Simple and Multiple Linear Regression

Fatemeh Asgarinejad

February 6, 2025

Announcements + schedule + links

- Lectures

  Tuesday/Thursday 2:30PM - 3:45PM. Keck center 130
- Professor Wen’s Office Hours

  Tuesday 10:30 am- 12:00 pm, Swenson Hall, Room N206
- GitHub (code + slides) | Google Colab
  - Slides: https://github.com/Fasgarinejad/Machine-Learning or scan

- Announcements: Quiz 1 will be on Monday February 11th

SCAN ME

Course Plan: Where We Are Now

Week	Date	Topic	Activities
1	Tuesday – Feb 4	Introduction to AI	
	Thursday – Feb 6	Linear regression	
2	Tuesday – Feb 11	Linear classifier	Quiz 1 – Feb 11
	Thursday – Feb 13		
3	Tuesday – Feb 18		
	Thursday – Feb 20		
4	Tuesday – Feb 25		Quiz 2– Feb 25
	Thursday – Feb 27		
5	Tuesday – March 4		
	Thursday – March 6		
6	Tuesday – March 11		Quiz 3– March 13
	Thursday – March 13		
7	Tuesday – March 18		Review-March 18
	Thursday – March 20		Exam1–March 20
8	Tuesday – March 25	Spring Break	
	Thursday – March 27		
9	Tuesday – April 1		
	Thursday – April 3		

Week	Date	Topic	Activities
10	Tuesday – April 8		Quiz 4– April 8
	Thursday – April 10		
11	Tuesday – April 15		
	Thursday – April 17		
12	Tuesday – April 22		Quiz 5– April 22
	Thursday – April 24		
13	Tuesday – April 29		
	Thursday – May 1		
14	Tuesday – May 6		Quiz 6– May 6
	Thursday – May 8		
15	Tuesday – May 13		
	Thursday – May 15		
16	Final		

Today's Learning Goals: Linear Regression

- Introduction to Linear Regression
- Simple Linear Regression
- Fitting the Linear Model
  - Parametrizing a Line
  - The Line Fitting Problem
- Model Evaluation:
  - Minimizing MSE Loss Function
  - R-Squared
- Multiple Linear Regression
- Regularization
- Assumptions in Linear Regression

