# SESA6085

Advanced Aerospace Engineering Management

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# Definitions

| | | | |
|---|---|---|---|
| $\boldsymbol{P(A)}$ | Probability of a general event A occurring. | $\boldsymbol{N}$ | Total number of equally likely possible outcomes in the sample space. |
| $\boldsymbol{n}$ | Number of favorable outcomes (ways in which event $A$ occurs) | $\boldsymbol{P(AB)}$ | Probability of events $\boldsymbol{A}$ **and** $\boldsymbol{B}$ occurring. |
| $\boldsymbol{P(A+B)}$ | Probability of events $\boldsymbol{A}$ **or** $\boldsymbol{B}$ occurring. | $\boldsymbol{P(A\|B)}$ | Probability of event $\boldsymbol{A}$ given event $\boldsymbol{B}$ has already occurred |
| $\boldsymbol{P(A)}$ | Probability of event $\boldsymbol{A}$ given event $\boldsymbol{B}$ has already occurred | $\boldsymbol{P\left(\overline{A}\right)}$ | Probability of event $\boldsymbol{A}$ not occurring. |
| **s-independent** | statistically independent events | **s-dependent** | statistically dependent events |
| $\boldsymbol{f(t)}$ | Probability Distribution Function (PDF) | $\boldsymbol{F(t)}$ | Cumulative Distribution Function (CDF) |
| $\boldsymbol{R(t)}$ | Reliability Function | $\boldsymbol{h(t)}$ | Hazard Function |
| $\boldsymbol{\mu}$ | Mean (Gaussian location parameter) | $\boldsymbol{\sigma}$ | Standard Deviation (Gaussian scaling parameter) |
| $\boldsymbol{\lambda}$ | Exponential Scaling Parameter | $\boldsymbol{\beta}$ | Weibull shape parameter |
| $\boldsymbol{\eta}$ | Weibull scaling parameter (characteristic life) | $\boldsymbol{\gamma}$ | Weibull location parameter (failure free time) |

# 1. Lecture 1

## 1.1. Probability Fundamentals, Rules and Notation

The most basic definition of the probability for a general event $A$ occurring **is the ratio of the number of favorable outcomes $n$ to the total number of equally likely possible outcomes $N$**, this is shown in a mathematical representation in **Eq. 1**.

$$P(A) = \frac{n}{N} \tag{1}$$

Where:
- $\boldsymbol{P(A)}$: The probability of outcome $A$.
- $\boldsymbol{N}$: Total number of equally likely possible outcomes in the sample space.
- $\boldsymbol{n}$: Number of favorable outcomes (ways in which event $A$ occurs)

Note that **Eq. 1** is only for events of equal probability, for example rolling a dice. Instead if **N is the number of experiments** then **Eq. 2** applies, implying that the larger the number of experiments the closer to **Eq. 1** the probability becomes.

$$P(A) = \lim_{N\to\infty}\left(\frac{n}{N}\right) \tag{2}$$

This module uses the following notation for the probability of combined events, these are:

- $\boldsymbol{P(A)}$: Probability of event $\boldsymbol{A}$ occurring.
- $\boldsymbol{P(AB)}$: Probability of events $\boldsymbol{A}$ **and** $\boldsymbol{B}$ occurring.
- $\boldsymbol{P(A+B)}$: Probability of events $\boldsymbol{A}$ **or** $\boldsymbol{B}$ occurring.
- $\boldsymbol{P(A|B)}$: Probability of event $\boldsymbol{A}$ given event $\boldsymbol{B}$ has already occurred.
- $\boldsymbol{P\left(\overline{A}\right)}$: Probability of event $\boldsymbol{A}$ **not** occurring (note that $P(A) = 1 - P\left(\overline{A}\right)$).

## 1.2. Statical Independence

If two events are **statistically independent** (s-independent) from one another (meaning that the probability of one event occurring is completely separate from another event happening or not happening), then **Eq. 3** is true.

$$\left.\begin{aligned} P(A|B) = P\left(A|\overline{B}\right) = P(A) \\ P(B|A) = P\left(B|\overline{A}\right) = P(B) \end{aligned}\right\} \text{s-independent} \tag{3}$$

Furthermore, the joint probability of two s-independent events can be represented in the forms shown in **Eq. 4** with the further expressions derived from subbing in **Eq. 3**, **Eq. 4** is also known as the **product or series rule**.

$$P(AB) = P(A)P(B)\} \text{ s-independent} \tag{4}$$

## 1.3. Statistical Dependence

If two events are instead **statistically dependent** (s-dependent) from one another (the probability of on event happening or not happening **does** have an effect of the probability of another event), then the adjoint probability of these two events is shown in **Eq. 5**

$$\left.\begin{array}{l} P(AB) = P(A)P(B|A) \\ P(AB) = P(A|B)P(B) \end{array}\right\} \text{s-dependent} \tag{5.1}$$

$$P(B|A) = \left.\frac{P(AB)}{P(A)}\right\} \text{s-dependent and } P(A) \neq 0 \tag{5.2}$$

## 1.4. Probability Fundamentals, Rules and Notation Cont.

Generally speaking the probability of one event **or** another event occurring , whether they are s-dependant or s-independent is given by equation **Eq. 6**.

$$P(A+B) = P(A) + P(B) - P(AB) \tag{6.1}$$

$$P(A+B) = P(A) + P(B) - P(A)P(B)\} \text{ s-independent} \tag{6.2}$$

Note that the $P(AB)$ in **Eq. 6** must be subtracted as it is counted twice in the first two terms.

## 1.5. Mutual Exclusivity

Two events can be said top be mutually exclusive if they **cannot occur at the same time as one another**. This means that the adjoint probability and or probability can be written in the form shown in **Eq. 7**.

$$\left.\begin{array}{l} P(AB) = 0 \\ P(A+B) = P(A) + P(B) \end{array}\right\} \text{If A and B are Mutually Exclusive} \tag{7}$$

If instead there are **multiple mutually exclusive events** which together yield the probability of another event, then then probability of that event can be written in the form given by **Eq. 8**.

$$P(A) = \sum_i P(AB_i) = \sum_i P(A|B_i)P(B_i)\}\text{If A and all Bs are Mutually Exclusive} \tag{8}$$

## 1.6. Sequence Diagrams

Sequence diagrams act as an easy way of visualizing complex interactions and can be used to calculate overall probabilities, an example of a sequence diagram is shown in **Figure 1**.

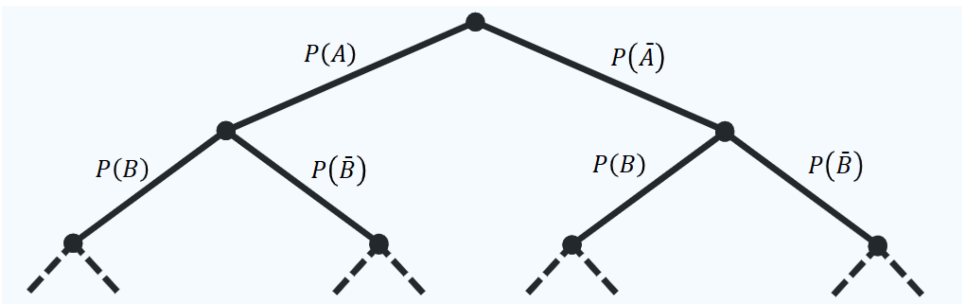


Figure 1: Example of a sequence diagram.

Probabilities down a leg are **and** probabilities and are therefore multiplied given that they are **statistically independent**. **Or** probabilities can be calculated by adding together subsequent probabilities.

## 1.7. Baye's Theorem

By rearranging **Eq. 5** a simple form of **Baye's theorem** which is shown in **Eq. 9**.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \qquad \text{Given } P(B) \neq 0 \tag{9}$$

**Eq. 9** can be further developed by substituting in **Eq. 8** which yields the **generalized Baye's theorem** shown **Eq. 10**.

$$P\left(A_j \, |B\right) = \frac{P(B|A_i)P(A_i)}{\sum_j P\left(B|A_j\right)P\left(A_j\right)} \tag{10}$$

Note that in **Eq. 10** $A_j$ is the jth event effecting the event $B$. If the probability of event $B$ depends on the probability of event $A$ both happening and not happening then **Eq. 10** simplifies down to a form called the **binary partition** form, shown in **Eq. 11**.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P\left(B|\overline{A}\right)P\left(\overline{A}\right)} \tag{11}$$

# 2. Lecture 2

## 2.1. Frequency Histograms

A **frequency histogram** is a type of bar chart which is used to represent the distribution of data, on the **x axis** are bins of data and the **y axis** represents the frequency that occurs within that bin, an example of a frequency h istogram is shown in **Figure 2**.

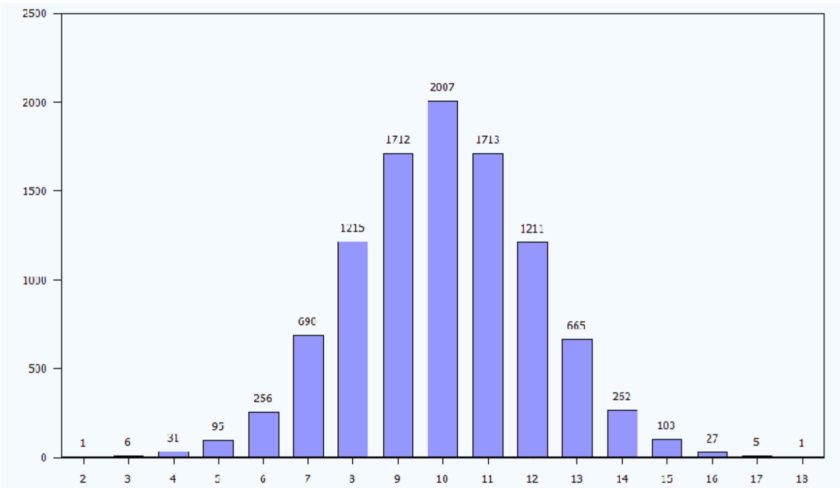


Figure 2: Example of a frequency histogram.

As we increase the number os samples taken, effectively decreasing the width of each bin, then the data will approach a smooth curve.

## 2.2. Definition of a PDF

As the frequency histogram bin width approaches an infinitesimal width, the histogram approaches a continuous curve known as the **Probability Density Function** (PDF). A PDF has one criteria in that the **area under the curve must be equal to 1**, the mathematical definition of a PDF is shown in **Eq. 12**.

$$\int_{-\infty}^{\infty} f(t)dt = 1 \tag{12}$$

PDFs can be used to find the probability that a certain value $t$ is that value. In terms of reliability engineering its the **probability that a component fails** at the time $t$.

