Markov, Bayes Filter I

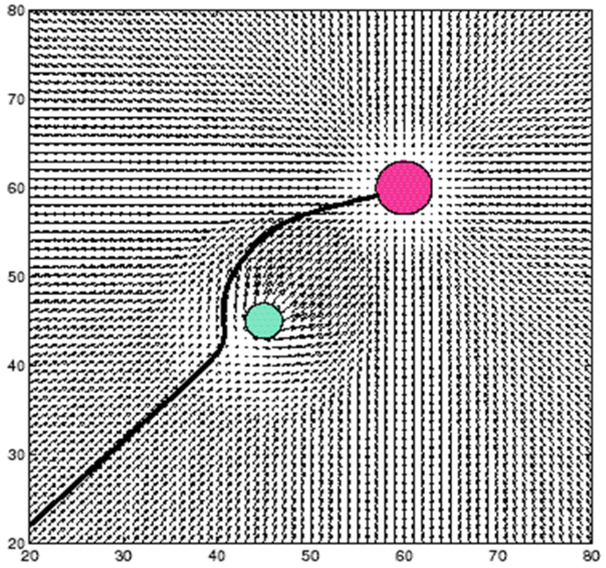
Fast Robots, ECE4160/5160, MAE 4190/5190

E. Farrell Helbling, 3/20/25

Class Action Items

- Lab 7: Kalman Filtering: please do not leave this to the weekend.
- Lab 8: Stunts. The lab is posted, you have two options, the flip or the drift. Please try to get this done next week before spring break (or do it after spring break). I don’t recommend taking the robot on a plane!
- Lab 3 regrade requests will close on **Thursday midnight.** Submit requests in canvas.
- Lab 4 grades delayed, found an error in the spreadsheet, will hopefully post grades later today!

Modeling path planning as a graph search problem

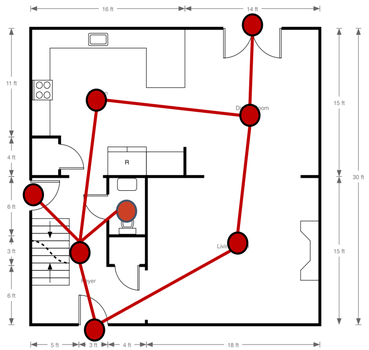


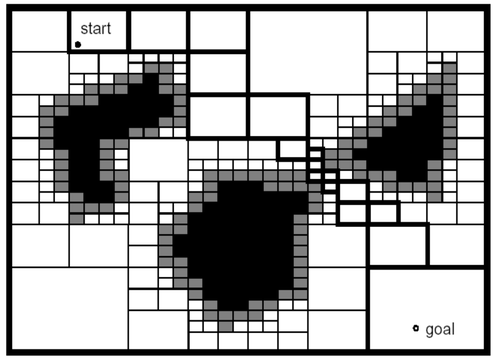
Common alternatives

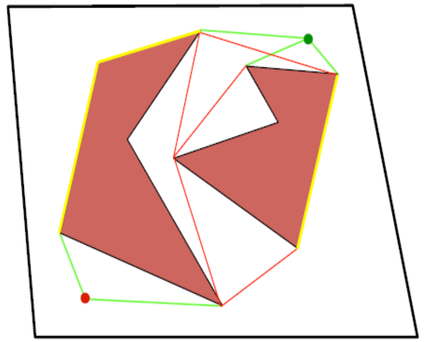
- Optimal control
- Potential fields

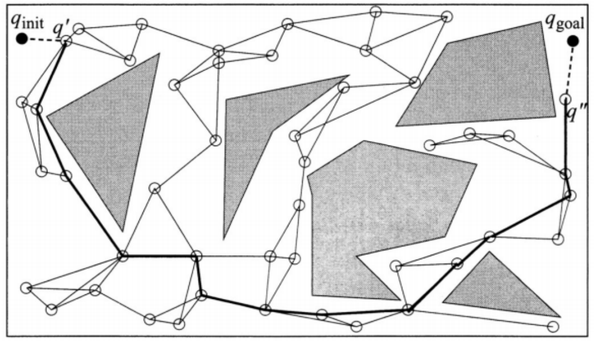
Modeling path planning as a graph search problem

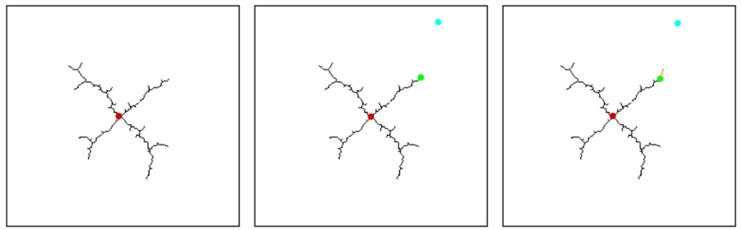
Real World → Configuration Space → Map Representation → **Graph Construction** → Graph Search











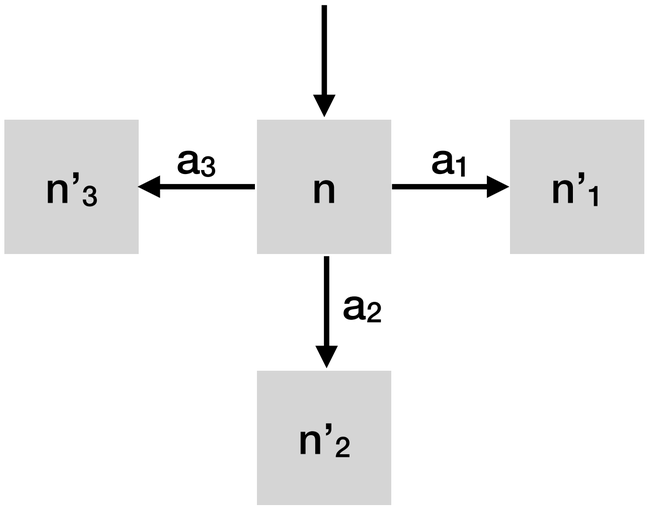
- Geometry-based graphs
  - Topological Graphs
  - Cell decomposition
  - Visibility Graphs
- Sampling-based graphs
  - RRT
  - PRM

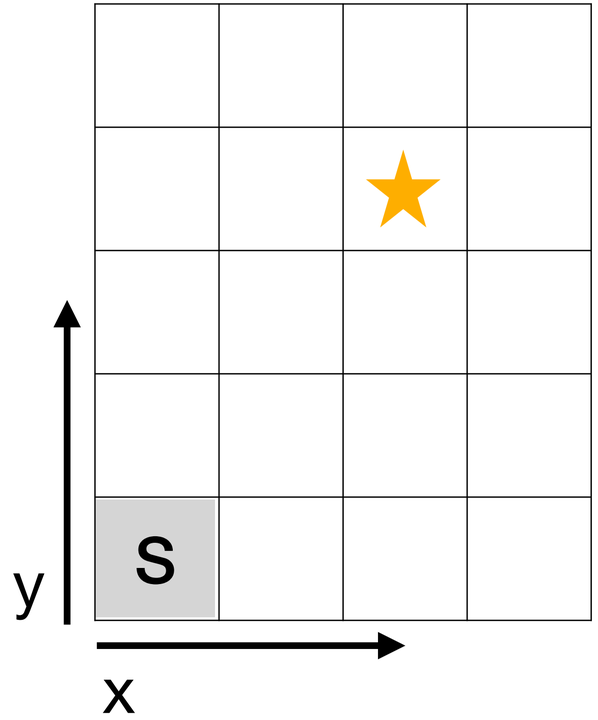
Modeling path planning as a graph search problem

- Uninformed Searches
  - Breadth First
  - Depth First
  - Dijkstra's (LCF)
- Informed Searches
  - Greedy
  - A*

Search Algorithms, General

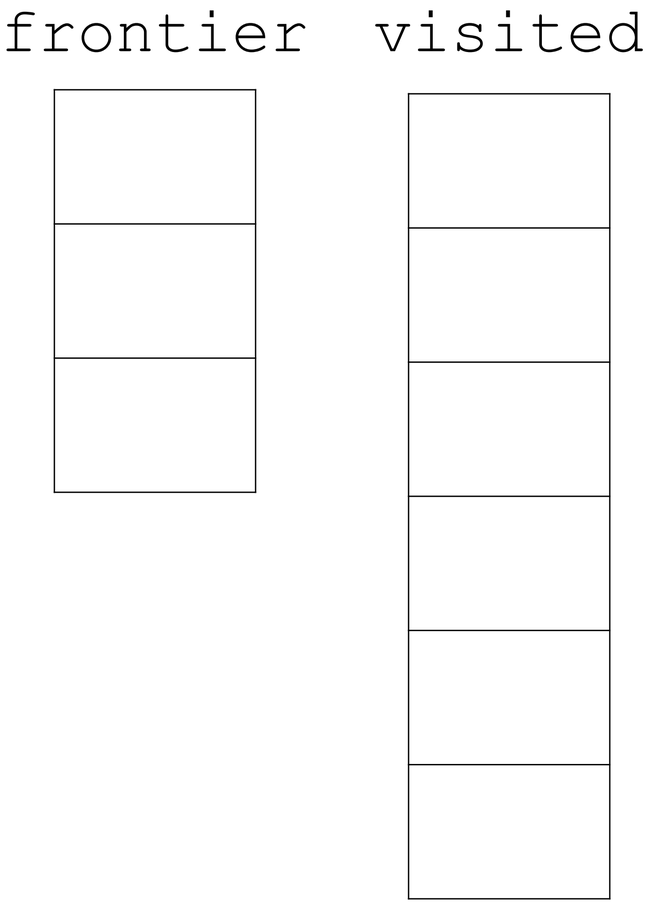
- For every node, n
- There is a set of actions, a
- That moves you to a new node, n’

