SESA 6071

Spacecraft Propulsion

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Definitions

$\boldsymbol{I_t}$	Total Impulse (Ns)	$\boldsymbol{I_{sp}}$	Specific Impulse (s)
$\boldsymbol{F}$	Rocket Thrust (N)	$\boldsymbol{g_0}$	Standard Gravitational Accel (m/s^2)
$\boldsymbol{\dot{m}}$	Propellent mass flow rate (kg/s)	$\boldsymbol{m_p}$	Expelled propellent mass (kg)
$\boldsymbol{c}$	Effective exhaust velocity (m/s)	$\boldsymbol{\eta_T}$	Power Conversion Efficiency
$\boldsymbol{P_{in}}$	Input Power (W)	$\boldsymbol{m}$	Spacecraft or launch vehicle mass (kg)
$\boldsymbol{\alpha}$	Specific power plant mass (kg/W)	$\boldsymbol{M_{pow}}$	Power plant mass (kg)
$\boldsymbol{v_e}$	Exhaust velocity (m/s)	$\boldsymbol{P_e}$	Exhaust pressure (Pa)
$\boldsymbol{P_a}$	Atmospheric pressure (Pa)	$\boldsymbol{A_e}$	Exhaust area (m^2)
$\boldsymbol{c^*}$	Characteristic velocity (m/s)	$\boldsymbol{P_c}$	Chamber pressure (Pa)
$\boldsymbol{A_t}$	Throat area (m^2)	**M**	Mass fraction
$\boldsymbol{M_0}$	Initial mass (kg)	$\boldsymbol{M_P}$	Propellent mass (kg)
$\boldsymbol{M_f}$	Fuel mass (kg)	$\boldsymbol{\Delta V}$	Change in velocity (m/s)
$\boldsymbol{\alpha}$	Angle of attack (° or rad)	$\boldsymbol{\delta}$	Gimbal angle (° or rad)
γ	Flight path angle (° or rad)	$\boldsymbol{\theta}$	Pitch angle (° or rad)
$\boldsymbol{D}$	Drag (N)	$\boldsymbol{c_p}$	Specific heat at a constant pressure (J/kgK)
$\boldsymbol{c_v}$	Specific heat at a constant volume (J/kgK)	$\boldsymbol{\theta}$	Pitch angle (° or rad)
$\boldsymbol{D}$	Drag (N)	$\boldsymbol{c_p}$	Specific heat at a constant pressure (J/kgK)
$\boldsymbol{C_F}$	Coefficient of thrust	$\boldsymbol{h}$	Enthalpy (J/ mol)
$\boldsymbol{k}$	Ratio or specific heats		

1. Lecture 1

START OF WEEK 1

1.1. What is Rocket Propulsion ?

Propulsion itself is the **act of changing the motion of a body**, typically by using newtons third law and it can be classified in various types of ways. A more colloquial way of defining rocket propulsion is as **mass drivers**, throwing out mass one way to yield an acceleration in the other.

1.2. Rocket Propulsion Family Tree

In **Figure 1** the rocket propulsion types are grouped by the energy source.

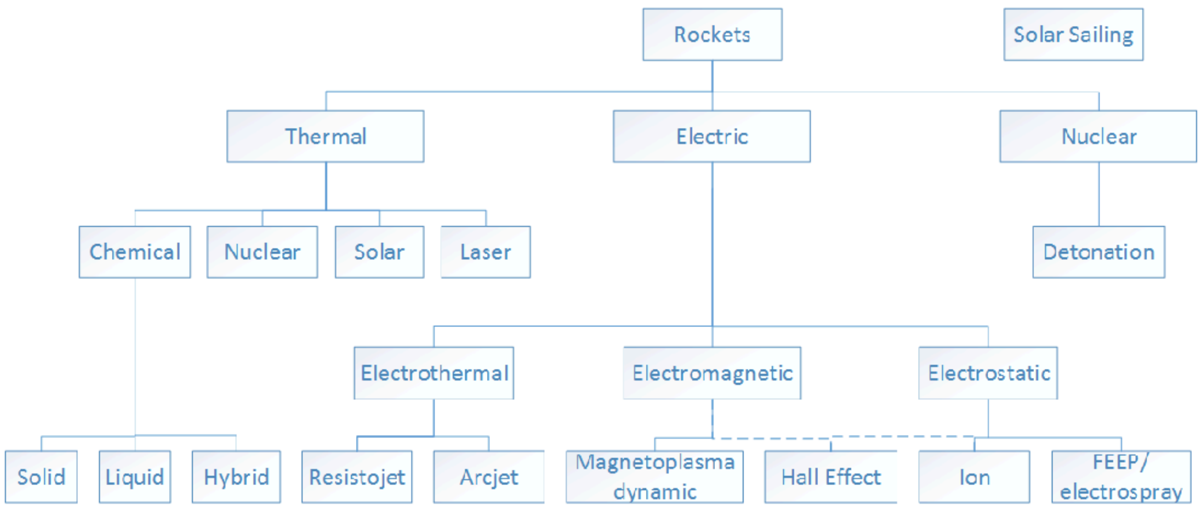

Figure 1: Flowchart of the rocket propulsion family tree

1.2.1. Chemical Rockets

These utilize either a chemical reaction or decomposition to generate energy. Gas is heated to between **700°C - 1300°C** and to speeds between **1.5 km/s - 4.5 km/s**. These require a **fuel and oxidizer** and come in the following types:

- **Solid:** Fuel and oxidizer mixed within into a solid grain which cannot stop burning once ignited. feature **high thrust with low performance**.
- **Liquid:** Burn a liquid fuel and oxidizer allowing for repeated firings and variable thrust. Feature **high performance and thrust with high complexity**.
- **Hybrid:** Have a liquid oxidizer but a solid fuel allowing for better performance than solid with lower complexity.

1.2.2. Electric Rockets

These use electrical energy to generate thrust without utilizing combustion. Typically have very high exhaust velocities ($\sim$ **60,000 m/s**) and therefore **very high performance** at the costs of **high complexities and very low thrust**. The four distinct groups are:

- **Electrothermal:** Uses electrical energy to heat a propellent (Resistojet). Are **simple to build** at the cost of **low thrust**.
- **Electrostatic:** Uses electrical energy to accelerate ionized fuel across an electric fields. Feature **good performance** at the cost of **being expensive and low thrust**.
- **Electromagnetic:** Accelerates an ionized fuel using a magnetic field. Fall issue to **low efficiency unless power input is high**.
- **Hall Effect Thruster:** Uses a mixture of both electrostatic and electromagnetic propulsion methods to accelerate propellent. These are the most **commonly used**.

1.2.3. Nuclear Rockets

Broadly speaking there are two types of nuclear rockets, these are:

- **Nuclear Detonation:** Use the shockwave produced when nuclear bombs are detonated to produce thrust (Orion Drive). **High performance and thrust** but are **very dangerous and have limited testing**.
- **Nuclear Thermal:** Uses the heat energy produced during nuclear fission to heat a propellent (typically hydrogen) which is then exhausted. These have **high performance and thrust** but are **dangerous and have limited testing**.

1.2.4. Solar and Laser Rockets

These systems use large diameter telescopes to focus in a laser or solar radiation to heat up a propellent. These systems feature **high theoretical performance and moderate thrust** but are **very complex and lack any real testing**.

1.2.5. Solar Sails

These systems use no propellent at all and instead produce thrust through the momentum gained when a photon is incident on the sail. These systems feature **good performance with no fuel** but fall victim to **low thrust and engineering complexity**.

1.3. Rocket Propulsion Applications

Instead of grouping together rocket propulsion methods using the energy source, the rocket application can also be used, for example:

- **High Thrust/Maneuverability:** Typically have the cost of **low performance** and use **chemical or solid** propulsion methods.
- **High Performance:** Typically have the cost of **low thrust** and use **electrical** propulsion methods.
- **Balanced Thrust and Performance:** Typically the middle ground is **nuclear thermal**.

