SESM6047

Finite Element Analysis in Solid Mechanics

Author: Yusaf Sultan
Lecturer: Alex Dickinson
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Definitions

$\mathbf{\Pi}$	Total potential energy (J)	$\boldsymbol{U}$	Stored elastic strain energy(J)
$\boldsymbol{V}$	Potential energy (J)	$\boldsymbol{F}$	Force (N)
$\boldsymbol{k}$	Stiffness (N/m)	$\boldsymbol{q}$	Generalized degree of freedom
$[\boldsymbol{K}]$	Stiffness matrix	$\{\boldsymbol{q}\}$	Displacement vector

0. Lecture 0

0.1. Matrix Terminology

A matrix is a structured way of organizing data in a rectangular array with rows and columns. For this module the notation of a matrix and reference to matrix terms is shown in **Eq. 1**.

$$[K]_{i\times j} = \begin{bmatrix} k_{11} & k_{12} & ... & k_{1j} \\ k_{21} & k_{22} & ... & k_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ k_{i1} & k_{i2} & ... & k_{ij} \end{bmatrix}_{i\times j} \quad (1)$$

Where:
- $\boldsymbol{[K]_{i\times j}}$: The matrix $[K]$ (Note for this module a matrix will be defined by the [] notation).
- $\boldsymbol{i}$: The row index (indexed from 1).
- $\boldsymbol{j}$: The column index (indexed from 2).
- $\boldsymbol{i\times j}$: The shape of the matrix (i number of rows, j number of columns).
- $\boldsymbol{k_{ij}}$: The item in the ith row and the jth column.

A special type of commonly used matrix is a **diagonal matrix** shown in **Eq. 2**. This type of matrix only features terms that sit on the indexes where the row index is equal to the column index ($i = j$).

$$[K]_{i\times j} = \begin{bmatrix} k_{11} & 0 & ... & 0 \\ 0 & k_{22} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & k_{ij} \end{bmatrix}_{i\times j} \quad (2)$$

0.2. Vector Terminology

A vector is a quantity that has a **magnitude** as well as a **direction**. A vector is like one column out of a matrix, the number of terms signifies the dimensions of a vector, the notation used in this module for a vector is shown in **Eq. 3**.

$$\{q\}_{i\times 1} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_i \end{Bmatrix}_{i\times 1} \quad (3)$$

0.3. Matrix Addition and Subtraction

Matrixes with the same shape can be added or subtracted. Matrix addition or subtraction is a **commutative** property and the process to do so is shown in **Eq. 4**.

$$[A]_{i\times j} \pm [B]_{i\times j} \quad (4.1)$$

$$= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} \end{bmatrix}_{i\times j} \pm \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} \\ b_{21} & b_{22} & \dots & b_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ b_{i1} & b_{i2} & \dots & b_{ij} \end{bmatrix}_{i\times j} = \begin{bmatrix} a_{11}\pm b_{11} & a_{12}\pm b_{12} & \dots & a_{1j}\pm b_{1j} \\ a_{21}\pm b_{21} & a_{22}\pm b_{22} & \dots & a_{2j}\pm b_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1}\pm b_{i1} & a_{i2}\pm b_{i2} & \dots & a_{ij}\pm b_{ij} \end{bmatrix}_{i\times j} \tag{4.2}$$

$$[A]_{i\times j} \pm [B]_{i\times j} = [B]_{i\times j} \pm [A]_{i\times j} \tag{4.3}$$

0.4. Matrix Multiplication

0.4.1. Scalar × Matrix Multiplication

A scalar can be multiplied with a matrix or vector of any shape, the scalar value is multiplied with every term within the matrix, as shown in **Eq. 5**.

$$k[A]_{i\times j} = k \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} \\ a_{21} & a_{22} & \dots & a_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} \end{bmatrix}_{i\times j} = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1j} \\ ka_{21} & ka_{22} & \dots & ka_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{i1} & ka_{i2} & \dots & ka_{ij} \end{bmatrix}_{i\times j} \tag{5}$$

0.4.2. Matrix × Matrix Multiplication

Any two matrices can be multiplied as long as **the number of columns of the first matrix is the same as the number of rows in the second matrix**. Given this criteria is satisfied the formula to then multiply two matrices together is shown in **Eq. 6**.

$$[A]_{i\times k}[B]_{k\times j} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ik} \end{bmatrix}_{i\times k} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} \\ b_{21} & b_{22} & \dots & b_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \dots & b_{kj} \end{bmatrix}_{k\times j} \tag{6.1}$$

$$= \begin{bmatrix} \sum_{r:1}^{k}(a_{1r}b_{r1}) & \sum_{r:1}^{k}(a_{1r}b_{r2}) & \dots & \sum_{r:1}^{k}(a_{1r}b_{rj}) \\ \sum_{r:1}^{k}(a_{2r}b_{r1}) & \sum_{r:1}^{k}(a_{2r}b_{r2}) & \dots & \sum_{r:1}^{k}(a_{2r}b_{rj}) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{r:1}^{k}(a_{ir}b_{r1}) & \sum_{r:1}^{k}(a_{ir}b_{r2}) & \dots & \sum_{r:1}^{k}(a_{ir}b_{rj}) \end{bmatrix}_{i\times j} \tag{6.2}$$