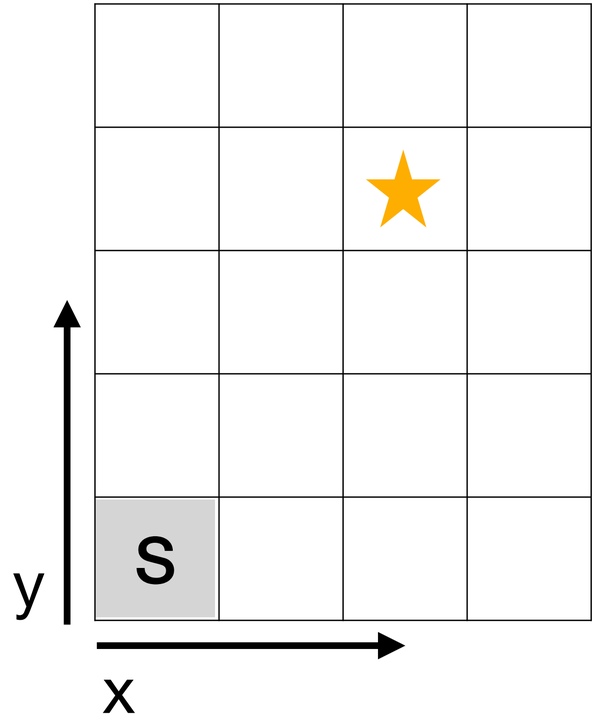




Uninformed Algorithms, General

```
n = state(init)
frontier.append(n)
while(frontier not empty)
  n = pull state from frontier
  append n to visited
  if n = goal, return solution
  for all actions in n
    n' = a(n)
    if n' not visited
      append n' to frontier
```





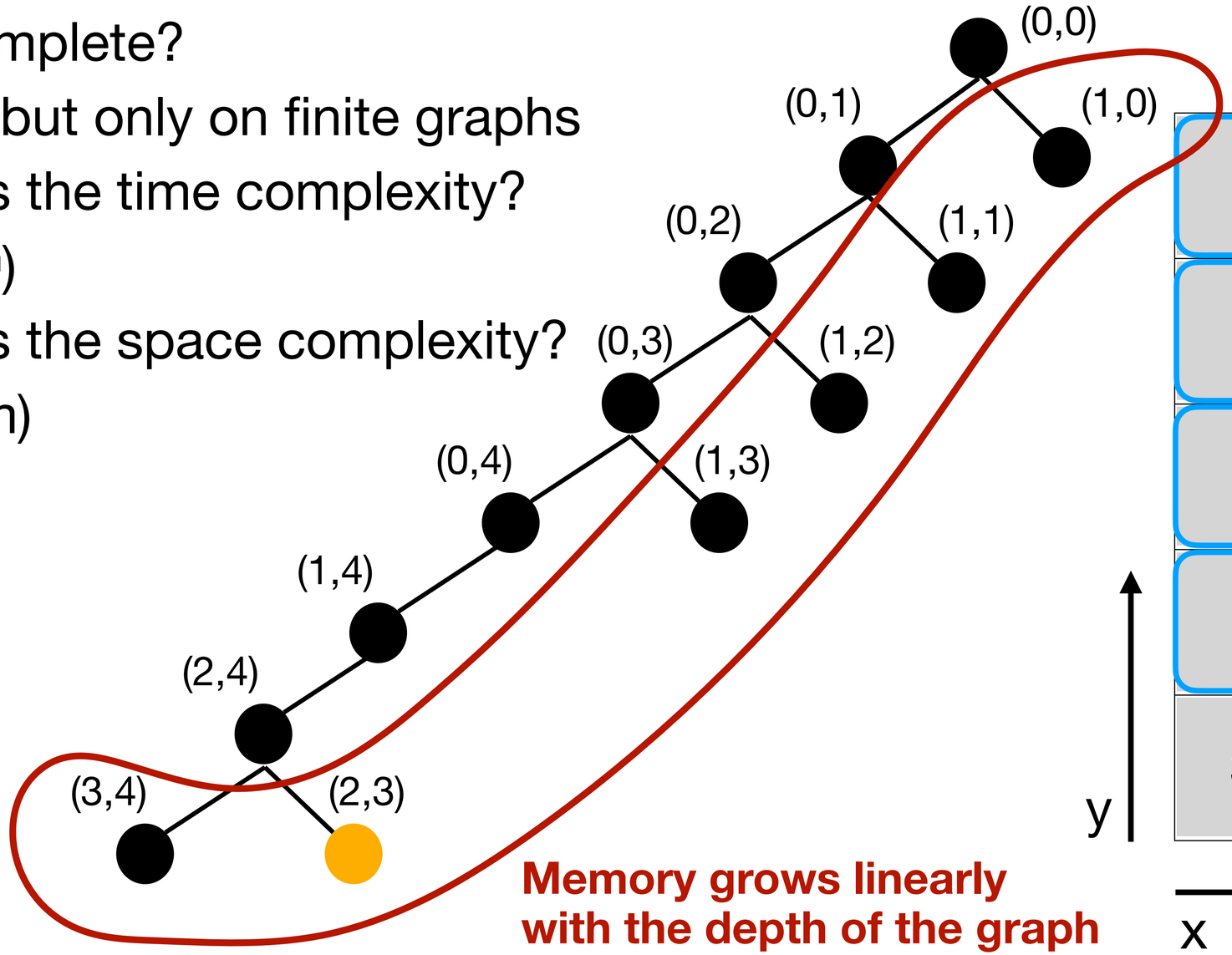
DFS: Last-In First-Out (LIFO)

BFS: First-In First-Out (FIFO)

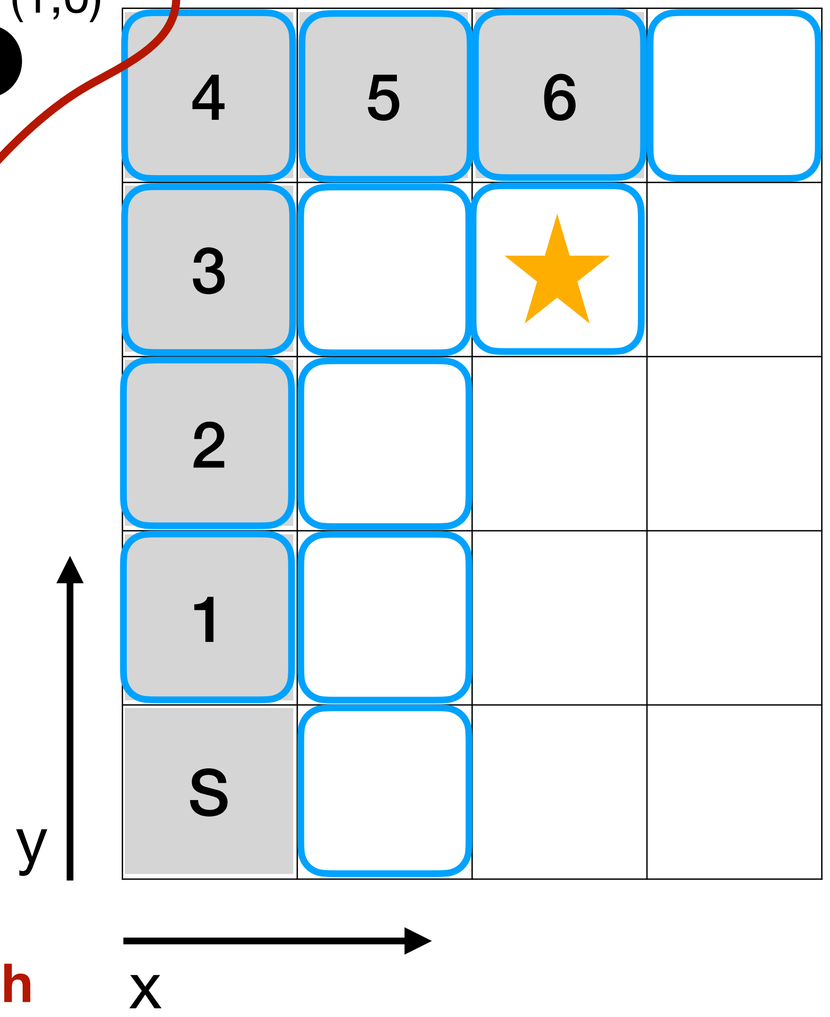
LCFS: Prioritize cost

Depth First Search

- Is it complete?
  - Yes, but only on finite graphs
- What is the time complexity?
  - $O(b^m)$
- What is the space complexity?
  - O(bm)



