

SESM6047

Finite Element Analysis in Solid Mechanics

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Definitions

Π	Total potential energy (J)	U	Stored elastic strain energy(J)
V	Potential energy (J)	F	Force (N)
k	Stiffness (N/m)	q	Generalized degree of freedom
$[K]$	Stiffness matrix	$\{q\}$	Displacement vector

0. Lecture 0

0.1. Matrix Terminology

A matrix is a structured way of organizing data in a rectangular array with rows and columns. For this module the notation of a matrix and reference to matrix terms is shown in **Eq. 1**.

$$[K]_{i \times j} = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1j} \\ k_{21} & k_{22} & \dots & k_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ k_{i1} & k_{i2} & \dots & k_{ij} \end{bmatrix}_{i \times j} \quad (1)$$

Where:

- $[K]_{i \times j}$: The matrix $[K]$ (Note for this module a matrix will be defined by the $[]$ notation).
- i : The row index (indexed from 1).
- j : The column index (indexed from 2).
- $i \times j$: The shape of the matrix (i number of rows, j number of columns).
- k_{ij} : The item in the i th row and the j th column.

A special type of commonly used matrix is a **diagonal matrix** shown in **Eq. 2**. This type of matrix only features terms that sit on the indexes where the row index is equal to the column index ($i = j$).

$$[K]_{i \times j} = \begin{bmatrix} k_{11} & 0 & \dots & 0 \\ 0 & k_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k_{ij} \end{bmatrix}_{i \times j} \quad (2)$$

0.2. Vector Terminology

A vector is a quantity that has a **magnitude** as well as a **direction**. A vector is like one column out of a matrix, the number of terms signifies the dimensions of a vector, the notation used in this module for a vector is shown in **Eq. 3**.

$$\{q\}_{i \times 1} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_i \end{Bmatrix}_{i \times 1} \quad (3)$$

0.3. Matrix Addition and Subtraction

Matrixes with the same shape can be added or subtracted. Matrix addition or subtraction is a **commutative** property and the process to do so is shown in **Eq. 4**.

$$[A]_{i \times j} \pm [B]_{i \times j} \quad (4.1)$$

$$= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} \end{bmatrix}_{i \times j} \pm \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} \\ b_{21} & b_{22} & \dots & b_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ b_{i1} & b_{i2} & \dots & b_{ij} \end{bmatrix}_{i \times j} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \dots & a_{1j} \pm b_{1j} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \dots & a_{2j} \pm b_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} \pm b_{i1} & a_{i2} \pm b_{i2} & \dots & a_{ij} \pm b_{ij} \end{bmatrix}_{i \times j} \quad (4.2)$$

$$[A]_{i \times j} \pm [B]_{i \times j} = [B]_{i \times j} \pm [A]_{i \times j} \quad (4.3)$$

0.4. Matrix Multiplication

0.4.1. Scalar \times Matrix Multiplication

A scalar can be multiplied with a matrix or vector of any shape, the scalar value is multiplied with every term within the matrix, as shown in **Eq. 5**.

$$k[A]_{i \times j} = k \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} \end{bmatrix}_{i \times j} = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1j} \\ ka_{21} & ka_{22} & \dots & ka_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{i1} & ka_{i2} & \dots & ka_{ij} \end{bmatrix}_{i \times j} \quad (5)$$

0.4.2. Matrix \times Matrix Multiplication

Any two matrices can be multiplied as long as **the number of columns of the first matrix is the same as the number of rows in the second matrix**. Given this criteria is satisfied the formula to then multiply two matrices together is shown in **Eq. 6**.

$$[A]_{i \times k} [B]_{k \times j} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ik} \end{bmatrix}_{i \times k} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} \\ b_{21} & b_{22} & \dots & b_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \dots & b_{kj} \end{bmatrix}_{k \times j} \quad (6.1)$$

$$= \begin{bmatrix} \sum_{r=1}^k (a_{1r} b_{r1}) & \sum_{r=1}^k (a_{1r} b_{r2}) & \dots & \sum_{r=1}^k (a_{1r} b_{rj}) \\ \sum_{r=1}^k (a_{2r} b_{r1}) & \sum_{r=1}^k (a_{2r} b_{r2}) & \dots & \sum_{r=1}^k (a_{2r} b_{rj}) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{r=1}^k (a_{ir} b_{r1}) & \sum_{r=1}^k (a_{ir} b_{r2}) & \dots & \sum_{r=1}^k (a_{ir} b_{rj}) \end{bmatrix}_{i \times j} \quad (6.2)$$

Note that matrix multiplication has some key properties associated with it, these are:

- **Non commutable** $([A]_{i \times k} [B]_{k \times j} \neq [B]_{k \times j} [A]_{i \times k})$.
- **Associative** $([A]_{i \times k} [B]_{k \times l} [C]_{l \times j} = [A]_{i \times k} ([B]_{k \times l} [C]_{l \times j}))$
- **Distributable** $[A]_{i \times k} ([B]_{k \times j} + [C]_{k \times j}) = [A]_{i \times k} [B]_{k \times j} + [A]_{i \times k} [C]_{k \times j}$

0.5. Identity Matrix

The identity matrix is a special matrix with some key and is defined in **Eq. 7**.

$$I_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad [I]_{i \times j} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{i \times j} \quad (7)$$

Identity matrices **are** commutable and multiplication by an appropriately sized identity matrix will not itself change the value of the matrix, these properties are shown in.

$$[I]_{i \times i} [K]_{i \times j} = [K]_{i \times j} = [K]_{i \times j} [I]_{j \times j} \quad (8)$$

0.6. Matrix Transposition

The transpose of a matrix is signified by the T symbol and will flip the matrix along the diagonal. The generalized expression for transposition is shown in **Eq. 9**.

$$[A]_{i \times j} = [a_{ij}] \rightarrow [A]_{j \times i}^T = [a_{ji}] \quad \text{where} \quad ([A]^T)_{ij} = a_{ji} \quad (9)$$

For example the transpose of a general 3×2 matrix is shown in **Eq. 10**.

$$[A]_{3 \times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3 \times 2} \rightarrow [A]_{2 \times 3}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}_{2 \times 3} \quad (10)$$

0.7. Matrix Determinant

The determinant of a matrix is a scalar-valued function of a **square matrix** (matrix must be square for there to exist a determinant). Determinants are symbolized via a $|$ and the general form of a determinant is shown in **Eq. 11**.

$$\text{let } [A]_{n \times n} = [a_{ij}] \text{ then } \det([A]) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(M_{ij}) \quad (11.1)$$

$$\text{where } M_{ij} \text{ is the minor of } a_{ij} \quad (11.2)$$

Determinants can be long and repetitive to calculate for larger matrices, the determinant for a 2×2 matrix is shown in **Eq. 12**.

$$[A]_{2 \times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2} \rightarrow |[A]_{2 \times 2}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \quad (12)$$

Just for completion, the determinant for a 3×3 matrix is shown in **Eq. 13**.

$$[A]_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3} \rightarrow |[A]_{3 \times 3}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad (13.1)$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad (13.2)$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \quad (13.3)$$

0.8. Minor of an Element, Adjoint Matrix and Cofactor Matrix

The **adjoint** or **adjugate** matrix is a useful mathematical tool which will eventually allow us to be able to form the inverse of a matrix. Before defining the adjoint matrix, the minor of an element and the cofactor matrix must first be understood.

0.8.1. Minor of an Element

To find the minor of a given element, **the i th row and j th column of that element are deleted** and the **determinant of the remaining submatrix is the minor**. As an example, the minor of the a_{12} element in a 3×3 matrix is shown in **Eq. 14**.

$$[A]_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3} \rightarrow M_{12} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23} \quad (14)$$

0.8.2. Cofactor Matrix

The cofactor matrix utilizes the minor of every element within a matrix to form a new matrix, the general mathematical definition for what a cofactor matrix is, is shown in **Eq. 15**.

$$\text{Cof}([A]_{i \times j}) = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1j} \\ C_{21} & C_{22} & \dots & C_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ C_{i1} & C_{i2} & \dots & C_{ij} \end{bmatrix}_{i \times j} \quad \text{Where } C_{ij} = (-1)^{i+j} \det(M_{ij}) \quad (15)$$

Note that this equation is effectively calculating the minor of every element within a matrix and then adding the sign in a checkerboard form as shown in **Eq. 16**.

$$\begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (16)$$

The cofactor matrix for a general 2×2 matrix is shown in **Eq. 17**

$$[A]_{2 \times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2} \quad (17.1)$$

$$\text{Cof}([A])_{2 \times 2} = \begin{bmatrix} +M_{11} & -M_{12} \\ -M_{21} & +M_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}_{2 \times 2} \quad (17.2)$$

The cofactor matrix for a general 3×3 matrix is shown in **Eq. 18**

$$[A]_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3} \quad (18.1)$$

$$\text{Cof}([A])_{3 \times 3} = \begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}_{3 \times 3} = \begin{bmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}_{3 \times 3} \quad (18.2)$$

$$= \begin{bmatrix} +(a_{22}a_{33} - a_{23}a_{32}) & -(a_{21}a_{33} - a_{23}a_{31}) & +(a_{21}a_{32} - a_{22}a_{31}) \\ -(a_{12}a_{33} - a_{13}a_{32}) & +(a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) \\ +(a_{12}a_{23} - a_{13}a_{22}) & -(a_{11}a_{23} - a_{13}a_{21}) & +(a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}_{3 \times 3} \quad (18.3)$$

0.8.3. Adjoint Matrix

To finish off, the adjoint matrix is just the **transposition of the cofactor matrix**, the general mathematical definition of the adjoint matrix is shown in **Eq. 19**. Note that the adjoint and cofactor operations can only be done on **square matrices**.

$$\text{Adj}([A]_{i \times j}) = \text{Cof}([A]_{i \times j})^T \quad (19)$$

For completeness the adjoint of a general 2×2 matrix is shown in **Eq. 20** and for a 3×3 matrix in **Eq. 21**.

$$[A]_{2 \times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2} \rightarrow \text{Adj}([A]_{2 \times 2}) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}_{2 \times 2} \quad (20)$$

$$[A]_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3} \quad (21.1)$$

$$\text{Adj}([A]_{3 \times 3}) = \begin{bmatrix} (a_{22}a_{33} - a_{23}a_{32}) & -(a_{12}a_{33} - a_{13}a_{32}) & (a_{12}a_{23} - a_{13}a_{22}) \\ -(a_{21}a_{33} - a_{23}a_{31}) & (a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{23} - a_{13}a_{21}) \\ (a_{21}a_{32} - a_{22}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) & (a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}_{3 \times 3} \quad (21.2)$$