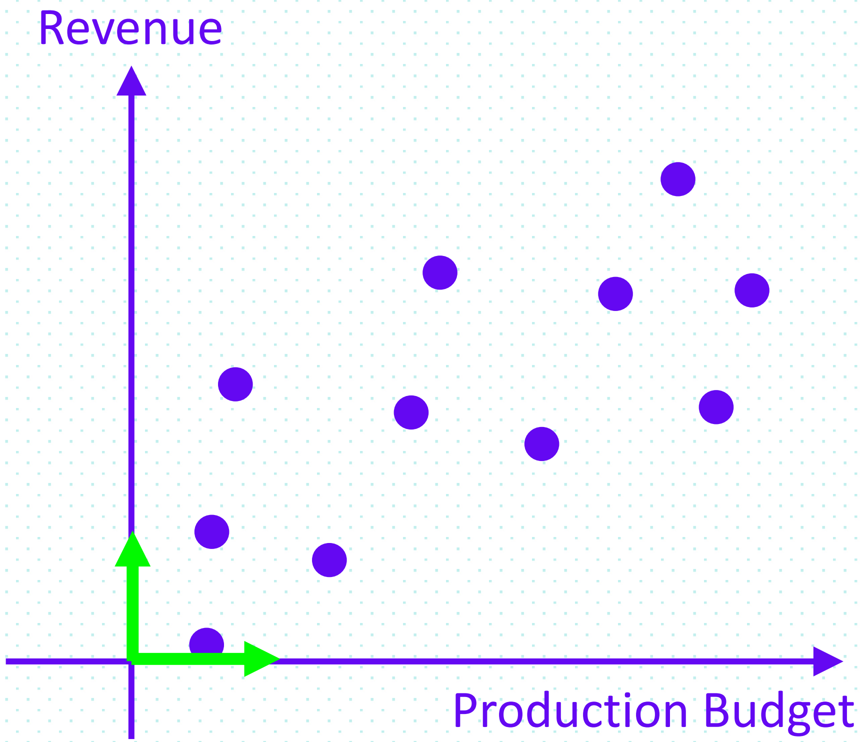
Data and Python Libraries

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
from sklearn.linear_model import LinearRegression
from sklearn.model_selection import train_test_split
from sklearn.metrics import mean_squared_error
```

Data is from THE NUMBERS

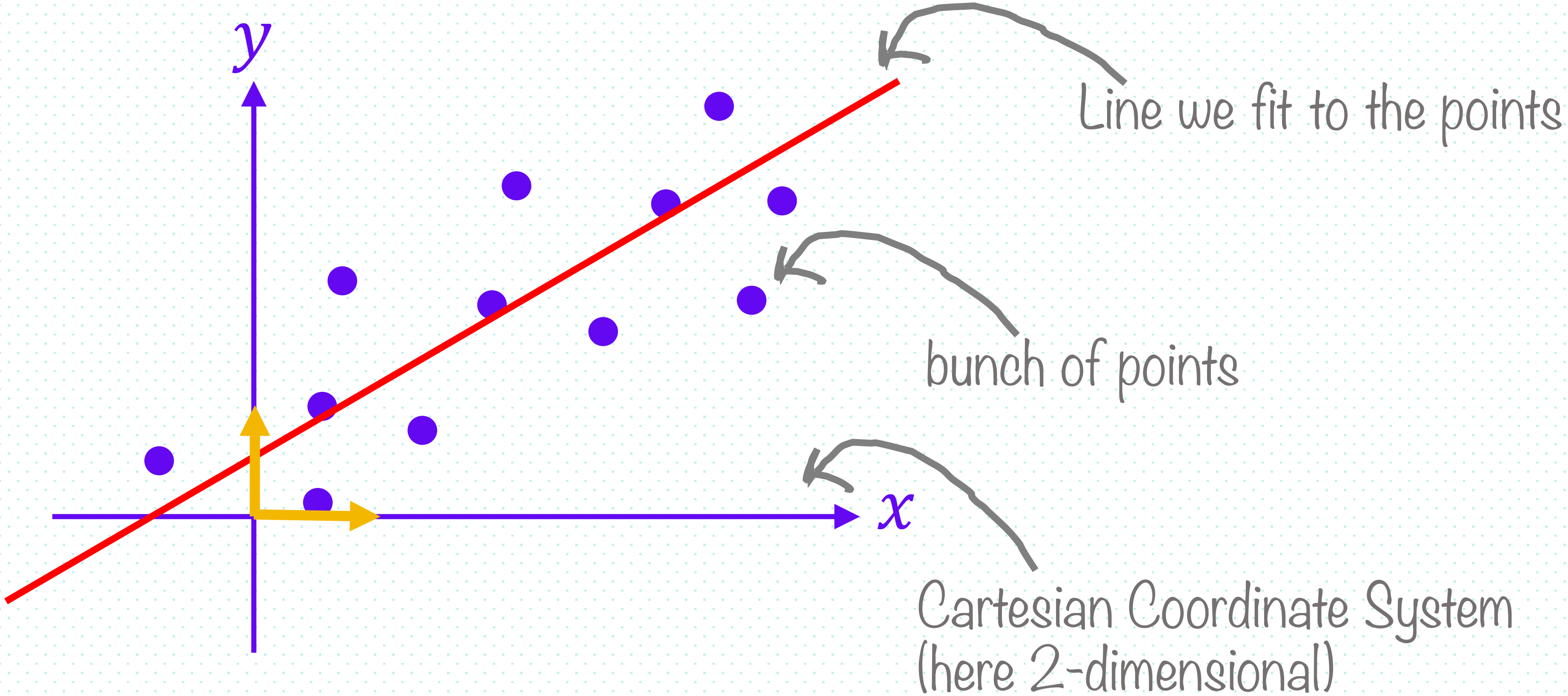
the-numbers.com

Can the Production Budget for movies be used to predict what their revenue will be like?



Linear Regression

Linear Regression is all about fitting a line to a bunch of points.



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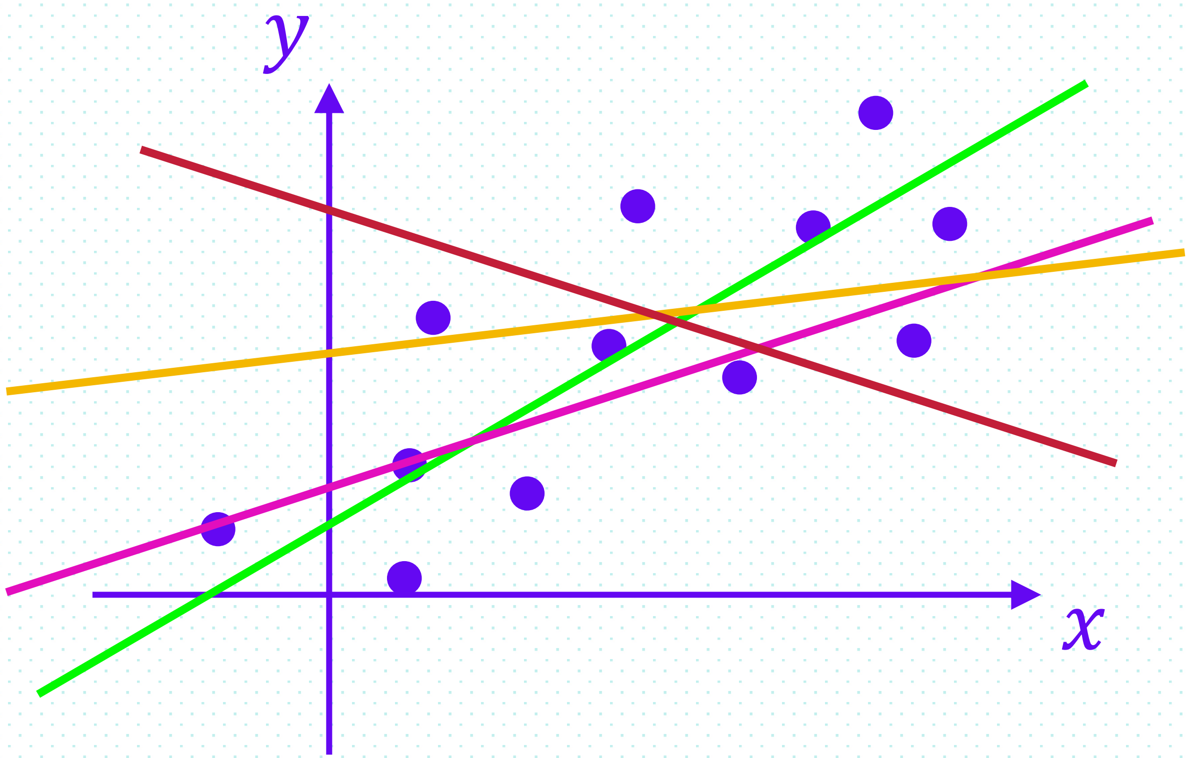

What are the x-axis and y-axis in the cartesian coordinate system?

What these points represent?

How to fit a line? And why should I care about fitting a line?

How to know the fitted line is good? And what it means to have a good line.

We want to do prediction: example (Movie Revenue)

	Release Date	Movie	Worldwide Revenue
1	Dec 16, 2015	Star Wars Ep. VII: The Force Awakens	$2,056,046,835
2	Dec 9, 2022	Avatar: The Way of Water	$2,315,589,775
3	Jun 28, 2023	Indiana Jones and the Dial of Destiny	$383,963,057
4	Apr 23, 2019	Avengers: Endgame	$2,748,242,781
5	May 21, 2025	Mission: Impossible—The Final Reckoning	
6	May 20, 2011	Pirates of the Caribbean: On Stranger Tides	$1,045,713,802
7	Apr 22, 2015	Avengers: Age of Ultron	$1,395,316,979
8	May 17, 2023	Fast X	$714,375,114
9	May 23, 2018	Solo: A Star Wars Story	$393,151,347
10	Apr 25, 2018	Avengers: Infinity War	$2,048,359,754
11	May 24, 2007	Pirates of the Caribbean: At World's End	$960,996,492
12	Nov 13, 2017	Justice League	$655,945,209
13	Jul 11, 2023	Mission: Impossible Dead Reckoning Part One	$566,644,143
14	Dec 14, 2016	Rogue One: A Star Wars Story	$1,055,083,596
15	Dec 18, 2019	Star Wars: The Rise of Skywalker	

Predict the worldwide Revenue for "Star Wars: The rise of Skywalker" movie

Without further information, we might predict the **mean** `617,650,554.57`.

What is the error of this prediction?

$$error = actual\ revenue\ - predicted\ revenue$$

What is the squared error of this prediction?

$$squared\ error = error^2$$
$$= (actual\ revenue\ - predicted\ revenue)^2$$

$$squared\ error$$
$$= (1{,}069{,}951{,}814 - 617{,}650{,}554.57)^2$$
$$= 1.8769e{+}16$$

data from: https://www.the-numbers.com/movie/budgets/all

Better predictions with more information

	Release Date	Movie	Production Budget	Worldwide Revenue
1	Dec 16, 2015	Star Wars Ep. VII: The Force Awakens	$533,200,000	$2,056,046,835
2	Dec 9, 2022	Avatar: The Way of Water	$460,000,000	$2,315,589,775
3	Jun 28, 2023	Indiana Jones and the Dial of Destiny	$402,300,000	$383,963,057
4	Apr 23, 2019	Avengers: Endgame	$400,000,000	$2,748,242,781
5	May 21, 2025	Mission: Impossible—The Final Reckoning	$400,000,000	