Note that matrix multiplication has some key properties associated with it, these are:

- **Non commutable** ($[A]_{i\times k}[B]_{k\times j} \neq [B]_{k\times j}[A]_{i\times k}$).
- **Associative** ($([A]_{i\times k}[B]_{k\times l})[C]_{l\times j} = [A]_{i\times k}([B]_{k\times l}[C]_{l\times j})$)
- **Distributable** $[A]_{i\times k}\big([B]_{k\times j} + [C]_{k\times j}\big) = [A]_{i\times k}[B]_{k\times j} + [A]_{i\times k}[C]_{k\times j}$

0.5. Identity Matrix

The identity matrix is a special matrix with some key and is is defined in **Eq. 7**.

$$I_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad [I]_{i\times j} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{i\times j} \tag{7}$$

Identity matrices **are** commutable and multiplication by an appropriately sized identity matrix will not itself change the value of the matrix, these properties are shown in.

$$[I]_{i\times i}[K]_{i\times j} = [K]_{i\times j} = [K]_{i\times j}[I]_{j\times j} \tag{8}$$

0.6. Matrix Transposition

The transpose of a matrix is signified by the T symbol and will flip the matrix along the diagonal. The generalized expression for transposition is shown in **Eq. 9**.

$$[A]_{i\times j} = \left[a_{ij}\right] \quad \rightarrow \quad [A]^T_{j\times i} = \left[a_{ji}\right] \quad \text{where} \quad \left([A]^T\right)_{ij} = a_{ji} \tag{9}$$

For example the transpose of a general 3×2 matrix is shown in **Eq. 10**.

$$[A]_{3\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2} \quad \rightarrow \quad [A]^T_{2\times 3} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}_{2\times 3} \tag{10}$$

0.7. Matrix Determinant

The determinant of a matrix is a scalar-valued function of a **square matrix** (matrix must be square for there to exist a determinant). Determinants are symbolized via a | and the general form of a determinant is shown in **Eq. 11**.

$$\text{let } [A]_{n\times n} = \left[a_{ij}\right] \text{ then} \quad \det([A]) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det\left(M_{ij}\right) \tag{11.1}$$

$$\text{where } M_{ij} \text{ is the minor of } a_{ij} \tag{11.2}$$

Determinants can be long and repetitive to calculate for larger matrices, the determinant for a 2×2 matrix is shown in **Eq. 12**.

$$[A]_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2} \quad \rightarrow \quad |[A]_{2\times 2}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{12}$$

Just for completion, the determinant for a 3×3 matrix is shown in **Eq. 13**.

$$[A]_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \quad \rightarrow \quad |[A]_{3\times 3}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{13.1}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{13.2}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \tag{13.3}$$

0.8. Minor of an Element, Adjoint Matrix and Cofactor Matrix

The **adjoint** or **adjugate** matrix is a useful mathematical tool which will eventually allow us to be able to form the inverse of a matrix. Before defining the adjoint matrix, the minor of an element and the cofactor matrix must first be understood.

0.8.1. Minor of an Element

To find the minor of a given element, **the ith row and jth column of that element are deleted** and the **determinant of the remaining submatrix is the minor.** As an example, the minor of the a_{12} element in a 3×3 matrix is shown in **Eq. 14**.

$$[A]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \quad \rightarrow \quad M_{12} = \begin{vmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23} \tag{14}$$

0.8.2. Cofactor Matrix

The cofactor matrix utilizes the minor of every element within a matrix to form a new matrix, the general mathematical definition for what a cofactor matrix is, is shown in **Eq. 15**.

$$\text{Cof}\big([A]_{i\times j}\big) = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1j} \\ C_{21} & C_{22} & \dots & C_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ C_{i1} & C_{i2} & \dots & C_{ij} \end{bmatrix}_{i\times j} \qquad \text{Where} \quad C_{ij} = (-1)^{i+j} \det\big(M_{ij}\big) \tag{15}$$

Note that this equation is effectively calculating the minor of every element within a matrix and then adding the sign in a checkerboard form as shown in **Eq. 16**.

$$\begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{16}$$

The cofactor matrix for a general 2×2 matrix is shown in **Eq. 17**

$$[A]_{2\times2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2} \tag{17.1}$$

$$\text{Cof}([A])_{2\times2} = \begin{bmatrix} +M_{11} & -M_{12} \\ -M_{21} & +M_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}_{2\times2} \tag{17.2}$$

