Calculus and Analytical Geometry

⊛ $\dfrac{6}{x-1} \geq 5 \implies \dfrac{11-5x}{x-1} \geq 0$

$\implies$ Put $11-5x = 0$, $x-1=0$

$x = \frac{11}{5}$, $x = 1$ (Free boundary value)

$(-\infty, 1)$ $(1, \frac{11}{5})$ $(\frac{11}{5}, +\infty)$

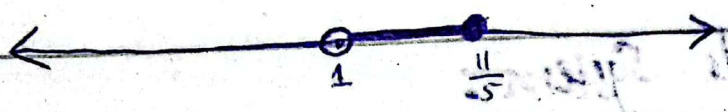
intervals	test point	$\frac{11-5x}{x-1} \geq 0$	Conclusion
$(-\infty, 1)$	0	$\frac{11}{-1} < 0$	Not satisfy
$(1, \frac{11}{5})$	2	$\frac{1}{1} \geq 0$	satisfy
$(\frac{11}{5}, +\infty)$	3	$-\frac{4}{2} \leq 0$	Not satisfy

The interval $(1, \frac{11}{5})$ is satisfied the given inequality, Also $x = \frac{11}{5}$ satisfied,

$S.S = (1, \frac{11}{5}) \cup \{\frac{11}{5}\}$

$\boxed{S.S = (1, \frac{11}{5}]}$

Graphically:



Quadratic inequalities:

⊛ Find all those numbers whose squares are greater or equal to two less than three times the number.

Therefore, $x^2 \geq 3x - 2$

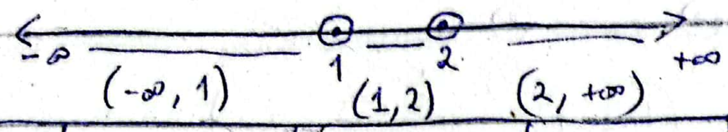
$$x^2 - 3x + 2 \geq 0$$

$$x^2 - 2x - x + 2 \geq 0$$

$$(x-2)(x-1) \geq 0$$

$x - 2 = 0$, $x - 1 = 0$

$\boxed{x=2}$, $\boxed{x=1}$ (these are boundary points)



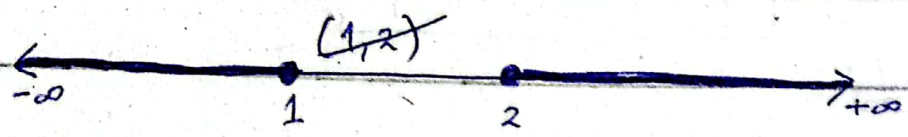
Intervals	test point	$(x-2)(x-1) \geq 0$	Conclusion
$(-\infty, 1)$	0	$(-2)(-1) \geq 0$	Satisfied
$(1, 2)$	1.5	$(-0.5)(0.5) \leq 0$	Not satisfy
$(2, +\infty)$	3	$(1)(2) \geq 0$	Satisfied

The intervals, $(-\infty, 1)$ and $(2, +\infty)$ satisfied the inequality, also $x = 1$, $x = 2$ satisfied.

$$S.S = (-\infty, 1) \cup (2, +\infty) \cup \{1\} \cup \{2\}$$

$$\boxed{S.S = (-\infty, 1] \cup [2, +\infty)}$$

Graphically:



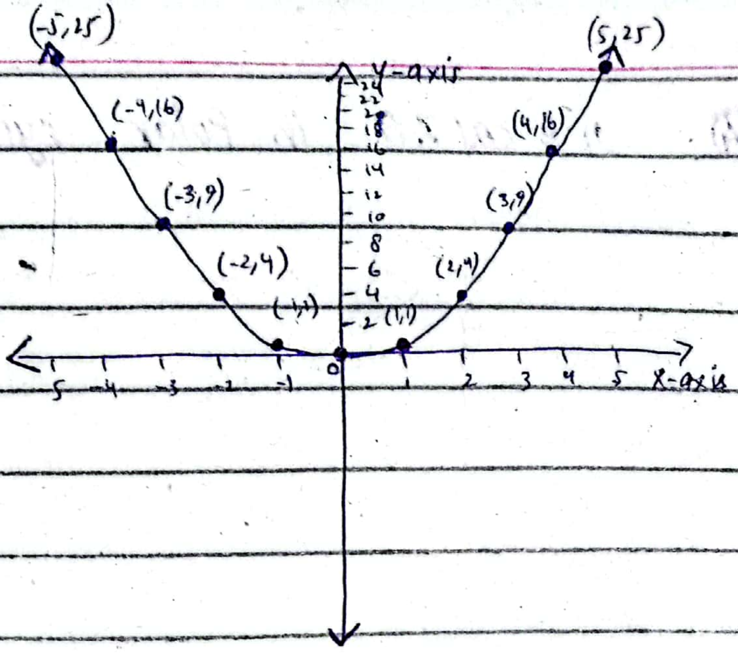
- **Co-ordinate System:**

if equation is given and interval is not given, then take it $[-5, 5]$

e.g $y = x^2$

$y = x^2$

x	y	x,y
-5	25	(-5,25)
-4	16	(-4,16)
-3	9	(-3,9)
-2	4	(-2,4)
-1	1	(-1,1)
0	0	(0,0)
5	25	(5,25)
4	16	(4,16)
3	9	(3,9)
2	4	(2,4)
1	1	(1,1)



Horizontal Scale:

$$= \frac{5-(-5)}{11} = \frac{10}{11} = 0.9 \approx 1 \quad \Rightarrow \frac{\text{greatest} - \text{lowest}}{n}$$

Vertical Scale:

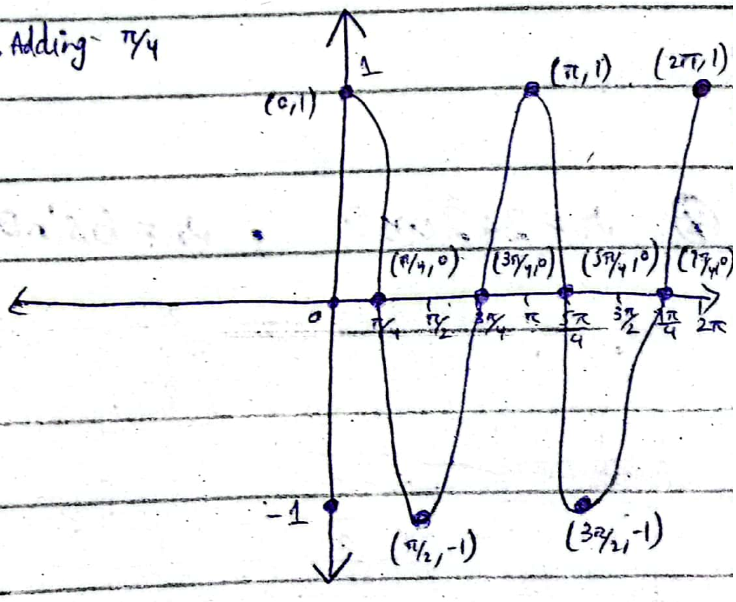
$$= \frac{25-0}{11} = 2.2 \approx 2$$

$$= \frac{5-(-5)}{11}$$

⊛ $y = \cos 2x$

∴ Adding $\pi/4$

x	y	(x,y)
0	1	(0,1)
$\pi/4$	0	$(\pi/4,0)$
$\pi/2$	-1	$(\pi/2,-1)$
$3\pi/4$	0	$(3\pi/4,0)$
π	1	$(\pi,1)$
$5\pi/4$	0	$(5\pi/4,0)$
$3\pi/2$	-1	$(3\pi/2,-1)$
$7\pi/4$	0	$(7\pi/4,0)$
2π	1	$(2\pi,1)$

