CS 2150 GRAPH THEORY
RESEARCH CYCLE - 1
PROJECT REPORT

GROUP A1

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DEFINITIONS AND THEOREMS

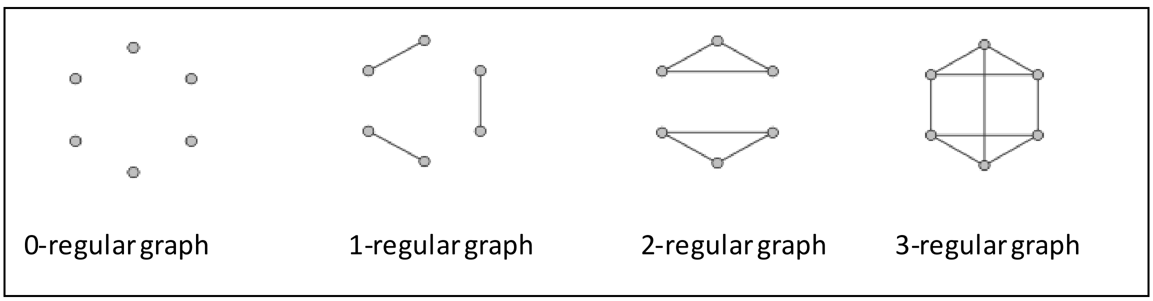
1. Graph

A graph G = (V, E) is an ordered pair of finite sets. Elements of V are called vertices or nodes, and elements of E are called edges or arcs. We refer to V as the vertex set of G, with E being the edge set. (Definition from [1])

2. Regular Graph

In graph theory, a regular graph is a graph where each vertex has the same number of neighbors; i.e. every vertex has the same degree or valency. A regular directed graph must also satisfy the stronger condition that the indegree and outdegree of each vertex are equal to each other.

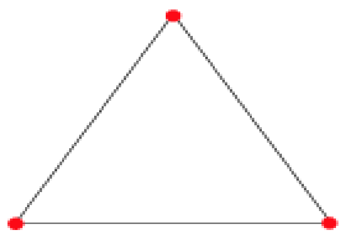
(Definition from [2])



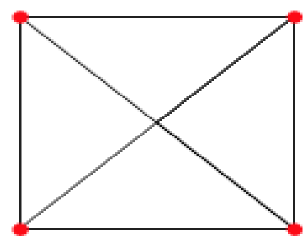
3. Complete Graph (Kn)

A complete graph is a graph in which each pair of vertices is connected by an edge. The complete graph with n vertices is denoted by Kn. (Definition from [3])

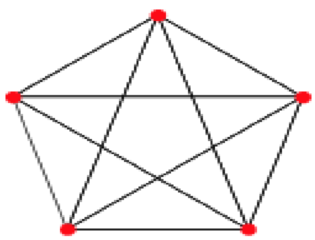
K_2



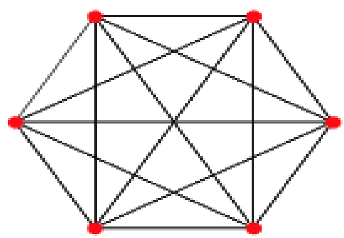
K_3



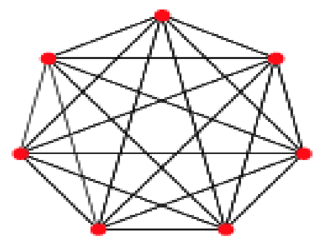
K_4



K_5



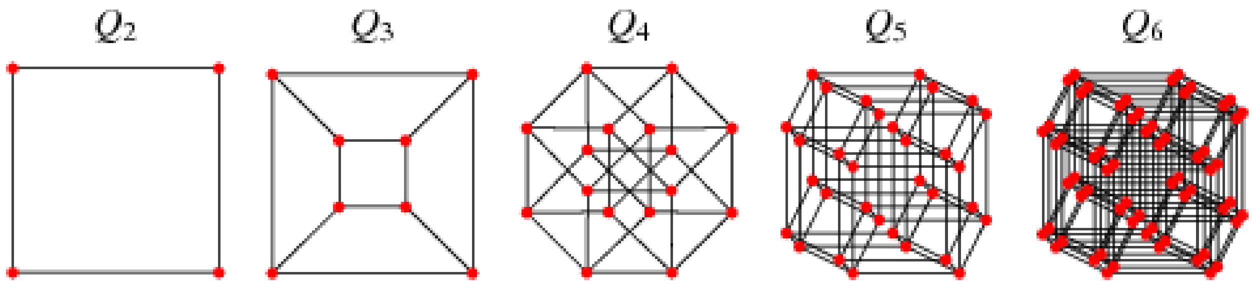
K_6



K_7

4. Hypercube Graph (Qn)

The hypercube graph is a regular graph with 2n vertices, 2n−1n edges, and n edges touching each vertex. It is denoted by Qn. (Definition from [4])



