LINEAR ALGEBRA

BSE: 4th Semester

24/6/24

Function:

$f(x) = 2x$

function, Input, output

→ float f(float x)

```
{
    return 2*x;
}
```

→ $f(x,y) = 2x - 4y$

```
float f(float x, float y)
{
    return 2*x - 4*y;
}
```

→ $f(x, y, z) = (2x + y, x + y + z)$

```
float * f(float x, float y, float z)
{
    float * r = new float[2];
        r[0] = 2*x+y;
        r[1] = x+y+z;
        return r;
}
```

⊛ General form:

$$[\text{output}] = \begin{bmatrix}\text{Matrix} \\ \text{of constants}\end{bmatrix}[\text{input}]$$

E.g:

→ $[2x] = [2][x]$

→ $[2x - 4y] = [2 \ -4]\begin{bmatrix} x \\ y \end{bmatrix}$

→ $\begin{bmatrix} 2x + y \\ x + y + z \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

⊙ $f(x, y) = 4x^2 - xy$

$[4x^2 - xy] = [\ \ ]\begin{bmatrix} x \\ y \end{bmatrix}$ ✗ : Not possible

- $f(x,y) = 4x + 2y + 1$

$$\begin{bmatrix} 4x+2y+1 \end{bmatrix} = \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ ∴ Not Possible

- **Linear Function:**

That can be written as matrix multiplication.

→ Linear Algebra → Matrices

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$ → complete information of linear function

∴ Matrix gives symbol free environment

→ Real world Problems which are valuable can be easily dealt with linear algebra.

- **Affine Function:**

Function which Loses Linearity because of addition of Constant. E.g: $f(x,y) = 2x + 4y + 1$ (1: constant)

Matrix-Vector Multiplication:

→ $\begin{bmatrix} 4 & 1 & 3 \\ 0 & 1 & 1 \\ 2 & 3 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ 35 \end{bmatrix}$

: mVr (matrix vector right)

→ $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 & 3 \\ 0 & 1 & 1 \\ 2 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 10 & 12 & 32 \end{bmatrix}$

: mVl

Matrix-Matrix Multiplication:

→ $\underbrace{\boxed{\begin{bmatrix} 1 & 4 & 9 \\ 6 & 1 & 2 \end{bmatrix}}}_{M} \begin{bmatrix} 3 & 4 & 0 & 9 \\ 1 & 1 & 1 & 2 \\ 1 & 3 & 0 & 3 \end{bmatrix}$

$V_1 \quad V_2 \quad V_3 \quad V_4$

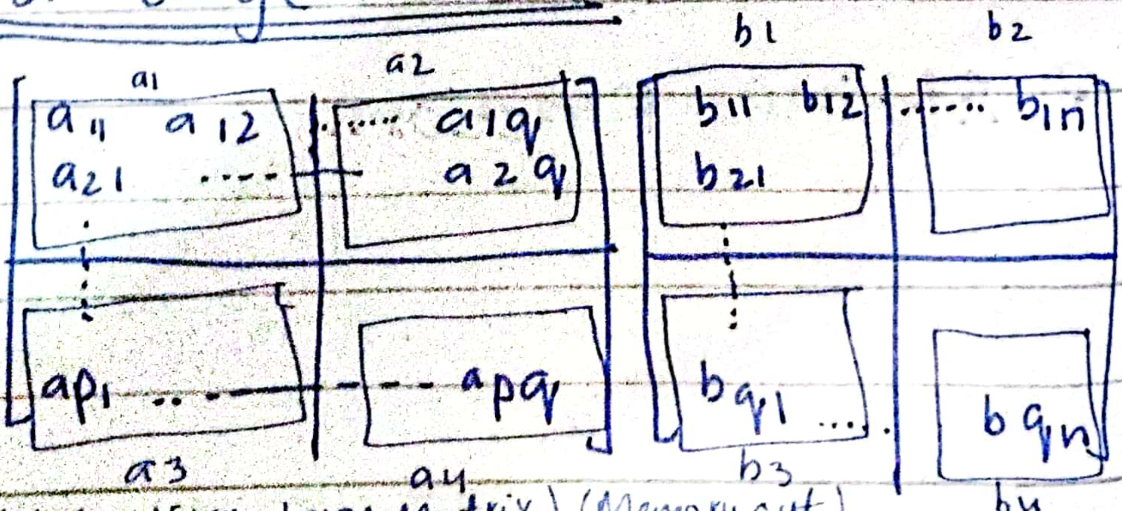
: do it by reuse mVr

: mVl

→ $\begin{array}{cc} v_1 \\ v_2 \end{array} \begin{bmatrix} 1 & 4 & 9 \\ 6 & 1 & 2 \end{bmatrix} \underbrace{\boxed{\begin{bmatrix} 3 & 4 & 0 & 9 \\ 1 & 1 & 1 & 2 \\ 1 & 3 & 0 & 3 \end{bmatrix}}}_{M}$