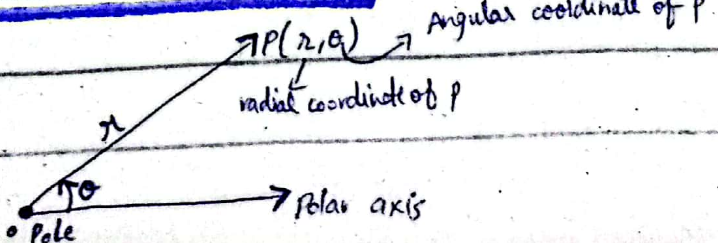


• Relations b/w Rectangular and Polar System:

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

$$\theta = \tan^{-1}(y/x) \quad , \quad r^2 = x^2 + y^2$$

• Polar Coordinate System:

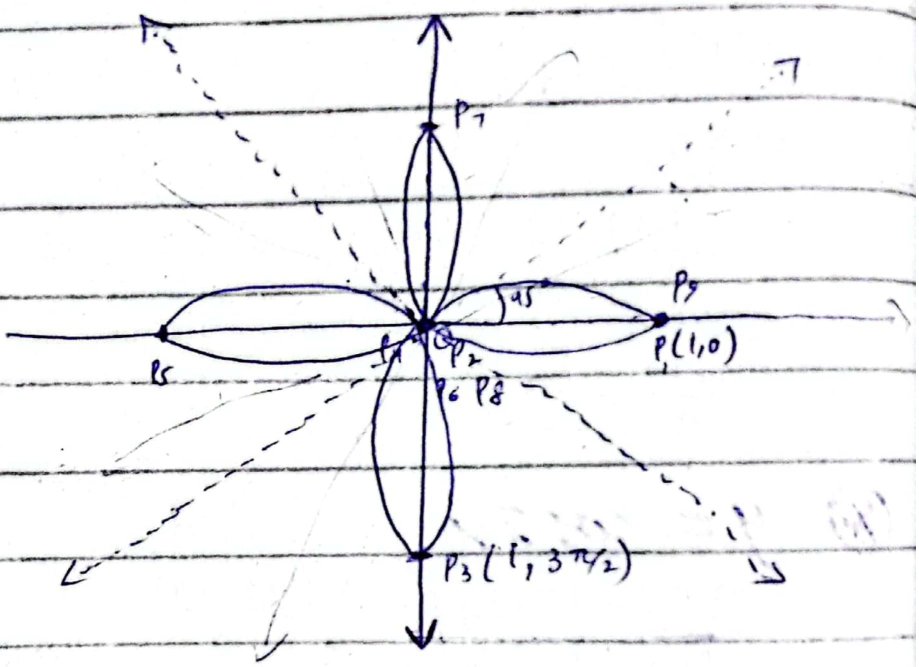


⊛ $r = \cos 2\theta$ in Polar system

same table as previous , r is distance, so we will change negative values to +ve. and add a π to θ.

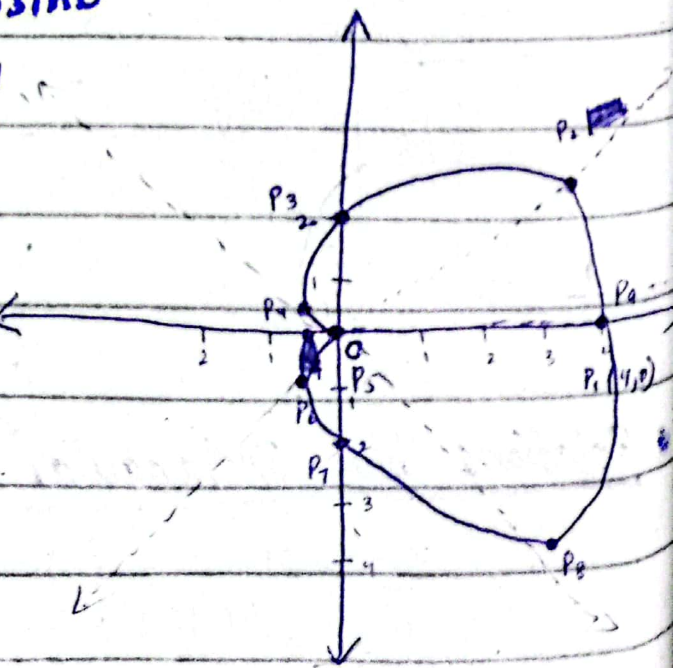
	(r, θ)
1	$(1, 0)$
2	$(0, \pi/4)$
3	$(1, 3\pi/2)$
4	$(0, 3\pi/4)$
5	$(1, \pi)$
6	$(0, 3\pi/4)$
7	$(1, 5\pi/2)$
8	$(0, 3\pi/4)$
9	$(1, 2\pi)$

∴ change $x \Rightarrow \theta$
$y \Rightarrow r$



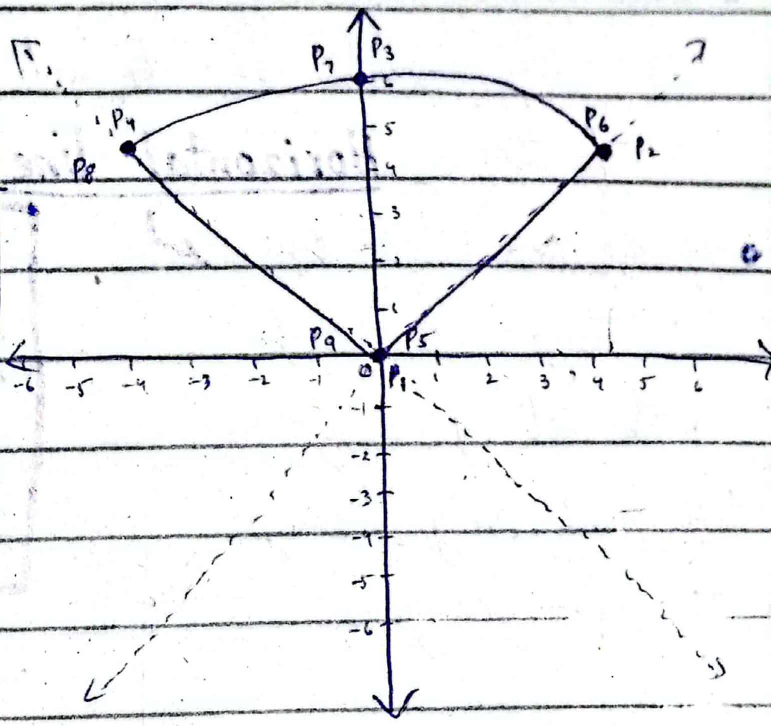
⊛ $r = 2 + 2\cos\theta$, $r = 6\sin\theta$

θ	$r = 2 + 2\cos\theta$	r, θ
0	$r = 2 + 2\cos 0 = 4$	$(4, 0)$
$\pi/4$	$r = 2 + 2\cos \pi/4 = 3.41$	$(3.4, \pi/4)$
$\pi/2$	$r = 2 + 2\cos \pi/2 = 2$	$(2, \pi/2)$
$3\pi/4$	$r = 2 + 2\cos 3\pi/4 = 0.58$	$(0.58, 3\pi/4)$
π	$r = 2 + 2\cos \pi = 0$	$(0, \pi)$
$5\pi/4$	$r = 2 + 2\cos 5\pi/4 = 0.58$	$(0.58, 5\pi/4)$
$3\pi/2$	$r = 2 + 2\cos 3\pi/2 = 2$	$(2, 3\pi/2)$
$7\pi/4$	$r = 2 + 2\cos 7\pi/4 = 3.41$	$(3.41, 7\pi/4)$
2π	$r = 2 + 2\cos 2\pi = 4$	$(4, 2\pi)$



④ $r = 6\sin\theta$

θ	$r = 6\sin\theta$	r, θ
0	$r = 6\sin\theta = 0$	(0,0)
$\pi/4$	$r = 4.2$	$(4.2, \pi/4)$
$\pi/2$	6	$(6, \pi/2)$
$3\pi/4$	4.2	$(4.2, 3\pi/4)$
π	0	$(0, \pi)$
$5\pi/4$	-4.2	$(4.2, 9\pi/4)$
$3\pi/2$	-6	$(6, 5\pi/2)$
$7\pi/4$	-4.2	$(4.2, 11\pi/4)$
2π	0	$(0, 2\pi)$



Functions

$$f : X \xrightarrow{} Y$$

domain → X, co-domain → Y

$f(x) = y$; $\forall x \in X$

y → image of x under f

- x cannot be repeated, if its a function.
- All images of f are collectively called Range.

