Regla de Simpson 3/8:

Aquí, el integrando se cambia por un polinomio interpolador. En esta regla en específico, se toban subintervalos con 4 puntos (de modo que hayan 3 “áreas” de integración en cada subintervalo).

$$f(x) \approx p(x) = \mathcal{L}_0(x)f(x_i) + \mathcal{L}_1(x)f(x_{i+1}) + \mathcal{L}_2(x)f(x_{i+2}) + \mathcal{L}_3(x)f(x_{i+3})$$

De modo que

$$\int_{x_i}^{x_{i+3}} f(x)dx = \int_{x_i}^{x_{i+3}} p(x)dx$$

$$= f(x_i)\int_{x_i}^{x_{i+3}} \mathcal{L}_0(x)dx + f(x_{i+1})\int_{x_i}^{x_{i+3}} \mathcal{L}_1(x)dx + f(x_{i+2})\int_{x_i}^{x_{i+3}} \mathcal{L}_2(x)dx + f(x_{i+3})\int_{x_i}^{x_{i+3}} \mathcal{L}_3(x)dx$$

Donde:

$\mathcal{L}_0 = \dfrac{(x-x_{i+1})(x-x_{i+2})(x-x_{i+3})}{(x_i-x_{i+1})(x_i-x_{i+2})(x_i-x_{i+3})}$	$\mathcal{L}_1 = \dfrac{(x-x_i)(x-x_{i+2})(x-x_{i+3})}{(x_{i+1}-x_i)(x_{i+1}-x_{i+2})(x_{i+1}-x_{i+3})}$
$\mathcal{L}_2 = \dfrac{(x-x_{i+1})(x-x_i)(x-x_{i+3})}{(x_{i+2}-x_{i+1})(x_{i+2}-x_i)(x_{i+2}-x_{i+3})}$	$\mathcal{L}_3 = \dfrac{(x-x_{i+1})(x-x_{i+2})(x-x_i)}{(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})(x_{i+3}-x_i)}$

Teniendo en cuenta que x se puede escribir como $x = x_i + th$

Haciendo ese cambio de variable y simplificando se tiene:

$$\mathcal{L}_0 = \frac{(x-x_{i+1})(x-x_{i+2})(x-x_{i+3})}{(-h)(-2h)(-3h)} = \frac{h(t-1)h(t-2)h(t-3)}{-6h^3}$$

$$\mathcal{L}_1 = \frac{(x-x_i)(x-x_{i+2})(x-x_{i+3})}{(h)(-h)(-2h)} = \frac{th * h(t-2)h(t-3)}{2h^3}$$

$$\mathcal{L}_2 = \frac{(x-x_{i+1})(x-x_i)(x-x_{i+3})}{(h)(2h)(-h)} = \frac{th * h(t-1)h(t-3)}{-2h^3}$$

$$\mathcal{L}_3 = \frac{(x-x_{i+1})(x-x_{i+2})(x-x_i)}{(2h)(h)(3h)} = \frac{th * h(t-1)h(t-2)}{6h^3}$$

Los límites de integración para la nueva variable t son

$$x_i = x_i + th \rightarrow t = 0$$

$$x_{i+3} = x_i + 3h = x_i + th \rightarrow t = 3$$

De 0 a 3

Y $dx = hdt$

De modo que la integral ahora se escribe como

$$= -\frac{f(x_i)h}{6}\int_0^3 (t-1)(t-2)(t-3)dt + \frac{f(x_{i+1})h}{2}\int_0^3 t(t-2)(t-3)dt$$
$$-\frac{f(x_{i+2})h}{2}\int_0^3 t(t-1)(t-3)dt + \frac{f(x_{i+3})h}{6}\int_0^3 t(t-1)(t-2)dt$$

La primera integral corresponde a:

$$-\frac{f(x_i)h}{6}\int_0^3 (t^3 - 6t^2 + 11t - 6)dt = \left[\frac{t^4}{4} - 2t^3 + \frac{11}{2}t^2 - 6t\right]_0^3 \left(-\frac{f(x_i)h}{6}\right)$$
$$= \left(-\frac{9}{4}\right)\left(-\frac{f(x_i)h}{6}\right)$$

La segunda integral corresponde a:

$$\frac{f(x_{i+1})h}{2}\int_0^3 (t^3 - 5t^2 + 6t)dt = \left[\frac{t^4}{4} - \frac{5}{3}t^3 + 3t^2\right]_0^3 \left(\frac{f(x_{i+1})h}{2}\right) = \frac{9}{4}\left(\frac{f(x_{i+1})h}{2}\right)$$

La tercera integral corresponde a:

$$-\frac{f(x_{i+2})h}{2}\int_0^3 (t^3 - 4t^2 + 3t)dt = \left[\frac{t^4}{4} - \frac{4}{3}t^3 + \frac{3}{2}t^2\right]_0^3 \left(-\frac{f(x_{i+2})h}{2}\right)$$
$$= \left(-\frac{9}{4}\right)\left(-\frac{f(x_{i+2})h}{2}\right)$$

La cuarta integral corresponde a:

$$\frac{f(x_{i+3})h}{6}\int_0^3 (t^3 - 3t^2 + 2t)dt = \left[\frac{t^4}{4} - t^3 + t^2\right]_0^3 \left(\frac{f(x_{i+3})h}{6}\right) = \frac{9}{4}\left(\frac{f(x_{i+3})h}{6}\right)$$

Comprobando entonces

$$\int_{x_i}^{x_{i+3}} f(x)dx = \frac{3}{8}hf(x_i) + \frac{9}{8}hf(x_{i+1}) + \frac{9}{8}hf(x_{i+2}) + \frac{3}{8}hf(x_{i+3})$$

$$\int_{x_i}^{x_{i+3}} f(x)dx = \frac{3}{8}h\big(f(x_i) + 3f(x_{i+1}) + 3f(x_{i+2}) + f(x_i)\big)$$