2. Lecture 2

2.1. Definitions and Fundamentals

To develop a empirical measure of performance we should first consider **Eq. 1**.

$$I_t = \int_0^t F \ \ dt \tag{1}$$

Where:
- $\boldsymbol{I_t}$: Total Impulse (Ns)
- $\boldsymbol{F}$: Thrust Force (N)
- $\boldsymbol{t}$: Burn Duration (s)

Note that for **Eq. 1**, if F is constant then the equation simplified to $I_t = Ft$. A more useful measure of performance for rocket engines is shown in **Eq. 2**.

$$I_{sp} = \frac{\int_0^t F \ \ dt}{g_0 \int_0^t \dot{m} \ \ dt} = \frac{I_t}{g_0 \int_0^t \dot{m} \ \ dt} \tag{2}$$

Where:
- $\boldsymbol{I_{sp}}$: Specific Impulse (s)
- $\boldsymbol{g_0}$: Standard Gravitational Accel (m/s^2) = 9.81 m/s^2
- $\boldsymbol{\dot{m}}$: Propellent mass flow rate (kg/s)

There is no concrete reason on why g_0 is present in this equation, however one common theory is that it allows $I_{sp}$to be in seconds instead of featuring a length unit which would eliminate any error in conversion from metric to imperial. If F and $\dot{m}$ are both constant over the t then **Eq. 2** simplifies to **Eq. 3**.

$$I_{sp} = \frac{I_t}{g_0 m_p} \tag{3}$$

Where:
- $\boldsymbol{m_p}$: Expelled propellent mass (kg) = $\dot{m}t$

Another useful parameter for defining engine performance is shown in **Eq. 4**.

$$c = \frac{F}{\dot{m}} \tag{4}$$

Where:
- $\boldsymbol{c}$: Effective exhaust velocity (m/s)

The exhaust velocity is called as such as the **velocity profile of the exhaust is not uniform**, this is most seen in chemical rockets due to the **no slip condition** but is slightly seen in electrical rockets too. Rearranging all of the previous equations together yields a definition for I_{sp} in terms of c.

$$I_{sp} = \frac{c}{g_0} \tag{5}$$

Typical I_{sp} values for the rocket engine types defined in the previous lecture are shown in **Table 1**.

Rocket Engine Type	$I_{sp}(s)$	Thrust (N)	Efficiency	Propellent
Chemical bi-propellent	200 - 450	$\leq 10MN$	0.8	Liquid or Solid Propellents
Chemical mono-propellent	150 - 250	0.03 - 100	0.9	N_2H_4
Thermal Nuclear Fission	500 - 860	$\leq 10MN$	0.5	H_2
Resistojet - electrothermal	150 - 350	0.01 - 10	0.4	N_2H_4 , NH_3 , H_2
Ion Thruster - electrostatic	1500-8000	$10^{-5} - 0.5$	0.65	Xe
Hall Effect Thruster	1500-2000	$10^{-5} - 2$	0.55	Xe

Table 1: Typical values of I_{sp}

2.2. Maximum Chemical Performance

A typical chemical reaction used in chemical rockets is combustion shown in **Eq. 6**.

$$H_2 + 1/2\ O_2 \longrightarrow H_2O + \text{Energy} \tag{6}$$

Combustion as shown in **Eq. 6** nis an exothermic reaction as the energy of the reactants is more than the energy of the products, allowing for an excess of energy after the reaction. To estimate an effective upper limit to the energy released during combustion, the bond energies shown in **Table 2** can be used.

Chemical	Bond Energy (kJ/mol)
H_2	436
O_2	498
H_2O	428
	498.7

Table 2: Respective bond energies of reactants and products in combustion.

Note that there are two bond energies in **Table 2** due to the OH and the OH - H bonds. The maximum energy can be calculated and are shown in **Figure 2**.

Energy Per Kilogram Released During Combustion

$$M_{Water} := 18.01528 \frac{gm}{mol} \qquad BE_{H2} := 436 \frac{kJ}{mol} \qquad BE_{O2} := 498 \frac{kJ}{mol}$$

$$BE_{OH} := 428 \frac{kJ}{mol} \qquad BE_{OHH} := 498.7 \frac{kJ}{mol}$$

$$E_{kg} := \frac{\left(BE_{OH} + BE_{OHH}\right) - \left(BE_{H2} + \frac{BE_{O2}}{2}\right)}{M_{Water}} = 13416388.75 \frac{J}{kg}$$

Maximum Chemical Performance

$$\eta := 0.9 \qquad g_0 := 9.81 \frac{m}{s^2}$$

$$c := \sqrt[2]{2\ \eta \cdot E_{kg}} = 4914.21 \frac{m}{s} \qquad I_{sp} := \frac{c}{g_0} = 500.94\ s$$

Figure 2: Calculations for maximum chemical rocket engine performance

Note that in this calculation, the bond energy of oxygen is halved as per **Eq. 6** and the equation for effective exhaust velocity comes from the kinetic energy equation and noting that $E_{kg} = Energy/mass$.

2.3. Comparative Electric Performance

To compare the efficiency of chemical propulsion to electric propulsion consider an electrostatic propulsion system shown in **Figure 3**.

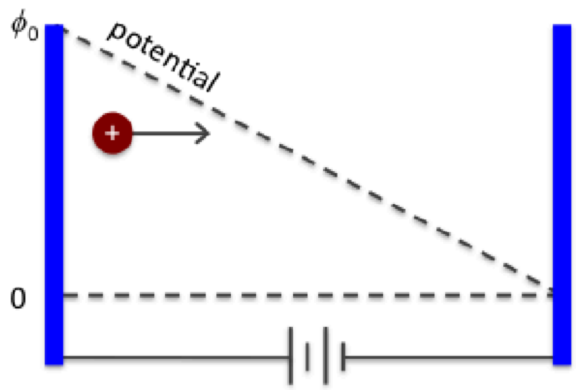

Figure 3: Basic principle of an electrostatic propulsion system.

A charged ion (assumed for these calculations to be a water ion) enters an electric field which causes it be accelerated to the more negative (lower potential plate). By setting the electric potential energy gained by the ion equal to the kinetic energy ($\eta E_p = E_k$) then the I_{sp} can be calculated, shown in **Figure 4**.

Voltage Required for an Isp of 500s Using Ionized Water

$$M_{Water} := 18.01528 \ \frac{gm}{mol} \qquad \eta := 0.9 \qquad g_0 := 9.81 \ \frac{m}{s^2} \qquad N_a := 6.023 \cdot 10^{23} \ mol^{-1}$$

$$I_{sp} := 500 \ s \qquad q := 1.6 \cdot 10^{-19} \ C$$

$$V := \frac{\frac{1}{2} \frac{M_{Water}}{N_a} \left(I_{sp} \cdot g_0\right)^2}{\eta \cdot q} = 2.5 \ V$$

Figure 4: Comparative electrical propulsion system voltage calculations.

As shown in **Figure 4** the voltage required to match the performance of a chemical system is very low and easily achievable, in reality electrostatic systems can achieve efficiencies in excess of 10,000s.

2.4. Nuclear Performance

To estimate the performance of a thermal nuclear rocket engine, Uranium-235 fission is considered, where the energy released in one fission event is immediately transferred to a water molecule, this calculation is shown in **Figure 5**.

Energy Transfered to One Water Molecule During One Nuclear Fission Event

$$E_{U235} := 180 \ MeV = \left(2.88 \cdot 10^{-11}\right) \ J \qquad M_{Water} := 18.01528 \ \frac{gm}{mol} \qquad N_a := 6.023 \cdot 10^{23} \ mol^{-1}$$

$$m_{Water} := \frac{M_{Water}}{N_a} = 0 \ kg$$ <- is 2.99x10^-26 but is too small for mathcad to show

$$E_{kg} := \frac{E_{U235}}{m_{Water}} = \left(9.63 \cdot 10^{14}\right) \ \frac{J}{kg}$$

Performance of a Nuclear Thermal System

$$\eta := 0.9 \qquad g_0 := 9.81 \ \frac{m}{s^2}$$

$$c := \sqrt[2]{2 \ \eta \cdot E_{kg}} = 41631151.15 \ \frac{m}{s} \qquad I_{sp} := \frac{c}{g_0} = \left(4.24 \cdot 10^6\right) \ s$$

Figure 5: Maximum nuclear thermal propulsive system performance.

Note that this I_{sp} is a theoretical upper limit and in reality the true performance is much lower and is limited by material limits due to heat.