Onto-function:

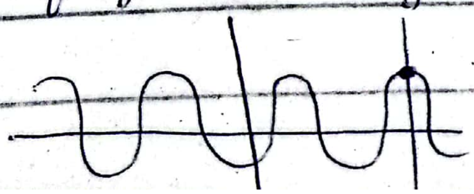
if Range = codomain

one-to-one function:

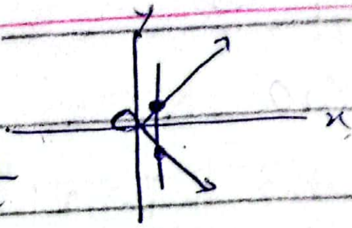
Every x has a distinct image.
(Koi aik y two x ka na ho)

e.g

$Y = \sin x$ is a function because every vertical line intersects the graph of $y = \sin x$ exactly at one point.

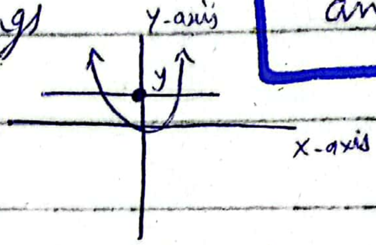


- since, vertical line intersects at two points. So, its not a function.



Horizontal line test

- If the horizontal line intersecting the graph meets at y-axis, then that point on y belongs to the range.

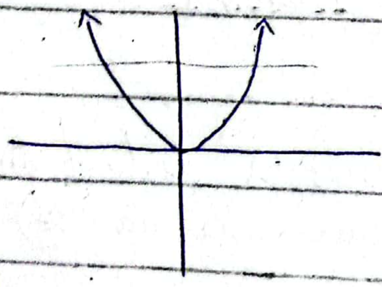


function not involving, fractions $\left(\frac{1}{x}\right)$ or square root $(\sqrt{x+1})$ then, domain and ranges lie in real l...

Onto : if every horizontal line intersects the graph (y-axis = graph)

One-to-One : if ~~every~~ horizontal lines that intersects the graph exactly at one point.

(*) $y = x^2$

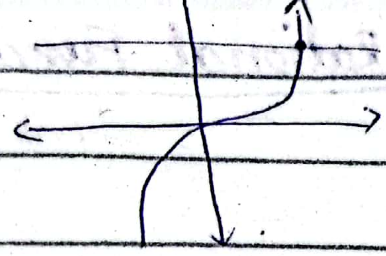


Visit 'Geogebra' to make graphs

- Domain = R
- Range = $[0, +\infty)$

Not onto.
Not one-to-one

⊗ $y = x^3$



- Domain = R
- Range = R

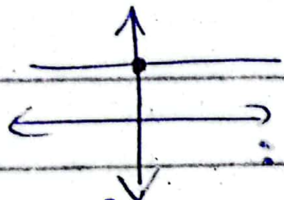
Both, one-to-one and onto.

Basic Functions

• Constant Function:

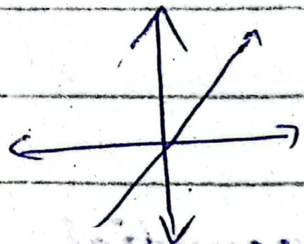
$f(x) = c$, e.g. $f(x) = 2$

- Not onto
- Not one-to-one



• Identity Function:

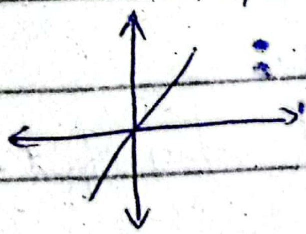
$f(x) = x$



- Both one-to-one and onto
- it is also linear

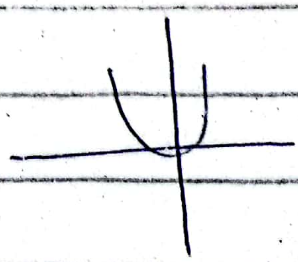
• Linear Function:

$f(x) = ax + b$, power of $x = 1$



• Quadratic Function:

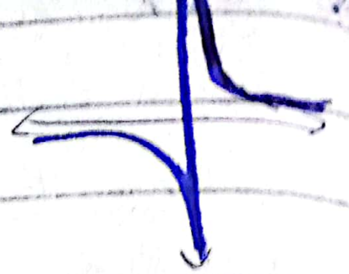
$f(x) = ax^2$,



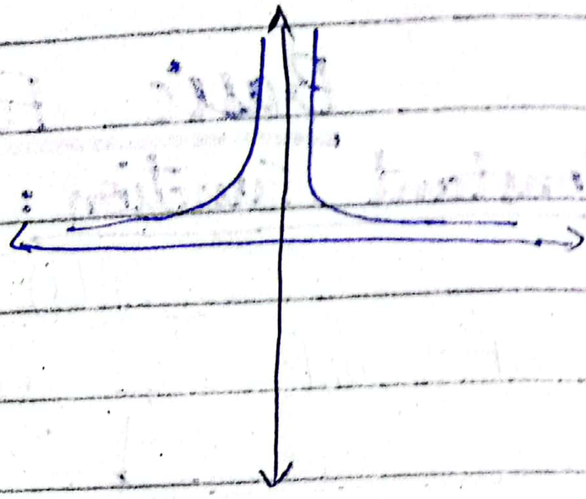
- Not onto.
- Not one-to-one

- **Rational Function:**

$$f(x) = \frac{p(x)}{q(x)} \quad , \quad f(x) = \frac{1}{x}$$



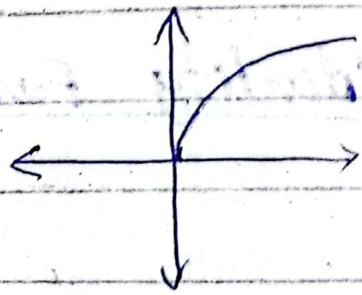
- $f(x) = \frac{1}{x^2}$
  - Domain $= R - \{0\}$
  - Range $=$



- **Square Root Function:**

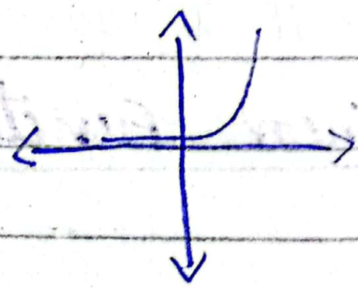
$$f(x) = \sqrt{x}$$

- y should not be -ve.



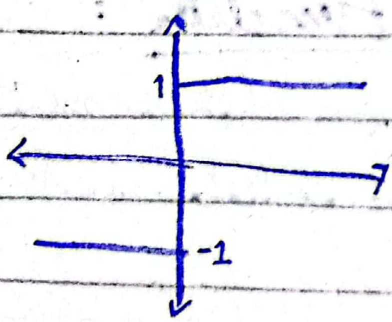
- **Exponential Function:**

$$f(x) = e^x$$



Piece-wise Function:

For different values of x, different formulas. e.g $f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$



Functions Representations

i) Formulas $y = x^2$, $f(x) = x^2$

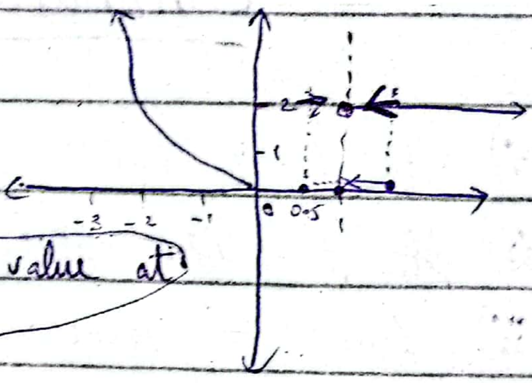
ii) Graphically

iii) Tables

iv) Words

Limit and Continuity

Limit : Prediction at a specific point.

