Function. A *function* from a set X to a set Y is a rule f which assigns to ever element x of X a unique element y of Y.

- The set X is called the *domain* of f and the set Y is called the *codomain* of f.
- y is called *image* of x under f, and is usually represented in terms of a formula $y = f(x)$.
- The set of all images is called the *range* of f.
- The variable x which represents all the elements of the domain of f is called the *independent variable*.
- The variable y which represents all the elements of the range of f is called the *dependent variable*.
- If $range(f) = Y$, the full codomain, then f is said to be an *onto* function.
- f is said to be a *one-to-one* function if distinct elements of domain has distinct images in codomain.

Note.

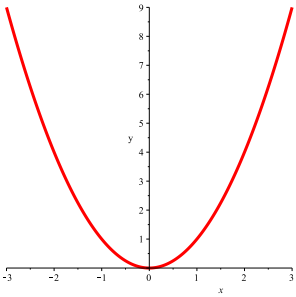
1. Domains and ranges of all our functions will be subsets of real numbers.
2. The domain of f is all real numbers if there is no square root or fraction in the formula.

Geometric Approach.

- **Vertical Line Test:** Every vertical line intersects the graph of a function f exactly at one point.
- If a horizontal line intersecting the graph meets the y-axis at the point y, then y belongs to the range of f. The set of all such y points form the range of f.
- f is onto if every horizontal line intersects the graph of f.
- f is one-to-one if every horizontal line intersects the graph of f exactly at one point.

Example 1. Sketch the function $f(x) = x^2$ and find its domain and range. Check wether it is one/onto or not.

Solution.
Step 1. (Graph)



Graph of $f(x) = x^2$

Step 2. (Domain and Range)

Domain: Since square of every real number is possible, the $domain(f) = \mathbb{R}$.

Range: Since all the horizontal lines that cut the graph lie on and above the x-axis, the $range(f) = [0, +\infty)$.

Step 3. (One-to-One and Onto)

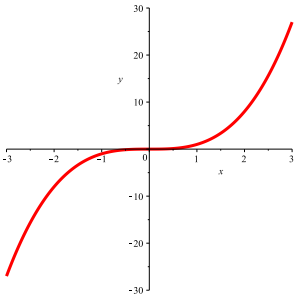
One-One: Since each horizontal line that lies above the x-axis intersects the graph at TWO points, f is not one-to-one.

Onto: Since the lines that lie below the x-axis do not intersect the graph, f is not onto.

———————————*Solution Ends*———————————

Example 2. Sketch the function $f(x) = x^3$ and find its domain and range. Check wether it is one/onto or not.

Solution.
Step 1. (Graph)



Graph of $f(x) = x^3$

Step 2. (Domain and Range)

Domain: Since cube of every real number is possible, the $domain(f) = \mathbb{R}$.

Range: Since all the horizontal lines intersect the graph, the $range(f) = \mathbb{R}$.

Step 3. (One-to-One and Onto)

One-One: Since each horizontal line intersects the graph exactly at one point, f is one-to-one.

Onto: Since all horizontal lines intersect the graph, f is onto.

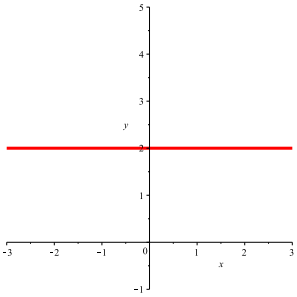
———————————*Solution Ends*———————————

Some Basic Functions.

1. **Constant Function.** The function of the form $f(x) = c, c \in \mathbb{R}$, is called the *constant function.*

   Graph: The graph of the constant function is always a horizontal line passing the y-axis at point c.

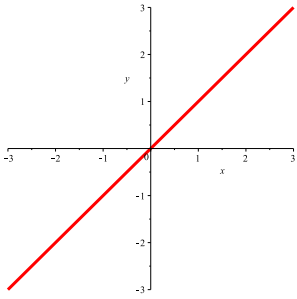
   Example: $y = 2$



2. **Identity Function.** The function of the form $f(x) = x, x \in \mathbb{R}$, is called the *identity function.*

   Graph: The graph of the identity function is always a straight line through the origin, making the angle of 45^o with the positive x-axis.

   Example: $f(x) = x$

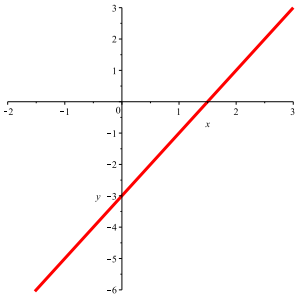


3. **Linear Function.** The function of the form $f(x) = ax + b, x \in \mathbb{R}$ and a, b are fixed real numbers, is called the *linear function.*

   Graph: The graph of a linear function is always a straight line with slope a (the coefficient of x).

