# CALCULUS AND ANALYTICAL GEOMETRY

BSE : 3rd Semester

22/1/24

- Calculus (study of change) and Geometry (study of shapes)

Calculator (Latin word means "small peebles") ← → Calculation

* Analysis the change and explain it in the graph. (Def.)
* Calculus (Everything changes with respect to others) (weather changing) (Baby Growth) (Person walking)
* Change may produce collection or may produce dispersion.

* Derivation (change w.r.t other)
* Integration (How much speed the things change)

* Calculus → ① Number System (Intervals (Line, ray), Inequalities)

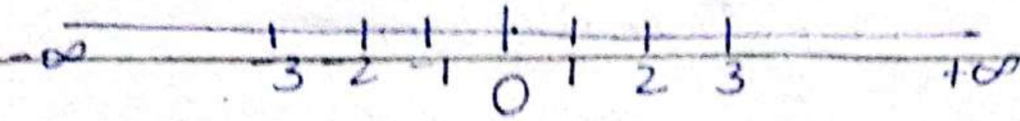
→ Natural numbers, Whole numbers, Integers, Rational, Irrational ($\frac{22}{7}$, $\sqrt{2}$), Real Numbers.
(If we divide & then not converted in original numbers.)

→ Real Numbers : (Minus infinity and plus infinity are not numbers but symbols) : $(-\infty, +\infty)$

## ⊛ Real Line:

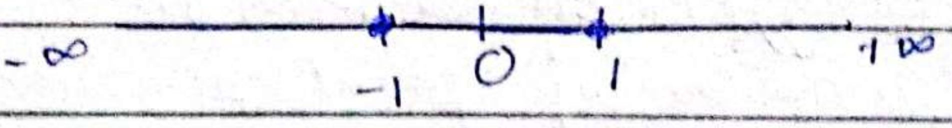
Line, Line Segment (Part of line)
ray (one point coutinous . other line).

(0.001

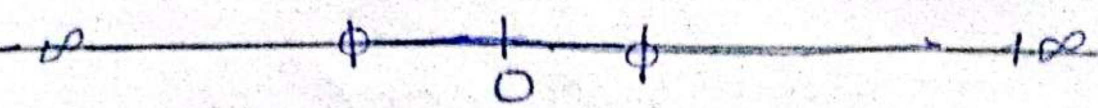



## ⊛ Intervals: (Subset of real line).

→ $[a, b]$ = closed interval $[-1, 1]$ $-1 \le 0.5 \le 1$ $-1 \le x \le 1$
Set form : $\{x \mid -1 \le x \le 1\}$
Graphical representation :

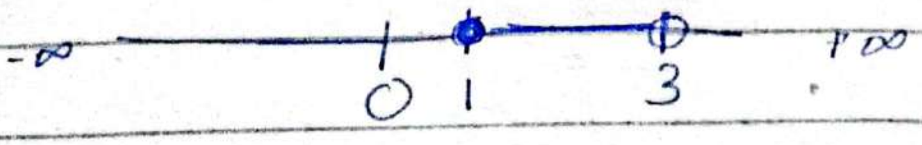



→ $(a, b)$ = open interval $(-1, 1)$
$-1 < x < 1$
Set form : $\{x \mid -1 < x < 1\}$
Graphical representation :




→ $[a, b)$ = semi open/closed $[1, 3)$
Set form : $\{x \mid 1 \le x < 3\}$

Graphical representation:




→ (a, b] = Semi closed/open (-10, 0.5]

set form: $\{x \mid -10 < x \leq 0.5\}$

Graphical representation:


⊛ Finite Intervals
(Both end points are defined).

⊛ Infinite Intervals:
(At least one end-point is defined).

⊙ **Inequalities:** (Relationship b/w algebraic expressions)
Comparison
(May be =, >, <)

⊛ $x, y$

$x < y$ (exactly less), $x > y$, $x = y$, $x \geq y$ (restricted), $x \leq y$

⊛ **Rules:** ⊙ $\frac{a}{2} < \frac{b}{3}$ (Inequality holds in addition & subtraction).

$2+5 < 3+5$

$2-5 < 3-5$

: Engineering
S.E(Agile)
RDCT
Requirement Design Cost Time

- $5 < 6$ ; $5 < 6$ ; $c < 0$

$c > 0$ ; $2 \times 5 < 2 \times 6$ ; $-2 \times 5 > -2 \times 6$

$10 < 12$ ; $-10 > -12$

- $a < b$ ; $a, b > 0$ ; $a, b < 0$ ; $ab > 0$

$\frac{a}{ab} < \frac{b}{ab}$

$\frac{1}{b} < \frac{1}{a}$

or $\frac{1}{a} > \frac{1}{b}$

## Solving Inequalities:
(Finding unknown value).

→ **Types:**

1. Linear (Power one of unknown)
2. Rational
3. Quadratic

Eg: - $3x - 5 \leq 3 - x$

$3x + x \leq 3 + 5$

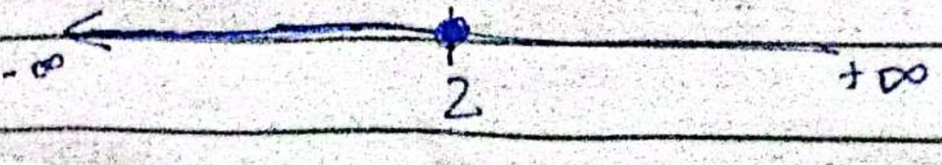
$4x \leq 8$

$x \leq 8/4$

$x \leq 2$

Sol. Set $= (-\infty, 2]$

→ Graphically:



$2x-1 < x+3$

$2x-x = 3+1$

$x < 4$

sol. set $= (-\infty, 4)$

→ Graphically:

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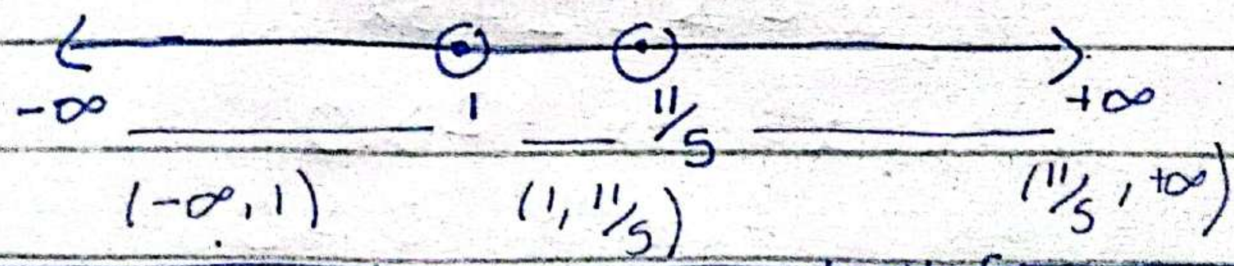
$$\frac{6}{x-1} \geq 5$$

⊙ $\dfrac{11-5x}{x-1} \geq 0$

$\Rightarrow$ Put $11-5x=0$ , $x-1=0$

$\Rightarrow x = \dfrac{11}{5}$ , $x = 1$ (Free boundary value). $\dfrac{1}{0}$ = undefined $= \infty$



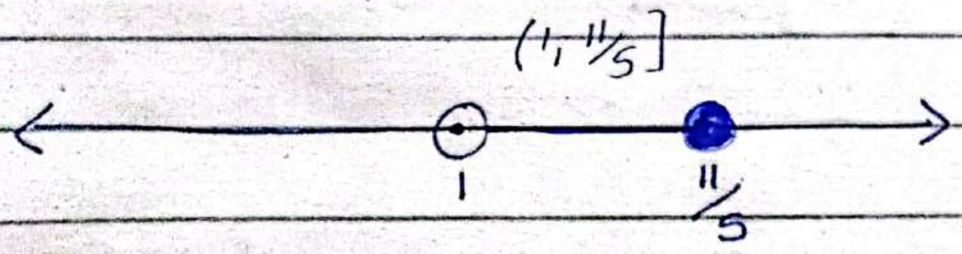

| Intervals | Test Point | $\frac{11-5x}{x-1} \geq 0$ | Conclusion |
|---|---|---|---|
| $(-\infty, 1)$ | 0 | $\frac{11}{-1} < 0$ | Not satisfied |
| $(1, \frac{11}{5})$ | 2 | $\frac{11-10}{2-1} > 0$ | Satisfied |
| $(\frac{11}{5}, +\infty)$ | 3 | $\frac{11-15}{3-1} < 0$ | Not satisfied |

The interval $\left(1, \frac{11}{5}\right)$ is satisfied the given inequality. Also $x = 11/5$ is satisfying the given inequality.

$$\text{Solution set} = \left(1, \frac{11}{5}\right) \cup \left\{\frac{11}{5}\right\}$$

$$= \left(1, \frac{11}{5}\right]$$

⊙ Graphically:




⊙ Quadratic Inequalities:

-> Word Problem:

$$x^2 \geq -2 + 3x$$

$$\Rightarrow x^2 - 3x + 2 \geq 0$$

$$\Rightarrow x^2 - 2x - x + 2 \geq 0$$

$$x(x-2) - 1(x-2) \geq 0$$

$$\Rightarrow (x-1)(x-2) \geq 0$$

$$x - 1 \geq 0 \quad , \quad x - 2 \geq 0$$

$$x \geq 1 \quad , \quad x \geq 2$$

Sol:- Let $x$ be the number that satisfied the given condition.

Therefore,

$$x^2 \geq 3x-2$$

$$\Rightarrow x^2-3x+2 \geq 0$$

$$\Rightarrow x^2-2x-x+2 \geq 0$$

$$x(x-2)-1(x-2) \geq 0$$

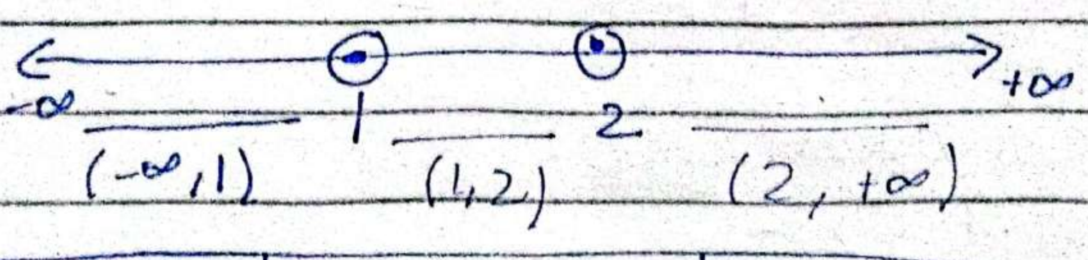
$$\Rightarrow (x-2)(x-1) \geq 0$$

$$\Rightarrow x-2=0 \text{ and } x-1=0$$

$$x=2,\ x=1$$ (these are boundary values)

∵ Free boundary points ar always found in denomin in fracti by putti value zer

⊙ Graphically:




| Intervals | Test Point | $(x-1)(x-2) \geq 0$ | Conclusion |
|---|---|---|---|
| $(-\infty,1)$ | 0 | $(-1)(-2)>0$ | Satisfied |
| $(1,2)$ | 1.5 | $(0.5)(-0.5)<0$ | Not satisfied |
| $(2,+\infty)$ | 3 | $(2)(1)>0$ | Satisfied |

The intervals $(-\infty, 1)$ and $(2, +\infty)$ are satisfying the inequalities, then $x = 1, 2$ satisfied the given inequality.

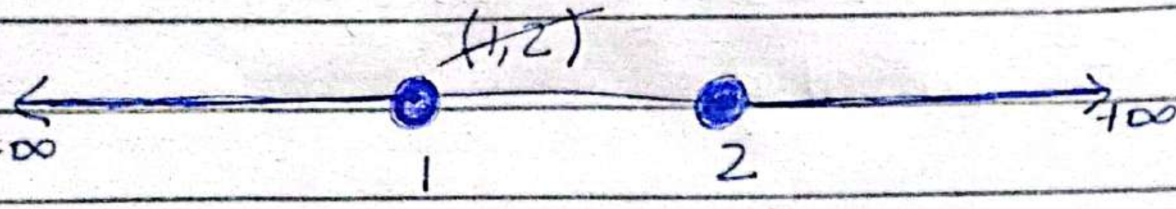
$$\text{Sol. Set} = (-\infty, 1) \cup \{1\} \cup \{2\} \cup (2, +\infty)$$

$$= (-\infty, 1] \cup [2, +\infty)$$

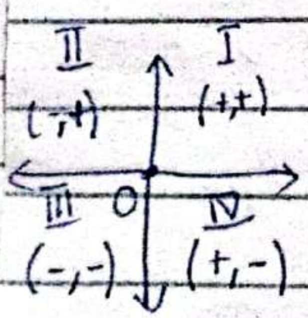
OR

$$= \underline{R - (1, 2)}$$

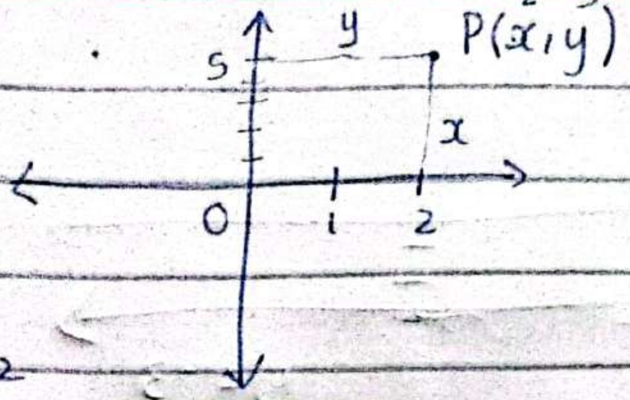
: Between 1 & 2 part not included that is why not $(-\infty, +\infty)$




## ⊙ Co-ordinate System:




- ⊙ Two lines perpendicular
- ⊙ Both lines meet is origin point.
- ⊙ Horizantal Line (x-axis), Vertical Line (y-axis).
- Divided into 4 Quadrants. (I, II, III, IV).