Search Order: N, E, S, W

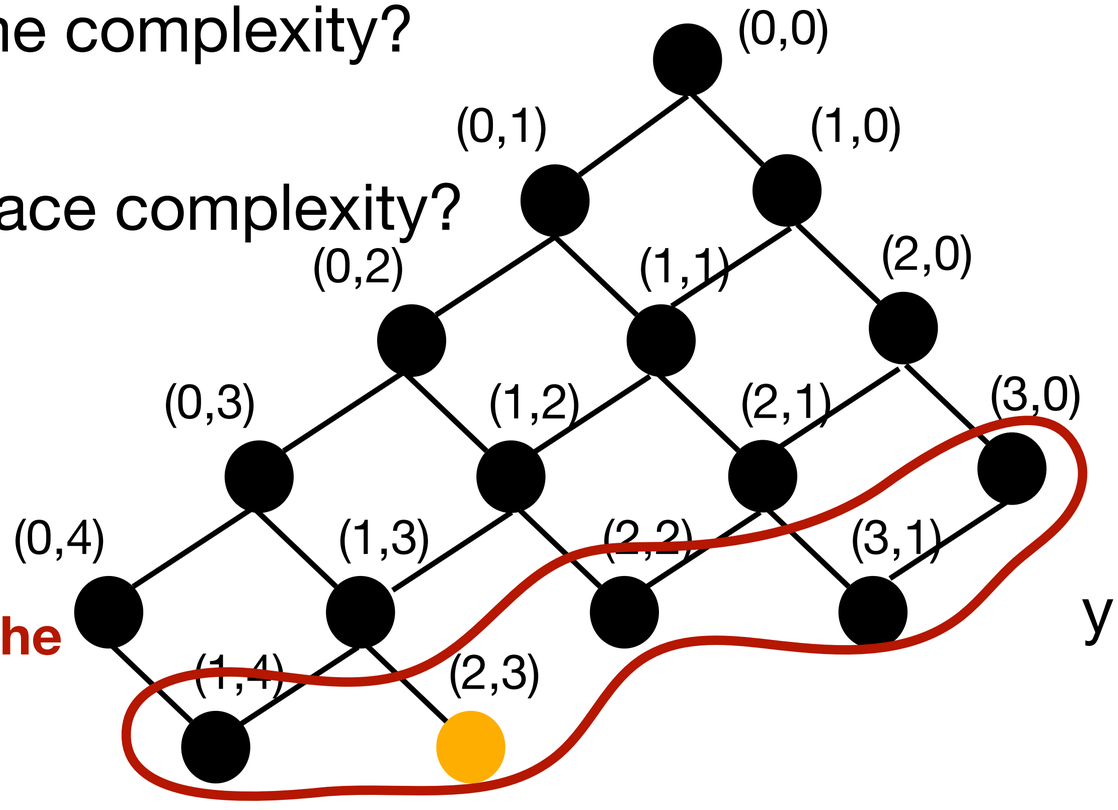


Memory grows linearly with the depth of the graph

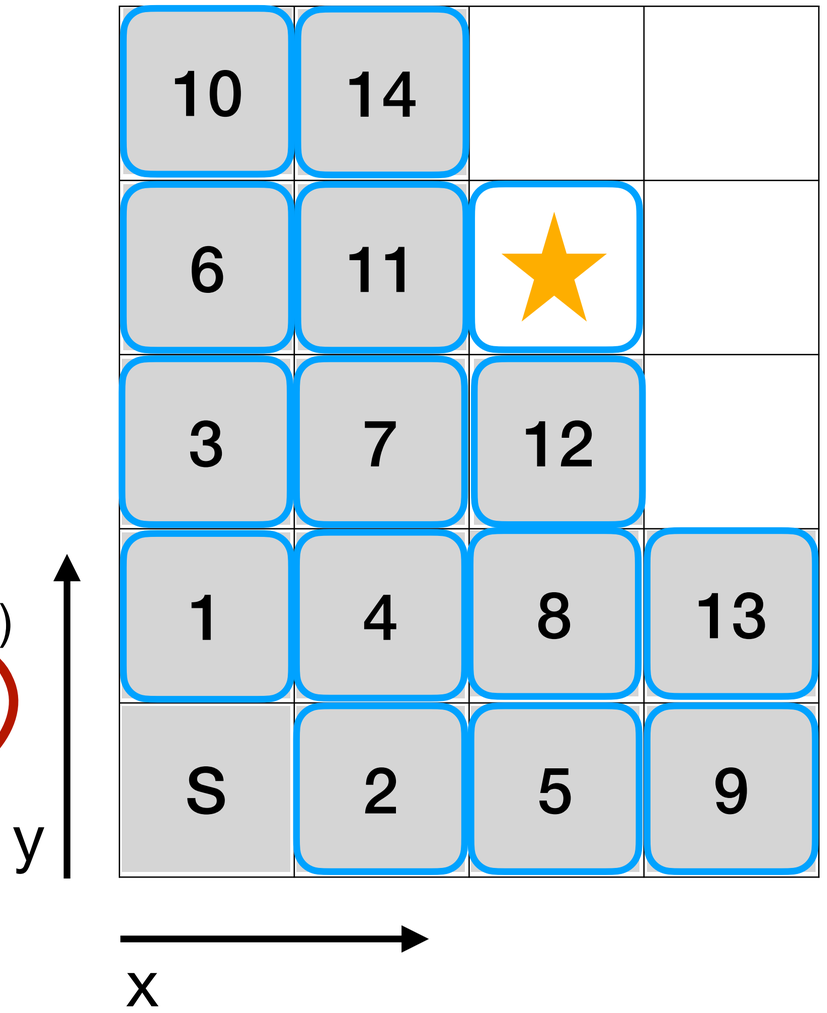
Breadth First Search

- Is it complete?
  - Yes, as long as b is finite
- Is it optimal?
  - Yes
- What is the time complexity?
  - $O(b^m)$
- What is the space complexity?
  - $O(b^m)$

Memory grows exponentially with the depth of the graph



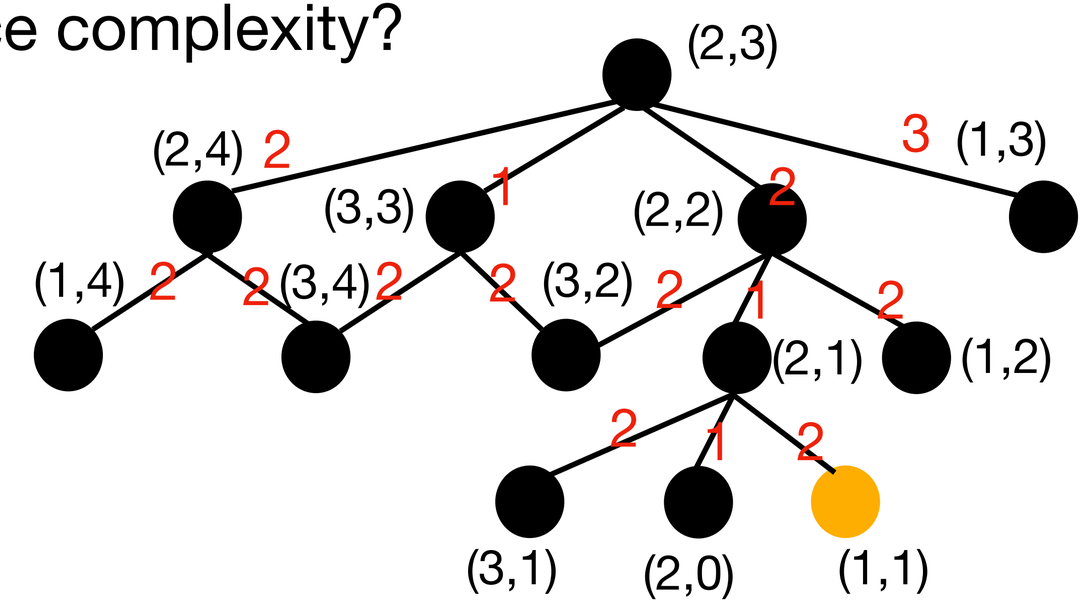
Search Order: N, E, S, W



Lowest-Cost First Search (LCFS)

Consider parent cost!

- Is it complete?
  - Yes, as long as path costs are positive
- What is the time complexity?
  - $O(b^{1+C/c})$
- What is the space complexity?
  - $O(b^{1+C/c})$



- Go straight, cost 1
- Turn one quadrant, cost 1

	(1,4)	(2,4)	(3,4)
	(1,3)	R	(3,3)
	(1,2)	(2,2)	(3,2)
	G	(2,1)	(3,1)
		(2,0)	

Uninformed Search Algorithms

Criterion	BFS	DFS	LCFS
Complete	Yes (finite)	No (finite)	Yes (positive cost)
Time	$O(b^m)$	$O(b^m)$	$O(b^{1+C/c})$
Space	$O(b^m)$	$O(bm)$	$O(b^{1+C/c})$
Optimal	Yes (identical cost)	No	Yes
When to use	• Memory is a nonissue • Shallow solutions • Minimal branching factors • Shortest path needed	• Memory is restricted • Deep solutions	• Care about cost over length of path

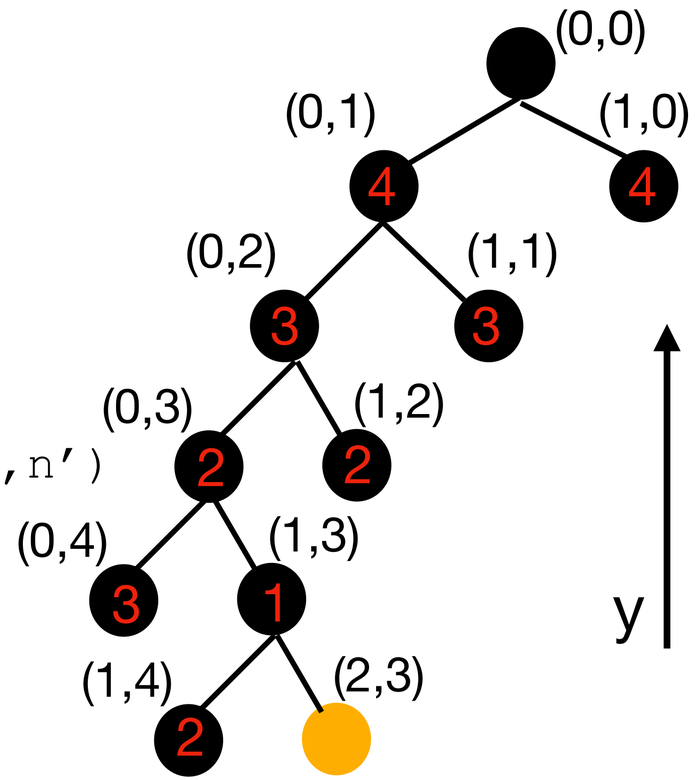
Informed Search

Greedy Search

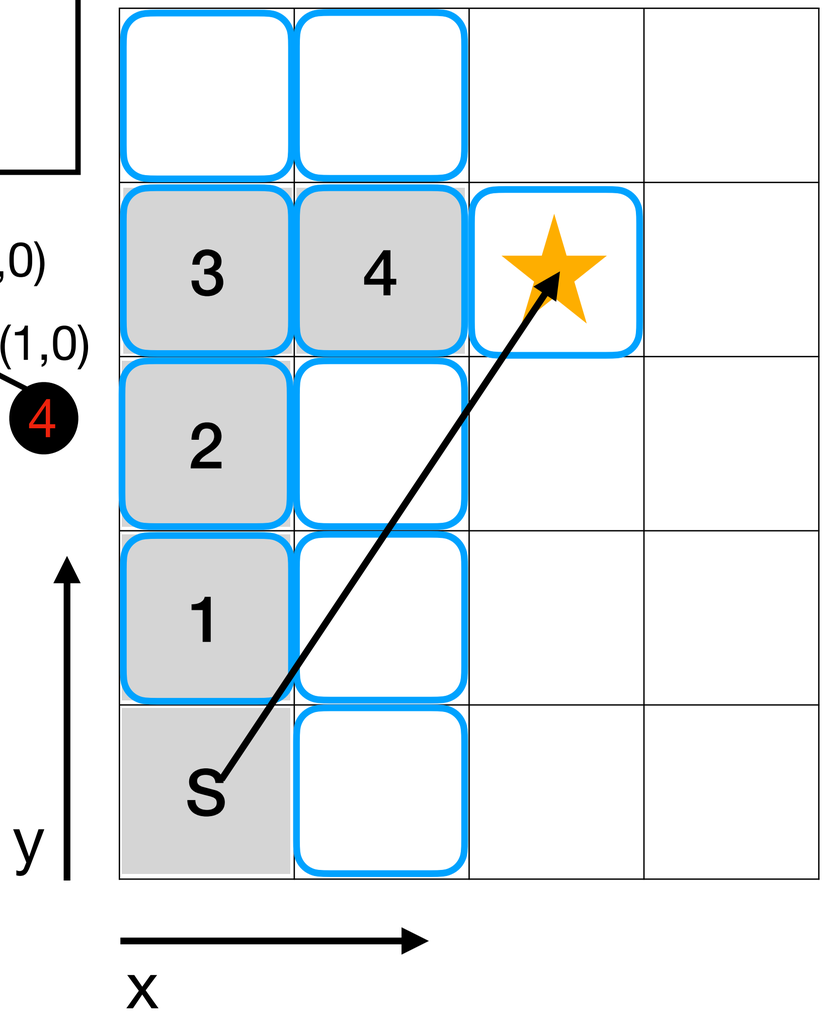
```
n = state(init)
frontier.append(n)
while(frontier not empty)
  n = pull state from frontier
  visited.append(n)
  if n = goal, return solution
  for all actions in n
    n' = a(n)
    if n' not visited
        priority = heuristic(goal,n')
        frontier.append(priority)
```

