Derivative
and its
Applications

What does the Derivative tell us?

- The Derivative of a function $f(x)$ gives us ***another function*** $f'(x)$ which can be referred as "SLOPE FUNCTION"
- For Example if $f(x) = \frac{1}{3}x^3, then\ f'(x) = x^2$.
- $f'(x) = x^2$ is itself a function but it tells us the slope of tangent lines to the curve of function at any point x.

What does the Derivative tell us?

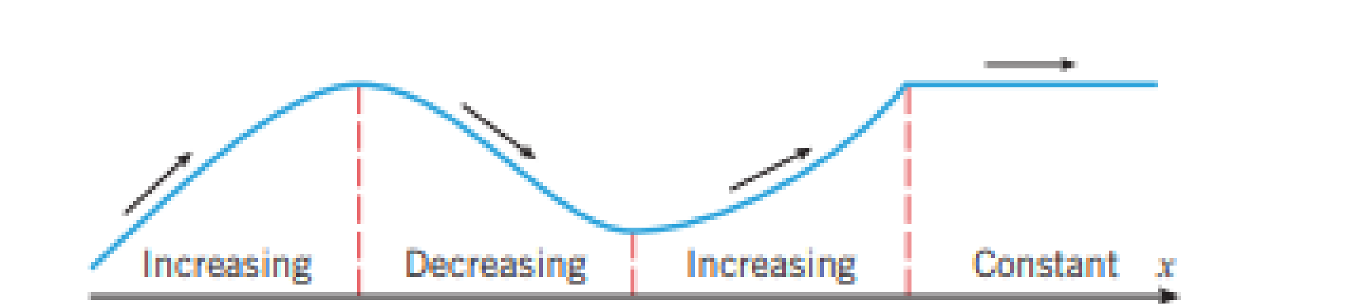
- Let us demonstrate it on a program made on GeoGebra
- https://www.geogebra.org/m/BDYnGhbt

What does Derivative tell us?

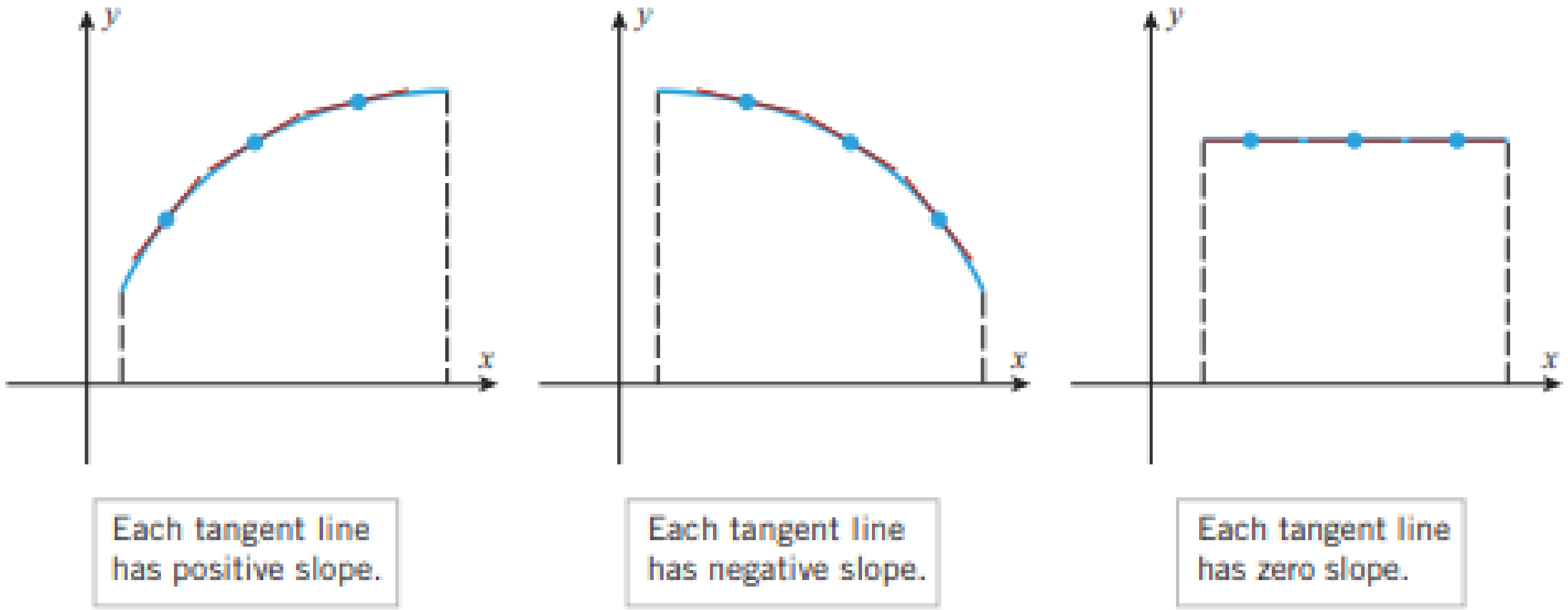
- Its tells us about the SLOPE of the Function.

