SESM6047

Finite Element Analysis in Solid Mechanics

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0. Lecture 0

0.1. Matrix Terminology

A matrix is a structured way of organizing data in a rectangular array with rows and columns. For this module the notation of a matrix and reference to matrix terms is shown in **Eq. 1**.

$$[K]_{i\times j} = \begin{bmatrix} k_{11} & k_{12} & ... & k_{1j} \\ k_{21} & k_{22} & ... & k_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ k_{i1} & k_{i2} & ... & k_{ij} \end{bmatrix}_{i\times j} \tag{1}$$

Where:

- $\boldsymbol{[K]_{i\times j}}$: The matrix $[K]$ (Note for this module a matrix will be defined by the [] notation).
- $\boldsymbol{i}$: The row index (indexed from 1).
- $\boldsymbol{j}$: The column index (indexed from 2).
- $\boldsymbol{i \times j}$: The shape of the matrix (i number of rows, j number of columns).
- $\boldsymbol{k_{ij}}$: The item in the ith row and the jth column.

A special type of commonly used matrix is a **diagonal matrix** shown in **Eq. 2**. This type of matrix only features terms that sit on the indexes where the row index is equal to the column index ($i = j$).

$$[K]_{i\times j} = \begin{bmatrix} k_{11} & 0 & ... & 0 \\ 0 & k_{22} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & k_{ij} \end{bmatrix}_{i\times j} \tag{2}$$

0.2. Vector Terminology

A vector is a quantity that has a **magnitude** as well as a **direction**. A vector is like one column out of a matrix, the number of terms signifies the dimensions of a vector, the notation used in this module for a vector is shown in **Eq. 3**.

$$\{q\}_{i\times 1} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_i \end{Bmatrix}_{i\times 1} \tag{3}$$

0.3. Matrix Addition and Subtraction

Matrixes with the same shape can be added or subtracted. Matrix addition or subtraction is a **commutative** property and the process to do so is shown in **Eq. 4**.

$$[A]_{i\times j} \pm [B]_{i\times j} \tag{4.1}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} \end{bmatrix}_{i\times j} \pm \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} \\ b_{21} & b_{22} & \dots & b_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ b_{i1} & b_{i2} & \dots & b_{ij} \end{bmatrix}_{i\times j} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \dots & a_{1j} \pm b_{1j} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \dots & a_{2j} \pm b_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} \pm b_{i1} & a_{i2} \pm b_{i2} & \dots & a_{ij} \pm b_{ij} \end{bmatrix}_{i\times j} \tag{4.2}$$

$$[A]_{i\times j} \pm [B]_{i\times j} = [B]_{i\times j} \pm [A]_{i\times j} \tag{4.3}$$

0.4. Matrix Multiplication

0.4.1. Scalar × Matrix Multiplication

A scalar can be multiplied with a matrix or vector of any shape, the scalar value is multiplied with every term within the matrix, as shown in **Eq. 5**.

$$k[A]_{i\times j} = k \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} \end{bmatrix}_{i\times j} = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1j} \\ ka_{21} & ka_{22} & \dots & ka_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{i1} & ka_{i2} & \dots & ka_{ij} \end{bmatrix}_{i\times j} \tag{5}$$

0.4.2. Matrix × Matrix Multiplication

Any two matrices can be multiplied as long as **the number of columns of the first matrix is the same as the number of rows in the second matrix**. Given this criteria is satisfied the formula to then multiply two matrices together is shown in **Eq. 6**.

$$[A]_{i\times k}[B]_{k\times j} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ik} \end{bmatrix}_{i\times k} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} \\ b_{21} & b_{22} & \dots & b_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \dots & b_{kj} \end{bmatrix}_{k\times j} \tag{6.1}$$

$$= \begin{bmatrix} \sum_{r:1}^{k}(a_{1r}b_{r1}) & \sum_{r:1}^{k}(a_{1r}b_{r2}) & \dots & \sum_{r:1}^{k}(a_{1r}b_{rj}) \\ \sum_{r:1}^{k}(a_{2r}b_{r1}) & \sum_{r:1}^{k}(a_{2r}b_{r2}) & \dots & \sum_{r:1}^{k}(a_{2r}b_{rj}) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{r:1}^{k}(a_{ir}b_{r1}) & \sum_{r:1}^{k}(a_{ir}b_{r2}) & \dots & \sum_{r:1}^{k}(a_{ir}b_{rj}) \end{bmatrix}_{i\times j} \tag{6.2}$$

Note that matrix multiplication has some key properties associated with it, these are:

- **Non commutable** ($[A]_{i\times k}[B]_{k\times j} \neq [B]_{k\times j}[A]_{i\times k}$).
- **Associative** ($([A]_{i\times k}[B]_{k\times l})[C]_{l\times j} = [A]_{i\times k}([B]_{k\times l}[C]_{l\times j})$)
- **Distributable** $[A]_{i\times k}\big([B]_{k\times j} + [C]_{k\times j}\big) = [A]_{i\times k}[B]_{k\times j} + [A]_{i\times k}[C]_{k\times j}$

0.5. Identity Matrix

The identity matrix is a special matrix with some key and is is defined in **Eq. 7**.

$$I_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad [I]_{i\times j} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{i\times j} \tag{7}$$

Identity matrices **are** commutable and multiplication by an appropriately sized identity matrix will not itself change the value of the matrix, these properties are shown in.

$$[I]_{i\times i}[K]_{i\times j} = [K]_{i\times j} = [K]_{i\times j}[I]_{j\times j} \tag{8}$$

