+1}$ **83.** $-\frac{2x+h}{x^2(x+h)^2}$

85. $-\frac{1}{(x+1)(x+h+1)}$ **87.** $-\frac{4}{(2x-1)(2x+2h-1)}$

89. $\frac{y+2x}{x^2y-xy^2}$ **91.** $\frac{x^2y^2+2}{4y^2-3x}$

Problem Set 0.6 (page 73)

1. 9 **3.** 5 **5.** $\frac{6}{7}$ **7.** $-\frac{3}{2}$ **9.** $2\sqrt{6}$

11. $4\sqrt{7}$ **13.** $-6\sqrt{11}$ **15.** $\frac{3\sqrt{5}}{2}$

17. $2x\sqrt{3}$ **19.** $8x^2y^3\sqrt{y}$ **21.** $\frac{9y^3\sqrt{5x}}{7}$

23. $4\sqrt[3]{2}$ **25.** $2x\sqrt[3]{2x}$ **27.** $2x\sqrt[4]{3x}$

29. $\frac{2\sqrt{3}}{5}$ **31.** $\frac{\sqrt{14}}{4}$ **33.** $\frac{4\sqrt{15}}{5}$

35. $\frac{3\sqrt{2}}{7}$ **37.** $\frac{\sqrt{15}}{6x^2}$ **39.** $\frac{2\sqrt{15a}}{5ab}$

41. $\frac{3\sqrt[3]{2}}{2}$ **43.** $\frac{\sqrt[3]{18x^2y}}{3x}$ **45.** $12\sqrt{3}$ **47.** $3\sqrt{7}$

49. $\frac{11\sqrt{3}}{6}$ **51.** $-\frac{89\sqrt{2}}{30}$ **53.** $48\sqrt{6}$

55. $10\sqrt{6}+8\sqrt{30}$ **57.** $3x\sqrt{6y}-6\sqrt{2xy}$

59. $13+7\sqrt{3}$ **61.** $30+11\sqrt{6}$ **63.** 16

65. $x+2\sqrt{xy}+y$ **67.** $a-b$ **69.** $3\sqrt{5}-6$

71. $\sqrt{7}+\sqrt{3}$ **73.** $\frac{-2\sqrt{10}+3\sqrt{14}}{43}$

75. $\frac{x+\sqrt{x}}{x-1}$ **77.** $\frac{x-\sqrt{xy}}{x-y}$ **79.** $\frac{6x+7\sqrt{xy}+2y}{9x-4y}$

81. $\frac{2}{\sqrt{2x+2h}+\sqrt{2x}}$ **83.** $\frac{1}{\sqrt{x+h-3}+\sqrt{x-3}}$

91. $4x^2$ **93.** $y^2\sqrt{3y}$ **95.** $2m^4\sqrt{7}$

97. $3d^3\sqrt{2d}$ **99.** $4n^{10}\sqrt{5}$

Problem Set 0.7 (page 79)

1. 7 **3.** 8 **5.** -4 **7.** 2 **9.** 64

11. 0.001 **13.** $\frac{1}{32}$ **15.** 2 **17.** $15x^{7/12}$

19. $y^{5/12}$ **21.** $64x^{3/4}y^{3/2}$ **23.** $4x^{4/15}$

25. $\frac{7}{a^{1/12}}$ **27.** $\frac{16x^{4/3}}{81y}$ **29.** $\frac{y^{3/2}}{x}$ **31.** $8a^{9/2}x^2$

33. $\sqrt[4]{8}$ **35.** $\sqrt[12]{x^7}$ **37.** $xy\sqrt[4]{xy^3}$ **39.** $a\sqrt[12]{a^5b^{11}}$

41. $4\sqrt[6]{2}$ **43.** $\sqrt[6]{2}$ **45.** $\sqrt{2}$ **47.** $x\sqrt[12]{x^7}$

49. $\frac{5\sqrt[3]{x^2}}{x}$ **51.** $\frac{\sqrt[6]{x^3y^4}}{y}$ **53.** $\frac{\sqrt[20]{x^{15}y^8}}{y}$