: General point (Intersection point).

: xy plane (2-D)

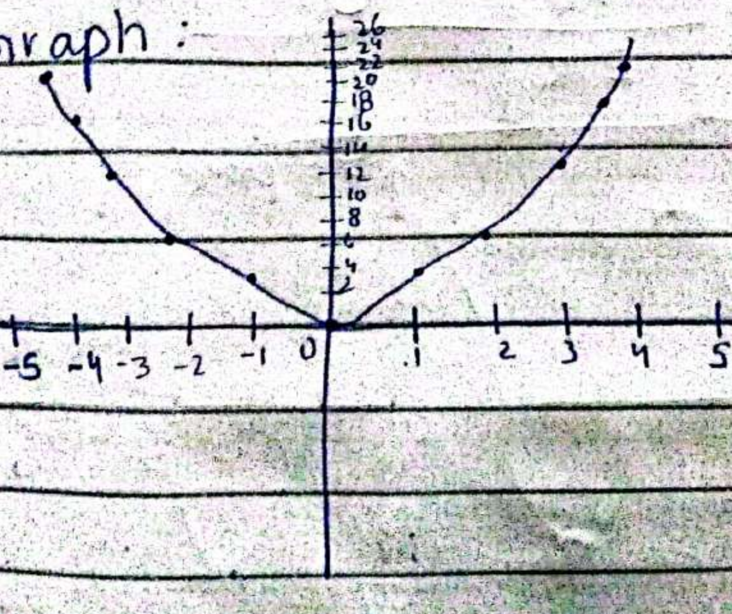
: $y = x^2$

1. Interval
2. x-axis, y-axis form
3. Ordered pair form (x, y)

• $y = x^2$ [-5, 5]

| $x$ | $y = x^2$ | $(x, y)$ |
|---|---|---|
| -5 | $25 = (-5)^2$ | (-5, 25) |
| -4 | 16 | (-4, 16) |
| -3 | 9 | (-3, 9) |
| -2 | 4 | (-2, 4) |
| -1 | 1 | (-1, 1) |
| 0 | 0 | (0, 0) |
| 1 | 1 | (1, 1) |
| 2 | 4 | (2, 4) |
| 3 | 9 | (3, 9) |
| 4 | 16 | (4, 16) |
| 5 | 25 | (5, 25) |

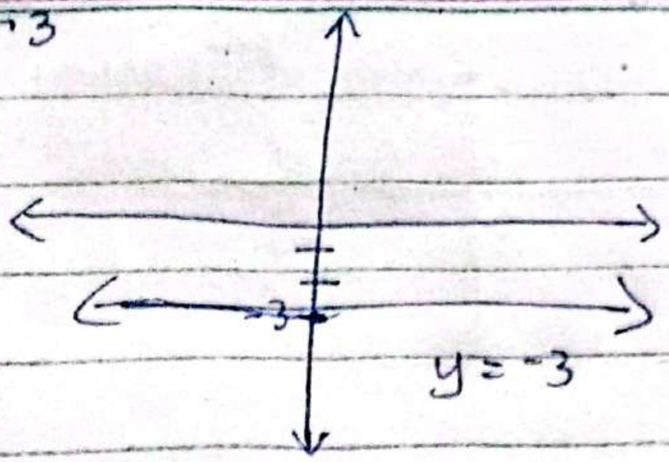
→ Graph:




Horizontal Scale
$= \frac{5-(-5)}{11} = \frac{10}{11} = 0.9$
$\approx 1$

Vertical Scale
$\frac{25-0}{11} = 2.2 \approx 2$

⊙ $y = -3$




| $x$ | $y=-3$ |
|---|---|
| -5 | -3 |
| -4 | -3 |
| -3 | -3 |
| -2 | -3 |
| -1 | -3 |
| 0 | -3 |

: $y = e^{-x}$

$y = 2\ln x$

⊙ $y = \cos 2x$

: Adding $\frac{\pi}{4}$

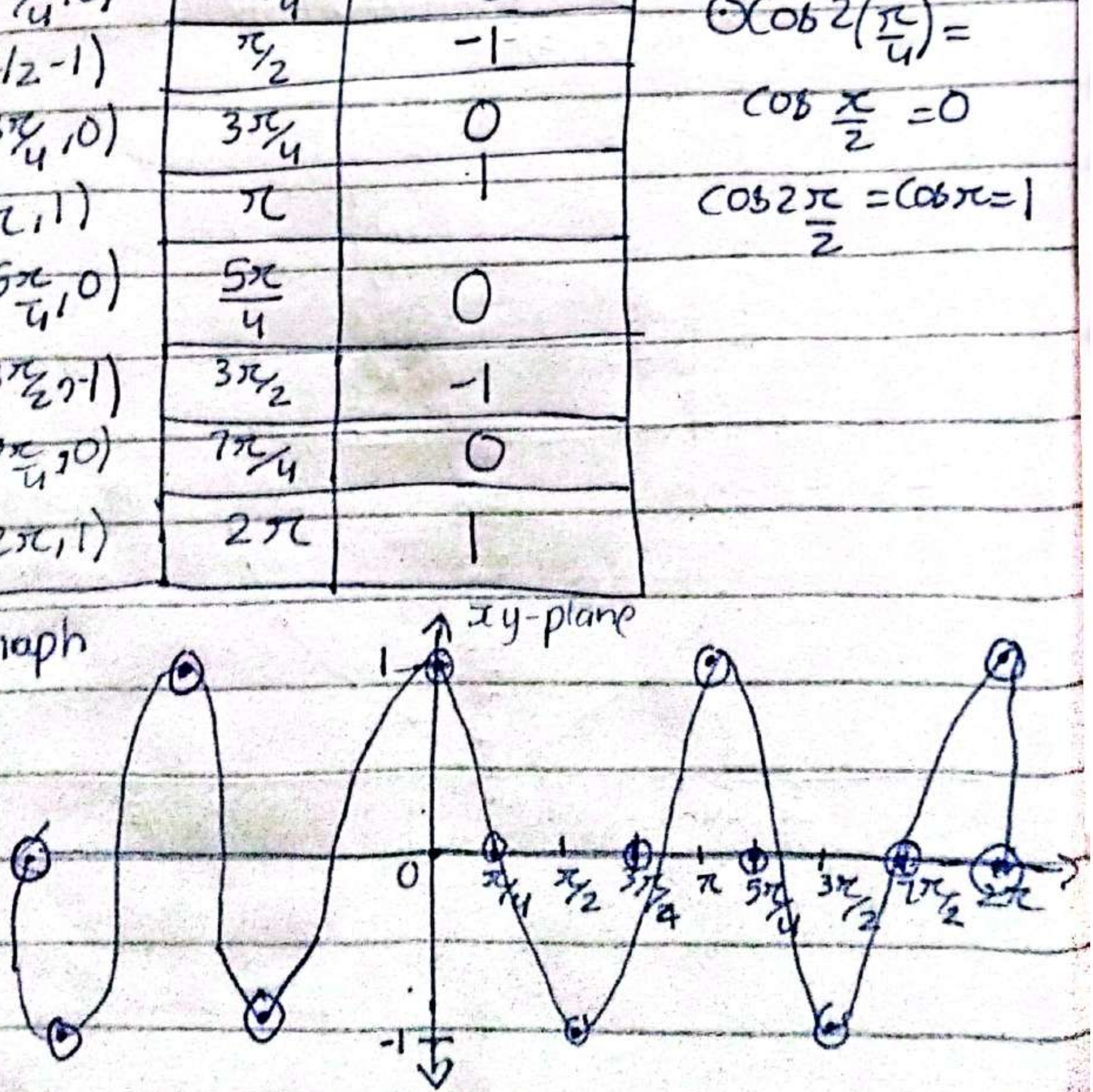
| $x, y$ | $x$ | $y=\cos 2x$ |
|---|---|---|
| $(0, 1)$ | $0$ | $1$ |
| $(\pi/4, 0)$ | $\pi/4$ | $0$ |
| $(\pi/2, -1)$ | $\pi/2$ | $-1$ |
| $(3\pi/4, 0)$ | $3\pi/4$ | $0$ |
| $(\pi, 1)$ | $\pi$ | $1$ |
| $(\frac{5\pi}{4}, 0)$ | $\frac{5\pi}{4}$ | $0$ |
| $(3\pi/2, -1)$ | $3\pi/2$ | $-1$ |
| $(\frac{7\pi}{4}, 0)$ | $7\pi/4$ | $0$ |
| $(2\pi, 1)$ | $2\pi$ | $1$ |

⊙ $\cos 2(\frac{\pi}{4}) =$

$\cos \frac{\pi}{2} = 0$

$\cos \frac{2\pi}{2} = \cos\pi = 1$

⊙ Graph



## (*) Rectangular and Polar Coordinates System: (Relation)

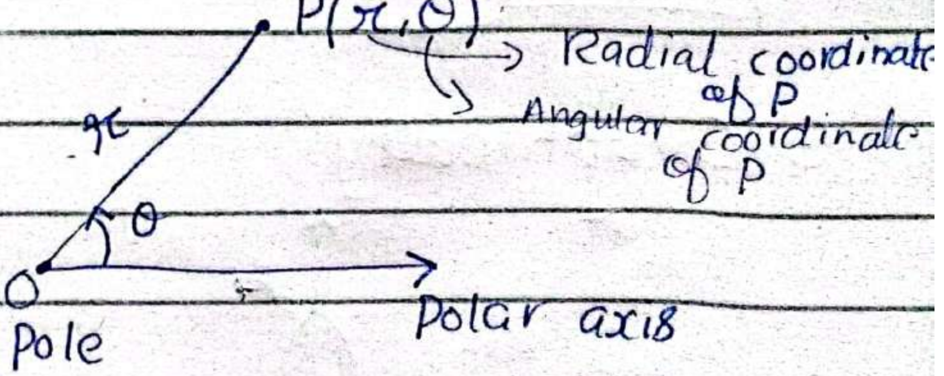
(*) $x = r\cos\theta$

$y = r\sin\theta$

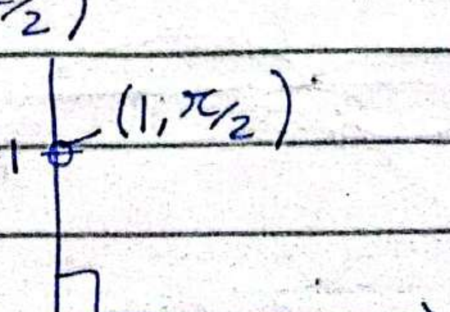
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

(*) $r^2 = x^2 + y^2$

## ⊙ **Polar Coordinate System:**




⊙ $(1, \pi/2)$




: First $\frac{\pi}{2}$ (90°) made on polar axis
: then $r$ on (1).

⊙ Draw graph $r = \cos 2\theta$ in Polar coordinate System:

(Same table as $y = \cos 2x$)

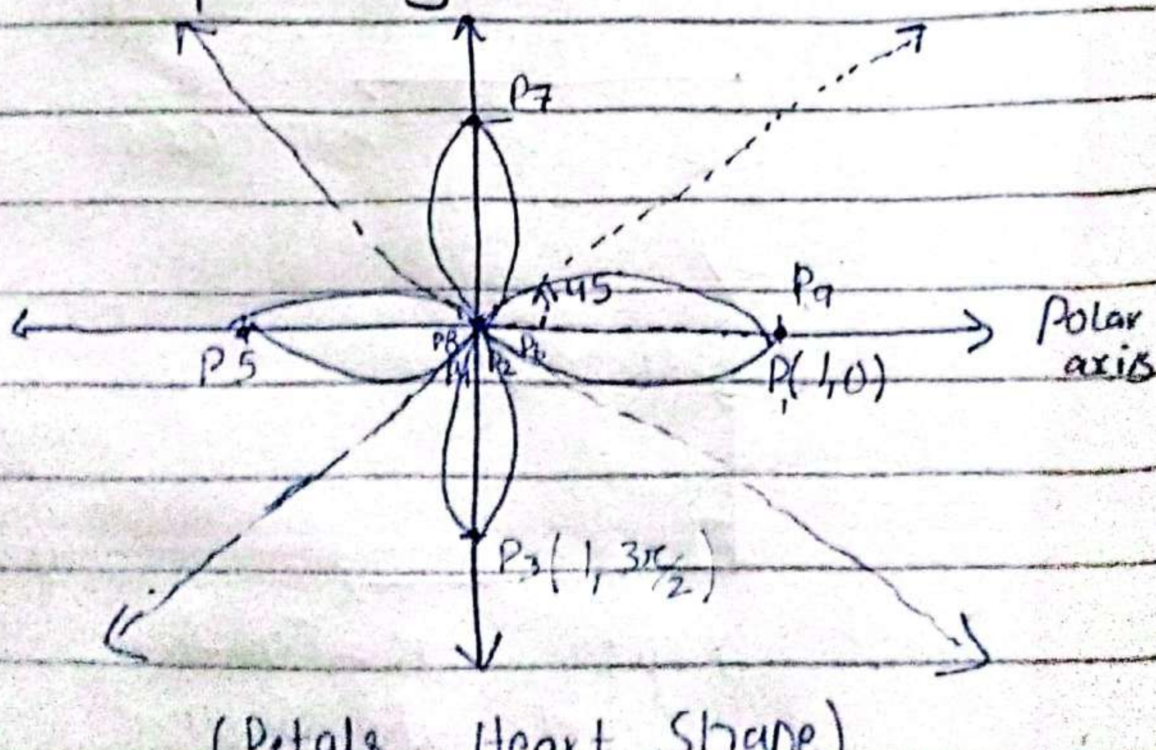
: If $r < 0$ then add $\pi$ in it.