0.9. Inverse Matrix

The inversion of a matrix is a key tool within linear algebra and will act as a backbone for FEA in general. The mathematical definition to invert a matrix is shown in **Eq. 22**.

$$[A]_{i \times j}^{-1} = \frac{1}{|[A]_{i \times j}|} \text{Adj}([A]_{i \times j}) \quad (22)$$

As **Eq. 22** uses the reciprocal of the determinant of the matrix, it also means that the **value of the determinant cannot itself be zero**. A matrix with a determinant of zero is said to be **singular**, **degenerate** or **rank deficient**. Examples for the inverses of a 2×2 matrix and a 3×3 matrix are shown in **Eq. 23** and **Eq. 24** respectively.

$$[A]_{2 \times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2} \rightarrow [A]_{2 \times 2}^{-1} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}_{2 \times 2} \quad (23)$$

$$[A]_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3} \quad (24.1)$$

$$[A]_{3 \times 3}^{-1} = \frac{1}{(a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}))} \dots \quad (24.2)$$

$$\dots \begin{bmatrix} (a_{22}a_{33} - a_{23}a_{32}) & -(a_{12}a_{33} - a_{13}a_{32}) & (a_{12}a_{23} - a_{13}a_{22}) \\ -(a_{21}a_{33} - a_{23}a_{31}) & (a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{23} - a_{13}a_{21}) \\ (a_{21}a_{32} - a_{22}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) & (a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}_{3 \times 3} \quad (24.3)$$

0.10. Solving Systems of Equations Using matrices

Using the steps outlined before, it is possible to use matrices to solve a system of simultaneous equations. Lets say we have n **linear** equations, and therefore n number of unknowns, then the system of equations can be written in the form shown in **Eq. 25**.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases} \quad (25)$$

The system of equations shown in **Eq. 25** can then be written in the matrix form shown in **Eq. 26**.

$$[A]_{n \times n} [X]_{n \times 1} = [B]_{n \times 1} \quad (26.1)$$

$$\text{where } [A]_{n \times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n \times n} \quad [X]_{n \times 1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \quad [B]_{n \times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n \times 1} \quad (26.2)$$

Assuming that the $[X]_{n \times 1}$ matrix is the set of unknowns, then the solution of this system can be found by inverting the $[A]_{n \times n}$ matrix (this can be done as $[A]_{n \times n} [A]_{n \times n}^{-1} = [A]_{n \times n}^{-1} [A]_{n \times n} = [I]_{n \times n}$), the form of this expression is shown in.

$$[X]_{n \times 1} = [B]_{n \times 1} [A]_{n \times n}^{-1} \quad \text{Given } |[A]_{n \times n}| \neq 0 \quad (27)$$

0.11. Matrix Zero-Padding

It is possible to use many of the aforementioned tools to “pad” a matrix with zeroes, with some degree of control. Let there be a submatrix $[K]_{n \times n}$ that we want to expand into a new matrix \bar{K} to a size $m \times m$, with the K matrix being in the top left of the \bar{K} matrix, then the multiplication shown in can be used **Eq. 28**.

$$[K]_{n \times n} = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1j} \\ k_{21} & k_{22} & \dots & k_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ k_{i1} & k_{i2} & \dots & k_{ij} \end{bmatrix}_{n \times n} \rightarrow [\bar{K}]_{m \times m} = \begin{bmatrix} k_{11} & k_{12} & \dots & 0 \\ k_{21} & k_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{m \times m} \quad (28.1)$$

$$\text{Then } [A] \text{ must have shape } [A]_{n \times m} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{n \times m} \quad (28.2)$$

$$\therefore [\bar{K}]_{m \times m} = [A]_{m \times n}^T [K]_{n \times n} [A]_{n \times m} \quad (28.3)$$

As an example a 2×2 matrix is embedded into a 5×5 matrix using **Eq. 28**, this is shown in **Eq. 29**.

$$[K]_{2 \times 2} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}_{2 \times 2} \quad [A]_{2 \times 5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}_{2 \times 5} \quad (29.1)$$

$$[\bar{K}]_{5 \times 5} = [A]_{5 \times 2}^T [K]_{2 \times 2} [A]_{2 \times 5} \quad (29.2)$$

$$[\bar{K}]_{5 \times 5} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{5 \times 2} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}_{2 \times 2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}_{2 \times 5} = \begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 \\ k_{21} & k_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5 \times 5} \quad (29.3)$$

Note that the placement of the $[K]$ matrix is dependant on where the ones are within the $[A]$ matrix. The further along the array the cascading ones are, the further down the diagonal the inserted $[K]$ matrix is. This process can be used to make a **banded matrix**, which is shown in **Eq. 30**.

$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 \\ k_{21} & k_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5 \times 5} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k_{11} & k_{12} & 0 & 0 \\ 0 & k_{21} & k_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5 \times 5} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{11} & k_{12} & 0 \\ 0 & 0 & k_{21} & k_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5 \times 5} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{11} & k_{12} \\ 0 & 0 & 0 & k_{21} & k_{22} \end{bmatrix}_{5 \times 5} \quad (30.1)$$

$$= \begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 \\ k_{21} & k_{22} + k_{11} & k_{12} & 0 & 0 \\ 0 & k_{21} & k_{22} + k_{11} & k_{12} & 0 \\ 0 & 0 & k_{21} & k_{22} + k_{11} & k_{12} \\ 0 & 0 & 0 & k_{21} & k_{22} \end{bmatrix}_{5 \times 5} \quad (30.2)$$

0.12. Matrix Cutting

In a similar way to how a matrix can be padded, it can also be cut to reduce the shape. This process has the following steps:

1. Start with an identity matrix the same size as the matrix to shorten.
2. Delete the columns you want to remove from the main matrix, keeping it square (this also will delete the rows with the same index).
3. Pre-multiply by the transpose of this reduced identity matrix and post-multiply by the original matrix to obtain the reduced form.

Eq. 31 Is an example of this where the 2nd, 3rd and 4th rows are removed from $[B]$.

$$[B]_{5 \times 5} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}_{5 \times 5} \quad (31.1)$$

$$[I]_{5 \times 5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5 \times 5} \rightarrow [C]_{5 \times 3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5 \times 5} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{5 \times 2} \quad (31.2)$$

$$[\bar{B}]_{2 \times 2} = [C]_{2 \times 5}^T [B]_{5 \times 5} [C]_{5 \times 3} \quad (31.3)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{2 \times 5} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}_{5 \times 5} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{5 \times 2} = \begin{bmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{bmatrix}_{2 \times 2} \quad (31.4)$$

0.13. Vector Insertion and Cutting

0.13.1. Vector Insertion

Vector insertion is easier than matrices padding, the following steps are used:

- 1) Define a matrix with zeros that has the following properties:
 - a) Number of rows equals the final major dimension of the vector
 - b) Number of columns equals the current major dimension of the vector
- 2) Add ones within the columns to specify where the zeroes will be inserted (see **Eq. 32**).
- 3) Pre-multiply by this matrix to obtain the modified column vector.

An example of this is shown in **Eq. 32** where zeroes are added before and after q_1 yielding the modified $\{\bar{q}\}_{5 \times 1}$.

$$\{q\}_{3 \times 1} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3 \times 1} \quad [C]_{5 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{5 \times 3} \quad (32.1)$$

$$\{\bar{q}\}_{5 \times 1} = [C]_{5 \times 3} \{q\}_{3 \times 1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{5 \times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3 \times 1} = \begin{Bmatrix} 0 \\ q_1 \\ 0 \\ q_2 \\ q_3 \end{Bmatrix}_{5 \times 1} \quad (32.2)$$

0.13.2. Vector Cutting

This process is similar to insertion with the key difference being pre-multiplication by a differently shaped $[C]$ matrix:

- 1) Define a matrix with zeros that has the following properties:
 - a) Number of **columns** equals the final major dimension of the vector
 - b) Number of **rows** equals the current major dimension of the vector
- 2) Add ones within the columns to specify which values will be kept after the cut (see **Eq. 33**).
- 3) Pre-multiply by this matrix to obtain the modified column vector.

An example of this is shown in **Eq. 33** where the size of the vector is reduced by removing the first and third terms.

$$\{q\}_{5 \times 1} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{Bmatrix}_{5 \times 1} \quad [C]_{5 \times 3} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{3 \times 5} \quad (33.1)$$

$$\{\bar{q}\}_{3 \times 1} = [C]_{3 \times 5} \{q\}_{5 \times 1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{3 \times 5} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{Bmatrix}_{5 \times 1} = \begin{Bmatrix} q_2 \\ q_4 \\ q_5 \end{Bmatrix}_{3 \times 1} \quad (33.2)$$

0.14. Eigenvalues and Eigenvectors

Eigenvectors are vectors with a key property, multiplying them with a matrix yields a vector with the same direction but with an altered magnitude (scales the vector). Mathematically we can write this in the form shown in **Eq. 34**.

$$[A]_{n \times n} \{X\}_{n \times 1} = \lambda \{X\}_{n \times 1} \quad (34)$$

Eq. 35 can be used to find the value of the eigenvalues, which will then allow for independent solutions for the eigenvectors.

$$|[A]_{n \times n} - \lambda [I]_{n \times n}| = 0 \quad (35)$$

Note that the determinant exists here as the solution to this equation itself has to be singular. An example calculation of the eigenvalues for a 2×2 matrix is shown in **Eq. 36**.

$$[A]_{n \times n} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \rightarrow |[A]_{n \times n} - \lambda [I]_{n \times n}| = 0 \quad (36.1)$$

$$\left| \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 4 - \lambda & 1 \\ 1 & 3 - \lambda \end{vmatrix} = (4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0 \quad (36.2)$$

$$\lambda_1 = 5 \quad \lambda_2 = 2 \quad (36.3)$$

The results for the eigenvalues can be plugged back into the following formula to yield the eigenvectors **Eq. 37**.

$$([A]_{n \times n} - \lambda [I]_{n \times n})[X]_{n \times 1} = 0 \quad (37)$$

1. Lecture 1

START OF WEEK 1

1.1. Principle of Minimum Total Potential Energy (PMTPE)

The principle of minimum total potential energy is a concept that underpins the whole of FEA. It states that a structure will deform or displace to a position which minimizes the total potential energy of that structure. To build up to a mathematical definition of PMTPE, the total potential energy can be written as **Eq. 38**.

$$\Pi = U + V \quad (38)$$

Where:

- Π : The total potential energy (J).
- U : The stored elastic strain energy (J).
- V : The potential energy (negative of the work done $-W$ by external forces) (J).