2.5. Definitions and Fundamentals Cont.

For propulsion systems, efficiency can be defined in terms of the fraction of source power that is converted to jet power, this efficiency is shown in **Eq. 7**.

$$\eta_T = \frac{\dot{m}c^2}{2P_{in}} \tag{7.1}$$

$$P_{in} = \frac{\dot{m}c^2}{2\eta_T} = \frac{Fc}{2\eta_T} \tag{7.2}$$

$$\frac{P_{in}}{m} = \frac{F}{m}\frac{c}{2\eta_T} = a\frac{c}{2\eta_T} \tag{7.3}$$

Where:

- $\boldsymbol{\eta_T}$: Power conversion efficiency
- $\boldsymbol{a}$: Acceleration (m/s^2)
- $\boldsymbol{P_{in}}$: Input or Source power (W)
- $\boldsymbol{m}$: Spacecraft mass (kg)

Note that for electrical systems P_{in}, the power must come from a source e.g., solar panel array. **Eq. 7** Also shows that **for a fixed specific power:** $(\frac{P_{in}}{m})$ **a high effective exhaust speed** (c) **means a low acceleration.** It is also useful to define a specific power plant mass as shown in **Eq. 8**.

$$\alpha = \frac{M_{pow}}{P_{in}} \tag{8}$$

Where:

- $\boldsymbol{\alpha}$: Specific power plant mass (kg/W)
- $\boldsymbol{M_{pow}}$: Power plant mass (kg)

By manipulating equations **Eq. 8** and **Eq. 7**, as well as assuming that $\eta_T \approx 1$ and $M_{pow} \approx 0.1m$ then the acceleration can be written as **Eq. 9**.

$$a = \frac{0.2}{\alpha c} \quad \begin{cases} M_{pow} \approx 0.1m \\ \eta_T \approx 1 \end{cases} \tag{9}$$

Eq. 9 shows that a and c are inversely proportional from one another, meaning a high acceleration will typically mean a low effective exhaust velocity and vice versa. A showing how performance varies with acceleration is shown in **Figure 6**.

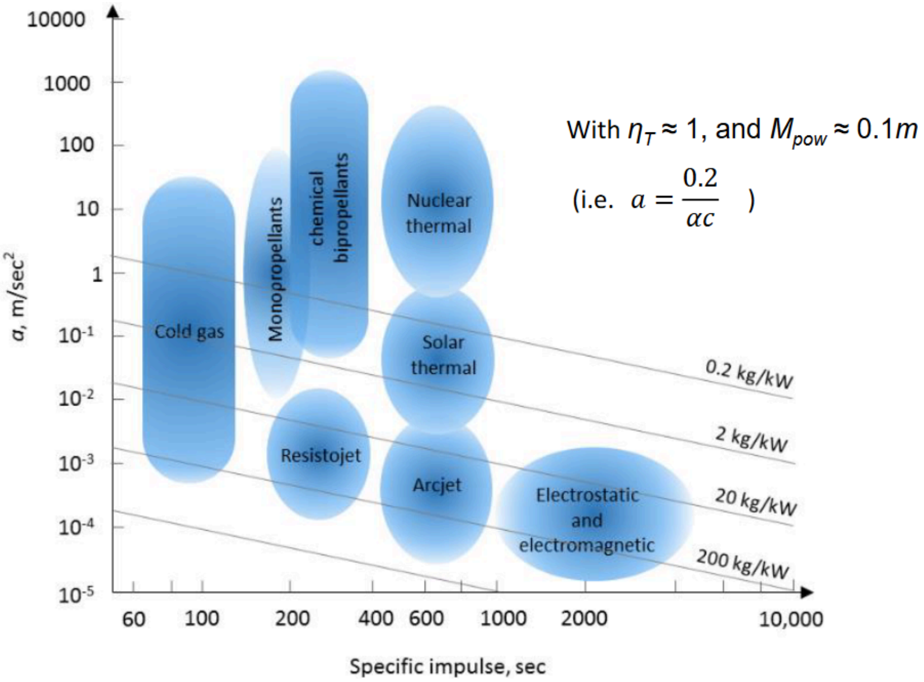

Figure 6: Variation of spacecraft acceleration against performance.

Note that for electrical propulsion systems shown in **Figure 6** a higher I_{sp} means a lower acceleration as $I_{sp} \propto c \propto \frac{1}{a}$. Different power sources have different values of α, for example:

- Nuclear Reactors $\Rightarrow 2kg/kW$
- Solar Panels $\Rightarrow 20kg/kW$
- RTGs $\Rightarrow 200kg/kW$

2.6. Thrust Fundamentals

By apply Newton's second law to a rocket nozzle, considering the difference the atmospheric and exhaust pressure as well as using the equations derived in the previous sections, **Eq. 10** can be derived.

$$F = \dot{m}v_e + (P_e - P_a)A_e \tag{10.1}$$

$$c = v_e + \frac{(P_e - P_a)A_e}{\dot{m}} \tag{10.2}$$

$$I_{sp} = \frac{1}{g_0}\left(v_e + \frac{(P_e - P_a)A_e}{\dot{m}}\right) \tag{10.3}$$

Where:

- $\boldsymbol{v_e}$: Exhaust velocity (m/s)
- $\boldsymbol{A_e}$: Exhaust Area (m^2)
- $\boldsymbol{P_e}$: Exhaust Pressure (Pa)
- $\boldsymbol{P_a}$: Atmospheric Pressure (Pa)

One key thing to note about **Eq. 10** is that the thrust is made up of two parts, the first part being the **momentum thrust** accounting for the majority of the thrust (90-70%) and the second part is the **pressure thrust** (10-30%).

Crucially, as $P_a(h)$ then the I_{sp} and c vary with the height, typically being lower at lower altitudes and increasing up an reaching their maximums in the thinner sections of the atmosphere.

Another impartial performance parameter for chemical rockets which does not depend on the altitude is shown in **Eq. 11**.

$$c^* = \frac{P_c A_t}{\dot{m}} \tag{11}$$

Where:
- $\boldsymbol{c^*}$: Characteristic velocity (m/s)
- $\boldsymbol{P_c}$: Chamber pressure (Pa)
- $\boldsymbol{A_t}$: Throat area (m^2)

Typical values of c^* are 1500 m/s for a solid rocket and 2500 for H_2/O_2 liquid bi-propelled rocket.

2.7. Tsiolkovsy Rocket Equation

One way to represent the quantity of propellent to the structure of the rocket is by using the **propellent mass fraction** shown in **Eq. 12**.

$$\mu = \frac{M_P}{M_0} \tag{12}$$

Where:
- $\boldsymbol{\mu}$: Mass fraction
- $\boldsymbol{M_P}$: Propellent mass (kg)
- $\boldsymbol{M_0}$: Structure Mass (kg)

For a well designed rocket $\mu \approx 0.8 - 0.85$. The famous rocket equation is derived by starting with Newtons second law and considering the momentum of the fuel leaving the engine and integrating that equation, this yields .

$$\Delta V = c \ln\left(\frac{M_0}{M_f}\right) = I_{sp} g_0 \ln\left(\frac{M_0}{M_f}\right) \tag{13}$$

Where:
- $\boldsymbol{\Delta V}$: Change in velocity (m/s)
- $\boldsymbol{M_f}$: Final mass (kg)

The ΔV and the M_0/M_f are plotted against one another in **Figure 7**. Note that for a single stage rocket $M_0/M_f \approx 20$ and the ΔV required to reach LEO is 9.5 km/s and so a single stage to rocket is on the boundary of being possible using a chemical bi-propellent rocket.

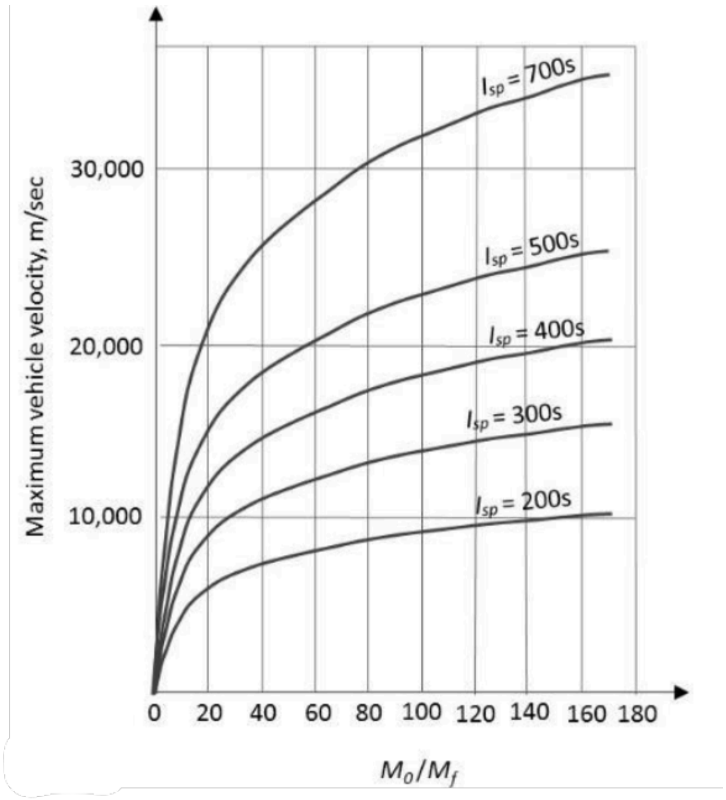

Figure 7: Plot of ΔV against M_0/M_f

3. Lecture 3

3.1. Rocket Staging

The typical ΔVs required for different manoeuvre are shown in **Table 3**.

