Daily Integral

Hard Derivatives

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1. 05/02/2026: $f(x) = \frac{\arctan^5(x)}{\sqrt{\arccos^4(x)}} + \frac{\sin^2(x)}{\cot^2(x)},\ f'(0.7) = ?$

To start with the derivative is defined and split into two separate derivatives using the **sum rule**, this is shown in **Eq. 1**.

$$f'(x) = \frac{d}{dx}(f(x)) = \frac{d}{dx}\left(\frac{\arctan^5(x)}{\sqrt{\arccos^4(x)}}\right) + \frac{d}{dx}\left(\frac{\sin^2(x)}{\cot^2(x)}\right) \tag{1}$$

The first section of the derivative can then be solved using a combination of the **quotient rule** and **chain rule**, the u and vs are shown in **Eq. 2**.

$$\text{Let } u = \arctan^5(x) \quad v = \sqrt{\arccos^4(x)} \tag{2}$$

To differentiate u and v, chain rule can be used. To obtain $\frac{du}{dx}$ the chain rule is applied in **Eq. 3**.

$$u = \arctan^5(x) \longrightarrow \text{Let } z = \arctan(x) \therefore u = z^5 \tag{3.1}$$

$$\frac{dz}{dx} = \frac{1}{1+x^2} \quad \frac{du}{dz} = 5z^4 \tag{3.2}$$

$$\frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx} = \frac{5z^4}{1+x^2} = \frac{5\arctan^4(x)}{1+x^2} \tag{3.3}$$

To obtain an expression for $\frac{dv}{dx}$ chain rule can be used twice, this is shown in **Eq. 4**.

$$v = \sqrt{\arccos^4(x)} \longrightarrow \text{Let } z = \arccos^4(x) \therefore v = \sqrt{z} \tag{4.1}$$

$$z = \arccos^4(x) \longrightarrow \text{Let } q = \arccos(x) \therefore z = q^4 \tag{4.2}$$

$$\frac{dz}{dq} = 4q^3 \quad \frac{dq}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \frac{dv}{dz} = \frac{1}{2\sqrt{z}} \tag{4.3}$$

$$\frac{dv}{dx} = \frac{dv}{dz} \cdot \frac{dz}{dx} = \frac{dv}{dz} \cdot \frac{dz}{dq} \cdot \frac{dq}{dx} = \frac{4q^3}{2\sqrt{z}\left(\sqrt{1-x^2}\right)} = \frac{4\arccos^3(x)}{2\sqrt{\arccos^4(x)}\left(\sqrt{1-x^2}\right)} = \ldots \tag{4.4}$$

$$\ldots = \frac{2\arccos(x)}{\sqrt{1-x^2}} \tag{4.5}$$

Now that the expressions for $\frac{du}{dx}$ and $\frac{dv}{dx}$ have been obtained, teh quotient rule can be applied to obtain $f'(x)$ for the first term, this is shown in **Eq. 5**.

$$\text{Let } u = \arctan^5(x) \quad v = \sqrt{\arccos^4(x)} \tag{5.1}$$

$$\frac{du}{dx} = \frac{5\arctan^4(x)}{1+x^2} \quad \frac{dv}{dx} = \frac{2\arccos(x)}{\sqrt{1-x^2}} \tag{5.2}$$

$$A' = \frac{1}{v^2}\left[\frac{du}{dx}v - u\frac{dv}{dx}\right] = \frac{1}{\arccos^4(x)}\left[\frac{5\arctan^4(x)\arccos^2(x)}{1+x^2} + \frac{2\arctan^5(x)\arccos(x)}{\sqrt{1-x^2}}\right] \tag{5.3}$$

The second section of the expression can first be simplified using trigonometric expressions and then the quotient rule can be applied. The initial trig simplification is shown in **Eq. 6**.

$$B = \frac{\sin^2(x)}{\cot^2(x)} = \frac{\sin^4(x)}{\cos^2(x)} \quad \text{As } \cot x = \frac{\cos x}{\sin x} \tag{6}$$

Eq. 6 can be differentiated using a mixture of chain rule and quotient rule, the u and v are defined in **Eq. 7**

$$\text{Let } u = \sin^4(x) \quad v = \cos^2(x) \tag{7}$$

An expression for $\frac{du}{dx}$ can be generated by using the chain rule shown in **Eq. 8**.

$$u = \sin^4(x) \longrightarrow \text{Let } z = \sin(x) \therefore u = z^4 \tag{8.1}$$

$$\frac{dz}{dx} = \cos(x) \quad \frac{du}{dz} = 4z^3 \tag{8.2}$$

$$\frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx} = 4\cos(x)z^3 = 4\cos(x)\sin^3(x) \tag{8.3}$$