Define a heuristic to target:

- Manhattan Distance
- $abs(x_s-x_g)+abs(y_s-y_g)$



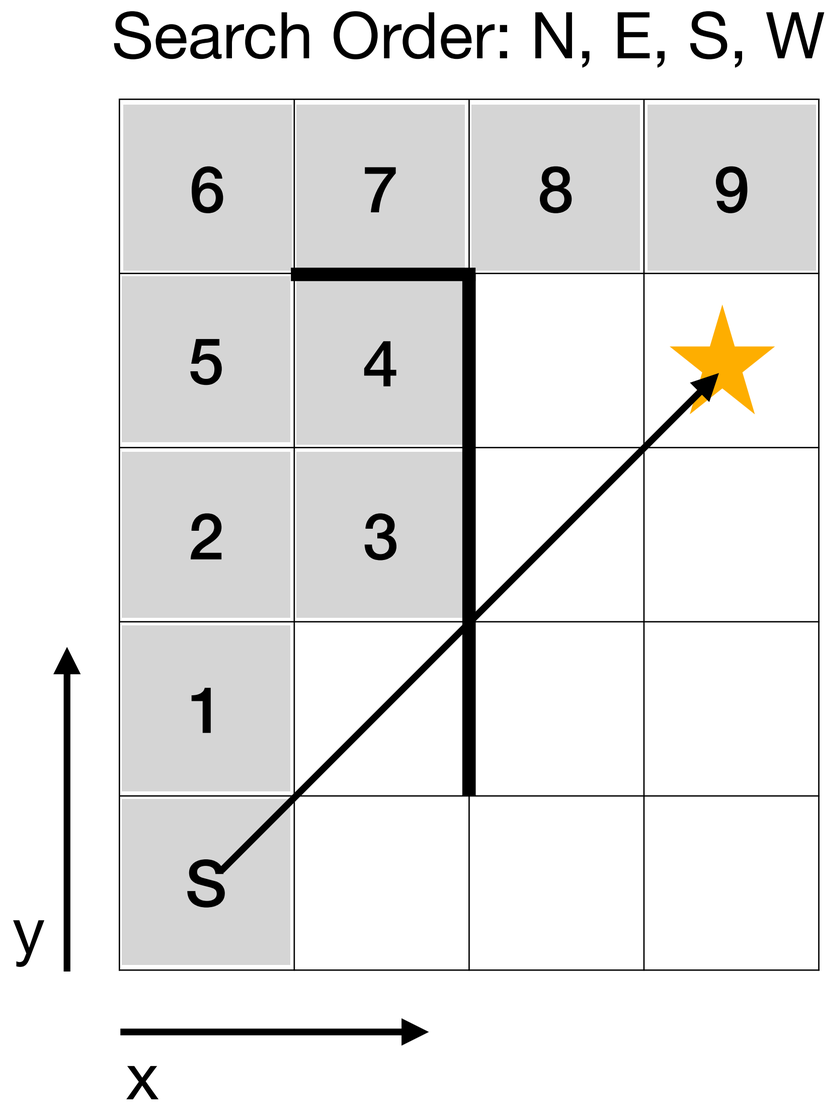
Search Order: N, E, S, W



Informed Search

Greedy Search

- Is it complete?
  - No
- What is the time complexity?
  - $O(b^m)$
- What is the space complexity?
  - $O(b^m)$
- Optimal?
  - No…



Search Algorithms, general

- Breadth First Search
  - Complete and optimal
  - …but searches everything

Can we do better? A*

- Lowest-Cost First Algorithm
  - Complete and optimal
  - … but it wastes time exploring in directions that aren't promising

Considers parent cost

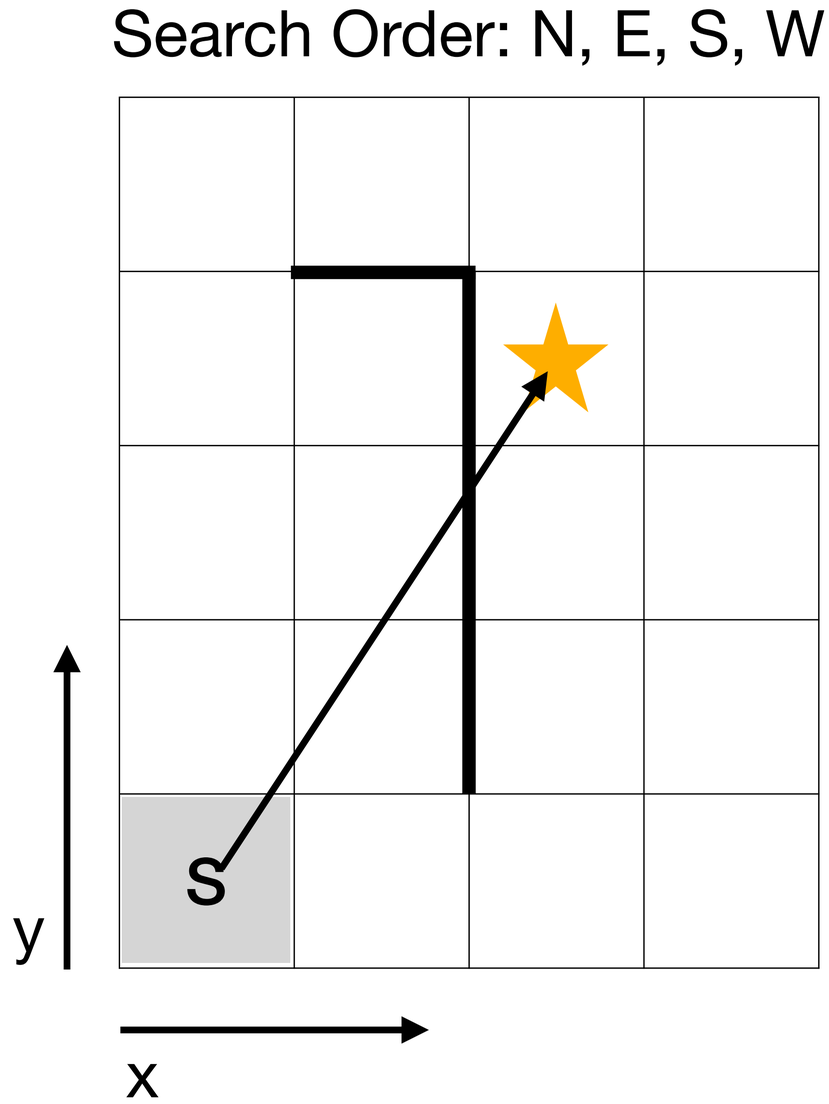
- Greedy Search
  - Complete (in most cases)
  - … only explores promising directions

Considers goal

Informed Search

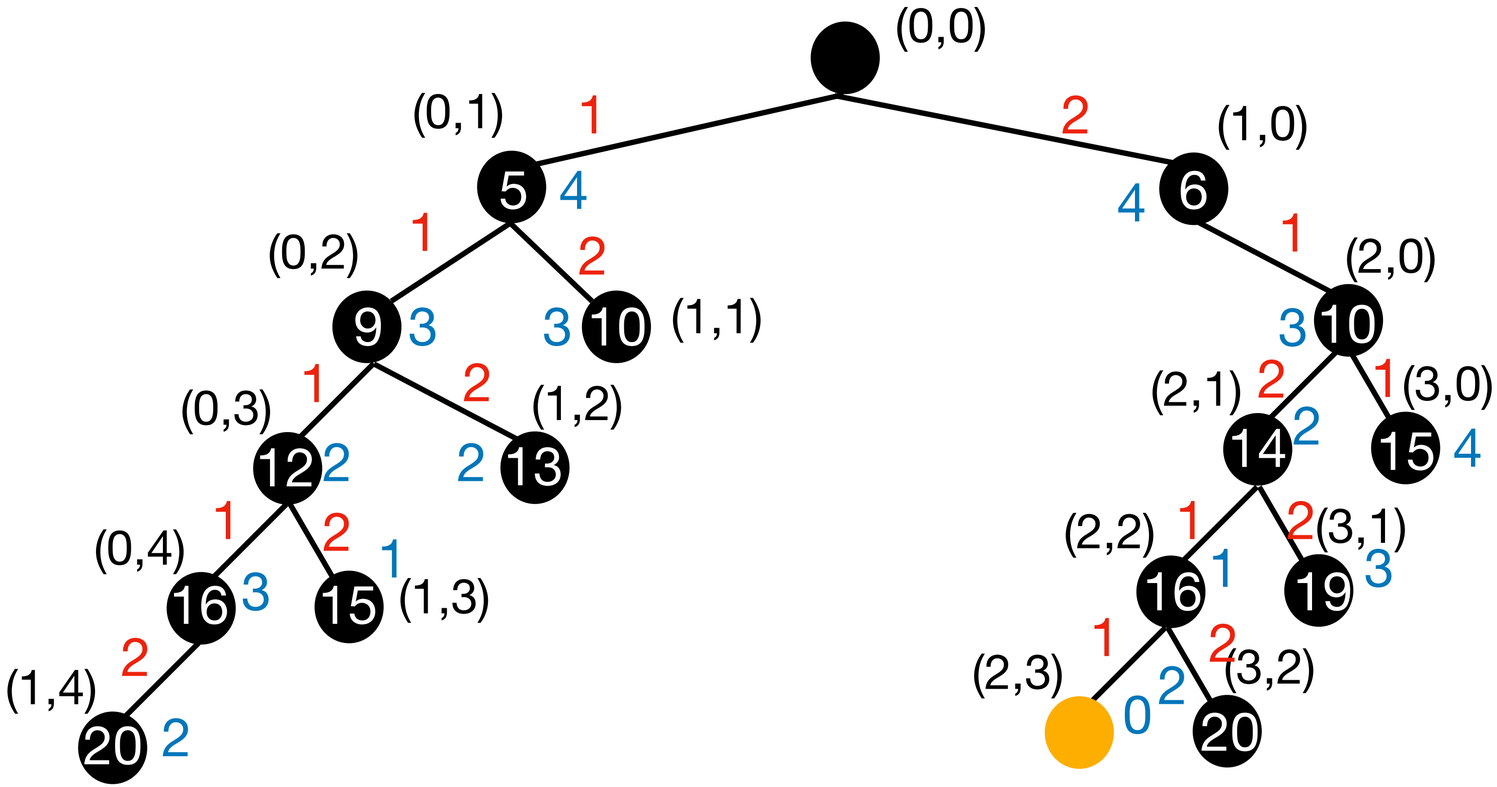
A* (A-star)

```
n = state(init)
frontier.append(n)
while(frontier not empty)
  n = pull state from frontier
  if n = goal, return solution
  for all actions in n
    n' = a(n)
    if (n' not visited)
        priority = heuristic(goal,n')+cost
        frontier.append(priority)
    if (visited and n'.cost < n_old.cost)
        visited.append(n')
```