The cofactor matrix for a general 3×3 matrix is shown in **Eq. 18**

$$[A]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \tag{18.1}$$

$$\text{Cof}([A])_{3\times3} = \begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}_{3\times3} = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}_{3\times3} \tag{18.2}$$

$$= \begin{bmatrix} +(a_{22}a_{33} - a_{23}a_{32}) & -(a_{21}a_{33} - a_{23}a_{31}) & +(a_{21}a_{32} - a_{22}a_{31}) \\ -(a_{12}a_{33} - a_{13}a_{32}) & +(a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) \\ +(a_{12}a_{23} - a_{13}a_{22}) & -(a_{11}a_{23} - a_{13}a_{21}) & +(a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}_{3\times3} \tag{18.3}$$

0.8.3. Adjoint Matrix

To finish off, the adjoint matrix is just the **transposition of the cofactor matrix**, the general mathematical definition of teh adjoint matrix is shown in **Eq. 19**. Note that the adjoint and cofactor operations can only be done on **square matrices**.

$$\text{Adj}\big([A_{i\times j}]\big) = \text{Cof}\big([A_{i\times j}]\big)^T \tag{19}$$

For completeness the adjoint of a general 2×2 matrix is shown in **Eq. 20** and for a 3×3 matrix in **Eq. 21**.

$$[A]_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2} \quad \rightarrow \quad \text{Adj}([A]_{2\times 2}) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}_{2\times 2} \tag{20}$$

$$[A]_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \tag{21.1}$$

$$\text{Adj}([A]_{3\times 3}) = \begin{bmatrix} (a_{22}a_{33} - a_{23}a_{32}) & -(a_{12}a_{33} - a_{13}a_{32}) & (a_{12}a_{23} - a_{13}a_{22}) \\ -(a_{21}a_{33} - a_{23}a_{31}) & (a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{23} - a_{13}a_{21}) \\ (a_{21}a_{32} - a_{22}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) & (a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}_{3\times 3} \tag{21.2}$$

0.9. Inverse Matrix

The inversion of a matrix is a key tool within linear algebra and will act as a backbone for FEA in general. The mathematical definition to invert a matrix is shown in **Eq. 22**.

$$[A]^{-1}_{i\times j} = \frac{1}{|[A]_{i\times j}|}\text{Adj}\big([A]_{i\times j}\big) \tag{22}$$

As **Eq. 22** uses the reciprocal of the determinant of the matrix, it also means that the **value of the determinant cannot itself be zero**. A matrix with a determinant of zero is said to be **singular**, **degenerate** or **rank deficient**. Examples for the inverses of a 2×2 matrix and a 3×3 matrix are shown in **Eq. 23** and **Eq. 24** respectively.

$$[A]_{2\times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2} \quad \rightarrow \quad [A]^{-1}_{2\times 2} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}_{2\times 2} \tag{23}$$

$$[A]_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \tag{24.1}$$

$$[A]^{-1}_{3\times 3} = \frac{1}{(a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}))}... \tag{24.2}$$

$$... \begin{bmatrix} (a_{22}a_{33} - a_{23}a_{32}) & -(a_{12}a_{33} - a_{13}a_{32}) & (a_{12}a_{23} - a_{13}a_{22}) \\ -(a_{21}a_{33} - a_{23}a_{31}) & (a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{23} - a_{13}a_{21}) \\ (a_{21}a_{32} - a_{22}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) & (a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}_{3\times 3} \tag{24.3}$$

0.10. Solving Systems of Equations Using matrices

Using the steps outlined before, it is possible to use matrices to solve a system of simultaneous equations. Lets say we have n **linear** equations, and therefore n number of unknowns, then the system of equations can be written in the form shown in **Eq. 25**.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{cases} \tag{25}$$

The system of equations shown in **Eq. 25** can then bbe written in the matrix form shown in **Eq. 26**.

$$[A]_{n\times n}[X]_{n\times 1} = [B]_{n\times 1} \tag{26.1}$$

$$\text{where} \quad [A]_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}_{n\times n} \quad [X]_{n\times 1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \quad [B]_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n\times 1} \tag{26.2}$$

Assuming that the $[X]_{n\times 1}$ matrix is the set of unknowns, then the solution of this system can be found by inverting the $[A]_{n\times n}$ matrix (this can be done as $[A]_{n\times n}[A]^{-1}_{n\times n} = [A]^{-1}_{n\times n}[A]_{n\times n} = [I]_{n\times n}$), the form of this expression is shown in.

$$[X]_{n\times 1} = [B]_{n\times 1}[A]^{-1}_{n\times n} \qquad \text{Given } |[A]_{n\times n}| \neq 0 \tag{27}$$

0.11. Matrix Zero-Padding

It is possible to use many of the aforementioned tools to "pad" a matrix with zeroes, with some degree of control. Let there be a submatrix $[K]_{n\times n}$ that we want to expand into a new matrix $\overline{K}$ to a size $m \times m$, with the K matrix being in the top left of the $\overline{K}$ matrix, then the multiplication shown in can be used **Eq. 28**.

