SESM6047

Finite Element Analysis in Solid Mechanics

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0. Lecture 0

0.1. Matrix Terminology

A matrix is a structured way of organizing data in a rectangular array with rows and columns. For this module the notation of a matrix and reference to matrix terms is shown in **Eq. 1**.

$$[K]_{i\times j} = \begin{bmatrix} k_{11} & k_{12} & ... & k_{1j} \\ k_{21} & k_{22} & ... & k_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ k_{i1} & k_{i2} & ... & k_{ij} \end{bmatrix}_{i\times j} \tag{1}$$

Where:
- $\boldsymbol{[K]_{i\times j}}$: The matrix $[K]$ (Note for this module a matrix will be defined by the [] notation).
- $\boldsymbol{i}$: The row index (indexed from 1).
- $\boldsymbol{j}$: The column index (indexed from 2).
- $\boldsymbol{i \times j}$: The shape of the matrix (i number of rows, j number of columns).
- $\boldsymbol{k_{ij}}$: The item in the ith row and the jth column.

A special type of commonly used matrix is a **diagonal matrix** shown in **Eq. 2**. This type of matrix only features terms that sit on the indexes where the row index is equal to the column index ($i = j$).

$$[K]_{i\times j} = \begin{bmatrix} k_{11} & 0 & ... & 0 \\ 0 & k_{22} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & k_{ij} \end{bmatrix}_{i\times j} \tag{2}$$

0.2. Vector Terminology

A vector is a quantity that has a **magnitude** as well as a **direction**. A vector is like one column out of a matrix, the number of terms signifies the dimensions of a vector, the notation used in this module for a vector is shown in **Eq. 3**.

$$\{q\}_{i\times 1} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_i \end{Bmatrix}_{i\times 1} \tag{3}$$

0.3. Matrix Addition and Subtraction

Matrixes with the same shape can be added or subtracted. Matrix addition or subtraction is a **commutative** property and the process to do so is shown in **Eq. 4**.

$$[A]_{i\times j} \pm [B]_{i\times j} \tag{4.1}$$

$$= \begin{bmatrix} a_{11} & a_{12} & ... & a_{1j} \\ a_{21} & a_{22} & ... & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & ... & a_{ij} \end{bmatrix}_{i\times j} \pm \begin{bmatrix} b_{11} & b_{12} & ... & b_{1j} \\ b_{21} & b_{22} & ... & b_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ b_{i1} & b_{i2} & ... & b_{ij} \end{bmatrix}_{i\times j} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & ... & a_{1j} \pm b_{1j} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & ... & a_{2j} \pm b_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} \pm b_{i1} & a_{i2} \pm b_{i2} & ... & a_{ij} \pm b_{ij} \end{bmatrix}_{i\times j} \tag{4.2}$$

$$[A]_{i\times j} \pm [B]_{i\times j} = [B]_{i\times j} \pm [A]_{i\times j} \tag{4.3}$$

0.4. Matrix Multiplication

0.4.1. Scalar × Matrix Multiplication

A scalar can be multiplied with a matrix or vector of any shape, the scalar value is multiplied with every term within the matrix, as shown in **Eq. 5**.

$$k[A]_{i\times j} = k \begin{bmatrix} a_{11} & a_{12} & ... & a_{1j} \\ a_{21} & a_{22} & ... & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & ... & a_{ij} \end{bmatrix}_{i\times j} = \begin{bmatrix} ka_{11} & ka_{12} & ... & ka_{1j} \\ ka_{21} & ka_{22} & ... & ka_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{i1} & ka_{i2} & ... & ka_{ij} \end{bmatrix}_{i\times j} \tag{5}$$

0.4.2. Matrix × Matrix Multiplication

Any two matrices can be multiplied as long as **the number of columns of the first matrix is the same as the number of rows in the second matrix**. Given this criteria is satisfied the formula to then multiply two matrices together is shown in **Eq. 6**.