| $\theta$ | $r = \cos 2\theta$ | $r, \theta$ |
|---|---|---|
| 0 | 1 | $(1, 0)$ |
| $\pi/4$ | 0 | $(0, \pi/4)$ |
| $\pi/2$ | $-1$ | $(1, 3\pi/2)$ |
| $3\pi/4$ | 0 | $(0, 3\pi/4)$ |
| $\pi$ | 1 | $(1, \pi)$ |
| $5\pi/4$ | 0 | $(0, 5\pi/4)$ |
| $3\pi/2$ | $-1$ | $(1, 5\pi/2)$ |
| $7\pi/4$ | 0 | $(0, 7\pi/2)$ |
| $2\pi$ | 1 | $(1, 2\pi)$ |

$\pi + \pi/2$

$\pi + \frac{3\pi}{2}$

## Graphically:




(Petals, Heart Shape)

## HomeWork Questions:

$$\boxed{\begin{aligned} r &= 2 + 2\cos\theta \\ r &= 6\sin\theta \end{aligned}}$$

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# * Function:

$f: X \rightarrow Y$ (set) (Relation) (Cartesian Product)

: $A \times B = \{1, 2\} \times \{3, 4\}$

$A \times B = (1,3), (1,4), (2,3), (2,4)$

$\frac{a}{b}$ $R_1 = \{\}$ : Completely divides

$R_2 = \{\frac{b}{a}\}$

① Every element
② Uniqueness (Image)

$R_2 = \{(1,3), (1,4), (2,4)\}$

$1 \longrightarrow 3$ (Relation)
$1 \xrightarrow{X} 4$ But not function

Rule

$f: X \longrightarrow Y$

Domain ($X$) → Co-domain ($Y$)

: $y = f(x)$

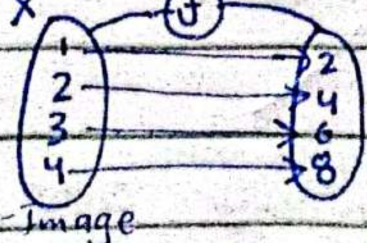
$f(x) = y \quad , \forall x \in X$

Image of $x$ under $f$

(*) Onto : co-domain = Range

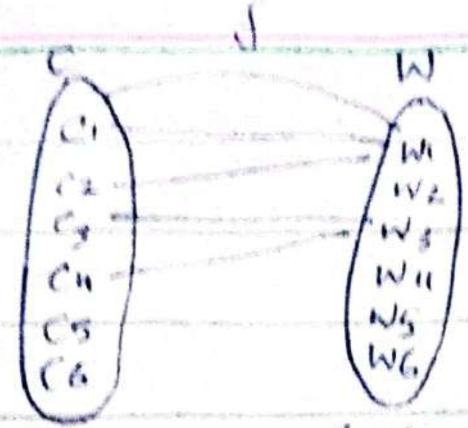
(*) $f: X \rightarrow Y$

$f(x) = 2x$

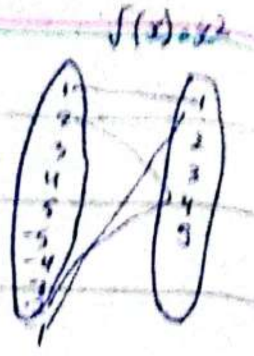



: Preimages unique

One to one function. (If Image is related unique by pre-Image then one-one function).




: one-one



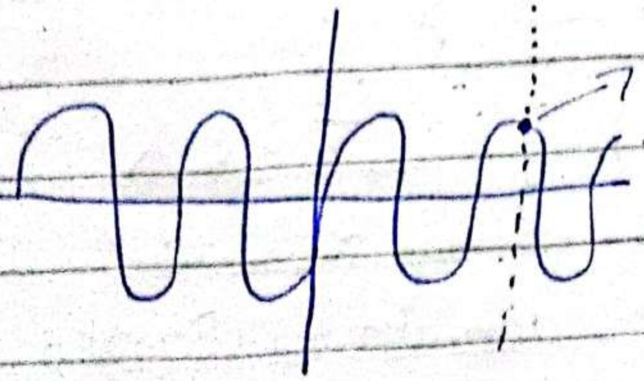

(Unique) ( May or may not unique)

$f(c_i) = w_i \quad \forall \ c_i \in C$

# ⊛ Geometric Approach:

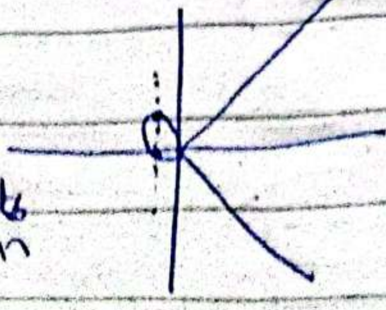
## → Vertical Line Test: (Intersect graph at one point)

⊙ $y = \sin x$



Since every vertical line intersects the graph of $y = \sin x$ exactly at one point. So, it is a graph of a function.

⊙



: y-axis represents Range :x-axis represents Domain

: Since the vertical line intersects the graph not exactly at one point, so, it is not a graph of a function.

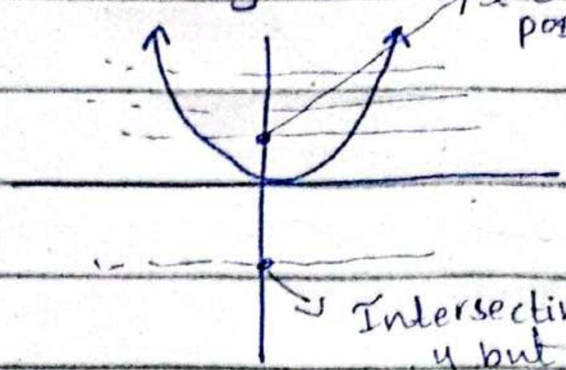
⊛ If rational number and square root not involved then its range is always real line and domain also real.

$\frac{1}{x}$ $\sqrt{x}$

## Horizontal Line :

If Horizontal line intersect the graph and meet y-axis, then that point belong to range.

$y = x2$




Range of $f(x)$
$= [0, +\infty)$

Intersecting y but not graph so it is not onto

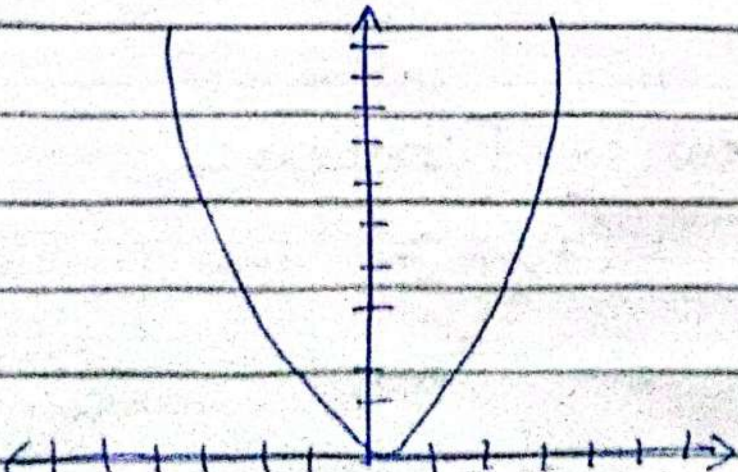
: onto Range equal to real numbers

: one-one (Horizontally at one point)

: One to one function because horizontal line intersects at one point.

: Geogebra (calculator) search on Mobile or computer

$y = x^2$



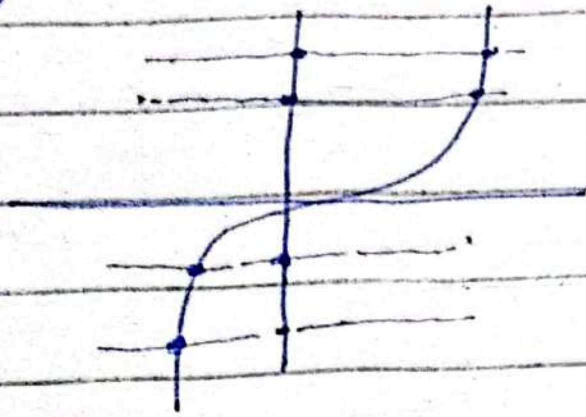
: Domain (all axis) (R) (all squars are defined).

: Range $[0, +\infty)$

It is not one-one function because horizontal line not intersects at one point only.

* It is not onto because every horizontal line not intersects at one point of graph. (Down graph not intersects).

* $y = x^3$



: Both $f(x)$ onto and one-one

: Range $f(x) = R$

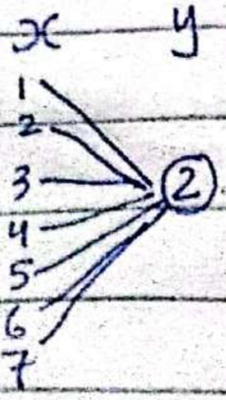
Domain $f(x) = R$

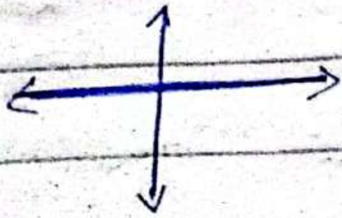
## * Basic Functions:

→ Constant Function:

$f(x) = 2$

Horizontal graph Line



: Range: output (2)

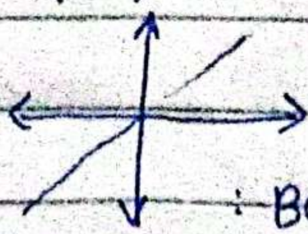
: Domain: All real numbers

: Not onto & not one-one.

→ Identity Function:

$f(x) = x$

: Straight Line graph



: Domain = R

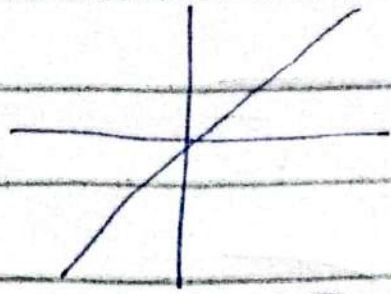
Range = R

: Both one-one and onto

→ Linear Function: 

: Straight line graph

Identity function is also linear function. Power of $x$ should be 1. $f(x) = x$

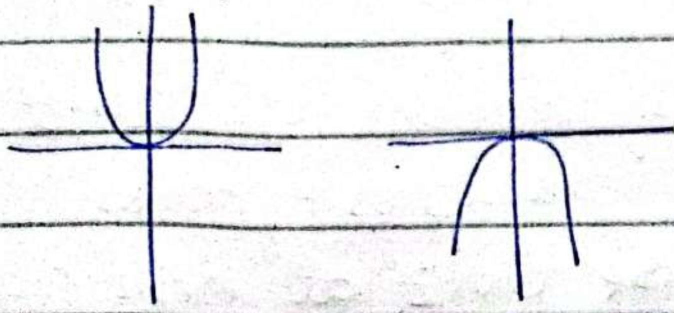


→ Quadratic Function:

$f(x) = ax^2$

: Parabola graph

: Not onto and one-one.



→ Rational Function:

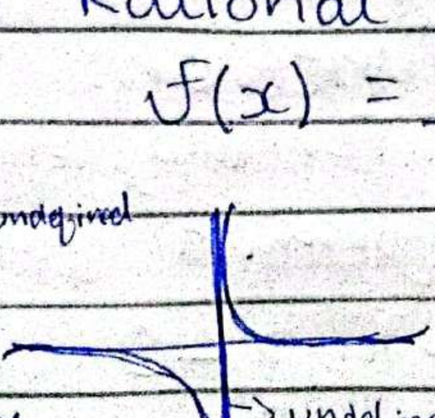
$f(x) = \frac{p(x)}{q(x)}$ → polynomial

: $\frac{1}{0}$ = undefined

(Point undefined (Bound check) or not)

: Domain $= R - \{0\}$

Range =




$f(x) = \frac{1}{x}$

: $f(x) = \frac{1}{(x-1)(x-3)}$

Domain of $f(x) = R - \{1,3\}$

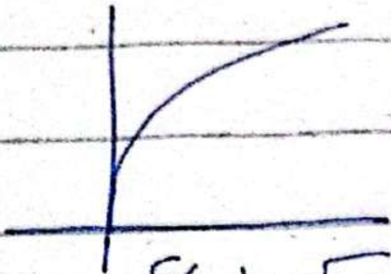
$= (-\infty, 1) \cup (1,3) \cup (3, +\infty)$

→ Square root function:

$f(x) = \sqrt{x}$

: Curve graph

⊙



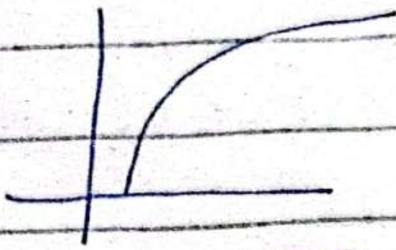
$f(x) = \sqrt{x}$

: Domain$(0, +\infty)$

: Range$(0, +\infty)$

: Only one-one function

⊙



$f(x) = \sqrt{x-1}$

: $\sqrt{-3-1}$

$\sqrt{-4}$ ✗

Not defined

$x - 1 \geq 0$

: Domain: $x \geq 1$

: Range: $(0, +\infty)$

: One-one but not onto

: 0.1 → function all solved examples + notes

(Suprise Quiz at next class)

## ⊛ Piece-Wise Function:

(Domain cutting)

(Formula will change dependent on $x$)

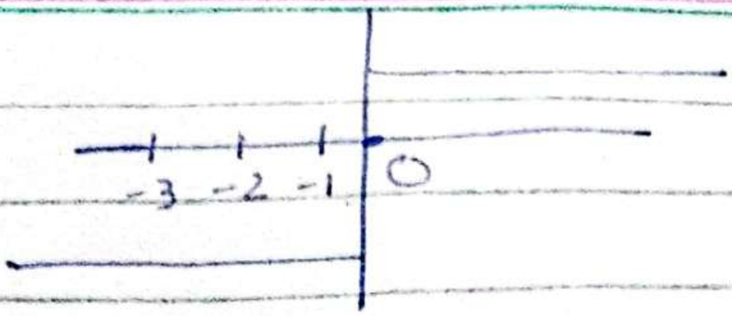
For different values of $x$, different formula given.

