SESA6085

Advanced Aerospace Engineering Management

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Definitions

$P(A)$	Probability of a general event A occurring.	N	Total number of equally likely possible outcomes in the sample space.
n	Number of favorable outcomes (ways in which event A occurs)	$P(AB)$	Probability of events A **and** B occurring.
$P(A+B)$	Probability of events A **or** B occurring.	$P(A|B)$	Probability of event A given event B has already occurred
$P(A)$	Probability of event A given event B has already occurred	$P(\overline{A})$	Probability of event A not occurring.
s-independent	statistically independent events	**s-dependent**	statistically dependent events
$f(t)$	Probability Distribution Function (PDF)	$F(t)$	Cumulative Distribution Function (CDF)
$R(t)$	Reliability Function	$h(t)$	Hazard Function
μ	Mean (Gaussian location parameter)	σ	Standard Deviation (Gaussian scaling parameter)
λ	Exponential Scaling Parameter	β	Weibull shape parameter
η	Weibull scaling parameter (characteristic life)	γ	Weibull location parameter (failure free time)

1. Lecture 1

1.1. Probability Fundamentals, Rules and Notation

The most basic definition of the probability for a general event A occurring **is the ratio of the number of favorable outcomes n to the total number of equally likely possible outcomes N**, this is shown in a mathematical representation in **Eq. 1**.

$$P(A) = \frac{n}{N} \tag{1}$$

Where:
- $P(A)$: The probability of outcome A.
- N: Total number of equally likely possible outcomes in the sample space.
- n: Number of favorable outcomes (ways in which event A occurs)

Note that **Eq. 1** is only for events of equal probability, for example rolling a dice. Instead if **N is the number of experiments** then **Eq. 2** applies, implying that the larger the number of experiments the closer to **Eq. 1** the probability becomes.

$$P(A) = \lim_{N \to \infty} \left(\frac{n}{N} \right) \tag{2}$$

This module uses the following notation for the probability of combined events, these are:

- $P(A)$: Probability of event A occurring.
- $P(AB)$: Probability of events A **and** B occurring.
- $P(A + B)$: Probability of events A **or** B occurring.
- $P(A|B)$: Probability of event A given event B has already occurred.
- $P(\overline{A})$: Probability of event A **not** occurring (note that $P(A) = 1 - P(\overline{A})$).

1.2. Statical Independence

If two events are **statistically independent** (s-independent) from one another (meaning that the probability of one event occurring is completely separate from another event happening or not happening), then **Eq. 3** is true.

$$\left. \begin{array}{l} P(A|B) = P\left(A|\overline{B}\right) = P(A) \\ P(B|A) = P\left(B|\overline{A}\right) = P(B) \end{array} \right\} \text{ s-independent} \tag{3}$$

Furthermore, the joint probability of two s-independent events can be represented in the forms shown in **Eq. 4** with the further expressions derived from subbing in **Eq. 3**, **Eq. 4** is also known as the **product or series rule**.

$$P(AB) = P(A)P(B) \} \text{ s-independent} \tag{4}$$

1.3. Statistical Dependence

If two events are instead **statistically dependent** (s-dependent) from one another (the probability of on event happening or not happening **does** have an effect of the probability of another event), then the adjoint probability of these two events is shown in **Eq. 5**

$$\left.\begin{array}{l} P(AB) = P(A)P(B|A) \\ P(AB) = P(A|B)P(B) \end{array}\right\} \text{ s-dependent} \tag{5.1}$$

$$P(B|A) = \tfrac{P(AB)}{P(A)} \Big\} \text{ s-dependent and } P(A) \neq 0 \tag{5.2}$$

1.4. Probability Fundamentals, Rules and Notation Cont.

Generally speaking the probability of one event **or** another event occurring , whether they are s-dependant or s-independent is given by equation **Eq. 6**.

$$P(A + B) = P(A) + P(B) - P(AB) \tag{6.1}$$

$$P(A + B) = P(A) + P(B) - P(A)P(B) \} \text{ s-independent} \tag{6.2}$$

Note that the $P(AB)$ in **Eq. 6** must be subtracted as it is counted twice in the first two terms.