$$[A]_{i\times k}[B]_{k\times j} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1k} \\ a_{21} & a_{22} & ... & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & ... & a_{ik} \end{bmatrix}_{i\times k} \begin{bmatrix} b_{11} & b_{12} & ... & b_{1j} \\ b_{21} & b_{22} & ... & b_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & ... & b_{kj} \end{bmatrix}_{k\times j} \tag{6.1}$$

$$= \begin{bmatrix} \sum_{r:1}^{k}(a_{1r}b_{r1}) & \sum_{r:1}^{k}(a_{1r}b_{r2}) & ... & \sum_{r:1}^{k}(a_{1r}b_{rj}) \\ \sum_{r:1}^{k}(a_{2r}b_{r1}) & \sum_{r:1}^{k}(a_{2r}b_{r2}) & ... & \sum_{r:1}^{k}(a_{2r}b_{rj}) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{r:1}^{k}(a_{ir}b_{r1}) & \sum_{r:1}^{k}(a_{ir}b_{r2}) & ... & \sum_{r:1}^{k}(a_{ir}b_{rj}) \end{bmatrix}_{i\times j} \tag{6.2}$$

Note that matrix multiplication has some key properties associated with it, these are:

- **Non commutable** ($[A]_{i\times k}[B]_{k\times j} \neq [B]_{k\times j}[A]_{i\times k}$).
- **Associative** ($([A]_{i\times k}[B]_{k\times l})[C]_{l\times j} = [A]_{i\times k}([B]_{k\times l}[C]_{l\times j})$
- **Distributable** $[A]_{i\times k}\big([B]_{k\times j} + [C]_{k\times j}\big) = [A]_{i\times k}[B]_{k\times j} + [A]_{i\times k}[C]_{k\times j}$

0.5. Identity Matrix

The identity matrix is a special matrix with some key and is is defined in **Eq. 7**.

$$I_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad [I]_{i\times j} = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 1 \end{bmatrix}_{i\times j} \tag{7}$$

Identity matrices **are** commutable and multiplication by an appropriately sized identity matrix will not itself change the value of the matrix, these properties are shown in.

$$[I]_{i\times i}[K]_{i\times j} = [K]_{i\times j} = [K]_{i\times j}[I]_{j\times j} \tag{8}$$

0.6. Matrix Transposition

The transpose of a matrix is signified by the T symbol and will flip the matrix along the diagonal. The generalized expression for transposition is shown in **Eq. 9**.

$$[A]_{i\times j} = \left[a_{ij}\right] \quad \rightarrow \quad [A]^T_{j\times i} = \left[a_{ji}\right] \quad \text{where} \quad \left([A]^T\right)_{ij} = a_{ji} \tag{9}$$

For example the transpose of a general 3×2 matrix is shown in **Eq. 10**.

$$[A]_{3\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2} \quad \rightarrow \quad [A]^T_{2\times 3} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}_{2\times 3} \tag{10}$$

0.7. Matrix Determinant

The determinant of a matrix is a scalar-valued function of a **square matrix** (matrix must be square for there to exist a determinant). Determinants are symbolized via a | and the general form of a determinant is shown in **Eq. 11**.

$$\text{let } [A]_{n\times n} = \left[a_{ij}\right] \text{ then} \quad \det([A]) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det\left(M_{ij}\right) \tag{11.1}$$

$$\text{where } M_{ij} \text{ is the minor of } a_{ij} \tag{11.2}$$

Determinants can be long and repetitive to calculate for larger matrices, the determinant for a 2×2 matrix is shown in **Eq. 12**.

$$[A]_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2} \quad \rightarrow \quad |[A]_{2\times 2}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{12}$$

Just for completion, the determinant for a 3×3 matrix is shown in **Eq. 13**.

$$[A]_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \quad \rightarrow \quad |[A]_{3\times 3}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{13.1}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{13.2}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \tag{13.3}$$

0.8. Minor of an Element, Adjoint Matrix and Cofactor Matrix

The **adjoint** or **adjugate** matrix is a useful mathematical tool which will eventually allow us to be able to form the inverse of a matrix. Before defining the adjoint matrix, the minor of an element and the cofactor matrix must first be understood.