55. $\frac{5\sqrt[12]{x^9y^8}}{4x}$ **57. a.** $\sqrt[6]{2}$ **c.** $\sqrt{x}$

61. $\frac{2x-2}{(2x-1)^{3/2}}$ **63.** $\frac{x}{(x^2+2x)^{3/2}}$ **65.** $\frac{4x}{(2x)^{4/3}}$

69. a. 13.391 **c.** 2.702 **e.** 4.304

Problem Set 0.8 (page 88)

1. $13+8i$ **3.** $3+4i$ **5.** $-11+i$ **7.** $-1-2i$

9. $-\frac{3}{20}+\frac{5}{12}i$ **11.** $\frac{7}{10}-\frac{11}{12}i$ **13.** $4+0i$

15. $3i$ **17.** $i\sqrt{19}$ **19.** $\frac{2}{3}i$ **21.** $2i\sqrt{2}$

23. $3i\sqrt{3}$ **25.** $3i\sqrt{6}$ **27.** $18i$ **29.** $12i\sqrt{2}$

31. $-2-i\sqrt{3}$ **33.** $-1-i\sqrt{2}$ **35.** $\frac{4+i\sqrt{5}}{2}$

37. -8 **39.** $-\sqrt{6}$ **41.** $-2\sqrt{5}$ **43.** $-2\sqrt{15}$

45. $-2\sqrt{14}$ **47.** 3 **49.** $\sqrt{6}$ **51.** $-21+0i$

53. $8+12i$ **55.** $0+26i$ **57.** $53-26i$

59. $10-24i$ **61.** $-14-8i$ **63.** $-7+24i$

65. $-3+4i$ **67.** $113+0i$ **69.** $13+0i$

71. $-\frac{8}{13}+\frac{12}{13}i$ **73.** $1-\frac{2}{3}i$ **75.** $0-\frac{3}{2}i$

77. $\frac{22}{41}-\frac{7}{41}i$ **79.** $-1+2i$ **81.** $-\frac{17}{10}+\frac{1}{10}i$

83. $\frac{5}{13}-\frac{1}{13}i$ **89. a.** $2+11i$ **c.** $-11+2i$ **e.** $-7-24i$
b. $-2-2i$ **d.** $-4+0i$ **f.** $4-4i$

Chapter 0 Review Problem Set (page 100)

1. $\frac{1}{125}$ **2.** $-\frac{1}{81}$ **3.** $\frac{16}{9}$ **4.** $\frac{1}{9}$ **5.** -8 **6.** $\frac{3}{2}$

7. $-\frac{1}{2}$ **8.** $\frac{1}{6}$ **9.** 4 **10.** -8 **11.** $12x^2y$

12. $-30x^{7/6}$ **13.** $\frac{48}{a^{1/6}}$ **14.** $\frac{27y^{3/5}}{x^2}$ **15.** $\frac{4y^5}{x^5}$

16. $\frac{8y}{x^{7/12}}$ **17.** $\frac{16x^6}{y^6}$ **18.** $-\frac{a^3b^1}{3}$ **19.** $4x-1$