## 2.3. Definition of a CDF

A **Cumulative Distribution Function** (CDF) yields the probability that a given value will fall between the limits of $-\infty$ and $t_1$, its mathematical definition is shown in **Eq. 13**.

$$F(t) = \int_{-\infty}^{t_1} f(t)dt \tag{13}$$

## 2.4. Reliability Function

Reliability is the probability that a component will survive from a time $t = 0$ to a time $t = t_1$ and its mathematical definition is shown in **Eq. 14**, with a graph depicting the reliability function shown in **Figure 3**.

$$R(t) = 1 - F(t) = 1 - \int_{-\infty}^{t_1} f(t)dx \equiv \int_{x}^{\infty} f(t)dx \tag{14}$$

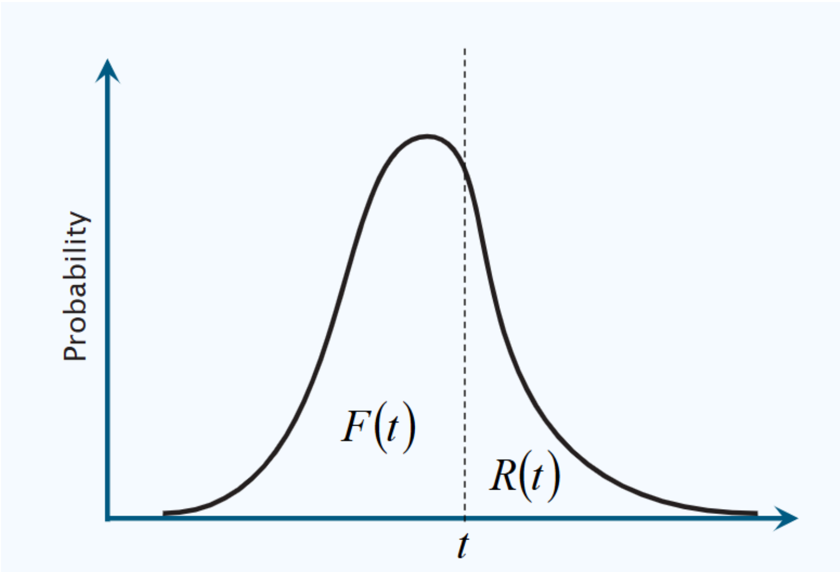


Figure 3: Graph of the reliability function and CDF on a PDF.

## 2.5. Hazard Function

Also known as the hazard rate the **hazard function** gives the probability of failure at a time $t$, given that there has not already been a failure. The mathematical definition for the hazard function is shown in **Eq. 15**.

$$h(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1 - R(t)} \tag{15}$$

The hazard function can be though of as a measure of the probability of eminent failure at $t$ or the proneness of failure after $t$. Note that there does exist a cumulative hazard function which is not assessed as well as methods to rearrange between all of these functions.

## 2.6. Continuous Distribution

A continuous PDF is a smooth curve representing how the probability varies with an area under the curve being equal to one. Effectively there is an infinite number of probability distributions as long as they satisfy the conditions set above, some of the most common are shown below.

### 2.6.1. Uniform Distribution Function

The most simplest distribution function assumes, that the distribution is zero and then one fixed value for a set time period. The PDF and CDF are defined in **Eq. 16**.

$$f(t) = \begin{cases} \frac{1}{b-a} & t \in [a, b] \\ 0 & \text{Otherwise} \end{cases} \tag{16.1}$$

$$F(t) = \begin{cases} 0 & t < a \\ \frac{t-a}{b-a} & t \in [a, b] \\ 1 & t > b \end{cases} \tag{16.2}$$

Where:
- ***a***: Start of non-zero probability.
- ***b***: End of non-zero probability.

The PDF and CDF for a uniform probability distribution are shown in graphically **Figure 4**.

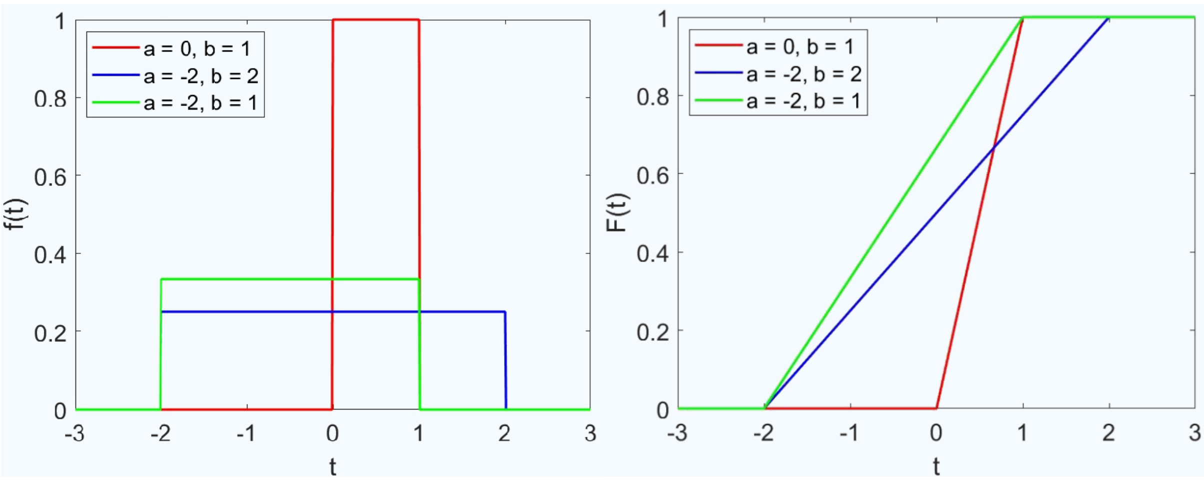


Figure 4: Plots of uniform PDFs [left] and plots of uniform CDFs [right].

### 2.6.2. Triangular Distribution Functions

Triangular distribution functions are slightly more complex than the aforementioned uniform distribution functions. Their PDF and CDF are shown in **Eq. 17**.

$$f(t) = \begin{cases} \frac{2(t-a)}{(c-a)(b-a)} & a \leq t \leq b \\ \frac{2(c-t)}{(c-a)(c-b)} & b \leq t \leq c \\ 0 & t < a, t > c \end{cases} \tag{17.1}$$

$$F(t) = \begin{cases} 0 & t < a \\ \frac{(t-a)^2}{(b-a)*(c-a)} & a \leq t \leq b \\ 1 - \frac{(c-t)^2}{(c-a)*(c-b)} & b \leq t \leq c \\ 1 & t > c \end{cases} \tag{17.2}$$

Where:
- ***a***: Start of non-zero probability.
- ***b***: Probability peak.
- ***c***: End of non-zero probability.

The PDF and CDF for a triangular probability distribution are shown graphically in **Figure 5**.

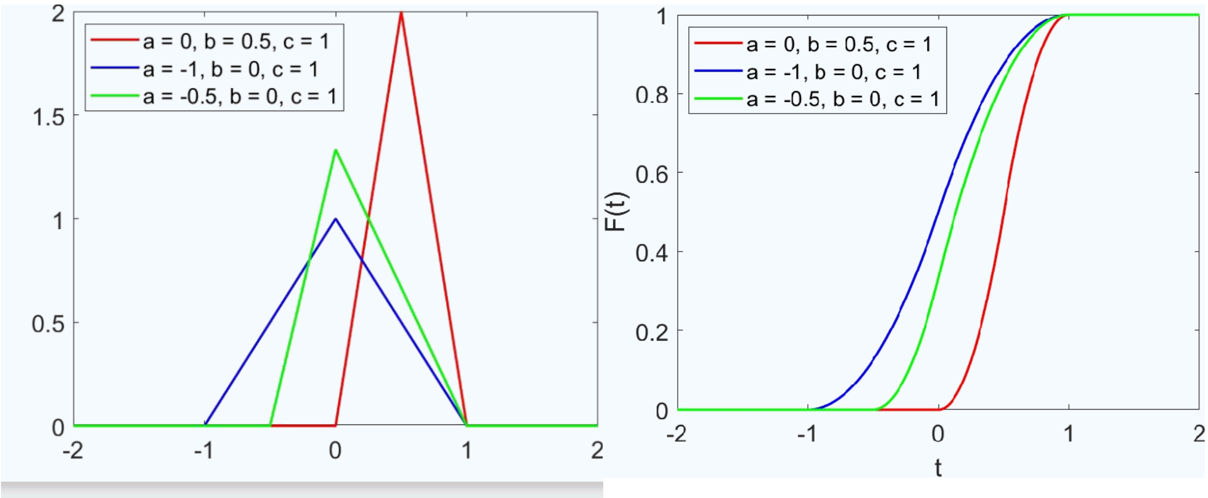


Figure 5: Plots of triangular PDFs [left] and plots of triangular CDFs [right].

### 2.6.3. Gaussian Distribution

Also known as the the **Normal Distribution** is the most commonly used probability distribution function. The PDF is shown in **Eq. 18** (**Note no close form CDF exists**).

$$f(t) = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right) \tag{18}$$

Where:

- $\boldsymbol{\mu}$: Mean (Location parameter).
- $\boldsymbol{\sigma}$: Standard Deviation (Scaling parameter).

The PDF and CDF for a Gaussian probability distribution are shown graphically in **Figure 6**.

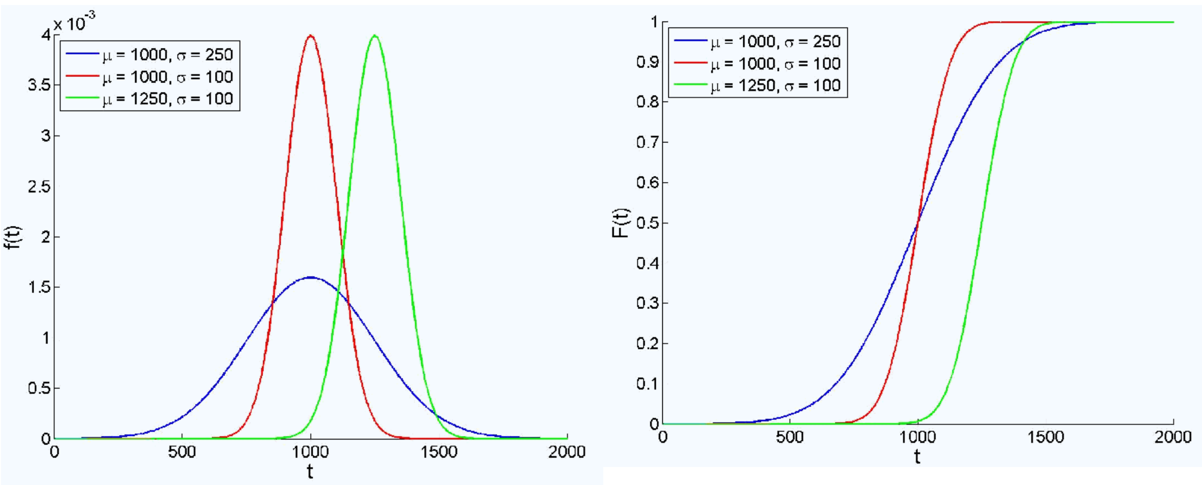


Figure 6: Plots of Gaussian PDFs [left] and plots of Gaussian CDFs [right].

The Gaussian distribution whilst being **symmetrical** also exhibits these key properties:

- **68.26%** of data is within 1 standard deviation of the mean$(\boldsymbol{\sigma})$.
- **95.44%** of data is within 2 standard deviations of the mean $(\boldsymbol{2\sigma})$.
- **99.74%** of data is within 3 standard deviations of the mean $(\boldsymbol{3\sigma})$.

### 2.6.4. Log Normal Distribution

A more versatile version of the Gaussian distribution that is better suited at modelling reliability data. The PDf is shown in **Eq. 19** (**Note no CDF exists**).

$$f(t) = \begin{cases} \frac{1}{t\sigma(2\pi)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right) & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{19}$$

Where:
- $\boldsymbol{\mu}$: Mean (Location parameter).
- $\boldsymbol{\sigma}$: Standard Deviation (Scaling parameter).

The PDF and CDF for a log normal probability distribution are shown graphically in **Figure 7**.