PMTE then states that if we have an infinite number of possible responses to a system, the correct solution will have the lowest energy out of the infinite solutions. Mathematically, we are looking for a minimum for the value of Π which yields the **Eq. 39**.

$$\delta\Pi = 0 \quad \therefore \quad U = -V \quad (39)$$

1.2. Deriving the Deformation Behavior of Spring

We can apply PMTPE to a 1D spring in order to yield the equation $F = kq$. To start the scenario is defined in **Figure 1**.

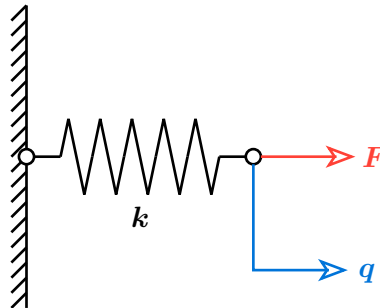


Figure 1: General 1D spring attached to wall.

Where:

- k : Spring stiffness ($\frac{N}{m}$)
- F : Force (N)
- q : Displacement (m)

Note that q is described as a **generalized coordinate** or a **degree of freedom** (DoF). From the scenario defined in **Figure 1**, an expression for the total potential energy of the system can be generated, shown in **Eq. 40**.

$$U = \frac{1}{2}kq^2 \quad V = -W = -Fq \quad (40.1)$$

$$\therefore \quad \Pi = \frac{1}{2}kq^2 - Fq \quad (40.2)$$

PMTPE implies that the true solution for the equation is given when the expression for the total potential energy is derived and then set to zero, this is shown in **Eq. 41**.

$$\frac{\partial \Pi(q)}{\partial q} = 0 \rightarrow \frac{\partial \Pi(q)}{\partial q} = \frac{1}{2}k \times 2q - F = 0 \quad (41.1)$$

$$F = kq \quad (41.2)$$

1.3. Deriving the Deformation Behavior of Two Springs

Similarly to when we had one spring, we can apply PMTPE to two springs joined together and see their behavior under a pair of loads, this scenario is defined in **Figure 2**.

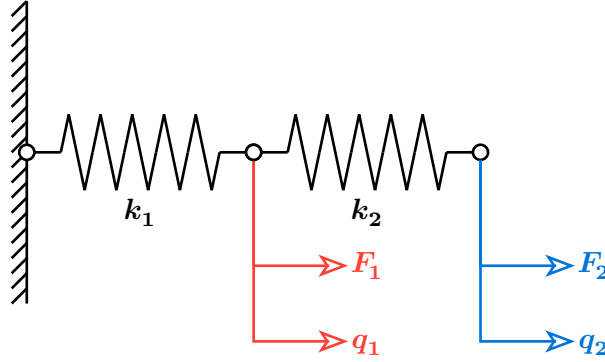


Figure 2: Two 1D spring attached to one another and a wall.

Now there are two unknowns of the system (q_1, q_2) , which means there must be two equations to solve this scenario fully. This is generated by applying the PMTPE to both of the unknowns. First the values for U and V are defined, shown in **Eq. 42**.

$$U = \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2 (q_2 - q_1)^2 \quad V = -F_1 q_1 - F_2 q_2 \quad (42.1)$$

$$\therefore \Pi(q_1, q_2) = \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2 (q_2 - q_1)^2 - (F_1 q_1 + F_2 q_2) \quad (42.2)$$

Note that in **Eq. 42**, the strain energy for the second spring has a subtracted q_1 term within it, this is due to the displacement q_2 being offset by q_1 when spring 1 is deformed. The solution to **Eq. 42** can be found by applying the PMTPE and deriving with respect to the unknown deformation variable, these steps are shown in **Eq. 43**.

$$\frac{\partial \Pi}{\partial q_1} = k_1 q_1 + k_2 (q_2 - q_1) - F_1 = 0 \rightarrow F_1 = (k_1 + k_2)q_1 + k_2 q_2 \quad (43.1)$$

$$\frac{\partial \Pi}{\partial q_2} = 0 + k_2 (q_2 - q_1) - F_2 = 0 \rightarrow F_2 = -k_2 q_1 + k_2 q_2 \quad (43.2)$$

Eq. 43 is a pair of linear simultaneous equations and therefore the matrix method can be used to solve this system of equations shown in **Eq. 44**.

$$\begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2 \times 1} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}_{2 \times 1} \rightarrow [K]_{2 \times 2} \{q\}_{2 \times 1} = \{F\}_{2 \times 1} \quad (44)$$

Where:

- $[K]$: Stiffness matrix

- $\{q\}$: Displacement vector
- $\{F\}$: Force vector

Note that **Eq. 44** can then be solved using matrix inversion $\{q\}_{2 \times 1} = [K]_{2 \times 2}^{-1} \{F\}_{2 \times 1}$.

1.4. Deriving deformation Behavior of a General Spring

More useful than the previous two examples, we can derive an expression for the deformation of a general spring in space, this scenario is shown in

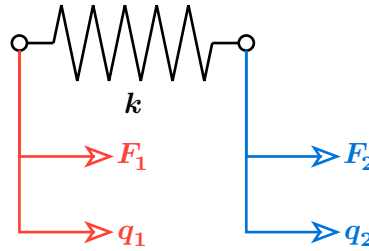


Figure 3: Entirely general 1D spring attached in free space.

Now that the scenario is defined, it is possible to write the expression for the strain energy of the system, this is shown in **Eq. 45**.

$$U = \frac{1}{2} k_1 (q_2 - q_1)^2 \quad \rightarrow \quad U = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{1 \times 2}^T \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2 \times 1} \quad (45.1)$$

$$U = \frac{1}{2} \{q\}_{1 \times 2}^T [K]_{2 \times 2} \{q\}_{2 \times 1} \quad (45.2)$$

Note that the equation for U can also be written in the matrix form shown on the left of **Eq. 45**. This form can be written in shorthand as $U = \frac{1}{2} \{q\}_{1 \times 2}^T [K]_{2 \times 2} \{q\}_{2 \times 1}$. Even though **Eq. 45** is written as a set of matrices, it is still a **scalar**, which makes sense as U is an energy.

2. Lecture 2

START OF WEEK 2

2.1. Formulating Problems using PMTPE

The finite element method allows for the simplification of complex problems by **assembling** together different elements. Take the scenario shown in **Figure 4**.

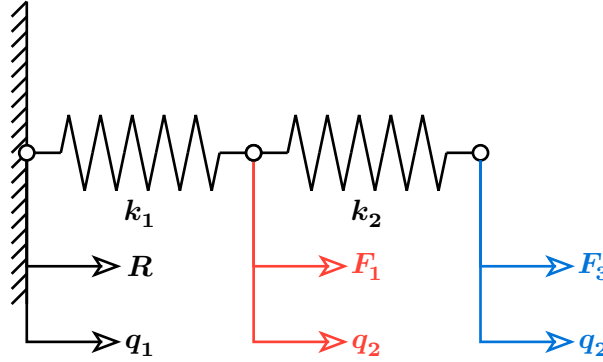


Figure 4: Two 1D springs attached to one another and a wall with boundary conditions defined.

Now that the scenario is defined, the strain energies for the two springs can be written in their matrix form, this is shown in .

$$U_1 = \frac{1}{2}k_1(q_2 - q_1)^2 \rightarrow U_1 = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{1 \times 2}^T \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2 \times 1} \quad (46.1)$$

$$U_2 = \frac{1}{2}k_2(q_3 - q_2)^2 \rightarrow U_2 = \frac{1}{2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_{1 \times 2}^T \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_{2 \times 1} \quad (46.2)$$

To find the total strain energy, these two components must be summed up. To achieve this, the matrix padding method shown in **Eq. 30** to ensure that the matrices are the same size as well as that the vector matrixes are the same as one another. This process is shown in **Eq. 47**.

$$U_1 = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{1 \times 2}^T \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2 \times 1} \rightarrow U_1 = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{1 \times 3}^T \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3 \times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3 \times 1} \quad (47.1)$$

$$U_2 = \frac{1}{2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_{1 \times 2}^T \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_{2 \times 1} \rightarrow U_2 = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{1 \times 3}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}_{3 \times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3 \times 1} \quad (47.2)$$

$$\therefore U = U_1 + U_2 = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{1 \times 3}^T \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}_{3 \times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3 \times 1} \quad (47.3)$$

Note that here, the stiffness matrix is called the **assembled stiffness matrix**. The potential energy of the system can be written in the form shown in **Eq. 48**.

$$V = -(q_1 R + q_2 F_1 + q_3 F_2) \rightarrow V = -\{R \ F_1 \ F_2\}_{1 \times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3 \times 1} = -\{F\}_{1 \times 3}^T \{q\}_{3 \times 1} \quad (48)$$

Note that **Eq. 48** is still a scalar value which makes sense as V is an energy. At this point the PMTPE condition can be applied, $\frac{\partial \Pi(q_i)}{\partial q_i}$, to obtain a solution. Alternatively, the **governing equation of equilibrium** can be applied the system to obtain a solution, shown in **Eq. 49**.

$$[K]\{q\} = \{F\} \rightarrow \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}_{3 \times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3 \times 1} = \begin{Bmatrix} R \\ F_1 \\ F_2 \end{Bmatrix}_{3 \times 1} \quad (49)$$

The next step to generate a solution is to invert the assembled matrix to find solutions for the $\{q\}$ matrix. However, currently the $[K]$ matrix is **singular**, as shown in **Eq. 50**.

$$\{q\} = [K]^{-1}\{F\} \rightarrow |K| = \begin{vmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{vmatrix} = 0 \quad (50)$$

$[K]$ being singular (determinant is equal to zero) means the inverse matrix cannot be calculated. This is effectively because the scenario is unbounded and so turns into a dynamics problem as there is nothing containing or holding down the springs. We can fix this by applying a **boundary condition**, for **Figure 4** we will apply $q_1 = 0$ as the boundary condition, this effectively eliminates those rows from the governing equation of equilibrium and makes the determinant non-zero, this is shown in **Eq. 51**.

$$\text{If } q_1 = 0 \text{ then } \rightarrow \begin{bmatrix} \cancel{k_1} & \cancel{-k_1} & \cancel{0} \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}_{3 \times 3} \begin{Bmatrix} \cancel{q_1} \\ q_2 \\ q_3 \end{Bmatrix}_{3 \times 1} = \begin{Bmatrix} \cancel{R} \\ F_1 \\ F_2 \end{Bmatrix}_{3 \times 1} \quad (51.1)$$

$$\therefore \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_{2 \times 1} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}_{2 \times 1} \quad (51.2)$$

Note that the **crossed out rows and columns are not deleted, just temporarily ignored** and can be re-used once a solution for $\{q\}$ has been obtained to get a value for R . The assembled stiffness matrix with the boundary conditions applied shown in **Eq. 51** is no longer singular meaning that the scenario can be solved for $\{q\}$. The steps taken to solve any question using the finite element method and PMTPE are:

1. Write expressions for the elastic strain energies U and potential energies V per element.
2. Combine all of the U s and V s using padding and matrix form to create a global $[K]$.
3. Apply boundary conditions to the system to make $[K]$ non-singular.
4. Use $\{q\} = [K]^{-1}\{F\}$ with $\{F\}$ from global V equation and $[K]$ from global U equation.
5. Obtain solutions for $\{q\}$.