1.5. Mutual Exclusivity

Two events can be said top be mutually exclusive if they **cannot occur at the same time as one another**. This means that the adjoint probability and or probability can be written in the form shown in **Eq. 7**.

$$\left.\begin{array}{l} P(AB) = 0 \\ P(A + B) = P(A) + P(B) \end{array}\right\} \text{ If A and B are Mutually Exclusive} \tag{7}$$

If instead there are **multiple mutually exclusive events** which together yield the probability of another event, then then probability of that event can be written in the form given by **Eq. 8**.

$$P(A) = \sum_i P(AB_i) = \sum_i P(A|B_i)P(B_i) \Big\} \text{If A and all Bs are Mutually Exclusive} \tag{8}$$

1.6. Sequence Diagrams

Sequence diagrams act as an easy way of visualizing complex interactions and can be used to calculate overall probabilities, an example of a sequence diagram is shown in **Figure 1**.
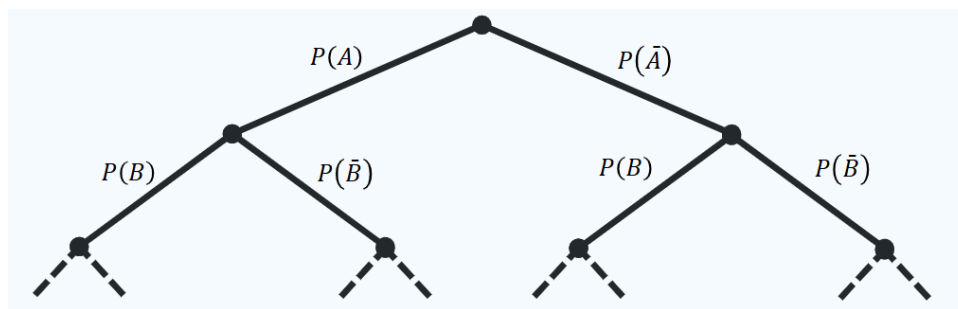

Figure 1: Example of a sequence diagram.

Probabilities down a leg are **and** probabilities and are therefore multiplied given that they are **statistically independent**. **Or** probabilities can be calculated by adding together subsequent probabilities.

1.7. Baye's Theorem

By rearranging **Eq. 5** a simple form of **Baye's theorem** which is shown in **Eq. 9**.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \qquad \text{Given } P(B) \neq 0 \tag{9}$$

Eq. 9 can be further developed by substituting in **Eq. 8** which yields the **generalized Baye's theorem** shown **Eq. 10**.

$$P(A_j \,|B) = \frac{P(B|A_i)P(A_i)}{\sum_j P(B|A_j)P(A_j)} \tag{10}$$

Note that in **Eq. 10** A_j is the jth event effecting the event B. If the probability of event B depends on the probability of event A both happening and not happening then **Eq. 10** simplifies down to a form called the **binary partition** form, shown in **Eq. 11**.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P\left(B|\overline{A}\right)P\left(\overline{A}\right)} \tag{11}$$