⊙ $f(x) = \begin{cases} -1 & x < 0 \\ 1 & 0 \leq x \end{cases}$

$x < 0$

| $x$ | $f(x) = -1$ | |
|---|---|---|
| -1 | -1 | (-1, -1) |
| -2 | -1 | (-2, -1) |
| -3 | -1 | (-3, -1) |
| -4 | -1 | (-4, -1) |

| $x$ | $f(x) = 1$ |
|---|---|
| 0 | 1 |
| 1 | 1 |
| 2 | 1 |
| 3 | 1 |




: We can't estimate which number is after or before zero

: Not one-one & onto

$$f(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

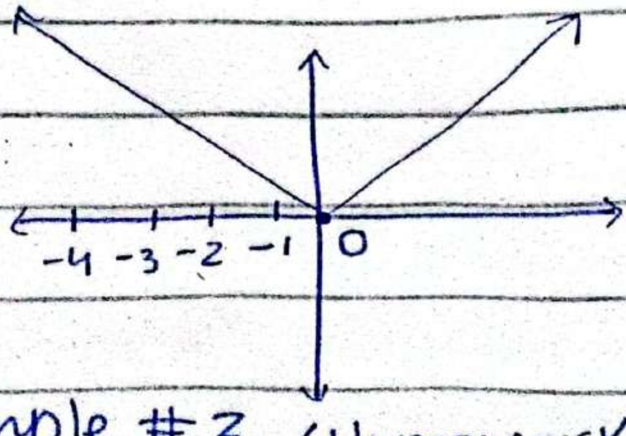
$x < 0$

| $x$ | $f(x) = -x$ |
|---|---|
| -1 | 1 |
| -2 | 2 |
| -3 | 3 |
| -4 | 4 |

$x > 0$

| $x$ | $f(x) = x$ |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 3 | 3 |
| 4 | 4 |

Straight line graph




: Domain (Real line)

: Range $(0, +\infty)$

: onto & one-one function.

: Example #3 (Homework)

31/1/24

## Function Representation:

1. By Formula:

   $y=x^2$, $f(x)=x^2$, $C=2\pi r$

2. By Graphically (Geometrically).

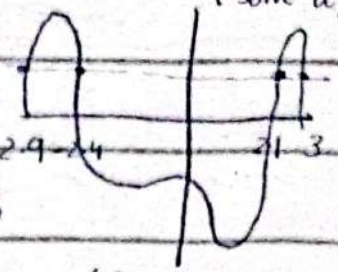
③ By Table
(We make table for $x$ and $y$).

(We study later) ★ ④ By Words (Verbally)
(Newton's Law etc....).

## EXERCISE 0.1

: Q1-10, 19-24 (Important) (Solve it).

⊛ Given that




- $y = -1 \rightarrow x = -2.9, -2.4, 2.1, 3$ ($f(x)$)
- $y = 3 \rightarrow x =$ no value $= \{\}$
- $x = 3 \rightarrow y = 0$
- $y \leq 0 = [0, -3] \longrightarrow x = [-1.8, 2.1]$
- (max) $(x, y) = (-2.5, 2.5)$
- (min) $(x, y) = (1.2, -2.2)$

## ⊛ LIMIT AND CONTINUITY

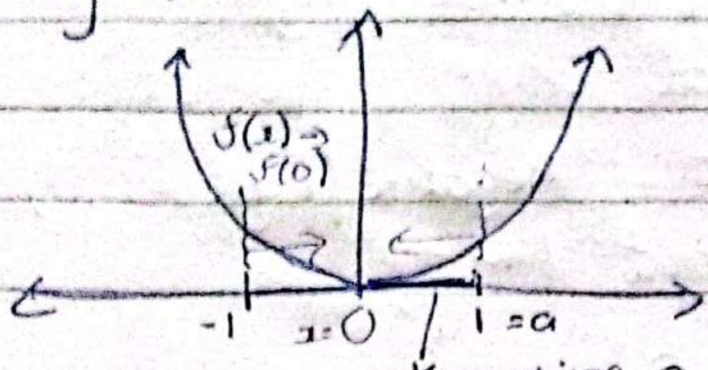
**Limit:** (Best Prediction)(About Missing)(Prediction about weather)(Before and after)

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \text{.......}\ 0$

: $\frac{1}{n}$ (approaching toward zero).

(It will not be equal to zero at any ~~maximum value~~)

(*) $y = x^2$




$\lim_{x \to 0} x^2 = ?$

$x \to 0$

$f(x) \to l$

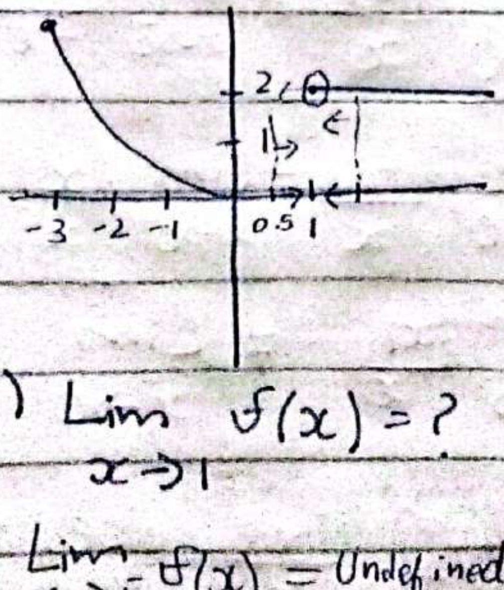
(i) $\lim_{x \to 0} x^2 = ?$

$\lim_{x \to 0^+} x^2 = 0$ , $\lim_{x \to 0^-} x^2 = 0$

Since $\lim_{x \to 0^+} x^2 = \lim_{x \to 0^-} x^2 = 0$

$\therefore$ $\lim_{x \to 0} x^2 = 0$

(*)




(ii) $\lim_{x \to 1} f(x) = ?$

$\lim_{x \to 1^-} f(x) = \text{Undefined}$ $\lim_{x \to 1^+} f(x) = 2$ : (value of y)

Function Value $= 1$

$f(1) =$ Does not exist

(*) If left and right limit not equal then function Limit not exist. Negative (left limit) and Positive (Right limit).

(*) $\lim_{x \to a} f(x) = ?$

(i) $\lim_{x \to a^-} f(x) = 3$

(ii) $\lim_{x \to a^+} f(x) = 1$

: $f(a) = 2$
Function value
~~(filled point)~~

(*) $\lim_{x \to 0} \frac{|x|}{x} = ?$

(i) $\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{|-x|}{-x} = \lim_{x \to 0^-} \frac{x}{-x} = -1$

(ii) $\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$

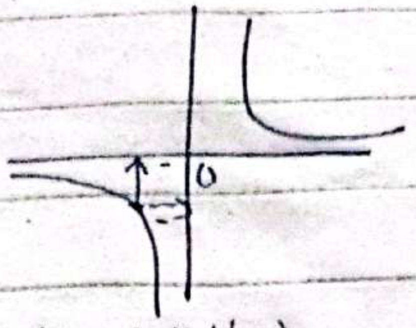
Since $\lim_{x \to 0^-} \frac{|x|}{x} \neq \lim_{x \to 0^+} \frac{|x|}{x}$

Therefore $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

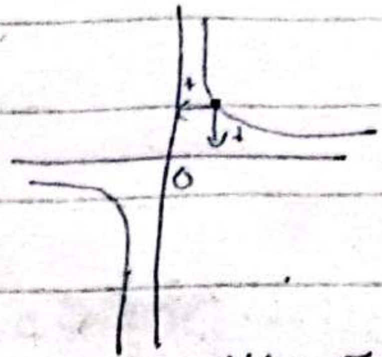
(*) Continuity:
1. Limit exist
2. Function value exist
3. Limit = Function value

: Then this function is continuous

(*) $f(x) = \frac{1}{x}$     $\lim_{x \to 0} f(x) = \frac{1}{x}$




(Negative infinity)




(Positive Infinity)

## 12/2/24

## * Limit and Continuity:

(Prediction at a specific point)

(i) $\lim_{x \to a} f(x) = L$

$\rightarrow \lim_{x \to a^-} f(x) = L_1$

$\rightarrow \lim_{x \to a^+} f(x) = L_2$

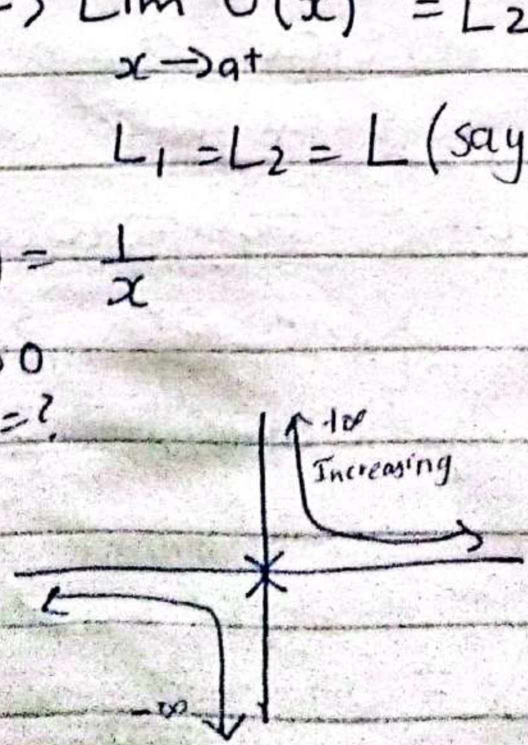
: left ≠ right (Limit not exists)

$L_1 = L_2 = L$ (say)

(*) $f(x) = \frac{1}{x}$

$x \to 0$

$\frac{1}{x} = ?$



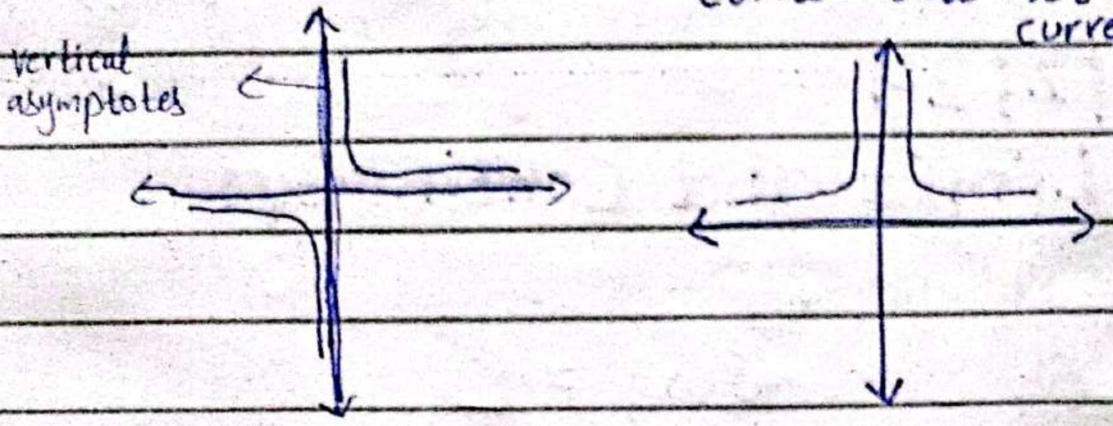
→ $\lim_{x \to 0} \frac{1}{x} = ?$

(i) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ , $\lim_{x \to 0^+} \frac{1}{x} = +\infty$

(By Graph).

## ⊛ Asymptotes:

(The line that move along curves but not intersect curves)




⊛ It is a straight line which continuously approaches curve but not intersects it till infinity. Vertical asymptotes which moves vertically with the curve.

⊛ Vertical line asymptotes lies where given function is undefined.

⊛ Rational function: $\frac{p(x)}{q(x)}$

and $p(x)$ and $q(x)$ polynomials and $q(x) \neq 0$.

## → Vertical Asymptotes:

$$f(x) = \frac{p(x)}{q(x)}$$

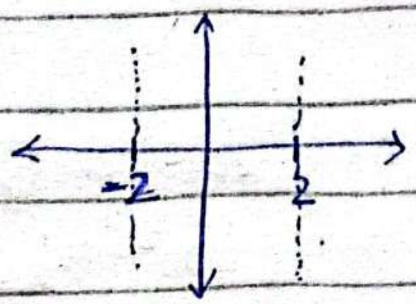
$: q(x) = 0$

⊙ $f(x) = \frac{1}{x^2-4}$

put $x^2-4=0$

$x = \pm 2$

$\boxed{x=-2}$, $\boxed{x=2}$




→ $f(x) = \frac{p(x)}{q(x)}$

$\mathrm{Deg}(P(x)) = n$

$\mathrm{Deg}(q(x)) = m$

: Leading coefficients with higher power.

(i) $n < m$
→ Horizontal asymptotes: $y=0$

(ii) $n = m$
→ Horizontal asymptotes $\therefore y = \frac{a_n}{b_m}$

(iii) $n > m$
→ No horizontal asymptotes exist.

(i) $f(x) = \dfrac{5x+2}{x^2+2x+1}$

: H.A = Horizontal asymptotes.

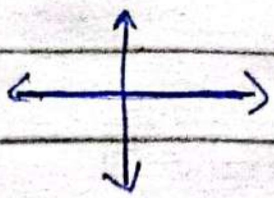
**H.A :**

$n = \text{Deg}(5x+2) = 1$

$m = \text{Deg}(x^2+2x+1) = 2$

Since $n < m$

$\therefore$ H.A $y = 0$



(ii) $f(x) = \dfrac{2x^2+x+1}{3x^2-4}$

**H.A :**

$n = \text{Deg}(2x^2+x+1) = 2$

$m = \text{Deg}(3x^2-4) = 2$

: Also draw HA on each graph.