Informed Search

A* (A-star)



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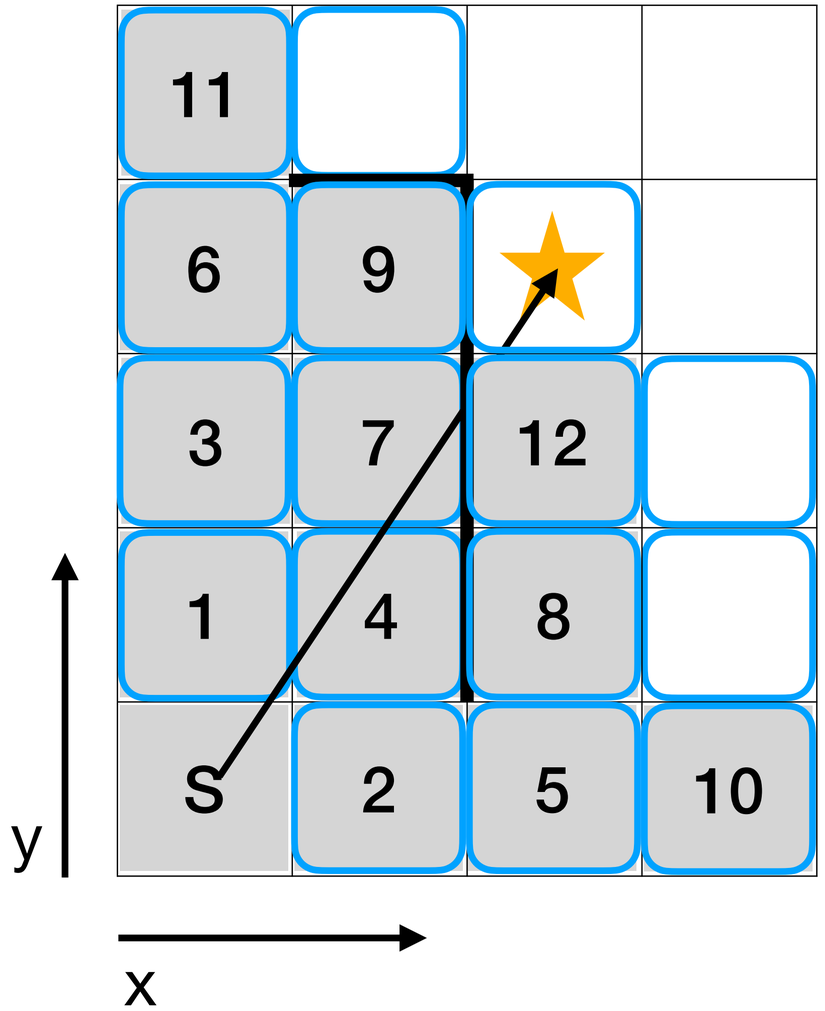


A* Search

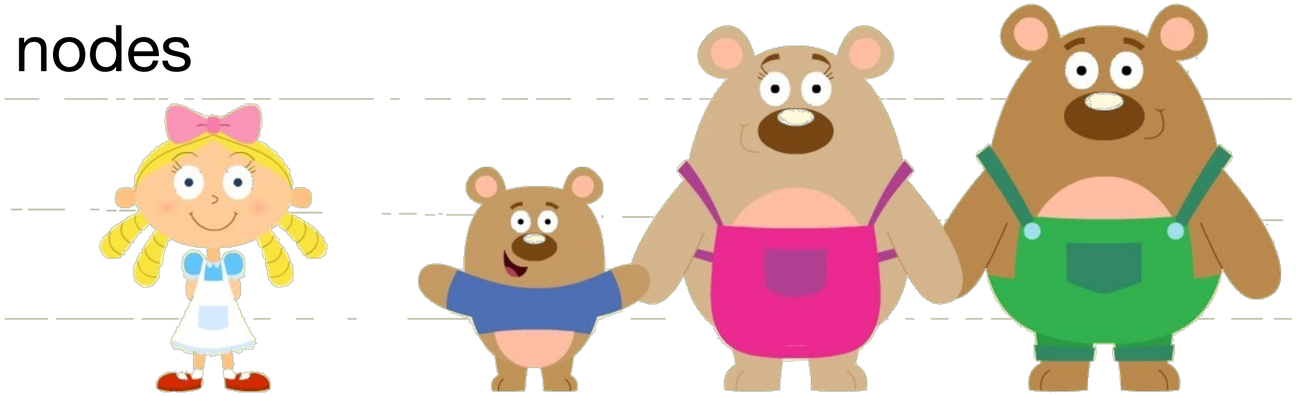
- What if the heuristic is too optimistic?
  - Estimated cost < true cost

Admissible heuristic

- What if the heuristic is too pessimistic?
  - Estimated cost > true cost
  - No longer guaranteed to be optimal

Inadmissible heuristic

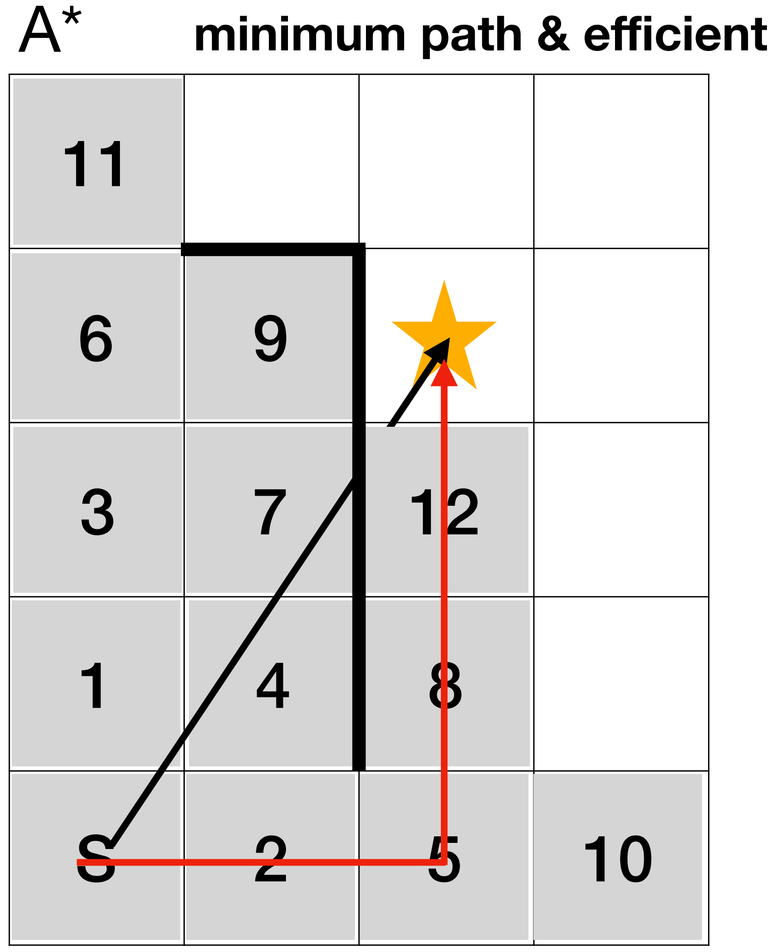
- What if the heuristic is just right?
  - Pre-compute the cost between all nodes
  - Feasible for you?



Informed Search

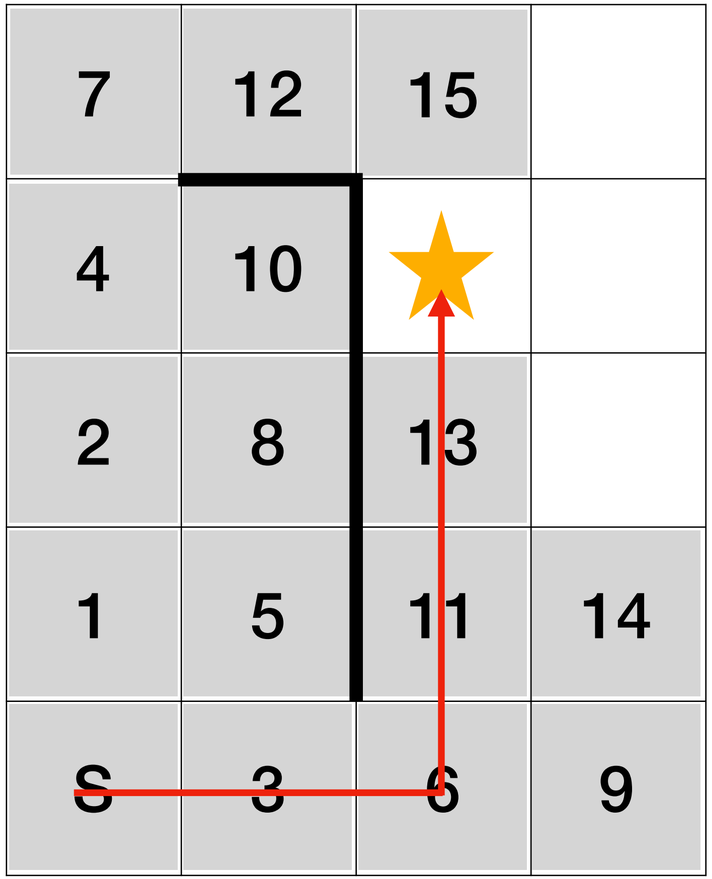
A* (A-star)

- Is it complete?
  - Yes!
- What is the time complexity?
  - $O(b^m)$
- What is the space complexity?
  - $O(b^m)$
- Optimal?
  - Yes, if the heuristic is admissible!