2. Lecture 2

2.1. Frequency Histograms

A **frequency histogram** is a type of bar chart which is used to represent the distribution of data, on the **x axis** are bins of data and the **y axis** represents the frequency that occurs within that bin, an example of a frequency h istogram is shown in **Figure 2**.
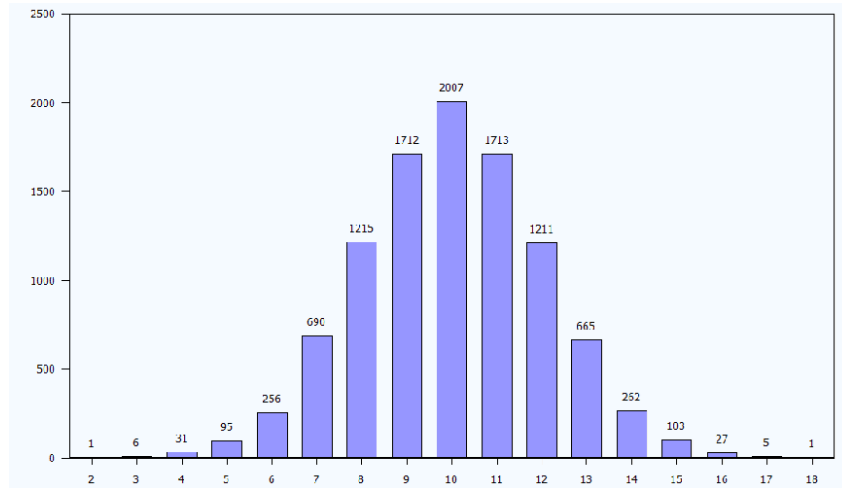

Figure 2: Example of a frequency histogram.

As we increase the number os samples taken, effectively decreasing the width of each bin, then the data will approach a smooth curve.

2.2. Definition of a PDF

As the frequency histogram bin width approaches an infinitesimal width, the histogram approaches a continuous curve known as the **Probability Density Function** (PDF). A PDF has one criteria in that the **area under the curve must be equal to 1**, the mathematical definition of a PDF is shown in **Eq. 12**.

$$\int_{-\infty}^{\infty} f(t)dt = 1 \tag{12}$$

PDFs can be used to find the probability that a certain value t is that value. In terms of reliability engineering its the **probability that a component fails** at the time t.

2.3. Definition of a CDF

A **Cumulative Distribution Function** (CDF) yields the probability that a given value will fall between the limits of $-\infty$ and t_1, its mathematical definition is shown in **Eq. 13**.

$$F(t) = \int_{-\infty}^{t_1} f(t)dt \tag{13}$$

2.4. Reliability Function

Reliability is the probability that a component will survive from a time $t = 0$ to a time $t = t_1$ and its mathematical definition is shown in **Eq. 14**, with a graph depicting the reliability function shown in **Figure 3**.

$$R(t) = 1 - F(t) = 1 - \int_{-\infty}^{t_1} f(t)dx \equiv \int_x^{\infty} f(t)dx \tag{14}$$
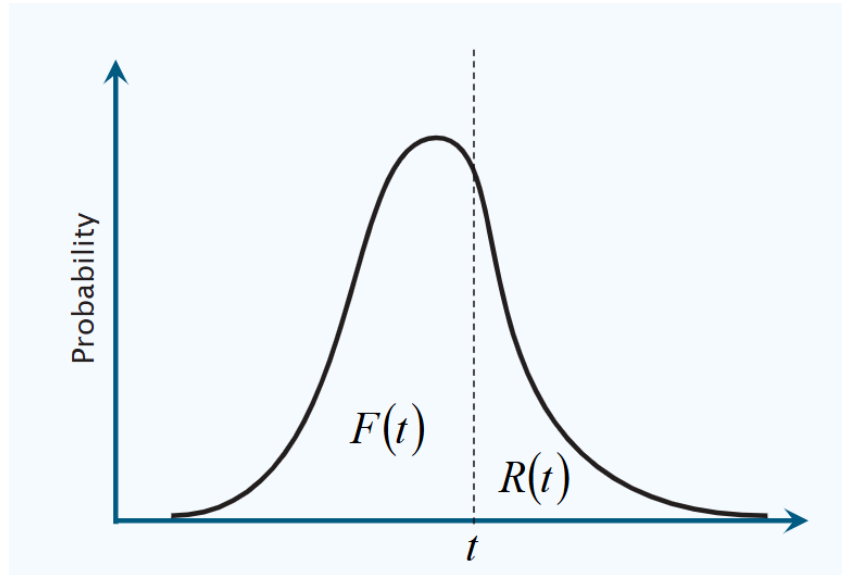

Figure 3: Graph of the reliability function and CDF on a PDF.