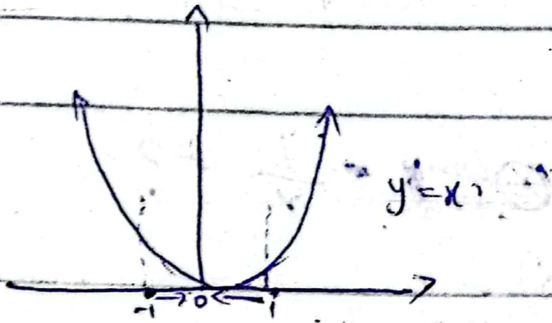


i) $\lim_{x \to 1} f(x) = ?$

ii) $\lim_{x \to 1^-} f(x) =$ not exist

iii) $\lim_{x \to 1^+} f(x) = 2$ ∴ corresponding y value at graph at $x = 1$

$f(1) =$ does not exist because point at 1 is not filled [o].



i) $\lim_{x \to 0^-} x^2 = 0$

ii) $\lim_{x \to 0^+} x^2 = 0$

iii) $\lim_{x \to 0} x^2 = 0$

- If left and right limits not equal then function ~~[illegible]~~ limit not exist, but value may exist.
- function value is the point of line vertical from point.

$$\lim_{x \to 0} \frac{|x|}{x} = ?$$

i) $\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{|-x|}{-x} = -1$

ii) $\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{|x|}{x} = 1$

iii) $\lim_{x \to 0^+} \neq \lim_{x \to 0^-}$

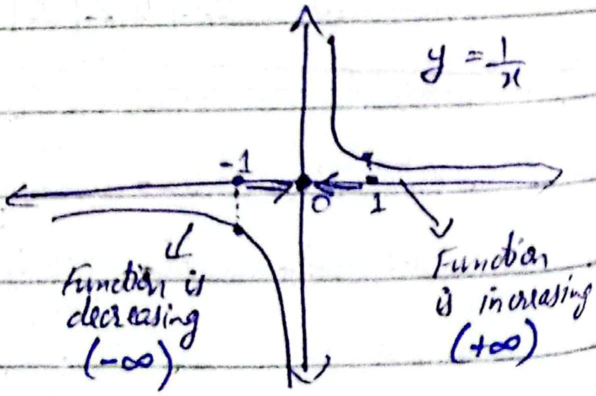
$\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Continuity of Function

- i) If Limit exist
- ii) If function value exist
- iii) Limit = value

Then, Continuous function

$$\lim_{x \to 0} \frac{1}{x} = ?$$



i) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

ii) $\lim_{x \to 0^+} \frac{1}{x} = +\infty$

iii) $\lim_{x \to 0} \frac{1}{x}$ does'nt exist

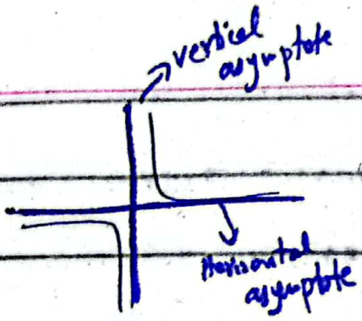
Asymptotes

A straight line which continuously move along the curve but never intersects it.

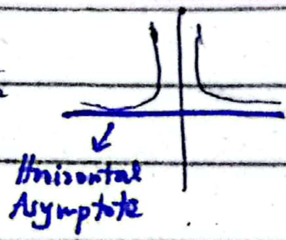
- vertical asymptotes
- Horizontal asymptotes

Ex : 1.1 ⇒ Q#1-12

$f(x) = \frac{1}{x}$



$f(x) = \frac{1}{x^2}$



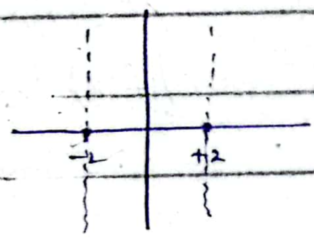
- vertical Asymptote lies where given function is undefined.
- Asymptotes are only of Rational functions. $\frac{P(x)}{Q(x)}$
- For vertical asymptote $Q(x)=0$. To find it, make $Q(x)=0$ and find x.

e.g $f(x) = \frac{1}{x^2-4}$

∴ Put $x^2-4=0$

$x^2 = 4$

$\boxed{x = \pm 2}$



- To Find horizontal asymptote, we compare degrees of $P(x)$ and $Q(x)$. Let, $Deg(P(x)) = n$

$Deg(Q(x)) = m$

if ⇒ $n < m$	$n = m$	$n > m$
$y = 0$	$y = \frac{a_n}{b_m}$	No H.A exist

∴ a_n, b_m are leading coefficients with heighest degree

e.g $f(x) = \frac{5x+2}{x^2+2x+1}$

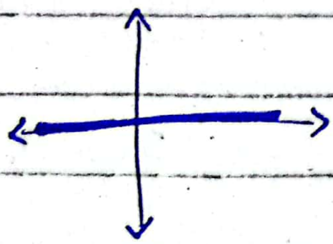
H.A

$n = Deg(5x+2) = 1$

$m = Deg(x^2+2x+1) = 2$

Since, $n < m$

∴ H.A ⇒ $y = 0$



$f(x) = \frac{2x^2+x+1}{3x^2-4}$

H.A $n = 2$

$m = 2$

Since, $n = m$

$\boxed{1.2 \Rightarrow \text{Odd No.} + \text{Questions} : 31, 32, 37 \to 40}$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \sqrt{x+13} = 4$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 - 5 = 4$

Exercise : 1.2

QNo. 1 :

When, $\lim_{x \to a} f(x) = 2$

a) $\lim_{x \to a} [f(x) + 2g(x)]$

$= \lim_{x \to a} f(x) + 2 \cdot \lim_{x \to a} g(x)$

$= 2 + 2 \cdot (-4)$

$\boxed{= -6}$

$\frac{x}{x-3}$ $\quad$ $\frac{x}{x-3} \times \frac{x+3}{x+3}$

$\frac{y+6}{y^2-36}$ $\quad$ $\frac{x(x+3)}{x^2-9}$

$\frac{\cancel{y+6}}{\cancel{(y+6)}(y-6)}$ $\quad$ $\frac{1}{y-6}$

QNo. 3 : $\lim_{x \to 2} x(x-1)(x+1)$

$= \lim_{x \to 2} x \cdot \lim_{x \to 2} (x-1) \cdot \lim_{x \to 2} (x+1)$

$= 2(2-1)(2+1)$

$= 6$

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$
$$=a^3+a^2b+ab^2-a^2b-ab^2-b^3$$
$$\boxed{a^3-b^3=a^3+ab^2-ab^2-b^3}$$
$$\boxed{a^3+b^3=a^3-a^2b-ab^2+b^3}$$

$$\frac{x^3+3x^2-12x+4}{x+1}$$

$$\cancel{= x^2(x+3)-4}$$

$$=\frac{x^3-5x^2+2x^2-10x-2x+4}{x+1}$$

$$\cancel{= x^3-5x(x+2)}$$

Q13) $$\lim_{x\to 2}\frac{t^3-12t+3t^2+4}{t^3-4t}$$

$$=\lim_{x\to 2}\frac{t^3-10t-2t+5t^2-2t^2+4}{t^3-4t}$$

$$=\frac{t^3+5t(t-2)-2(t-2)-2t^2}{t^3-4t}$$

$$=\frac{t^2(t-2)+5t(t-2)-2(t-2)}{t^3-4t}$$

$$=\frac{(t-2)(t^2+5t-2)}{t(t^2-2^2)}=\frac{\cancel{(t-2)}(t^2+5t-2)}{t(t+2)\cancel{(t-2)}}$$

$$=\lim_{x\to 2}\frac{t^2+5t-2}{t(t+2)}=\frac{4+10-2}{8}=\frac{\cancel{12}\,3}{\cancel{8}\,2}=\boxed{\frac{3}{2}}$$