3. Lecture 3

3.1. Elastic Rods

An elastic rod is another 1D element similar in nature to a spring element, however there are a few differences which make them more complex. Similar to springs, elastic rods can only deform axially in **tension or compression**. A general elastic rod element is shown in **Figure 5**.

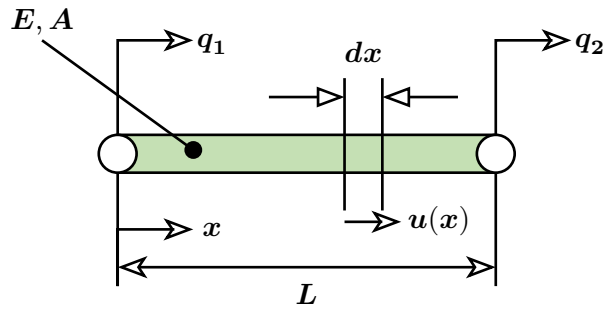


Figure 5: General elastic rod element.

Where:

- E : Young's Modulus (Pa).
- A : Cross sectional area (m^2).
- L : Length of bar (m).
- q_1 : Axial displacement of LHS end (m).
- q_2 : Axial displacement of RHS end (m).
- x : Distance from LHS along the bar (m).
- dx : Infinitesimal slice of the bar.
- $u(x)$: Deformation of the bar at slice (m).

3.2. Strain Energy for Elastic Rods

Similar to the spring element, the first step to generate a general expression for the elastic rod is to derive an expression for the strain energy U . This is shown in **Eq. 52**.

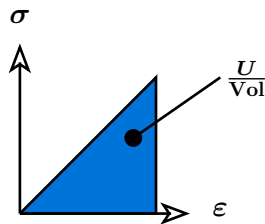


Figure 6: Strain energy plot.

Assuming 1D uniaxial elasticity: (52.1)

$$U = \frac{1}{2} \int \sigma_x \epsilon_x dVol \quad (52.2)$$

Given that the slice of the bar is a cylinder: (52.3)

$$U = \frac{1}{2} \int \sigma_x \epsilon_x A dx \quad (52.4)$$

As we are assuming that the rod is behaving purely elastically, then the stress (σ_x) can be rewritten as shown in **Eq. 53**.

$$\sigma_x = E \epsilon_x \rightarrow U = \frac{1}{2} \int EA \epsilon_x^2 dx \quad (53)$$

Finally, the strain itself ϵ_x can be rewritten in terms of the deformation seen over the infinitesimal length slice, this is shown in **Eq. 54**.

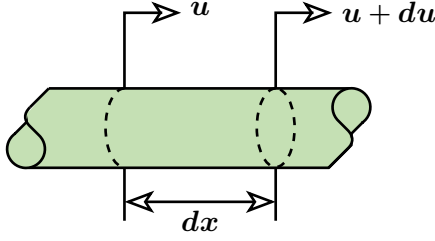


Figure 7: Deformation over slice.

The strain ε_x can be rewritten as: (54.1)

$$\varepsilon_x = \frac{\Delta L}{L} = \frac{(u + du) - u}{dx} = \frac{du}{dx} \quad (54.2)$$

Substituting **Eq. 54** into **Eq. 53** yields a final expression for the strain energy shown in **Eq. 55**.

$$U = \frac{1}{2} \int EA \varepsilon_x^2 dx = \frac{1}{2} \int EA \left(\frac{du}{dx} \right)^2 dx = \frac{1}{2} \int EA u'^2 dx \quad (55)$$

3.3. Shape Functions for Elastic Rods

One issue with **Eq. 55** is that the strain energy is written in terms of the deformation which currently has no relation to the displacements q_1 and q_2 . Therefore a more rigorous definition of $u(x)$ must be developed, and this is done through utilizing shape functions, shown in **Eq. 56**.

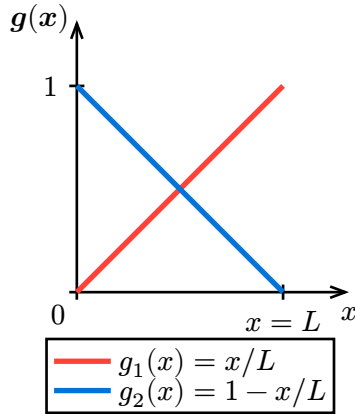


Figure 8: Shape functions for an elastic rod.

$$u(x) = g_1(x)q_1 + g_2(x)q_2 \quad (56.1)$$

$$g_1(x) = 1 - \frac{x}{L} \quad g_2(x) = \frac{x}{L} \quad (56.2)$$

$$\therefore u(x) = \left(1 - \frac{x}{L}\right)q_1 + \left(\frac{x}{L}\right)q_2 \quad (56.3)$$

The **shape function** controls the amount of **influence** each displacement receives and is effectively an interpolation. At $x = 0$ the displacement is given by q_1 and at $x = L$ the displacement is given by q_2 . Using the expression for deformation shown in **Eq. 56**, expressions for the strain and stress can be derived as shown in **Eq. 57**.

$$\varepsilon(x) = \frac{du}{dx} = \left(-\frac{1}{L}\right)q_1 + \left(\frac{1}{L}\right)q_2 \quad \sigma(x) = E\varepsilon(x) = \left(-\frac{E}{L}\right)q_1 + \left(\frac{E}{L}\right)q_2 \quad (57.1)$$

$$\varepsilon(x) = \left\{-\frac{1}{L} \quad \frac{1}{L}\right\}_{1 \times 2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2 \times 1} \quad \sigma(x) = E \left\{-\frac{1}{L} \quad \frac{1}{L}\right\}_{1 \times 2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2 \times 1} \quad (57.2)$$

Note that these can also be written in a matrix form as is shown in **Eq. 57**. The expressions for the stress and strain in **Eq. 57** don't actually depend on x and so the stress and strain are constant throughout the bar.

4. Lecture 4

START OF WEEK 3

4.1. Stiffness Matrix for Elastic Rods

To derive an expression which will include the stiffness matrix, the strain energy equation for elastic rods (shown in **Eq. 55**) can be combined with the equation for strain of the elastic rod (shown in **Eq. 57**) to yield **Eq. 58**.

$$U = \frac{1}{2} \int EA \varepsilon_x^2 dx \rightarrow U = \frac{1}{2} \int EA \left[\left(-\frac{1}{L} \right) q_1 + \left(\frac{1}{L} \right) q_2 \right]^2 dx \quad (58)$$

The equation shown in **Eq. 58** can be simplified and then integrated to yield **Eq. 59.1**. The resulting quadratic form can be rewritten in matrix form to include the stiffness matrix, also shown in **Eq. 59.2**.

$$U = \frac{1}{2} \left(\frac{EA}{L^2} \right) (-q_1 + q_2)^2 \int_0^L 1 dx = \frac{1}{2} \left(\frac{EA}{L} \right) (-q_1 + q_2)^2 = \frac{1}{2} \left(\frac{EA}{L} \right) (q_1^2 - 2q_1 q_2 + q_2^2) \quad (59.1)$$

$$U = \frac{1}{2} \left(\frac{EA}{L} \right) \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{1 \times 2}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2 \times 1} \quad \text{or} \quad U = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{1 \times 2}^T \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2 \times 1} \quad (59.2)$$

Where the **stiffness matrix** is shown in the bottom right equation with dimensions 2×2 .

4.2. Solving a Simple Elastic Rod scenario

The process of solving a question with elastic rods is similar to the spring scenario shown earlier, consider the scenario defined in **Figure 9**.

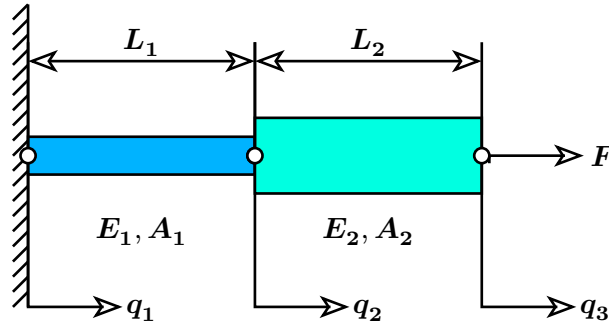


Figure 9: Scenario with two elastic rods joined together.