2.5. Hazard Function

Also known as the hazard rate the **hazard function** gives the probability of failure at a time t, given that there has not already been a failure. The mathematical definition for the hazard function is shown in **Eq. 15**.

$$h(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1 - R(t)} \tag{15}$$

The hazard function can be though of as a measure of the probability of eminent failure at t or the proneness of failure after t. Note that there does exist a cumulative hazard function which is not assessed as well as methods to rearrange between all of these functions.

2.6. Continuous Distribution

A continuous PDF is a smooth curve representing how the probability varies with an area under the curve being equal to one. Effectively there is an infinite number of probability distributions as long as they satisfy the conditions set above, some of the most common are shown below.

2.6.1. Uniform Distribution Function

The most simplest distribution function assumes, that the distribution is zero and then one fixed value for a set time period. The PDF and CDF are defined in **Eq. 16**.

$$f(t) = \begin{cases} \frac{1}{b-a} & t \in [a, b] \\ 0 & \text{Otherwise} \end{cases} \tag{16.1}$$

$$F(t) = \begin{cases} 0 & t < a \\ \frac{t-a}{b-a} & t \in [a, b] \\ 1 & t > b \end{cases} \tag{16.2}$$

Where:
- **a**: Start of non-zero probability.
- **b**: End of non-zero probability.

The PDF and CDF for a uniform probability distribution are shown in graphically **Figure 4**.
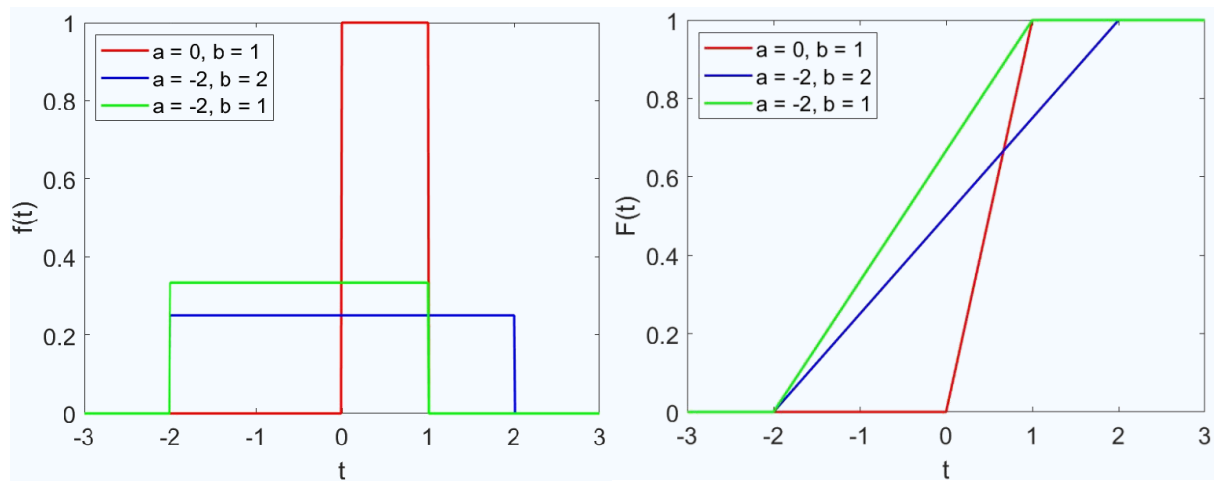

Figure 4: Plots of uniform PDFs [left] and plots of uniform CDFs [right].

2.6.2. Triangular Distribution Functions

Triangular distribution functions are slightly more complex than the aforementioned uniform distribution functions. Their PDF and CDF are shown in **Eq. 17**.

$$f(t) = \begin{cases} \frac{2(t-a)}{(c-a)(b-a)} & a \leq t \leq b \\ \frac{2(c-t)}{(c-a)(c-b)} & b \leq t \leq c \\ 0 & t < a, t > c \end{cases} \tag{17.1}$$

$$F(t) = \begin{cases} 0 & t < a \\ \frac{(t-a)^2}{(b-a)*(c-a)} & a \leq t \leq b \\ 1 - \frac{(c-t)^2}{(c-a)*(c-b)} & b \leq t \leq c \\ 1 & t > c \end{cases} \tag{17.2}$$

Where:
- **a**: Start of non-zero probability.
- **b**: Probability peak.
- **c**: End of non-zero probability.