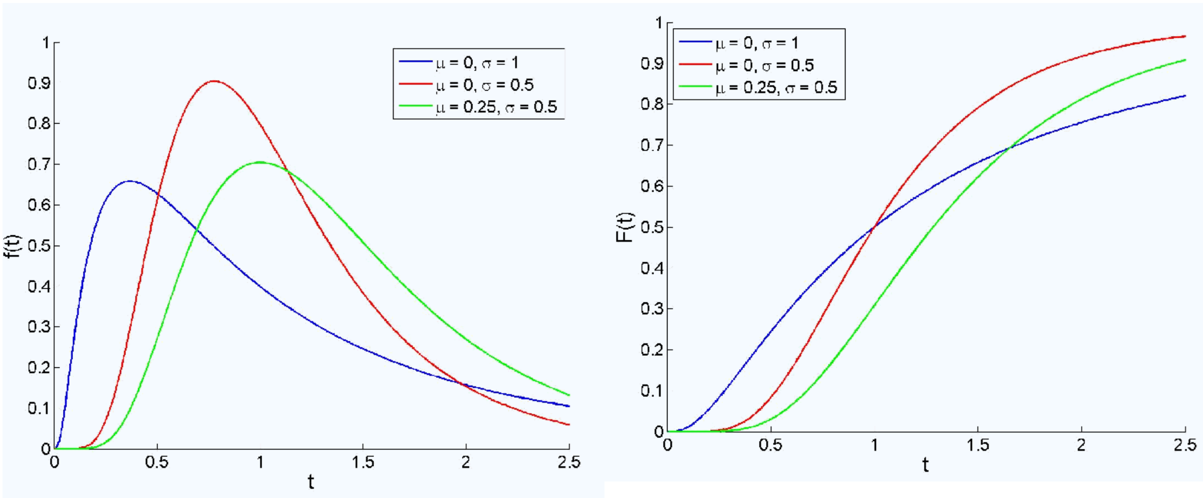


Figure 7: Plots of log normal PDFs [left] and plots of log normal CDFs [right].

### 2.6.5. Exponential Distribution

These distributions feature a **constant hazard rate** which is useful to model some processes. The PDf and CDF are shown mathematically in **Eq. 20**.

$$f(t) = \begin{cases} \lambda \exp(-\lambda t) & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{20.1}$$

$$F(t) = \begin{cases} 0 & t < 0 \\ 1 - \exp(-\lambda t) & t \geq 0 \end{cases} \tag{20.2}$$

Where:
- $\boldsymbol{\lambda}$: Scaling parameter (Also is the constant hazard rate)

It is important to note that $1/\lambda$ is the mean time to failure (MTTF). The PDF and CDF for an exponential distribution are shown graphically in **Figure 8**.

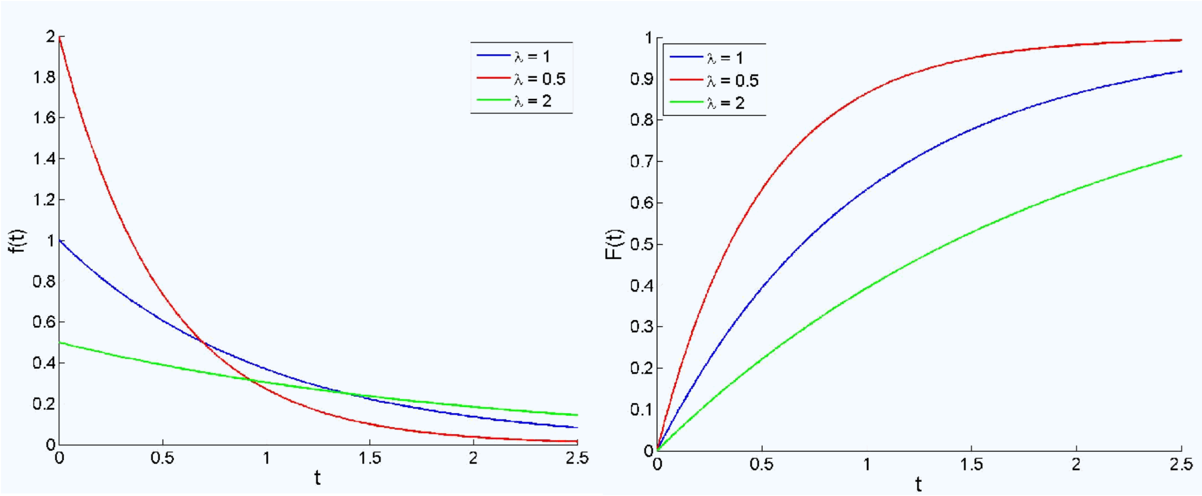


Figure 8: Plots of exponential PDFs [left] and plots of exponential CDFs [right].

### 2.6.6. Weibull Distribution

Is one of the most extensible and useful distributions out there, and can be used to model a lot of different distributions. The PDF, CDF and hazard rate are shown in **Eq. 21**.

$$f(t) = \begin{cases} \frac{\beta}{\eta^\beta} t^{\beta-1} \exp\left(-\left(\frac{t}{\eta}\right)^\beta\right) & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{21.1}$$

$$F(t) = 1 - \exp\left(-\left(\frac{t}{\eta}\right)^\beta\right) \qquad h(t) = \frac{\beta}{\eta^\beta} t^{\beta-1} \tag{21.2}$$

Where:

- $\boldsymbol{\beta}$: Shape parameter
- $\boldsymbol{\eta}$: Scaling parameter (characteristic life)

$\eta$ is also the point at which 63.2% of the population have failed. Weibull distributions are so versatile as the $\beta$ parameter changes the shape into different distributions:

- $\beta = 1$: Constant hazard function (exponential dist)
- $\beta < 1$: Decreasing hazard rate
- $\beta > 1$: Increasing hazard rate
- $\beta = 3.5$: Normal distribution

The PDF and CDF for various Weibull distributions are shown graphically in **Figure 9**.

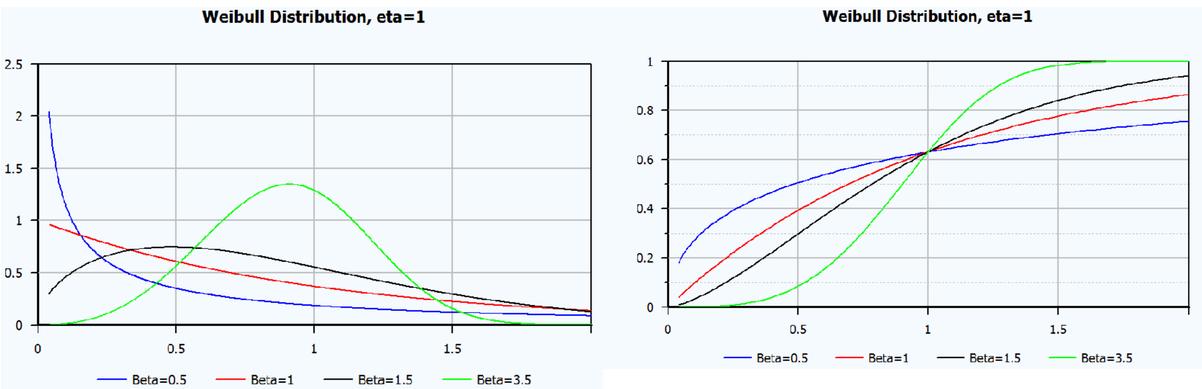


Figure 9: Plots of Weibull PDFs [left] and plots of Weibull CDFs [right] for various $\beta$s.

#### 2.6.7. Three Parameter Weibull Distribution

Introduces a new parameter $\gamma$ which is used to switch on the probability, its useful if the failures only start after a set time. The PDF, CDF and hazard rate are shown in **Eq. 22**.

$$f(t) = \begin{cases} \frac{\beta}{\eta^\beta}(t-\gamma)^{\beta-1}\exp\left(-\left(\frac{t-\gamma}{\eta}\right)^\beta\right) & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{22.1}$$

$$F(t) = 1 - \exp\left(-\left(\frac{t-\gamma}{\eta}\right)^\beta\right) \qquad h(t) = \frac{\beta}{\eta^\beta}(t-\gamma)^{\beta-1} \tag{22.2}$$

Where:

- $\boldsymbol{\beta}$: Shape parameter
- $\boldsymbol{\eta}$: Scaling parameter (characteristic life)
- $\boldsymbol{\gamma}$: Location parameter (failure free time)

#### 2.6.8. Other Distribution Functions (Non-Examinable)

Like was stated previously, there are an infin ite number of PDFs as the only criteria is for the area under the curve to sum to 1. Some other common distributions and their purposes are mentioned below:

- **Rayleigh Distribution**: Similar to exponential but with a linearly increasing hazard rate.
- **Gamma Distribution**: Similar to Weibull in that it can model a wide number of distributions by varying the parameters.
- **Beta Distribution**: A complex distribution which uses multiple gamma distributions to ensure that the life is limited to a set interval.
- **Inverse Gamma Distribution**
- **Log-logistic Distribution**
- **Birnbaum-Saunders Distribution**

### 2.7. Discrete Distributions

Whereas continuous distributions can model the probability over time, discrete distributions model the probability per an $n$ number of events, some common discrete distributions are shown below.

#### 2.7.1. Binomial Distribution

Used where the outcome of each discrete event is either pass or fail, the PDF for a binomial distribution function is defined by **Eq. 23**.

$$f(x) = \binom{n}{x} p^x q^{n-x} \tag{23.1}$$

$$\frac{n!}{x!(n-x)!} \equiv \binom{n}{x} \tag{23.2}$$

Where:

- $\boldsymbol{x}$: The number of passes

- $\boldsymbol{n}$: Total number of trails
- $\boldsymbol{p}$: Probability of success
- $\boldsymbol{q}$: Probability of failure
- $\binom{\boldsymbol{n}}{\boldsymbol{x}}$: Binomial coefficient

Note that the **binomial coefficient** is a parameter that will appear often and is read as `"n choose x"`. Usefully, it also represents the **number of possible combinations of $\boldsymbol{n}$ from $\boldsymbol{x}$**.

### 2.7.2. Other Discrete Distributions

Some other commonly used discrete distributions are:

- **Poisson's Distribution**: Represents an event occurring at a constant rate and can approximate the binomial distribution.
- **Hypergeometric Distribution** Models the probability if there are no replacements.

# 3. Lecture 3

## 3.1. Parameter Estimation

This is the process of estimating the key parameters within a given PDF ($\mu, \sigma$ for Gaussian $\lambda$ for exponential etc) by using the current set of data. for all parameter estimation techniques, the following must be true:

- **Unbiased:**: The estimator should not consistently under or overestimate the true value of the parameter.
- **Consistent**: The estimator should converge to the true value as the sample size increases.
- **Efficient**: The estimator should be consistent with a standard deviation in that estimate smaller than any other estimator for the same population.
- **Sufficient**: The estimator should use all of the information about the parameter that the data sample possesses.

All parameter estimation models depend on the quality of the data used. The three most common methods are the **method of moments**, the **maximum likelihood method** and the **least squares method**.

## 3.2. Maximum Likelihood Estimation

Maximum Likelihood Estimation (MLE) is a parameter estimation processed based on the most likely estimate of the data, and roughly consists of the following steps:

1. Define the formulation of the PDF.
2. Define the parameter to estimate.
3. Define the natural log of the PDF
4. Define the log likelihood function.
5. Define it's derivative(s).
6. Equate the derivative(s) to zero and solve for the parameter(s).