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8	May 17, 2023	Fast X	$340,000,000	$714,375,114
9	May 23, 2018	Solo: A Star Wars Story	$330,400,000	$393,151,347
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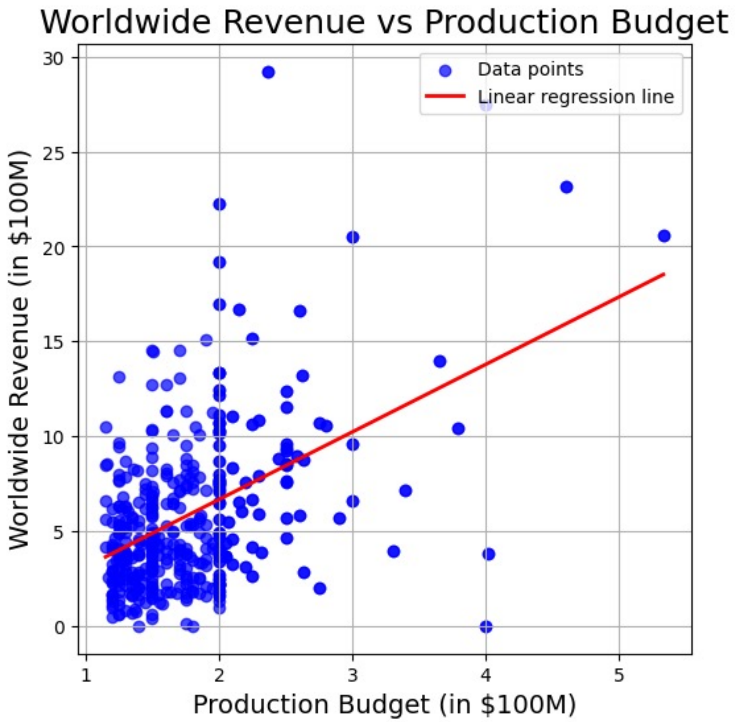
12	Nov 13, 2017	Justice League	$300,000,000	$655,945,209
13	Jul 11, 2023	Mission: Impossible Dead Reckoning Part On	$290,000,000	$566,644,143
14	Dec 14, 2016	Rogue One: A Star Wars Story	$280,200,000	$1,055,083,596
15	Dec 18, 2019	Star Wars: The Rise of Skywalker	$275,000,000	

We also get Production Budget of movies



This is a **regression problem** with:

Predictor variable: Production Budget

Response variable: Worldwide Revenue

data from: https://www.the-numbers.com/movie/budgets/all

Better predictions with more information

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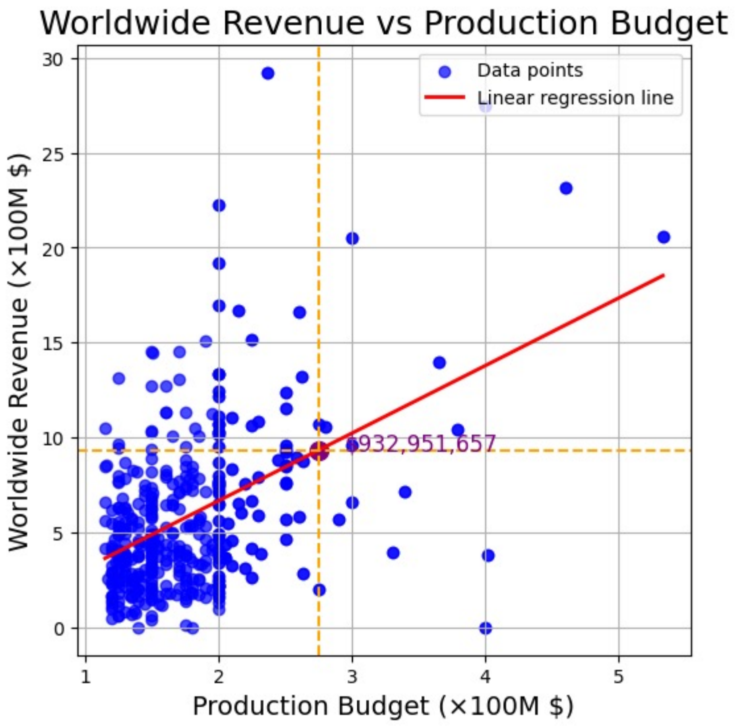
Prediction using linear regression



$$\textit{squared error} = (1{,}069{,}951{,}814 - 932{,}951{,}657)^2 = 1.8769043\text{e+}16 \quad \ll \; 2.0457643\text{e+}17$$

Better predictions with more information

	Release Date	Movie	Production Budget	Worldwide Revenue
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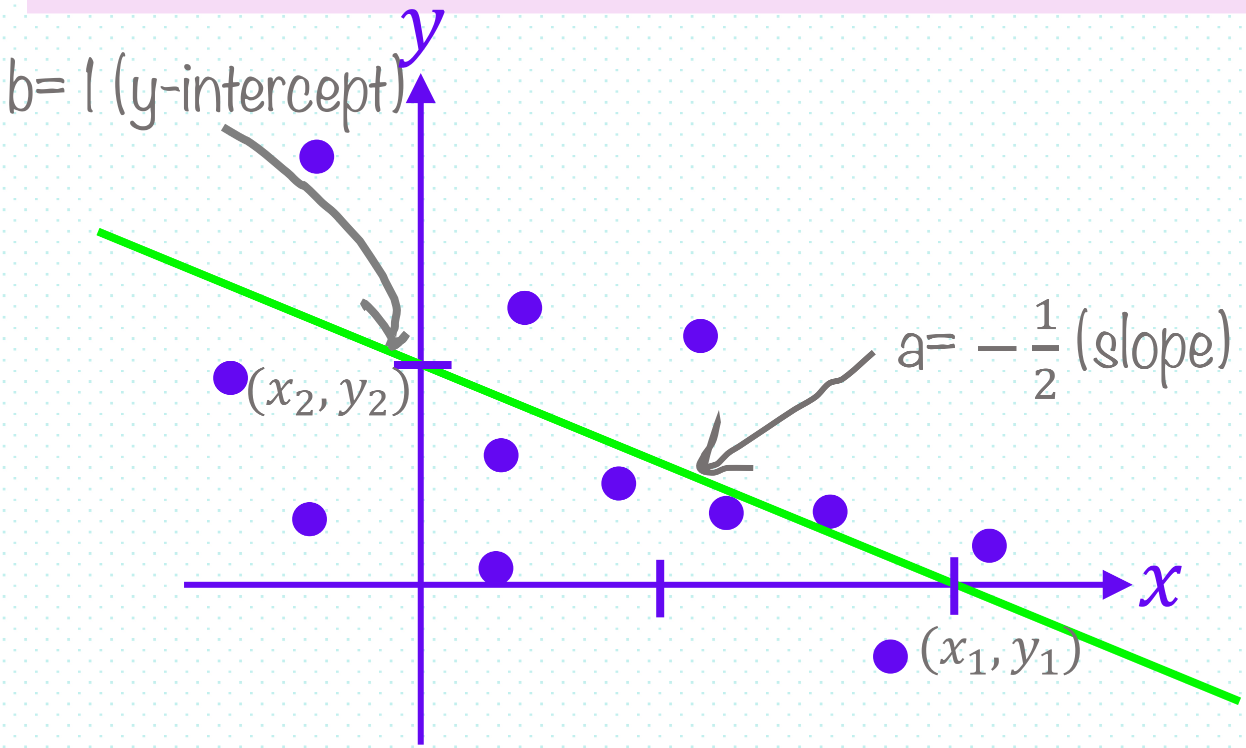
Parametrizing a Line

A line can be parametrized as $\boldsymbol{y = ax + b}$ $(a\!: slope, b\!: intercept)$

What is the corresponding equation for the following line?



Note that slope = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{0-1}{2-0} = -\frac{1}{2}$

$$\boldsymbol{y = ax + b}\ (a\!: slope, b\!: intercept)$$

$$\boldsymbol{y = -\frac{1}{2}x + 1}$$

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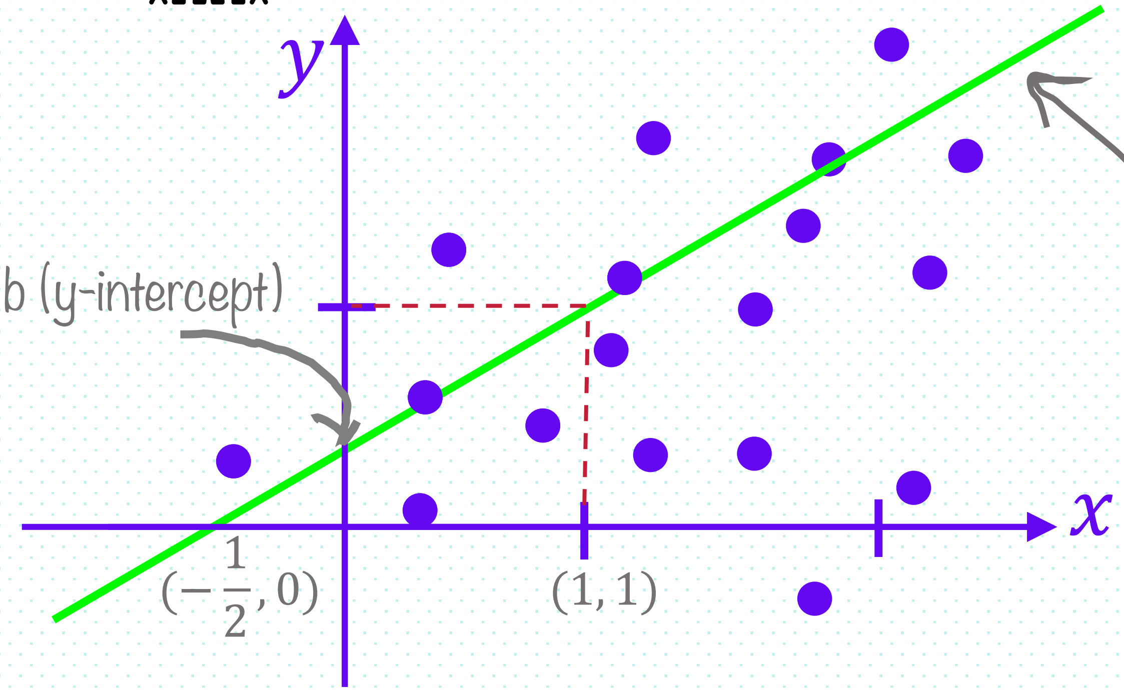
What is the corresponding equation for the following line?



a= $\frac{2}{3}$

$$a = \text{slope} = \frac{2}{3} = \frac{y_{intercept} - 1}{0-1} \rightarrow b = \frac{1}{3}$$

$$\boldsymbol{y = ax + b}\ (a{:}\, slope, b{:}\, intercept)$$

$$\boldsymbol{y = \frac{2}{3}x + \frac{1}{3}}$$

Simple Linear Regression

Linear Regression is all about fitting a line to a bunch of points.

$$\hat{y} = w_1 x_1 + w_0$$

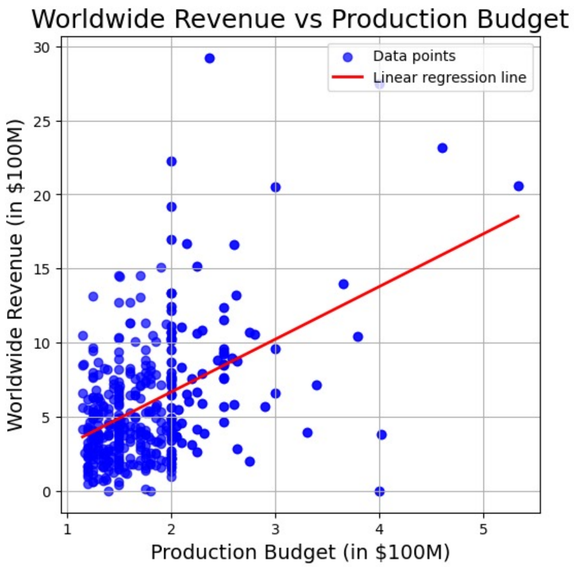
Dependent variable (y-hat)

Slope coefficient

Independent variable

y-intercept

$$\textbf{Worldwide Gross} = 3.5613 \times \textbf{Production Budget} - 4.631763366$$



Parametrizing a Line

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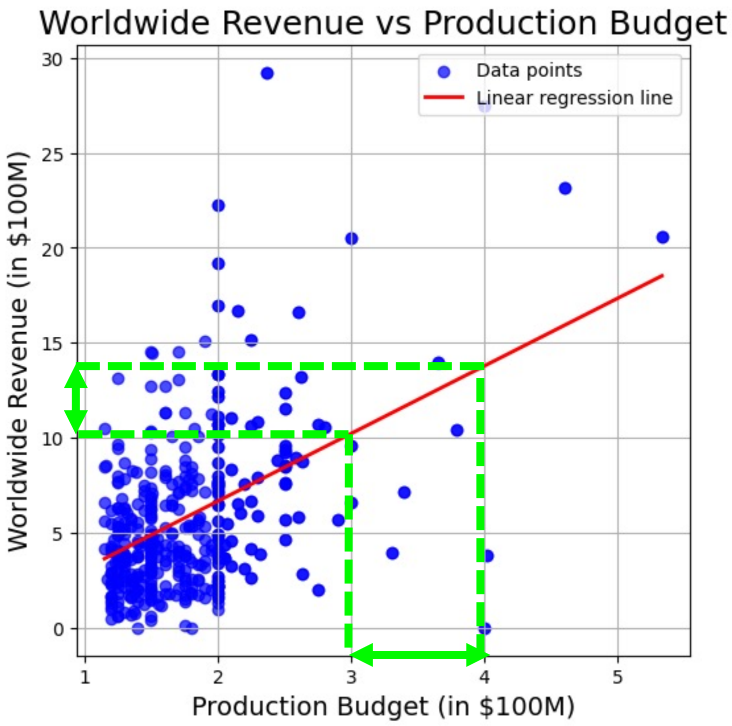
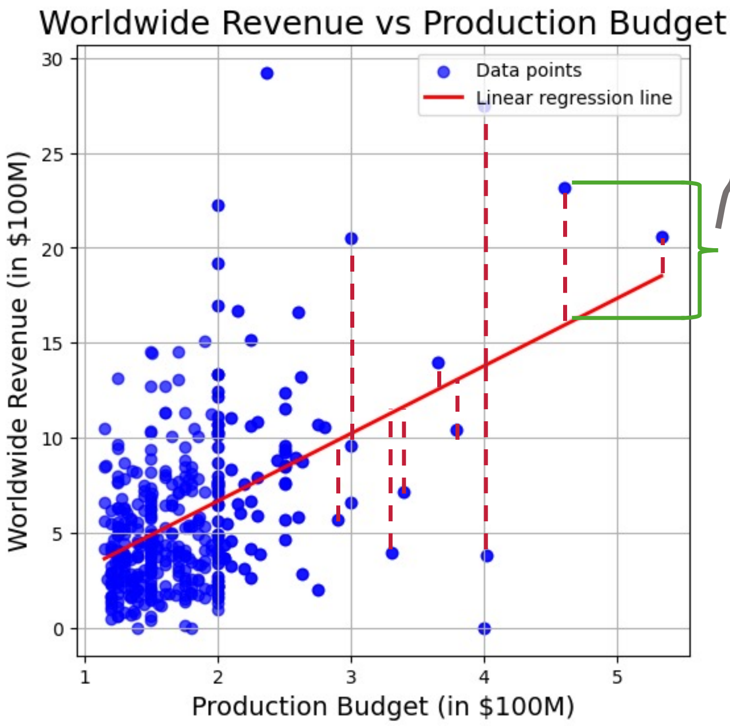


The above linear equation means that, for any unit increase in Production Budget, there will be 3.5613 unit increase in Worldwide Revenue

The Line Fitting Problem

What is a good line $\hat{\boldsymbol{y}} = \boldsymbol{w_1} \times \boldsymbol{x_1} + \boldsymbol{w_0}$ to fit to the data?



residual (error): $\epsilon_{data\ i} = y_{data\ i} - \widehat{y_{data\ i}}$