The PDF and CDF for a triangular probability distribution are shown graphically in **Figure 5**.
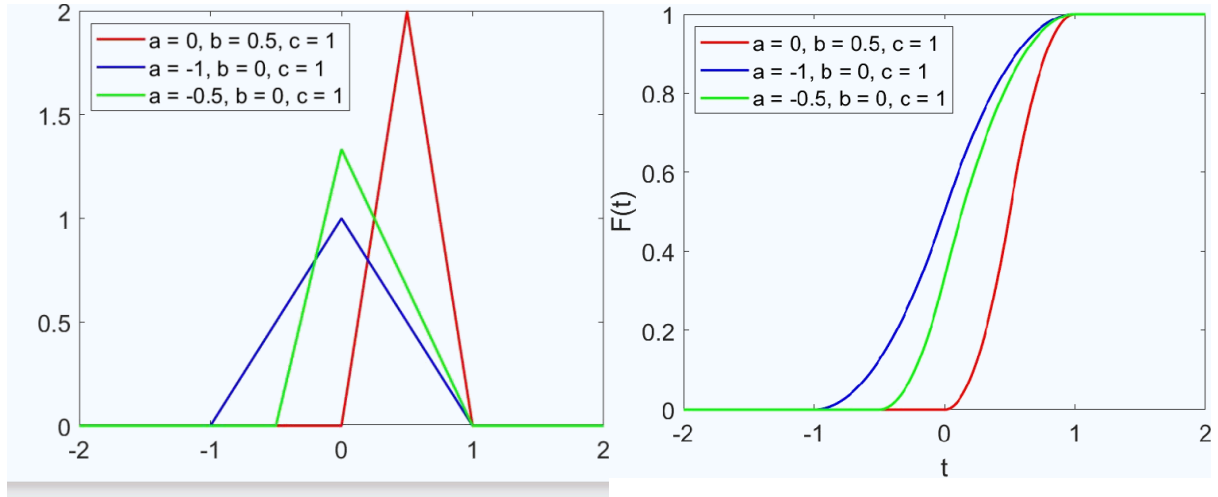
Figure 5: Plots of triangular PDFs [left] and plots of triangular CDFs [right].

2.6.3. Gaussian Distribution

Also known as the the **Normal Distribution** is the most commonly used probability distribution function. The PDF is shown in **Eq. 18** (**Note no CDF exists**).

$$f(t) = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right) \tag{18}$$

Where:
- μ: Mean (Location parameter).
- σ: Standard Deviation (Scaling parameter).

The PDF and CDF for a Gaussian probability distribution are shown graphically in **Figure 6**.
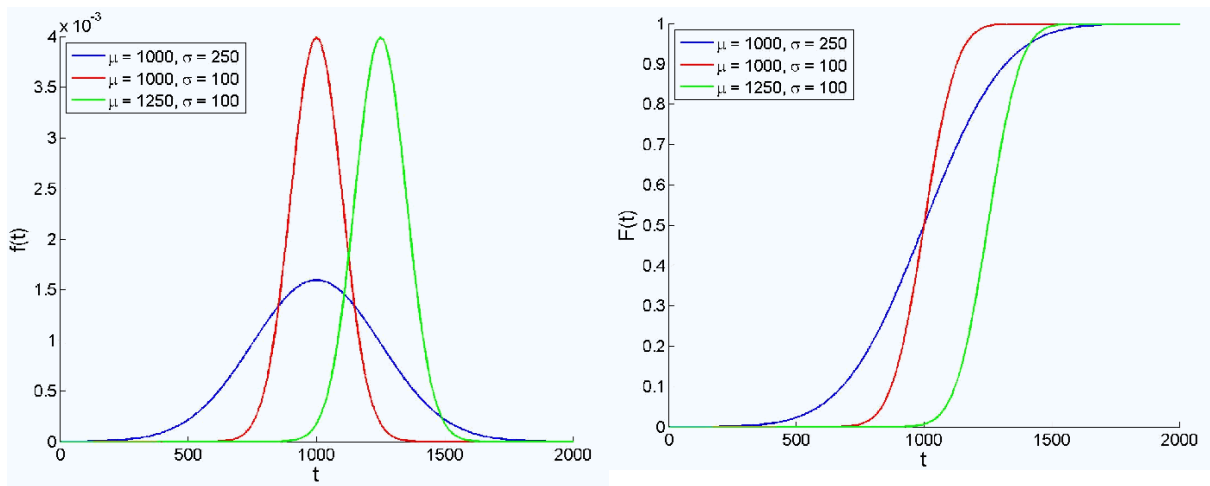