## 3.3. How to Do MLE

For MLE we assume we already know which PDF form we are using (Gaussian, exponential, Weibull etc). The PDF as well as the parameters are written in the form shown in **Eq. 24**, known as the **density function**.

$$f(t; \theta_1, \theta_2, ..., \theta_m) \equiv f(t : \theta) \tag{24}$$

Where:

- $\boldsymbol{\theta}$: PDF parameters $(\theta_1, \theta_2, ..., \theta_m)$

For example, $\theta_1$ and $\theta_2$ for a Gaussian distribution would be $\mu$ and $\sigma$. Following on from **Eq. 24**, the likelihood is mathematically defined in **Eq. 25**.

$$L(\theta) = \prod_{i=1}^{n} f(t_i : \theta) \tag{25}$$

Where $L(\theta)$ is the **likelihood** for a given PDF parameter. Effectively, the likelihood itself will hel to fit the correct PDF parameters to the given form of the data. The definition of

$L(\theta)$ has its issues however, mainly that the product of the probabilities reduce very quickly and so instead the log-likelihood is used, defined in **Eq. 26**.

$$l(\theta) = \log L(\theta) \tag{26}$$

Where $l(\theta)$ is the log-likelihood and gets around the issue of very small probabilities. The maximum likelihood estimators of the PDF parameters are written in the notation shown in **Eq. 27**.

$$\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_m \tag{27}$$

The values of **Eq. 27** can be obtained using an optimization algorithm, or can be obtained via analytical equations using the criteria shown in **Eq. 28**.

$$\frac{\partial l}{\partial \theta_j} = 0 \qquad j = 1, 2, ..., m \tag{28}$$

#### 3.3.1. MLE for a Gaussian Distribution

Recall that the form that the PDF takes for a Gaussian distribution is given in **Eq. 18**. Currently, $\mu$ & $\sigma$ are unknown for a given set of data, MLE can be used to obtain $\hat{\mu}$ & $\hat{\sigma}$. For a given observation $t_i$, the log likelihood of the Guassian PDF is given in **Eq. 29**.

$$\ln(f(t_i)) = -\ln(\sigma) - \frac{1}{2}\ln(2\pi) - \frac{1}{2}\left(\frac{t_i - \mu}{\sigma}\right)^2 \tag{29}$$

Moving on from **Eq. 29**, the MLE parameters are defined and the equation is re-written in **Eq. 30**.

$$l(\mu, \sigma) = \ln(L(\mu, \sigma)) = \ln\left(\prod_{i=1}^{n} f(t_i : \mu, \sigma)\right) = \sum_{i=1}^{n} \ln(f(t_i : \mu, \sigma)) \tag{30}$$

Note that in **Eq. 30**, using the log likelihood here is advantageous as it converts the products into much more manageable sums. Applying the sums in **Eq. 30** to **Eq. 29** yields **Eq. 31**.

$$\ln(f(t_i)) = -n\ln(\sigma) - \frac{n}{2}\ln(2\pi) - \sum_{i=1}^{n}\frac{1}{2}\left(\frac{t_i - \mu}{\sigma}\right)^2 \tag{31}$$

Taking the partial derivative of the log likelihood function with respect to the PDF parameters yields **Eq. 32**, note that when these equations are set to zero and rearranged, the MLE parameters are obtained.

$$\frac{\partial l}{\partial \mu} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}(\mu - t_i)^2 \quad \rightarrow \quad \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} t_i \tag{32.1}$$

$$\frac{\partial l}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(t_i - \mu)^2 \quad \rightarrow \quad \hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(t_i - \mu)^2} \tag{32.2}$$

Note that the MLE params for a Guassian distribution are the mean and standard deviation.

#### 3.3.2. MLE for an Exponential Distribution

Recall that the PDF for a exponential distribution is defined in **Eq. 20**. Currently $\lambda$ is unknown for a set of data, MLE can find $\hat{\lambda}$. Taking logs of the PDF, for a given observation $t_i$, the log likelihood is given in **Eq. 33**.

$$\ln(f(t_i)) = \ln(\lambda \exp(-\lambda t_i)) = \ln(\lambda) - t_i \tag{33}$$

Taking the product of the probability of all of the events yields **Eq. 34**. This equation is then differentiated with respect to $\lambda$, set to zero and rearranged for the MLE parameter.

$$l(\lambda) = n \exp(\lambda) - \lambda \sum_{i=1}^{n} t_i \tag{34.1}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} t_i \quad \rightarrow \quad \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} t_i} \tag{34.2}$$

Note that $\hat{\lambda}$ is the $1/\mu$ and is also the reaction that $1/\lambda$ is the mean time to failure (MTTF).

### 3.4. Parameter Confidence

Now that the MLE parameters have been defined, the confidence in the value of these parameters must also be quantified. It is clear that **as the data set size increases**, the **confidence in the calculated MLE parameter should increase**. To quantify parameter confidence, a Fisher information matrix is used, shown in **Eq. 35**.

$$\text{The } \ I_{ij} \text{ component} \rightarrow I_{ij} = E\left[-\frac{\partial^2 l(t:\theta)}{\partial \theta_i \partial \theta_j}\right] \tag{35}$$

Calculating and constructing the Fisher information matrix and then inverting it yields the **covariance** matrix which allows for the calculation of the variance of a specific MLE parameter and the covariance between two MLE parameters, this is shown in

$$I^{-1} = \begin{bmatrix} \mathrm{Var}(\theta_1) & \mathrm{Cov}(\theta_1, \theta_2) & \dots & \mathrm{Cov}(\theta_1, \theta_k) \\ \mathrm{Cov}(\theta_2, \theta_1) & \mathrm{Var}(\theta_2) & \dots & \mathrm{Cov}(\theta_2, \theta_k) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(\theta_k, \theta_1) & \mathrm{Cov}(\theta_k, \theta_2) & \dots & \mathrm{Var}(\theta_k) \end{bmatrix}_{k \times k} \tag{36}$$

#### 3.4.1. Parameter Confidence for Guassian Distribution

To form the Fisher information matrix for a Guassian distribution, the second partial derivative as well as the mixed partial derivative must be calculated, this is shown in **Eq. 37**. Note that $E$ in the Fisher matrix **represents the expected value**, allowing the moments of a normal to be applied

$$\text{Moments of a Guassian Dist:} \begin{cases} E[t_i - \mu] = 0 \\ E[\mu - t_i] = 0 \\ E\left[(t_i - \mu)^2\right] = \sigma^2 \\ E\left[(\mu - t_i)^2\right] = -\sigma^2 \end{cases} \tag{37.1}$$

$$E\left[-\frac{\partial^2 l}{\partial \mu^2}\right] = E\left[-\frac{\partial}{\partial \mu}\left(-\frac{1}{\sigma^2}\sum_{i=1}^{n}(\mu - t_i)\right)\right] = \frac{\boldsymbol{n}}{\boldsymbol{\hat{\sigma}}} \tag{37.2}$$

$$E\left[-\frac{\partial^2 l}{\partial \sigma^2}\right] = E\left[-\frac{\partial}{\partial \sigma}\left(-\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(t_i - \mu)^2\right)\right] = E\left[-\frac{n}{\sigma^2} + \frac{3}{\sigma^4}\sum_{i=1}^{n}(t_i - \mu)^2\right] \tag{37.3}$$

$$= -\frac{n}{\sigma^2} + \frac{3}{\sigma^4}(\sigma^2) = \frac{\boldsymbol{2n}}{\boldsymbol{\hat{\sigma}^2}} \tag{37.4}$$

$$E\left[-\frac{\partial^2 l}{\partial \mu \partial \sigma}\right] = \left[-\frac{\partial}{\partial \sigma}\left(\frac{\partial}{\partial \mu}\right)\right] = E\left[\frac{2}{\sigma^3}\sum_{i=1}^{n}(\mu - t_i)\right] = 0 \tag{37.5}$$

Now the Fisher information matrix can be constructed, noting that both of the covariance terms are zero, this is shown in **Eq. 38** alongside the inverted matrix.

$$I_{2\times2} = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{bmatrix}_{2\times2} \quad \rightarrow \quad I_{2\times2}^{-1} = \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^2}{2n} \end{bmatrix}_{2\times2} \tag{38.1}$$

$$\therefore \mathrm{Var}(\hat{\mu}) = \frac{\hat{\sigma}^2}{n} \quad \mathrm{Var}(\hat{\sigma}) = \frac{\hat{\sigma}^2}{2n} \tag{38.2}$$

This means that each of the MLE parameters themselves have a normal distribution associated with themselves where for $\hat{\mu} : \mu = \hat{\mu}, \sigma = \sqrt{\mathrm{Var}(\hat{\mu})}$ and for $\hat{\sigma} : \mu = \hat{\sigma}, \sigma = \sqrt{\mathrm{Var}(\hat{\sigma})}$. The confidence in a given MLE parameter is therefore written in the form show in **Eq. 39**.

$$P\left(\theta_l \leq \hat{\theta}, \leq \theta_u\right) = \gamma \tag{39}$$

Where $\theta_l$ & $\theta_u$ are the upper and lower bounds of the MLE parameter and $\gamma$ is the confidence level. For example if $\gamma = 0.95$, then 95% of the time, $\theta$ is within the upper and lower bounds. The upper and lower bounds are calculated using the formula shown in.

$$\theta_l = \hat{\theta} - Z_{\alpha/2}\sqrt{\mathrm{Var}\left(\hat{\theta}\right)} \tag{40.1}$$

$$\theta_u = \hat{\theta} + Z_{\alpha/2}\sqrt{\mathrm{Var}\left(\hat{\theta}\right)} \tag{40.2}$$

Where $\alpha = 1 - \gamma$ and $Z$ is the the standard normal statistic (negative of the inverse CDF of a normal distribution with mean of 0 and a standard deviation of 1 for the probability $\alpha/2$).

# 4. Lecture 4

## 4.1. Multivariate Models

In many scenarios, more than one factor will effect the failure and therefore the reliability and most of the time, these factors are not independent from one another. For univariate data, the CDF can be used to calculate the probability in the manner shown in **Eq. 41**.

$$P(a < x \leq b) = F(b) - F(a) \tag{41}$$

For multivariate models, multiple occurrence are observed at the same time. The probability two variables fall between two bounds is shown in **Eq. 42**.

$$P(a_1 < x < b_1, a_2 < y < b_2) = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2) \tag{42}$$

#### 4.1.1. CDF of a Multivariate Model

In a similar method to univariate model, the CDF of a given multivariate PDF is the integral between minus and positive infinity of that PDF with respect to each variable, this is mathematically written in **Eq. 43**

$$f(x_1, x_2, ..., x_n) \tag{43.1}$$

$$F(a_1, a_2, ..., a_n) = \int_{-\infty}^{a_n} ... \int_{-\infty}^{a_2} \int_{-\infty}^{a_1} f(x_1, x_2, ..., x_n) dx_1 dx_2 ... dx_n \tag{43.2}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f(x_1, x_2, ..., x_n) dx_1 dx_2 ... dx_n = 1 \tag{43.3}$$

## 4.2. Multivariate Normal Distribution

A multivariate normal distribution has a different vectorized form from the equation shown in **Eq. 18**, the multivariate form is shown in **Eq. 44**.

$$f(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^p \ |\boldsymbol{\Sigma}|}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu})\right) \tag{44}$$

Where:

- $\boldsymbol{x}$: Vector of variables.
- $\boldsymbol{p}$: Number of variables
- $\boldsymbol{\Sigma}$: Covariance matrix between variables
- $\boldsymbol{\mu}$: Vector of means for each variable

Note that when **Eq. 44** has $p = 1$, the equation reduces to the univariate case. Some example plots of bivariate normal distributions are shown in **Figure 10**.