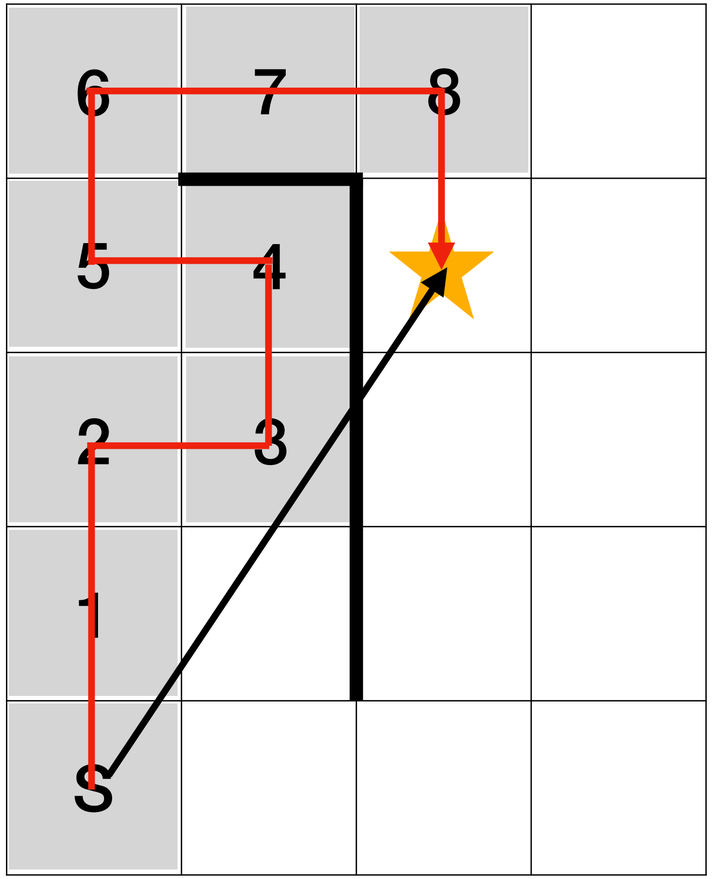


Summary

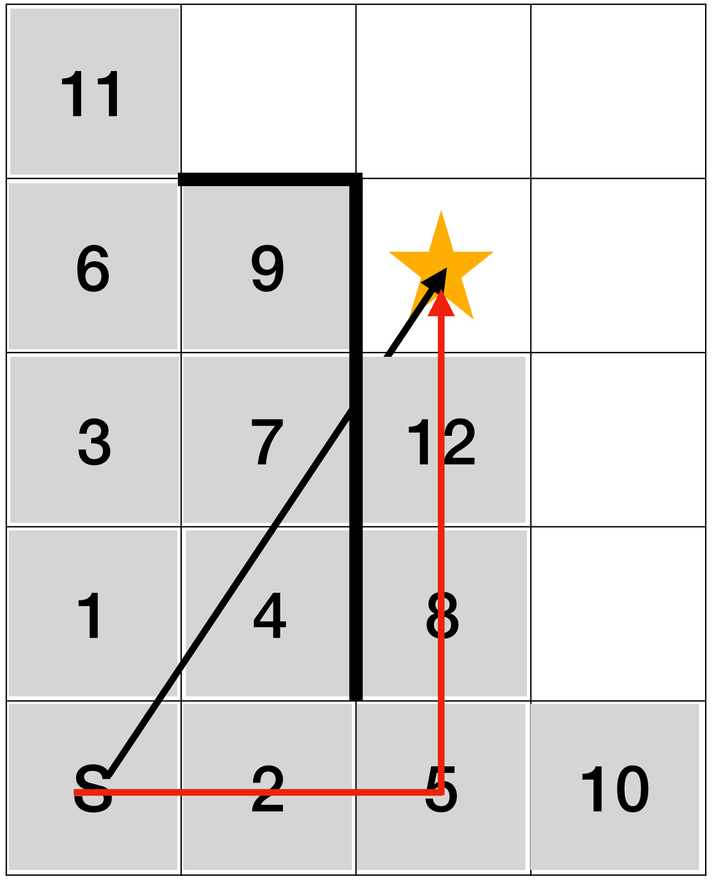
LCFS **minimum path**



Greedy



A* **minimum path & efficient**



Bayes Theorem

+

Robot-Environment Model

+

Markov Assumption

=

Bayes Filter

Bayes Theorem

+

Robot-Environment Model

+

Markov Assumption

=

Bayes Filter

Recap

ECE 3100 Intro to Probability and Inference

- Random variable
  - $X : \Omega \rightarrow \mathbb{R}$
- The probability that the random variable X has value x
  - $P(X = x)$ or $p(x)$
- Probabilities sum to 1
  - $\Sigma_x P(X = x) = 1$
- Probabilities are always greater than 0
  - $P(X = x) \geq 0$
- Joint distribution Y
  - $p(x, y) = P(X = x \text{ and } Y = y)$
- Conditional probability
  - $p(x \mid y) = \dfrac{p(x, y)}{p(y)}$
- Independence
  - $p(x, y) = p(x)p(y)$
  - $p(x \mid y) = p(x) = \dfrac{p(x, y)}{p(y)}$
- If X and Y are conditionally independent given Z=z, then
  - $p(x, y \mid z) = p(x \mid z)\ p(y \mid z)$
- Marginal probability
  - $p(x) = \Sigma_y p(x \mid y)p(y)$

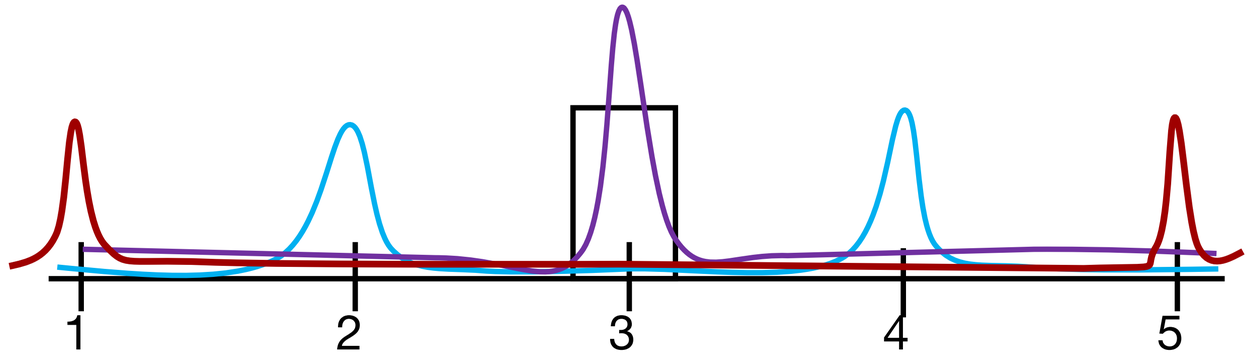
Bayesian Inference

$$\underbrace{P(x \mid z)}_{\text{Posterior}} = \frac{\overbrace{P(z \mid x)}^{\text{Likelihood}} \overbrace{P(x)}^{\text{Prior}}}{\underbrace{P(z)}_{\text{Marginal Likelihood (constant)}}}$$

x: robot pose

z: sensor data

- Lost robot example
  - Sensor measures distance to the door
  - $p(X_0 = 1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5) = 1/5$
  - $p(x \mid z)$ can be hard to compute
  - What is $p(z \mid x)$?
  - If $Z = 1$, where are you most likely to be?
  - If $Z = 0$, where are you most likely to be?
  - If $Z = 2$, where are you most likely to be?



Bayes Theorem

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Robot-Environment Model

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Markov Assumption

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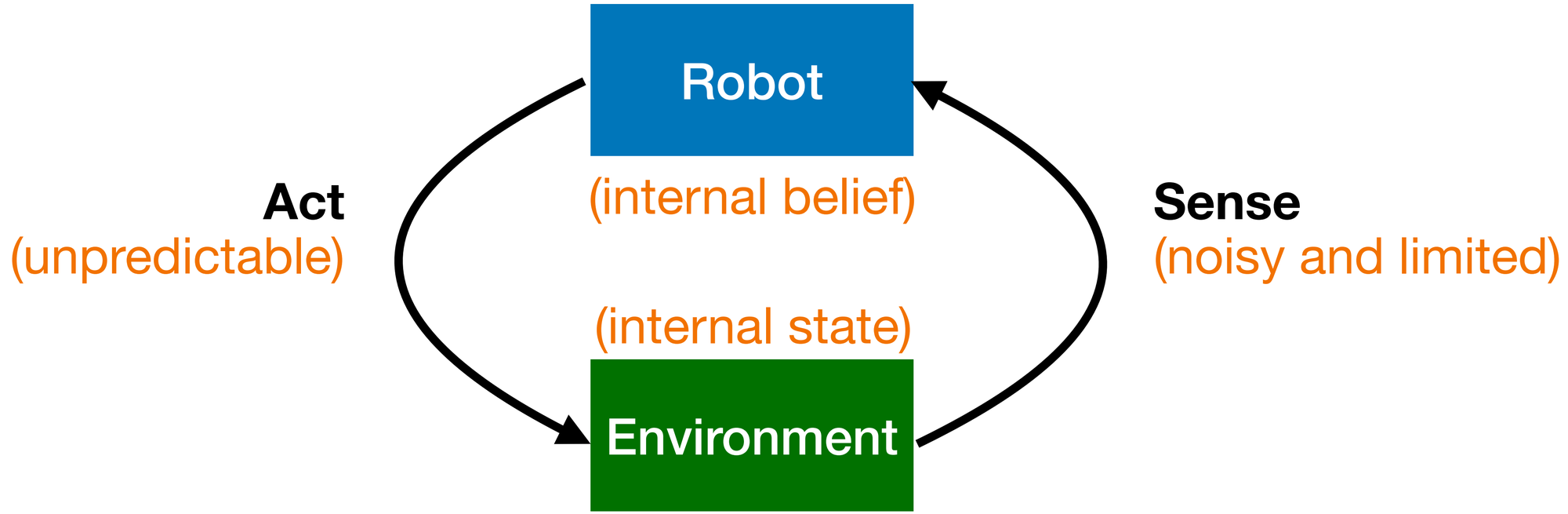
Bayes Filter

Robot-Environment Interaction



- Two fundamental types of interaction between a robot and its environment:
  - Sensor measurements/ observations
  - Control actions

Robot-Environment Model

- Helps us express a robot-environment interaction using probability
  - Typically modeled as a discrete time system
    - The **state** at time t will be denoted as x_t
    - A **sensor measurement** at time t will be denoted as z_t
    - A **control action** will be denoted as u_t
      - Induces a transition from x_{t-1} to x_t

Conventions as per Siegwart, R., Nourbakhsh, I.R. and Scaramuzza, D., 2011. Introduction to autonomous mobile robots. MIT press.