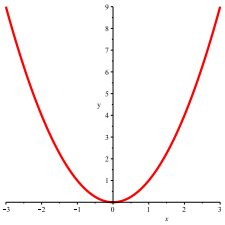
   Example:

   $f(x) = 2x - 3$

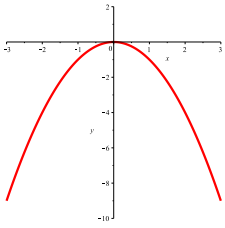
   

4. **Quadratic Function.** The function of the form $f(x) = ax^2 + bx + c, x \in \mathbb{R}$ and a, b, c are fixed real numbers, is called the *quadratic function.*

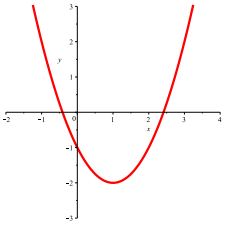
   Graph: The graph of a quadratic function is always a parabola. If a is positive, parabola opens upward; if a is negative, parabola opens downward.

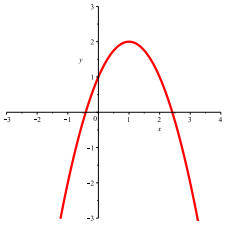
   $y = x^2$

   $y = -x^2$

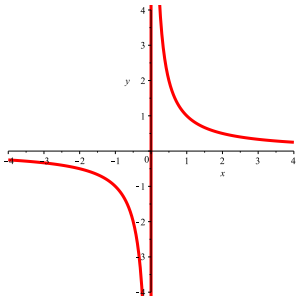
   

   $y = x^2 - 2x - 1$

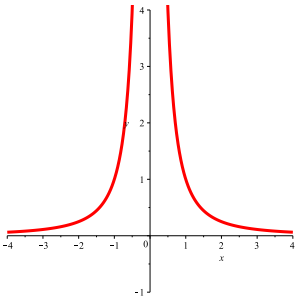
   

   $y = -x^2 + 2x + 1$

5. **Rational Function.** A function of the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials, is called the *rational function.*

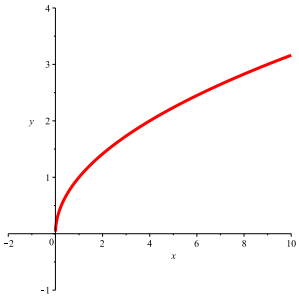


$f(x) = \frac{1}{x}$

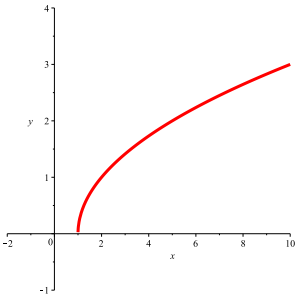


$f(x) = \frac{1}{x^2}$

6. **Square Root Function.** A function of the form $f(x) = \sqrt{x}$ is called the *root function*.

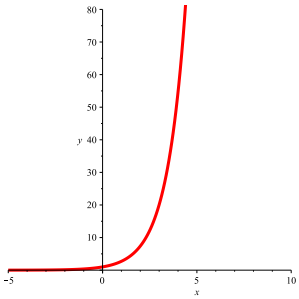


$f(x) = \sqrt{x}$

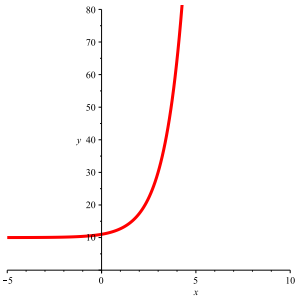


$f(x) = \sqrt{x-1}$

7. **Exponential Function.** A function of the form $f(x) = e^x$ is called the *exponential function*.

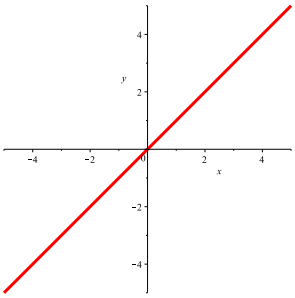


$f(x) = e^x$

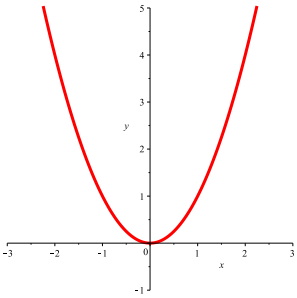


$f(x) = e^x + 10$

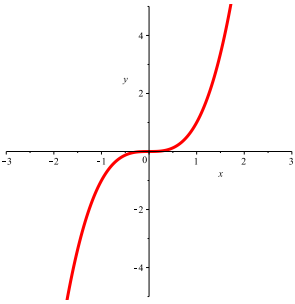
8. **Power Function.** A function of the form $f(x) = x^n$, where $n \in \mathbb{Z}_+$ is called the *power function*.