$$[K]_{n\times n} = \begin{bmatrix} k_{11} & k_{12} & ... & k_{1j} \\ k_{21} & k_{22} & ... & k_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ k_{i1} & k_{i2} & ... & k_{ij} \end{bmatrix}_{n\times n} \quad \rightarrow \quad \left[\overline{K}\right]_{n\times n} = \begin{bmatrix} k_{11} & k_{12} & ... & 0 \\ k_{21} & k_{22} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 0 \end{bmatrix}_{m\times m} \tag{28.1}$$

$$\text{Then [A] must have shape} \qquad [A]_{n\times m} = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 0 \end{bmatrix}_{n\times m} \tag{28.2}$$

$$\therefore \quad \left[\overline{K}\right]_{n\times n} = [A]^T_{m\times n}[K]_{n\times n}[A]_{n\times m} \tag{28.3}$$

As an example a 2×2 matrix is embedded into a 5×5 matrix using **Eq. 28**, this is shown in **Eq. 29** .

$$[K]_{2\times 2} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}_{2\times 5} \qquad [A]_{2\times 5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}_{2\times 5} \tag{29.1}$$

$$[\overline{K}]_{5\times5} = [A]^T_{5\times2}[K]_{2\times2}[A]_{2\times5} \tag{29.2}$$

$$[\overline{K}]_{5\times5} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{5\times2} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}_{2\times5} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}_{2\times5} = \begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 \\ k_{21} & k_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times5} \tag{29.3}$$

Note that the placement of the $[K]$ matrix is dependant on where the ones are within the $[A]$ matrix. The further along the array the cascading ones are, the further down the diagonal the inserted $[K]$ matrix is. This process can be used to make a **banded matrix**, which is shown in **Eq. 30**.

$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 \\ k_{21} & k_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times5} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k_{11} & k_{12} & 0 & 0 \\ 0 & k_{21} & k_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times5} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{11} & k_{12} & 0 \\ 0 & 0 & k_{21} & k_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times5} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{11} & k_{12} \\ 0 & 0 & 0 & k_{21} & k_{22} \end{bmatrix}_{5\times5} \tag{30.1}$$

$$= \begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 \\ k_{21} & k_{22}+k_{11} & k_{12} & 0 & 0 \\ 0 & k_{21} & k_{22}+k_{11} & k_{12} & 0 \\ 0 & 0 & k_{21} & k_{22}+k_{11} & k_{12} \\ 0 & 0 & 0 & k_{21} & k_{22} \end{bmatrix}_{5\times5} \tag{30.2}$$

0.12. Matrix Cutting

In a similar way to how a matrix can be padded, it can also be cut to reduce the shape. This process has the following steps:

1. Start with an identity matrix the same size as the matrix to shorten.
2. Delete the columns you want to remove from the main matrix, keeping it square (this also will delete the rows with the same index).
3. Pre-multiply by the transpose of this reduced identity matrix and post-multiply by the original matrix to obtain the reduced form.

Eq. 31 Is an example of this where the 2nd, 3rd and 4th rows are removed from $[B]$.

$$[B]_{5\times5} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}_{5\times5} \tag{31.1}$$

$$[I]_{5\times5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times2} \rightarrow [C]_{5\times3} = \begin{bmatrix} 1 & \cancel{0} & 0 & \cancel{0} & \cancel{0} \\ 0 & \cancel{1} & 0 & \cancel{0} & \cancel{0} \\ 0 & \cancel{0} & 1 & \cancel{0} & \cancel{0} \\ 0 & \cancel{0} & 0 & \cancel{1} & \cancel{0} \\ 0 & \cancel{0} & 0 & \cancel{0} & \cancel{1} \end{bmatrix}_{5\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{5\times2} \tag{31.2}$$

$$\left[\overline{B}\right]_{2\times 2} = [C]^T_{2\times 5}[B]_{5\times 5}[C]_{5\times 3} \tag{31.3}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{2\times 5} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}_{5\times 5} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{5\times 2} = \begin{bmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{bmatrix}_{2\times 2} \tag{31.4}$$

0.13. Vector Insertion and Cutting

0.13.1. Vector Insertion

Vector insertion is easier than matrices padding, the following steps are used:

1) Define a matrix with zeros that has the following properties:

    a) Number of rows equals the final major dimension of the vector
    b) Number of columns equals the current major dimension of the vector

2) Add ones within the columns to specify where the zeroes will be inserted (see **Eq. 32**).

3) Pre-multiply by this matrix to obtain the modified column vector.

An example of this is shown in **Eq. 32** where zeroes are added before and after q_1 yielding the modfied $\{\overline{q}\}_{5\times 1}$.

$$\{q\}_{3\times 1} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times 1} \quad [C]_{5\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{5\times 3} \tag{32.1}$$

$$\{\overline{q}\}_{5\times 1} = [C]_{5\times 3}\{q\}_{3\times 1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{5\times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times 1} = \begin{Bmatrix} 0 \\ q_1 \\ 0 \\ q_2 \\ q_3 \end{Bmatrix}_{5\times 1} \tag{32.2}$$

0.13.2. Vector Cutting

This process is similar to insertion with the key difference being pre-multiplication by a differently shaped $[C]$ matrix:

1) Define a matrix with zeros that has the following properties:

    a) Number of **columns** equals the final major dimension of the vector
    b) Number of **rows** equals the current major dimension of the vector

2) Add ones within the columns to specify which values will be kept after the cut (see **Eq. 33**).