#### 4.2.0.1. Fitting Data to Multivariate Normal Distributions

The MLE process can be applied to obtain the MLE parameters for $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, these are both shown in **Eq. 45** note that $\hat{\boldsymbol{\Sigma}}_{ij}$ is the same as the sample covariance matrix.

$$\hat{\boldsymbol{\mu}} = \left[\frac{1}{n}\sum_{i=1}^{n} x_{1_i}, \frac{1}{n}\sum_{i=1}^{n} x_{2_i}, ..., \frac{1}{n}\sum_{i=1}^{n} x_{k_i}\right] \quad \hat{\boldsymbol{\Sigma}}_{ij} = (\boldsymbol{x_i} - \mu_i)^T (\boldsymbol{x_j} - \mu_j)/n \tag{45}$$

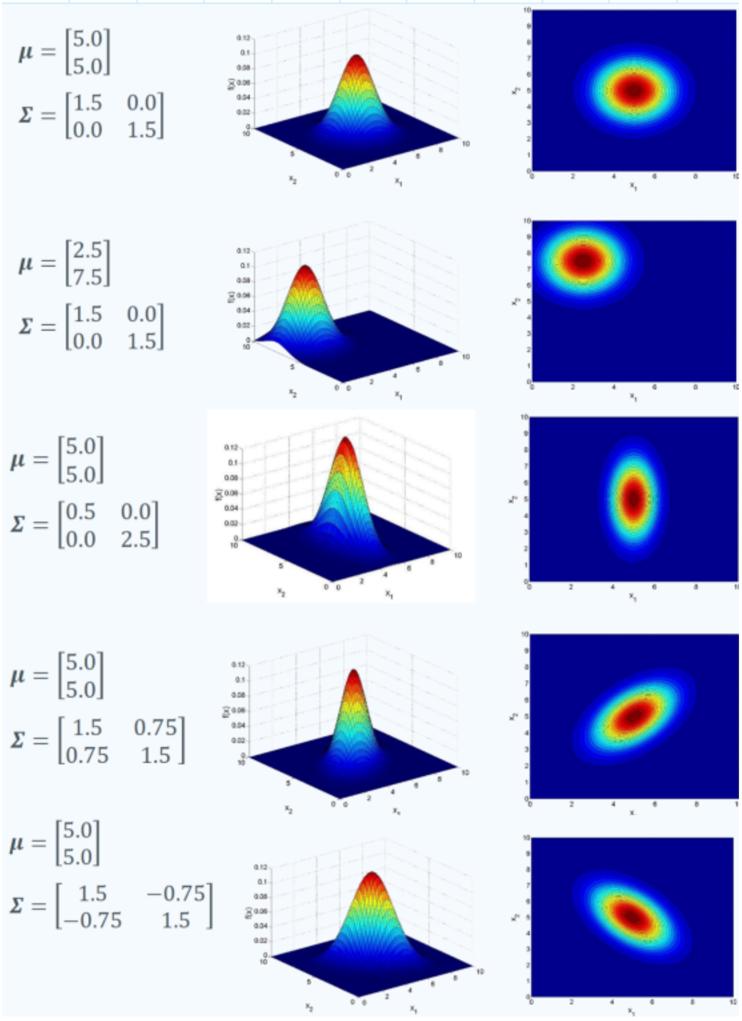

Figure 10: Various bivariate normal distributions with variations in the $\boldsymbol{\mu}$ vector and $\boldsymbol{\Sigma}$ matrix

## 4.3. Joint Distribution Functions

Joint distributions can be created for **two variables that are independent from one another** given that they satisfy a few conditions, these conditions are shown in **Eq. 46**

$$F(-\infty, -\infty, ..., -\infty) = 0 \qquad F(\infty, \infty, ..., \infty) = 0 \tag{46.1}$$

$$\text{If } a < b \text{ and } c < d \text{ then } F(a, c) < F(b, d) \tag{46.2}$$

If the conditions in **Eq. 46** are satisfied, then the $n$ number of joinable distributions, the PDF and CDF can be written in the form shown in **Eq. 47**.

$$F(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} F_i(x) \tag{47.1}$$

$$f(x_1, x_2, ..., x_n) = \frac{\partial^n f}{\partial x_1 \partial x_2 ... \partial x_\emptyset} \tag{47.2}$$

#### 4.3.1. Bivariate Exponential Distribution

Assuming that two distributions are statistically independent from one another, they can be joined in the manner defined in the previous section. The CDF for the bivariate exponential distribution is shown in **Eq. 48**.

$$F(x) = 1 - \exp(-\lambda x) \tag{48.1}$$

$$F(x_1, x_2) = F_1(x)F_2(x) = \begin{cases} (1 - \exp(-\lambda_1 x_1))(1 - \exp(-\lambda_2 x_2)) & x_{1,2} \geq 0 \\ 0 & x_{1,2} < 0 \end{cases} \tag{48.2}$$

The resulting PDF can be calculating by applying **Eq. 47** to **Eq. 48**, this yields **Eq. 49**.

$$f(x_1, x_2) = \frac{\partial^2 F}{\partial x_1 \partial x_2} \tag{49.1}$$

$$f(x_1, x_2) = \begin{cases} \lambda_1 \lambda_2 \exp(-\lambda_1 x_1 - \lambda_2 x_2) & x_{1,2} \geq 0 \\ 0 & x_{1,2} < 0 \end{cases} \tag{49.2}$$

An example bivariate exponential distribution with $\lambda_1 = 5, \lambda_2 = 2.5$ is shown in **Figure 11**

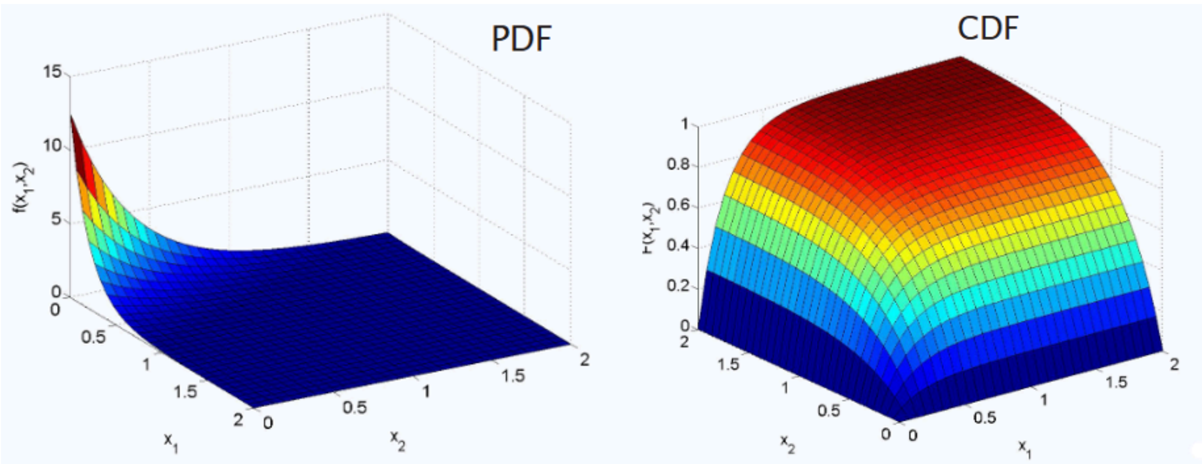


Figure 11: A bivariate exponential distributions PDF [Left] and CDF [Right]

# 5. Lecture 5

## 5.1. Censored Data

Often data sets are not presented in a whole manner, instead data is presented in a censored or incomplete manner. There are three main ways data can be presented **right censored**, **left censored** and **interval censored**.

### 5.1.1. Right Censored Data

Also known as **type 1** censored data and occurs when the failure condition of components exceeds the observation time. An example of right censored data is shown in **Table 1** for 10 servos over the span of 72 hours.

| Component Number | Component Failure Time (Hrs) |
|---|---|
| Servo #1 | 30 |
| Servo #2 | 32.5 |
| Servo #3 | 40 |
| Servo #4 | 41.0 |
| Servo #5 | 43.0 |
| Servo #6 | 50.6 |
| Servo #7 | 57.2 |
| Servo #8 | 67 |
| Servo #9 and Servo #10 | >72 |

Table 1: Right censored data example, failure times of 10 servos over 72 hours.

Note that in **Table 1**, the failure time of servos 9 and 10 is unknown as they did not fail within the observation time.

### 5.1.2. Left Censored Data

Left censored data is where components fail after teh start of a test but before the start of observation. Another example with 10 servos over a 10 hour span is shown in **Table 2**.

| Component Number | Component Failure Time (Hrs) |
|---|---|
| Servo #1 and Servo #2 | $< 24$ |
| Servo #3 | 31 |
| Servo #4 | 35 |
| Servo #5 | 43.7 |
| Servo #6 | 50.3 |
| Servo #7 | 56.1 |
| Servo #8 | 65.8 |
| Servo #9 | 70.3 |
| Servo #10 | 80.1 |

Table 2: Left censored data example, failure times of 10 servos over 72 hours.

Note that in **Table 2** the exact failure times of servos 1 and 2 are unknown as they were before the observation time.

### 5.1.3. Interval Censored Data

Interval censored data is where the exact failure time ius unknown, just the interval within which the failure occurred. With this type of censoring there is an associated upper and lower bound for each interval, an example of this is shown in **Table 3**.

| Time Interval (Hrs) | Number of Observed Failures |
|---|---|
| 0-10 | 0 |
| 11-20 | 5 |
| 21-30 | 4 |
| 31-40 | 6 |
| 41-50 | 10 |
| 51-60 | 2 |

Table 3: Interval censored data example, failure times of 10 servos over 72 hours.

## 5.2. Censored Data Notation

To develop equations and expressions to include censored data within the statistical models, the following notations will be used:

- $\boldsymbol{U}$ : Uncensored data subset.
- $\boldsymbol{C}$ : Censored data subset.
  - $\boldsymbol{C_R}$ : Right censored data subset.
  - $\boldsymbol{C_L}$ : Left censored data subset.
  - $\boldsymbol{C_I}$ : Interval censored data subset.

## 5.3. Likelihood for Right Censored Data

For a right censored set of data, despite the exact failure time of components beyond the observation time is unknown, the probability of them failing can be calculated. The probability that a component fails above a certain time $T_R$ is shown in **Figure 12** and given by **Eq. 50**.

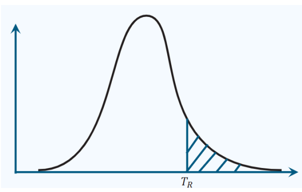

Figure 12: PDF where the probability of failure of a component over a time $T_R$ is shaded.

$$P(t > T_R) = 1 - \int_0^{T_R} f(t)dt = R(T_R) \quad (50)$$

Note that in **Eq. 50** the probability that a components fails after the cutoff time $T_R$ is given by the **reliability function**. This mathematical representation for the right censored data can then be used to rewrite the likelihood expression, this is shown in **Eq. 51**.

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(t_i : \boldsymbol{\theta}) \quad \rightarrow \quad L(\boldsymbol{\theta}) = \left\{ \prod_{i \in U} f(t_i : \boldsymbol{\theta}) \right\} \left\{ \prod_{i \in C_R} R(T_R : \boldsymbol{\theta}) \right\} \quad (51)$$

## 5.4. Likelihood for Left Censored Data

Though, the exact failure time of the components before the start of the observation time is unknown, the probability of them failing in this timeframe can be calculated. The probability that a component fails before $T_L$ is shown in **Figure 13** and given by **Eq. 52**.