An expression for $\frac{dv}{dx}$ can be generated by using the chain rule shown in **Eq. 9**.

$$u = \cos^2(x) \longrightarrow \text{Let } z = \cos(x) \therefore u = z^2 \tag{9.1}$$

$$\frac{dz}{dx} = -\sin(x) \quad \frac{du}{dz} = 2z \tag{9.2}$$

$$\frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx} = -2\sin(x)z = -2\sin(x)\cos(x) \tag{9.3}$$

Now that an expression for $\frac{du}{dx}$ and $\frac{dv}{dx}$ have been defined, the quotient rule can be applied, as shown in **Eq. 10**

$$\text{Let } u = \sin^4(x) \quad v = \cos^2(x) \tag{10.1}$$

$$\frac{du}{dx} = 4\sin^3(x)\cos(x) \quad \frac{dv}{dx} = -2\sin(x)\cos(x) \tag{10.2}$$

$$B' = \frac{1}{v^2}\left[\frac{du}{dx}v - u\frac{dv}{dx}\right] = \frac{1}{\cos^2(x)}[4\sin^3(x)\cos^3(x) + 2\sin^5(x)\cos(x)] = ... \tag{10.3}$$

$$... = 4\sin^3(x)\cos(x) + \frac{2\sin^5(x)}{\cos(x)} \tag{10.4}$$

Combining the A' and B' terms together yield a final expression for $f'(x)$ can be constructed, this is shown in **Eq. 10**.

$$f'(x) = \frac{1}{\arccos^4(x)}\left[\frac{5\arctan^4(x)\arccos^2(x)}{1+x^2} + \frac{2\arctan^5(x)\arccos(x)}{\sqrt{1-x^2}}\right] + ... \tag{11.1}$$

$$.. + 4\sin^3(x)\cos(x) + \frac{2\sin^5(x)}{\cos(x)} \tag{11.2}$$

To solve the question, $x = 0$ is substituted into $f'(0)$, which is shown in **Eq. 11**.

$$f'(0) = 2.498(0.300 + 0.189) + 0.818 + 0.2901 \approx 2.32 \quad (2 \text{ d.p}) \tag{12}$$

2. 06/02/2026: $\sqrt[5]{\ln(3+\cos(e^{x^2}))}\tan\left(\arctan(x^3)+\sqrt{x+2}\right),\ f'(0) = ?$

To solve this derivative a combination of **product rule** and **chain rule** can be used, the u and vs are shown in **Eq. 13**.

$$\text{Let } u = \sqrt[5]{\ln(3+\cos(e^{x^2}))} \quad v = \tan\left(\arctan(x^3)+\sqrt{x+2}\right) \tag{13}$$

To obtain an expression for $\frac{du}{dx}$ chain rule can be repeatedly used, this is shown in **Eq. 14**.

$$u = \sqrt[5]{\ln(3+\cos(e^{x^2}))} \longrightarrow \text{Let } A = \ln\left(3+\cos\left(e^{x^2}\right)\right) \therefore u = \sqrt[5]{A} \tag{14.1}$$

$$A = \ln\left(3+\cos\left(e^{x^2}\right)\right) \longrightarrow \text{Let } B = 3+\cos\left(e^{x^2}\right) \therefore A = \ln(B) \tag{14.2}$$

$$B = 3+\cos\left(e^{x^2}\right) \longrightarrow \text{Let } C = e^{x^2} \therefore B = 3+\cos(C) \tag{14.3}$$

$$C = e^{x^2} \longrightarrow \text{Let } D = x^2 \therefore C = e^D \tag{14.4}$$

$$\frac{du}{dA} = \frac{1}{5\left(\sqrt[5]{A}\right)^4} \quad \frac{dA}{dB} = \frac{1}{B} \quad \frac{dB}{dC} = -\sin(C) \quad \frac{dC}{dD} = e^D \quad \frac{dD}{dx} = 2x \tag{14.5}$$

$$\frac{du}{dx} = \frac{dA}{dB}\cdot\frac{dB}{dC}\cdot\frac{dC}{dD}\cdot\frac{dD}{dx}\cdot\frac{du}{dA} = \dots \tag{14.6}$$

$$\dots = \frac{1}{B}\cdot -\sin(C)\cdot e^D\cdot 2x\cdot\frac{1}{5\left(\sqrt[5]{A}\right)^4} = \frac{-2xe^{x^2}\sin\left(e^{x^2}\right)}{5\left(\sqrt[5]{\ln(3+\cos(e^{x^2}))}\right)^4(3+\cos(e^{x^2}))} \tag{14.7}$$

The v term can be differentiated using chain rule, however each term within the tan brackets will need to have chain rule applied separately, this is done for the $\arctan(x^3)$ term in **Eq. 15**.