3) Pre-multiply by this matrix to obtain the modified column vector.

An example of this is shown in **Eq. 33** where the size of the vector is reduced by removing the first and third terms.

$$\{q\}_{5\times1} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{Bmatrix}_{5\times1} \quad [C]_{5\times3} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{3\times5} \tag{33.1}$$

$$\{\overline{q}\}_{3\times1} = [C]_{3\times5}\{q\}_{5\times1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{3\times5} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{Bmatrix}_{5\times1} = \begin{Bmatrix} q_2 \\ q_4 \\ q_5 \end{Bmatrix}_{3\times1} \tag{33.2}$$

0.14. Eigenvalues and Eigenvectors

Eigenvectors are vectors with a key property, multiplying them with a matrix yields a vector with the same direction but with an altered magnitude (scales the vector). Mathematically we can write this in the form shown in **Eq. 34**.

$$[A]_{n\times n}\{X\}_{n\times1} = \lambda\{X\}_{n\times1} \tag{34}$$

Eq. 35 can be used to find the value of the eigenvalues, which will then allow for independent solutions for the eigenvectors.

$$|[A]_{n\times n} - \lambda[I]_{n\times n}| = 0 \tag{35}$$

Note that the determinant exists here as the solution to this equation itself has to be singular. An example calculation of the eigenvalues for a 2×2 matrix is shown in **Eq. 36**.

$$[A]_{n\times n} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \quad \rightarrow \quad |[A]_{n\times n} - \lambda[I]_{n\times n}| = 0 \tag{36.1}$$

$$\left| \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 4-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0 \tag{36.2}$$

$$\lambda_1 = 5 \qquad \lambda_2 = 2 \tag{36.3}$$

The results for the eigenvalues can be plugged back into the following formula to yield the eigenvectors **Eq. 37**.

$$([A]_{n\times n} - \lambda[I]_{n\times n})[X]_{n\times1} = 0 \tag{37}$$

1. Lecture 1

START OF WEEK 1

1.1. Principle of Minimum Total Potential Energy (PMTPE)

The principle of minimum total potential energy is a concept that underpins the whole of FEA. It states that a structure will deform or displace to a position which minimizes the total potential energy of that structure. To build up to a mathematical definition of PMTPE, the total potential energy can be written as **Eq. 38**.

$$\Pi = U + V \tag{38}$$

Where:

- **Π**: The total potential energy (J).
- **U**: The stored elastic strain energy (J).
- **V**: The potential energy (negative of the work done $-W$ by external forces) (J).

PMTE then states that if we have an infinite number of possible responses to a system, the correct solution will have the lowest energy out of the infinite solutions. Mathematically, we are looking for a minimum for the value of Π which yields the **Eq. 39**.

$$\delta\Pi = 0 \quad \therefore \quad U = -V \tag{39}$$

1.2. Deriving the Deformation Behavior of Spring

We can apply PMTPE to a 1D spring in order to yield the equation $F = kq$. To start the scenario is defined in **Figure 1**.

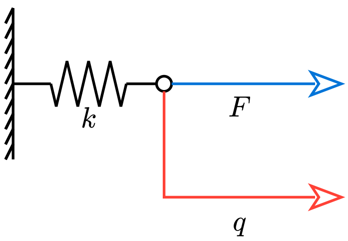

Figure 1: General 1D spring attatched to wall.

Where:
- **k**: Spring stiffness $(\frac{N}{m})$
- **F**: Force (N)
- **q**: Displacement (m)

Note that q is described as a **generalized coordinate** or a **degree of freedom** (DoF). From the scenario defined in **Figure 1**, an expression for the total potential energy of the system can be generated, shown in **Eq. 40**.

$$U = \frac{1}{2}kq^2 \quad V = -W = -Fq \tag{40.1}$$

$$\therefore \quad \Pi = \frac{1}{2}kq^2 - Fq \tag{40.2}$$

PMTPE implies that the true solution for the equation is given when the expression for the total potential energy is derived and then set to zero, this is shown in **Eq. 41**.

$$\frac{\partial \Pi(q)}{\partial q} = 0 \quad \rightarrow \quad \frac{\partial \Pi(q)}{\partial q} = \frac{1}{2}k \times 2q - F = 0 \tag{41.1}$$

$$F = kq \tag{41.2}$$

1.3. Deriving the Deformation Behavior of Two Springs

Similarly to when we had one spring, we can apply PMTPE to two springs joined together and see their behavior under a pair of loads, this scenario is defined in **Figure 2**.

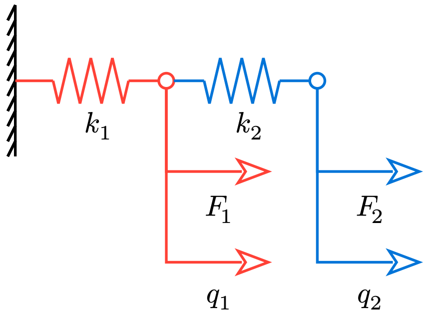

Figure 2: Two 1D spring attached to one another and a wall.