20. $-3x+8$ **21.** $12a-19$ **22.** $20x^2-11x-42$

23. $-12x^2+17x-6$ **24.** $-35x^2+22x-3$

25. $x^3+x^2-19x-28$ **26.** $6x^3-x^2+10x+6$

27. $25x^2-30x+9$ **28.** $9x^2+42x+49$

29. $8x^3-12x^2+6x-1$

30. $27x^3+135x^2+225x+125$

31. $x^4 + 2x^3 - 6x^2 - 22x - 15$
32. $2x^4 + 11x^3 - 16x^2 - 8x + 8$ **33.** $-4x^2y^3 + 8xy^2$
34. $-7y + 9xy^2$ **35.** $(3x + 2y)(3x - 2y)$
36. $3x(x + 5)(x - 8)$ **37.** $(2x + 5)^2$
38. $(x - y + 3)(x - y - 3)$ **39.** $(x - 2)(x - y)$
40. $(4x - 3y)(16x^2 + 12xy + 9y^2)$
41. $(3x - 4)(5x + 2)$ **42.** $3(x^3 + 12)$
43. Not factorable **44.** $3(x + 2)(x^2 - 2x + 4)$
45. $(x + 3)(x - 3)(x + 2)(x - 2)$
46. $(2x - 1 - y)(2x - 1 + y)$ **47.** $\frac{2}{3y}$ **48.** $\frac{-5a^2}{3}$
49. $\frac{3x + 5}{x}$ **50.** $\frac{2(3x - 1)}{x^2 + 4}$ **51.** $\frac{29x - 10}{12}$
52. $\frac{x - 38}{15}$ **53.** $\frac{-6n + 15}{5n^2}$ **54.** $\frac{-3x - 16}{x(x + 7)}$
55. $\frac{3x^2 - 8x - 40}{(x + 4)(x - 4)(x - 10)}$ **56.** $\frac{8x - 4}{x(x + 2)(x - 2)}$
57. $\frac{3xy - 2x^2}{5y + 7x^2}$ **58.** $\frac{3x - 2}{4x + 3}$ **59.** $-\frac{6x + 3h}{x^2(x + h)^2}$
60. $\frac{12}{(x^2 + 2)^{3/2}}$ **61.** $20\sqrt{3}$ **62.** $6x\sqrt{6x}$
63. $2xy\sqrt[3]{4xy^2}$ **64.** $\sqrt{3}$ **65.** $\frac{\sqrt{10x}}{2y}$
66. $\frac{15 - 3\sqrt{2}}{23}$ **67.** $\frac{24 - 4\sqrt{6}}{15}$ **68.** $\frac{3x + 6\sqrt{xy}}{x - 4y}$
69. $\sqrt[6]{5^5}$ **70.** $\sqrt[12]{x^{11}}$ **71.** $x^2\sqrt[6]{x^5}$
72. $x\sqrt[10]{xy^9}$ **73.** $\sqrt[6]{5}$ **74.** $\frac{\sqrt[12]{x^{11}}}{x}$ **75.** $-11 - 6i$
76. $-1 - 2i$ **77.** $1 - 2i$ **78.** $21 + 0i$
79. $26 - 7i$ **80.** $-25 + 15i$ **81.** $-14 - 12i$
82. $29 + 0i$ **83.** $0 - \frac{5}{3}i$ **84.** $-\frac{6}{25} + \frac{17}{25}i$
85. $0 + i$ **86.** $-\frac{12}{29} - \frac{30}{29}i$ **87.** $10i$
88. $2i\sqrt{10}$ **89.** $16i\sqrt{5}$ **90.** -12
91. $-4\sqrt{3}$ **92.** $2\sqrt{2}$
93. 600,000,000 **94.** 800,000

Chapter 0 Test (page 102)

1. a. $-\frac{1}{49}$ **b.** $\frac{8}{27}$ **c.** $\frac{8}{27}$ **d.** $\frac{3}{4}$ **2.** $-\frac{15}{x^4y^2}$
3. $-12x - 8$ **4.** $-30x^2 + 32x - 8$
5. $3x^3 + 4x^2 - 11x - 14$ **6.** $64x^3 - 48x^2 + 12x - 1$
7. $9x^3y + 12x^4y^2$ **8.** $3x(2x + 1)(3x - 4)$
9. $(5x + 2)(6x - 5)$ **10.** $8(x + 2)(x^2 - 2x + 4)$
11. $(x - 2)(x + y)$ **12.** $\frac{21x^5}{20}$ **13.** $\frac{x + 3}{x^2 + 2x + 4}$
14. $\frac{n - 8}{12}$ **15.** $\frac{23x + 6}{6x(x - 3)(x + 2)}$ **16.** $\frac{8 - 13n}{2n^2}$
17. $\frac{2y^2 - 5xy}{3y^2 + 4x}$ **18.** $12x^2\sqrt{7x}$ **19.** $\frac{5\sqrt{2}}{6}$
20. $\frac{4\sqrt{3} + 3\sqrt{2}}{5}$ **21.** $2xy\sqrt[3]{6xy^2}$ **22.** $-4 - 3i$
23. $34 - 18i$ **24.** $85 + 0i$ **25.** $\frac{1}{10} + \frac{7}{10}i$

This page contains answers for this chapter only