C·P

1.3 + Solved examples

Q 32) $g(t) = \begin{cases} t-2 & ; t<0 \\ t^2 & 0 \le t \le 2 \\ 2t & t>2 \end{cases}$

i) $\lim_{t\to 0} g(t)$

Sol: $\lim_{t\to 0^-} g(t) = t-2 = -2$

$\lim_{t\to 0^+} g(t) = t^2 = 0$

$\lim_{t\to 0} g(t)$ does not exist.

ii) $\lim_{t\to -96} g(t)$

Sol: $\lim_{t\to -96} g(t) = t-2 = -98$

Limits at $x \to \infty$

$\boxed{\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = 2.7183 = e}$ ⇐ Formula Imp.

• $\lim_{x\to\infty} \left(1+\frac{1}{2x}\right)^x$

$= \lim_{x\to\infty} \left(1+\frac{1}{2x}\right)^{2x \times 1/2}$

$= \left[\lim_{x\to\infty} \left(1+\frac{1}{2x}\right)^{2x}\right]^{1/2}$

$= e^{1/2}$ Ans. $\because \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

• $\lim_{x\to +\infty} 2x^5 = 2(+\infty)^5 = +\infty$

$\lim_{x\to -\infty} 2x^5 = 2(-\infty)^5 = -\infty$

$\lim_{x\to+\infty} e = +\infty$	$\lim_{x\to+\infty} e^{-x} = 0$	$\lim_{x\to0^+} \ln x = -\infty$
$\lim_{x\to-\infty} e^{x} = 0$	$\lim_{x\to-\infty} = +\infty$.	$\lim_{x\to+\infty} \ln x = +\infty$

⇒ if function is fractional, at $x \to \infty$

⊛ $= \lim_{x\to+\infty} \dfrac{3x+5}{6x-8}$

Pick highest power 'x' of denomenator and divide it by every term.

$$= \lim_{x\to+\infty} \frac{\frac{3x}{x} + 5/x}{6x/x - 8/x}$$

$$= \lim_{x\to+\infty} \frac{3 + 5/x}{6 - 8/x} = \frac{3 + 5/\infty}{6 - 8/\infty}$$

$$= \frac{3+0}{6-0} = \frac{1}{2} \text{ Ans.}$$

Short Method.

⊛ $\lim_{x\to-\infty} \dfrac{4x^2 - x}{2x^3 - 5}$ ⟹ we can also select highest term of denomenator & numerator. $\lim_{x\to-\infty} \dfrac{4x^2}{2x^3} = 0$

$$= \lim_{x\to-\infty} \frac{\frac{4x^2}{x^3} - x/x^3}{\frac{2x^3}{x^3} - 5/x^3}$$

$$= \lim_{x\to-\infty} \frac{\frac{4}{x} - \frac{1}{x^2}}{2 - 5/x^3} = \frac{\frac{4}{-\infty} - \frac{1}{-\infty}}{2 - 5/(-\infty)^3} = 0$$

⊛ $\lim_{x\to+\infty} \dfrac{5x^3 - 2x^2 + 1}{1 - 3x}$ ⟹ $\lim_{x\to+\infty} \dfrac{5x^3}{-3x} = -\infty$

Short method

$$= \lim_{x\to+\infty} \frac{\frac{5x^3}{x} - \frac{2x^2}{x} + \frac{1}{x}}{\frac{1}{x} - \frac{3x}{x}}$$

$$= \lim_{x\to+\infty} \frac{5x^2 - 2x + 1/x}{\frac{1}{x} - 3} = \frac{5(\infty)^2 - 2(\infty) + \frac{1}{\infty}}{\frac{1}{\infty} - 3}$$

whole value is ∞

$$= \frac{\{\cancel{\infty} - \cancel{\infty} + 0\}}{0 - 3} = \frac{\infty}{-3} \boxed{= -\infty}$$

Ans

• $f(x) = \dfrac{x^2-4}{x-2}$, $x = 2$, Find value and limit :

i) value $\Rightarrow f(2) = \dfrac{(2)^2-4}{2-2} = \dfrac{0}{0}$ = undefined

∴ To find value, we cannot simplify equation

ii) $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2} \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = 4$

iii) value $\neq$ limit, not continuous function.

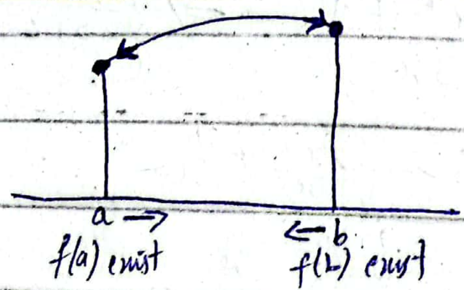
• $g(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$, at $x = 2$

i) $g(2) = 3$

ii) $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} = 4$

iii) value $\neq$ limit, not continuous.

Continuity on an Interval



Interval will continuous if,

- $f(a), f(b)$ exist and continuous. (should only exi[st] / not equal)
- f is continuous from right of a
- f is continuous from left of b

What can you say about continuity of

$$f(x) = \sqrt{9-x^2}$$

i) $f(x)$ is defined on the interval $[-3,3]$.

ii) $\lim\limits_{x\to a} \sqrt{9-x^2}$ exists $\forall\ a \in [-3,3]$

iii) $f(a) = \lim\limits_{x\to a} f(x)\ \forall\ a \in [-3,3]$

So, $f(x)$ is continuous at $[-3,3]$

Show that $|x|$ is continuous everywhere,

$$|x| = \begin{cases} x & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -x & \text{if } x<0 \end{cases}$$

$\therefore$ At $x=1$

i) $f(1) = |x| = 1$

ii) $\lim\limits_{x\to 1^+} |x| = 1$

$\lim\limits_{x\to 1^-} |-x| = 1$

Q:29 a) $f(x) = \begin{cases} 7x-2 \ ; & x \le 1 \\ Kx^2 \ , & x>1 \end{cases}$

Since, $f(x)$ is continuous every where

$\therefore \lim\limits_{x\to c} f(x) = f(c)$

$\therefore$ Let, take $x=1$ to find K.

$\lim\limits_{x\to 1^+} Kx^2 = K(1)^2 = K$

$\lim\limits_{x\to 1^-} 7x-2 = 7(1)-2 = 5$

Since, $f(x)$ is continuous,

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^-} f(x)$

$\boxed{K=5}$

Question : 31

Let, $x = -1$

$\lim_{x \to -1^+} m(x+1)+K = m(-1+1)+K = K$

$\lim_{x \to -1^-} 2x^3+x+7 = 2(-1)^3+(-1)+7 = 4$

Since, $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x)$

$\boxed{K = 4}$

Similarly, at $x = 2$

$\lim_{x \to 2^+} x^2+5 = 9$

$\lim_{x \to 2^-} m(x+1)+K = 3m+4$

Since, $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

$9 = 3m+4$

$5 = 3m$

$\boxed{m = \frac{5}{3}}$

Continuity of Trigonometric Functions

In trignometric functions, we apply limit on angle.

$$\lim_{x \to c} \sin x = \sin \lim_{x \to c} x$$
$$= \sin c$$

$\lim_{x \to 1} \cos\left(\frac{x^2-1}{x-1}\right)$

$$= \cos \lim_{x \to 1} \frac{x^2-1}{x-1}$$
$$= \cos \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$$
$$= \cos \lim_{x \to 1} (x+1)$$
$$= \cos 2$$