Since $n = m$

Therefore $y = \dfrac{a_n}{b_n} = \dfrac{2}{3} = 0.666 = 0.67$

(iii) $f(x) = \dfrac{8x^4+x^2+2x^2+x+1}{2x^3+3x^2+1}$

**H.A :**

$n = \text{Deg}(8x^4+x^2+2x^2+x+1) = 4$

$m = \text{Deg}(2x^3+3x^2+1) = 3$

Since $n > m$

Therefore, there does not exist horizontal asymptotes.

(*) Exercise 1.1 (1-12) (Questions) (Solve it).

(*) Basic limits:

- $\lim_{x \to a} K = K$
- $\lim_{x \to 0^-} \frac{1}{x} = -\infty$
- $\lim_{x \to a} x = a$
- $\lim_{x \to 0^+} \frac{1}{x} = +\infty$

(*) Applying limits on mixed limits form in addition, subtraction, division and multiplication:

(*) $\lim_{x \to a} \ln(f(x)) = \ln\left(\lim_{x \to a} f(x)\right)$

(*) Find $\lim_{x \to 5} (x^2 - 4x + 3)$

Solution:

$$\lim_{x \to 5} (x^2 - 4x + 3) = \lim_{x \to 5} x^2 - \lim_{x \to 5} 4x + \lim_{x \to 5} 3$$

$$= \lim_{x \to 5} x^2 - 4\lim_{x \to 5} x + \lim_{x \to 5} 3$$

$$= 5^2 - 4(5) + 3$$

$$= 25 - 20 + 3$$

$$\boxed{\lim_{x \to 5} (x^2 - 4x + 3) = 8}$$

(*) $\lim_{x \to 1} (x^7 - 2x^3 + 1)^{35}$

$\left(\lim_{x \to 1} (x^7 - 2x^3 + 1)\right)^{35}$

$= \left(\lim_{x \to 1} x^7 - 2\lim_{x \to 1} x^3 + \lim_{x \to 1} 1\right)^{35}$

$= \left((1)^7 - 2(1) + 1\right)^{35}$

$= (1 - 2 + 1)^{35} = (0)^{35} = 0$

(*) $\lim_{x \to 3} \frac{x^2 - 6x + 9}{x - 3}$

First we will factorize (if any)

$= \lim_{x \to 3} \frac{x^2 - 3x - 3x + 9}{x - 3}$

$= \lim_{x \to 3} \frac{(x-3)(x-3)}{x-3}$

$= \lim_{x \to 3} (x - 3) \Rightarrow \lim_{x \to 3} x - \lim_{x \to 3} 3$

$= 3 - 3 = 0$

(*) $x - 1$

$(\sqrt{x})^2 - 1$

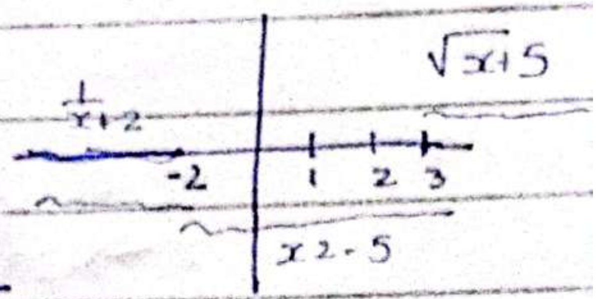
$= (\sqrt{x} - 1)(\sqrt{x} + 1)$

★ : $a^2 - b^2 = (a - b)(a + b)$

(f) Piece-Wise function:

$$f(x) = \begin{cases} \frac{1}{x+2} & x < -2 \\ x^2 - 5 & -2 < x \le 3 \\ \sqrt{x+13} & x > 3 \end{cases}$$

(a) $\lim_{x \to -2} f(x)$




(i) $\lim_{x \to -2^-} f(x) = \lim_{x \to -2} \frac{1}{x+2}$

$$= \frac{1}{-2+2} = \frac{1}{0} = \infty$$

(ii) $\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} x^2 - 5$

$$= (-2)^2 - 5$$
$$= 4 - 5$$
$$= -1$$

(b) $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 - 5$

$$= \lim_{x \to 0} x^2 - \lim_{x \to 0} 5$$

$$= 0 - 5 = -5$$

(c) $\lim_{x \to 3} f(x)$

(i) $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 5)$

$= \lim_{x \to 3^-} x^2 - \lim_{x \to 3^-} 5$

$= 9 - 5$

$= 4$

(ii) $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \left(\sqrt{x+13}\right)$

$= \sqrt{3+13} = \sqrt{16}$

$= 4$

: Q :11
(V.V.Imp)

*** ~~Quiz till studied~~ in Next class.

14/2/24

(*) Quiz :

Q1: Draw Graph

Q2: Graphically Limit

(one-one)(onto)(Bijective)

(Left or right Limit)(Function value) · (Intervals).

## EXERCISE 1.2

**(ii)** $\lim_{x \to a} [f(x) + 2g(x)]$

Sol.-

$$= \lim_{x \to a} f(x) + 2 \lim_{x \to a} g(x)$$

$$= 2 + 2(-4)$$

$$= 2 - 8 = -6$$

$\because \lim_{x \to a} f(x) = 2$

$\lim_{x \to a} g(x) = -4$

## QUESTION: 2

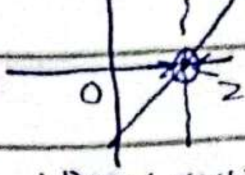
⊙ $\lim_{x \to 2} (f(x) + g(x))$

$$= \lim_{x \to 2} f(x) + \lim_{x \to 2} g(x)$$

$$= 0 + 0$$

$$= 0$$

$x \to 2$
$f(x) \to ?$
(y value)



: Draw vertical check through graph and x-axis

⊙ $\lim_{x \to 1} \dfrac{1+g(x)}{f(x)}$

$$= \frac{\lim_{x \to 1} (1+g(x))}{\lim_{x \to 1} f(x)}$$

$$= \frac{1 + \lim_{x \to 1} g(x)}{\lim_{x \to 1} f(x)} \quad \text{......} \ (1)$$

First we will find $\lim_{x \to 1} g(x)$ and

odd no.: 31+32 + 37-40
3, 5, 9, 11, 13, 7

$\lim_{x \to 1} f(x)$.

$\rightarrow \lim_{x \to 1} g(x) = 1$ $\rightarrow \lim_{x \to 1} f(x) = -1$

putting in 1

$$= \frac{1 + \lim_{x \to 1} g(x)}{\lim f(x)}$$

$$= \frac{1+1}{-1} = \frac{2}{-1} = -2 \text{ Ans}$$

**Q:11** $\lim_{x \to -1} \frac{2x^2 + x - 1}{x+1}$

$$\lim_{x \to -1} \frac{2x^2 + 2x - x - 1}{x+1}$$

$$\lim_{x \to -1} \frac{2x(x+1) - 1(x+1)}{x+1}$$

$$\lim_{x \to -1} \frac{(2x-1)(x+1)}{x+1}$$

$$\lim_{x \to -1} 2x - 1$$

$$\lim_{x \to -1} 2(-1) - 1$$

$$= -2 - 1 = -3$$

$\because a^3 - b^3$
$= (a-b)(a^2+ab+b^2)$
$= a^3 - a^2b + ab^2 - a^2b - ab^2 - b^3$
$= a^3 - 2a^2b - b^3$

**Q:3**

$$\lim_{x \to 2} x(x-1)(x+1)$$

Sol:-

$$= \lim_{x \to 2} x \cdot \lim_{x \to 2} (x-1) \cdot \lim_{x \to 2} (x+1)$$

$$= 2 \cdot (2-1)(2+1)$$

$$= 2 \cdot (1) \cdot (3)$$

$$= 6 \quad \text{Ans.}$$

**Q5:** $\lim \dfrac{x^2 - 2x}{x+1}$

$$= \lim_{x \to 3} \frac{x(x-2)}{x+1}$$

$$= \frac{\lim_{x \to 3} x \cdot \lim_{x \to 3} (x-2)}{\lim_{x \to 3} (x+1)}$$

$$= \frac{3 \cdot (3-2)}{3+1} = \frac{3}{4}$$

**Q13:**

$$\lim_{t \to 2} \frac{t^3 + 3t^2 - 12t + 4}{t^3 - 4t}$$

$\lim_{t \to 2}$

Q:19 $\lim_{x \to 2^-} \frac{x}{x^2-4}$

Sol:-

$$= \lim_{x \to 2^-} \frac{x}{(x+2)(x-2)}$$

$$= \frac{\lim_{x \to 2^-} x}{\lim_{x \to 2^-} (x+2) \cdot \lim_{x \to 2^-} (x-2)}$$

$$= \frac{2}{(2+2)(2-2)} = \frac{2}{0} = -\infty$$

19/2/24

Q:13 $\lim_{t \to 2} \frac{t^3+3t^2-12t+4}{t^3-4t}$ 3 (5,2) 2 (10,2)

$$\lim_{t \to 2} \frac{t^3+5t^2-2t^2-10t-2t+4}{t^3-4t}$$

$$\lim_{t \to 2} \frac{t^3+5t^2-2t-2t^2-10t+4}{t(t^2-2^2)}$$

$$= \lim_{t \to 2} \frac{t(t^2+5t-2)-2(t^2+5t-2)}{t(t+2)(t-2)}$$

$$= \lim_{t \to 2} \frac{(t^2+5t-2)(t-2)}{t(t+2)(t-2)}$$

$$= \lim_{t\to 2} \frac{t^2+5t-2}{t(t+2)}$$

$$= \frac{(2)^2+5(2)-2}{2(2+2)}$$

$$= \frac{4+10-2}{8} = \frac{\cancel{12}\ 3}{\cancel{8}\ 2}$$

$$= \frac{3}{2}$$

**Q32:**

$$g(t) = \begin{cases} t-2 & t<0 \\ t^2 & 0\le t\le 2 \\ 2t & t>2 \end{cases}$$

(i) $\lim_{t\to 0} g(t) = ?$

$\rightarrow \lim_{t\to 0^-} g(t) = \lim_{t\to 0^-} t-2$

$= 0-2 = -2$

$\rightarrow \lim_{t\to 0^+} g(t) = t^2$

$= 0^2 = 0$

Since

$\lim_{t\to 0^-} g(t) \neq \lim_{t\to 0^+} g(t)$

Therefore $\lim_{t \to 0} g(t)$ doesnot exist

(b) $\lim_{t \to -96} g(t) = ?$

$\rightarrow \lim_{t \to -96} g(t) = \lim_{t \to -96} t-2$

$= -96 - 2$

$= -98$

** (*) Ex: $1.3^{(1.2)}$ + Solved Examples

(C.P : Mark 5) (Next class Quiz)

21|2|24

* $x \rightarrow a$ (real number)

$f(x) \rightarrow l$

$x \rightarrow \pm\infty$ (Infinity)

$f(x) \rightarrow ?$

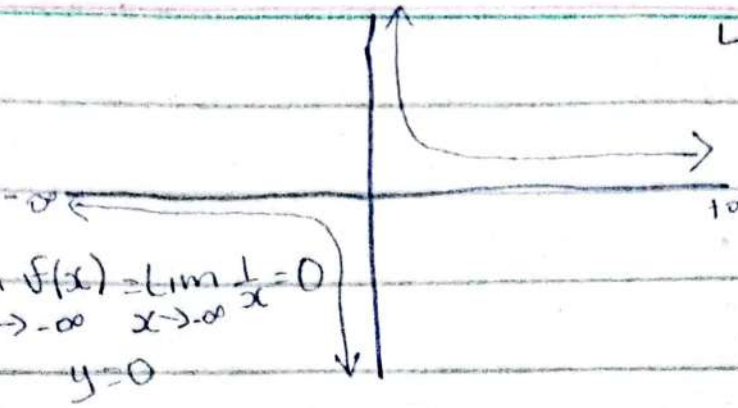
* $f(x) = \frac{1}{x}$

(i) $\lim_{x \to +\infty} \frac{1}{x} = \frac{1}{\infty} = 0$

$\lim_{x \to -\infty} \frac{1}{x} = \frac{1}{-\infty} = 0$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{1}{x} = 0$$

$y = 0$




$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{x} = 0$$

$y = 0$

∴ Horizontal asymptotes along curve gives value of function

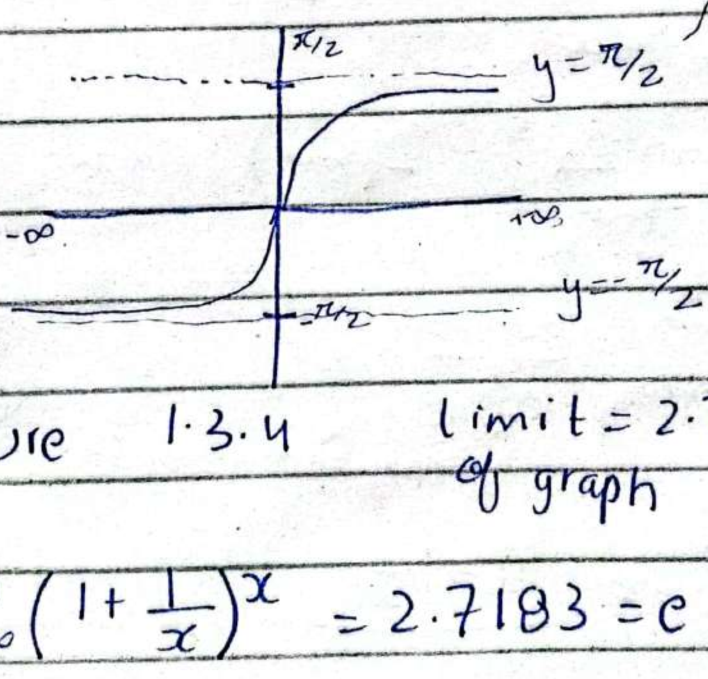
☆ Monday (Rational functions) (C.P)

# * Example : 2

(i) $\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$

$\lim_{x \to \infty} \tan^{-1} x = -\pi/2$

(ii)




$\lim_{x \to +\infty} \tan^{-1} x = H.A = \pi/2$

$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$

(*) Figure 1.3.4 limit of graph $= 2.7183$

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = 2.7183 = e$$

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = 2.7183 = e$$

(*) $$\lim_{x \to +\infty} \frac{1}{x^n} = \left(\lim_{x \to +\infty} \frac{1}{x}\right)^n$$

$$= \left(\frac{1}{+\infty}\right)^n$$

$: \left(1+\frac{1}{x}\right)^x = e$
$: \left(1+\frac{1}{2x}\right)^{2x} = e$

$= (0)^n \Rightarrow 0$

⊙ $\lim_{x \to +\infty} \left(1+\frac{1}{2x}\right)^x$ $\quad : \frac{1}{2x} \to 2x$

Sol:- $\lim_{x \to +\infty} \left(1+\frac{1}{2x}\right)^{2x \times \frac{1}{2}}$

$= \left[\lim_{x \to +\infty} \left(1+\frac{1}{2x}\right)^{2x}\right]^{\frac{1}{2}}$ (By power rule)

$= e^{\frac{1}{2}}$ $\quad : \lim_{x \to \infty} \left(x+\frac{1}{x}\right)^x = e$

⊙ $\lim_{x \to +\infty} 2x^5 = 2(+\infty)^5 = +\infty$

$\lim_{x \to -\infty} 2x^5 = 2(-\infty)^6 = +\infty$

: Decreasing graph/curve $(-\infty)$
: Increasing graph or curve $(+\infty)$

⊙ $\lim_{x \to +\infty} \frac{3x+5}{6x-8}$

multiply and divide it.