Manoeuvre	Req ΔV (km/s)
Surface of Earth to LEO (inc drag and grav losses)	9.5
LEO to GEO (impulsive no plane change)	3.95
LEO to GEO (low thrust no plane change)	4.71
LEO to Lunar (impulsive)	3.9
LEO to Lunar (low thrust)	8
LEO to Mars (impulsive)	5.7
GEO station keeping	50 m/s /year
LEO station keeping	< 25 m/s /year

Table 3: Typical ΔV values for different manoeuvre

For a conventional chemical rocket, to reach LEO from the surface of the Earth, assuming an ideal mass ratio ($I_{sp} \approx 450s, \quad \Delta V \approx 9.5km/s, \quad M_o/M_f \approx 8.6$) then the mass fraction μ would have to be $\approx 90\%$, leaving 10% for the payload itself. This is mitigated through using **rocket staging**. Stages offer various benefits, the most prominent of which is the gain in ΔV when compared with one stage. The expression of the ΔV of a multistage rocket is shown in **Eq. 14**.

$$\Delta V_{\text{Total}} = \Delta V_{\text{Stage 1}} + \Delta V_{\text{Stage 2}} + ... + \Delta V_{\text{Stage } n} \tag{14.1}$$

$$\Delta V_{\text{Total}} = I_{sp \text{ Stage 1}} g_0 \left(\frac{M_{0 \text{ Stage 1}}}{M_{1 \text{ Stage 1}}} \right) \tag{14.2}$$

$$+ I_{sp \text{ Stage 2}} g_0 \left(\frac{M_{0 \text{ Stage 2}}}{M_{1 \text{ Stage 2}}} \right) \tag{14.3}$$

$$+... \tag{14.4}$$

$$+ I_{sp \text{ Stage } n} g_0 \left(\frac{M_{0 \text{ Stage } n}}{M_{1 \text{ Stage } n}} \right) \tag{14.5}$$

An image depicting the payload fraction against delta V is shown in **Figure 8**.

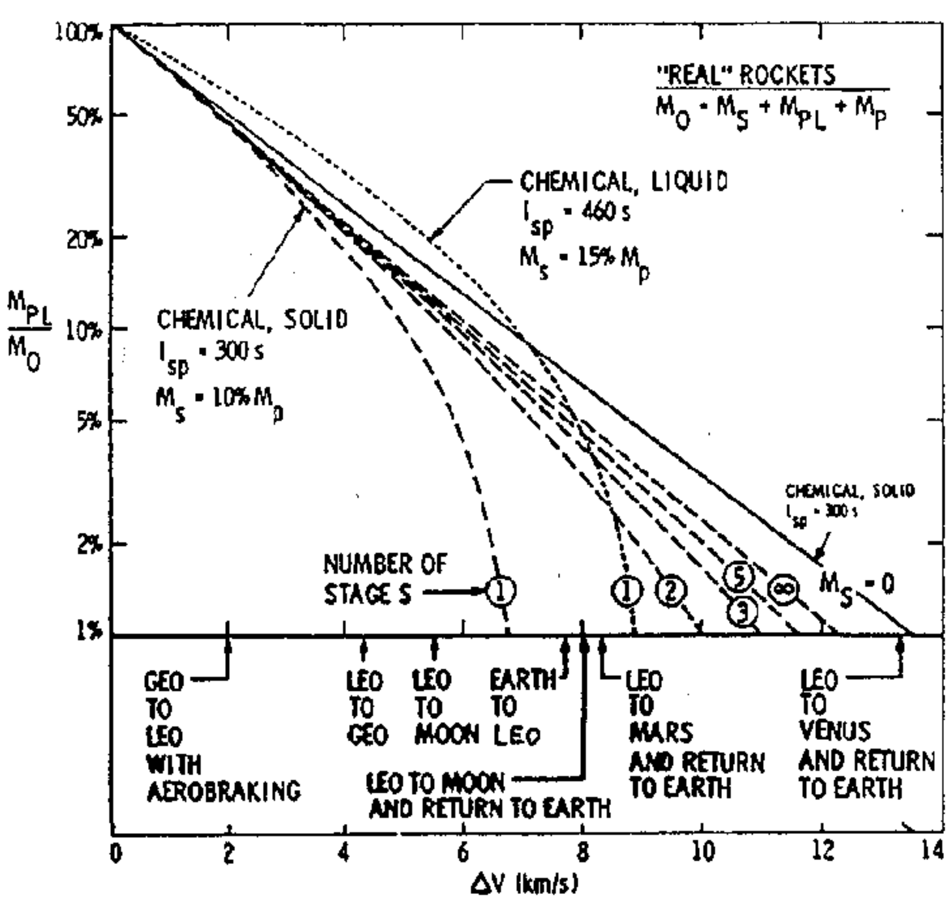

Figure 8: Plot depicting the effect of staging on the ΔV for a given payload fraction.

3.2. Launch Vehicle Dynamics

The key forced acting on a launch vehicle during launch are shown in **Figure 9**.

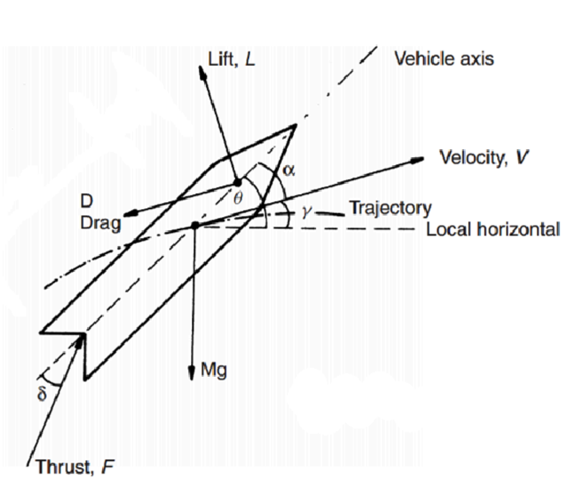

Figure 9: Plot illustrating the forces present on a lunch vehicle.

Taking the forces shown in **Figure 9**, a differential expression can be generated for the motion of the craft, using Newton's second law, this is shown in

$$M\left(\frac{dV}{dt}\right) = F\cos(\alpha - \delta) - Mg\sin(\gamma) - D \tag{15}$$

Where:

- $\boldsymbol{M}$: Total launch vehicle mass (kg)
- $\boldsymbol{F}$: Thrust (N)
- $\boldsymbol{\delta}$: Gimbal angle (° or rad)
- $\boldsymbol{\theta}$: Pitch angle (° or rad) $= \gamma + \alpha$
- $\boldsymbol{V}$: Spacecraft velocity (m/s)
- $\boldsymbol{\alpha}$: Angle of attack (° or rad)
- $\boldsymbol{\gamma}$: Flight path angle (° or rad)
- $\boldsymbol{D}$: Drag (N)

Note that within **Eq. 15**, many of the terms depend on the time as well as on one another. These equations can be rearranged and manipulated to yield **Eq. 16** (assuming $V_0 \approx 0, \alpha \approx 0, \delta \approx 0$).

$$\Delta V = \Delta V_{\text{ideal}} - \Delta V_g - \Delta V_D \tag{16.1}$$

$$\Delta V_{\text{ideal}} = \bar{c}\ln\left(\frac{M_0}{M_f}\right) \tag{16.2}$$

$$\Delta V_g = \int_0^{t_b} g\sin(\gamma)dt \tag{16.3}$$

$$\Delta V_D = \int_0^{t_b} \frac{D/M_0}{1 - \mu t/t_p}dt \tag{16.4}$$

Note that for **Eq. 16**, $\Delta V_g \approx 1.1km/s, \Delta V_D \approx 0.2km/s$. Additionally a boost of 0.5 km/s can be gained by launching at the equator. Note that $\bar{c}$ is an averaged effective exhaust velocity.