The first step for generating a solution to this scenario is to write expressions for the strain energies and then assemble these expressions into a global strain energy expression, this process is shown in **Eq. 60**.

$$U_1 = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{1 \times 2}^T \begin{bmatrix} \frac{E_1 A_1}{L_1} & -\frac{E_1 A_1}{L_1} \\ -\frac{E_1 A_1}{L_1} & \frac{E_1 A_1}{L_1} \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2 \times 1} \quad U_2 = \frac{1}{2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_{1 \times 2}^T \begin{bmatrix} \frac{E_2 A_2}{L_2} & -\frac{E_2 A_2}{L_2} \\ -\frac{E_2 A_2}{L_2} & \frac{E_2 A_2}{L_2} \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_{2 \times 1} \quad (60.1)$$

$$U = U_1 + U_2 = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{1 \times 3}^T \begin{bmatrix} \frac{E_1 A_1}{L_1} & -\frac{E_1 A_1}{L_1} & 0 \\ -\frac{E_1 A_1}{L_1} & \frac{E_1 A_1}{L_1} + \frac{E_2 A_2}{L_2} & -\frac{E_2 A_2}{L_2} \\ 0 & -\frac{E_2 A_2}{L_2} & \frac{E_2 A_2}{L_2} \end{bmatrix}_{3 \times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3 \times 1} \quad (60.2)$$

Note that addition of the strain energies in **Eq. 60** can only be done through padding of each of the strain energies. The next step is to write the potential energy V for this scenario, this expression is shown in **Eq. 61**.

$$V = -\{R \ 0 \ F\}_{1 \times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3 \times 1} \quad (61)$$

Skipping going through partial differentiation and directly going to the governing equation of equilibrium yields **Eq. 62**. This equation can then have boundary conditions applied to generate values for the displacements, and these can be used to work out reactions.

$$[K]\{q\} = \{F\} \rightarrow \begin{bmatrix} \frac{E_1 A_1}{L_1} & -\frac{E_1 A_1}{L_1} & 0 \\ -\frac{E_1 A_1}{L_1} & \frac{E_1 A_1}{L_1} + \frac{E_2 A_2}{L_2} & -\frac{E_2 A_2}{L_2} \\ 0 & -\frac{E_2 A_2}{L_2} & \frac{E_2 A_2}{L_2} \end{bmatrix}_{3 \times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3 \times 1} = \begin{Bmatrix} R \\ 0 \\ F \end{Bmatrix}_{3 \times 1} \quad (62)$$

4.3. Combining Elastic Rods and Springs

Like lego pieces, elastic rods and springs can be combined to simulate more complex behavior, one common example of this is introducing a **lumped spring** which is commonly used to simulate a connection of a structure to a fixed wall, this configuration is shown in

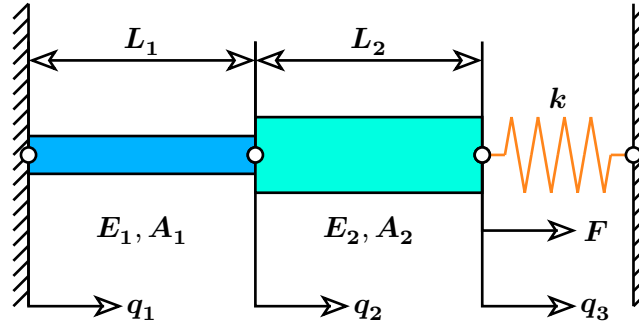


Figure 10: Scenario with two elastic rods joined together with a lumped spring.

This change only has a small effect on the overall behavior of the system, as the force and displacement vectors remain unchanged, the altered governing equation of equilibrium is shown in **Eq. 63**.

$$\begin{bmatrix} \frac{E_1 A_1}{L_1} & -\frac{E_1 A_1}{L_1} & 0 \\ -\frac{E_1 A_1}{L_1} & \frac{E_1 A_1}{L_1} + \frac{E_2 A_2}{L_2} & -\frac{E_2 A_2}{L_2} \\ 0 & -\frac{E_2 A_2}{L_2} & \frac{E_2 A_2}{L_2} + k \end{bmatrix}_{3 \times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3 \times 1} = \begin{Bmatrix} R \\ 0 \\ F \end{Bmatrix}_{3 \times 1} \quad (63)$$

Note that the only difference here is the stiffness in the bottom right cell. Alternatively the spring could be unbounded as shown in **Figure 11** where it is situated between the two elastic rods.

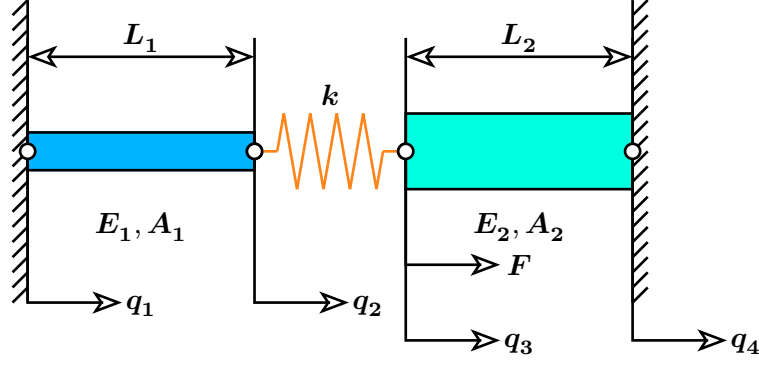


Figure 11: Scenario with two elastic rods joined together with a lumped spring.

This change will effect the different matrices in the following way shown in **Eq. 64**.

$$U_1 = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{1 \times 2}^T \begin{bmatrix} \frac{E_1 A_1}{L_1} & -\frac{E_1 A_1}{L_1} \\ -\frac{E_1 A_1}{L_1} & \frac{E_1 A_1}{L_1} \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2 \times 1} \quad U_2 = \frac{1}{2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_{1 \times 2}^T \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_{2 \times 1} \quad (64.1)$$

$$U_3 = \frac{1}{2} \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix}_{1 \times 2}^T \begin{bmatrix} \frac{E_2 A_2}{L_2} & -\frac{E_2 A_2}{L_2} \\ -\frac{E_2 A_2}{L_2} & \frac{E_2 A_2}{L_2} \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix}_{2 \times 1} \quad (64.2)$$

$$U = U_1 + U_2 + U_3 = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_{1 \times 4}^T \begin{bmatrix} \frac{E_1 A_1}{L_1} & -\frac{E_1 A_1}{L_1} & 0 & 0 \\ -\frac{E_1 A_1}{L_1} & \frac{E_1 A_1}{L_1} + k_1 & -k_1 & 0 \\ 0 & -k_1 & \frac{E_2 A_2}{L_2} + k_1 & -\frac{E_2 A_2}{L_2} \\ 0 & 0 & -\frac{E_2 A_2}{L_2} & \frac{E_2 A_2}{L_2} \end{bmatrix}_{4 \times 4} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_{4 \times 1} \quad (64.3)$$

$$V = -\{R_1 \ 0 \ F \ R_2\}_{1 \times 4} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_{4 \times 1} \quad (64.4)$$

4.4. Distributed Load

To extend the functionality of the elastic rod, imagine a distributed load applied to one of the two elements, shown in **Figure 12**.

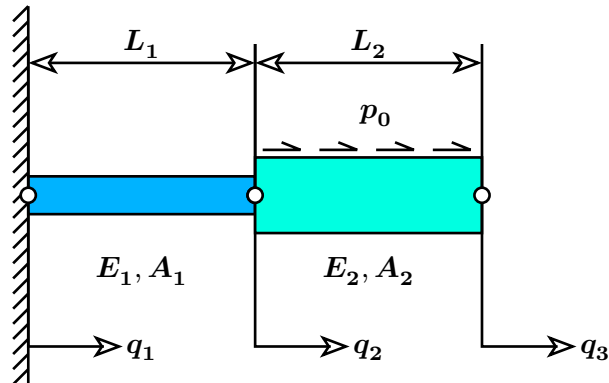


Figure 12: Scenario with two elastic rods joined together with a distributed force applied to one of the rods.

One issue with a distributed load is that the resulting force must be applied at the nodes, even though the distribution is across the whole element. To generate an expression for the distributed load, an infinitesimal slice of the bar is considered as shown in **Eq. 65**.

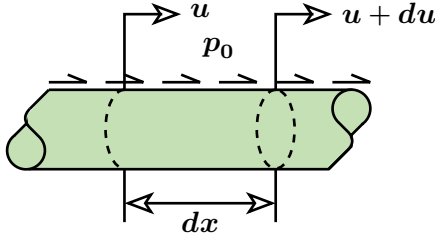


Figure 13: Deformation over slice with distributed load.

$$V = -W = -F \times d = - \int_{x=0}^{x=L_E} p_0 dx \times u(x) \quad (65.1)$$

$$\text{As } u(x) = \left(1 - \frac{x}{L_E}\right) q_2 + \left(\frac{x}{L_E}\right) q_3 \quad \text{Then} \quad (65.2)$$

$$V = -\left(P_0 \frac{L_E}{2}\right) q_2 - \left(P_0 \frac{L_E}{2}\right) q_3 \quad (65.3)$$

Eq. 65 can also be written in the force matrix form, to allow for easier substitution into the governing equation of equilibrium, this is shown in **Eq. 66**.

$$V = -\left(P_0 \frac{L_E}{2}\right) q_2 - \left(P_0 \frac{L_E}{2}\right) q_3 \rightarrow V = -\left\{R \ P_0 \frac{L_E}{2} \ P_0 \frac{L_E}{2}\right\}_{1 \times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3 \times 1} \quad (66.1)$$

$$V = -\{F\}_{1 \times 3}^T \{q\}_{3 \times 1} \quad (66.2)$$

This definition can easily be expanded for more complex loading cases. Take the scenario defined in **Figure 12**, and apply a force at an intermediate node, as shown in **Figure 14**.

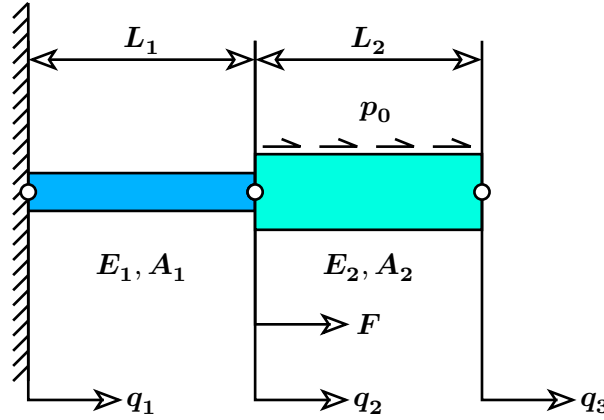


Figure 14: Scenario with two elastic rods joined together with a distributed force applied to one of the rods and an intermediate point force applied.

Adding this force would only produce a small change in the force vector, shown in **Eq. 67**.

$$V = -\left\{R \ P_0 \frac{L_E}{2} + F \ P_0 \frac{L_E}{2}\right\}_{1 \times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3 \times 1} = -\{F\}_{1 \times 3}^T \{q\}_{3 \times 1} \quad (67)$$

Just to summarize, the typical process for solving a question with elastic rods is:

1. Draw a diagram.
2. Idealize the elements, calculating equivalent nodal forces.
3. Assemble the global stiffness matrix and equilibrium equation.

