Daily Integral

Limit

Author: Yusaf Sultan

Contents

1. 17/02/26: $\lim\limits_{x\to 0}\left(\dfrac{\cos(x)-\sqrt{1-x^2}}{\sin^2(2x)+x^2}\right)$

This limit can be evaluated by utilizing **Taylor series** for sin and cos as well as the **general binomial formula**. The general binomial formula is shown in **Eq. 1**

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... \tag{1}$$

Initially consider the first order approximations for the expression, shown in **Eq. 2**.

$$\cos(x) \approx 1 \quad \sin(x) \approx x \therefore \sin^2(2x) \approx 4x^2 \quad -\sqrt{1-x^2} \approx -1 \tag{2.1}$$

$$\therefore \lim_{x\to 0}\left(\frac{\cos(x)-\sqrt{1-x^2}}{\sin^2(2x)+x^2}\right) \approx \lim_{x\to 0}\left(\frac{1-1}{5x^2}\right) \approx \frac{0}{0} \tag{2.2}$$

The first order approximation is undefined, therefore a second order approximation up to O^2 is shown in **Eq. 3**, using **Eq. 1** for the $-\sqrt{1-x^2}$ term.

$$\therefore -\sqrt{1-x^2} \approx -\left(1+\left(\frac{1}{2}\right)(-x^2)\right) = -1 + \frac{x^2}{2} \tag{3.1}$$

$$\cos(x) \approx 1 - \frac{x^2}{2} \quad \sin(x) \approx x \therefore \sin^2(2x) \approx 4x^2 \tag{3.2}$$

$$\therefore \lim_{x\to 0}\left(\frac{\cos(x)-\sqrt{1-x^2}}{\sin^2(2x)+x^2}\right) \approx \lim_{x\to 0}\left(\frac{1-\frac{x^2}{2}-1+\frac{x^2}{2}}{5x^2}\right) \approx \frac{0}{0} \tag{3.3}$$

The second order approximation up to O^2 is also undefined so a third order approximation up to O^4 is shown in **Eq. 4**, using **Eq. 1** for the $-\sqrt{1-x^2}$ term.

$$\therefore -\sqrt{1-x^2} \approx -\left(1+\left(\frac{1}{2}\right)(x^2)+\left(\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}\right)\left(-x^2\right)^2\right) = -1 + \frac{x^2}{2} + \frac{x^4}{8} \tag{4.1}$$

$$\cos(x) \approx 1 - \frac{x^2}{2} + \frac{x^4}{24} \quad \sin(x) \approx x - \frac{x^3}{6} \therefore \sin^2(2x) \approx 4x^2 - \frac{16}{3}x^4 (\text{to } O^4) \tag{4.2}$$

$$\therefore \lim_{x\to 0}\left(\frac{\cos(x)-\sqrt{1-x^2}}{\sin^2(2x)+x^2}\right) \approx \lim_{x\to 0}\left(\frac{1-\frac{x^2}{2}+\frac{x^4}{24}-1+\frac{x^2}{2}+\frac{x^4}{8}}{4x^2-\frac{16}{3}x^4+x^2}\right) \approx \lim_{x\to 0}\left(\frac{\frac{x^4}{6}}{5x^2-\frac{16}{3}x^4}\right) \tag{4.3}$$

The final simplified limit shown in **Eq. 4** can be rearranged and evaluated to obtain a solution, shown in **Eq. 5**.

$$\lim_{x\to 0}\left(\frac{\frac{x^4}{6}}{5x^2-\frac{16}{3}x^4}\right) = \lim_{x\to 0}\left(\frac{\frac{x^4}{6}(1)}{\frac{x^4}{6}\left(\frac{30}{x^2}-32\right)}\right) = \lim_{x\to 0}\left(\frac{1}{\frac{30}{x^2}-32}\right) = \frac{1}{\infty - 32} = \frac{1}{\infty} = 0 \tag{5}$$