$\begin{bmatrix} \vdots & \vdots \end{bmatrix}$

→ <u>For large Matrix:</u>



(Very Very Large Matrix) (Memory out)

: Divide it into parts:

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$: a_1. rows = b_1. cols

→ $2x + 3y = 10$ → $3x - y = 0$ (satisfy both) (complex one)

$2(2) + 3(2) = 10$

$2(5) + 3(0) = 10$

: $(2,2)$, $(5,0)$ → sol. set.

→ If we set more constraints, solution may not be possible.

(*) $2x_1 + 3x_2 + 4x_3 + x_4 + 5x_9 + 69x_{100} = 5$

$3x_1 + 4x_2 + \ldots\ldots + 3x_{100} = 9$

$\vdots$: lot of constraints

$49x_1 + \ldots\ldots + 4x_{100} = 0$

→ $$\boxed{\begin{aligned} 2x + 3y - z &= 4 \\ 5x + 6y + 6z &= 9 \end{aligned}}$$

: Linear Systems $$\boxed{\begin{bmatrix} 2 & 3 & -1 \\ 5 & 6 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}}$$

(*) First Assignment Linear system (C++)

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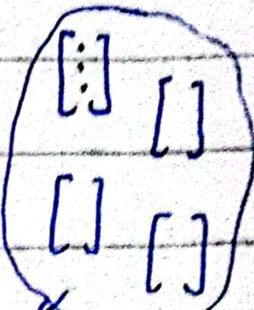
⊙ Linear System:

$$2x + 3y - z = 9$$
$$4x + 4y + 3z = -1$$
$$6x + 7y + 2z = 8$$

: Unknowns satisfy both
: more constraints more compl.

$\Rightarrow (2x+4x) + (3y+4y) + (-z+3z) = 9-1$

$\Rightarrow (2x+3y-z) + (4x+4y+3z) = 9-1$



Solution space

(Sometimes more constraints donot effect solution set)
(E.g: car & specifications).

: MC↑ LSS↓
(More constraints) (less solution space).

$\Rightarrow 2x + 3y - z = 9$
$4x + 6y - 2z = 8$

$\rightarrow 2(2x+3y-z) = 8$

$\underbrace{2(2x+3y-z)}_{9} \neq 8$

: Sometime we have contradictory constraints

$\Rightarrow$ Equation swap doesnot change the solution.

$$\begin{matrix} 2x+3y+4z = 9 \\ x+y-z = 4 \\ 2x-y-z = 0 \end{matrix} \rightarrow \begin{matrix} x+y-z = 4 \\ 2x+3y+4z = 9 \\ 2x-y-z = 0 \end{matrix}$$

$eq_1 \leftrightarrow eq_2$

⇒ Multiplying or Equation scaling doesnot change solution ($n>0$) (non-zero)

: $2x+3y+4z=9$

$x+y-z=4$

$2(2x-y-z)=2(0)$

: Solution will be same

⇒ Adding number into equation doesnot change solution :

: $2x+3y+4z=9$

$(x+y-3)+9=4+9$

$2x-y-3=0$

: solution will be same

: $(x+y-3)+(2x+3y+4z)$ $=4+9$

⇒ If we add one equation into other solution doesnot change the solution.

$(x+y-3)+(2x+3y+4z)=4+9$

: Adding scalar multiple = doesnot change the solution.

⊛ 3 to 4 rules of solution which do not effect solution :

⊛ Echelon form of Matrix.