4. Invert the stiffness matrix and solve for nodal displacements.
5. Solve for displacements across elements.
6. Solve for strain across elements.
7. Solve for stress across elements.

4.5. Solving a Complex Elastic Rod question

As an example of the process to fully solve a complex elastic rod type question, consider the scenario defined in **Figure 15**.

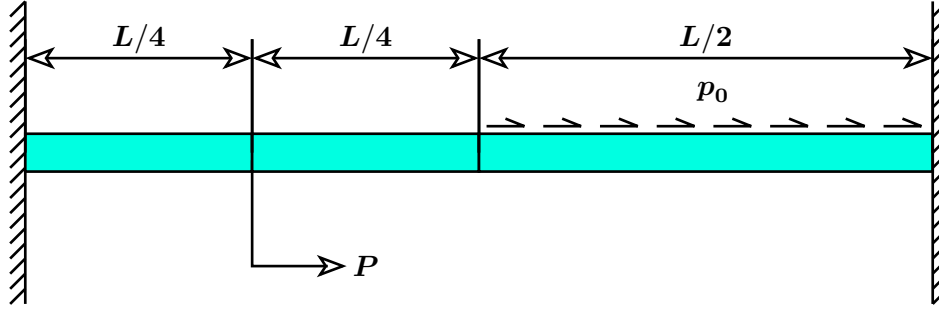


Figure 15: Complex elastic rod scenario

Currently **Figure 15** is not an FE scenario yet, to make it so, nodes, nodal displacements and forces must be defined, these are shown in **Figure 16**.

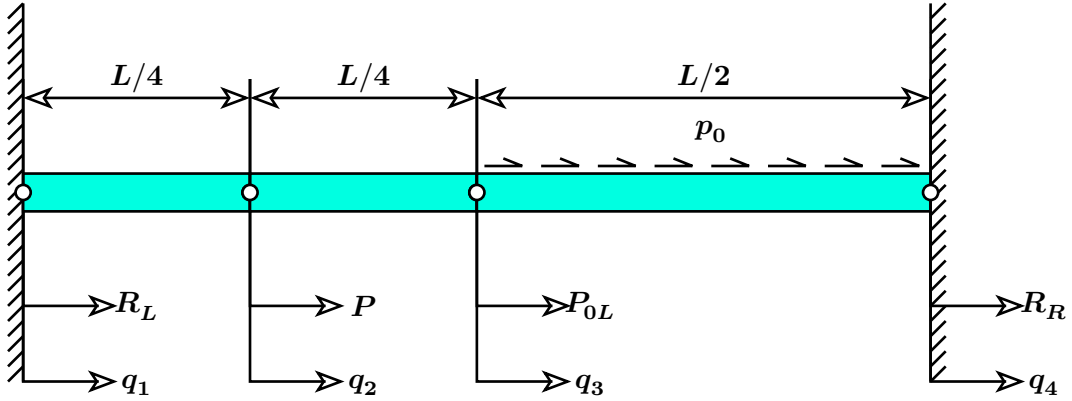


Figure 16: Complex elastic rod scenario broken down using FE.

Now that the scenario is defined, the strain energy per element can be defined and a global assembled stiffness matrix can be obtained, as shown in **Eq. 68**.

$$U_1 = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{1 \times 2}^T \frac{EA}{L} \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2 \times 1} \quad (68.1)$$

$$U_2 = \frac{1}{2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_{1 \times 2}^T \frac{EA}{L} \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_{2 \times 1} \quad (68.2)$$

$$U_3 = \frac{1}{2} \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix}_{1 \times 2}^T \frac{EA}{L} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_3 \\ q_4 \end{Bmatrix}_{2 \times 1} \quad (68.3)$$

$$U = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_{1 \times 4}^T \frac{EA}{L} \begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 4+4 & -4 & 0 \\ 0 & -4 & 4+2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}_{4 \times 4} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_{4 \times 1} \quad (68.4)$$

After obtaining the expression for the strain energy in **Eq. 68**, a similar expression can be developed for the potential energy shown in **Eq. 69**.

$$V = -W = -\left(R_L q_1 + P q_2 + \frac{p_0 L}{4} q_3 + \frac{p_0 L}{4} q_4 + R_R q_4\right) \quad (69.1)$$

$$V = -\left\{R_L \ P \ \frac{p_0 L}{4} \ \frac{p_0 L}{4} + R_R\right\}_{1 \times 4} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_{4 \times 1} \quad (69.2)$$

Finally, the assembled stiffness matrix in **Eq. 68** can be combined with the force vector shown in **Eq. 69** in the governing equation of equilibrium to yield **Eq. 70**, boundary conditions at the walls can also be applied to yield **Eq. 70** a reduced non-singular problem.

$$\frac{EA}{L} \begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 4+4 & -4 & 0 \\ 0 & -4 & 4+2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}_{4 \times 4} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_{4 \times 1} = \begin{Bmatrix} R_L \\ P \\ \frac{p_0 L}{4} \\ \frac{p_0 L}{4} + R_R \end{Bmatrix}_{4 \times 1} \quad (70.1)$$

$$\text{Applying BCs : } \frac{EA}{L} \begin{bmatrix} 8 & -4 \\ -4 & 6 \end{bmatrix}_{2 \times 2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_{2 \times 1} = \begin{Bmatrix} P \\ \frac{p_0 L}{4} \end{Bmatrix}_{2 \times 1} \quad (70.2)$$

After solving for the displacements, its possible to further solve for the stress and strain, for this example, these plots are shown in **Figure 17** and **Figure 18**.

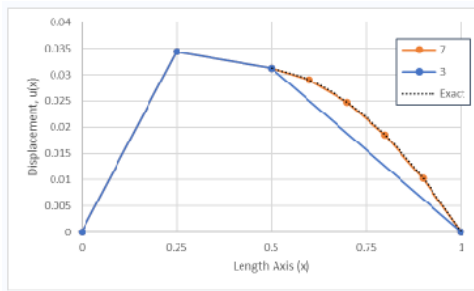


Figure 17: Plot of nodal displacement along the rod.

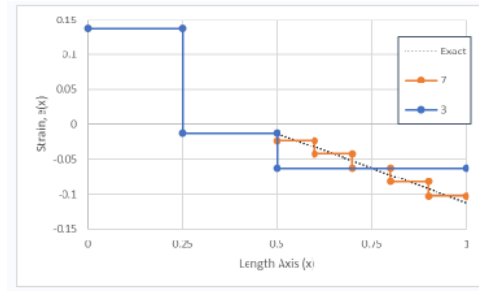


Figure 18: Plot of strain along the rod.

Notice that the strain is the derivative of the displacement curve. Further note that for both plots, increasing the number of nodes within the section with the distributed force gets closer and closer to a smooth curve, this is because the force of the distributed load is only being approximated to be at the nodes.

5. Lecture 5

START OF WEEK 4

5.1. Motivation for Using Shape Functions

In continuum mechanics, there are an infinite number of DoFs as you can go down to the atomic scale. The FE process approximates a continuum as a finite number of nodes where **shape functions** are used to approximate the behavior between the nodes. Shape functions should provide continuity between adjoining elements, as shown in **Figure 19** for multiple elastic rods.

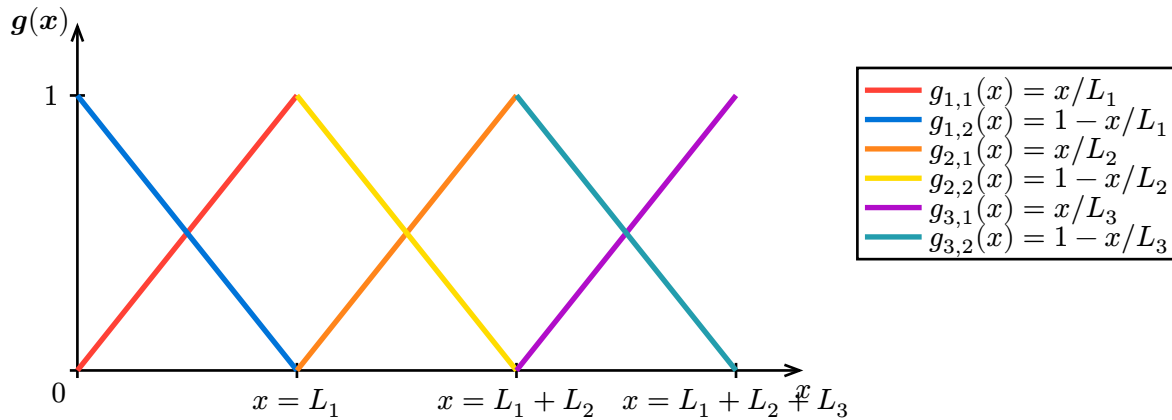


Figure 19: Plot showing continuity of shape functions.

5.2. Beam Element

A beam element is a 1D element which only deforms in flexure, where the displacement (denoted by $w(x)$) is perpendicular to the element axis. A general 1D beam element is shown in **Figure 20**.

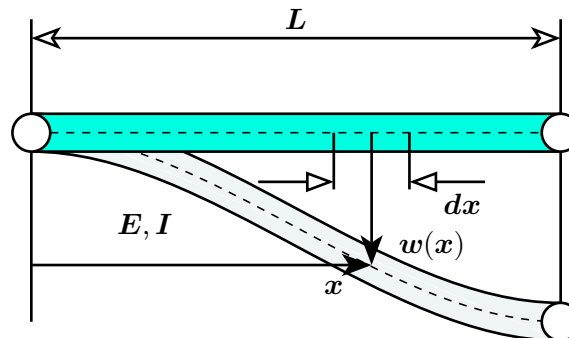


Figure 20: General 1D beam element

Where:

- E : Young's Modulus (Pa)
- I : Second Moment of Area (m^4)

Note that the Euler-Bernoulli hypothesis assumptions still hold true for beam elements, these are:

- That the **cross section remains constant** and does not change during bending.
- That the **cross section remains perpendicular** to the neutral axis during bending

For beams the strain energy is given by **Eq. 71**, note the similarities to the rod strain energy given by **Eq. 55**

$$U = \frac{1}{2} \int_0^L EI(w(x''))^2 dx \quad (71)$$

The governing differential equation for transverse loading is more complex than the one for longitudinal loading, both are shown in **Eq. 72**.

$$\text{Governing Transverse Diff Eq } EAu'' = \text{Longitudinal Load Intensity} \quad (72.1)$$

$$\text{Governing Flexure Diff Eq } EIw'''' = \text{Transversal Load Intensity} \quad (72.2)$$

The order of the shape function required is one less than the order of the derivative of the governing differential equation, this is why for a rod element, the shape function is linear. This means that for a beam element, the shape function must be a **cubic**. Using a cubic shape function, allows for beam elements to encompass the following behaviors:

- Capture a linearly varying bending moment $M(x) = EI \frac{d^2 w(x)}{dx^2}$
- Therefore, capture linearly varying stress $\sigma = \frac{M(x)y}{I}$
- Capture constant shear force $V(x) = EI \frac{d^3 w(x)}{dx^3}$

5.2.1. Hermite Cubics

The cubic shape functions used for beam elements are called the **Hermite cubics** and encompass a group of four cubics defined over the length of the element. These are shown in **Figure 21**.