Robot-Environment Model

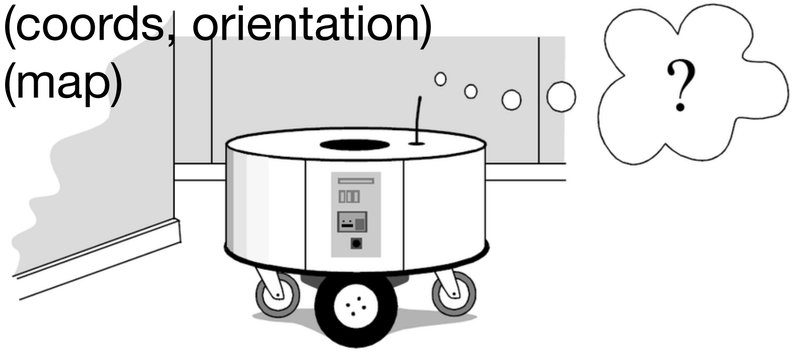
Assumptions (arbitrary)

- The robot executes a control action u_t first and then takes a measurement z_t
- There is one control action u per time step t
  - This includes the legal action “do-nothing”
- There is only one measurement z per time step t
- Shorthand notation: $x_{t1:t2} = x_{t1}, x_{t1+1}, x_{t1+2}, \ldots, x_{t2}$

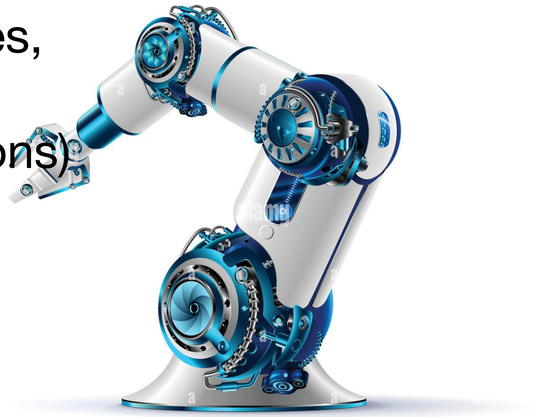
Robot State

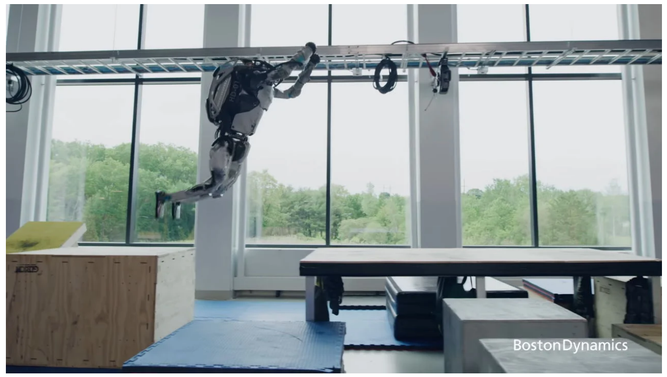
- Robot-specific:
  - Pose, velocity, sensor status, etc.
- Environment-specific:
  - Static variables: locations of walls, etc.
  - Dynamic variables: people, etc.
- … context specific

(coords, orientation)
(map)



(joint angles, velocities, accelerations)
(objects, texture)





Sensor Measurements/ Observations

- Tend to increase the robot's knowledge

balance

Control Actions

- … change the state of the world
- Carry information about the change of robot state in the time interval $(t-1:t]$
- Tends to induce loss of knowledge

Probabilistic Generative Laws

- The evolution of state and measurements is governed by probabilistic laws
  - State: How is x_t generated stochastically?
  - Measurements: How is z_t generated stochastically?
- State generation
  - x_t depends on $x_{0:t-1}$, $z_{1:t-1}$, and $u_{1:t}$
  - $p(x_t \mid x_{0:t-1},\ z_{1:t-1},\ u_{1:t})$ **…intractable!**

Bayes Theorem

+

Robot-Environment Model

+

Markov Assumption

=

Bayes Filter

Markov Assumption

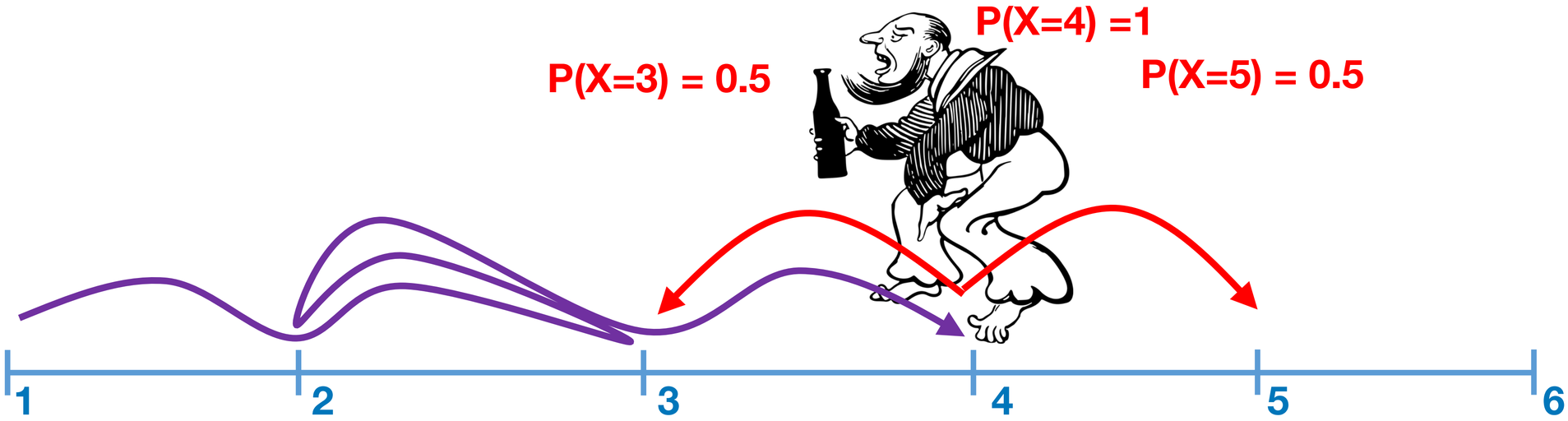
The Markov assumption postulates that past and future data are independent if one knows the current state

- A stochastic model/ process that obeys the Markov assumption is a Markov model
  - This does not mean that x_t is deterministic based on x_{t-1}
- **If** we can model our robot as a Markov process…
  - We can recursively estimate x_t using:
    - x_{t-1}, z_t, and u_t
    - **But not** $x_{0:t-1}$, $z_{1:t-1}$, and $u_{1:t}$!
    - Tractable!

Andrey Markov (1856–1922) was a Russian mathematician best known for his work on stochastic processes

Drunkards Walk

- Random walk on the number line
  - At each step, the position may change by +1 or -1 with equal probability
- The transition probabilities depend only on the current position, not on the manner in which the position was reached
- This is a Markov process!



Coin purse

- Contents
  - 5 quarters (25¢)
  - 5 dimes (10¢)
  - 5 nickels (5¢)
- Draw coins randomly, one at a time and place them on a table
- X_n is the total value of coins on the table after n draws
- The sequence $\{X_n : n \in \mathbb{N}\}$ is a stochastic process

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow$$

- Example:
  - First, I draw a nickel, what is $X_1 = 5¢$
  - Next I draw a dime, what is $X_2 = 15¢$

Coin purse

- Suppose…
  - In the first six draws, you pick all 5 nickels and 1 quarter, $X_6 = 50¢$
  - What can we say about X_7?
    - $P(X_7 \geq 0.55) = 1$
  - Can we do better?
  - Can you draw a nickel on the 7th draw?
    - $P(X_7 \geq 0.6) = 1$

- Contents
  - 5 quarters (25¢)
  - 5 dimes (10¢)
  - 5 nickels (5¢)

- Is this a Markov Model?
- Can we tweak the definition of X_n to make it one?