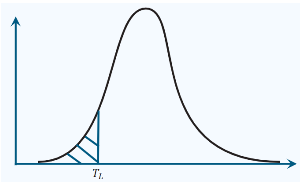

Figure 13: PDF where the probability of failure of a component under a time $T_L$ is shaded.

$$P(t > T_R) = \int_0^{T_L} f(t)dt \tag{52}$$

The mathematical expression shown in **Eq. 52** can be used within the likelihood expression shown in **Eq. 53**

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(t_i : \boldsymbol{\theta}) \quad \rightarrow \quad L(\boldsymbol{\theta}) = \left\{ \prod_{i \in U} f(t_i : \boldsymbol{\theta}) \right\} \left\{ \prod_{i \in C_L} F(T_L : \boldsymbol{\theta}) \right\} \tag{53}$$

## 5.5. Likelihood for Interval Censored Data

Although the exact failure time of a component within the interval is unknown, the probability that a component does fail within that time span can be calculated.The probability that a component fails in the interval $T_{1_{LB}} \rightarrow T_{1_{UB}}$ is shown in **Figure 14** and given by **Eq. 54**.

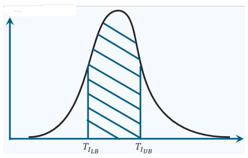

Figure 14: PDF where the probability of failure of a component within the interval $T_{1_{LB}} \rightarrow T_{1_{UB}}$ is shaded.

$$P\left(T_{I_{LB}} < t < T_{I_{UB}}\right) = \int_0^{T_{I_{UB}}} f(t)dt - \int_0^{T_{I_{LB}}} f(t)dt \tag{54}$$

The mathematical expression shown in **Eq. 54** can be used within the likelihood expression shown in **Eq. 55**, note that more intervals can be added by multiplying the on to the end.

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(t_i : \boldsymbol{\theta}) \quad \rightarrow \quad L(\boldsymbol{\theta}) = \left\{ \prod_{i \in U} f(t_i : \boldsymbol{\theta}) \right\} \left\{ \prod_{i \in C_I} F\left(T_{I_{UB}} : \boldsymbol{\theta}\right) - F\left(T_{I_{LB}} : \boldsymbol{\theta}\right) \right\} \tag{55}$$

## 5.6. Likelihood With Multiple Types of Censoring

Multiple types of censoring are shown in **Eq. 56**. Note, it is still possible to use MLE to find optimum parameters, use the Fisher matrix to find the confidence intervals and form a censored joint distribution function.

$$L(\boldsymbol{\theta}) = \left\{ \prod_{i \in U} f(t_i : \boldsymbol{\theta}) \right\} \left\{ \prod_{i \in C_L} F(T_L : \boldsymbol{\theta}) \right\} \left\{ \prod_{i \in C_I} F\left(T_{I_{UB}} : \boldsymbol{\theta}\right) - F\left(T_{I_{LB}} : \boldsymbol{\theta}\right) \right\} \left\{ \prod_{i \in C_R} R(T_R : \boldsymbol{\theta}) \right\} \tag{56}$$

# 6. Lecture 6

## 6.1. Deterministic Vs Stochastic Simulation

A **deterministic simulation** is a simulation where **there is no randomness** and the same input will always yield the same output. A **stochastic simulation** however, is a simulation where there **is an element of randomness** based on some probability. Consider the following stochastic and deterministic simulation parameters for an aerofoil:

**Deterministic Simulation Parameters**
- $R_e = 6 \times 10^6$
- $\alpha = 2°$
- $M = 0.2$

**Output Parameter**
- $C_D = 7.85 \times 10^{-3}$

Performing a stochastic simulation of the aerofoil allows for design in a much more robust manner, as the aerofoil can be optimized for the mean drag it will face in use or it can be optimized for the peak drag. Alternately teh design can be optimized to reduce the variation of $C_d$ over the range of variable values.

**Stochastic Simulation Parameters**
- $5.5 \times 10^6 < R_e < 6.5 \times 10^6$
- $1.5° < \alpha < 2.5°$
- $0.15 < M < 0.25$

**Output Parameter**

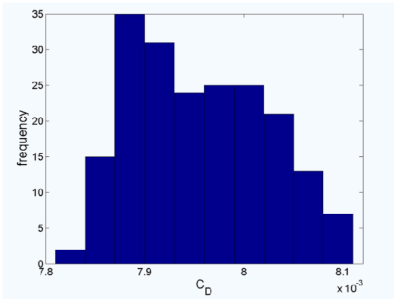


Figure 15: Output of a stochastic aerofoil simulation.

## 6.2. Distributed Random Numbers

CDFs can be used to map a uniformly distributed set of random numbers into a distributed set according to the associated PDF for the CDF. This is possible due to the CDF linking toi the original PDF as well as the CDF having no repeating values from 0 to 1. In order to generate a set of distributed random numbers, the uniform set is fed into the inverse of the CDF, this is shown for a normal and exponential distribution in **Figure 16** and **Figure 17**.

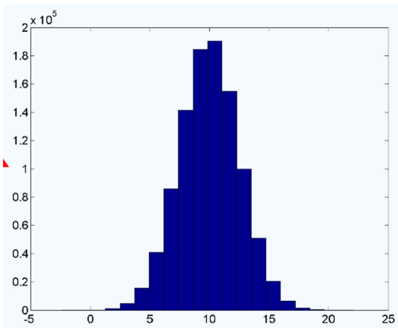


Figure 16: Normally distributed random numbers.

$$t = \mu + \sigma\sqrt{2}\mathrm{erf}^{-1}(2F(t) - 1) \tag{57}$$

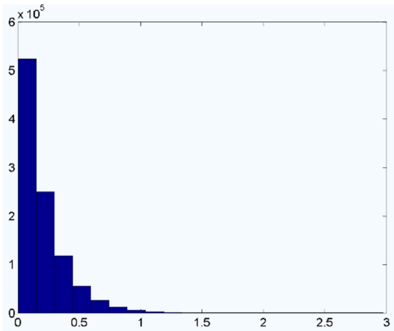


Figure 17: Exponentially distributed random numbers.

$$t = \frac{\ln(1 - F(t))}{-\lambda} \tag{58}$$

## 6.3. Monte Carlo Simulation

A monte carlo simulation is a method of stochastic simulation where thousands of trials are ran with input values sourced from distributed random numbers. Each input variable is randomly distributed via an inverse CDF, each CDF being carefully chosen to mimic the real world conditions of the structure. An example monte carlo simulation is ran on the aerofoil example shown previously and the resulting histogram for the $C_d$ is shown in **Figure 18**.

- $R_e \sim N(\mu : 6 \times 10^6, \sigma : 0.25 \times 10^6)$
- $\alpha \sim N(\mu : 2°, \sigma : 0.25°)$
- $M \sim N(\mu : 0.2, \sigma : 0.025)$

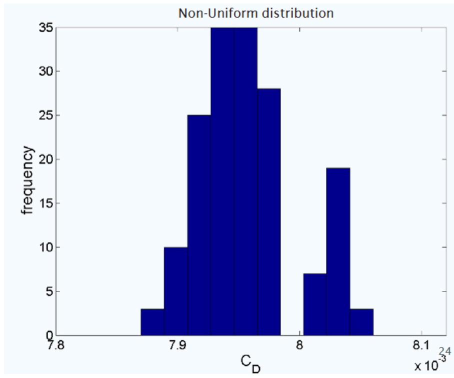


Figure 18: Monte Carlo simulation for an aerofoil.

## 6.4. Monte Carlo Convergence

The number of trails required for a monte carlo simulation to converge is case specific and there is no set equation to calculate it. The number of trails depends on:

- The **complexity of the underlying simulation**, a higher complexity will need more runs.
- The **required accuracy,** a more accurate output will require more runs.
- The **variance in the input** or output, a large input variance will mean a high number of runs.
- The **number of inputs**, more inputs require more runs.

As a monte carlo analysis is a statistical measure, the error in the distribution mean can be found by using **Eq. 59**.

$$\mathrm{Er}(\mu) = \frac{Z_{\alpha/2}\sigma}{\sqrt{N}} \tag{59}$$

Note that in this equation, $\sigma$ is initially unknown. Critically, this equation also shows that for a **one order of magnitude improvement in accuracy** we need a **two order of magnitude increase in the number of runs**.

## 6.5. Pseudo-random Vs Quasi-random Numbers

Normal monte carlo simulations make used of **pseudo-random** numbers. Pseudo-random numbers attempt to mimic real world randomness and are generated via a random number generator. One issue with pseudo-random numbers however, is that they are not very uniform, they tend to cluster and leave gaps. For better convergence of the monte carlo simulation, quasi-random numbers are used, which have a focus on uniformity rather than randomness, an example of pseudo-random and quasi-random numbers is shown in **Figure 19**.

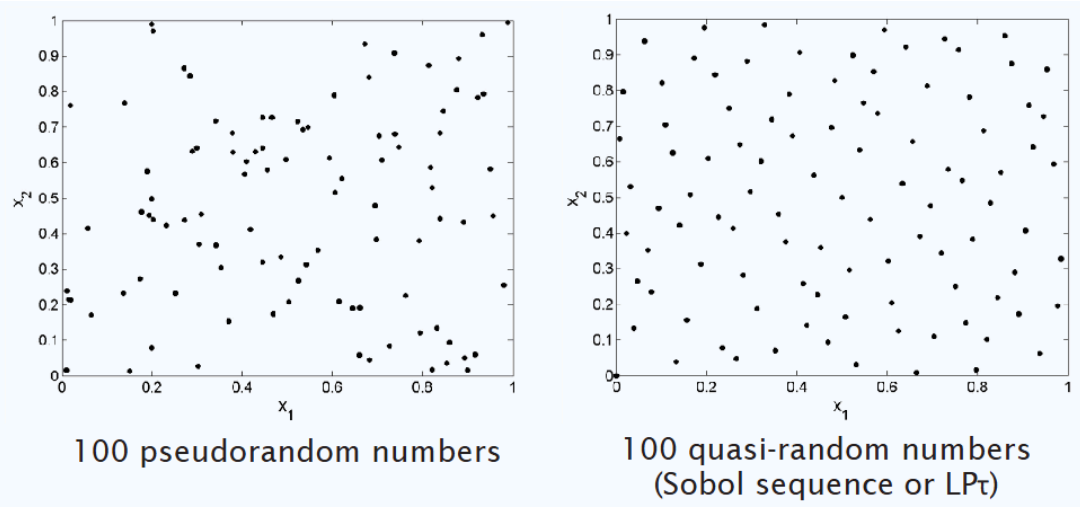


Figure 19: Pseudo-random vs Quasi-random numbers.

## 6.6. Quasi-Monte Carlo Analysis

Quasi-Monte carlo analysis makes use of quasi-random numbers instead of pseudo-random numbers, allowing for better converge than a normal monte-carlo analysis ($O\left(1/\sqrt{N}\right)$ vs $O(1/N)$). An image showing teh converge rates of a monte-carlo and quasi-monte carlo analysis is shown in **Figure 20**.