We want b_0 and b_1 in $\hat{\boldsymbol{y}} = \boldsymbol{w_1} \times \boldsymbol{x_1} + \boldsymbol{w_0}$ such that the residual^2 $(squared\ error) = (y_{data\ i} - \widehat{y_{data\ i}})^2$ is minimized.

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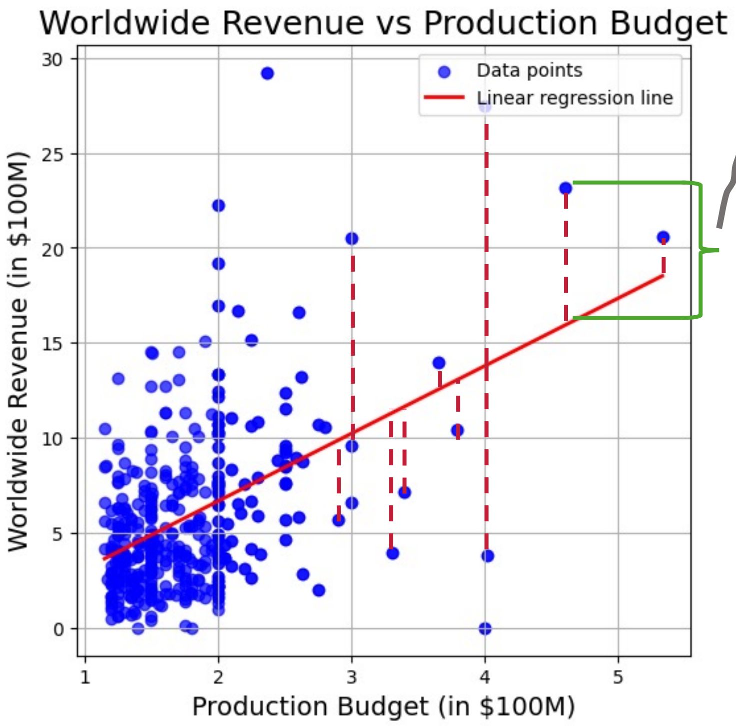

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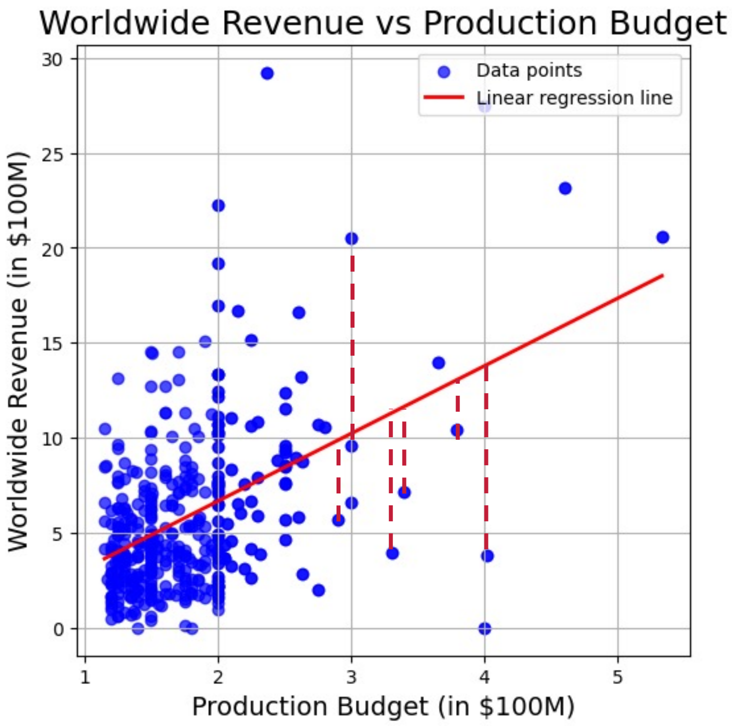
And we have n points, so we average residual^2$(squared\ error)s$

$$Mean\ Squared\ Error\ (MSE) = \frac{1}{n}\sum_{i=1}^{n}(y_{data\ i} - \widehat{y_{data\ i}})^2$$

Why are we squaring residuals?

The Line Fitting Problem

What is a good line $\hat{\boldsymbol{y}} = \boldsymbol{w_1} \times \ \boldsymbol{x_1} + \boldsymbol{w_0}$ to fit to the data?



$$Mean\ Squared\ Error\ (MSE) = \frac{1}{n}\sum_{i=1}^{n}(y_{data\ i} - \widehat{y_{data\ i}})^2$$

We choose MSE as our Loss Function and our goal is to minimize it.

Loss Function measures the difference between a model's predicted output and the actual target and evaluates how well our model fits our dataset.

Minimizing MSE Loss Function

$$Mean\ Squared\ Error\ (MSE) = \frac{1}{n}\sum_{i=1}^{n}(y_{data\ i} - \widehat{y_{data\ i}})^2$$

We choose MSE as our Loss Function and our goal is to minimize it.

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(y_{data\ i} - \widehat{y_{data\ i}})^2 = \frac{1}{n}\sum_{i=1}^{n}\left(y_{data\ i} - (w_1 x_1^i + w_0)\right)^2 \quad Remember\ \ \hat{\boldsymbol{y}} = \boldsymbol{w_1} \times \boldsymbol{x_1} + \boldsymbol{w_0}$$

To minimize MSE (Loss **L**), Compute the gradient (vector of partial derivatives, set it to zero, and solve:

$$\frac{d\boldsymbol{L}}{dw_1} = \frac{d\boldsymbol{L}}{dw_0} = 0$$

$$Remember\ \frac{du^2}{dx} = 2u\frac{du}{dx}$$

Minimizing Loss Function

$$Mean\ Squared\ Error\ (MSE) = \frac{1}{n}\sum_{i=1}^{n}(y_{data\ i} - \widehat{y_{data\ i}})^2 = \frac{1}{n}\sum_{i=1}^{n}\left(y_{data\ i} - (w_1x_1^i + w_0)\right)^2$$

To minimize MSE (Loss **L**), compute the gradient (vector of partial derivatives, set it to zero, and solve:

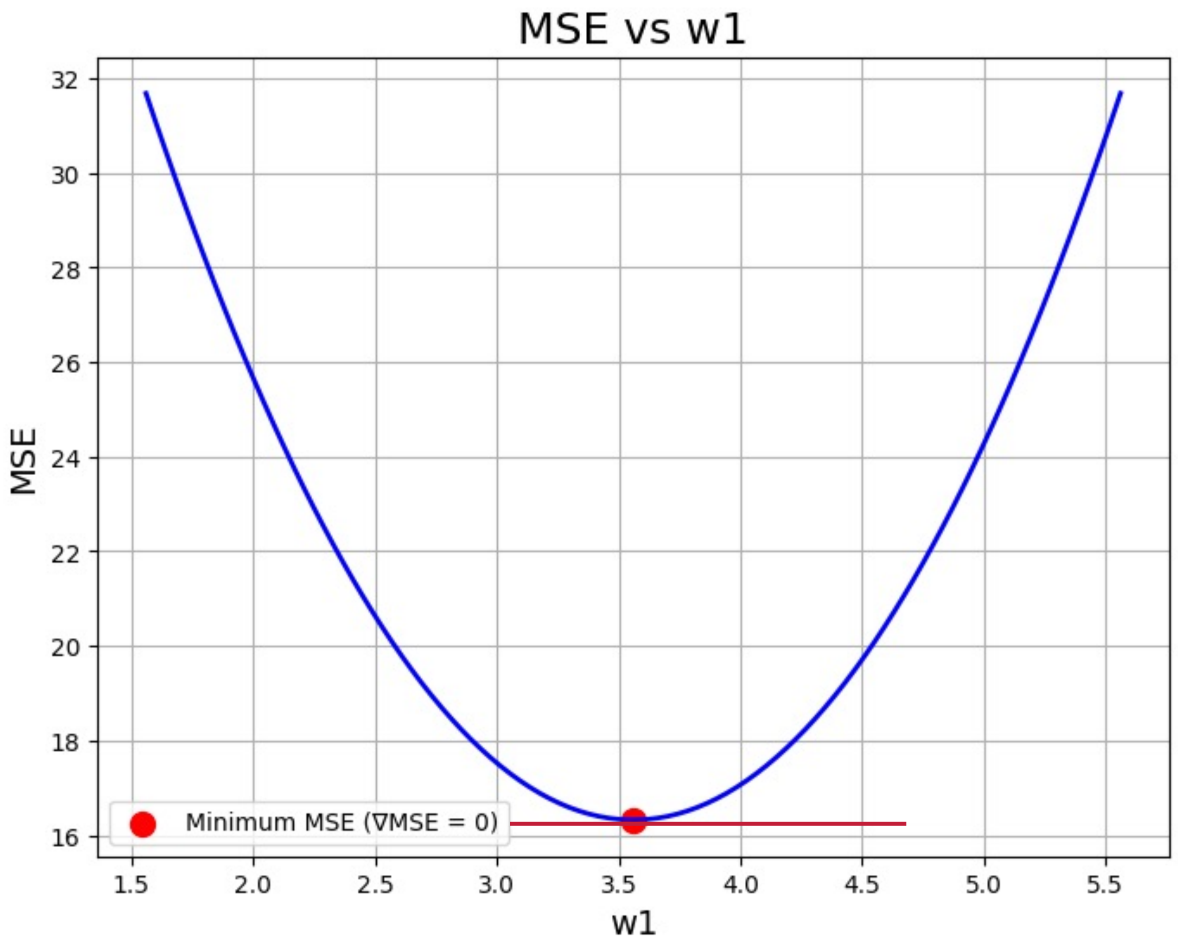
$$\frac{d\boldsymbol{L}}{dw_1} = \frac{d\boldsymbol{L}}{dw_0} = 0$$

$$\frac{d\boldsymbol{L}}{dw_0} = \frac{2}{n}\sum_{i=1}^{n}\left(y_i - (w_1x_1^i + w_0)\right)\times(-1) = 0 \quad \rightarrow \sum_{i=1}^{n} y_i = w_1\times\sum_{i=1}^{n} x_i + nw_o \quad \rightarrow w_0 = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{1}{n}w_1\times\sum_{i=1}^{n} x_i$$

$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ $\rightarrow \boldsymbol{w_0} = \bar{\boldsymbol{y}} - \mathbf{w_1}\,\bar{\boldsymbol{x}}$ (**best intercept** we can get in linear regression)

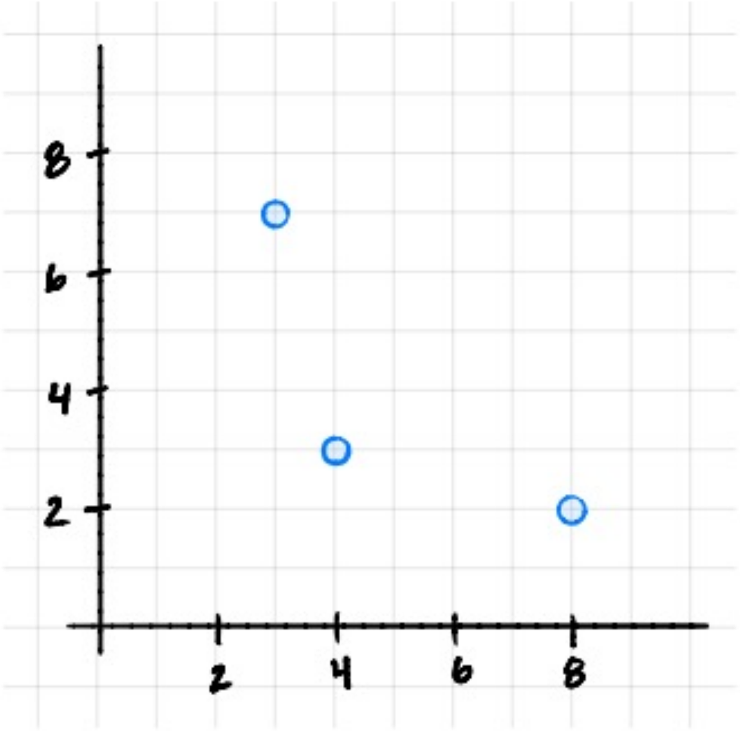