However, Slope *(rate of change of function)* tells us a lot of things:

1. When a function is increasing or decreasing,
2. How rapidly its increasing or decreasing,
3. Or when a function is not changing at all..
4. Or when it changes its behavior.
5. What are the highest/lowest points of a function…



Using Derivative



4.1.2 THEOREM *Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).*

(a) *If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.*

(b) *If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.*

(c) *If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.*

Using Derivative

- When is a function not changing?

when $f'(x) = 0$

$Slope = 0$ means y is not changing w.r.t x

Such points $\left(x_0, f(x_0)\right)$ where $f'(x_0) = 0$ are known as

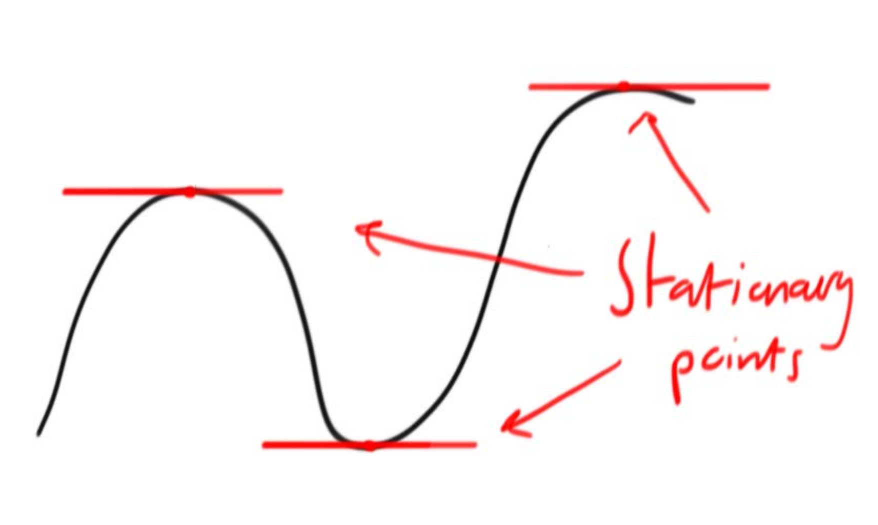
STATIONARY POINTS

Stationary Points

- A train **station** is a railway facility where trains **stop** to load or unload passengers.

Stationary Points

- Similarly, the points where the function momentarily stop or where $f'(x) = 0$ are called **Stationary Points** of the curve.

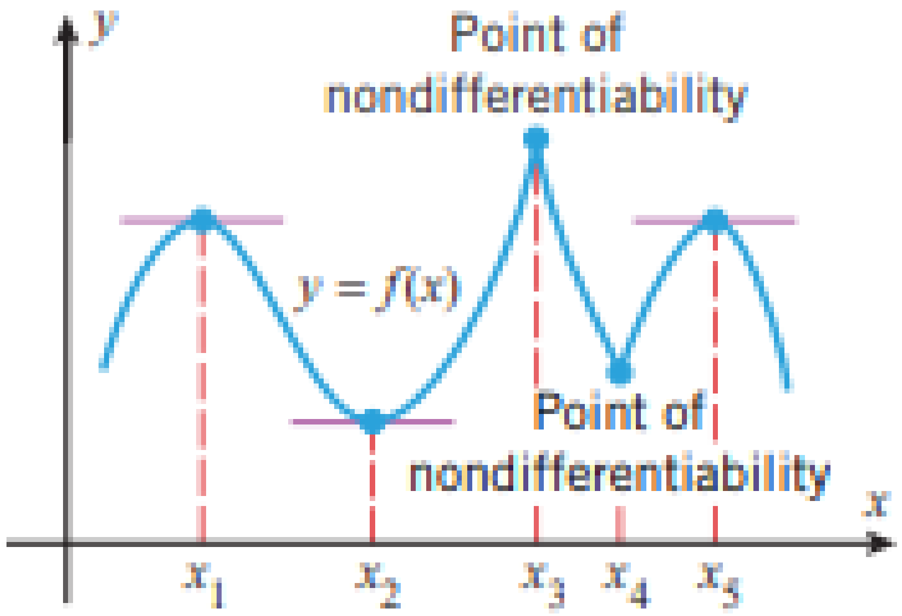


Critical Points

- Stationary points are all ***critical points***. But all critical points are not stationary points. (See definition)

For any function f, a point p in the domain of f where $f'(p) = 0$ or $f'(p)$ is undefined is called a **critical point** of the function. In addition, the point $(p, f(p))$ on the graph of f is also called a critical point. A **critical value** of f is the value, $f(p)$, of the function at a critical point, p.

Not all Critical Points are Stationary Points?



▲ **Figure 4.2.3** The points x_1, x_2, x_3, x_4, and x_5 are critical points. Of these, x_1, x_2, and x_5 are stationary points.

But we will mostly deal with critical points that are stationary points. We will discuss SMOOTH CURVES

Why are Stationary(critical points) important?

- Observe at stationary points, a curve takes a turn.. A U-turn.
- The slope of the function changes direction.
- These points are also **local MAXIMA** or **local MINIMA** of the curve.