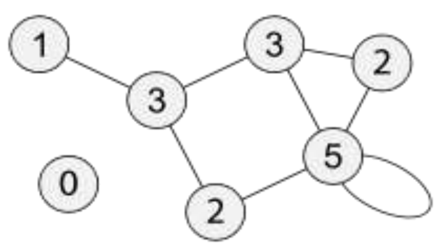
5. Degree of a vertex Δ(G)

The degree of vertex in a graph is number of edges that enter or exit from the vertex. A loop contributes 2 to the degree of its vertex.

Directed graphs have two types of degrees, known as the indegree and the outdegree.

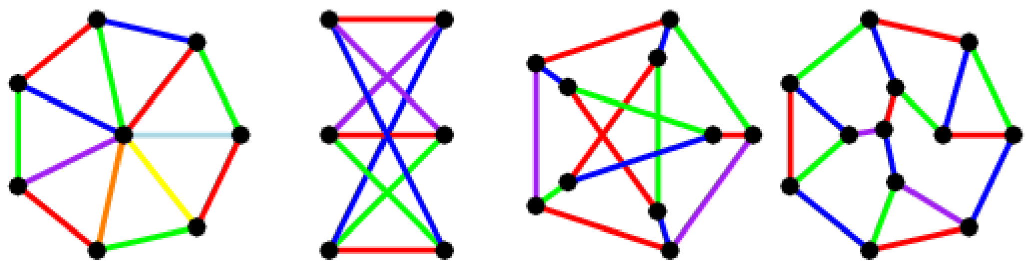
- Indegree - The number of inward directed graph edges from a given graph vertex
- Outdegree - The number of outward directed graph edges from a given In the following figure, degree of vertex is shown inside vertex.

(Definition from [5])



6. Proper Edge Colouring

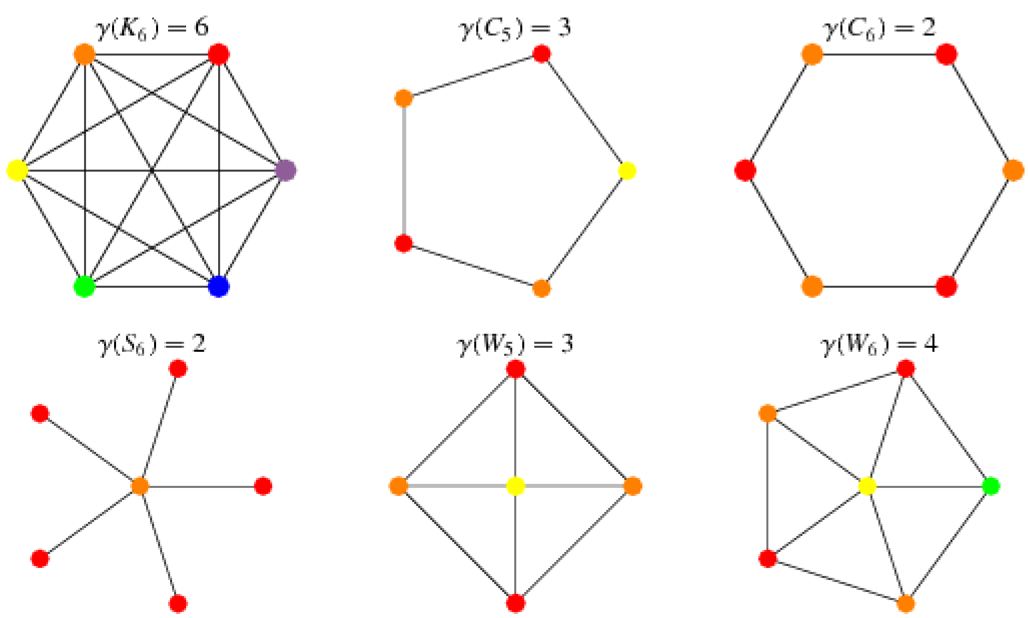
An edge colouring of a graph is an assignment of colours to edges of a graph so that no vertex is incident to two edges of the same colour. An edge colouring with k colours is called a **k-edge-colouring**. (Definition from [6])



7. Chromatic Number

The chromatic number of a graph G is the smallest number of colours needed to colour the vertices of G so that no two adjacent vertices share the same colour.