$\frac{d\boldsymbol{L}}{dw_1} = 0 \ \rightarrow\ w_1 = \frac{\sum_{i=1}^{n}(\boldsymbol{x_i} - \bar{\boldsymbol{x}})(\boldsymbol{y_i} - \bar{\boldsymbol{y}})}{\sum_{i=1}^{n}(\boldsymbol{x_i} - \bar{\boldsymbol{x}})^2}$ (**best coefficient** we can get in linear regression)

Minimizing MSE Loss Function using Gradient Descent



Example of finding the best line to fit data

What is the best line $\hat{y} = w_1 \times x_1 + w_0$ to fit to the following data?



Hints: $w_0 = \bar{y} - w_1 \bar{x}$ — **best intercept**

$w_1 = \dfrac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$ — **best coefficient**

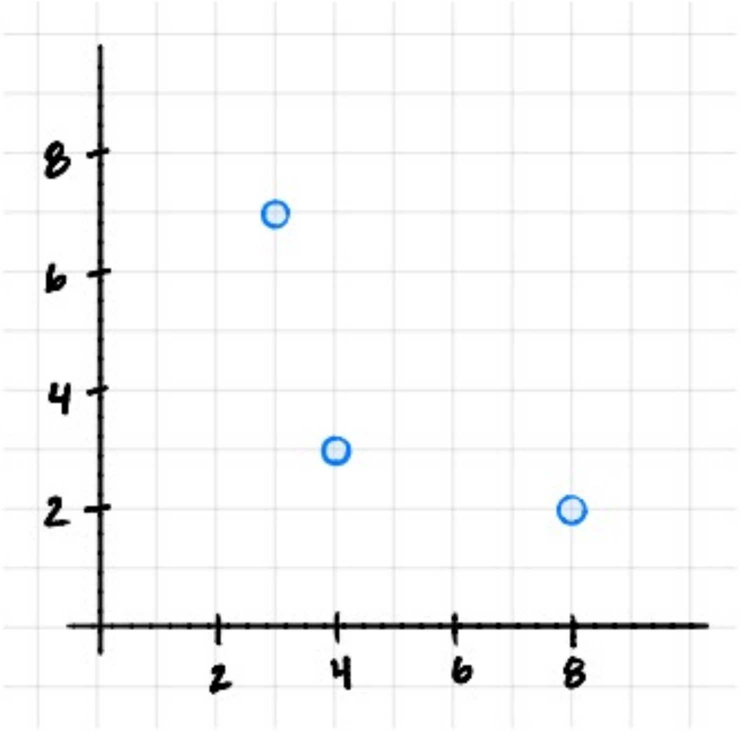
$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{3}(3+4+8) = 5$$

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{3}(7+3+2) = 4$$

x_i	y_i	$(x_i - \bar{x})$	$(y_i - \bar{y})$	$(x_i - \bar{x})(y_i - \bar{y})$	$(x_i - \bar{x})^2$
3	7				
4	3				
8	2				

Example of finding the best line to fit data

What is a best line $\hat{\boldsymbol{y}} = \boldsymbol{w_1} \times \boldsymbol{x_1} + \boldsymbol{w_0}$ to fit to the following data?



Hints: $\boldsymbol{w_0} = \bar{\boldsymbol{y}} - \mathbf{w_1}\,\bar{\boldsymbol{x}}$ **best intercept**

$\boldsymbol{w_1} = \frac{\sum_{i=1}^{n}(\boldsymbol{x_i} - \bar{\boldsymbol{x}})(\boldsymbol{y_i} - \bar{\boldsymbol{y}})}{\sum_{i=1}^{n}(\boldsymbol{x_i} - \bar{\boldsymbol{x}})^2}$ **best coefficient**

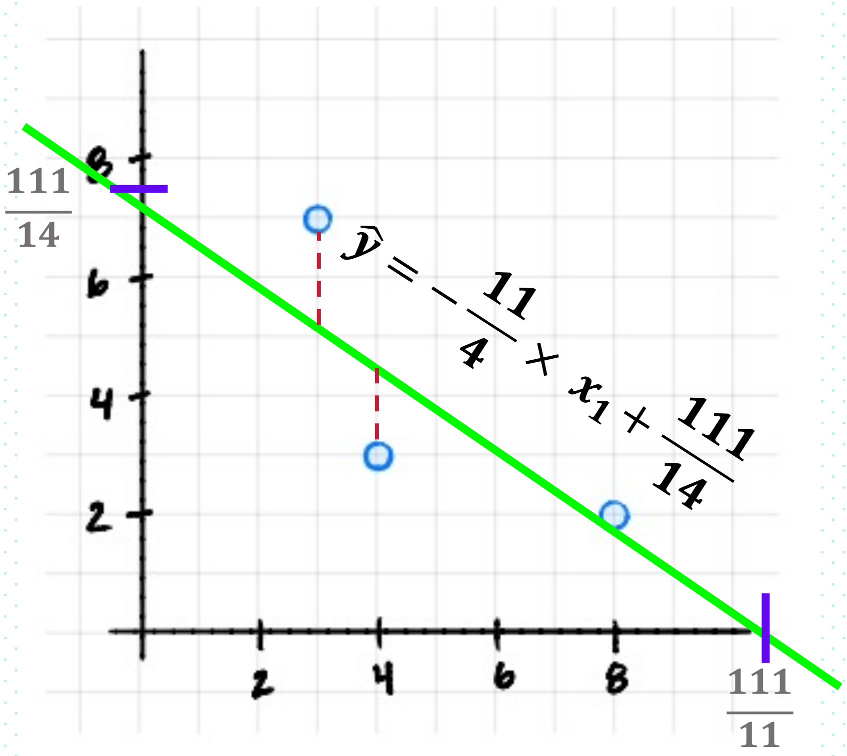
$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{3}(3+4+8) = 5 \qquad \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{3}(7+3+2) = 4$$

$\mathbf{x_i}$	$\boldsymbol{y_i}$	$(\boldsymbol{x_i} - \bar{\boldsymbol{x}})$	$(\boldsymbol{y_i} - \bar{\boldsymbol{y}})$	$(\boldsymbol{x_i} - \bar{\boldsymbol{x}})\,(\boldsymbol{y_i} - \bar{\boldsymbol{y}})$	$(\boldsymbol{x_i} - \bar{\boldsymbol{x}})^2$
3	7	(3-5)	(7-4)	(3-5)(7-4)=-6	4
4	3	(4-5)	(3-4)	(4-5)(3-4)=1	1
8	2	(8-5)	(2-4)	(8-5)(2-4)=-6	9

$$w_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{= (-6+1-6)}{4+1+9} = -\frac{11}{14} \qquad w_0 = \bar{y} - \mathrm{w}_1\,\bar{x} = 4 - \left(-\frac{11}{14}\right)\times 5 = \frac{111}{14} \quad \rightarrow \quad \hat{\boldsymbol{y}} = -\frac{\mathbf{11}}{\mathbf{4}} \times \boldsymbol{x_1} + \frac{\mathbf{111}}{\mathbf{14}}$$

Example of finding the best line to fit data

What is a best line $\widehat{\boldsymbol{y}} = \boldsymbol{w_1} \times \boldsymbol{x_1} + \boldsymbol{w_0}$ to fit to the following data?



Let's plot $\widehat{\boldsymbol{y}} = -\frac{\mathbf{11}}{\mathbf{4}} \times \boldsymbol{x_1} + \frac{\mathbf{111}}{\mathbf{14}}$

Slope coefficient

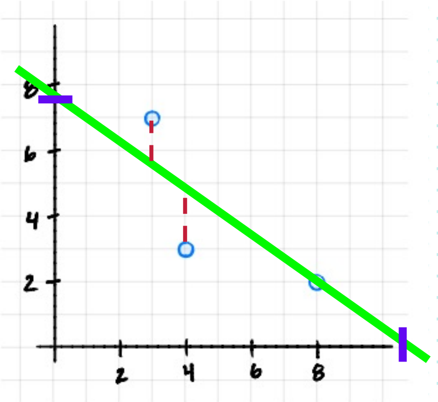
y-intercept

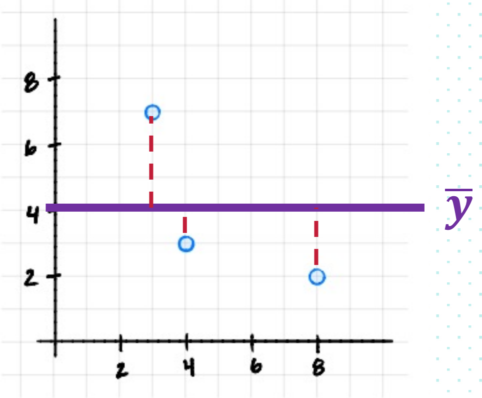
Other Metrics: R^2 $(R - squared)$

$$R^2(y, \hat{y}) = 1 - \frac{Residual\ Sum\ of\ Squares}{Total\ Sum\ of\ Squares}$$

$$R^2(y, \hat{y}) = 1 - \frac{\sum_{i=1}^{n}(y_i, \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i, \bar{y})^2}$$