Figure 6: Plots of Gaussian PDFs [left] and plots of Gaussian CDFs [right].

The Gaussian distribution whilst being **symmetrical** also exhibits these key properties:

- **68.26%** of data is within 1 standard deviation of the mean(σ).
- **95.44%** of data is within 2 standard deviations of the mean (2σ).
- **99.74%** of data is within 3 standard deviations of the mean (3σ).

2.6.4. Log Normal Distribution

A more versatile version of the Gaussian distribution that is better suited at modelling reliability data. The PDf is shown in **Eq. 19** (**Note no CDF exists**).

$$f(t) = \begin{cases} \frac{1}{t\sigma(2\pi)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right) & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{19}$$

Where:
- μ: Mean (Location parameter).
- σ: Standard Deviation (Scaling parameter).

The PDF and CDF for a log normal probability distribution are shown graphically in **Figure 7**.
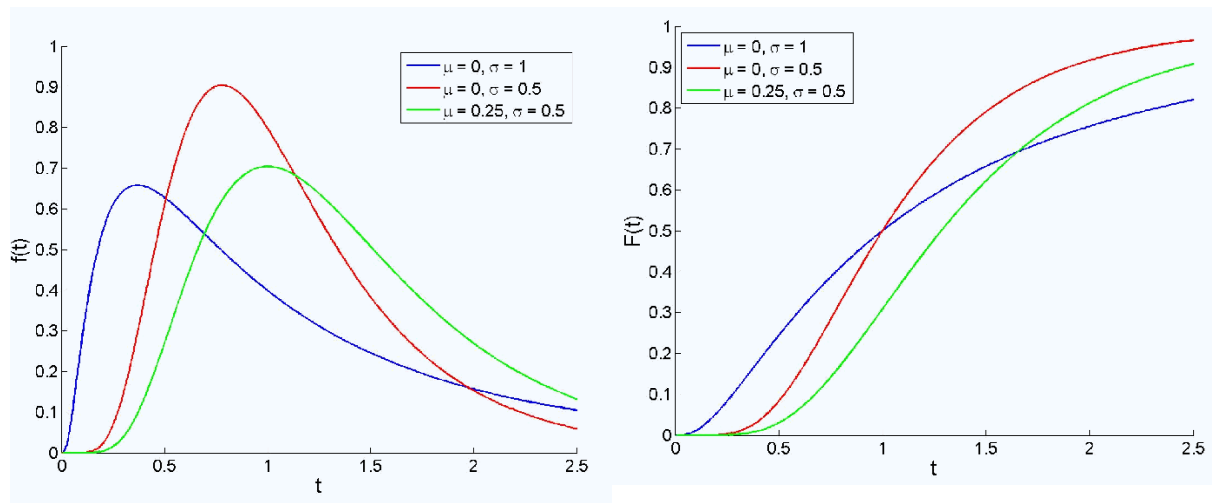

Figure 7: Plots of log normal PDFs [left] and plots of log normal CDFs [right].