3.3. Converging Diverging Nozzle

START OF WEEK 2

All of the thermal rockets that were shown in **Figure 1** will most likely use a converging diverging nozzle (De-Laval nozzle) to accelerate the hot exhaust gas and increase the thrust of the engine. They effectively **convert the gases thermal energy to kinetic energy**. Note that when considering gaseous or liquid flow in this module, the following assumptions will be made:

- The fluid used are homogeneous.
- The species are gaseous.
- No heat transfer across the rocket walls (adiabatic assumption).
- No friction and all boundary layer effects effects negligible
- No shock waves or discontinuities in the nozzle
- Gas composition does not change in the noz zle (frozen flow) (not necessarily true but will assume for simplification that all reactions occur in the combustion chamber)

A plot of how the temperature, pressure, velocity and Mach number change over the nozzle is shown in **Figure 10**.

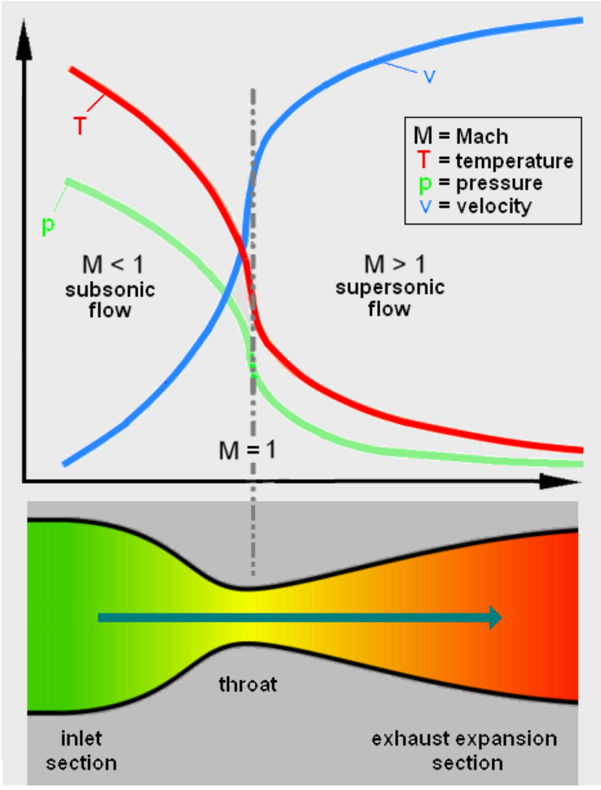

Figure 10: Plot of pressure, temperature, velocity and Mach number over a De-Laval nozzle.

3.4. Exit Velocity Equation

Utilizing the isentropic flow equations it is possible to derive equations for many of the nozzle and engine parameters that have been previously stated. To derive an expression for the **exit velocity** v_e from isentropic flow equations, we first start with the expression for stagnation enthalpy and apply the following criteria shown in **Eq. 17**.

$$h_0 = h_e + \frac{v_e^2}{2} \tag{17.1}$$

$$c_p T_0 = c_p T_e + \frac{v_e^2}{2} \quad \begin{cases} 1. & \text{Ideal gas} \\ 2. & c_p \text{ is constant at a given } T \end{cases} \tag{17.2}$$

Where:

- $\boldsymbol{h_0}$: Stagnation enthalpy (J/mol).
- $\boldsymbol{h_e}$: Enthalpy at nozzle exit (J/mol).
- $\boldsymbol{c_p}$: Specific heat at a constant pressure (J/mol).
- $\boldsymbol{T_0}$: Stagnation temperature (T).
- $\boldsymbol{T_e}$: Temperature at nozzle exit (T).

This equation can be further developed by **assuming isentropic flow** from the stagnation point to the exhaust point. This allows for the isentropic flow equations to apply, which are shown in **Eq. 18**.

$$\frac{T_0}{T_e} = \left(\frac{P_0}{P_e}\right)^{\frac{k-1}{k}} = \left(\frac{\rho_0}{\rho_e}\right)^{k-1} \tag{18}$$

Where:

- $\boldsymbol{P_0}$: Stagnation pressure (pa).
- $\boldsymbol{\rho_0}$: Stagnation density (kg/m^3).
- $\boldsymbol{k}$: Ratio of specific heats.
- $\boldsymbol{P_e}$: Pressure at nozzle exit (pa).
- $\boldsymbol{\rho_e}$: Density at nozzle exit (kg/m^3).

Finally the last equation that is needed for a useful expression for v_e is the equation for the specific heat capacity at a constant pressure c_p, this is shown in **Eq. 19**.

$$c_p = \frac{R}{W}\frac{k}{k-1} \tag{19}$$

Where:

- $\boldsymbol{R}$: Molar gas constant ($J/(\text{mol}\ K)$).
- $\boldsymbol{W}$: Molecular weight (kg/mol).

Using **Eq. 19**, **Eq. 18** and **Eq. 17**, a useful expression for the exhaust velocity v_e can be derived, this is shown in **Eq. 20**.

$$v_e = \sqrt{\frac{R}{W}\frac{2k}{k-1}T_0\left(1-\left(\frac{P_e}{P_0}\right)^{\frac{k-1}{k}}\right)} \tag{20}$$

Where T_0, P_0 can be assumed to be the combustion conditions. Alternatively, **Eq. 20** can also be used to define the I_{sp}, shown in **Eq. 21** (**assuming ideal expansion**).

$$I_{sp} = \frac{1}{g_0}\sqrt{\frac{R}{W}\frac{2k}{k-1}T_0\left(1-\left(\frac{P_e}{P_0}\right)^{\frac{k-1}{k}}\right)} \tag{21}$$

To see what parameters effect the value of v_e and what need sto be maximized, various plots are shown in **Figure 11**.

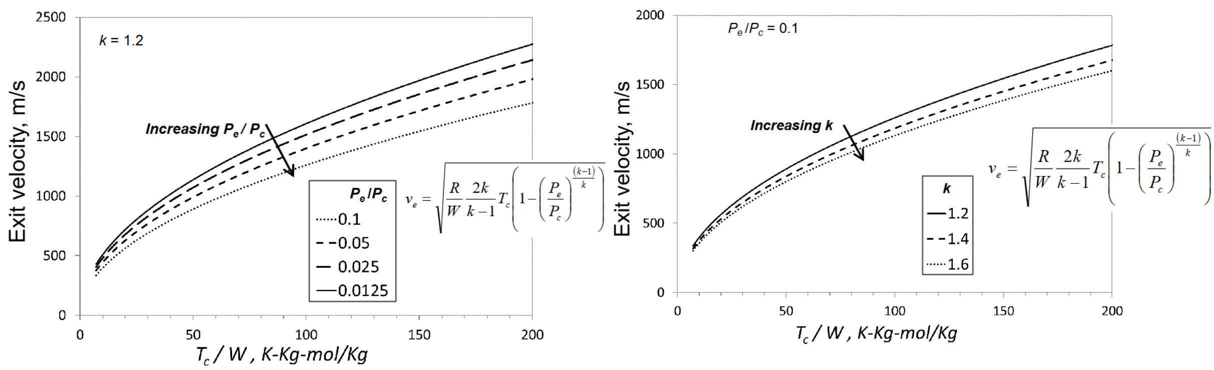

Figure 11: Plot of exit velocity for increasing P_e/P_c ratios [Left], Plot of exit velocity for increasing k ratios [Right]

From **Figure 11** it is clear to see that to maximize the value of v_e the following optimizations of parameters must occur:

- **Minimizing the molecular weight M of the reactants** will have a substantial effect on v_e.
- **Maximizing the combustion temperature T_c** will have a substantial effect on v_e.
- **Decreasing the ratio of P_e/P_c** will have a small impact on v_e.

- **Decreasing the ratio of** k will have a small impact on v_e.

3.5. Mass Flow Rate Equation

Assuming chocked flow (M_a@ Throat ≈ 1), the mass flow rate $\dot{m}$ is given by the expression shown in **Eq. 22**.

$$\dot{m} = \rho_t A_t v_t \tag{22}$$

Where:
- $\boldsymbol{\rho_t}$: Density at the throat (kg/m^3)
- $\boldsymbol{A_t}$: Area of the throat (m^2)
- $\boldsymbol{v_t}$: Velocity at the throat (m/s).