0.8.1. Minor of an Element

To find the minor of a given element, **the ith row and jth column of that element are deleted** and the **determinant of the remaining submatrix is the minor.** As an example, the minor of the a_{12} element in a 3×3 matrix is shown in **Eq. 14**.

$$[A]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \quad \rightarrow \quad M_{12} = \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23} \tag{14}$$

0.8.2. Cofactor Matrix

The cofactor matrix utilizes the minor of every element within a matrix to form a new matrix, the general mathematical definition for what a cofactor matrix is, is shown in **Eq. 15**.

$$\mathrm{Cof}\big([A]_{i\times j}\big) = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1j} \\ C_{21} & C_{22} & \dots & C_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ C_{i1} & C_{i2} & \dots & C_{ij} \end{bmatrix}_{i\times j} \qquad \text{Where} \quad C_{ij} = (-1)^{i+j}\ \det\big(M_{ij}\big) \tag{15}$$

Note that this equation is effectively calculating the minor of every element within a matrix and then adding the sign in a checkerboard form as shown in **Eq. 16**.

$$\begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{16}$$

The cofactor matrix for a general 2×2 matrix is shown in **Eq. 17**

$$[A]_{2\times2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2} \tag{17.1}$$

$$\mathrm{Cof}([A])_{2\times2} = \begin{bmatrix} +M_{11} & -M_{12} \\ -M_{21} & +M_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}_{2\times2} \tag{17.2}$$

The cofactor matrix for a general 3×3 matrix is shown in **Eq. 18**

$$[A]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \tag{18.1}$$

$$\mathrm{Cof}([A])_{3\times3} = \begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}_{3\times3} = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}_{3\times3} \tag{18.2}$$

$$= \begin{bmatrix} +(a_{22}a_{33} - a_{23}a_{32}) & -(a_{21}a_{33} - a_{23}a_{31}) & +(a_{21}a_{32} - a_{22}a_{31}) \\ -(a_{12}a_{33} - a_{13}a_{32}) & +(a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) \\ +(a_{12}a_{23} - a_{13}a_{22}) & -(a_{11}a_{23} - a_{13}a_{21}) & +(a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}_{3\times3} \tag{18.3}$$

0.8.3. Adjoint Matrix

To finish off, the adjoint matrix is just the **transposition of the cofactor matrix**, the general mathematical definition of teh adjoint matrix is shown in **Eq. 19**. Note that the adjoint and cofactor operations can only be done on **square matrices**.

$$\text{Adj}\big([A_{i\times j}]\big) = \text{Cof}\big([A_{i\times j}]\big)^T \tag{19}$$

For completeness the adjoint of a general 2×2 matrix is shown in **Eq. 20** and for a 3×3 matrix in **Eq. 21**.

$$[A]_{2\times2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2} \quad \rightarrow \quad \text{Adj}([A]_{2\times2}) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}_{2\times2} \tag{20}$$

$$[A]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \tag{21.1}$$

$$\text{Adj}([A]_{3\times3}) = \begin{bmatrix} (a_{22}a_{33} - a_{23}a_{32}) & -(a_{12}a_{33} - a_{13}a_{32}) & (a_{12}a_{23} - a_{13}a_{22}) \\ -(a_{21}a_{33} - a_{23}a_{31}) & (a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{23} - a_{13}a_{21}) \\ (a_{21}a_{32} - a_{22}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) & (a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}_{3\times3} \tag{21.2}$$

0.9. Inverse Matrix

The inversion of a matrix is a key tool within linear algebra and will act as a backbone for FEA in general. The mathematical definition to invert a matrix is shown in **Eq. 22**.

$$[A]^{-1}_{i\times j} = \frac{1}{|[A]_{i\times j}|}\text{Adj}\big([A]_{i\times j}\big) \tag{22}$$

As **Eq. 22** uses the reciprocal of the determinant of the matrix, it also means that the **value of the determinant cannot itself be zero**. A matrix with a determinant of zero is said to be **singular**, **degenerate** or **rank deficient**. Examples for the inverses of a 2×2 matrix and a 3×3 matrix are shown in **Eq. 23** and **Eq. 24** respectively.