2.6.5. Exponential Distribution

These distributions feature a **constant hazard rate** which is useful to model some processes. The PDf and CDF are shown mathematically in **Eq. 20**.

$$f(t) = \begin{cases} \lambda \exp(-\lambda t) & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{20.1}$$

$$F(t) = \begin{cases} 0 & t < 0 \\ 1 - \exp(-\lambda t) & t \geq 0 \end{cases} \tag{20.2}$$

Where:
- λ: Scaling parameter (Also is the constant hazard rate)

It is important to note that $1/\lambda$ is the mean time to failure (MTTF). The PDF and CDF for an exponential distribution are shown graphically in **Figure 8**.
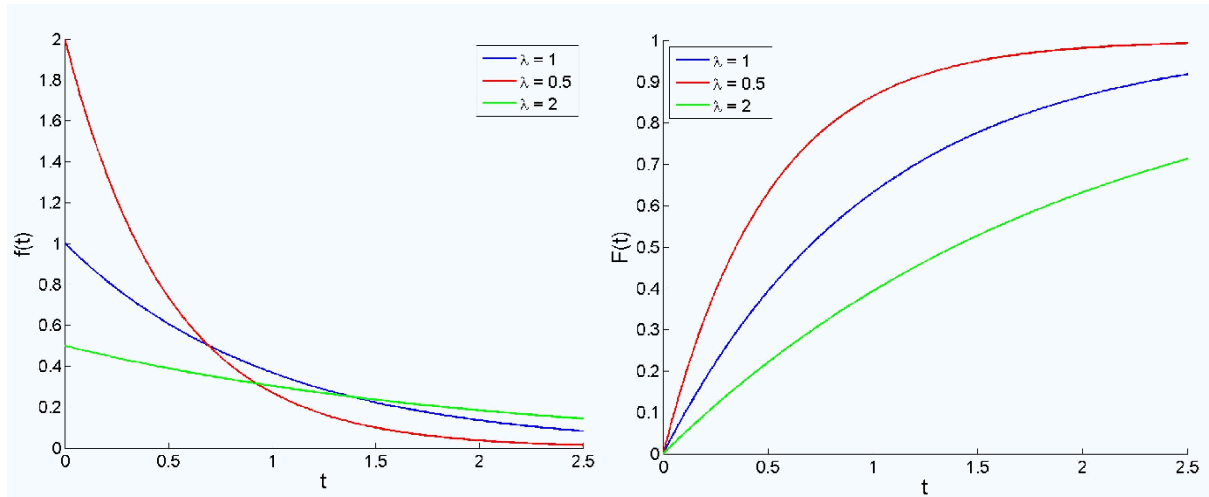
Figure 8: Plots of exponential PDFs [left] and plots of exponential CDFs [right].

2.6.6. Weibull Distribution

Is one of the most extensible and useful distributions out there, and can be used to model a lot of different distributions. The PDF, CDF and hazard rate are shown in **Eq. 21**.

$$f(t) = \begin{cases} \frac{\beta}{\eta^\beta} t^{\beta-1} \exp\left(-\left(\frac{t}{\eta}\right)^\beta\right) & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{21.1}$$

$$F(t) = 1 - \exp\left(-\left(\frac{t}{\eta}\right)^\beta\right) \qquad h(t) = \frac{\beta}{\eta^\beta} t^{\beta-1} \tag{21.2}$$

Where:

- β: Shape parameter
- η: Scaling parameter (characteristic life)

η is also the point at which 63.2% of the population have failed. Weibull distributions are so versatile as the β parameter changes the shape into different distributions:

- $\beta = 1$: Constant hazard function (exponential dist)
- $\beta < 1$: Decreasing hazard rate
- $\beta > 1$: Increasing hazard rate
- $\beta = 3.5$: Normal distribution