Ideally **Eq. 22** should be expressed in terms of chamber parameters. The first substitution that can be made is an expression for the velocity at the throat v_t using the speed of sound equation, this equation is shown in **Eq. 23**. **Eq. 17** can then be used to yield an expression for the stagnation/chamber pressure, shown again in **Eq. 23**.

$$v_t = a = \sqrt{\frac{kRT_t}{W}} \tag{23.1}$$

$$T_0 = T_t + \frac{v_t^2}{2c_p} = T_t + \frac{\left(M_a\sqrt{\frac{kRT_t}{W}}\right)^2}{2c_p} \tag{23.2}$$

Where:

- $\boldsymbol{a}$: Speed of sound (m/s)
- $\boldsymbol{M_a}$: Mach number

The next goal is to find expressions for the throat temperature and densities as this will then eliminate them from the equation. By using **Eq. 19**, **Eq. 18** and assuming that $M_t \approx 1$ **Eq. 24** can be derived for T_t as well as for ρ_t.

$$T_t = \frac{2T_c}{k+1} \qquad \rho_t = \rho_c\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} \tag{24}$$

Finally, **Eq. 24** and **Eq. 23** can be substituted into **Eq. 22** to yield **Eq. 25**.

$$\dot{m} = \frac{A_t \rho_c k}{\sqrt{\frac{kRT_c}{W}}}\sqrt{\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}} \tag{25}$$

4. Lecture 4

4.1. Nozzle Expansion Ratio Equation

Momentum conservation can be applied between the exhaust and the throat to yield an expression including A_t and A_e, this expression is shown in **Eq. 26**.

$$\dot{m} = \rho_t A_t v_t = \rho_e A_e v_e \quad \rightarrow \quad \frac{A_t}{A_e} = \frac{\rho_e v_e}{\rho_t v_t} \tag{26}$$

Equations for v_e, v_t and ρ_e/ρ_t have already been defined and so by substituting **Eq. 23**, **Eq. 20** and **Eq. 18** into **Eq. 26** will yield **Eq. 27**.

$$\frac{A_t}{A_e} = \left(\frac{k+1}{2}\right)^{\frac{1}{k-1}} \left(\frac{P_e}{P_c}\right)^{\frac{1}{k}} \sqrt{\frac{k+1}{k-1}\left(1-\frac{P_e}{P_c}\right)^{\frac{k-1}{k}}} \tag{27}$$

Note that for low altitude rockets $\frac{A_e}{A_t} \approx 3-25$ and for high altitude rockets $\frac{A_e}{A_t} \approx 40-200$.

4.2. Characteristic Velocity Equation

The characteristic velocity was first defined in **Eq. 11**. It can be rewritten in terms of the equations that have been previously defined to yield **Eq. 28**.

$$c^* = \frac{P_c A_t}{\dot{m}} = \frac{\sqrt{\frac{kRT_c}{W}}}{k\sqrt{\frac{2}{k+1}^{\frac{k+1}{k-1}}}} \tag{28}$$

Note that for a liquid oxygen, liquid hydrogen bipropellant rocket, $c^* \approx 2300m/s$ and for an ammonium perchlorate + polymer + Al solid rocket, $c^* \approx 1590m/s$.

4.3. Thrust Equation

Similarly to characteristic velocity, the thrust can be written in terms of the equations that have just been derived, mainly **Eq. 20** and **Eq. 25** to yield **Eq. 29**.

$$F = \dot{m}v_e + (P_e - P_a)A_e = A_t P_c \sqrt{\frac{2k^2}{k-1}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}\left(1-\left(\frac{P_e}{P_c}\right)^{\frac{k-1}{k}}\right)} + (P_e - P_a)A_e \tag{29}$$

4.4. Coefficient of Thrust Equation

A useful parameter when quantifying the performance of a nozzle is the coefficient of thrust C_F. The definition of C_F as well as the equation after substituting **Eq. 29** ito it are shown in **Eq. 30**.

$$C_F = \frac{F}{P_c A_t} = \sqrt{\frac{2k^2}{k-1}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}\left(1-\left(\frac{P_e}{P_c}\right)^{\frac{k-1}{k}}\right)} + \frac{(P_e - P_a)A_e}{P_c A_t} \tag{30}$$

Values of $C_F \approx 0.8 - 1.9$ with a higher value meaning better thrust amplification. C_F is a peak when there is ideal expansion ($P_e = P_a$) at a constant P_a/P_c. Note that the equation has a **momentum part** and a **pressure part** similar to the thrust itself. The behavior of the C_F against the area and pressure ratios is shown in **Figure 12**

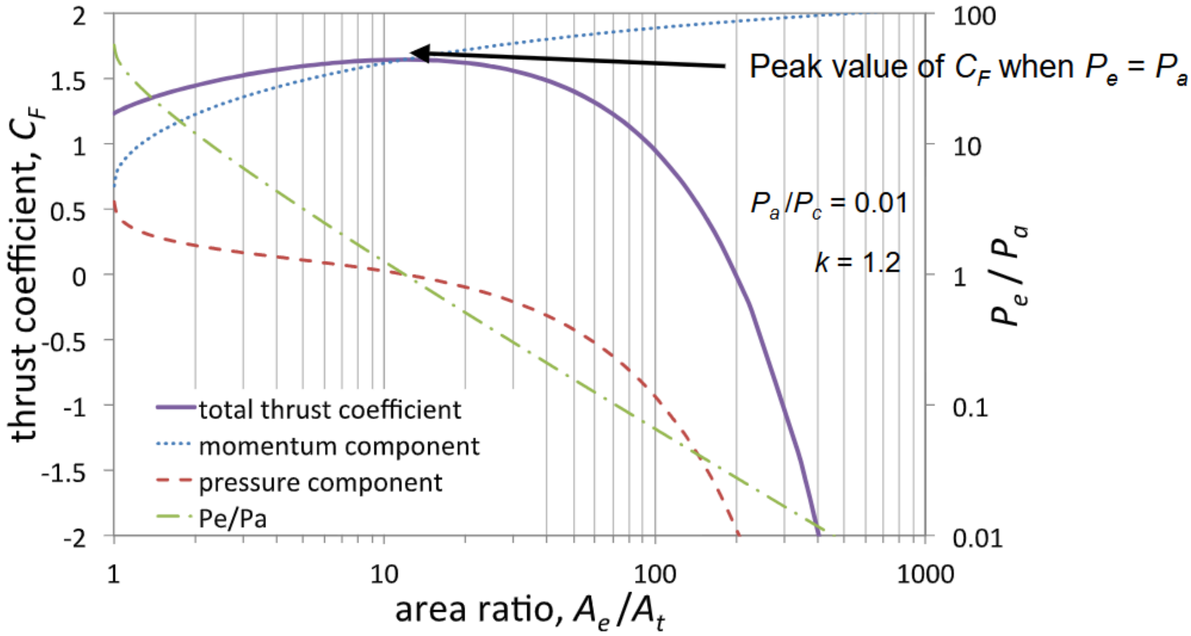

Figure 12: Plot of C_F against area and pressure ratios

Note that as the area ratio increases the momentum component increases but the pressure component decreases. This is interesting as the area ratio **does not appear in the momentum section of the equation**. In reality there is a still a dependency as area ratio depends on pressure ratio which is present in the area ratio equation. Another plot is shown in **Figure 13**.