$\Rightarrow$

$$2x+3y+4z=9$$
$$x+y-z=4$$
$$2x-y-z=0$$

Sol:-

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & +1 & -1 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc:c} 2 & 3 & 4 & 9 \\ 1 & 1 & -1 & 4 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc:c} 2 & 1 & 3 & 4 & 6 \\ 9 & 1 & 1 & 3 & 2 \\ 6 & 1 & -1 & 0 & 0 \end{array}\right]$$

$$2a+b+3c+4d=6$$
$$9a+b+c+3d=2$$
$$6a+b-c=0$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 4 \\ 0 & 9 & 7 & 3 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

$$a+2b+3c=4$$
$$9b+7c=3$$
$$4c=8$$

: We can calculate value of 'c' and find others easily.

$$\left[\begin{array}{ccc:c} 2 & 3 & 4 & 9 \\ 1 & 1 & -1 & 4 \\ 2 & -1 & -1 & 0 \end{array}\right] \xrightarrow{R_3+(-2R_2)} \left[\begin{array}{ccc:c} 2 & 3 & 4 & 9 \\ 1 & 1 & -1 & 4 \\ 0 & -3 & 1 & -8 \end{array}\right]$$

Now swap

swap R_1 and R_2

$-2 \; -2 \; 2 \; -8$

$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ 2 & 3 & 4 & 9 \\ 0 & -3 & 1 & -8 \end{bmatrix} \xrightarrow{R_2+(-2R_1)} \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & 6 & 1 \\ 0 & -3 & 1 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & 6 & 1 \\ 0 & -3 & 1 & -8 \end{bmatrix} \xrightarrow{R_3+(+3R_2)} \begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 19 & -5 \end{bmatrix}$$

$\therefore a + b - c = 4$

$b + 6c = 1$

$19c = -5$

$\therefore \boxed{c = -\frac{5}{19}}$

$\therefore b + 6\left(-\frac{5}{19}\right) = 1$

$\therefore b = 1 + \frac{30}{19}$

$\boxed{b = \frac{49}{19}}$

$\therefore a + \frac{49}{19} + \frac{5}{19} = 4$

$\therefore a = 4 - \frac{54}{19}$ $\qquad$ $76-54$

$\boxed{a = \frac{22}{19}}$

put in eq. 1 $\qquad$ put in eq. 3

$\frac{22}{19} + \frac{49}{19} + \frac{5}{19} = 4$ $\qquad$ $\therefore 19\left(\frac{-5}{19}\right) = -5$

$\frac{\cancel{76}^{4}}{\cancel{19}_{1}} = 4$ $\qquad$ $-5 = -5$ (satisfy)

$4 = 4$ (satisfy)

⊙ Magic:

-6 3 -18 -9

$$\begin{bmatrix} 2 & -1 & 6 & 3 \\ 4 & 1 & 0 & 1 \\ 6 & 0 & 6 & 5 \end{bmatrix} \xrightarrow{R_3 + (-3R_1)} \begin{bmatrix} 2 & -1 & 6 & 3 \\ 4 & 1 & 0 & 1 \\ 0 & 3 & -12 & -4 \end{bmatrix}$$

$R_2 + (-2R_1)$

-4 2 -12 -6

$R_3 + (-1R_2)$

$$\begin{bmatrix} 2 & -1 & 6 & 3 \\ 0 & 3 & -12 & -5 \\ 0 & 3 & -12 & -4 \end{bmatrix} \xrightarrow{R_3 - (R_2)} \begin{bmatrix} 2 & -1 & 6 & 3 \\ 0 & 3 & -12 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$2a - b + 6c = 3$$
$$3b - 12c = -5$$
$$\boxed{0 = 1} \text{ Contradiction}$$

No Solution

: If complete zero last row then re tundan

→ Row Echelon form:

Rules:

① The pivot of a non-zero row is the first non-zero entry from left to right.

② Every pivot must contain(s) zeros below it.

③ First come first serve pivots. $\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$

④ zero row must be at the end.