$: \lim_{x \to +\infty} \frac{3x}{6x}$
$= \lim_{x \to +\infty} \frac{1}{2} = \frac{1}{2}$

$\lim_{x \to +\infty} \frac{\frac{3x}{x} + \frac{5}{x}}{\frac{6x}{x} - \frac{8}{x}}$

$= \lim_{x \to +\infty} \frac{3+\frac{5}{x}}{6-8/x}$

$= \frac{3+\frac{5}{\infty}}{6-\frac{8}{\infty}} = \frac{3+0}{6-0} = \frac{3}{6} = \frac{1}{2}$ Ans

$$\lim_{x\to-\infty} \frac{4x^2-x}{2x^3-5}$$

: Behaviour same (overall behaviour matches with)

$$\lim_{x\to-\infty} \frac{4x^2}{2x^3} = \lim_{x\to-\infty} \frac{2}{x}$$

$$= \frac{2}{-\infty} = 0$$

$$\lim_{x\to-\infty} \frac{\frac{4x^2}{x^3} - x/x^3}{\frac{2x^3}{x^3} - 5/x^3}$$

$$\lim_{x\to-\infty} \frac{\frac{4}{x} - \frac{1}{x^2}}{2 - 5/x^3}$$

$$= \frac{\frac{4}{-\infty} - \frac{1}{-\infty}}{2 - 5/(-\infty)^3} = \frac{0-0}{2-0} = 0$$

$$\lim_{x\to+\infty} \frac{5x^3-2x^2+1}{1-3x}$$

: Divide all by highest denominator term

$$\lim_{x\to+\infty} \frac{\frac{5x^3}{x} - \frac{2x^2}{x} + \frac{1}{x}}{\frac{1}{x} - \frac{3x}{x}}$$

: $\lim_{x\to+\infty} \frac{5x^3}{-3x}$ (short method)

$$\lim_{x\to+\infty} \frac{5x^2 - 2x + \frac{1}{x}}{\frac{1}{x} - 3}$$

$$= \frac{5(+\infty)^2 - 2(+\infty) + \frac{1}{+\infty}}{\frac{1}{+\infty} - 3}$$

$$= \frac{0 - 0 + 0}{0 - 3} = \frac{0}{-3} = -\infty$$

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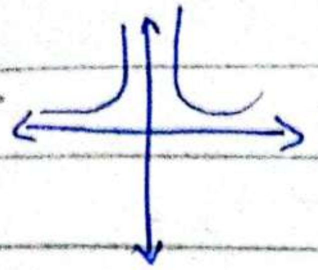
: $f(x) = \frac{1}{x2}$



→ Function Discoutinuous if limit is $+\infty$ and $-\infty$. Also if function value or limit not exist or not equal, then it is also discoutinuous.

→ Coutinuous:

→ Limit defined
→ Function value
→ Limit = Function value.

④ $f(x) = \frac{x^2-4}{x-2}$ $x=2$

Sol:-

(i) $f(2) = \frac{(2)^2-4}{2-2} = \frac{0}{0}$ = un defined

(ii) $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2}$

$= \lim_{x\to 2}(x+2) = 2+2 = 4$

(iii) Since $f(2)$ is undefined, therefore $f(x) = \frac{x^2-4}{x-2}$ is discontinuous at $x=2$.

(*) $g(x) = \begin{cases} \frac{x^2-4}{x-2} & , x \neq 2 \\ 3 & , x = 2 \end{cases}$

(i) $f(2) = 3$

(ii) $\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$

(iii) Since $f(2) \neq \lim_{x \to 2} \frac{x^2-4}{x-2}$

Therefore $f(x)$ is continous at $x=2$.

(*) $h(x) = \begin{cases} \frac{x^2-4}{x-2} & , x \neq 2 \\ 4 & ; x = 2 \end{cases}$

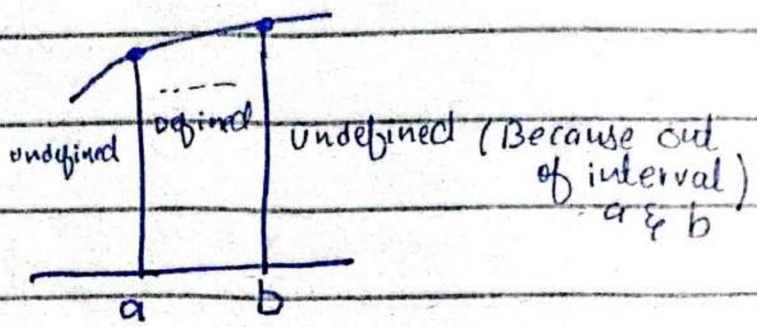
Sol:-

(i) $f(2) = 4$

(4) Real Life Examples (Signals)
(Radio, Data etc...).

(*) Continuous on an Interval:

$[-\infty, +\infty]$




→ Continuous on open-interval or closed-interval.

(*) What can you say about the continuity of function:

$$f(x) = \sqrt{9-x^2}$$

(i) $f(x)$ is defined on the interval:
$[-3, 3]$

(ii) $\lim_{x \to a} \sqrt{9-x^2}$ exists $\forall\, a \in [-3, 3]$

(iii) $f(a) = \lim_{x \to a} f(x)$ $\forall\, a \in [-3, 3]$

$f(x)$ is continuous at $[-3, 3]$

: Discontinuous at $R - [-3, 3]$

(*) $f(x) = \dfrac{x^2-9}{x^2-5x+6}$

put $x^2-5x+6=0$

$x^2-2x-3x+6=0$

$x(x-2)-3(x-2)=0$

$(x-2)(x-3)=0$

$x=2\ ,\ x=3$ function is discontinuous.

(*) $|x|$ is continuous. (show that)

Since

$$|x| = \begin{cases} x & x>0 \\ 0 & x=0 \\ -x & x<0 \end{cases}$$ (By definition).

(i) $f(0) = 0$

(ii) $\lim\limits_{x\to 0} |x|$ :

$\lim\limits_{x\to 0^+} |x| = 0$

$\lim\limits_{x\to 0^-} |x| = -0 = 0$

(i) at $x=1$

$f(1) = |x|_{x=1}$

$= 1$ (exactly for $x>0$)

(ii) $\lim\limits_{x\to 1} |x| = 1$ $\quad \lim\limits_{x\to 1} |-x| = |-1| = 1$

* ⊙ Composite function Def (Learn it).

⊙ Exercise 1.5 :

$\rightarrow x2+4=0$

$x2=-2$

$x=\pm 2i$ (continuous everywhere)

$\rightarrow \frac{3}{x}+\frac{x-1}{x2-1}$

$x(x2-1)$ $\because (-1,1,0)$ Discontinuous

$\rightarrow x^4+x=0$

$x(x3+1)=0$

$x=0 \quad , x3+1=0$

$x3=-1$ : ~~1-32~~ (Practice).

$x=-1$

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**Q: 29**

Since $f(x)$ is continuous everywhere :

$\therefore \lim_{x\to c} f(x)=f(c) \quad : \forall c\in(-\infty,+\infty)$

To find out K.

**(b)** At $x=2$

⊙ $\lim_{x\to 2^+} 2x+K = 2(2)+K = 4+K \quad \ldots\ldots (i)$

⊙ $\lim_{x \to 2^-} Kx^2 = (2)^2 \cdot K = 4K$ ........ (ii)

Since $f(x)$ is continuous :

$\therefore \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

$\Rightarrow 4 + K = 4K$ : From (i) and (ii)

$\Rightarrow 4 = 3K$

$\Rightarrow \boxed{K = 4/3}$

**Q31:**

~~Since $f(x)$ is continuous.~~

: $\lim_{x \to c} f(x) = f(c) \quad \forall c \in (-\infty, +\infty)$

To find out K and m

At $x = -1$

⊙ $\lim_{x \to -1^+} m(x+1) + K = m(-1+1) + K = K$ ....(i)

⊙ $\lim_{x \to -1^-} 2x^3 + x + 7 = 2(-1)^3 + (-1) + 7$

$= -2 - 1 + 7 = 4$ ...... (ii)

⊙ Since

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x)$

$\boxed{K = 4}$ : From (i) and (ii)

⊙ similarly

at $x = 2$

$$\therefore \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 + 5$$

$$= (2)^2 + 5 = 4 + 5$$

$$= 9 \quad \ldots (iii)$$

$$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} m(x+1) + K$$

$$= m(2+1) + K$$

$$= 3m + K$$

$$= 3m + 4 \quad \ldots (iv)$$

$$\therefore \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$$

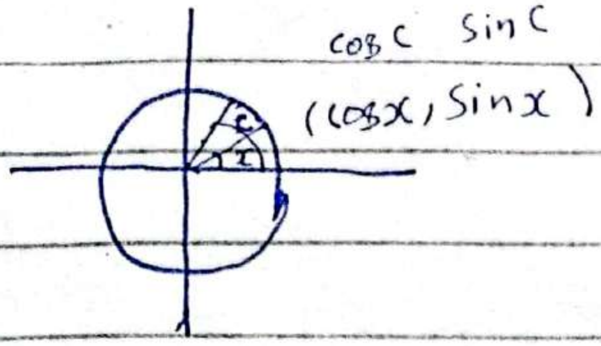
$$9 = 3m + 4$$

$$3m = 5$$

$$\boxed{m = \frac{5}{3}}$$

## ⊙ CONTINUITY OF TRIGNOMETRIC FUNCTIONS:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$




$$\lim_{x \to c} \sin x = \sin \lim_{x \to c} x$$

$$= \sin c$$

(*) $$\lim_{x \to 1} \cos\left(\frac{x^2-1}{x-1}\right)$$

$$= \cos \lim_{x \to 1} \frac{x^2-1}{x-1}$$

$$= \cos \lim_{x \to 1} \frac{(x+1)(x-1)}{x-1}$$

$$= \cos(1+1)$$

$$= \cos 2$$

: Limit would be applied on angle.

## ⊙ CONTINUITY OF INVERSE FUNCTIONS:

$f$ → one-to-one, continuous

: Inverse is possible of one-to-one and continuous function.

: $f^{-1}$ continuous on its Domain ($f$ → Range)

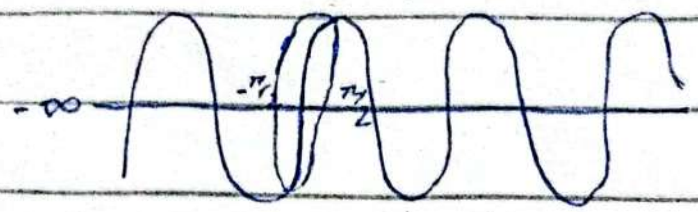
⊙ $\sin^{-1} x$ :

: Trignometric function are not one-to-one but its part is one-to-one

Sol:-

$\sin x$ is continuous at points $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Range of $\sin x = [-1, 1]$




: By 1.6.2 Theorem

Therefore $\sin^{-1}x$ is continuous at $[-1,1]$ as $[-1,1]$ is Range of $\sin x$.

## ⊙ Exponential function:

$b^x$ — $(-\infty, +\infty)$

$e^x$ $\quad$ $2^x$

$\log x, \ln x$ - $(0, +\infty)$

⊙ Example 3:

$$f(x) = \frac{\tan^{-1}x + \ln x}{x^2 - 4}$$

$f(x)$ = function

(i) $\tan^{-1}x$ is continuous everywhere i.e $(-\infty, +\infty)$

(ii) $\ln x$ is continuous when $x > 0$ or $(0, +\infty)$

(ii) $x^2 - 4$, being a polynomial, is continuous everywhere.