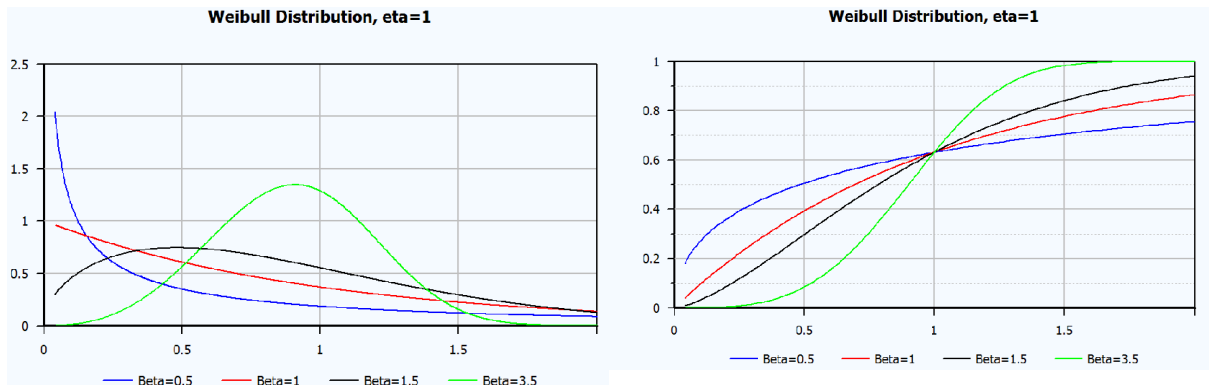The PDF and CDF for various Weibull distributions are shown graphically in **Figure 9**.

Figure 9: Plots of Weibull PDFs [left] and plots of Weibull CDFs [right] for various βs.

2.6.7. Three Parameter Weibull Distribution

Introduces a new parameter γ which is used to switch on the probability, its useful if the failures only start after a set time. The PDF, CDF and hazard rate are shown in **Eq. 22**.

$$f(t) = \begin{cases} \frac{\beta}{\eta^\beta}(t-\gamma)^{\beta-1} \exp\left(-\left(\frac{t-\gamma}{\eta}\right)^\beta\right) & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{22.1}$$

$$F(t) = 1 - \exp\left(-\left(\frac{t-\gamma}{\eta}\right)^\beta\right) \qquad h(t) = \frac{\beta}{\eta^\beta}(t-\gamma)^{\beta-1} \tag{22.2}$$

Where:

- β: Shape parameter
- η: Scaling parameter (characteristic life)
- γ: Location parameter (failure free time)

2.6.8. Other Distribution Functions (Non-Examinable)

Like was stated previously, there are an infin ite number of PDFs as the only criteria is for the area under the curve to sum to 1. Some other common distributions and their purposes are mentioned below:

- **Rayleigh Distribution**: Similar to exponential but with a linearly increasing hazard rate.
- **Gamma Distribution**: Similar to Weibull in that it can model a wide number of distributions by varying the parameters.
- **Beta Distribution**: A complex distribution which uses multiple gamma distributions to ensure that the life is limited to a set interval.
- **Inverse Gamma Distribution**
- **Log-logistic Distribution**
- **Birnbaum-Saunders Distribution**

2.7. Discrete Distributions

Whereas continuous distributions can model the probability over time, discrete distributions model the probability per an n number of events, some common discrete distributions are shown below.

2.7.1. Binomial Distribution

Used where the outcome of each discrete event is either pass or fail, the PDF for a binomial distribution function is defined by **Eq. 23**.

$$f(x) = \binom{n}{x} p^x q^{n-x} \tag{23.1}$$

$$\frac{n!}{x!(n-x)!} \equiv \binom{n}{x} \tag{23.2}$$

Where:

- x: The number of passes

- n: Total number of trails
- p: Probability of success
- q: Probability of failure
- $\binom{n}{x}$: Binomial coefficient

Note that the **binomial coefficient** is a parameter that will appear often and is read as "n choose x". Usefully, it also represents the **number of possible combinations of n from x**.

2.7.2. Other Discrete Distributions

Some other commonly used discrete distributions are:

- **Poisson's Distribution**: Represents an event occurring at a constant rate and can approximate the binomial distribution.
- **Hypergeometric Distribution** Models the probability if there are no replacements.