Coin purse

- Markov model
  - $X_n = \{\text{number of quarters, number of dimes, number of nickels}\}$ drawn
  - First, you pick a nickel
    - $X_1 = \{0,0,1\}$
    - $X_6 = \{1,0,5\}$
  - Now, what can you say about X_7?
    - $P(X_7 \geq 0.6) = 1$
  - State space has 6^3 possible states
  - … but independent of the number of draws

- Contents
  - 5 quarters (25¢)
  - 5 dimes (10¢)
  - 5 nickels (5¢)

Bayes Theorem

+

Robot-Environment Model

+

Markov Assumption

=

Bayes Filter

State Generative Model

- x_t is generated stochastically from the state x_{t-1}
- x_t depends on $x_{0:t-1}$, $z_{1:t-1}$, and $u_{1:t}$, and $p(x_t \mid x_{0:t-1},\ z_{1:t-1},\ u_{1:t})$ … intractable!
- If state x_t is modeled under the **Markov Assumption**, then
  - $p(x_t \mid x_{0:t-1},\ z_{1:t-1},\ u_{1:t}) = p(x_t \mid x_{t-1},\ u_t)$
  - Knowledge of only the previous state x_{t-1} and control u_t is sufficient to predict x_t

Measurement Generative Model

- Similarly, the process by which measurements are generated are of importance
  - $p(z_t \mid x_{0:t-1},\ z_{1:t-1},\ u_{1:t})$
- If state x_t conforms to the **Markov Assumption**, then
  - $p(z_t \mid x_{0:t-1},\ z_{1:t-1},\ u_{1:t}) = p(z_t \mid x_t)$
  - The state x_t is sufficient to predict the (potentially noisy) measurements
  - Knowledge of any other variable, such as past measurements, controls, or even past states, is irrelevant under the Markov Assumption

Bayes Theorem

+

Robot-Environment Model

+

Markov Assumption

=

Bayes Filter

Robot Belief

- Probabilistic robotics represents beliefs through *posterior conditional probability distributions*
  - Probability distributions over state variables conditioned on available data
  - The belief of a robot is the posterior distribution over the state of the environment, given all past sensor measurements and all past controls
    - Belief over a state variable x_t is denoted by $bel(x_t)$:
      - $bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t})$
    - The (prior) belief is the belief before incorporating the latest measurement z_t:
      - $\overline{bel}(x_t) = p(x_t \mid z_{1:t-1}, u_{1:t})$

Bayes Filter

- A recursive algorithm that calculates the belief distribution from measurements and control data

1. **Algorithm Bayes_Filter** $(bel(x_{t-1}),\ u_t,\ z_t)$:
2. **for** all x_t do
3. $\overline{bel}(x_t) = \Sigma_{x_{t-1}}\ p(x_t \mid u_t,\ x_{t-1})\ bel(x_{t-1})$
4. $bel(x_t) = \eta\ p(z_t \mid x_t)\ \overline{bel}(x_t)$
5. end **for**
6. **return** $bel(x_t)$

Bayes Filter

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Transition probability/ action model

(Prediction step)

Bayes Filter

- A recursive algorithm that calculates the belief distribution from measurements and control data

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3. $\overline{bel}(x_t) = \Sigma_{x_{t-1}}\ p(x_t \mid u_t,\ x_{t-1})\ bel(x_{t-1})$ **(Prediction step)**
4. $bel(x_t) = \eta\ p(z_t \mid x_t)\ \overline{bel}(x_t)$ **(Update/measurement step)**
5. end **for**
6. **return** $bel(x_t)$

Transition probability/ action model

Measurement probability/ sensor model

Kalman Filter Implementation

Kalman Filter $(\mu(t-1),\ \Sigma(t-1),\ u(t),\ z(t))$

prediction

1. $\mu_p(t) = A\mu(t-1) + Bu(t)$
2. $\Sigma_p(t) = A\Sigma(t-1)A^T + \Sigma_u$

update

3. $K_{KF} = \Sigma_p(t)C^T(C\Sigma_p(t)C^T + \Sigma_z)^{-1}$
4. $\mu(t) = \mu_p(t) + K_{KF}(z(t) - C\mu_p(t))$
5. $\Sigma(t) = (I - K_{KF}C)\Sigma_p(t)$
6. Return $\mu(t)$ and $\Sigma(t)$

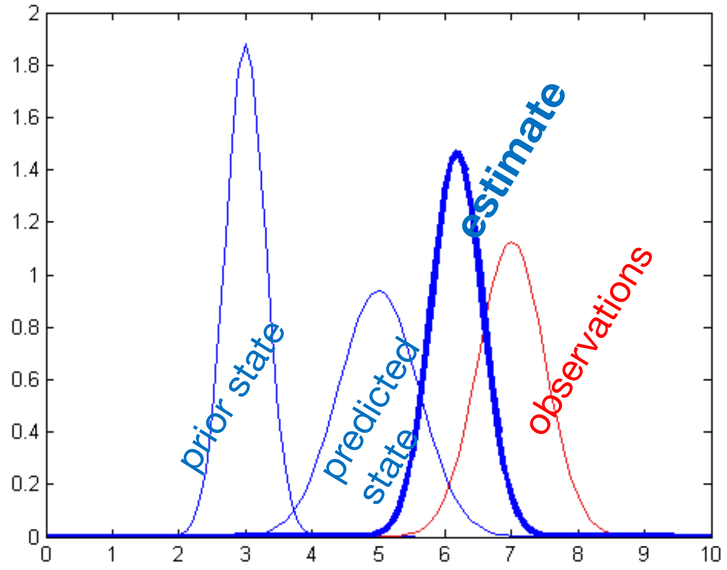
State estimate: $\mu(t)$

State uncertainty: $\Sigma(t)$

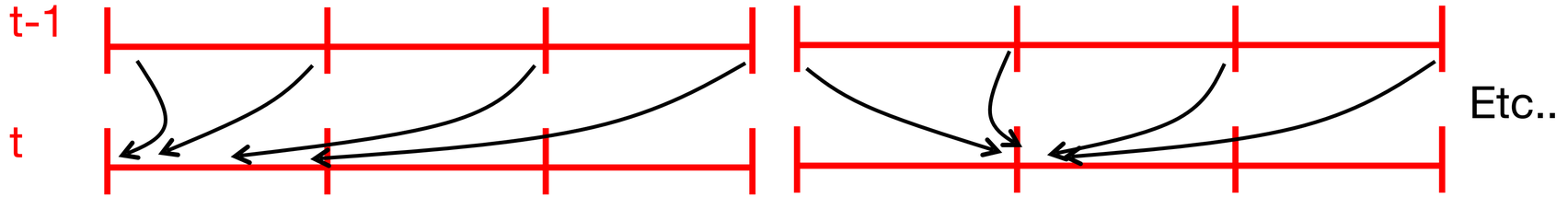
Process noise: Σ_u

Kalman filter gain: K_{KF}

Measurement noise: Σ_z



Bayes Filter



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Transition probability/ action model

Measurement probability/ sensor model

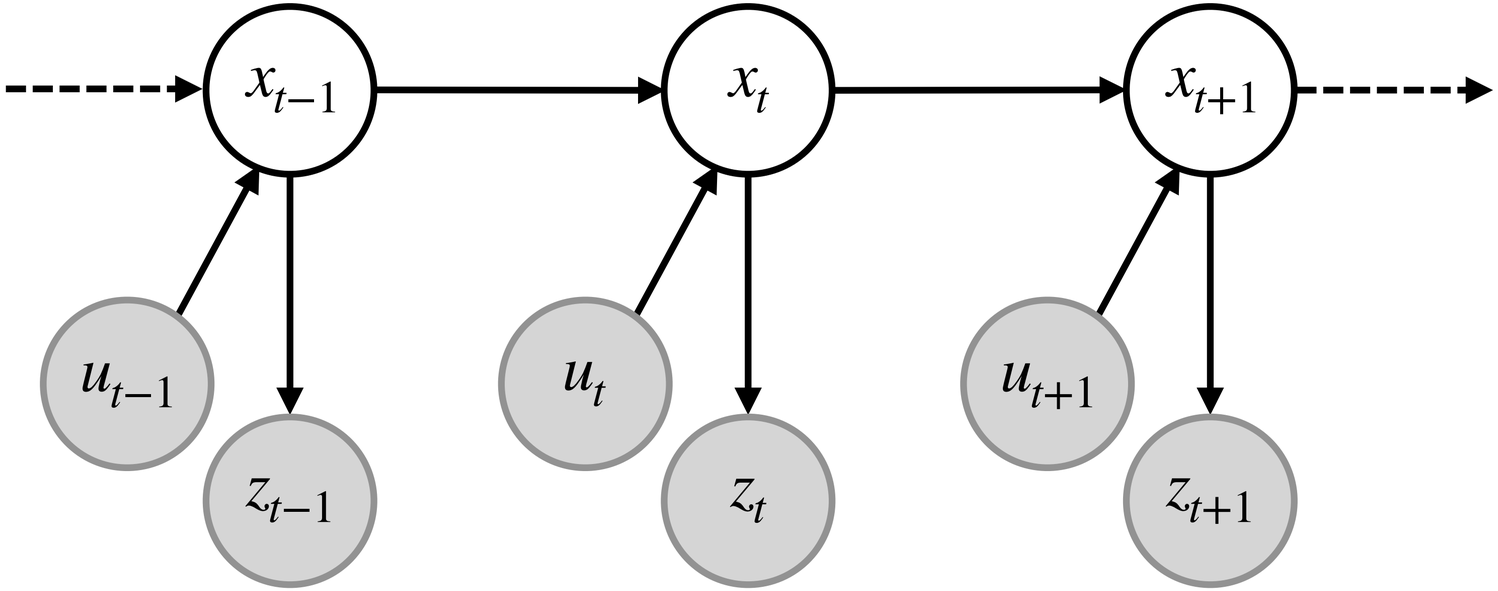
Bayes Filter

Dynamic stochastic model

- $p(x_t \mid x_{t-1}, u_t)$ is the **state transition probability**
  - How the robot state x_t evolves over time as a function of the control u_t
- $p(z_t \mid x_t)$ is the **measurement probability**
  - How measurements are generated from the robot state x_t
  - Informally, you can think of measurements as noisy projections of x_t
- Remember that these prediction are *stochastic and not deterministic*

Bayes Filter

Dynamic stochastic model



Dynamic Bayes Network that characterizes the evolution of controls, states, and measurements.

- $p(x_t \mid x_{t-1}, u_t)$ and $p(z_t \mid x_t)$ together describe the **dynamic stochastic system** of the robot and its environment

- This generative model is also known as a Hidden Markov Model (HMM) or Dynamic Bayes Network (DBN)

Bayes Filter

Initial conditions

- To compute the posterior belief recursively, the algorithm requires an initial belief $bel(x_0)$ at time $t = 0$

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5. end **for**
6. **return** $bel(x_t)$

Bayes Filter

Initial conditions

- To compute the posterior belief recursively, the algorithm requires an initial belief $bel(x_0)$ at time $t = 0$
- If we know the initial state with absolute certainty, we can initialize a point mass distribution that centers all probability mass on the correct value of x_0 and assign zero everywhere else
- If we are entirely ignorant of the initial state, we can initialize it with a uniform probability distribution over all the possible states

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