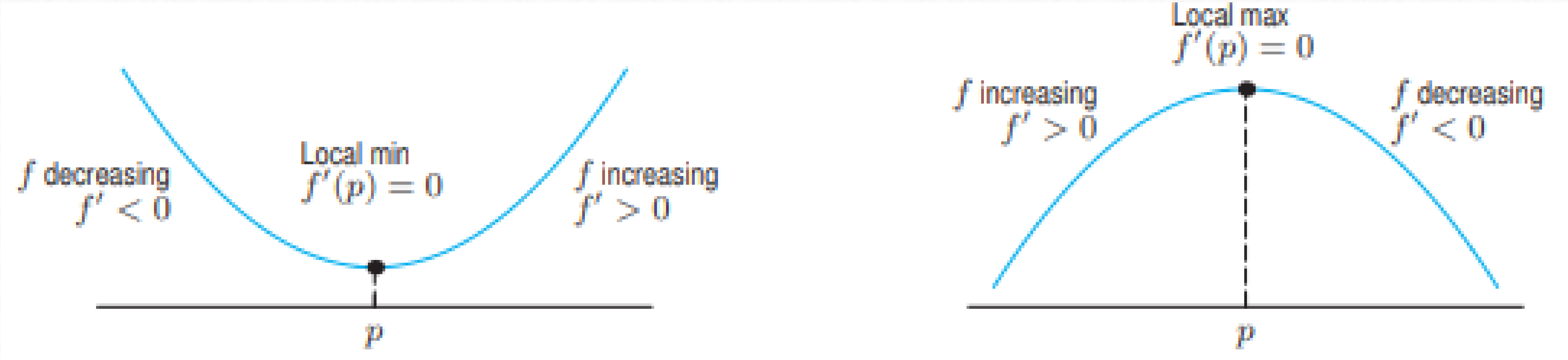


Figure 4.6: Changes in direction at a critical point, p: Local maxima and minima

Local Maxima/Minima Points- Extreme Points

Suppose p is a point in the domain of f:

- f has a **local minimum** at p if $f(p)$ is less than or equal to the values of f for points near p.
- f has a **local maximum** at p if $f(p)$ is greater than or equal to the values of f for points near p.

We use the adjective "local" because we are describing only what happens near p.

How to identify whether a stationary point is Maxima or Minima?

- Two Ways: But lets discuss the first way:

First Derivative Test for Local Maxima and Minima

Suppose p is a critical point of a continuous function f. Then, as we go from left to right:

- If f changes from decreasing to increasing at p, then f has a local minimum at p.
- If f changes from increasing to decreasing at p, then f has a local maximum at p.

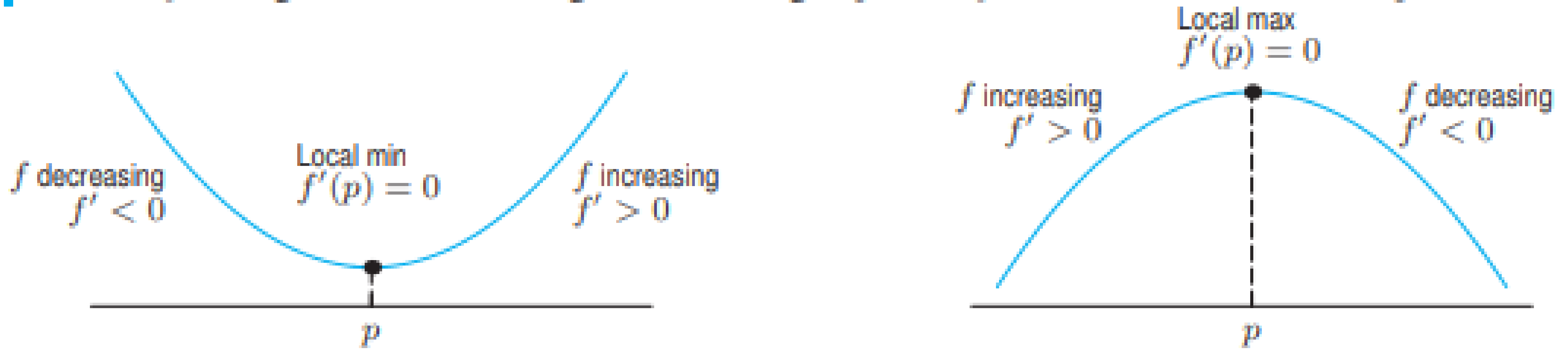


Figure 4.6: Changes in direction at a critical point, p: Local maxima and minima

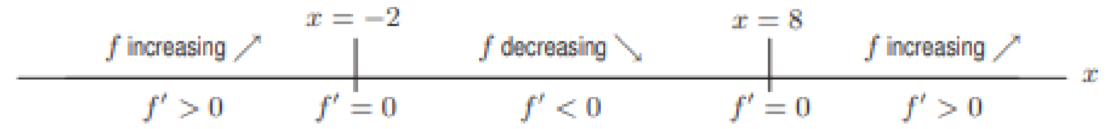
Example 1

- Let $f(x) = x^3 - 9x^2 - 48x + 52$ defined over the interval $(-\infty, +\infty)$

1. Find the stationary point of the curve.
2. In which intervals of the domain is the function increasing and/or decreasing.
3. Which of the stationary point is the local maxima and minima?
4. Sketch the graph of the function, by using the answers above. Also find the stationary values(function value) at maxima and minima point.