Since $x^2 - 4$ is a part of denominator of $f(x)$:

$x^2 - 4 = 0 \Rightarrow x = \pm 2$ are eliminated from $\mathbb{R}$:

$$(-\infty, -2) \cup (-2, 2) \cup (2, +\infty)$$

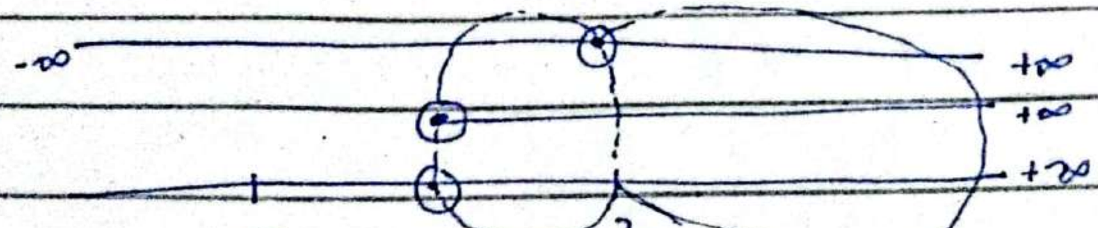
Therefore $f(x)$ is continous:

$$= (-\infty, +\infty) \cap (0, +\infty) \cap ((-\infty, 2) \cup (-2, 2) \cup (2, +\infty))$$

$$= (0, +\infty) \cap ((-\infty, -2) \cup (-2, 2) \cup (2, +\infty))$$

$$= (0, 2) \cup (2, +\infty)$$

☆




$$= (0, 2) \cup (2, +\infty)$$

## ④ SANDWICH THEOREM:

Proof: Given that: $f, g, h$

$$g(x) \leq f(x) \leq h(x)$$

$$\Rightarrow \lim_{x\to c} g(x) \leq \lim_{x\to c} f(x) \leq \lim_{x\to c} h(x)$$

$$\boxed{l \leq \lim_{x\to c} f(x) \leq l}$$

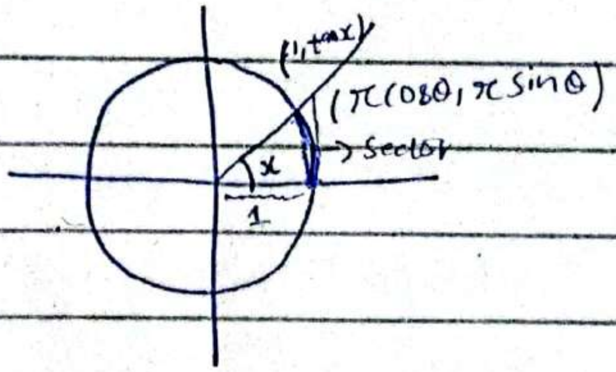
: $\lim_{x\to c} g(x) = l = \lim_{x\to c} h(x)$

$$\Rightarrow \lim_{x\to c} f(x) = l$$

: It may not be less or greater at one time so it is equal to 'l'.

$$\rightarrow \lim_{x\to 0} \frac{\sin x}{x} = 1$$

Proof:-




: As it is unit circle so $r = 1$

: $\frac{1}{2} 1 \cdot \tan x$

: $\frac{1}{2} r^2 \theta$

: Area of right triangle $\geq$ Area of vertex $\geq$ Area of triangle

: $\frac{1}{2} \cdot r^2 x$

: $\frac{1}{2} (1)^2 \cdot \sin x$

$$\frac{1}{2} \tan x \geq \frac{x}{2} \geq \frac{\sin x}{2}$$

: Multiplying by 2

$$\Rightarrow \tan x \geq x \geq \sin x$$

$$\Rightarrow \frac{\sin x}{\cos x} \geq x \geq \sin x$$

: Divide by $\sin x$

$$\Rightarrow \frac{1}{\cos x} \geq \frac{x}{\sin x} \geq 1$$

$$\Rightarrow \cos x \leq \frac{\sin x}{x} \leq 1$$

: Inverse it.

Applying Limit:

$$\Rightarrow \lim_{x\to 0} \cos x \leq \lim_{x\to 0} \frac{\sin x}{x} \leq \lim_{x\to 0} 1$$

$$\Rightarrow 1 \leq \lim_{x\to 0} \frac{\sin x}{x} \leq 1$$

$$\Rightarrow \lim_{x\to 0} \frac{\sin x}{x} = 1$$

: Practice other questions of exercises

: Quiz on Monday

$$\rightarrow \lim_{x\to 0} \frac{\sin 2\theta}{\theta}$$

: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$\Rightarrow \lim_{x\to 0} \frac{\sin 2\theta}{2\theta} \cdot 2$$

$$= 2 \cdot (1)$$

$$= 2$$

$$\rightarrow \lim_{x\to 0} \sin \frac{1}{x}$$

$$\lim_{x\to 0} \frac{\sin \frac{1}{x}}{\frac{1}{x}}$$

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## CHAPTER #2

### ⊛ Derivative:

Rate of change with

: Geomdrically or concept of tangent line respect to other.

## (i) Rate of Change:

$$\frac{\text{Rise}}{\text{Run}} \quad : \frac{\text{Function value}}{\text{Domain value}}$$

### (a) Average rate of change:

$$\frac{f(t_0)}{t_0} \qquad \frac{f(t_1)}{t_1} = \frac{f(t_1)-f(t_0)}{t_1-t_0}$$

(Interval or two points will be given.

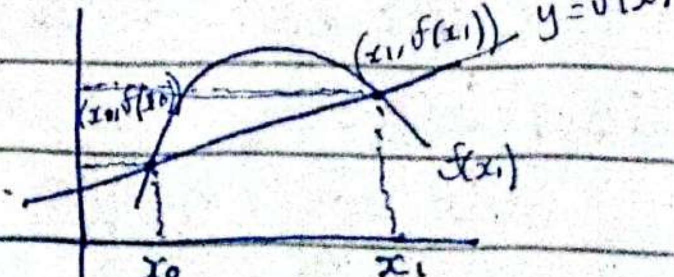
### (b) Instantaneous rate of change:

(At specific or one point change). : sudden or ایک دم

## Equation of Secant Line:

$$y-y_0 = \underline{m}(x-x_0)$$

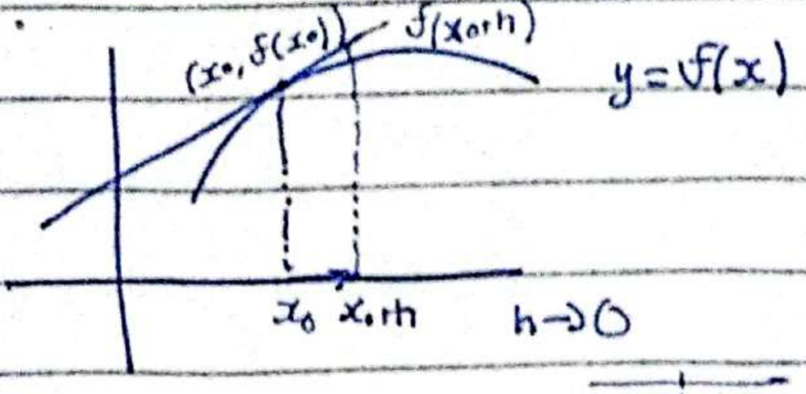
where



$$\boxed{m = \frac{f(x_1) - f(x_0)}{x_1 - x_0}}$$

## Equation of Tangent Line:

$$y - y_0 = m(x - x_0)$$




: slope represent constant rate of change.

Where

$$m_{tan} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{x_0 + h - x_0}$$

$\xrightarrow{h}$ $h = 0.0001$

: we have one point only

$$m_{tan} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

: $h = x_1$

OR

: let $x_0 + h = x_1$

$\Rightarrow h = x_1 - x_0$

$$m = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

## Example-1:

- Step-1: To find equation of tangent line $y - f(x_0) = m_{tan}(x - x_0)$
- Step-2: Given that:

  $y = x^2$ at $P(1,1)$, $x_0 = 1$,

$$\boxed{f(x_0) = 1 \quad , \; y_0 = 1}$$

⊙ Step-3:

$$m_{tan} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$\because y = x^2$
or $f(x) = x^2$
$f(x_1) = (x_1)^2$
$f(1) = (1)^2$

$$= \lim_{x_1 \to 1} \frac{f(x_1) - f(1)}{x_1 - 1}$$

$$= \lim_{x_1 \to 1} \frac{x_1^2 - (1)^2}{x_1 - 1}$$

$$m = \lim_{x_1 \to 1} \frac{\cancel{(x_1 - 1)}(x_1 + 1)}{\cancel{(x_1 - 1)}}$$

$$m = \lim_{x_1 \to 1} x_1 + 1$$

$$= 1 + 1$$

$$\boxed{m_{tan} = 2}$$

⊙ Equation of tangent line:

$$y - y_0 = m(x - x_0)$$
$$y - 1 = 2(x - 1)$$
$$y - 1 = 2x - 2$$
$$2x - y - 1 = 0$$

Example #2

$y = x^2$ at $P(1,1)$

$P(x_0, f(x_0))$

$$\text{Slope} = m_{tan} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$\because y = x^2$
$f(x) = x^2$

$$= \lim_{h\to 0} \frac{(1+h)^2 - (1)^2}{h}$$

$$= \lim_{h\to 0} \frac{1+2h+h^2 - 1}{h}$$

$$m_{tan} = \lim_{h\to 0} \frac{h(2+h)}{h}$$

$$= \lim_{h\to 0} 2+h$$

$$= 2+0$$

$\boxed{m_{tan} = 2}$ Ans.

→ **Example #3 :**

⊙ Step-I: To find equation of tangent line

$$y - f(x_0) = m_{tan}(x - x_0)$$

⊙ Step-2: Given that:

$y = \frac{2}{x}$ at $P(2,1)$

i.e: $x_0 = 2$, $f(x_0) = y_0 = 1$

⊙ Step 3:

$$m_{tan} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{2+h} - \frac{2}{2}}{1}$$

$$= \lim_{h \to 0} \frac{\frac{2-2-h}{2+h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{(2+h)h}$$

$$m_{tan} = \lim_{h \to 0} \frac{-1}{2+h}$$

$$\boxed{m_{tan} = \frac{-1}{2}}$$

→ **Example #4 :**

$$m_{tan} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Here

$f(x) = \sqrt{x}$ , $x_0 = 1$ , $f(x_0) = f(1) = \sqrt{1} = 1$

$$f(x_0+h) = f(1+h)$$
$$= \sqrt{1+h}.$$

$$\rightarrow m_{tan} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \times \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1}$$

$$= \lim_{h \to 0} \frac{(\sqrt{1+h})^2 - (1)^2}{h(\sqrt{1+h} + 1)}$$

$$= \lim_{h \to 0} \frac{1+h-1}{h(\sqrt{1+h} + 1)}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{1+h} + 1)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1}$$

$$m_{tan} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} = \frac{1}{2}$$

$$\boxed{m_{tan} = \frac{1}{2}}$$

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(1st principle rule)
: (By definition)

* $m = \text{slope} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= f'(x)$ : Differentiation

## * Differentiation:

Signs of differentiate:

(i) $f'(x)$

(ii) $\frac{df}{dx}$

(iii) $D(f)$.

→ **Rules:**

**(i)** $f(x) = c \qquad c \in R$

Derivate w.r.t $x$

$\frac{df}{dx} = \frac{d}{dx}[c]$

$= 0$

Eg:

$f(x) = 3^2$

$\frac{d}{dx} f(x) = \frac{d}{dx} 3^2$ : constant : 9

$= 0$

**(ii)** $y = x^n$

$\frac{dy}{dx} = \frac{dx^n}{dx} = n \cdot x^{n-1}$

(iii) $y = ax^n$

$y' = a\frac{dx^n}{dx}$ : $y'$ (Differentiation).

$= a(n.x^{n-1})$

$= anx^{n-1}$.

* (iv) $h(x) = f(x) \pm g(x)$

$h'(x) = f'(x) \pm g'(x)$

(v) $h(x) = f(x).g(x)$

$h'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$.

*(vi) $h(x) = \frac{f(x)}{g(x)}$ : Quotient Rule $\frac{\text{1st function}}{\text{2nd function}}$

$h'(x) = \frac{f'(x).g(x) - f(x).g'(x)}{(g(x))^2}$

* $f(x) = c$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ : $f(x) = c$ : $f(x+h) = c$ (constant value will come).

$= \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = 0$

∴ Slope = 1st derivative : $f(x) = -3$ $\quad f(x) = -\sqrt{2}$

∴ Horizontal Line: $f'(x) = 0$ $\quad f'(x) = 0$

Slope = 0.

## Example 6:

$$y = 3x^8 - 2x^5 + 6x + 1$$

$$\frac{dy}{dx} = \frac{d}{dx}[3x^8] - \frac{d}{dx}[2x^5] + \frac{d}{dx}[6x] + \frac{d}{dx}[1]$$

$$= 3\frac{d}{dx}(x^8) - 2\frac{d}{dx}x^5 + 6\frac{d}{dx} + 0$$

$$= 3(8x^7) - 2(5x^4) + 6(1x^0) + 0$$

$$= 24x^7 - 10x^4 + 6 \quad \underline{\text{Ans.}}$$

## Example #7:

$$y'(x) = 0$$

$$\frac{d}{dx}[x^3 - 3x + 4] = 0$$

$$3x^2 - 3 + 0 = 0$$

$$3(x^2 - 1) = 0$$

$$(x+1)(x-1) = 0$$

$$x + 1 = 0, \quad x - 1 = 0$$

$$x = -1, \quad x = 1$$

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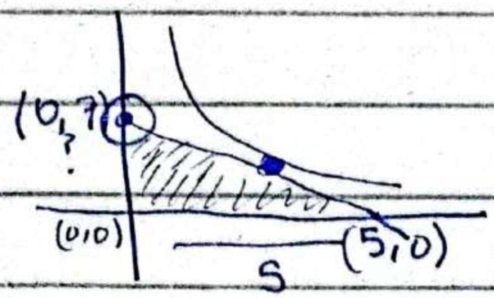
: $\frac{1}{2}$ base × height




Sol:-

Eq. of tangent line at (5,0) is:

$$y - y_0 = m(x - x_0)$$

$$y - 0 = m(x - 5) \quad \cdots\cdots (1)$$

Now here

$$m = \frac{dy}{dx}$$

$$m = \frac{dy}{dx}\left(5x^{-1} - \frac{1}{5}x\right)$$

$$m = -5x^{-2} - \frac{1}{5}$$

$$m\Big|_{(5,0)} = -5(5)^{-2} - \frac{1}{5}$$

$$= -\frac{1}{5} - \frac{1}{5}$$

$$m\Big|_{(5,0)} = -\frac{2}{5}$$

Eq. (1) becomes:

$$y = -\frac{2}{5}(x - 5)$$

$$y = -\frac{2}{5}x + 2$$

: $y = mx + c$
m = slope
c = y-intercept

Now, here y-intercept is 2.