$$[A]_{2\times2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2} \quad \rightarrow \quad [A]^{-1}_{2\times2} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}_{2\times2} \tag{23}$$

$$[A]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \tag{24.1}$$

$$[A]^{-1}_{3\times3} = \frac{1}{(a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}))}\ldots \tag{24.2}$$

$$\ldots\begin{bmatrix} (a_{22}a_{33} - a_{23}a_{32}) & -(a_{12}a_{33} - a_{13}a_{32}) & (a_{12}a_{23} - a_{13}a_{22}) \\ -(a_{21}a_{33} - a_{23}a_{31}) & (a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{23} - a_{13}a_{21}) \\ (a_{21}a_{32} - a_{22}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) & (a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}_{3\times3} \tag{24.3}$$

0.10. Solving Systems of Equations Using matrices

Using the steps outlined before, it is possible to use matrices to solve a system of simultaneous equations. Lets say we have n **linear** equations, and therefore n number of unknowns, then the system of equations can be written in the form shown in **Eq. 25**.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{cases} \tag{25}$$

The system of equations shown in **Eq. 25** can then bbe written in the matrix form shown in **Eq. 26**.

$$[A]_{n\times n}[X]_{n\times 1} = [B]_{n\times 1} \tag{26.1}$$

$$\text{where} \quad [A]_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}_{n\times n} \quad [X]_{n\times 1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \quad [B]_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n\times 1} \tag{26.2}$$

Assuming that the $[X]_{n\times 1}$ matrix is the set of unknowns, then the solution of this system can be found by inverting the $[A]_{n\times n}$ matrix (this can be done as $[A]_{n\times n}[A]^{-1}_{n\times n} = [A]^{-1}_{n\times n}[A]_{n\times n} = [I]_{n\times n}$), the form of this expression is shown in.

$$[X]_{n\times 1} = [B]_{n\times 1}[A]^{-1}_{n\times n} \qquad \text{Given } |[A]_{n\times n}| \neq 0 \tag{27}$$

0.11. Matrix Zero-Padding

It is possible to use many of the aforementioned tools to "pad" a matrix with zeroes, with some degree of control. Let there be a submatrix $[K]_{n\times n}$ that we want to expand into a new matrix $\overline{K}$ to a size $m \times m$, with the K matrix being in the top left of the $\overline{K}$ matrix, then the multiplication shown in can be used **Eq. 28**.

$$[K]_{n\times n} = \begin{bmatrix} k_{11} & k_{12} & ... & k_{1j} \\ k_{21} & k_{22} & ... & k_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ k_{i1} & k_{i2} & ... & k_{ij} \end{bmatrix}_{n\times n} \quad \rightarrow \quad \left[\overline{K}\right]_{n\times n} = \begin{bmatrix} k_{11} & k_{12} & ... & 0 \\ k_{21} & k_{22} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 0 \end{bmatrix}_{m\times m} \tag{28.1}$$

$$\text{Then [A] must have shape} \qquad [A]_{n\times m} = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 0 \end{bmatrix}_{n\times m} \tag{28.2}$$

$$\therefore \quad \left[\overline{K}\right]_{n\times n} = [A]^T_{m\times n}[K]_{n\times n}[A]_{n\times m} \tag{28.3}$$

As an example a 2×2 matrix is embedded into a 5×5 matrix using **Eq. 28**, this is shown in **Eq. 29** .

$$[K]_{2\times 2} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}_{2\times 5} \qquad [A]_{2\times 5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}_{2\times 5} \tag{29.1}$$

$$[\overline{K}]_{5\times5} = [A]^T_{5\times2}[K]_{2\times2}[A]_{2\times5} \tag{29.2}$$

$$[\overline{K}]_{5\times5} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{5\times2} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}_{2\times5} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}_{2\times5} = \begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 \\ k_{21} & k_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times5} \tag{29.3}$$

Note that the placement of the $[K]$ matrix is dependant on where the ones are within the $[A]$ matrix. The further along the array the cascading ones are, the further down the diagonal the inserted $[K]$ matrix is. This process can be used to make a **banded matrix**, which is shown in **Eq. 30**.