The chromatic number of a graph G is most commonly denoted $\chi(G)$.



Finding the chromatic number of a graph is really two problems. If we claim that $\chi(G) = n$, then we must show

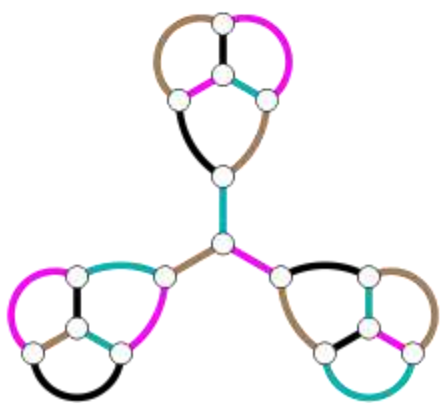
1. that G <u>cannot</u> be coloured with $n - 1$ colours and
2. that it <u>can</u> be coloured with n colours

(Definition from [7])

8. Edge Chromatic Number

The edge chromatic number, sometimes also called the chromatic index, of a graph G is fewest number of colours necessary to colour each edge of G such that no two edges incident on the same vertex have the same colour. In other words, it is the number of distinct colours in a minimum edge colouring.

The edge chromatic number of following figure is 4.



Finding the edge chromatic number of a graph is really two problems. If we claim that $\chi'(G) = n$, then we must show

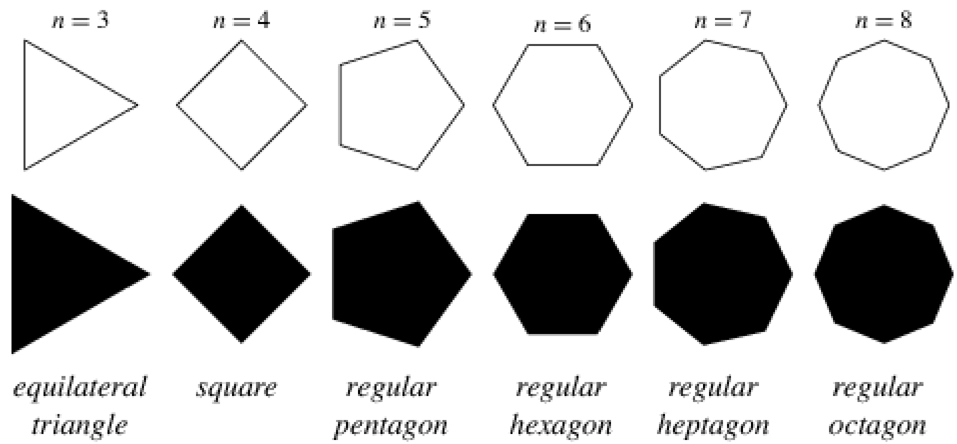
1. that edges of G cannot be coloured with n – 1 colours and

2. that edges can be coloured with n colours

(Definition from [8])

9. Regular Polygon

A regular polygon is an n-sided polygon in which the sides are all the same length and are symmetrically placed about a common center. (Definition from [9])



10. Platonic Solid

The Platonic solids, also called the regular solids or regular polyhedra, are convex polyhedra with equivalent faces composed of congruent convex regular polygons.

The faces are congruent, regular polygons, with the same number of faces meeting at each vertex. In platonic solid, all degree of vertex is same. There are five platonic solids. Each one is a polyhedron (a solid with flat faces). They are made from the regular polygons: equilateral triangle, square and regular pentagon. They are also convex. (Definition from [10])

Cube

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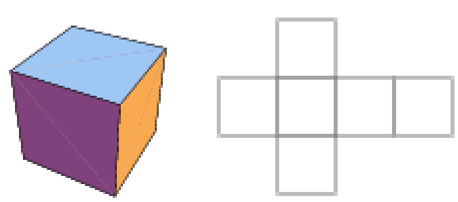
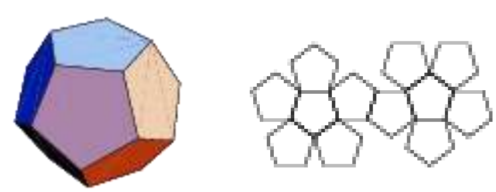
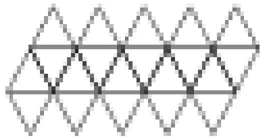