1.6.2 Theorem: If f is one-to-one and continuous, then f^{-1} is continuous at its domain but actually domain of f^{-1} is the range of f.

f → one-to-one, continuous $\Rightarrow$ f^{-1} is continuous at its Domain

Domain ↓ Range of f

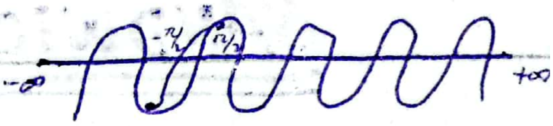
Example: 2

$\sin x$ is continuous at point $[-\frac{\pi}{2}, \frac{\pi}{2}]$ → domain of $\sin x$

Range of $\sin x = [-1, 1]$

∴ All periodic functions are not continuous but we are selecting a limited area

Therefore, $\sin^{-1}x$ is continuous at $[-1,1]$ as $[-1,1]$ is Range of $\sin x$



Exponential functions a (e^x, 2^x) are continuous at → $(-\infty, +\infty)$

Logarithmic functions ($\log x$, $\ln x$) are continuous at → $(0, +\infty)$

Example: 3 $f(x) = \dfrac{\tan^{-1}x + \ln x}{x^2 - 4}$

∴ Fractions are continuous if numerator and denominator are continuous and denominator is non-zero

i) $\tan^{-1}x$ is continuous everywhere i.e $(-\infty, +\infty)$

ii) $\ln x$ is continuous when $x > 0$ or $(0, +\infty)$

iii) $x^2 - 4$, being a polynomial, is continuous everywhere.

Since, $x-4$ is denominator of $f(x)$

$\therefore x^2-4=0 \Rightarrow x=\pm 2$ are eliminated

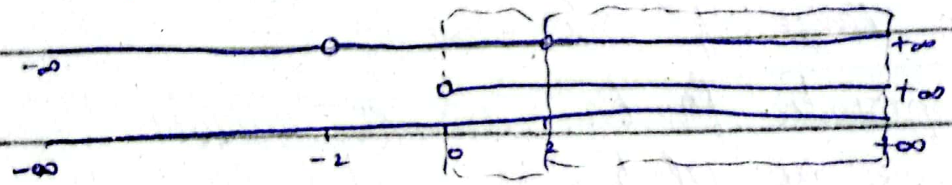
$x-4$ is continuous at,

$$\boxed{(-\infty,-2)\cup(-2,2)\cup(2,+\infty)}$$

Therefore $f(x)$ is continuous $=(-\infty,+\infty)\cap(0,+\infty)\cap\left((-\infty,-2)\cup(-2,2)\cup(2,+\infty)\right)$

$$=(0,+\infty)\cap\left((-\infty,-2)\cup(-2,2)\cup(2,+\infty)\right)$$

$$\boxed{=(0,2)\cup(2,+\infty)}$$



SANDWICH THEOREM

Given that, $g(x)\le f(x)\le h(x)$

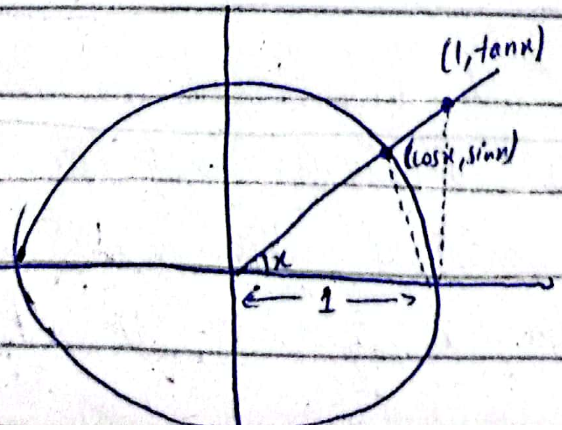
$\lim\limits_{x\to c} g(x) \le \lim\limits_{x\to c} f(x) \le \lim\limits_{x\to c} h(x)$ $\therefore \left(\lim\limits_{x\to c} g(x) = l = \lim\limits_{x\to c} h(x)\right)$

$l \le \lim\limits_{x\to c} f(x) \le l$ $\therefore$ it is not possible

$\Rightarrow \lim\limits_{x\to c} f(x) = l$

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x}=1}$$

Proof:



Area of right triangle $\geq$ Area of sector $\geq$ Area of triangle

$$\frac{1}{2} b \times h \geq \frac{1}{2} r^2 \theta \geq \frac{1}{2} b \times h$$

$$\frac{1}{2} 1 . \tan x \geq \frac{1}{2} (1)^2 x \geq \frac{1}{2} . 1 . \sin x$$

$$\frac{\tan x}{2} \geq \frac{x}{2} \geq \frac{\sin x}{2}$$

$$\tan x \geq x \geq \sin x$$

$$\frac{1}{\cos x} \geq \frac{x}{\sin x} \geq 1 \qquad \because (\text{dividing by } \sin x)$$

$$\cos x \leq \frac{\sin x}{x} \leq 1$$

$\because$ Applying Limit,

$$\Rightarrow \lim_{x \to 0} \cos x \leq \lim_{x \to 0} \frac{\sin x}{x} \leq \lim_{x \to 0} 1$$

$$1 \leq \lim_{x \to 0} \frac{\sin x}{x} \leq 1$$

$$\Rightarrow \boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

Chapter # 2

Derivative

"Rate of change w.r.t some other quantity."

$$= \frac{\text{Rise}}{\text{Run}}$$

i) Average rate of change:

$$\underset{t_0}{f(t_0)} \longrightarrow \underset{t_1}{f(t_1)} \qquad = \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

ii) Instantaneous rate of change:

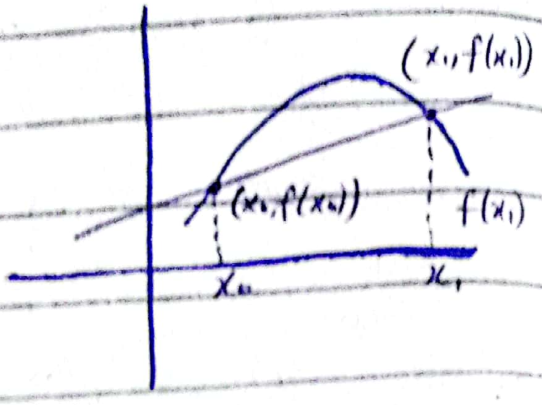
It represents 'Derivative'

$\Rightarrow$ At some specific one point.

Equation of secant line

"Line that intersects curve at two points"

$$y - y_0 = m(x - x_0)$$



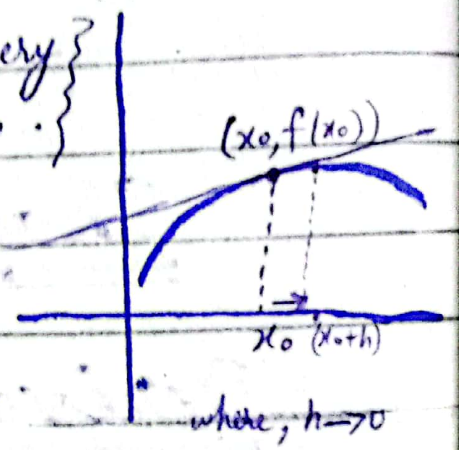
$$\text{slope} = m = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Equation of Tangent line

$$y - y_0 = m(x - x_0)$$

"Line that intersects curve at one point"

{we assumed a very small h near x_0.}



$$\text{slope} = m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{\cancel{x_0} + h - \cancel{x_0}}$$

where, $h \to 0$

$$\boxed{m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}$$ $\Rightarrow$ Use if said, find by definition

or

$$\boxed{m = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}}$$ $\therefore$ by letting, $x_0 + h = x_1$