$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 \\ k_{21} & k_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times5} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k_{11} & k_{12} & 0 & 0 \\ 0 & k_{21} & k_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times5} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{11} & k_{12} & 0 \\ 0 & 0 & k_{21} & k_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times5} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{11} & k_{12} \\ 0 & 0 & 0 & k_{21} & k_{22} \end{bmatrix}_{5\times5} \tag{30.1}$$

$$= \begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 \\ k_{21} & k_{22}+k_{11} & k_{12} & 0 & 0 \\ 0 & k_{21} & k_{22}+k_{11} & k_{12} & 0 \\ 0 & 0 & k_{21} & k_{22}+k_{11} & k_{12} \\ 0 & 0 & 0 & k_{21} & k_{22} \end{bmatrix}_{5\times5} \tag{30.2}$$

0.12. Matrix Cutting

In a similar way to how a matrix can be padded, it can also be cut to reduce the shape. This process has the following steps:

1. Start with an identity matrix the same size as the matrix to shorten.
2. Delete the columns you want to remove from the main matrix, keeping it square (this also will delete the rows with the same index).
3. Pre-multiply by the transpose of this reduced identity matrix and post-multiply by the original matrix to obtain the reduced form.

Eq. 31 Is an example of this where the 2nd, 3rd and 4th rows are removed from $[B]$.

$$[B]_{5\times5} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}_{5\times5} \tag{31.1}$$

$$[I]_{5\times5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times2} \rightarrow [C]_{5\times3} = \begin{bmatrix} 1 & \cancel{0} & 0 & \cancel{0} & \cancel{0} \\ 0 & \cancel{1} & 0 & \cancel{0} & \cancel{0} \\ 0 & \cancel{0} & 1 & \cancel{0} & \cancel{0} \\ 0 & \cancel{0} & 0 & \cancel{1} & \cancel{0} \\ 0 & \cancel{0} & 0 & \cancel{0} & \cancel{1} \end{bmatrix}_{5\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{5\times2} \tag{31.2}$$

$$\left[\overline{B}\right]_{2\times 2} = [C]^T_{2\times 5}[B]_{5\times 5}[C]_{5\times 3} \tag{31.3}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{2\times 5} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}_{5\times 5} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{5\times 2} = \begin{bmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{bmatrix}_{2\times 2} \tag{31.4}$$

0.13. Vector Insertion and Cutting

0.13.1. Vector Insertion

Vector insertion is easier than matrices padding, the following steps are used:

1) Define a matrix with zeros that has the following properties:

    a) Number of rows equals the final major dimension of the vector
    b) Number of columns equals the current major dimension of the vector

2) Add ones within the columns to specify where the zeroes will be inserted (see **Eq. 32**).

3) Pre-multiply by this matrix to obtain the modified column vector.

An example of this is shown in **Eq. 32** where zeroes are added before and after q_1 yielding the modfied $\{\overline{q}\}_{5\times 1}$.

$$\{q\}_{3\times 1} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times 1} \quad [C]_{5\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{5\times 3} \tag{32.1}$$

$$\{\overline{q}\}_{5\times 1} = [C]_{5\times 3}\{q\}_{3\times 1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{5\times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times 1} = \begin{Bmatrix} 0 \\ q_1 \\ 0 \\ q_2 \\ q_3 \end{Bmatrix}_{5\times 1} \tag{32.2}$$

0.13.2. Vector Cutting

This process is similar to insertion with the key difference being pre-multiplication by a differently shaped $[C]$ matrix:

1) Define a matrix with zeros that has the following properties:

    a) Number of **columns** equals the final major dimension of the vector
    b) Number of **rows** equals the current major dimension of the vector

2) Add ones within the columns to specify which values will be kept after the cut (see **Eq. 33**).

3) Pre-multiply by this matrix to obtain the modified column vector.