Now there are two unknowns of the system (q_1, q_2), which means there must be two equations to solve this scenario fully. This is generated by applying the PMTPE to both of the unknowns. First the values for U and V are defined, shown in **Eq. 42**.

$$U = \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2(q_2 - q_1)^2 \qquad V = -F_1 q_1 - F_2 q_2 \tag{42.1}$$

$$\therefore \qquad \Pi(q_1, q_2) = \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2(q_2 - q_1)^2 - (F_1 q_1 + F_2 q_2) \tag{42.2}$$

Note that in **Eq. 42**, the strain energy for the second spring has a subtracted q_1 term within it, this is due to the displacement q_2 being offset by q_1 when spring 1 is deformed. The solution to **Eq. 42** can be found by applying the PMTPE and deriving with respect to the unknown deformation variable, these steps are shown in **Eq. 43**.

$$\frac{\partial \Pi}{\partial q_1} = k_1 q_1 + k_2(q_2 - q_1) - F_1 = 0 \quad \rightarrow \quad F_1 = (k_1 + k_2)q_1 + k_2 q_2 \tag{43.1}$$

$$\frac{\partial \Pi}{\partial q_2} = 0 + k_2(q_2 - q_1) - F_2 = 0 \quad \rightarrow \quad F_2 = -k_2 q_1 + k_2 q_2 \tag{43.2}$$

Eq. 43 is a pair of linear simultaneous equations and therefore the matrix method can be used to solve this system of equations shown in **Eq. 44**.

$$\begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix}_{2\times2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2\times1} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}_{2\times1} \quad \rightarrow \quad [K]_{2\times2}\{q\}_{2\times1} = \{F\}_{2\times1} \tag{44}$$

Where:
- $[\boldsymbol{K}]$: Stiffness matrix
- $\{\boldsymbol{q}\}$: Displacement vector
- $\{\boldsymbol{F}\}$: Force vector

Note that **Eq. 44** can then be solved using matrix inversion $\{q\}_{2\times1} = [K]^{-1}_{2\times2}\{F\}_{2\times1}$.

1.4. Deriving deformation Behavior of a General Spring

More useful than the previous two examples, we can derive an expression for the deformation of a general spring in space, this scenario is shown in

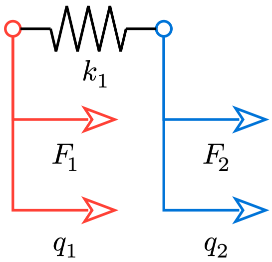

Figure 3: Entirely general 1D spring attached in free space.

Now that the scenario is defined, it is possible to write the expression for the strain energy of the system, this is shown in **Eq. 45**.

$$U = \frac{1}{2}k_1(q_2 - q_1)^2 \quad \rightarrow \quad U = \frac{1}{2}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T_{1\times 2} \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}_{2\times 2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2\times 1} \tag{45.1}$$

$$U = \frac{1}{2}\{q\}^T_{1\times 2}[K]_{2\times 2}\{q\}_{2\times 1} \tag{45.2}$$

Note that the equation for U can also be written in the matrix form shown on the left of **Eq. 45**. This form can be written in shorthand as $U = \frac{1}{2}\{q\}^T_{1\times 2}[K]_{2\times 2}\{q\}_{2\times 1}$. Even though **Eq. 45** is written as a set of matrices, it is still a **scalar**, which makes sense as U is an energy.

2. Lecture 2

START OF WEEK 2

2.1. Formulating Problems using PMTPE

The finite element method allows for the simplification of complex problems by **assembeling** together dioofferent elements. Take the scenario shown in **Figure 4**.

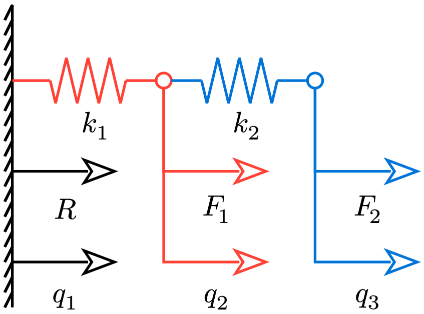

Figure 4: Two 1D springs attached to one another and a wall with boundary conditions defined.