Example 1

- Solution: On white-board



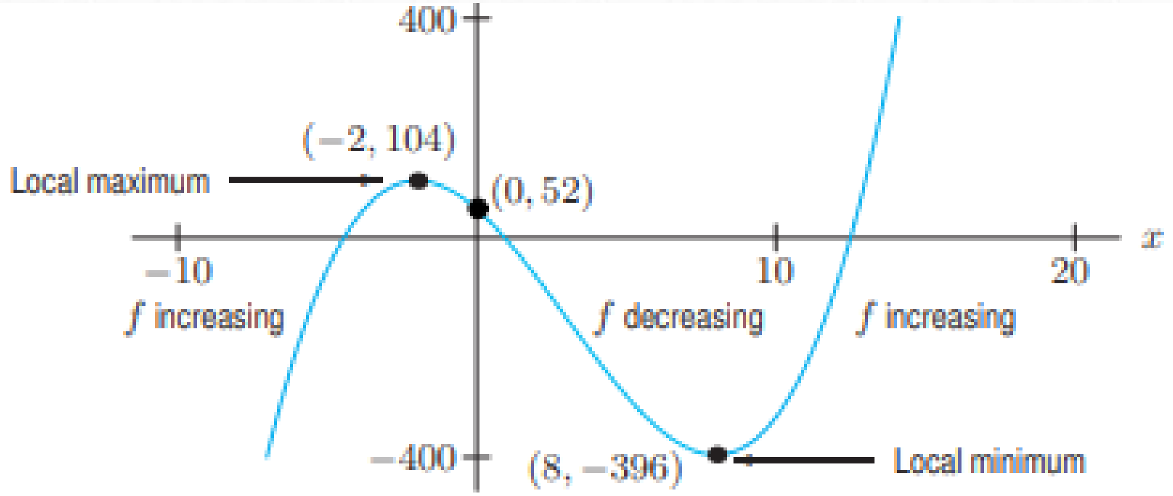


Figure 4.2: Useful graph of $f(x) = x^3 - 9x^2 - 48x + 52$. Notice that the scales on the x-and y-axes are different

Do it yourself

Practice Problems.
Find the intervals on which f is increasing and the intervals on which it is decreasing.

1. $f(x) = x^3 - 3x^2 + 1$
2. $f(x) = 3x^4 - 4x^3$.
3. $f(x) = x^3 - x^2 - 2x$.

Using a calculator or computer, graph the functions in Problems 10–15. Describe in words the interesting features of the graph, including the location of the critical points and where the function is monotonic (that is, increasing or decreasing). Then use the derivative and algebra to explain the shape of the graph.

10. $f(x) = x^3 - 6x + 1$

11. $f(x) = x^3 + 6x + 1$

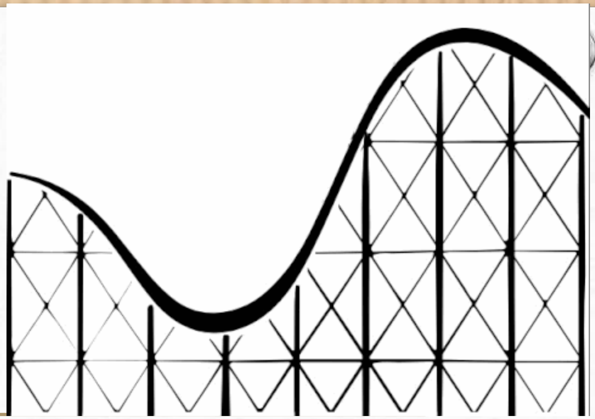
12. $f(x) = 3x^5 - 5x^3$

13. $f(x) = e^x - 10x$

14. $f(x) = x \ln x, \quad x > 0$

15. $f(x) = x + 2\sin x$

Activity



A brochure for a roller coaster says that, for the first 10 seconds of the ride, the height of the coaster can be determined by $h(t) = \frac{1}{3}t^3 - 5t^2 + 21t$, where t is the time in seconds and h is the height in feet.

- **What was the height of the coaster at $t = 8$ seconds?**
- **At what time instant(s) during the first 10 seconds , the coaster momentarily stopped?**
- **At which intervals, the coaster was going up and when was it coming down?**
- **Calculate the maximum and minimum height the rollercoaster can reach during the first 10 seconds.**

Second Derivative

For the function $y = f(x)$

Slope Function $\frac{dy}{dx} = f'(x)$:

- Tells the rate of change of function
- Where Function is increasing/decreasing.

$\frac{d^2y}{dx^2} = f''(x)$:

- Tell us the rate of change of slopes
- Where Slopes is increasing/ decreasing.
- **Or in other words, it tells us about CONCAVITY of the graph**

Concavity

4.1.3 DEFINITION If f is differentiable on an open interval, then f is said to be ***concave up*** on the open interval if f' is increasing on that interval, and f is said to be ***concave down*** on the open interval if f' is decreasing on that interval.

4.1.4 THEOREM *Let f be twice differentiable on an open interval.*

(*a*) *If $f''(x) > 0$ for every value of x in the open interval, then f is concave up on that interval.*

(*b*) *If $f''(x) < 0$ for every value of x in the open interval, then f is concave down on that interval.*