$1 -$





Do we aim to have a higher R^2 or a lower?

R^2 is a normalized version of *MSE*, and is often easier to interpret as it does not depend on the scale of the data.

Multiple Linear Regression

Linear Regression is all about fitting a line to a bunch of points.

$$\hat{y} = w_d x_d + \cdots + w_2 x_2 + w_1 x_1 + w_0$$

Dependent variable (y-hat)

Slope coefficient n

Independent variable n

Slope coefficient 1

Independent variable 1

y-intercept

Worldwide Gross = $w_d \times$ **Production Budget** $+ \cdots + w_2 \times \mathbf{Genre} + w_1 \times \mathbf{Theater\ Count} + w_0$

Multiple Linear Regression

For each movie we have:

6 features $x = (x_1, x_2, \dots, x_6)$

Production Budget, Awards/Nominations, Theater Count, Star Power, Release Date and Genre.

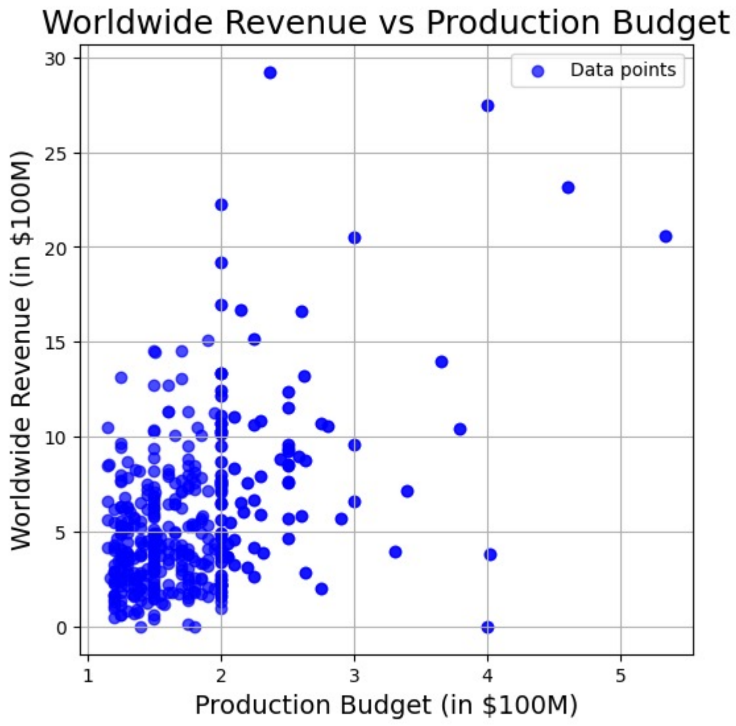
Worldwide Gross = $\boldsymbol{w_d \times}$ **Production Budget** $+ \cdots + \boldsymbol{w_2 \times \text{Genre} + w_1 \times \text{Theater Count} + w_0}$

This is also a **regression problem** with:

Predictor (independent) variables: Production Budget, Awards/Nominations, Theater Count, Star Power, Release Date and Genre. $\mathrm{x} \in \mathbb{R}^6$

Response (dependent) variable: Worldwide Revenue ($y \in \mathbb{R}$)

Back to the Movies Revenues



No predictor variables (average) $MSE =$ 2.0457643e+17

One predictor variables (Production Budget) $MSE =$ 1.8769043e+16

Exerceise: Two predictor variables (Production Budget, Theater Count) $MSE =$

Do more features result in better prediction?

Least-Squares Regression

Linear function of 6 variables: for x $\in \mathbb{R}^6$ where $w = (w_1, w_2, \ldots.., w_6\ )$

$$f(x) = y = w_6 \times x_6 + \cdots + w_2 \times x_2 + w_1 \times \mathrm{x}_1 + w_0 = \boldsymbol{w}.\boldsymbol{x} + \boldsymbol{w_0}$$

Minimizing the Least Squares

actual value

predicted value for dependent variable

$$L(w, w_0) = \sum_{i=1}^{n} (y_i - \widehat{y_i})^2 = \sum_{i=1}^{n} \left(y^{(i)} - (\boldsymbol{w}.\boldsymbol{x}^{(i)} + \boldsymbol{w_0})\right)^2 = \sum_{i=1}^{n} \left(y^{(i)} - (\tilde{w}.\tilde{x}^{(i)})\right)^2$$

$$\boldsymbol{w}.\boldsymbol{x}^{(i)} + \boldsymbol{w_0} = w_0 * 1 + w_1 * x_2^i + \cdots + w_d * x_d^i =$$

$$= (w_0, \ldots, w_d).\left(1, \ldots, x_d^i\right) = (w_0, w).(1, x) = \tilde{w}.\tilde{x}^{(i)}$$

Least-Squares Regression

Goal: Finding $\tilde{w} \in R^{n+1}$ that minimizes $L(\tilde{w}) = \sum_{i=1}^{n} \left(y^{(i)} - (\tilde{w}.\tilde{x}^{(i)})\right)^2$

$$x = \begin{bmatrix} \leftarrow & \tilde{x}^{(1)} & \rightarrow \\ \leftarrow & \tilde{x}^{(2)} & \rightarrow \\ & \vdots & \\ \leftarrow & \tilde{x}^{(n)} & \rightarrow \end{bmatrix} \qquad y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n)} \end{bmatrix}$$

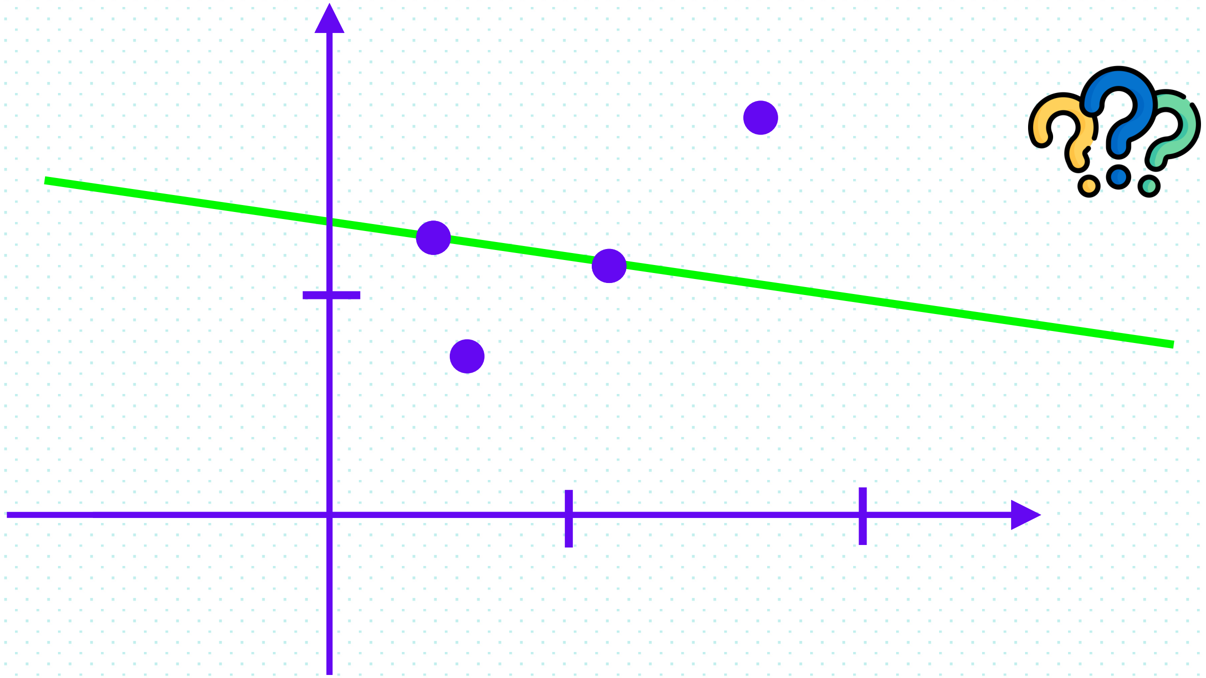