0.6. Matrix Transposition

The transpose of a matrix is signified by the T symbol and will flip the matrix along the diagonal. The generalized expression for transposition is shown in **Eq. 9**.

$$[A]_{i\times j} = \left[a_{ij}\right] \quad \rightarrow \quad [A]^T_{j\times i} = \left[a_{ji}\right] \quad \text{where} \quad \left([A]^T\right)_{ij} = a_{ji} \tag{9}$$

For example the transpose of a general 3×2 matrix is shown in **Eq. 10**.

$$[A]_{3\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2} \quad \rightarrow \quad [A]^T_{2\times 3} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}_{2\times 3} \tag{10}$$

0.7. Matrix Determinant

The determinant of a matrix is a scalar-valued function of a **square matrix** (matrix must be square for there to exist a determinant). Determinants are symbolized via a | and the general form of a determinant is shown in **Eq. 11**.

$$\text{let } [A]_{n\times n} = \left[a_{ij}\right] \text{ then } \quad \det([A]) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det\left(M_{ij}\right) \tag{11.1}$$

$$\text{where } M_{ij} \text{ is the minor of } a_{ij} \tag{11.2}$$

Determinants can be long and repetitive to calculate for larger matrices, the determinant for a 2×2 matrix is shown in **Eq. 12**.

$$[A]_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2} \quad \rightarrow \quad |[A]_{2\times 2}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{12}$$

Just for completion, the determinant for a 3×3 matrix is shown in **Eq. 13**.

$$[A]_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \quad \rightarrow \quad |[A]_{3\times 3}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{13.1}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{13.2}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \tag{13.3}$$

0.8. Minor of an Element, Adjoint Matrix and Cofactor Matrix

The **adjoint** or **adjugate** matrix is a useful mathematical tool which will eventually allow us to be able to form the inverse of a matrix. Before defining the adjoint matrix, the minor of an element and the cofactor matrix must first be understood.

0.8.1. Minor of an Element

To find the minor of a given element, **the ith row and jth column of that element are deleted** and the **determinant of the remaining submatrix is the minor.** As an example, the minor of the a_{12} element in a 3×3 matrix is shown in **Eq. 14**.

$$[A]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \quad \rightarrow \quad M_{12} = \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23} \tag{14}$$

0.8.2. Cofactor Matrix

The cofactor matrix utilizes the minor of every element within a matrix to form a new matrix, the general mathematical definition for what a cofactor matrix is, is shown in **Eq. 15**.

$$\mathrm{Cof}\big([A]_{i\times j}\big) = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1j} \\ C_{21} & C_{22} & \dots & C_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ C_{i1} & C_{i2} & \dots & C_{ij} \end{bmatrix}_{i\times j} \qquad \text{Where} \quad C_{ij} = (-1)^{i+j} \det\big(M_{ij}\big) \tag{15}$$

Note that this equation is effectively calculating the minor of every element within a matrix and then adding the sign in a checkerboard form as shown in **Eq. 16**.

$$\begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{16}$$

The cofactor matrix for a general 2×2 matrix is shown in **Eq. 17**

$$[A]_{2\times2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2} \tag{17.1}$$

$$\mathrm{Cof}([A])_{2\times2} = \begin{bmatrix} +M_{11} & -M_{12} \\ -M_{21} & +M_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}_{2\times2} \tag{17.2}$$

The cofactor matrix for a general 3×3 matrix is shown in **Eq. 18**

$$[A]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \tag{18.1}$$

$$\mathrm{Cof}([A])_{3\times3} = \begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}_{3\times3} = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}_{3\times3} \tag{18.2}$$

$$= \begin{bmatrix} +(a_{22}a_{33} - a_{23}a_{32}) & -(a_{21}a_{33} - a_{23}a_{31}) & +(a_{21}a_{32} - a_{22}a_{31}) \\ -(a_{12}a_{33} - a_{13}a_{32}) & +(a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) \\ +(a_{12}a_{23} - a_{13}a_{22}) & -(a_{11}a_{23} - a_{13}a_{21}) & +(a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}_{3\times3} \tag{18.3}$$

0.8.3. Adjoint Matrix

To finish off, the adjoint matrix is just the **transposition of the cofactor matrix**, the general mathematical definition of teh adjoint matrix is shown in **Eq. 19**. Note that the adjoint and cofactor operations can only be done on **square matrices**.

$$\mathrm{Adj}\big([A_{i\times j}]\big) = \mathrm{Cof}\big([A_{i\times j}]\big)^T \tag{19}$$

For completeness the adjoint of a general 2×2 matrix is shown in **Eq. 20** and for a 3×3 matrix in **Eq. 21**.

$$[A]_{2\times2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2} \quad \rightarrow \quad \mathrm{Adj}([A]_{2\times2}) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}_{2\times2} \tag{20}$$

$$[A]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \tag{21.1}$$

$$\mathrm{Adj}([A]_{3\times3}) = \begin{bmatrix} (a_{22}a_{33} - a_{23}a_{32}) & -(a_{12}a_{33} - a_{13}a_{32}) & (a_{12}a_{23} - a_{13}a_{22}) \\ -(a_{21}a_{33} - a_{23}a_{31}) & (a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{23} - a_{13}a_{21}) \\ (a_{21}a_{32} - a_{22}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) & (a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}_{3\times3} \tag{21.2}$$

0.9. Inverse Matrix

The inversion of a matrix is a key tool within linear algebra and will act as a backbone for FEA in general. The mathematical definition to invert a matrix is shown in **Eq. 22**.

$$[A]^{-1}_{i\times j} = \frac{1}{|[A]_{i\times j}|}\mathrm{Adj}\big([A]_{i\times j}\big) \tag{22}$$