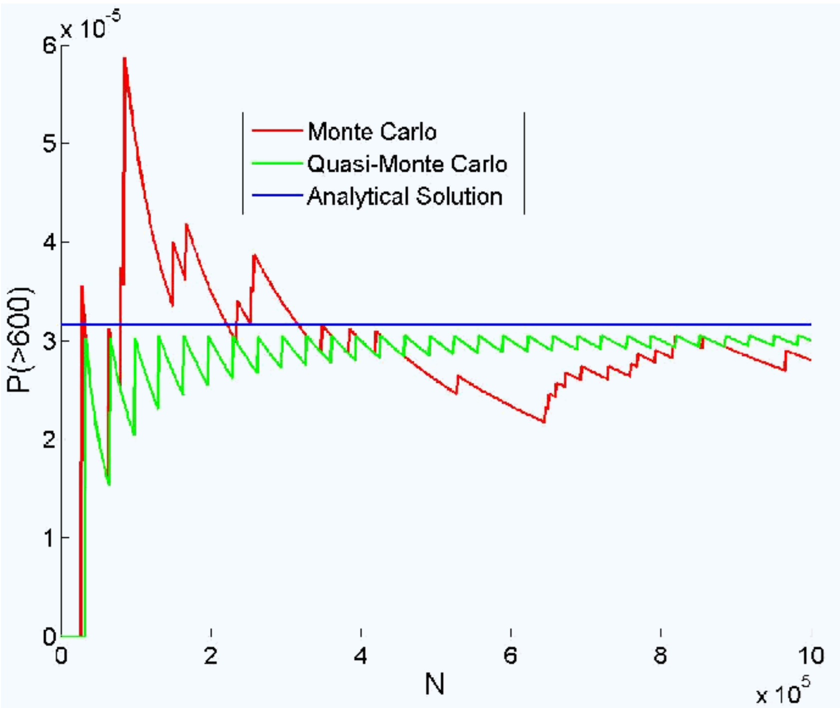


Figure 20: Convergence of quasi and normal monte carlo analyses

# 7. Lecture 7

## 7.1. Reliability New Products

An accurate understanding of the reliability of a new product is critical for many reasons, such as:

- Product must be more reliable than other competing products on the market for increased marketability.
- Allows for costing of warranty, spares, maintenance cost etc,
- When replacement/maintenance should occur for continual running of the system.

The issue is however, that a **reliability prediction can rarely be made with high accuracy or confidence**. This is due to each component have a level of uncertainty associated with its reliability prediction. As the system is then constructed of these components, the uncertainty stacks and the final system level reliability has a very high uncertainty. This is further compounded the lower the initial component level. Further to this, the human factor to reliability is almost impossible to include in a model, failure is defined differently by different stake holders and users can use products in unintended ways, effecting reliability.

## 7.2. Fundamental Limitations

Just as physical equations have limitations (ohms law not true near absolute zero), reliability predictions also have similar limitations. Reasonably credible reliability predictions can be made if:

1. The system is similar to systems developed, built and used previously.
2. The new system does not involve significant technological risk.

Further to this credible predictions must be made if:

1. The system will be manufactured in large quantities, or is very complex, or will be used for a long time, or a combination of these.
2. There is a strong commitment to the achievement of the reliability predicted.

If a prediction is not made in the above scenarios then this opens the door for massive recalls, user injury or other unwanted results.

## 7.3. Systems Reliability Models

To model the reliability of a system, a **Reliability Block Diagram** (RBD) is used. RBDs are a graphical diagram of the reliability of a system and allow for the calculation of the reliability of the whole system. A RBD works like a flow chart where if it is possible to traverse from one side of a network to another then the system is working. Each block has an associated probability which controls whether that block has failed or not and can therefore no longer be traversed through. The configuration of the RBD does not have to be in terms of the system's operational logic or functional partitioning, however this may help create the RBD

#### 7.3.1. Series Reliability Model

A basic series reliability model consists of two or more blocks connected in series with one another. Failure of any one component would result in the failure of the whole system. An example of this system is shown in **Figure 21**

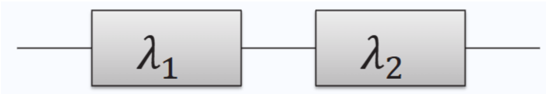


Figure 21: Series RBD with constant hazard rates.

Recall here that a hazard rate $h(x)$ instantaneous probability of failure at time $t$ given the item has survived until $t$. A constant hazard rate therefore means that the instantaneous failure probability never changes. The system reliability for the RBD shown in **Figure 21** is given by **Eq. 60**.

$$R_1 = \exp(-\lambda_1 t) \qquad R_2 = \exp(-\lambda_2 t) \tag{60.1}$$

$$R_{sys} = R_1 R_2 = \exp(-\lambda_1 t)\exp(-\lambda_2 t) = \exp(-(\lambda_1 + \lambda_2)t) \tag{60.2}$$

For an $n$ number of components which are connected in series with one another, the system reliability is shown in **Eq. 61**.

$$R_{sys} = \prod_{i=1}^{n} R_i \qquad \text{If } \lambda = \text{Const} \rightarrow \quad R_{sys} = \exp\left(-\left(\sum_{i=1}^{n} \lambda_i\right) t\right) \tag{61}$$

#### 7.3.2. Active Redundancy Model

An active redundancy model features multiple parallel pathways with components on each branch. If a single component fails, then the system itself may not fail. An example active redundancy model is shown in **Figure 22**.

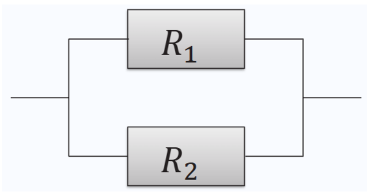


Figure 22: Active redundancy RBD with constant hazard rates.

The system reliability for a simple two component actively redundant system as shown in **Figure 22** is given by the expression shown in **Eq. 62**.

$$R_{sys} = R_1 + R_2 - R_1 R_2 = 1 - (1 - R_1)(1 - R_2) \tag{62.1}$$

$$R_{sys} = 1 - (1 - \exp(-\lambda_1 t))(1 - \exp(-\lambda_2 t)) \tag{62.2}$$

For an actively redundant system with $n$ number of branches which are connected in parallel with one another, the general system reliability is shown in **Eq. 63**.

$$R_{sys} = 1 - \prod_{i=1}^{n} (1 - R_i) \tag{63}$$

### 7.3.3. m-out-of-n Model

An m-out-of-n model states that at least m number of branches must work out of a total number of n branches for that section of the system to work. The probability of m-out-of-n system with s-independent components, each with equal unit realities is given as **Eq. 64** and shown in **Figure 23**.

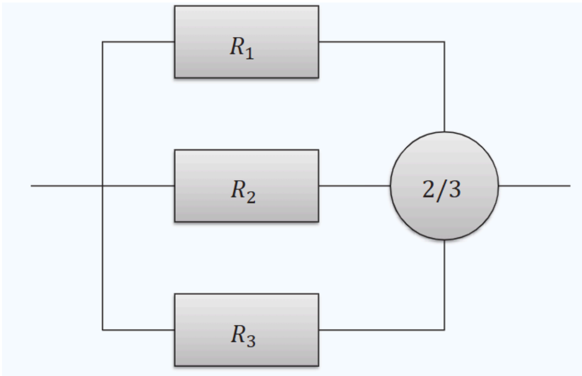


Figure 23: Example of an m-out-of-n RBD (2-out-of-3).

$$R_{sys} = 1 - \sum_{i=0}^{m-1} \binom{n}{i} R^i (1-R)^{i-1} \qquad \text{Where:} \quad \binom{n}{i} \equiv \frac{n!}{x!(n-x)!} \tag{64}$$

### 7.3.4. RBD Decomposition

Decomposition can be used to break complex RBDs into simpler ones, similar to resistors in circuits. An example of complex decomposition utilizing the RBDs covered in previous sections is shown in **Figure 24**.

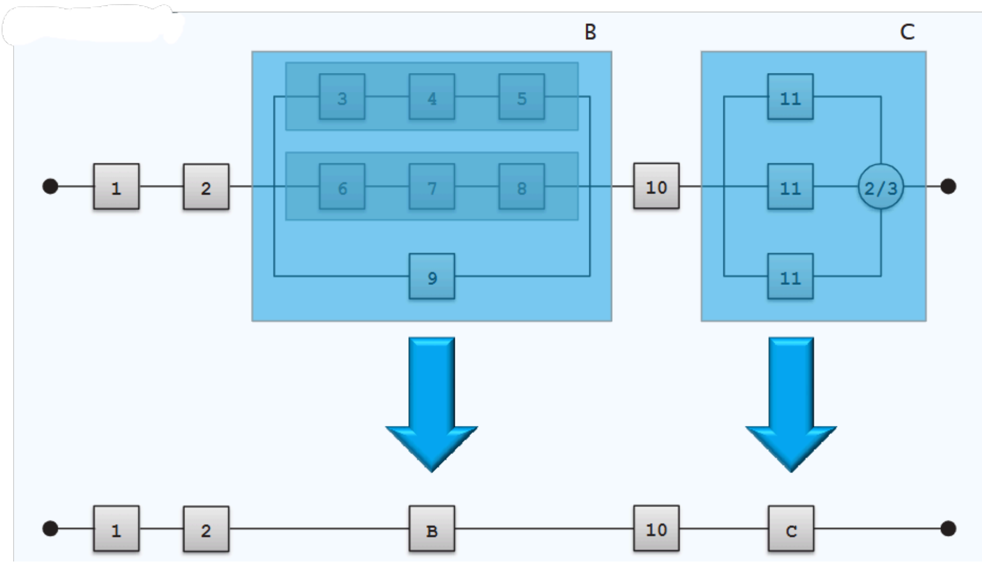


Figure 24: Decomposition of a complex RBD into a simpler system.

The decomposition process allows for easy identification of sections of the RBD which drive the systems reliability and further redesign of trouble areas to improve their reliability.

#### 7.3.5. Common Modes of Failure

Sometimes adding extra redundancy within a system does not increase its reliability as the failure of the system is driven by another section of it. Take the two RBDs shown in **Figure 25**.

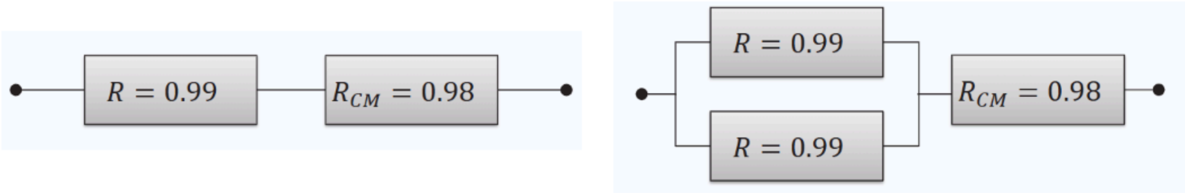


Figure 25: Series RBD [Left] composite series and active redundant system [Right]

The calculated system reliability for the LHS RBD is 0.9702 whereas the RHS RBD is 0.9799. There is only an increase of 0.0097 as the dominant failure point $R_{CM}$ has not changed.

# 8. Lecture 8

## 8.1. Balanced m-out-of-n Systems

A balanced system is where if one branch fails, another branch must be turned off to allow for the system to still function. An example would be a four thruster lander, if one thruster fails, the opposite thruster must then turn off for stability. The reliability of such a system is given in **Eq. 65**.

$$R_{sys} = 1 - P(Y = 0) = 1 - \prod_{i=1}^{n} P(X^i = 0) \tag{65}$$

The system reliability of a balanced m-out-of-n system will always be less than a regular m-out-of-n system as one branch needs to be turned off, this is shown in **Figure 26**.