∴ ⊛ Assignment (Read Matrix from text and convert to row echelon form) (till Monday)

: if(a==0) : if (abs(a) ≤ 0.000000001)

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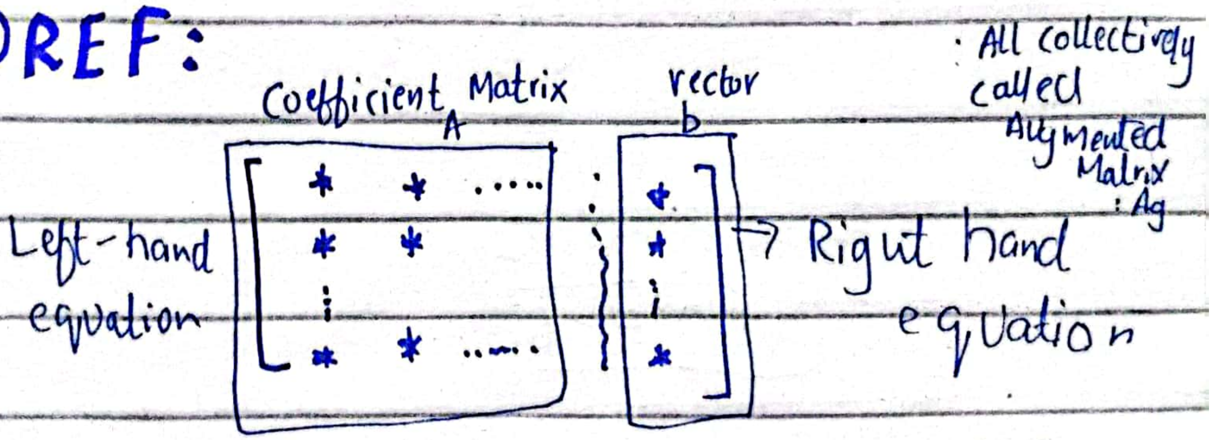
⊙ REF:



: Every Matrix can be convertible to Row Echelon Form.

⊛ $2x + 3y = 4$
$x - y = 7$

$$\underbrace{\underbrace{\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 4 \\ 7 \end{bmatrix}}_{b}}$$

$$A\boxed{x} = b$$

: Solution :
(i) Exist
(ii) Not exist
(iii) Infinite

$$\left[\boxed{A}\ \boxed{b}\right] = \left[\begin{array}{cc|c} 2 & 3 & 4 \\ 1 & -1 & 7 \end{array}\right]$$

$$\begin{bmatrix} \boxed{2} & 3 & 9 & 7 & 5 \\ 0 & \boxed{1} & 5 & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$2p + 3q + 9r + 7s = 5$$
$$q + 5r + 2s = 0$$
$$s = 3$$

: $q + 5r + 6 = 0$

$$\boxed{q + 5r = -6}$$

: $r = 0, q = -6$

: Infinite solutions

$$\begin{bmatrix} \boxed{} \\ \boxed{-6-5t} \\ \boxed{t} \\ \boxed{3} \end{bmatrix}$$

$r = t$

$q = -6 - 5t$

: $2p + 3(-6-5t) + 9t + 21 = 5$

$2p - 18 - 15t + 9t = 5 - 21$

$$p = \frac{5 - 21 + 18 + 15t - 9t}{2}$$

$$= \frac{2 + 6t}{2} = \boxed{1 + 3t}$$

⊙ $$\begin{bmatrix} 2 & 3 & 9 & 7 & 5 \\ 0 & 1 & 5 & 2 & 0 \\ 0 & 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$2a+3b+9c+7d=5$ ...... ①

$b+5c+7d=0$

$+2c+1d=3$

put $d=t$

$2c+t=3 \Rightarrow c=\frac{3-t}{2}$

$2\left(\frac{3-t}{2}\right)+t=3$

$\frac{6-2t}{2}+t=3$

$\frac{6-2t+2t}{2}=3$

$\frac{6}{2}=3$

$3=3$

put in eq. ①:

$2a+3b+9c+7d=5$

$2a+3b+9\left(\frac{3-t}{2}\right)+7t=5$

$2a+3b+\frac{27-9t}{2}+7t=5$

$4a+6b+27-9t+14t=5$

$4a+6b+27+5t=5$

$4a+6b+5t=-22$

⊙
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

$a+2b+3c+4d=5$

$+c+2d=3$

$+d=-6$

: $c+(-12)=3$

$\boxed{c=15}$

$a+2b+3(15)+4(-6)=5$

$a+2b=5-45+24$

$a+2b=-16$

⊙
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
: $t=$ free Variable

$a+2b+3c+4d=5$

$+c+2d=3$

put $d=t_1$

$c=3-2t_1$

$a+2b+3(3-2t_1)+4(t_1)=5$

$a+2b+9-6t_1+4t_1=5$

$a+2b+9-2t_1=5$

$a+2b-2t_1=-4$

$a+2b=2t_1-4$ $\quad$: $b=t_2$

$a+2t_2=2t_1-4$

$a=2t_1-2t_2-4$

$2t_1-2t_2-4+2t_2=2t_1-4$

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* Rank of a Matrix:

→ Rank (A) → Non-negative integer

↳ The total number of pivots in its REF (Row Echelon form)

→ $A \xrightarrow{REF} \begin{bmatrix} \boxed{5} & 3 & 4 & 9 \\ 0 & \boxed{1} & 3 & 2 \\ 0 & 0 & 0 & \boxed{6} \\ 0 & 0 & 0 & 0 \end{bmatrix}$: 3 pivots

→ $A=\begin{bmatrix} 1 & 5 & 3 & 2 & 7 \\ 6 & 1 & -1 & 3 & 2 \\ 0 & 1 & 9 & 6 & 1 \end{bmatrix}$

: $R_2 - 6R_1$

$$A = \begin{bmatrix} 1 & 5 & 3 & 2 & 7 \\ 0 & -29 & -19 & -9 & -40 \\ 0 & 1 & 9 & 6 & 1 \end{bmatrix}$$

: $29R_3 + R_2$

$$A = \begin{bmatrix} 1 & 5 & 3 & 2 & 7 \\ 0 & -29 & -19 & -9 & -12 \\ 0 & 0 & 270 & \text{(Not Zero)} & \text{(Not Zero)} \end{bmatrix}$$

Rank $(A) = 3$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \begin{matrix} R_1 \\ + \\ R_2 \\ = \\ 3 \end{matrix}$$

: Independant Rows (Number of pivots)

: $R_2 - 2R_1$

2 4 6 8

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

: Depends on first row.

$R_3 \sim R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 6 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q: Find out the number of linearly independant rows of a matrix A.

Ans:

A - Rank(A)

Q: $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{bmatrix}$

$\therefore$ Number of Independant Rows = 1

$\therefore$ Rank = 1

$\rightarrow$ $\boxed{\text{Rank}(A) = \text{Rank}(A^T)}$

(Prove it wrong).

* $\begin{bmatrix} 1 & 0 \\ -3 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix}$ (True) (Rank: 2)

* $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$ (True) (Rank: 1)

* $\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$ (True) (Rank: 3)

* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (True) (Rank: 3)

* $\boxed{\text{Rank}(A) = \text{Rank}(A)^t}$

(Always true).

**: Anton, Romers
(Linear Algebra and its application)

* Total independant rows is equal to total independant columns of a matrix.

* Questions in Paper:
  (i) Linear Systems
  (ii) Rank of Matrix.
  (iii) Program (C++) (Row Echelon Form related) (Rank using REF).

Coefficical Matrix

* $\underset{3\times3}{\boxed{A}}\ \underset{3\times1}{x} = \underset{3\times1}{b}$

: 3 Variables (x_1, x_2, x_3)
: 3 Rows
: Size of Matrix (3)

$$Ag = \underbrace{\left[\underset{3\times3}{A} \;\vdots\; \underset{3\times1}{b}\right]}_{3\times4}$$

→ Rank(Ag) $\underline{\geq}$ Rank(A)

$=$ $\boxed{\geq}$
$<$ $\leq$
$>$

(A rank will be 3 w.r.t columns but Ag rank can be 3 or greater than 3 w.r.t column

* $\boxed{\text{Rank}(Ag) = \text{Rank}(A)}$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 6 \\ 0 & 4 & 5 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
(consistent)

OR

Rank(Ag) = Rank(A) + 1

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 6 \\ 0 & 4 & 5 & 7 \\ 0 & 0 & 0 & 8 \end{array}\right]$$
(Inconsistent)

$$\overset{A}{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}\overset{B}{\begin{bmatrix} \\ \\ \end{bmatrix}} = \left[\begin{array}{cc:c} \multicolumn{2}{c}{A} & B \\ \bigcirc & & \bigcirc \\ \bigcirc & & \bigcirc \\ 0 & 0 & 0 \end{array}\right]$$

(Consistent) $A_g = A$

: If $A \neq A_g$ (System is inconsistent)

$$A x = b$$

$p \times q, \; q \times 1 \qquad p \times 1$

: p = equations

q = unknowns

(Exam Question)

$$\underline{\text{Rank}(A) = P}$$

Consistent or Not?