$$L(\tilde{w}) = \sum_{i=1}^{n} \left(y^{(i)} - (\tilde{w}.\tilde{x}^{(i)})\right)^2 = \|y - X\tilde{w}\|^2$$

The Loss function is minimized at $\tilde{w} = (X^T X)^{-1}(X^T y)$

Least-Squares Regression

Is training loss a good estimate of future performance?

If n is large enough: maybe
Otherwise: Probably an underestimate of future's error



n is small, training error is 0

If training error is not a good measure of future test error, how do we get a good estimate of future error?

Better error estimates

Recall: k-fold cross-validation

Divide the data into k equal-sized groups $S_1, S_2, \ldots, S_k$

For $i = 1$ to k:

Train a regressor on all data except S_i

Let E_i be its error on S_i

Error estimate: average of $E_1, E_2, \ldots, E_k$

If the training error is really not a very good measure of future error, why we use an optimization criterion **based exclusive on training error?**

Ridge Regression

Minimize Squared Loss **plus** a term that penalizes "complex" w**:**

$$L(w, w_0) = \sum_{i=1}^{n} \left(y^{(i)} - (w.x^{(i)} + w_0)\right)^2 + \lambda \|w\|^2$$

Adding a penalty term like this is called **regularization**.

$\lambda = 0 \rightarrow$ (Least Squares Solution)

$\lambda \rightarrow \infty \quad w \rightarrow 0$

$w = (X^T X + \lambda I)^{-1} (X^T y)$

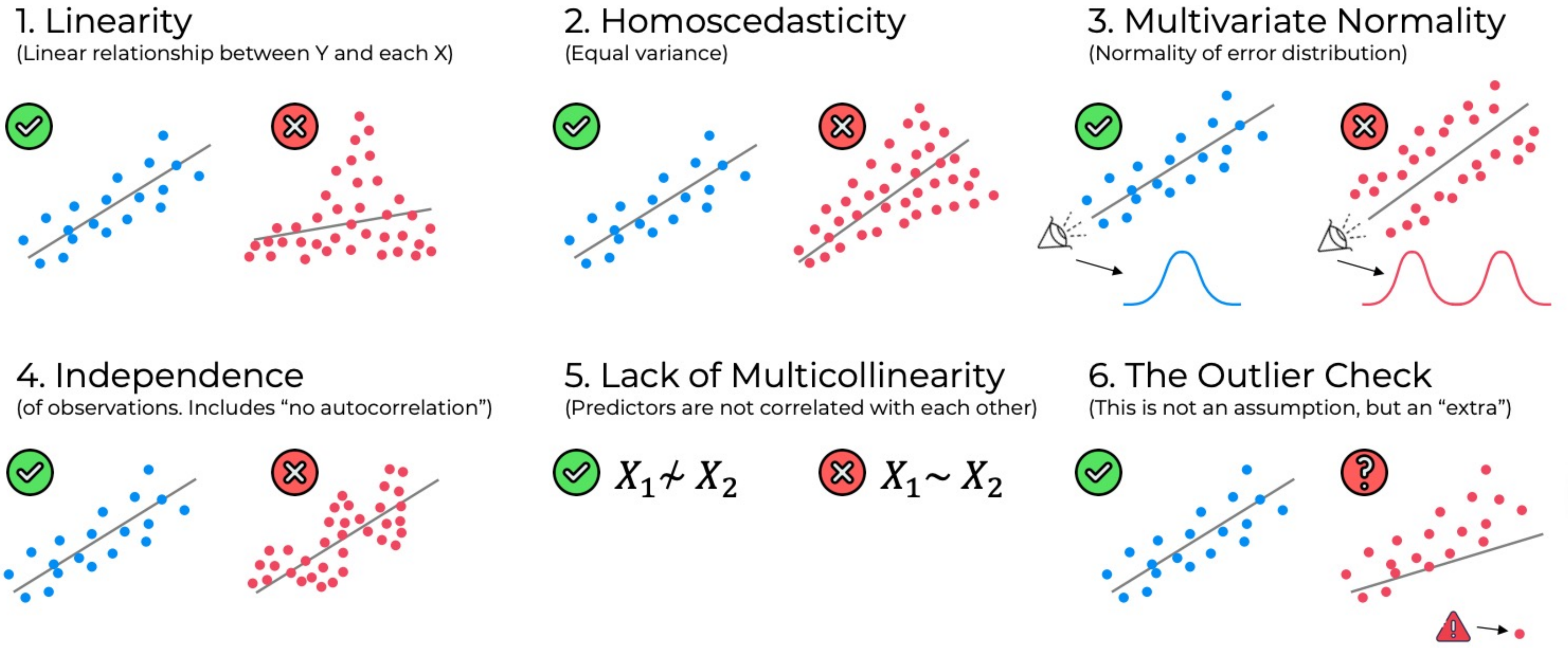
Lasso Regression

Minimize Squared Loss **plus** a term that penalizes “complex” w**:**

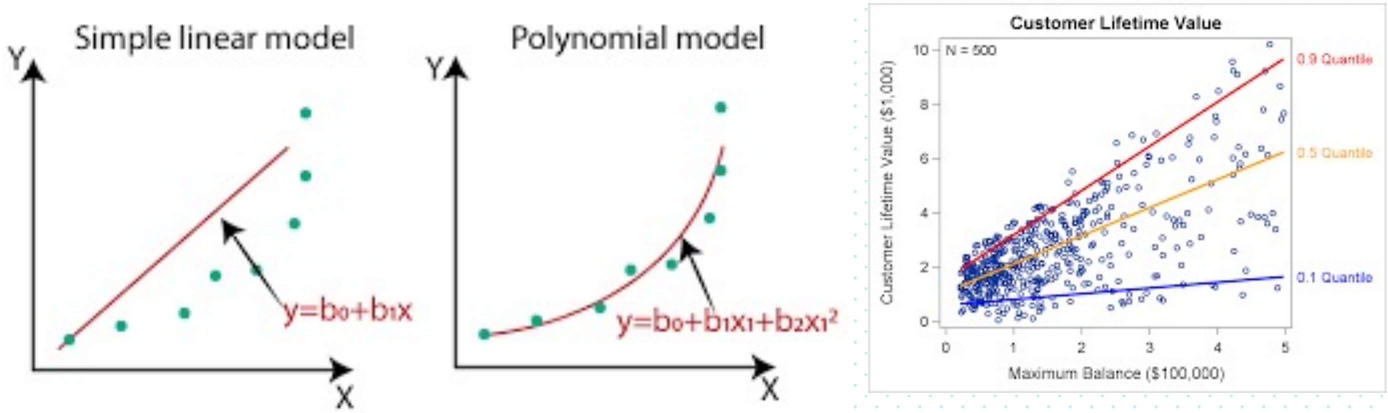
$$L(w, w_0) = \sum_{i=1}^{n} \left(y^{(i)} - (w.x^{(i)} + w_0)\right)^2 + \lambda \|w\|$$

Lasso tends to produce sparse w

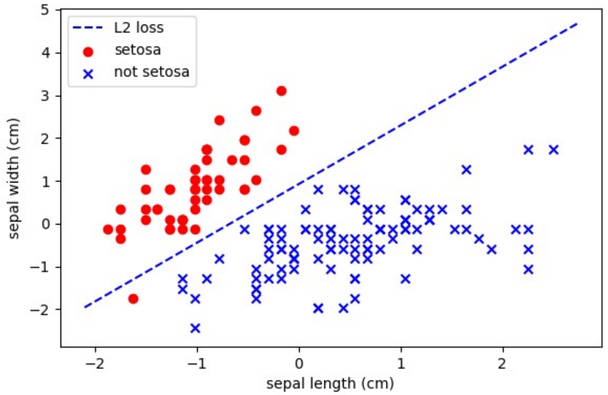
Assumptions of Linear Regression (SuperDataScience)



Going Beyond Linear Regression



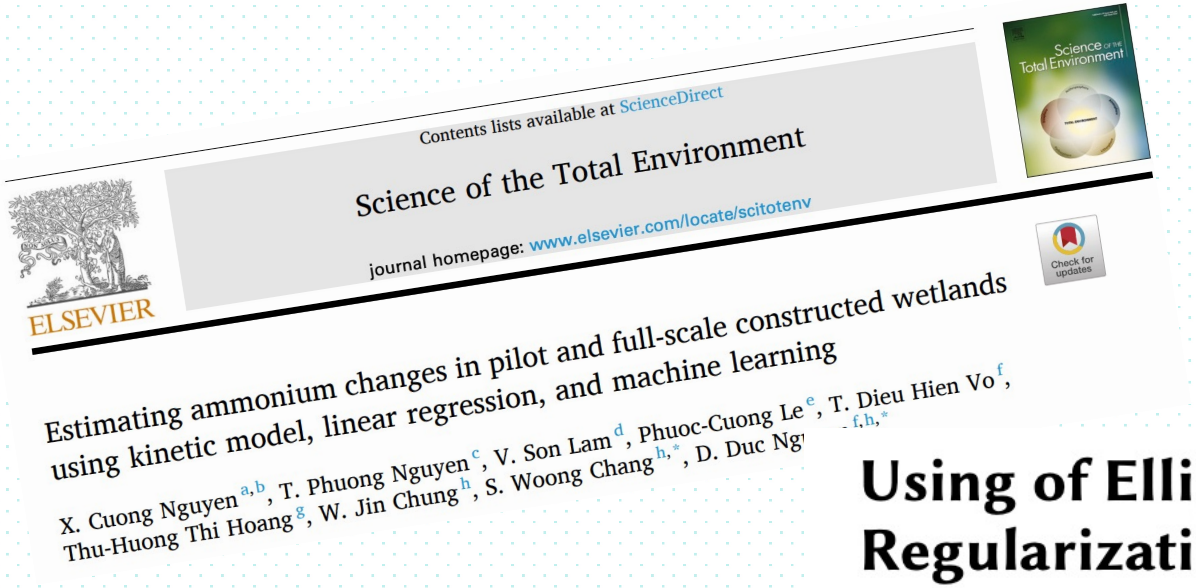
Some Variants of Linear Regression



Linear Models for classification

When the world is all about LLMs and GenAI, are we still using Linear Regression?

Yes! Linear regression is still widely used because it's interpretable, efficient, and works well for many problems where complex models are unnecessary.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X. Cuong Nguyen[a,b], T. Phuong Nguyen[c], V. Son Lam[d], Phuoc-Cuong Le[e], T. Dieu Hien Vo[f], Thu-Huong Thi Hoang[g], W. Jin Chung[h], S. Woong Chang[h,*], D. Duc Ng

Using of Ellipsoid Method and Linear Regression with L_1-Regularization for Medical Data Investigation

Petro Stetsyuk[1], Viktor Stovba[1], Ivan Senko[1], and Illya Chaikovsky[1]

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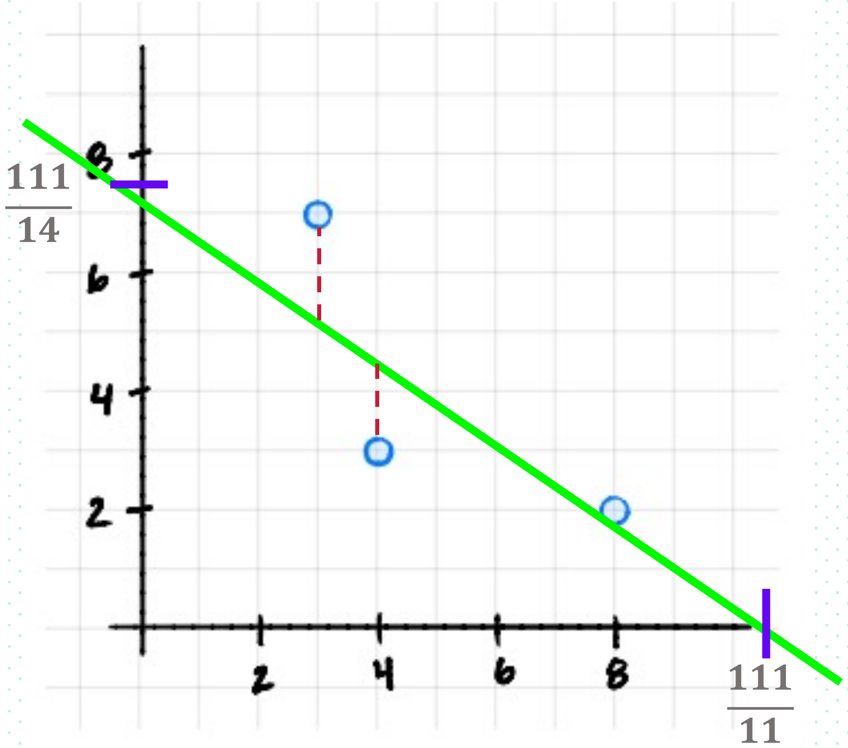


What is the R^2 of the above fitted line?

Inspired by Resources:

Fundamentals of Machine Learning, UC San Diego, Professor Sanjoy Dasgupta

Fundamentals of Machine Learning, Sharif University of Technology, Professor Ali Sharifi Zarchi