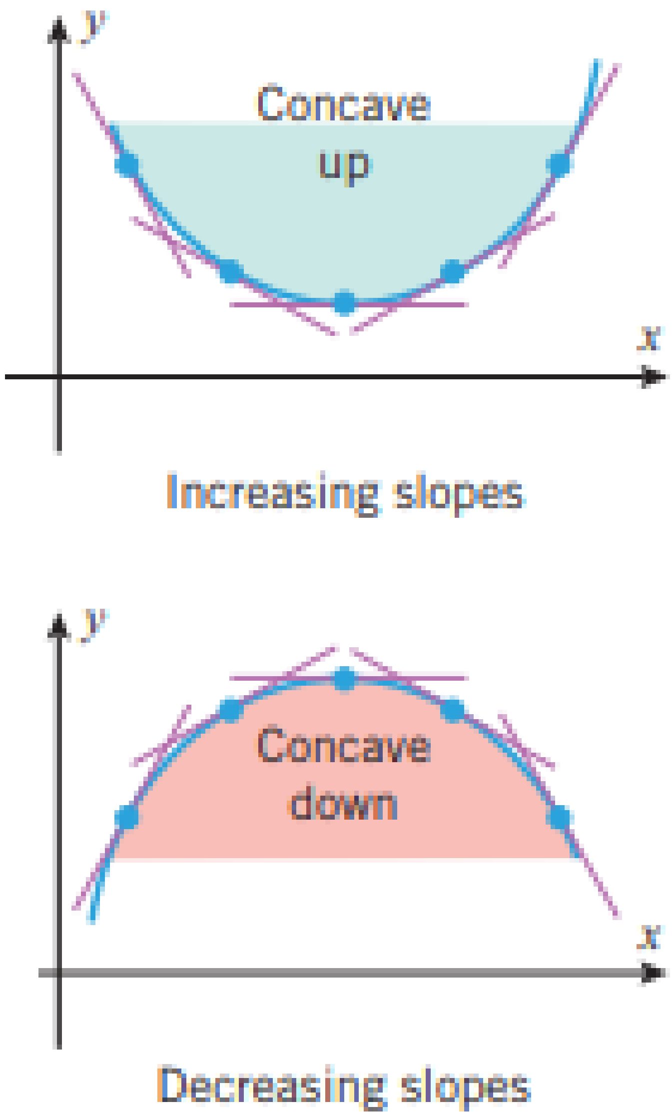


Figure 4.1.8

Observe!
At Maxima, the graph is concave down
At Minima, the graph is concave up.

How to identify a Stationary point is Maxima/Minima

Second Derivative Test for Local Maxima and Minima

Suppose p is a critical point of a continuous function f, and $f'(p) = 0$.

- If f is concave up at p, then f has a local minimum at p.
- If f is concave down at p, then f has a local maximum at p.

Inflection Point

A point at which the graph of a function f changes concavity is called an **inflection point** of f.

How Do You Locate an Inflection Point?

Since the concavity of the graph of f changes at an inflection point, the sign of f'' changes there: it is positive on one side of the inflection point and negative on the other. Thus, at the inflection point, f'' is zero or undefined. (See Figure 4.16.)

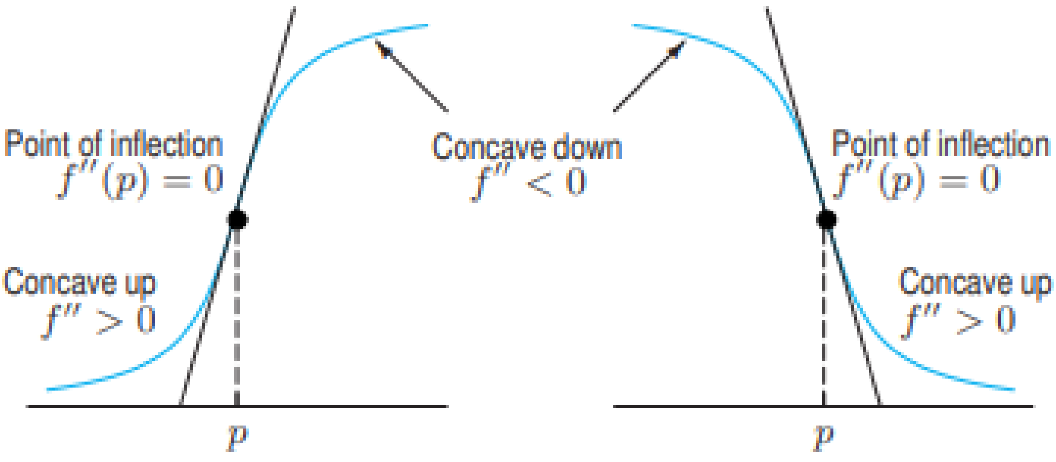


Figure 4.16: Change in concavity (from positive to negative or vice versa) at point p

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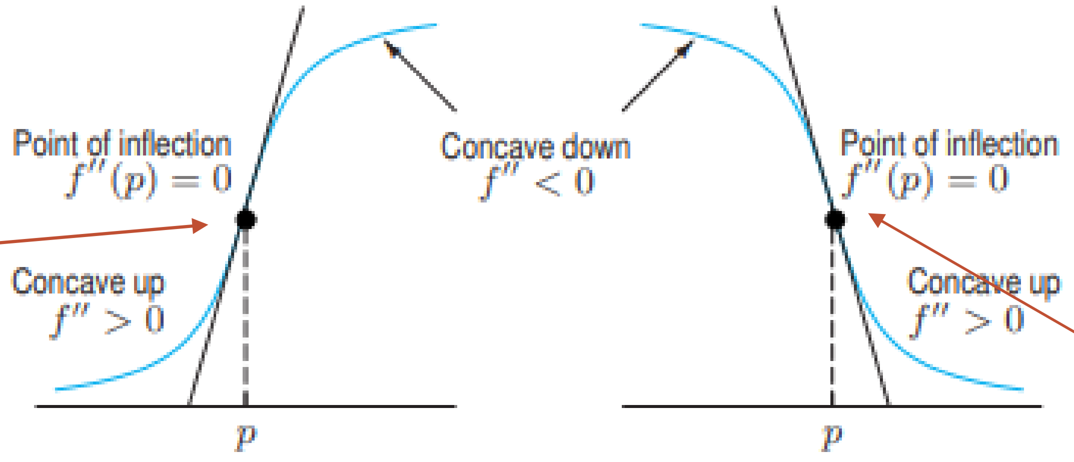
IMPORTANT!
The Inflection point is the point where most rapid change occurs:
Most rapid increases or decrease