Example : 1 : $y = x^2$ at $P(1,1)$

Step 1: To ~~#~~ find equation of Tangent line,

$$y - f(x_0) = m(x - x_0)$$

Step 2: $y = x^2$ at $P(1,1)$

$x_0 = 1$, $f(x_0) = 1$, $y_0 = 1$

Step 3 :

$$m = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \lim_{x_1 \to 1} \frac{f(x_1) - f(1)}{x_1 - 1}$$

$$= \lim_{x_1 \to 1} \frac{x_1^2 - (1)^2}{x_1 - 1}$$

$\because \left\{ \begin{array}{l} y = x^2 \\ f(x) = x^2 \\ f(x_1) = x_1^2 \\ f(1) = (1)^2 \end{array} \right\}$

$$= \lim_{x_1 \to 1} \frac{\cancel{(x_1 - 1)}(x_1 + 1)}{\cancel{(x_1 - 1)}}$$

$$= \lim_{x_1 \to 1} x_1 + 1$$

$$\boxed{m = 2}$$

$\therefore$ Eq. of tangent line

$$y - y_0 = m(x - x_0)$$

$$y - 1 = 2(x - 1)$$

$$y - 1 = 2x - 2 \Rightarrow 2x - y - 1 = 0$$

$$\boxed{y = 2x - 1}$$

Example:2 $y = x^2$ at $P(1,1)$, $\rightarrow$ given point is only one, then slope is of tangent to Find.

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - (1)^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h} = \lim_{h \to 0} \frac{h(2+h)}{h}$$

$$= \lim_{h \to 0} 2 + h$$

$$\boxed{m = 2}$$

Example :3

Step 2 : $y = \frac{2}{x}$ at $P(2,1)$

i-e $x_0 = 2$, $f(x_0) = y_0 = 1$

Step 3 :

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{2+h} - 1}{h} = \lim_{h \to 0} \frac{\frac{2-2-h}{2+h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{(2+h)h}$$

$$\boxed{m = \frac{-1}{2}}$$

∴ Putting 'm' in tangent line,

$$y - 1 = \frac{-1}{2}(x-2)$$

$$2y - 2 = -x + 2$$

$$\Rightarrow 2y = -x + 4$$

$$\boxed{y = \frac{4-x}{2}}$$

Example :4

$f(x) = \sqrt{x}$, $x_0 = 1$, $f(x_0) = f(1) = \sqrt{1} = 1$

$\Rightarrow f(x_0+h) = f(1+h)$

$= \sqrt{1+h}$

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

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Quiz → Monday $\boxed{\text{Continuity Full Chapter}}$

$$= \lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} \times \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$

$$= \lim_{h\to 0} \frac{(\sqrt{1+h})^2-(1)^2}{h(\sqrt{1+h}+1)} = \cancel{\lim_{h\to 0} \frac{(\sqrt{1+h}+1)(\sqrt{1+h}-1)}{h(\sqrt{1+h}-1)}}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{1+h}+1)} = \cancel{\lim_{h\to 0}} \frac{1}{\sqrt{1+0}+1}$$

$$= \cancel{\lim} \frac{1}{1+1} = \boxed{\frac{1}{2} = m}$$

Types of Functions

i) $f(x) = c \qquad c \in \mathbb{R}$

Derivate w.r.t x

$$\frac{d}{dx} f = \frac{d}{dx}(c)$$
$$= 0$$

ii) $y = x^n, \; f(x) =$

$$\frac{dy}{dx} = \frac{d}{dx}(x^n)$$
$$= n x^{n-1} \frac{d}{dx}$$

e.g. $y = ax^n$

$$y' = a(nx^{n-1})$$

iii) $h(x) = f(x) \pm g(x)$

$h'(x) = f'(x) \pm g'(x)$

iv) $h(x) = f(x)\, g(x)$

$h'(x) \quad f'(x)\, g(x) + f(x)\, g'(x)$

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v) $h(x) = \frac{f(x)}{g(x)}$

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Question:

$y = 3x^8 - 2x^5 + 6x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}[3x^8] - \frac{d}{dx}[2x^5] + \frac{d}{dx}[6x] + \frac{d}{dx}(1)$$

$$= 3[8.x^7] - 2[5x^4] + 6[1] + 0$$

$$\boxed{\frac{dy}{dx} = 24x^7 - 10x^4 + 6}$$

Horizontal tangent line in graph exist when slope (m) = 0. We find 'm' by taking 1^{st} derivative.

Question: $f(x) = x^3 - 3x + 4$, find point where horizontal tangent line exist.

$y'(x) = 0$

$$\frac{d}{dx}[x^3 - 3x + 4] = 0$$

$$3x^2 - 3 + 0 = 0$$

$$3(x^2 - 1) = 0$$

$\boxed{x = -1}$, $\boxed{x = 1}$

Example : 8

$$f(x) = 5x^{-1} - \frac{1}{5}x$$

$$f'(x) = \frac{d}{dx}\left[5x^{-1} - \frac{1}{5}x\right]$$

$$= \frac{-5}{x^2} - \frac{1}{5}$$

Question: $y = \left[5x^{-1} - \frac{1}{5}x\right]$

Find eq. of tangent line at $(5,0)$

$$y - y_0 = m(x - x_0)$$

$$y - 0 = m(x-5)$$

Now, here $m = \frac{dy}{dx}$

$$m = \frac{d}{dx}\left[5x^{-1} - \frac{1}{5}x\right]$$

$$= -5x^{-2} - \frac{1}{5}$$

$$m|_{(5,0)} = -5(+5)^{-2} - \frac{1}{5}$$

$$= -\frac{1}{5} - \frac{1}{5}$$

$$\boxed{m|_{(5,0)} = -\frac{2}{5}}$$

$\therefore$ eq (i) becomes

$$y = -\frac{2}{5}(x-5)$$

$$\boxed{y = -\frac{2}{5}x + 2}$$

$y = mx + c$
$\downarrow$
y intercept

Now, here y intercept is 2.

So, height $= 2$, base $= 5$

$$\text{Area} = \frac{1}{2}\text{ height} \times \text{base}$$

$$= \frac{1}{2}(2)(5)$$

$$\boxed{\text{Area} = 5}$$

Ex : 2.4 (1-31)

add Questions

Question : $f(x) = 3x^4 - 2x^3 - 4x + 2$

Find $f^n(x)$

Sol: $f'(x) = 12x^3 - 6x^2 - 4$

$f''(x) = 36x^2 - 12x$

$f'''(x) = 72x - 12$

$f^4(x) = 72$

$f^5(x) = 0$

$f^6(x) = 0$

$\boxed{f^n(x) = 0}$

* Find $f'''(2)$, where $f(x) = 3x^2 - 2$

$f'(x) = 6x$

$f''(x) = 6$

$f'''(x) = 0$

$\boxed{f'''(2) = 0}$

* Find $\left.\frac{d^2y}{dx^2}\right|_{x=1}$, where $y = 6x^5 - 4x^2$

$\frac{dy}{dx} = 30x^4 - 8x$

$\frac{d^2y}{dx^2} = 120x^3 - 8$

$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 120 - 8 \boxed{= 112}$

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$y = x\sin x$, find y'