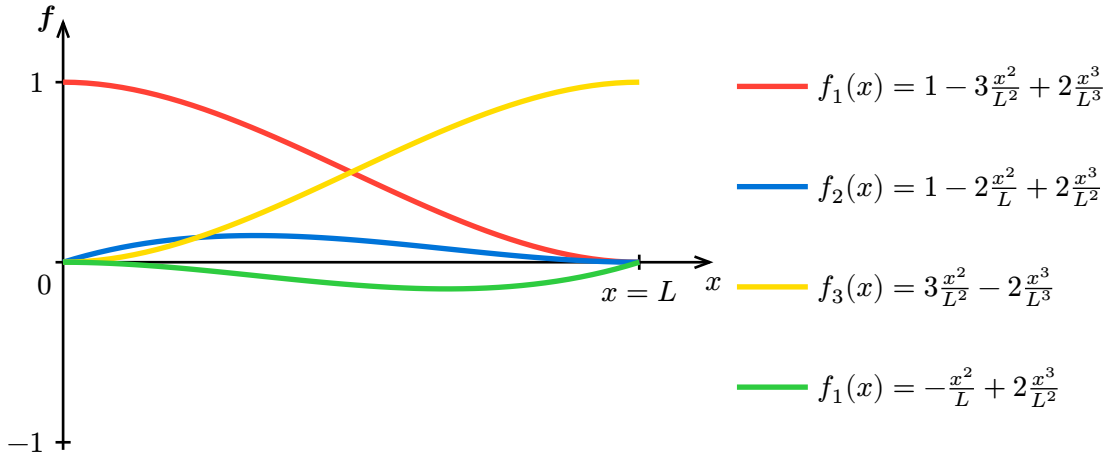


Figure 21: Definition of Hermite cubics.

The displacement is given as a combination of these 4 cubics applied to the four degrees of freedom of the beam element, shown in **Eq. 73**.

$$w(x) = f_1(x)q_1 + f_2(x)q_2 + f_3(x)q_3 + f_4(x)q_4 \quad (73)$$

5.2.2. DoF for Beam Elements

Unlike rod elements, beam elements have four degrees of freedom, two translational and two rotational. The positive sign conventions for these DoFs is shown in **Figure 23**.

If the flexural deformation of the element is small, then the derivative of the deformation (the slope) can be approximated as the rotation itself, this is shown in **Figure 22**.

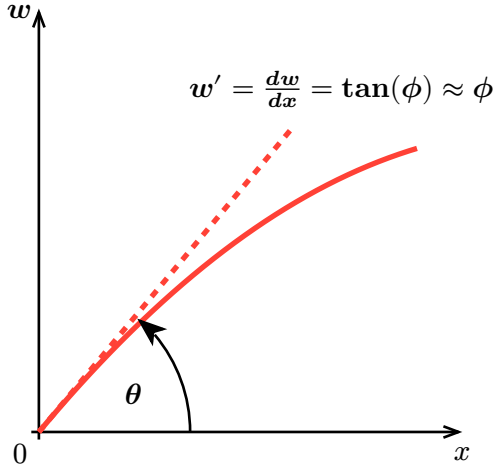


Figure 22: Plot showing for small strains, slope is approximately equal to rotation.



Figure 23: Plot showing positive sign conventions for a beam element

5.2.3. Deriving the Stiffness Matrix for a Beam Element

As **Eq. 71** and **Eq. 73** have both been defined earlier, the strain energy will therefore depend on the second derivative of the displacement function in **Eq. 73** this yields **Eq. 74**.

$$U = \frac{1}{2}EI \int_0^L [f_1''(x)q_1 + f_2''(x)q_2 + f_3''(x)q_3 + f_4''(x)q_4]^2 dx \quad (74)$$

Note that as the expression for the displacement ($w''(x)$) is squared, the fully expanded form will consist of 10 integrals. These 10 integrals can then be solved using the definitions of the Hermite cubics and assembled into a stiffness matrix form to yield **Eq. 75**.

$$U = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_{1 \times 4}^T \left(\frac{EI}{L^3} \right) \begin{bmatrix} 12 & 6L & -12 & -6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}_{4 \times 4} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_{4 \times 1} \quad (75.1)$$

$$U = \frac{1}{2} \left(\frac{EI}{L^3} \right) \{q\}_{1 \times 4}^T [K]_{4 \times 4} \{q\}_{4 \times 1} \quad (75.2)$$

6. Lecture 6

START OF WEEK 5

6.1. Solving Beam Questions

In a similar manner to solving rod and spring questions, an inversion of the stiffness matrix can be used to solve questions with beam elements. Take the scenario defined in **Figure 24**.

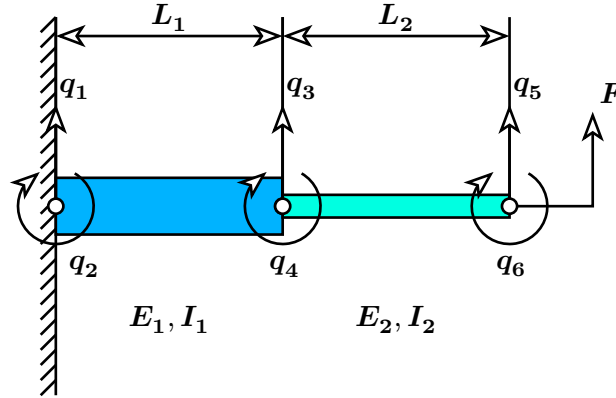


Figure 24: Scenario with two beam elements joined together.

The strain energies for the two elements defined in **Figure 24** is given by the expression shown in **Eq. 76**.

$$U_1 = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}^T \left(\frac{E_1 I_1}{L_1^3} \right) \begin{bmatrix} 12 & 6L_1 & -12 & -6L_1 \\ 6L_1 & 4L_1^2 & -6L_1 & 2L_1^2 \\ -12 & -6L_1 & 12 & -6L_1 \\ 6L_1 & 2L_1^2 & -6L_1 & 4L_1^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} \quad (76.1)$$

$$U_2 = \frac{1}{2} \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}^T \left(\frac{E_2 I_2}{L_2^3} \right) \begin{bmatrix} 12 & 6L_2 & -12 & -6L_2 \\ 6L_2 & 4L_2^2 & -6L_2 & 2L_2^2 \\ -12 & -6L_2 & 12 & -6L_2 \\ 6L_2 & 2L_2^2 & -6L_2 & 4L_2^2 \end{bmatrix} \begin{Bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} \quad (76.2)$$

The total strain energy will therefore be the sum of these two padded matrices, this is shown in **Eq. 77** where $U = U_1 + U_2$ and the EI/L^3 are expanded before addition.

$$U = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}^T \begin{bmatrix} \boxed{U_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boxed{U_2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} \quad (77)$$

The next thing to develop is the potential energy of the system, to do this consider the reactions at the wall, not only is there a **reaction force** but there is also a **reaction moment**, this is shown in **Eq. 78**.

$$V = -W = - \begin{matrix} \begin{matrix} R \\ M \\ 0 \\ 0 \\ 0 \\ F \end{matrix}^T \\ \begin{matrix} \\ \\ \\ \\ \\ \end{matrix} \\ \begin{matrix} \\ \\ \\ \\ \\ \end{matrix} \\ \begin{matrix} \\ \\ \\ \\ \\ \end{matrix} \\ \begin{matrix} \\ \\ \\ \\ \\ \end{matrix} \\ \begin{matrix} \\ \\ \\ \\ \\ \end{matrix} \end{matrix} \begin{matrix} \begin{matrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{matrix} \\ \begin{matrix} \\ \\ \\ \\ \\ \end{matrix} \end{matrix} \quad (78)$$

Substituting the force vector, stiffness matrix and displacement vector into the governing equation of equilibrium yields **Eq. 79**.

$$\begin{bmatrix} \text{4x4 matrix} \end{bmatrix} \begin{matrix} \begin{matrix} 0 \\ 0 \\ 0 \\ F \end{matrix} \\ \begin{matrix} \\ \\ \\ \end{matrix} \end{matrix} = \begin{matrix} \begin{matrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{matrix} \\ \begin{matrix} \\ \\ \\ \end{matrix} \end{matrix} \quad (79)$$

6.2. Calculating Intermittent Displacements

Say for the previous scenario, it was important to know the displacement at an intermittent point between two nodes, instead of formulating a new FE scenario, it is possible to use the shape functions as the displacements at the nodes are known, this is shown in **Eq. 80**.

$$w(x) = f_1(x)q_1 + f_2(x)q_2 + f_3(x)q_3 + f_4(x)q_4 \quad (80.1)$$

$$\rightarrow w\left(x = \frac{L}{2}\right) = f_1\left(\frac{L}{2}\right)q_1 + f_2\left(\frac{L}{2}\right)q_2 + f_3\left(\frac{L}{2}\right)q_3 + f_4\left(\frac{L}{2}\right)q_4 \quad (80.2)$$

6.3. Stress and Strain for Beam Elements

It was previously states that deflections for beam elements of this formulation are assumed to be small so $\tan(\phi) \approx \phi$. Despite the entire beam element only deflecting in flexure, there still has to be a **local longitudinal deflection** to allow for flexure itself. This local longitudinal deflection is shown mathematically in **Eq. 81** and graphically in **Figure 25**.