MOST RAPID INCREASE



MOST RAPID DECREASE

Figure 4.16: Change in concavity (from positive to negative or vice versa) at point p

Example 1

- Let $f(x) = x^3 - 9x^2 - 48x + 52$ defined over the interval $(-\infty, +\infty)$

1. Find the stationary point of the curve.
2. In which intervals of the domain is the function increasing and/or decreasing.
3. Which of the stationary point is the local maxima and minima?
4. Sketch the graph of the function, by using the answers above. Also find the stationary values(function value) at maxima and minima point.
5. **Redo part 3, by using 2^{nd} derivative test.**
6. **Find the inflection point.**

Example 2

(a) How many critical points and how many inflection points does the function $f(x) = xe^{-x}$ have?
(b) Use derivatives to find the critical points and inflection points exactly.
(c) With help of above answer, try to sketch the graph of the function over the interval $[0, 4]$.

Example-Word Problem

40. The quantity of a drug in the bloodstream t hours after a tablet is swallowed is given, in mg, by

$$q(t) = 20(e^{-t} - e^{-2t}).$$

(a) How much of the drug is in the bloodstream at time $t = 0$?

(b) When is the maximum quantity of drug in the bloodstream? What is that maximum?

(c) In the long run, what happens to the quantity?

40. a. $q(0) = 20\,(e^0 - e^0) = 20(1-1) = 0\ mg$

The answer is consistent with the statement/scenario of the given question. At $t = 0$, the tablet is not swalloled yet therefore there is $0mg$ drug in the bloodstream.

b. The local maxima (or local minima) which are essentially the stationary points of the function $f(x)$, and at stationary points, $f'(x) = 0$.