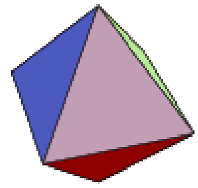
Dodecahedron

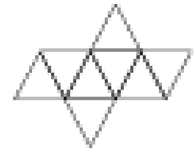


Icosahedrons

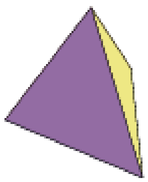


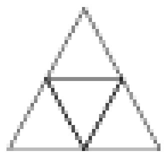
Octahedron





Tetrahedron





11. Map Colouring

Map colouring is the act of assigning different colours to different features on a map. There are two very different uses of this term. The first is in cartography, choosing the colours to be used when producing a map. The second is in mathematics, where the problem is to determine the minimum number of colours needed to colour a map so that no two adjacent features have the same colour. (Definition from [11])

12. Contrapositive Proof

In logic, the contrapositive of a conditional statement is formed by negating both terms and reversing the direction of inference. Explicitly, the contrapositive of the statement "if A, then B" is "if not B, then not A." A statement and its contrapositive are logically equivalent: if the statement is true, then its contrapositive is true, and vice versa. (Definition from [12])

PROBLEM – GROUP A1

a) Show that the number of colours required for a proper edge colouring of a graph G is $\geq$ the maximum degree of any vertex of G.

b) The graph Q_3 depicts the vertices and edges of the cube which is one of the five Platonic solids (solids whose faces are identical regular polygons).

i. Which platonic solid is represented by K_4? ii. Find the chromatic number, the edge chromatic number, and the minimum number of colours needed to colour the map represented by K_4

SOLUTIONS TO THE PROBLEM

Part a)

Show that the number of colours required for a proper edge colouring of a graph G is ≥ the maximum degree of any vertex of G.

Let’s prove the statement by Contra positive method.

Statement:

The number of colours required for a proper edge colouring of a graph G is ≥ The maximum degree of any vertex of G

Let,

- The number of colours required for proper edge colouring of G (Edge chromatic number) be ‘n’
- The maximum degree of any vertex of G be ‘x’.

Prepositions in the Statement:

P: Graph G is properly coloured

! P: Graph G is not properly coloured

Q: $n \geq x$ (number of colours required for proper edge colouring is greater than or equal to the

Maximum degree of any vertex of G)

!Q: $n < x$

To prove:

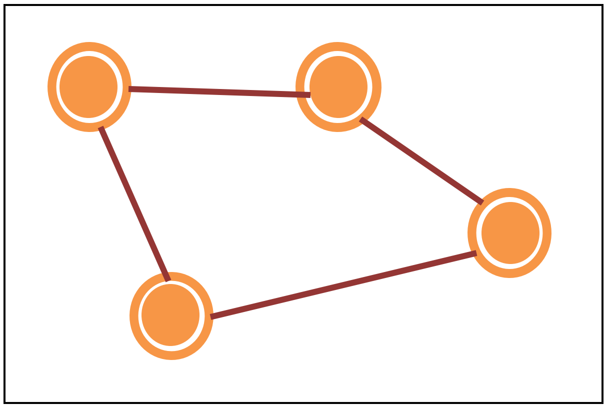
P => Q

By contra positive method, we have to prove that, **!Q =>!P**

That is, if $n < x$, then graph G is not properly coloured.

Proof:

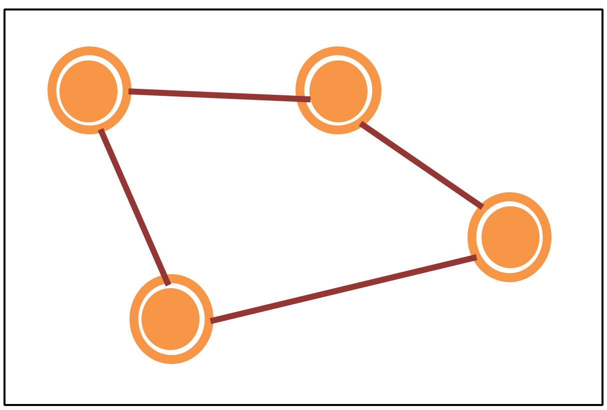
Lets take the following graph



Here maximum degree of any vertex = 2

So x = 2

For n<x to be true, we must be able to colour the edges using at most 1 colour.