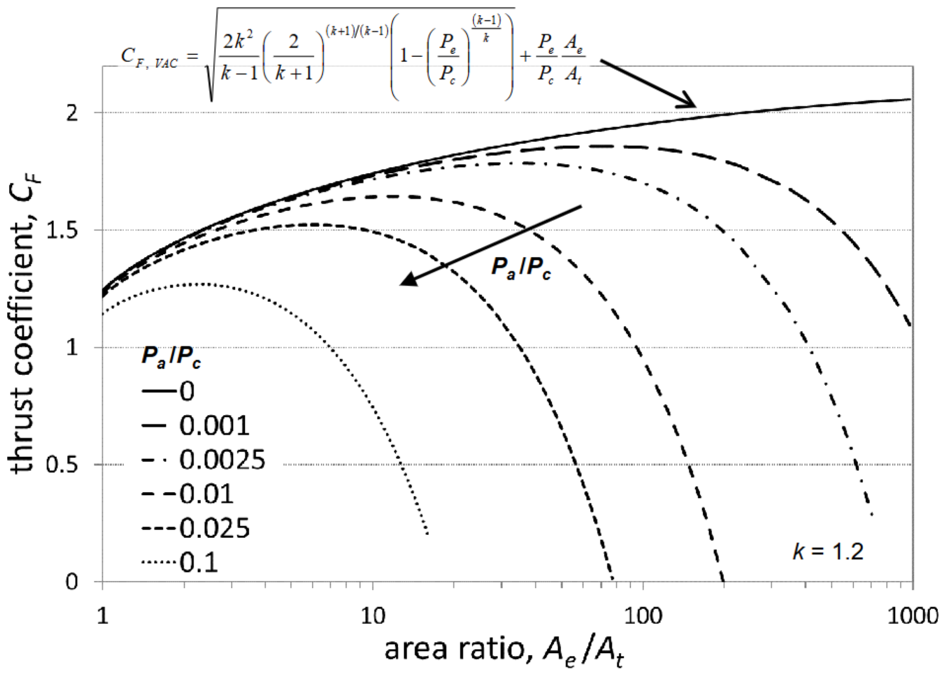

Figure 13: Plot of C_F against area ratio for varying pressure ratios

Note that in **Figure 13**, increasing the pressure ratio will decrease the thrust coefficient. The highest possible thrust coefficient is given when the pressure ratio is zero such as in a vacuum.

4.5. Summary of Equations

$$v_e(R, W, k, T_0, P_e, P_0) = \sqrt{\left(\frac{R}{W}\right)\frac{2k}{k-1}T_0\left(1-\left(\frac{P_e}{P_0}\right)^{\frac{k-1}{k}}\right)} \tag{31.1}$$

$$\dot{m}(A_t, \rho_c, k, R, T_c, W) = \frac{A_t\rho_c k}{\sqrt{\frac{kRT_c}{W}}}\sqrt{\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}} \tag{31.2}$$

$$\frac{A_t}{A_e}(k, P_e, P_c) = \left(\frac{k+1}{2}\right)^{\frac{1}{k-1}}\left(\frac{P_e}{P_c}\right)^{\frac{1}{k}}\sqrt{\frac{k+1}{k-1}\left(1-\left(\frac{P_e}{P_c}\right)^{\frac{k-1}{k}}\right)} \tag{31.3}$$

$$c^*(T_c, k, R, W) = \frac{\sqrt{\frac{kRT_c}{W}}}{k\sqrt{\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}}} \tag{31.4}$$

$$F(A_t, P_c, k, P_e, A_e, P_a) = A_tP_c\sqrt{\frac{2k^2}{k-1}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}\left(1-\left(\frac{P_e}{P_c}\right)^{\frac{k-1}{k}}\right)} + (P_e - P_a)A_e \tag{31.5}$$

$$C_{F(k,P_e,P_a,A_e,P_c,A_t)} = \sqrt{\frac{2k^2}{k-1}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}\left(1-\left(\frac{P_e}{P_c}\right)^{\frac{k-1}{k}}\right)} + \frac{(P_e - P_a)A_e}{P_cA_t} \tag{31.6}$$

4.6. Equations Involving Mach Relations

Many of the previous equations can be represented in terms of mach number, namely **Eq. 18**, which are shown in **Eq. 32**.

$$T_0 = T\left(1+\frac{1}{2}(k-1)M^2\right) \quad P_0 = P\left(1+\frac{1}{2}(k-1)M^2\right)^{\frac{k}{k-1}} \quad \rho_0 = \rho\left(1+\frac{1}{2}(k-1)M^2\right)^{\frac{1}{k-1}} \tag{32}$$

The Mach relations can be applied to **Eq. 27** to yield an expression fr the area ratio in terms of Mach number shown in **Eq. 33**.

$$\frac{A_y}{A_x} = \frac{M_x}{M_y}\sqrt{\left(\frac{1+\frac{1}{2}(k-1)M_y^2}{1+\frac{1}{2}(k-1)M_x^2}\right)^{\frac{k+1}{k-1}}} \tag{33}$$

Eq. 33 shows that area ratio is directly proportional top the Mach ratio. Furthermore this equation is also proportional to coefficient of thrust as was previously states, and this relation is also shown in **Figure 14**.

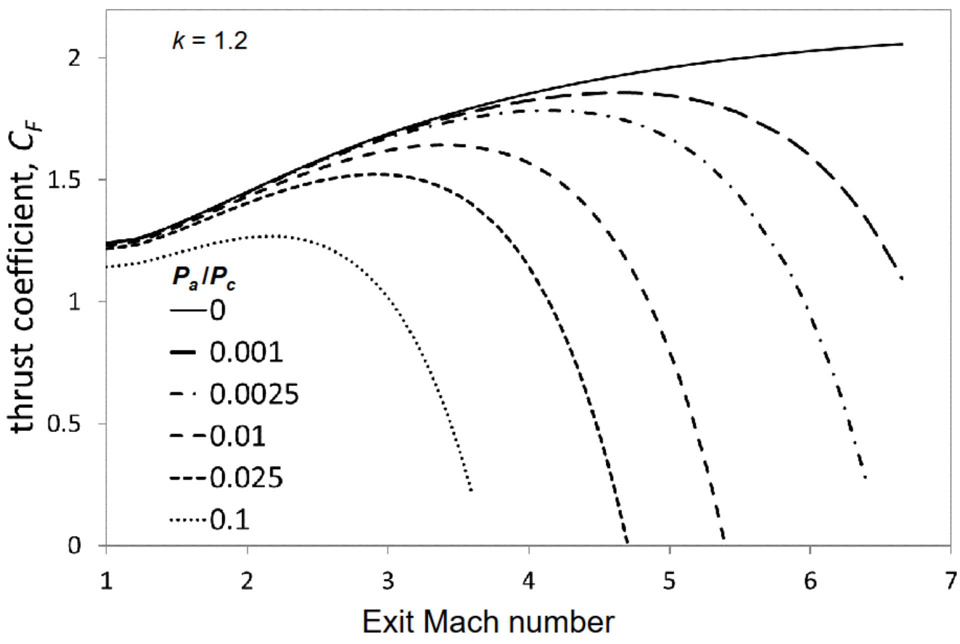

Figure 14: Plot of C_F against exit Mach number for varying pressure ratios

Figure 14 is effectively the same as **Figure 13** apart from altering the x-axis. A larger mach number will require a larger area ratio which will drive up the C_F as it depends on the pressure ratio which is proportional.

4.7. Coefficient of Thrust for Converging Nozzles

Figure 14 can be further edited to yield a neater plot. To get to this, consider the pressure equation in **Eq. 32** when there is no diverging nozzle. This would mean that $M_e = 1$ and **Eq. 32** can therefore be then written as **Eq. 34**.

$$\frac{P_c}{P_e} = \left(1 + \frac{1}{2}(k-1)M_e^2\right)^{\frac{k}{k-1}} \quad \text{If } M_e = 1 \rightarrow \quad \frac{P_e}{P_c} = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \tag{34}$$

Eq. 34 can be substituted into **Eq. 30** to yield an equation for C_F for the converging section of the nozzle, this is shown in **Eq. 35**.

$$C_{F\ \text{Converging}} = (k+1)\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} - \frac{P_a}{P_c} \tag{35}$$

Using **Eq. 35** a modified version of **Figure 14** can be plotted, this plot is shown in **Figure 15**. This plot now has a point where all lines roginiate, when the ratio of $C_F/C_{F\ \text{Converging}} = 1$ and $A_e/A_t = 1$ when there is no diverging section at all.