Let's find $q'(t)$

$$\frac{d}{dt}[20(e^{-t} - e^{-2t})]$$

$$= 20\frac{d}{dt}[(e^{-t} - e^{-2t})]$$

$$= 20\left(-1.e^{-t} - (-2)(e^{-2t})\right)$$

$$= 20(2e^{-2t} - e^{-t})$$

At stationary point, $q'(t) = 0$

$$20(2e^{-2t} - e^{-t}) = 0$$

$$2e^{-2t} - e^{-t} = 0$$

$$2e^{-2t} = e^{-t}$$

$$\frac{e^{-2t}}{e^{-t}} = \frac{1}{2}$$

$$e^{-t} = \frac{1}{2}$$

Taking antilog/natural log at both sides

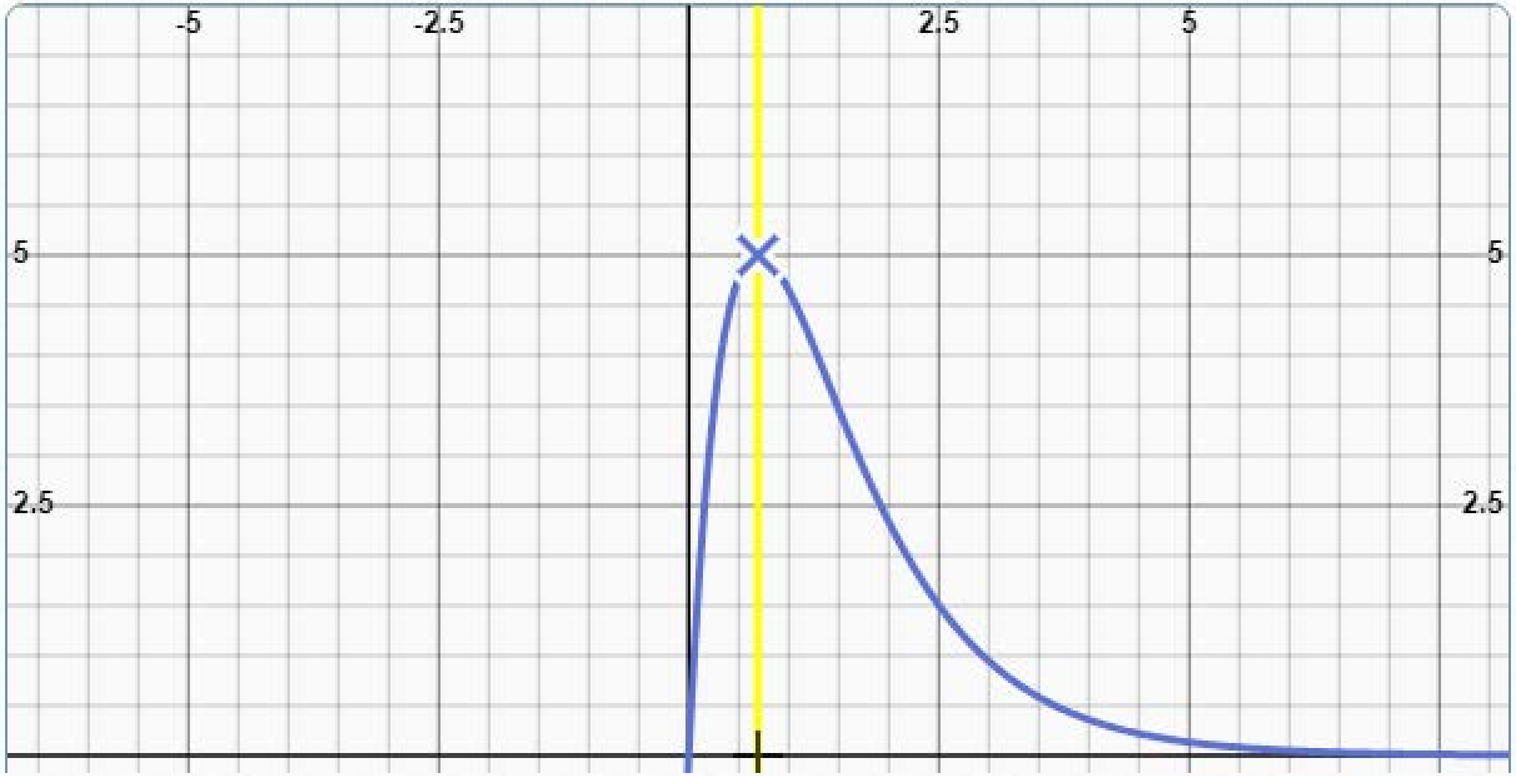
$$-t = \ln\frac{1}{2}$$

$$t = 0.693\ hr$$

To verify that it this time will give maximum quantity of drug and not minimum quantity, we should check that $q''(0.693) < 0$. (DIY, find $q''(t)$ and put $t = 0.693$)

The maximum value of drug is at $q(0.693) = 5\ mg$

c. In long run, find $\lim_{t \to \infty} q(t)$. It approaches to zero.



Exercise

10. For $f(x) = x^3 - 18x^2 - 10x + 6$, find the inflection point algebraically. Graph the function with a calculator or computer and confirm your answer.

In each of Problems 11–20, use the first derivative to find all critical points and use the second derivative to find all inflection points. Use a graph to identify each critical point as a local maximum, a local minimum, or neither.

11. $f(x) = x^2 - 5x + 3$

12. $f(x) = x^3 - 3x + 10$

13. $f(x) = 2x^3 + 3x^2 - 36x + 5$

14. $f(x) = \frac{x^3}{6} + \frac{x^2}{4} - x + 2$

Exercise-Applications

55. Where on the curve $y = (1 + x^2)^{-1}$ does the tangent line have the greatest slope?

56. Suppose that the number of bacteria in a culture at time t is given by $N = 5000(25 + te^{-t/20})$.

(a) Find the largest and smallest number of bacteria in the culture during the time interval $0 \leq t \leq 100$.

(b) At what time during the time interval in part (a) is the number of bacteria decreasing most rapidly?

55. (A past paper question) Solution:

Let's try to understand the question, it asks for value of x where it has greatest slope of tangent line.

At inflection point, the graph of the function shows most rapid increase or decrease. Most rapid increase means that the tangent line has the greatest slope at that point.

Therefore, to find this point, we find inflection point.

At inflection point $f''(x) = 0$.

Let's find the first derivative, $f'(x)$.

$$\frac{d}{dx}\left[\frac{1}{x^2+1}\right]$$

$$= -\frac{\frac{d}{dx}\left[x^2+1\right]}{\left(x^2+1\right)^2}$$

$$= -\frac{\frac{d}{dx}\left[x^2\right]+\frac{d}{dx}[1]}{\left(x^2+1\right)^2}$$

$$= -\frac{2x+0}{\left(x^2+1\right)^2}$$

$$= -\frac{2x}{\left(x^2+1\right)^2}$$

Now let's find the second derivative $f''(x)$

$$\frac{d}{dx}\left[-\frac{2x}{\left(x^2+1\right)^2}\right]$$

$$= -2\cdot\frac{d}{dx}\left[\frac{x}{\left(x^2+1\right)^2}\right]$$

$$= -2\cdot\frac{\frac{d}{dx}[x]\cdot\left(x^2+1\right)^2 - x\cdot\frac{d}{dx}\left[\left(x^2+1\right)^2\right]}{\left(\left(x^2+1\right)^2\right)^2}$$

$$= -\frac{2\left(1\left(x^2+1\right)^2 - 2\left(x^2+1\right)\cdot\frac{d}{dx}\left[x^2+1\right]\cdot x\right)}{\left(x^2+1\right)^4}$$

$$= -\frac{2\left(\left(x^2+1\right)^2 - 2\left(x^2+1\right)\left(\frac{d}{dx}\left[x^2\right]+\frac{d}{dx}[1]\right)x\right)}{\left(x^2+1\right)^4}$$

$$= -\frac{2\left(\left(x^2+1\right)^2 - 2\left(x^2+1\right)(2x+0)\,x\right)}{\left(x^2+1\right)^4}$$

$$= -\frac{2\left(\left(x^2+1\right)^2 - 4x^2\left(x^2+1\right)\right)}{\left(x^2+1\right)^4}$$

Rewrite/simplify:

$$= \frac{8x^2}{(x^2+1)^3} - \frac{2}{(x^2+1)^2}$$

Simplify/rewrite:

$$\frac{2(3x^2-1)}{(x^2+1)^3}$$

- To find inflection point, we know, $\frac{d^2y}{dx^2} = 0$

$$\frac{2(3x^2-1)}{(x^2+1)^3} = 0$$

$$2(3x^2-1) = 0$$

$$3x^2 - 1 = 0$$

$$x^2 = \frac{1}{3}$$

$$x = \frac{1}{\sqrt{3}} \approx 0.5773502691896258$$

$$x = -\frac{1}{\sqrt{3}} \approx -0.5773502691896258$$

- **Its not over yet!**

We got **two** values of x.