Now that the scenario is defined, the strain energies for the two springs can be written in their matrix form, this is shown in .

$$U_1 = \frac{1}{2}k_1(q_2 - q_1)^2 \quad \rightarrow \quad U_1 = \frac{1}{2}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T_{1\times2} \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}_{2\times2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2\times1} \tag{46.1}$$

$$U_2 = \frac{1}{2}k_1(q_3 - q_2)^2 \quad \rightarrow \quad U_2 = \frac{1}{2}\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}^T_{1\times2} \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}_{2\times2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_{2\times1} \tag{46.2}$$

To find the total strain energy, these two components must be summed up. To achieve this, the matrix padding method shown in **Eq. 30** to ensue that the matrices are the same size as well as that the vector matrixes are the same as one another. This process is shown in **Eq. 47**.

$$U_1 = \frac{1}{2}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^T_{1\times2} \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}_{2\times2} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2\times1} \quad \rightarrow \quad U_1 = \frac{1}{2}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}^T_{1\times3} \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times1} \tag{47.1}$$

$$U_2 = \frac{1}{2}\begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}^T_{1\times2} \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}_{2\times2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_{2\times1} \quad \rightarrow \quad U_2 = \frac{1}{2}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}^T_{1\times3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}_{3\times3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times1} \tag{47.2}$$

$$\therefore \quad U = U_1 + U_2 = \frac{1}{2}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}^T_{1\times3} \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}_{3\times3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times1} \tag{47.3}$$

Note that here, the stiffness matrix is called the **assembled stiffness matrix**. The potential energy of the system can be written in the form shown in **Eq. 48**.

$$V = -(q_1 R + q_2 F_1 + q_3 F_2) \quad \rightarrow \quad V = -\{R \;\; F_1 \;\; F_2\}_{1\times3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times1} = -\{F\}^T_{1\times3}\{q\}_{3\times1} \tag{48}$$

Note that **Eq. 48** is still a scalar value which makes sense as V is an energy. At this point the PMTPE condition can be applied, $\frac{\partial \Pi(q_i)}{\partial q_i}$, to obtain a solution. Alternatively, the **governing equation of equilibrium** can be applied the system to obtain a solution, shown in **Eq. 49**.

$$[K]\{q\} = \{F\} \quad \rightarrow \quad \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}_{3\times 3} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}_{3\times 1} = \begin{Bmatrix} R \\ F_1 \\ F_2 \end{Bmatrix}_{3\times 1} \tag{49}$$

The next step to generate a solution is to invert the assembled matrix to find solutions for the $\{q\}$ matrix. However, currently the $[K]$ matrix is **singular**, as shown in **Eq. 50**.

$$\{q\} = [K]^{-1}\{F\} \quad \rightarrow \quad |K| = \begin{vmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{vmatrix} = 0 \tag{50}$$

$[K]$ being singular (determinant is equal to zero) means the inverse matrix cannot be calculated. This is effectively because the scenario is unbounded and so turns into a dynamics problem as there is nothing containing or holding down the springs. We can fix this by applying a **boundary condition**, for **Figure 4** we will apply $q_1 = 0$ as the boundary condition, this effectively eliminates those rows from the governing equation of equilibrium and makes the determinant non-zero, this is shown in **Eq. 51**.

$$\text{If } q_1 = 0 \text{ then} \quad \rightarrow \quad \begin{bmatrix} \cancel{k_1} & \cancel{-k_1} & \cancel{0} \\ \cancel{-k_1} & k_1 + k_2 & -k_2 \\ \cancel{0} & -k_2 & k_2 \end{bmatrix}_{3\times 3} \begin{Bmatrix} \cancel{q_1} \\ q_2 \\ q_3 \end{Bmatrix}_{3\times 1} = \begin{Bmatrix} \cancel{R} \\ F_1 \\ F_2 \end{Bmatrix}_{3\times 1} \tag{51.1}$$

$$\therefore \quad \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}_{2\times 2} \begin{Bmatrix} q_2 \\ q_3 \end{Bmatrix}_{2\times 1} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}_{2\times 1} \tag{51.2}$$

Note that the **crossed out rows and columns are not deleted, just temporarily ignored** and can be re-used once a solution for $\{q\}$ has been obtained to get a value for R. The assembled stiffness matrix with the boundary conditions applied shown in **Eq. 51** is no longer singular meaning that the scenario can be solved for $\{q\}$. The steps taken to solve any question using the finite element method and PMTPE are:

1. Write expressions for the elastic strain energies U and potential energies V per element.
2. Combine all of the Us and Vs using padding and matrix form to create a global $[K]$.
3. Apply boundary conditions to the system to make $[K]$ non-singular.
4. Use $\{q\} = [K]^{-1}\{F\}$ with $\{F\}$ from global V equation and $[K]$ from global U equation.
5. Obtain solutions for $\{q\}$.

3. Lecture 3

3.1. Elastic Rods

An elastic rod is another 1D element similar in nature to a spring element, however there are a few differences which make them more complex. Similar to springs, elastic rods can only deform axially in **tension or compression**. A general elastic rod element is shown in **Figure 5**.

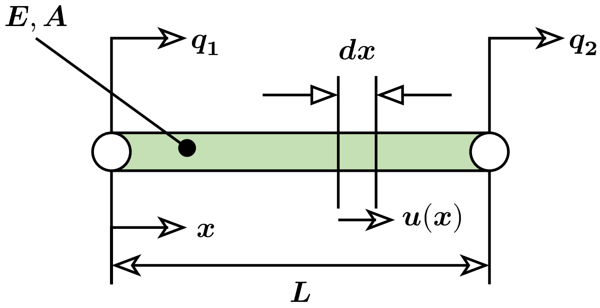

Figure 5: General elastic rod element.