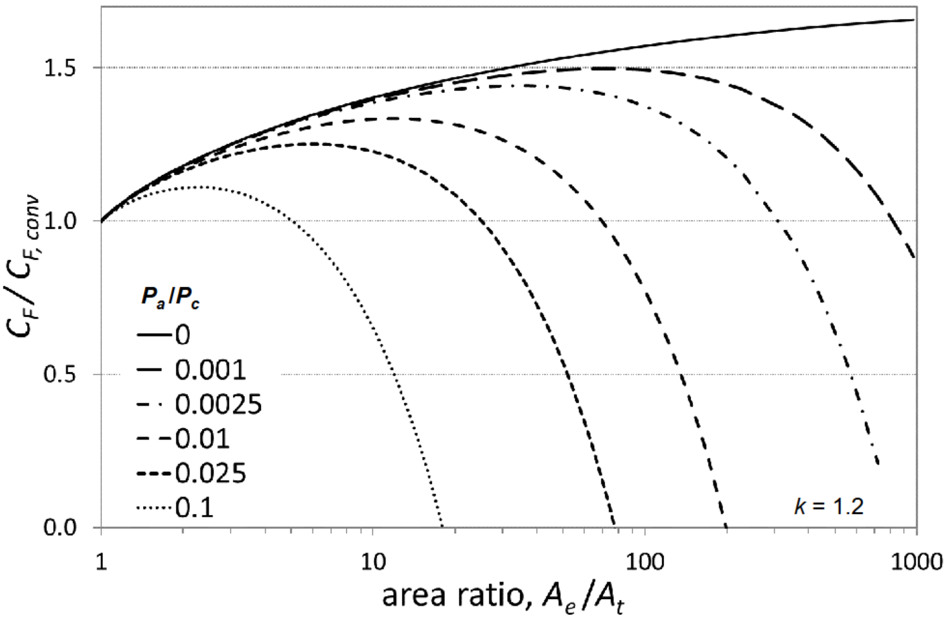

Figure 15: Plot of $C_F/C_{F\ \text{Converging}}$ against exit area ratio for varying pressure ratios

4.8. Under, Ideal and Over Expanded Nozzles

Depending on the relationship between the exit pressure P_e and the ambient pressure P_a, there are three cases of nozzle exhaust flow, these are:

- **Under-expanded** ($P_e > P_a$):
  - Typically occurs at **high altitudes** and happens when the **nozzle is too short** .Exhaust wasn't expanded enough and so expands out the back of the nozzle via expansion waves.
  - C_F and thrust are **below maximum**.

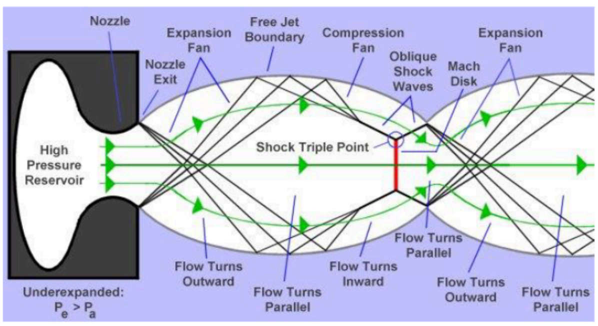

Figure 16: Under-expanded flow out of a nozzle

- **Ideally Expanded** ($P_e \approx P_a$):
  - **Nozzle is perfect length** and exhaust exits in a perfect rectangular plume with no losses or shocks.
  - C_F and thrust are **maximized**.
  - $v_e = c$, exhaust velocity is equal to effective exhaust velocity.

- **Over-expanded** ($P_e < P_a$):
  - Typically occurs at **low altitudes** and happens when the **nozzle is too long**. Exhaust is at a lower pressure than ambient causing shocks and possible flow separation within the nozzle.
  - C_F and thrust are **below maximum**.

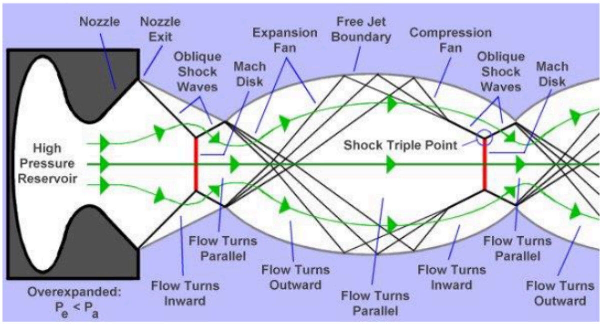

Figure 17: Over-expanded flow out of a nozzle

Plotting the behavior of the thrust coefficient against pressure ratio and the area ratio yields **Figure 18**. Note that the value of C_F is maximized when $P_e = P_a$ and $P_e/P_a = 1$.

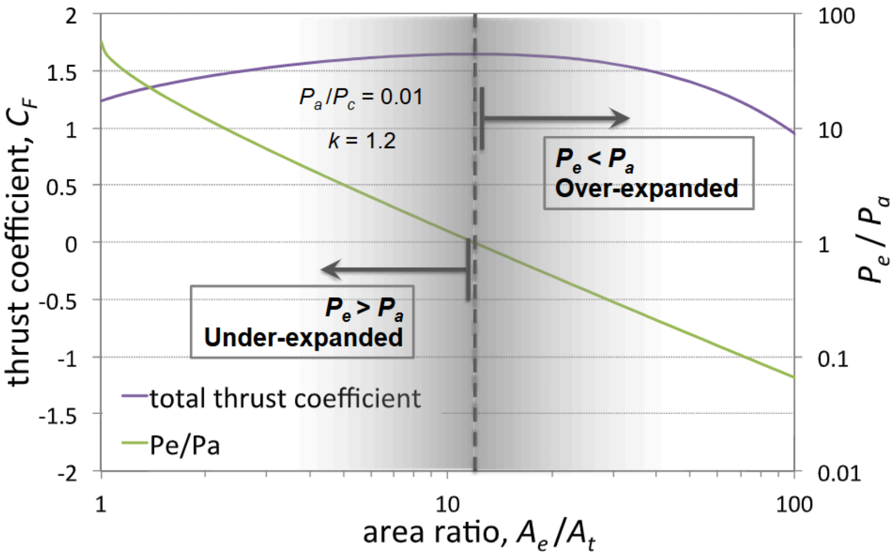

Figure 18: Plot of C_F against pressure ratio and area ratio

4.8.1. Summerfield Criterion

The Summerfield criterion applies to heavily over-expanded nozzles and describes when the flow is likely to separate from inside of the nozzle and create shocks. The criterion is shown in **Eq. 36**.

$$P_e < (0.25 \text{ to } 0.4)P_a \tag{36}$$

Eq. 36 as well as the line of ideal expansion can be applied to **Figure 15** to produce **Figure 19**.

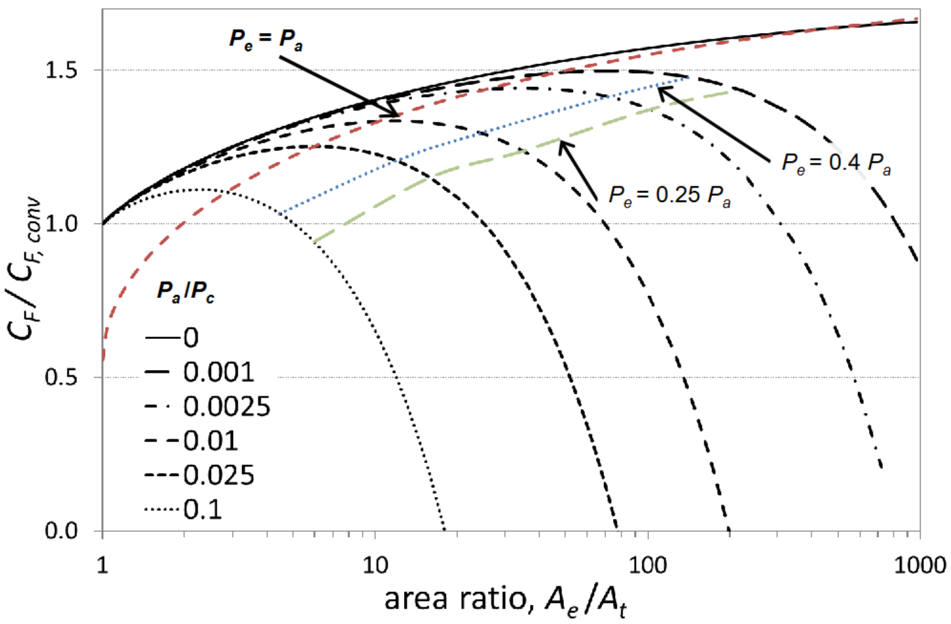

Figure 19: Plot of $C_F/C_{F\text{ Converging}}$ against exit area ratio for varying pressure ratios with summerfield criterion and ideal expansion line.

On **Figure 19**, the red dotted line represents ideal expansion. **Below this line** sits **over-expanded flow**. **Above this line** sits **under-expanded flow**. **Below the yellow and blue lines** sits **super over-expansion** when the Summerfield criterion applies. Note that a typical rocket fired at sea level will undergo the following movements through this graph:

1. Initially **over-expanded** at sea level.
2. As the altitudes rises the rocket engine moves vertically upwards on the graph and the engine becomes less and less over-expanded until it is **ideally-expanded**.
3. As the rocket ascends further, the engine starts to become **under-expanded** and thrust and C_F start to decrease.