This violates the definition of proper edge colouring because two edges incident on the same vertex have the same colour.

That is the graph G is not properly coloured. (!P)

Taking !Q as true, we have proved that !P. (!Q => !P).

By the contrapositive method, **P => Q**

If Graph G is properly coloured, then the number of colours required is ≥ maximum degree of vertex.

Part b)

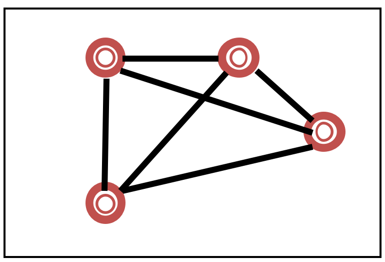
The graph Q_3 depicts the vertices and edges of the cube which is one of the five Platonic solids (solids whose faces are identical regular polygons).

i. Which platonic solid is represented by K_4?

Answer: Tetrahedron

Proof

By the definition of Kn, K_4 has 4 vertices and each and every pair of its vertices is connected with each other. This can be graphically represented as below.

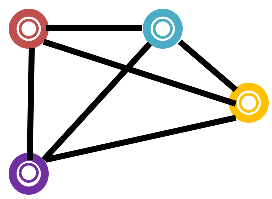


- Platonic solids have faces with regular polygons.
- K_4 has 6 edges and the corresponding platonic solid should have **6 edges.**
- Since there are 4 vertices in the graph, that platonic solid should also have **4 vertices**.
- **Triangular faces** can be obtained when each of the three vertices in the graph are considered.
- Consider that there are **equilateral triangular faces** in this platonic solid.
- The platonic solid which satisfies all the above details is a tetrahedron.

ii. Find the chromatic number, the edge chromatic number, and the minimum number of colours needed to colour the map represented by K_4

a) <u>Chromatic number</u>

Each and every vertex in the graph is adjacent to all other vertices of the graph. The definition of chromatic number states that no two adjacent vertices share the same colour. Therefore each edge should have a different colours. We need 4 colours and we cannot colour the graph 4 using 3 colours.



The chromatic number = 4

b) <u>Edge Chromatic number</u>

In a complete graph (Kn) , each vertex is connected with the rest of the vertices. Each vertex has (n-1) vertices connected with it. Thus, there are (n-1) edges incident on each vertex. The definition of proper edge colouring states that no two edges incident on the same vertex have the same colour. Therefore (n-1) colours are required to colour (n-1) edges incident on each vertex.

In K_4 graph, n=4. We need (4-1 =3) colours.

We have to use at least 3 colours to do proper edge colouring.

Edge Chromatic number = 3

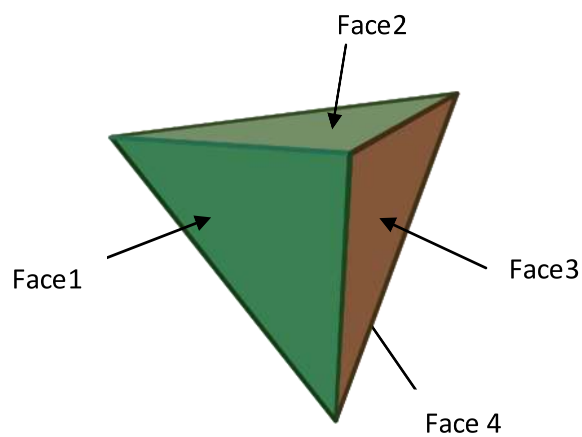
c) **<u>The minimum number of colours needed to colour the map</u>**

In order to implement map colouring, no two areas touching each other should have the same colour.

In a tetrahedron, there are 4 areas. Each triangular area touches the remaining 3 triangular faces.

Therefore all the 4 faces should have distinct colours.

Therefore we need at least 4 colours to colour the map.



QUESTIONS

1. State whether the following statements are true or false.
   a) The number of colours needed for proper edge colouring of a graph is equal to the maximum degree of any vertex of that graph
   b) The minimum number of colours needed need for map colouring of any platonic solid is always 4.

2. The map of Sri Lanka is divided into 9 provinces. Using the concepts of graph theory, how many colours are needed to properly map colour the 9 provinces.

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