So height = 2 , base = 5

Area of triangle $= \frac{1}{2}$ (Height)(base) $= \frac{1}{2}(2)(5)$

$= 5$

## ⊛ Higher Order Derivative:

$y = f(x)$

$y' = f'(x)$ $\frac{dy}{dx}$

$y'' = f''(x)$ $\frac{d^2y}{dx^2}$

$y''' = f'''(x)$ $\frac{d^3y}{dx^3}$

$y^{(4)} = f^{(4)}(x)$ $\frac{d^4y}{dx^4}$

$y^{(5)} = f^{(5)}(x)$

$\vdots$

$y^{(n)} = f^{(n)}(x)$ $\frac{d^ny}{dx^n}$

Q → $f(x) = 3x^4 - 2x^3 - 4x + 2$

find $f^{(n)}(x)$.

Sol:-

$f'(x) = 12x^3 - 6x^2 - 4$ ; $f^{(6)}(x) = 0$

$f''(x) = 36x^2 - 12x$ ; $f^{(7)}(x) = 0$

$f'''(x) = 72x - 12$ ; $\vdots$

$f^{(4)}(x) = 72$ ; $f^{(n)}(x) = 0$

$f^{(5)}(x) = 0$

Q→ Find $f'''(2)$, where $f(x)=3x^2-2$

Sol:-

$$f'(x) = 6x$$
$$f''(x) = 6$$
$$f'''(x) = 0$$
$$f'''(2) = 0$$

Q→ Find $\left.\frac{d^2y}{dx}\right|_{x=1}$, where $y = 6x^5 - 4x^2$

Sol:-

$$\frac{dy}{dx} = 30x^4 - 8x$$

: Ex: 2.4 (1-31) (Differentiate)

$$\frac{d^2y}{dx^2} = 120x^3 - 8$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 120(1)^3 - 8$$
$$= 120 - 8$$
$$= 112 \text{ Ans.}$$

Q→ $y = \sin x$

$$\frac{dy}{dx} = \frac{d}{dx}\sin x$$
$$= \cos x$$

; $y = \cos x$

$$\frac{dy}{dx} = \frac{d}{dx}\cos x$$
$$= -\sin x$$

(•) $y = \sec x$ ; $y = \tan x$

$y' = \sec x \cdot \tan x$ ; $y' = \sec^2 x$

**Q: Example:**

$y = x\sin x$ Find $y'$

$$y' = \frac{d}{dx}(x)\cdot\sin x + x\cdot\frac{d}{dx}(\sin x)$$

$$y' = (1\cdot\sin x) + x(\cos x)$$

$$y' = \sin x + x\cos x$$

$$y'\Big|_{\pi/2} = \sin\frac{\pi}{2} + \frac{\pi}{2}\left(\cos\frac{\pi}{2}\right)$$

$$= 1 + \frac{\pi}{2}(0)$$

**Q:** $y = \dfrac{\sin x}{1+\cos x}$ find $y'$ at $x = \frac{\pi}{2}$

Sol:-

$$y' = \frac{\left[\frac{d}{dx}\sin x\right](1+\cos x) - \sin x\cdot\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{\cos x(1+\cos x) - \sin x(0-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$y' = \frac{\cos x + 1}{(1+\cos x)^2}$$ $\because \cos^2 x + \sin^2 x = 1$

$$y' = \frac{1}{1+\cos x}$$

$$y'\Big|_{x=\pi/2} = \frac{1}{1+\cos \pi/2}$$

$$= \frac{1}{1+0}$$

$$= 1$$

$\because (\sin x)^2 = \sin^2 x$

**Ex :**

Find $f''(\frac{\pi}{4})$, where $f(x) = \sec x$

Sol:-

$$f'(x) = \sec x \cdot \tan x$$

$$f''(x) = \frac{d}{dx}(\sec x) \cdot \tan x + \sec x \cdot \frac{d}{dx} \tan x$$

$$= (\sec x \cdot \tan x) \overset{\because \tan x}{+} \sec x (\sec^2 x)$$

$$f''(\frac{\pi}{4}) = \sec x \cdot \tan^2 x + \sec^3 x$$

$$= \sec (\frac{\pi}{4}) \cdot \tan^2 (\frac{\pi}{4}) + \sec^3 (\frac{\pi}{4})$$

$$= (1.414) \cdot 1 + 2.82$$

$$= 4.234$$

| | |
|---|---|
| $y = e^x$ | $y = \ln x$ |
| $y' = e^x$ | $y' = \frac{1}{x}$ |

(Exponential same derivative and integration).

(*) $y = \ln(1+x)$

$$y' = \frac{1}{1+x}$$

**25|3|24**

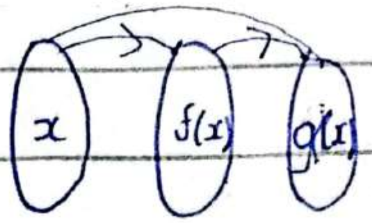
## (*) CHAIN RULE:

: continuous + differentiable (composite)

→ composite function:

$f, g$

$f(x)$




$f(g(x))$

1st Method

(i) $y = f(g(x))$

$y' = f'(g(x)) \cdot g'((x))$ ----- ①

2nd method

(ii) Let $g(x) = U$

$$y' = \frac{dy}{dU} \cdot \frac{dU}{dx} \quad —— ②$$

**Example:**

① if $y = \cos x^3$, find $y'$

Method I :-

$y = f(g(x))$

$y' = f'(g(x)) \cdot g'(x)$

$y = \cos(x^3)$

$y' = -\sin x^3 \cdot \frac{dx^3}{dx}$

$= -\sin x^3 \cdot 3x^2$

$= -3x^2 \sin x^3$

$\because \cancel{y = \cos x}$

$= -\sin x \cdot \frac{dx}{dx}$

$= -\sin x$

-> Method 2:

$y = \cos x^3$

Let $u = x^3$

$\frac{du}{dx} = 3x^2$ ..... (ii)

$\therefore y = \cos u$

$y' = \frac{dy}{du} \cdot \frac{du}{dx}$

$= \frac{d\cos\theta}{du} \cdot 3x^2$ (From ii)

$y' = -\sin u \cdot 3x^2$

$= -\sin(x^3) \cdot 3x^2$

$= -3x^2 \sin x^3$ Ans

② $y = (\tan x)^2$

: As $x$ value not simple

$y' = 2(\tan x)^{2-1} \cdot \frac{d}{dx}(\tan x)$

$y' = 2\tan x \cdot (\sec^2 x)$

: cosecx
$= -\operatorname{cosec}x \cdot \cot x$

(3) $y = \sqrt{x^3 + \operatorname{cosec}x}$

$y' = (x^3 + \operatorname{cosec}x)^{1/2}$

$y' = \frac{1}{2}(x^3 + \operatorname{cosec}x)^{-1/2} \cdot \frac{d}{dx}(x^3 + \operatorname{cosec}x)$

$y' = \frac{1}{2}(x^3 + \operatorname{cosec}x)^{-1/2} \cdot (3x^2 - \operatorname{cosec}x \cdot \cot x)$

$y' = \frac{1}{2} \frac{1}{\sqrt{x^3 + \operatorname{cosec}x}} \cdot (3x^2 - \operatorname{cosec}x \cdot \cot x)$

$$y' = \frac{3x^2 - \operatorname{cosec}x \cdot \cot x}{2\sqrt{x^3 + \operatorname{cosec}x}}$$

## ⊙ Example 6:

w: omega

Find $\frac{du}{dt}$, where $u = \sec\sqrt{wt}$

Sol:-

$u = \sec\sqrt{wt}$

$\frac{du}{dt} = \frac{d}{dt} \sec\sqrt{wt}$ : w = constant

$= \sec\sqrt{wt} \cdot \tan\sqrt{wt} \, \frac{d}{dt}\sqrt{wt}$

$= \sec\sqrt{wt} \cdot \tan\sqrt{wt} \cdot \left(\sqrt{w} \frac{d}{dt}(t)^{1/2}\right)$

$= \sec\sqrt{wt} \cdot \tan\sqrt{wt} \cdot \sqrt{w} \cdot \frac{1}{2\sqrt{t}}$

$= \frac{\sqrt{w}}{2\sqrt{t}} \cdot \sec\sqrt{wt} \cdot \tan\sqrt{wt}$ Ans

Ex: 2.6 (1-50)
(odd numbers).

- $y = \sin(\sqrt{1+\cos x})$

$y' = \cos(\sqrt{1+\cos x}) \cdot \frac{d}{dx}(\sqrt{1+\cos x})$

$= \cos(\sqrt{1+\cos x}) \cdot \frac{1}{2}(1+\cos x)^{-1/2} \cdot \frac{d}{dx}(1+\cos x)$

$= \frac{\cos\sqrt{1+\cos x} \cdot (-\sin x)}{2\sqrt{1+\cos x}}$

$= \frac{-\sin x \cdot \cos\sqrt{1+\cos x}}{2\sqrt{1+\cos x}}$ Ans

- If $y = \log_b x$, find $y'$

Sol:-

$y' = \frac{1}{x \cdot \ln b}$ $\because b > 0,\ b \neq 1$

$\because y = \ln x$, $y' = \frac{1}{x}$ | $y = \log_e x$, $y' = \frac{1}{x \cdot \ln e}$, $y' = \frac{1}{x}$ $\because \ln e = 1$

- $y = \ln(x^2+1)$

Differentiale w.r.t $x$ $\because$ Ex: 3.2

$y' = \frac{1}{x^2+1} \cdot \frac{d}{dx}(x^2+1)$

$= \frac{1}{x^2+1} \cdot 2x$

$y' = \frac{2x}{x^2+1}$ Ans

$$\frac{d}{dx}\left[\ln\left(\frac{x^2\sin x}{\sqrt{1+x}}\right)\right]$$

$$= \frac{1}{\frac{x^2\sin x}{\sqrt{1+x}}} \cdot \frac{d}{dx}\left(\frac{x^2\sin x}{\sqrt{1+x}}\right)$$

: long method

Short Method (log properties)

$$= \frac{d}{dx}\left[\ln(x^2\sin x) - \ln\sqrt{1+x}\right]$$

$$= \frac{d}{dx}\left[\ln x^2 + \ln\sin x - \ln(1+x)^{\frac{1}{2}}\right]$$

$$= \frac{d}{dx}\left[2\ln x + \ln\sin x - \frac{1}{2}\ln(1+x)\right]$$

$$= 2\frac{d}{dx}(\ln x) + \frac{d}{dx}\ln\sin x - \frac{1}{2}\frac{d}{dx}\ln(1+x)$$

: $\ln\frac{f}{g} = \ln f - \ln g$

$\ln fg = \ln f + \ln g$

$\ln f^n = n\ln f$

$$= 2\cdot\frac{1}{x} + \frac{1}{\sin x}\frac{d}{dx}\cos x - \frac{1}{2}\left(\frac{1}{1+x}\right)$$

$$= \frac{2}{x} + \cot x - \frac{1}{2(1+x)} \quad \underline{\text{Ans}}$$

(*) Mid-Term:

→ 80-90 marks

→ Q1-3 (Graphs) (xy co-ordinate not polar)

27/3/24

- Find derivation of:

$$y = \frac{x^2 \cdot \sqrt[3]{7x+14}}{(1+x^2)^4}$$

: Almost all methods are applied in it.
: It takes lot of time.

$$\left(y' = \frac{\frac{d}{dx}(x^2)\cdot\sqrt[3]{7x+14} - x^2\cdot\frac{d}{dx}\left(\sqrt[3]{7x+14}\right)}{}\right)$$

$$y' = \frac{\frac{d}{dx}\left(x^2\cdot\sqrt[3]{7x+14}\right)\cdot(1+x^2)^4 - \left(x^2\cdot\sqrt[3]{7x+14}\right)\frac{d}{dx}(1+x^2)^4}{\left((1+x^2)^4\right)^2}$$

(Chain Rule)

$$\rightarrow \quad y = \frac{x^2\cdot\sqrt[3]{7x+14}}{(1+x^2)^4}$$

: Taking log

$$\ln y = \ln\left(\frac{x^2\cdot\sqrt[3]{7x+14}}{(1+x^2)^4}\right)$$

$$= \ln\left(x^2\cdot(7x+14)^{1/3}\right) - \ln(1+x^2)^4$$

: $\ln f/g = \ln f - \ln g$

$$= \ln x^2 + \ln(7x+14)^{1/3} - \ln(1+x^2)^4$$

$$\ln y = 2\ln x + \frac{1}{3}\ln(7x+14) - 4\ln(1+x^2)$$

Differentiate w.r.t $x$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(2\cdot\ln x + \frac{1}{3}\ln(7x+14) - 4\ln(1+x^2)\right)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 2\cdot\frac{1}{x} + \frac{1}{3}\left(\frac{1}{7x+14}\frac{d}{dx}(7x+14)\right) - \frac{4}{1+x^2}\cdot\frac{d}{dx}(1+x^2)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2}{x} + \frac{7}{3(7x+14)} - \frac{8x}{1+x^2}$$

$$\frac{dy}{dx} = y \cdot \left[ \frac{2}{x} + \frac{7}{3(7x+14)} - \frac{8x}{1+x^2} \right]$$

$$= \frac{x^3 \cdot \sqrt[3]{7x+14}}{(1+x^2)^4} \left[ \frac{2}{x} + \frac{7}{3(7x+14)} - \frac{8x}{1+x^2} \right] \text{.... From } ①$$

(Further simplify will expand it much).