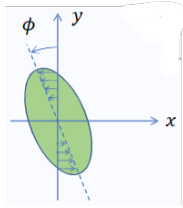


Figure 25: Plot of nodal displacement along the rod.

$$u = -y\phi = -y \frac{dw}{dx} \quad (81.1)$$

$$\varepsilon_x = \frac{du}{dx} = -y \frac{d^2w}{dx^2} \quad (81.2)$$

Taking the second derivative of the equation for flexural deformation $w(x)$ and rearranging it into a quasi matrix form yields **Eq. 82**.

$$\varepsilon_x = -y \left[\left(-\frac{6}{L^2} + \frac{12x}{L^3} \right) \left(-\frac{4}{L} + \frac{6x}{L^2} \right) \left(\frac{6}{L^2} - \frac{12x}{L^3} \right) \left(-\frac{2}{L} + \frac{6x}{L^2} \right) \right]_{1 \times 4} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_{4 \times 1} \quad (82.1)$$

$$\varepsilon_x = [B]_{1 \times 4} \{q\}_{4 \times 1} \quad \sigma_x = [D]_{1 \times 1} [B]_{1 \times 4} \{q\}_{4 \times 1} = E[B]_{1 \times 4} \{q\}_{4 \times 1} \quad (82.2)$$

In **Eq. 82**, the strain is written as the product of a $[B]$ matrix and $\{q\}$ matrix which is the common convention. To then calculate the resulting stress, a further pre-multiplication is done by a $[D]$ matrix, which in a simple 1D case is just the Young's Modulus E .

6.4. Distributed Load on Beam Elements

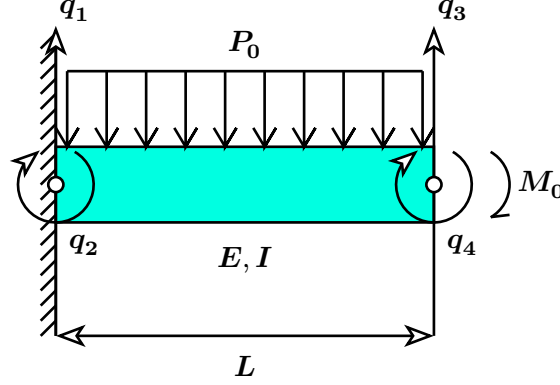


Figure 26: Distributed load applied on a beam element.

Consider the scenario defined in **Figure 26**, with a distributed load and a moment applied at the RHS of a beam element. In a similar way to the formulation of a distributed load on an elastic rod, the distributed force on a beam element is written in the following way **Eq. 83**.

$$\left(\frac{EI}{L^3}\right) \begin{bmatrix} 12 & 6L & -12 & -6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}_{4 \times 4} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_{4 \times 1} = \begin{Bmatrix} p_0 \int_0^L f_1(x) dx \\ p_0 \int_0^L f_2(x) dx \\ p_0 \int_0^L f_3(x) dx \\ p_0 \int_0^L f_4(x) dx \end{Bmatrix} + \begin{Bmatrix} R \\ M \\ 0 \\ -M_0 \end{Bmatrix} \quad (83)$$

6.5. Adding a Grounded Spring

Grounded springs can be added to the stiffness matrices to change the dynamics of systems, take a grounded spring added vertically to the scenario defined in **Figure 26**, shown in **Figure 27**.

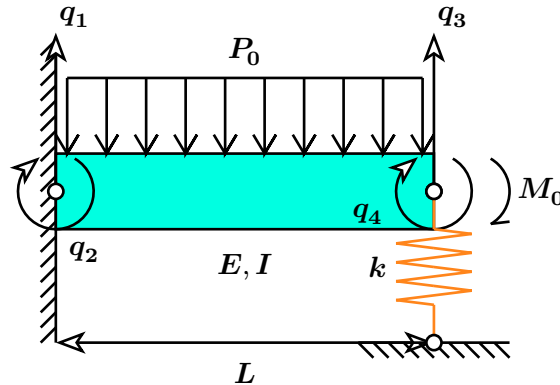


Figure 27: Translational grounded spring added to **Figure 26**

This change only results in a change to the stiffness matrix, not the force vector, the altered stiffness matrix is shown in **Eq. 84**.

$$U = U_1 + U_{\text{spring}} = U_1 + \frac{1}{2} k q_3^2 \quad (84.1)$$

$$U = \frac{1}{2} \left(\frac{EI}{L^3} \right) \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_{1 \times 4} \begin{bmatrix} 12 & 6L & -12 & -6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 + \frac{L^3}{EI} k & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}_{4 \times 4} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_{4 \times 1} \quad (84.2)$$

Note that the this spring stiffness is only applied to the element in the stiffness matrix which corresponds to the q_3^2 . Also note the factor it is pre-multiplied by, this cancels out the effect from EI/L^3 . Also consider a grounded rotational spring, shown in **Figure 28**.

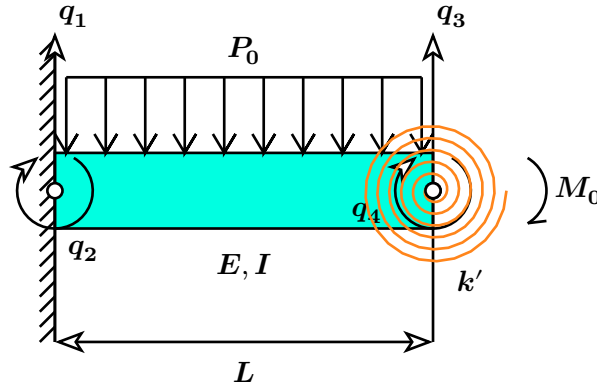


Figure 28: Rotational grounded spring added to **Figure 26**

Similar to **Figure 27**, the rotational spring only affects the stiffness matrix, this is shown in **Eq. 85**.

$$U = U_1 + U_{\text{spring}} = U_1 + \frac{1}{2} k' q_4^2 \quad (85.1)$$

$$U = \frac{1}{2} \left(\frac{EI}{L^3} \right) \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_{1 \times 4} \begin{bmatrix} 12 & 6L & -12 & -6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 + \frac{L^3}{EI} k' \end{bmatrix}_{4 \times 4} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_{4 \times 1} \quad (85.2)$$

6.6. Boundary Conditions

Boundary conditions allow for a solution to an FE problem, so far wall conditions have been considered where for beams $q_1 = R = 0$ and $q_2 = M = 0$. Note that intermediate boundary conditions are also possible, shown in **Figure 29**.

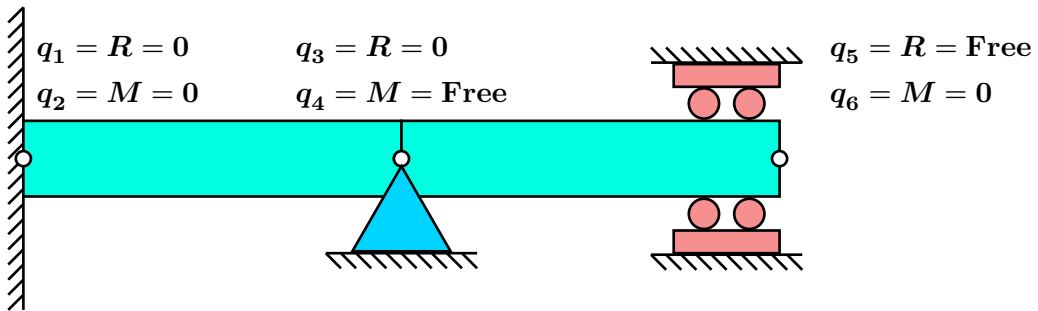


Figure 29: Translational grounded spring added to **Figure 26**

6.7. Beam-Rod Elements

Beam elements can only deform in flexure and rod elements in extension, a **beam-rod** (sometimes called beam-column) elements can do both by essentially placing a rod element over a beam element, shown in **Figure 30**, **Figure 31**.

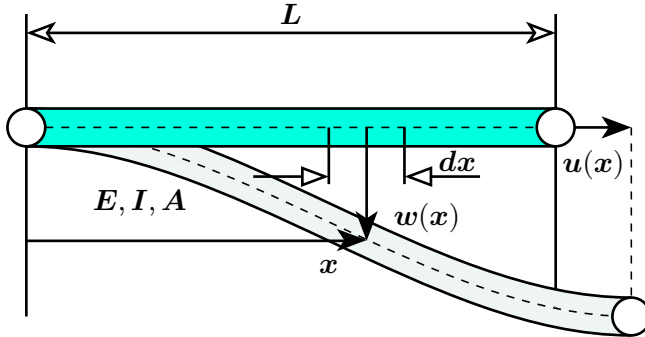


Figure 30: General 1D beam-rod element

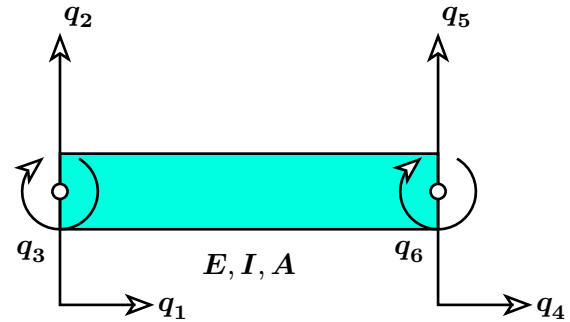


Figure 31: DoFs for a beam-rod element

The stiffness matrix for a beam-rod element is an assembly of both a rod and beam element, this is shown in **Eq. 86**, where the red sections are from a rod element and blue for a beam element.

$$U = \frac{1}{2} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}^T \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{EA}{L^3}(12) & \frac{EA}{L^3}(6L) & 0 & \frac{EA}{L^3}(-12) & \frac{EA}{L^3}(6L) \\ 0 & \frac{EA}{L^3}(6L) & \frac{EA}{L^3}(4L^2) & 0 & \frac{EA}{L^3}(-6L) & \frac{EA}{L^3}(2L^2) \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & \frac{EA}{L^3}(-12) & \frac{EA}{L^3}(-6L) & 0 & \frac{EA}{L^3}(12) & \frac{EA}{L^3}(-6L) \\ 0 & \frac{EA}{L^3}(6L) & \frac{EA}{L^3}(2L^2) & 0 & \frac{EA}{L^3}(-6L) & \frac{EA}{L^3}(4L^2) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} \quad (86)$$