Theoretical Foundations of Data Science, UC San Diego, Dr Albuyeh

Dataset: The-Numbers.com

$$Remember \quad w_0 = \bar{y} - \mathrm{w}_1 \bar{x}$$

Backup: Minimizing Loss Function

$$\frac{d\boldsymbol{L}}{dw_1} = \frac{2}{n}\sum_{i=1}^{n}\left(y_i - (w_1 x_1^i + w_0)\right)\times\left(x_1^i\right) = 0 \rightarrow \sum_{i=1}^{n} y_i x_1^i - w_1 \times \sum_{i=1}^{n}\left(x_1^i\right)^2 - w_0 \times \sum_{i=1}^{n} x_1^i = 0$$

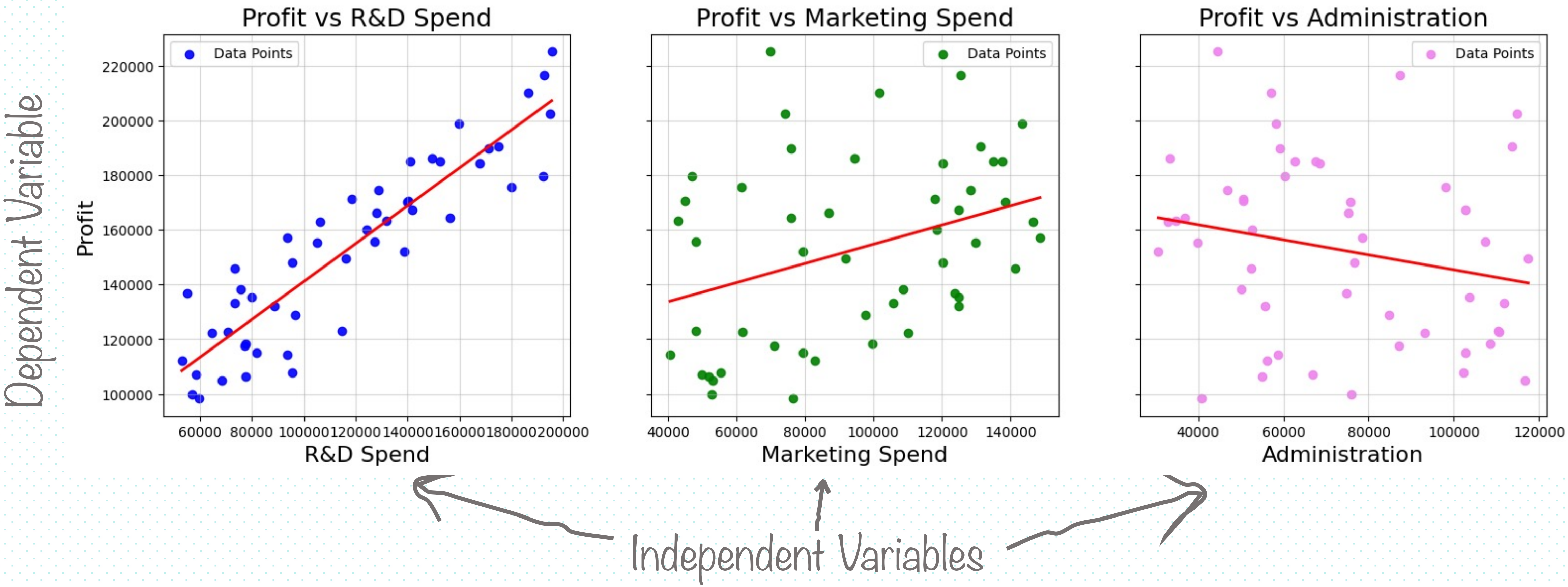
$$\sum_{i=1}^{n} y_i x_1^i - w_1 \times \sum_{i=1}^{n}\left(x_1^i\right)^2 - (\bar{y} - \mathrm{w}_1 \bar{x}) \times \sum_{i=1}^{n} x_1^i = 0 \rightarrow \sum_{i=1}^{n} y_i x_1^i - w_1 \times \sum_{i=1}^{n}\left(x_1^i\right)^2 - \bar{y}\sum_{i=1}^{n} x_1^i + \mathrm{w}_1 \bar{x}\sum_{i=1}^{n} x_1^i = 0$$

$$\sum_{i=1}^{n} y_i x_1^i - \bar{y}\sum_{i=1}^{n} x_1^i = w_1 \times \sum_{i=1}^{n}\left(x_1^i\right)^2 - \mathrm{w}_1 \bar{x}\sum_{i=1}^{n} x_1^i \qquad w_1 = \frac{\sum_{i=1}^{n} y_i x_1^i - \bar{y}\sum_{i=1}^{n} x_1^i}{\sum_{i=1}^{n}\left(x_1^i\right)^2 - \bar{x}\sum_{i=1}^{n} x_1^i}$$

$$w_1 = \frac{\sum_{i=1}^{n} y_i x_1^i - \bar{y}\sum_{i=1}^{n} x_1^i}{\sum_{i=1}^{n}\left(x_1^i\right)^2 - \bar{x}\sum_{i=1}^{n} x_1^i} = \frac{\sum_{i=1}^{n} y_i x_1^i - n\bar{y}\bar{x}}{\sum_{i=1}^{n}\left(x_1^i\right)^2 - n\bar{x}^2} = \frac{\boldsymbol{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}}{\boldsymbol{\sum_{i=1}^{n}(x_i - \bar{x})^2}} = \frac{\boldsymbol{Cov(x, y)}}{\boldsymbol{\sigma^2(x)}}$$

Linear Regression: Another Example

Linear Regression is all about fitting a line to a bunch of points.



Ridge Regression

Minimize Squared Loss **plus** a term that penalizes “complex” w**:**

$$MSE = \frac{1}{n}\sum_{i=1}^{n}\left(y^{(i)} - (w.x^{(i)} + w_0)\right)^2 + \lambda||w|$$

λ	training MSE	test MSE
0.00001	0.00	585.81
0.0001	0.00	564.28
0.001	0.00	404.08
0.01	0.01	83.48
0.1	0.03	19.26
1.0	0.07	7.02
10.0	0.35	2.84
100.0	2.40	5.79
1000.0	8.19	10.97
10000.0	10.83	12.63

Adding a penalty term lie this is called **regularization**.

$\lambda = 0$ (Least Squares Solution)

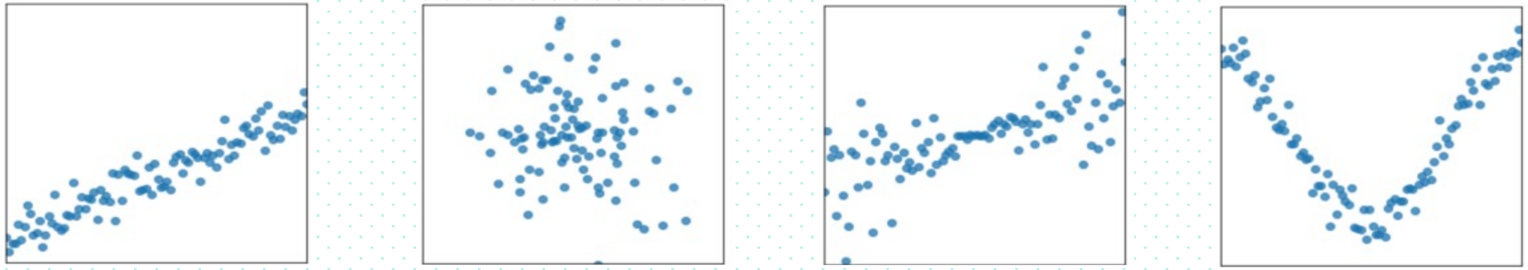
$\lambda \rightarrow \infty\, w = 0$

Put predictor vectors in matrix X and responses in vector y.

$$w = (X^T X + \lambda I)^{-1}(X^T y)$$

Pattens in Scatter Plots: Connection with Correlation

Correlation is the **linear association** of two variables x and y.



$r = 0.949$ $r = -0.121$ $r = 0.704$ $r = 0.052$

Correlation measures how tightly clustered a scatter plot is around a straight line.

Correlation ranges between -1 and 1.

Correlation is the average of the product of x and y, when both are in standard units.

$$r = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{\sigma_x}\right)\sum_{i=1}^{n}\left(\frac{y_i - \bar{y}}{\sigma_y}\right)$$

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