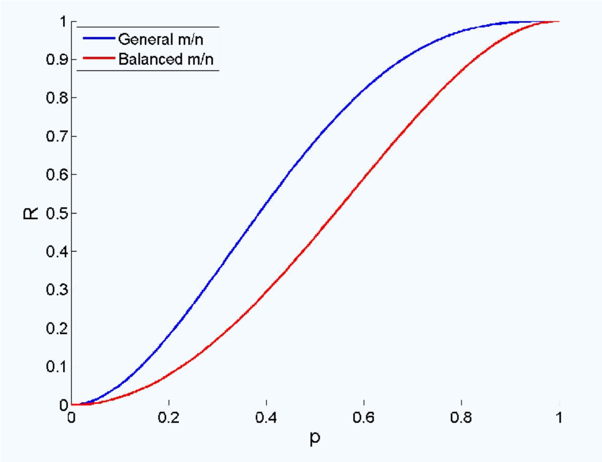


Figure 26: System reliability of a standard m-out-of-n system vs a balanced m-out-of-n system.

## 8.2. Active Vs Inactive Redundancy

In the previous lecture, the redundant system covered was an active one where both sections of the system are active and sharing the load. There is also the case of an inactive redundant system where one section only activates if the other fails. The RBD for both of these systems is shown in **Figure 27**.

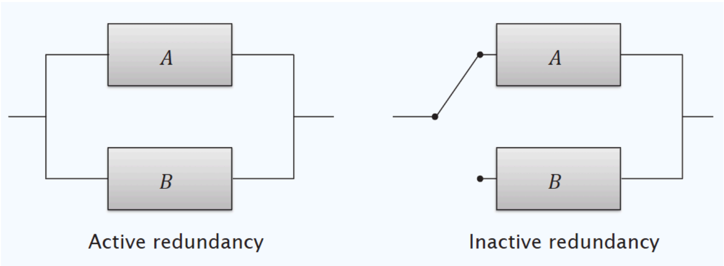


Figure 27: RBD for an active redundant system [Left] and an inactive redundant system [Right].

### 8.2.1. Types of Standby System

For inactive redundant systems, there are a few different types of standby systems, depending on how the failure rate of the back up component differs from the original component, these are:

- **Hot** standby:
  - Standby components have the same failure rate as the primary component.
  - $\lambda_{\text{hot}} = \lambda$
- **Warm** standby:
  - Standby components have a smaller failure rate than the primary.
  - $\lambda_{\text{warm}} < \lambda$
- **Cold** standby:
  - Standby components don't fail in standby and have a zero failure rate until activated
  - $\lambda_{\text{cold}} = 0$

### 8.2.2. Reliability of a Standby Redundant System

Consider a two component standby redundant system, shown in **Figure 27**. The reliability for this system is given by **Eq. 66**.

$$R_{sys} = R_A(t) + \int_{\tau=0}^{t} f_A(\tau) R_B(t-\tau) d\tau \tag{66}$$

Where $\tau$ is the failure time of component A. What **Eq. 66** does is effectively shift the PDF for component B along by $\tau$, this is graphically shown in **Figure 28**.

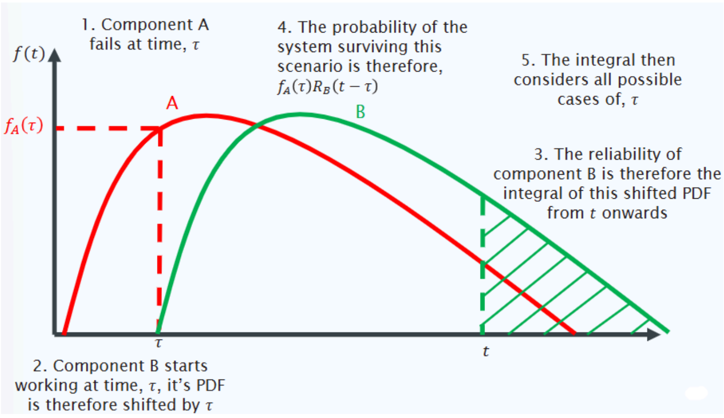


Figure 28: Graph illustrating the effect of active redundancy.

If it is assumed that both of the components have an exponential distribution, then a closed form solution for the system reliability can be determined, this is shown in **Eq. 67**. Note that the equation can be further simplified if the components are assumed to be identical $\lambda_A = \lambda_B = \lambda$.

$$R_{sys} = \exp(-\lambda_A t) + \frac{\lambda_A \exp(-\lambda_B t)}{\lambda_A - \lambda_B}(1 - \exp(-(\lambda_A + \lambda_B)t)) \tag{67.1}$$

$$R_{sys} = (1 + \lambda t)\exp(-\lambda t) \tag{67.2}$$

For multiple inactive redundant branches, all with the same component, assuming perfect switching and all having an exponential PDF, the general expression for the reliability is given by **Eq. 68**.

$$R_{sys} = \exp(-\lambda t)\left[1 + \lambda t + \frac{(\lambda t)^2}{2!} + ... + \frac{(\lambda t)^{n-1}}{(n-1)!}\right] \quad (68)$$

Typically, inactive redundant systems have a higher reliability than active ones however, these equations assuming perfect switching which may not be necessarily true. A graph depicting the comparative reliability of an active and inactive system is shown in **Figure 29**.

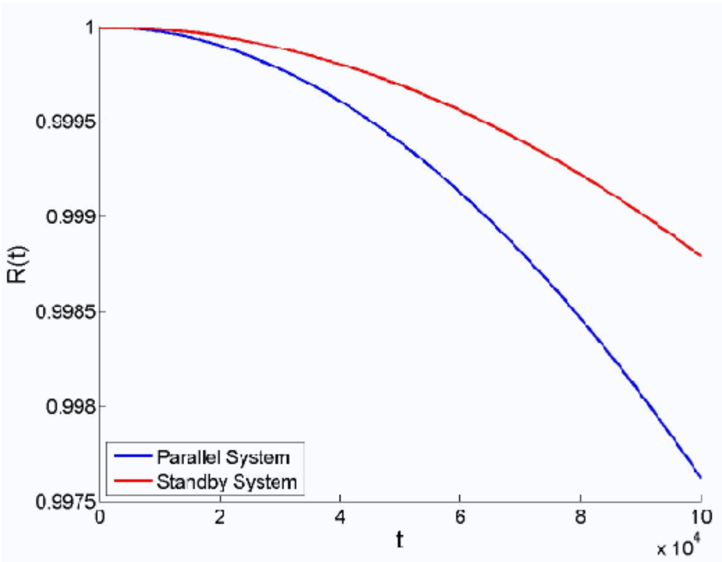


Figure 29: Graph illustrating reliability gain of an inactive dual redundant system over an active one.

## 8.3. Multistate Components

So far, all components have been modelled as existing in one of two states, working or not working. In reality components can have multiple states of operation or failure, and these are called multistate components. Consider a simple diode, it can exits in one of three states:

- Component operates normally.
- Components fails due to short circuit (zero resistance in both directions).
- Component fails due to open circuit (infinite resistance in both directions).

For a set of diodes in series or parallel, the circuit fails if **one of the diodes short circuits** or if **all of the diodes are in an open circuit**. The expression for the reliability of this system in series or parallel is shown in **Eq. 69**.

$$R_{(sys)_{\text{Series}}} = \prod_{i=1}^{n}(1-q_{oi}) - \prod_{i=1}^{n} q_{si} \qquad R_{(sys)_{\text{Parallel}}} = \prod_{i=1}^{n}(1-q_{si}) - \prod_{i=1}^{n} q_{oi} \quad (69)$$

Where $q_o$ is the probability of failure for an open circuit and $q_s$ is the probability of failure for a short circuit. Note that for more complex systems, a monte-carlo simulation is often used to calculate a numerical solution to the system reliability.

# 9. Lecture 9

## 9.1. Importance of a Component

A component within a system is more or less important based on the influence it has within the system. Components with a high amount of influence on the behavior of the system can be focused on more with less significant components given less focus. From a reliability standpoint the **most important component** is the one that has the **biggest impact on reliability** when removed or changed. Note that purely optimizing for reliability is unrealistic, performance, mass, cost, manufacturing constrains must all also be taken into account when creating a product. Identifying components importance is therefore a key metric in optimizing systems reliability. To assess the importance of a component within a system, the system's reliability is considered when:

- The component is working
- The component is not working

## 9.2. Importance Notation and Assumptions

For a system built up of $n$ components, a given $i^{th}$ component can either be working normally (denoted as $X^i = 1$) or it can be broken (denoted as $X_i = 0$). The probability therefore of a component not working is denoted as $P(X_i = 0) = q_i$. The global state of the system is denoted as $\phi(\boldsymbol{X})$ where $\phi(\boldsymbol{X}) = 1$ indicates a fully working system and $\phi(\boldsymbol{X}) = 0$ denotes a broken or inoperative system. Tying together all of this notation gives the expression in **Eq. 70**.

$$\phi(\boldsymbol{X}) \quad \text{Depends on the states} \quad X_i \quad \text{Where} \quad X_i \in [1, n] \tag{70}$$

Building on from **Eq. 70**, the system unreliability function can be defined and is shown in **Eq. 71**.

$$G(\boldsymbol{q}) = 1 - P[\phi(\boldsymbol{X}) = 1] \tag{71}$$

Where $\boldsymbol{q}$ is a function of the failure probability $q_i$ for all components $n$. From this the following two variants of teh unreliability function can be defined:

- $G(0_i, \boldsymbol{q})$ : System unavailability when the $i^{th}$ component is **operating normally** (failure probability is zero, $q_i = 0$).

- $G(1_i, \boldsymbol{q})$ : System unavailability when the $i^{th}$ component is **inoperative** (failure probability is one, $q_i = 1$).

## 9.3. Birnbaum's Importance Measure

This is an importance measure based on the system unavailability expressions denoted in the previous section. The expression for Birnbaum's importance measure is given as **Eq. 72**.

$$I_B^i(t) = \frac{\partial G(\boldsymbol{q}(t))}{\partial q_i(t)} = G(1_i, \boldsymbol{q}) - G(0_i, \boldsymbol{q}) \tag{72}$$

The higher the value of **Eq. 72**, the more important it is within the system. It is important to note that for series systems, the component with the lowest reliability will be the most important and for a parallel system the component with the highest reliability will be the most important.

## 9.4. Criticality Importance Measure

The criticality importance measure is the conditional probability that the system is in a state at time $t$ such that the $i^{th}$ component is critical and has failed by $t$. The expression for calculating the criticality importance measure is shown in **Eq. 73**.

$$I_{CR}^{i}(t) = \frac{\partial G(\boldsymbol{q}(t))}{\partial q_i(t)} \times \frac{q_i(t)}{G(\boldsymbol{q}(t))} = \frac{[G(1_i, \boldsymbol{q}) - G(0_i, \boldsymbol{q})] \times q_i(t)}{G(\boldsymbol{q}(t))} \tag{73}$$

Note that **Eq. 73** includes **Eq. 72** within it. Note that for some systems, the critically can be the same for all components even if teh reliability is not the same as each component may be equally critical.

## 9.5. Upgrade Function

The upgrade function is defined as the fractional reduction in the probability of the system failure when the failure rate of the $i^{th}$ component is reduced. The expression for the upgrade function is shown in **Eq. 74**, note the upgrade function can only be applied to systems where all components have an exponential reliability.

$$I_{UF}^{i}(t) = \frac{\lambda_i}{G(\boldsymbol{q}(t))} \times \frac{\partial G(\boldsymbol{q}(t))}{\partial \lambda_i} \tag{74}$$

Note that utilizing the upgrade function will yield a time varying value for the importance measure $I_{UF}$, which is useful to see how the component importance varies overtime.