An example of this is shown in **Eq. 33** where the size of the vector is reduced by removing the first and third terms.

$$\{q\}_{5\times1} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{Bmatrix}_{5\times1} \quad [C]_{5\times3} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{3\times5} \tag{33.1}$$

$$\{\overline{q}\}_{3\times1} = [C]_{3\times5}\{q\}_{5\times1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{3\times5} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{Bmatrix}_{5\times1} = \begin{Bmatrix} q_2 \\ q_4 \\ q_5 \end{Bmatrix}_{3\times1} \tag{33.2}$$

0.14. Eigenvalues and Eigenvectors

Eigenvectors are vectors with a key property, multiplying them with a matrix yields a vector with the same direction but with an altered magnitude (scales the vector). Mathematically we can write this in the form shown in **Eq. 34**.

$$[A]_{n\times n}\{X\}_{n\times1} = \lambda\{X\}_{n\times1} \tag{34}$$

Eq. 35 can be used to find the value of the eigenvalues, which will then allow for independent solutions for the eigenvectors.

$$|[A]_{n\times n} - \lambda[I]_{n\times n}| = 0 \tag{35}$$

Note that the determinant exists here as the solution to this equation itself has to be singular. An example calculation of the eigenvalues for a 2×2 matrix is shown in **Eq. 36**.

$$[A]_{n\times n} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \quad \rightarrow \quad |[A]_{n\times n} - \lambda[I]_{n\times n}| = 0 \tag{36.1}$$

$$\left| \left[\begin{smallmatrix} 4 & 1 \\ 1 & 3 \end{smallmatrix}\right] - \lambda\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \right| = \begin{vmatrix} 4-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0 \tag{36.2}$$

$$\lambda_1 = 5 \qquad \lambda_2 = 2 \tag{36.3}$$

The results for the eigenvalues can be plugged back into the following formula to yield the eigenvectors **Eq. 37**.

$$([A]_{n\times n} - \lambda[I]_{n\times n})[X]_{n\times1} = 0 \tag{37}$$

1. Lecture 1

1.1. Principle of Minimum Total Potential Energy (PMTPE)

The principle of minimum total potential energy is a concept that underpins the whole of FEA. It states that a structure will deform or displace to a position which minimizes the total potential energy of that structure. To build up to a mathematical definition of PMTPE, the total potential energy can be written as **Eq. 38**.

$$\Pi = U + V \tag{38}$$

Where:

- $\boldsymbol{\Pi}$: The total potential energy (J).
- $\boldsymbol{U}$: The stored elastic strain energy (J).
- $\boldsymbol{V}$: The potential energy (negative of the work done $-W$ by external forces) (J).

PMTE then states that if we have an infinite number of possible responses to a system, the correct solution will have the lowest energy out of the infinite solutions. Mathematically, we are looking for a minimum for the value of Π which yields the **Eq. 39**.

$$\delta\Pi = 0 \quad \therefore \quad U = -V \tag{39}$$

1.2. Deriving the Deformation Behavior of Spring

We can apply PMTPE to a 1D spring in order to yield the equation $F = kq$. To start the scenario is defined in **Figure 1**.

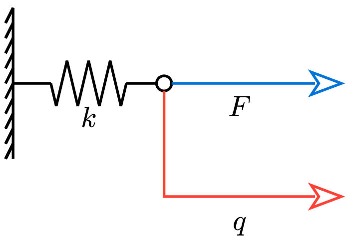

Figure 1: General 1D spring attatched to wall.

Where:
- $\boldsymbol{k}$: Spring stiffness $(\frac{N}{m})$
- $\boldsymbol{F}$: Force (N)
- $\boldsymbol{q}$: Displacement (m)

Note that q is described as a **generalized coordinate** or a **degree of freedom** (DoF). From the scenario defined in **Figure 1**, an expression for the total potential energy of the system can be generated, shown in **Eq. 40**.

$$U = \frac{1}{2}kq^2 \quad V = -W = -Fq \tag{40.1}$$

$$\therefore \quad \Pi = \frac{1}{2}kq^2 - Fq \tag{40.2}$$