From these two values, which value of x corresponds to the tangent line with greatest slope?

Because these values might include a value of $\boldsymbol{x}$ that gives least slope(a negative slope) !!!

The first derivative could help us!

The first derivative tells us about slope, whether its negative or positive.

If the first derivative at $\boldsymbol{x}$ gives a **positive value**, then it's the point of $\boldsymbol{x}$ where there is positive slope… and vice versa.

55. Solution(cont.)

- $f'(0.577) < 0$

Therefore at $x = 0.577$, the tangent line has negative slope and since it is a inflection point, then it will have the least slope(most rapid decrease) at this point.

- $f'(-0.577) > 0$

Therefore at $x = 0.577$, the tangent line has positive slope and since it is a inflection point, then it will have the greatest slope(most rapid increase) at this point.

Our Answer: $x = -\frac{1}{\sqrt{3}}$

Exercise-Application

39. The oxygen supply, S, in the blood depends on the hematocrit, H, the percentage of red blood cells in the blood: 4

$$S = aHe^{-bH} \quad \text{for positive constants } a, b.$$

(a) What value of H maximizes the oxygen supply? What is the maximum oxygen supply?

(b) How does increasing the value of the constants a and b change the maximum value of S?

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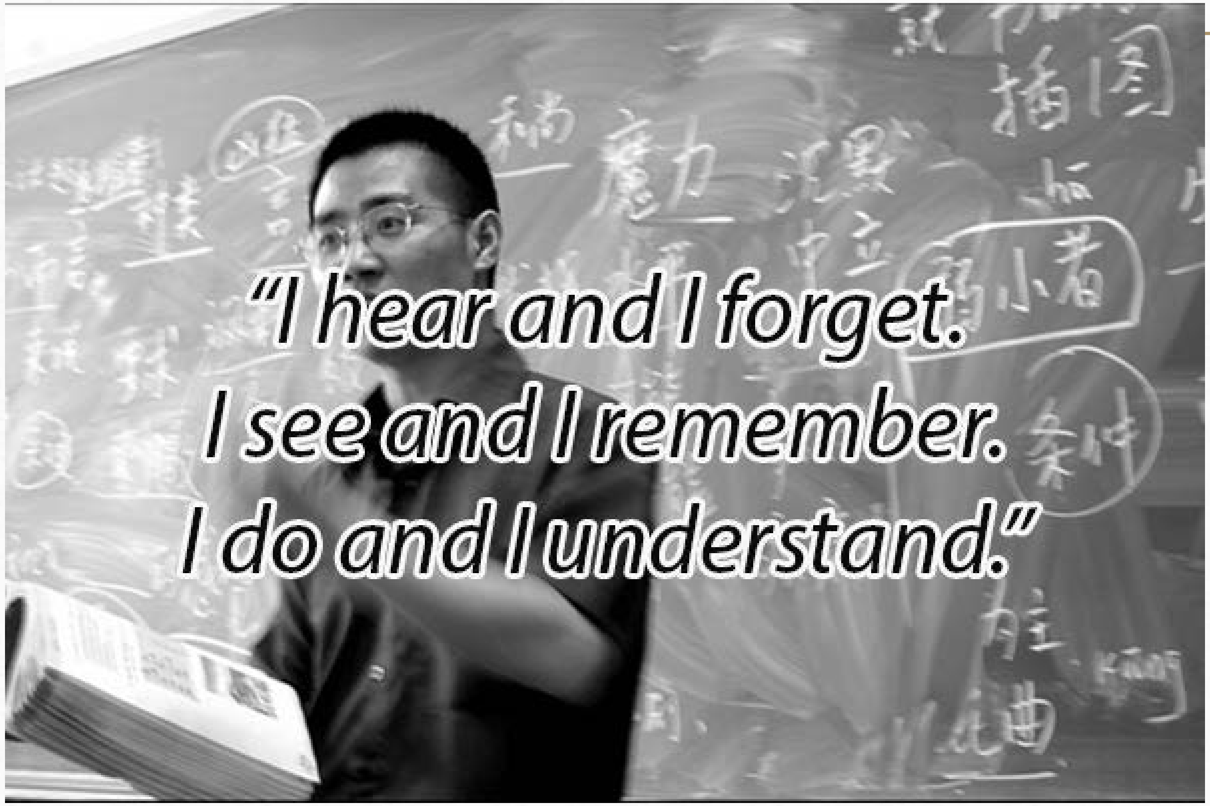
$$q(t) = 20(e^{-t} - e^{-2t}).$$

(a) How much of the drug is in the bloodstream at time $t = 0$?

(b) When is the maximum quantity of drug in the bloodstream? What is that maximum?

(c) In the long run, what happens to the quantity? 4


"I hear and I forget.
I see and I remember.
I do and I understand."