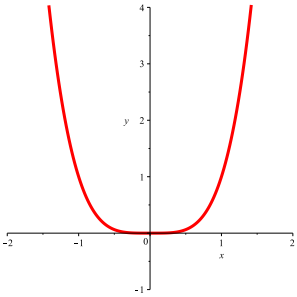
$f(x) = x$



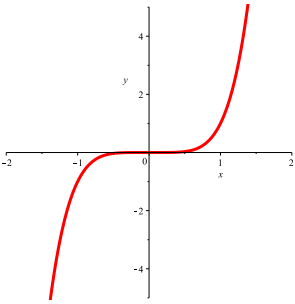
$f(x) = x^2$



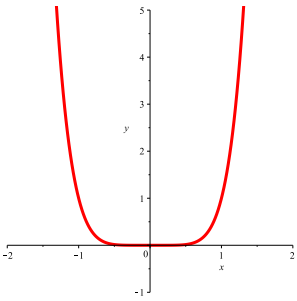
$f(x) = x^3$



$f(x) = x^4$

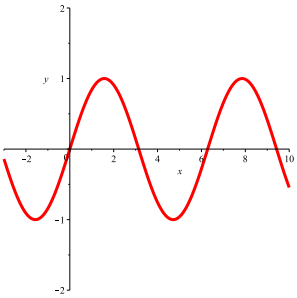


$f(x) = x^5$

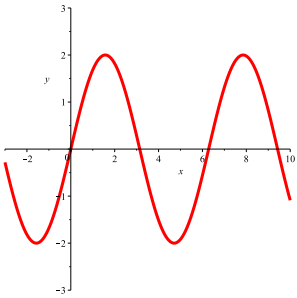


$f(x) = x^6$

9. **Sine Function.** A function of the form $f(x) = \sin x$ is called the s*ine function.*

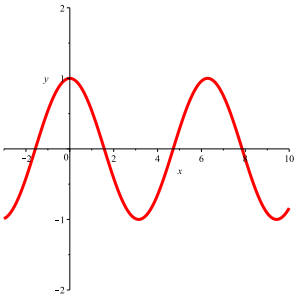


$f(x) = \sin x$

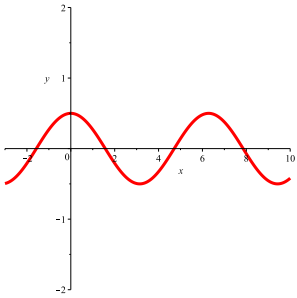


$f(x) = 2\sin x$

10. **Cosine Function.** A function of the form $f(x) = \cos x$ is called the *Cosine function.*

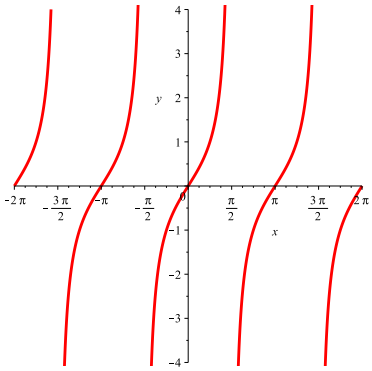


$f(x) = \cos x$



$f(x) = \frac{1}{2}\cos x$

11. **Tangent Function.** A function of the form $f(x) = \tan x$ is called the *tangent function.*

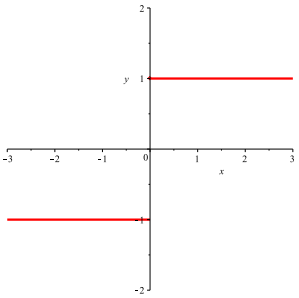


$f(x) = \tan x$

12. **Piecewise-Defined Function.** A function is represented by different formulas for different parts of its domain.

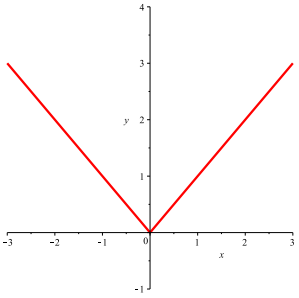
Example 1.

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & 0 \le x \end{cases}$$



Example 2.

$$f(x) = \begin{cases} -x & x < 0 \\ x & 0 \le x \end{cases}$$



Example 3.

$$f(x) = \begin{cases} -1 & x < -1 \\ -x & -1 \le x < 0 \\ x^2 & 0 \le x < 1 \\ 2 & 1 \le x \end{cases}$$