$y' = (1)(\sin x) + (x)(\cos x)$

$y' = \sin x + x\cos x$

$y'\big|_{x=\pi/2} = \sin\frac{\pi}{2} + \frac{\pi}{2}\cos\frac{\pi}{2}$

$= 1 + \frac{\pi}{2}(0)$

$\boxed{y'\big|_{x=\frac{\pi}{2}} = 1}$

$y = \frac{\sin x}{1+\cos x}$, y' at $x = \frac{\pi}{2}$

$$y' = \frac{\left(\frac{d}{dx}\sin x\right)(1+\cos x) - (\sin x)\left(\frac{d}{dx}(1+\cos x)\right)}{(1+\cos x)^2}$$

$$= \frac{\cos x(1+\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \overbrace{\cos^2 x + \sin^2 x}^{1}}{(1+\cos x)^2} = \frac{\cos x + 1}{(1+\cos x)^2}$$

$$y' = \frac{1}{1+\cos x}$$

$$y'\big|_{x=\frac{\pi}{2}} = \frac{1}{1+\cos\frac{\pi}{2}}$$

$$= \frac{1}{1+0} \quad \boxed{= 1}$$

Find $f''\left(\frac{\pi}{4}\right)$, $f(x) = \sec x$

$f'(x) = \sec x . \tan x$

$f''(x) = \sec x . (\sec^2 x) + \tan x(\sec x . \tan x)$

$= \sec^3 x + \sec x . \tan^2 x$

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$$f''\left(\frac{\pi}{4}\right) = \sec^3\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right) \cdot \tan^2\left(\frac{\pi}{4}\right)$$

$$= \sec\left(\frac{\pi}{4}\right)\left[\sec^2\left(\frac{\pi}{4}\right) + \tan^2\left(\frac{\pi}{4}\right)\right]$$

$$= \sec\left(\frac{\pi}{4}\right)(-1)$$

$$\boxed{f''\left(\frac{\pi}{4}\right) = -\sqrt{2}}$$

Chain Rule

$$f(g(x))$$

i) $y' = f'(g(x)) \cdot g'(x)$ ——— (1)

ii) Let, $g(x) = U$

$$y' = \frac{dy}{dU} \cdot \frac{dU}{dx} \quad \text{——— (2)}$$

(*) $y = \cos x^3$, find y'

Method 1:

$$y' = -\sin x^3 \cdot \frac{d}{dx}(x^3)$$

$$= -\sin x^3 \cdot 3x^2$$

$$= -3x^2 \sin x^3$$

Method 2:

Let, $U = x^3$

$$\frac{dU}{dx} = 3x^2$$

Now, $y = \cos U$

$$y' = \frac{dy}{dU} \cdot \frac{dU}{dx}$$

$$y' = -\sin U \cdot 3x^2$$

Ex: 2.6 ⇒ 1 − 50 (Odd Questions)

$\frac{d}{du} = -\sin u$

$= -\sin x^3 \cdot 2x^2$

$\frac{du}{dx} = 3x^2 \sin x^3$

$y' = -\sin x^3 \cdot 3x^2$

$y' = -3x^2 \cdot \sin x^3$

(8) $y = (\tan x)^2$

$y' = 2\tan x \cdot \sec^2 x$

(8) $y = \sqrt{x^3 + \operatorname{cosec} x}$

$y' = \frac{1}{2} \cdot (x^3 + \operatorname{cosec} x)^{-1/2} \frac{d}{dx}(x^3 + \operatorname{cosec} x)$

$= \frac{1}{2\sqrt{x^3 + \operatorname{cosec} x}} \cdot 3x^2 + (-\cot x \cdot \operatorname{cosec} x)$

$= \frac{3x^2 - \operatorname{cosec} x \cot x}{2\sqrt{x^3 + \operatorname{cosec} x}}$

Exercise 6: find $\frac{dU}{dt}$, where $U = \sec\sqrt{\omega t}$

$U' = \sec\sqrt{\omega t} \cdot \tan\sqrt{\omega t} \cdot \frac{d}{dt}(\sqrt{\omega t})$

$U' = \sec\sqrt{\omega t} \cdot \tan\sqrt{\omega t} \cdot \left(\sqrt{\omega} \cdot \frac{d}{dt}(t)^{1/2}\right)$

$U' = \sec\sqrt{\omega t}\, \tan\sqrt{\omega t} \cdot \sqrt{\omega} \cdot \frac{1}{2\sqrt{t}}$

$U' = \frac{\sqrt{\omega} \sec\sqrt{\omega t} \cdot \tan\sqrt{\omega t}}{2\sqrt{t}}$

(*) $y = \sin\sqrt{1 + \cos x}$

$y' = \cos\sqrt{1 + \cos x} \cdot \frac{d}{dx}(\sqrt{1 + \cos x})$

$y' = \cos\sqrt{1 + \cos x} \cdot \frac{1}{2\sqrt{1 + \cos x}}(-\sin x)$

$= -\sin x \cdot \cos\sqrt{1 + \cos x}$

EX: 3.2

#

- If $y = \log_b x$, Find y'

$$y' = \frac{1}{x \cdot \ln b} \quad ; \quad b > 0,\ b \neq 1$$

#

- $y = \ln x$

$$y' = \frac{1}{x}$$

- $\ln e = y$

$$y' = 1$$

- $y = \ln(x^2+1)$

$$y' = \frac{1}{x^2+1} \cdot (2x)$$

$$y' = \frac{2x}{x^2+1}$$

- $\frac{d}{dx}\left[\ln\left(\frac{x^2 \sin x}{\sqrt{1+x}}\right)\right] \Rightarrow$ Using Properties of ln.

Method-1:

$$= \frac{1}{\frac{x^2 \sin x}{\sqrt{1+x}}} \cdot \frac{d}{dx}\left(\frac{x^2 \sin x}{\sqrt{1+x}}\right)$$

⋮

continue

Method-2: short method, by ln properties

$$= \frac{d}{dx}\left(\ln(x^2 \sin x) - \ln\sqrt{1+x}\right)$$

$$= \frac{d}{dx}\left[\ln x^2 + \ln \sin x - \ln(1+x)^{1/2}\right]$$

$$= \frac{d}{dx}\left[2 \cdot \ln x + \ln \sin x - \frac{1}{2}\ln(1+x)\right]$$

$$= 2(1) + \frac{1}{\sin x} \cdot \cos x - \frac{1}{2(1+x)}$$

Ex : 3.2 ⇒ 1-50 (odd Numbers)

$$= 2 + \frac{\cos x}{\sin x} - \frac{1}{2(1+x)}$$

⊛ Find derivation of,

$$y = \frac{x^2 \cdot \sqrt[3]{7x+14}}{(1+x^2)^4}$$

$$y' = \frac{d}{dx}\left[\frac{x^2 \cdot \sqrt[3]{7x+14}}{(1+x^2)^4}\right]$$ ∴ using Quotient rule

$$= \frac{\frac{d}{dx}\left(x^2 \cdot \sqrt[3]{7x+14}\right) \cdot (1+x^2)^4 - \left(x^2 \cdot \sqrt[3]{7x+14}\right)\frac{d}{dx}\left[(1+x^2)^4\right]}{\left[(1+x^2)^4\right]^2}$$

⇒ This method is very difficult

Now,

∴ Taking ln on booth sides

$$\ln y = \ln\left(\frac{x^2 \cdot \sqrt[3]{7x+14}}{(1+x^2)^4}\right)$$

$$= \ln\left(x^2 \cdot (7x+14)^{1/3}\right) - \ln(1+x^2)^4$$

$$= \ln x^2 + \ln(7x+14)^{1/3} - \ln(1+x^2)^4$$

$$\ln y = 2\ln x + \frac{1}{3}\ln(7x+14) - 4\ln(1+x^2)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 2\cdot\frac{1}{x} + \frac{1}{3}\left(\frac{1}{7x+14}\frac{d}{dx}(7x+14)\right) - \frac{4}{1+x^2}\cdot\frac{d}{dx}(1+x^2)$$

$$= \frac{2}{x} + \frac{7}{3(7x+14)} - \frac{8x}{(1+x^2)}$$

$$\frac{dy}{dx} = y\cdot\left[\frac{2}{x} + \frac{7}{3(7x+14)} - \frac{8x}{(1+x^2)}\right]$$

$$= \frac{x^2\sqrt[3]{7x+14}}{(1+x^2)^4}\left[\frac{2}{x} + \frac{7}{3(7x+14)} - \frac{8x}{1+x^2}\right]$$