Where:

- E: Young's Modulus (Pa).
- A: Cross sectional area (m^2).
- L: Length of bar (m).
- q_1: Axial displacement of LHS end (m).
- q_2: Axial displacement of RHS end (m).
- x: Distance from LHS along the bar (m).
- dx: Infinitesimal slice of the bar.
- $u(x)$: Deformation of the bar at slice (m).

3.2. Strain Energy for Elastic Rods

Similar to the spring element, the first step to generate a general expression for the elastic rod is to derive an expression for the strain energy U. This is shown in **Eq. 52**.

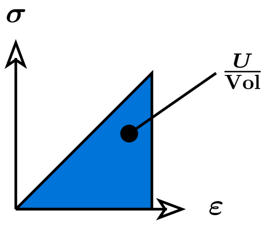

Figure 6: Strain energy plot.

$$\text{Assuming 1D uniaxial elasticity:} \tag{52.1}$$

$$U = \frac{1}{2}\int \sigma_x \varepsilon_x d\text{Vol} \tag{52.2}$$

$$\text{Given that the slice of the bar is a cylinder:} \tag{52.3}$$

$$U = \frac{1}{2}\int \sigma_x \varepsilon_x A dx \tag{52.4}$$

As we are assuming that the rod is behaving purely elastically, then the stress (σ_x) can be rewritten as shown in **Eq. 53**.

$$\sigma_x = E\varepsilon_x \quad \rightarrow \quad U = \frac{1}{2}\int EA\varepsilon_x^2 dx \tag{53}$$

Finally, the strain itself ε_x can be rewritten in terms of the deformation seen over the infinitesimal length slice, this is shown in **Eq. 54**.

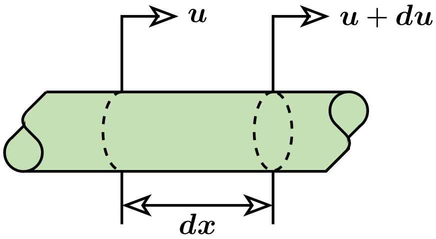

Figure 7: Deformation over slice.

The strain ε_x can be rewritten as: (54.1)

$$\varepsilon_x = \frac{\Delta L}{L} = \frac{(u - du) - u}{dx} = \frac{du}{dx} \tag{54.2}$$

Substituting **Eq. 54** into **Eq. 53** yields a final expression for the strain energy show in **Eq. 55**.

$$U = \frac{1}{2}\int EA\varepsilon_x^2 dx = \frac{1}{2}\int EA\left(\frac{du}{dx}\right)^2 dx = \frac{1}{2}\int EAu'^2 dx \tag{55}$$

3.3. Shape Functions for Elastic Rods

One issue with **Eq. 55** is that the strain energy is written in terms of the deformation which currently has no relation tio the displacements q_1 and q_2. Therefore a more rigorous definition of $u(x)$ must be developed, and this is done through utilizing shape functions, shown in **Eq. 56**.

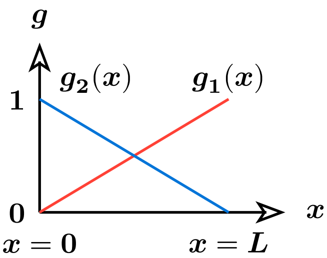

Figure 8: Shape functions for an elastic rod.

$$u(x) = g_1(x)q_1 + g_2(x)q_2 \tag{56.1}$$

$$g_1(x) = 1 - \frac{x}{L} \quad g_2(x) = \frac{x}{L} \tag{56.2}$$

$$\therefore u(x) = \left(1 - \frac{x}{L}\right)q_1 + \left(\frac{x}{L}\right)q_2 \tag{56.3}$$

The **shape function** controls the amount of **influence** each displacement receives and is effectively an interpolation. At $x = 0$ the displacement is given by q_1 and at $x = L$ the displacement is given by q_2. Using the expression for deformation shown in **Eq. 56**, expressions for the strain and stress can be derived as shown in **Eq. 57**.

$$\varepsilon(x) = \frac{du}{dx} = \left(-\frac{1}{L}\right)q_1 + \left(\frac{1}{L}\right)q_2 \qquad \sigma(x) = E\varepsilon(x) = \left(-\frac{E}{L}\right)q_1 + \left(\frac{E}{L}\right)q_2 \tag{57.1}$$

$$\varepsilon(x) = \left\{-\tfrac{1}{L} \ \ \tfrac{1}{L}\right\}_{1\times2}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2\times1} \qquad \sigma(x) = E\left\{-\tfrac{1}{L} \ \ \tfrac{1}{L}\right\}_{1\times2}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_{2\times1} \tag{57.2}$$

Note that these can also be written in a matrix form as is shown in **Eq. 57**. The expressions for the stress and strain in **Eq. 57** don't actually depend on x and so the stress and strain are constant throughout the bar.