$$A_1 = \arctan(x^3) \text{ Let } z = x^3 \therefore A_1 = \arctan(z) \tag{15.1}$$

$$\frac{dz}{dx} = 3x^2 \quad \frac{dA_1}{dz} = \frac{1}{1+z^2} \tag{15.2}$$

$$\frac{dA_1}{dx} = \frac{dA_1}{dz}\cdot\frac{dz}{dx} = \frac{3x^2}{1+x^6} \tag{15.3}$$

A similar process can be done for the second term ($\sqrt{x+2}$) which is shown in **Eq. 16**.

$$A_2 = \sqrt{x+2} \text{ Let } z = x+2 \therefore A_2 = \sqrt{z} \tag{16.1}$$

$$\frac{dz}{dx} = 1 \quad \frac{dA_2}{dz} = \frac{1}{2\sqrt{z}} \tag{16.2}$$

$$\frac{dA_2}{dx} = \frac{dA_2}{dz}\cdot\frac{dz}{dx} = \frac{1}{\sqrt{x+2}} \tag{16.3}$$

Now that expressions for $\frac{dA_1}{dx}$ and $\frac{dA_2}{dx}$ have been obtained chain rule can be constructed into an expression for $\frac{dv}{dx}$, shown in **Eq. 17**.

$$v = \tan\Big(\arctan(x^3) + \sqrt{x+2}\Big) \text{ Let } z = \arctan(x^3) + \sqrt{x+2} \therefore v = \tan(z) \tag{17.1}$$

$$\frac{dz}{dx} = \frac{dA_1}{dx} + \frac{dA_2}{dx} = \frac{3x^2}{1+x^6} + \frac{1}{\sqrt{x+2}} \quad \frac{dv}{dz} = \sec^2(z) \tag{17.2}$$

$$\frac{dv}{dx} = \frac{dv}{dz} \cdot \frac{dz}{dx} = \sec^2(z)\left[\frac{3x^2}{1+x^6} + \frac{1}{\sqrt{x+2}}\right] = ... \tag{17.3}$$

$$... = \frac{3x^2 \sec^2\Big(\arctan(x^3) + \sqrt{x+2}\Big)}{1+x^6} + \frac{\sec^2\Big(\arctan(x^3) + \sqrt{x+2}\Big)}{\sqrt{x+2}} \tag{17.4}$$

3. 17/02/2026: $f(x) = x^{x^{x^{x^{\cdots}}}}$, $f'\left(\frac{6}{5}\right) = ?$

To solve this question, **implicit differentiation** can be used in combination with an **iterative processes**. Initially the function can be simplified into an expression that can be differentiated, shown in **Eq. 18**.

$$y = x^{x^{x^{x^{\cdots}}}} \longrightarrow y = x^y \longrightarrow \ln(y) = y\ln(x) \tag{18}$$

Each term in **Eq. 18** can be differentiated, using product rule for the $x\ln(y)$ term, shown in **Eq. 19**.

$$A' = \frac{d}{dx}(\ln(y)) = \frac{1}{y}\frac{dy}{dx} \tag{19.1}$$

$$\text{Let } u = y \quad v = \ln(x) \tag{19.2}$$

$$\frac{du}{dx} = \frac{dy}{dx} \quad \frac{dv}{dx} = \frac{1}{x} \tag{19.3}$$

$$B' = u\frac{dv}{dx} + \frac{du}{dx}v = \frac{y}{x} + \frac{dy}{dx}\ln(x) \tag{19.4}$$

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{y}{x} + \frac{dy}{dx}\ln(x) \longrightarrow \frac{dy}{dx} = \frac{y^2}{x - xy\ln(x)} \tag{19.5}$$

An approximate value of y can be found by using the simplified form of the expression and iterating with an initial guess to find the true value of y when $x = \frac{6}{5}$, this is shown in **Eq. 20**.

$$y_{n+1} = x^{y_n} \quad \text{Let } x = \frac{6}{5} = 1.2,\ y_0 = 2 \tag{20.1}$$

$$y_1 = x^{y_0} = 1.2^2 = 1.44 \tag{20.2}$$

$$y_2 = x^{y_1} = 1.2^{1.44} = 1.30 \tag{20.3}$$

$$\ldots \tag{20.4}$$

$$y_7 = 1.2577 \therefore y \approx 1.26 \tag{20.5}$$

The values of x and y can then be inputted into $f'(x)$ to obtain a solution for the problem, shown in **Eq. 21**.

$$f'(x) = \frac{y^2}{x - xy\ln(x)} = \frac{1.26^2}{1.2 - 1.2 \times 1.26 \times \ln(1.2)} \approx 1.71 \quad \text{(2 d.p)} \tag{21}$$