Daily Integral

Hard Integrals

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1. 24/02/2026: $\int_0^1 \frac{x^{e-1}-1}{x-1-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-...}dx$

This integral can be initially simplified by observing that the denominator of the integral is a Taylor Series of $\ln(x)$, this simplification is applied in **Eq. 1**.

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... \quad \therefore \quad \ln(x) = x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - ... \tag{1.1}$$

$$\therefore \quad \int_0^1 \frac{x^{e-1}-1}{\ln(x)}dx \tag{1.2}$$

This integral can then be solved by taking the general case of this form of integral where x is raised to an arbitrary power. By applying the **Leibniz integral rule**, this general expression can be obtained, shown in **Eq. 2**.

$$\int_0^1 \frac{x^a-1}{\ln(x)}dx \quad \longrightarrow \quad \text{Let } I(a) = \int_0^1 \frac{x^a-1}{\ln(x)}dx \tag{2.1}$$

$$\frac{d}{da}(I(a)) = I'(a) = \frac{d}{da}\left(\int_0^1 \frac{x^a-1}{\ln(x)}dx\right) = \int_0^1 \frac{\partial}{\partial a}\left[\frac{x^a-1}{\ln(x)}\right]dx = ... \tag{2.2}$$

$$... = \int_0^1 \frac{1}{\ln(x)}\frac{\partial}{\partial a}[(x^a-1)]dx = \int_0^1 \frac{1}{\ln(x)} \cdot \ln(x)x^a dx = \int_0^1 x^a dx = ... \tag{2.3}$$

$$... = \left[\frac{x^{a+1}}{a+1}\right]_{x=0}^{x=1} = \left(\frac{1^{a+1}}{a+1}\right) - \left(\frac{0^{a+1}}{a+1}\right) = \frac{1}{a+1} \tag{2.4}$$

$$\therefore \quad I'(a) = \frac{1}{a+1} \tag{2.5}$$

Integrating $I'(a)$ will obtain an expression for $I(a)$, being careful to eliminate the constant of integration, this process is shown in **Eq. 3**.

$$I(a) = \int I'(a)da = \int \frac{1}{a+1}da = \ln|a+1| + C \tag{3.1}$$

$$I(0) = \ln|0+1| + C = C \quad \therefore \quad I(0) = C \tag{3.2}$$

$$\text{Using the original definition of } I(a) \longrightarrow I(a) = \int_0^1 \frac{x^a-1}{\ln(x)}dx \tag{3.3}$$

$$I(0) = \int_0^1 \frac{x^0-1}{\ln(x)}dx = \int_0^1 0dx = 0 \quad \therefore \quad C = 0 \tag{3.4}$$

$$\therefore \quad \int_0^1 \frac{x^a-1}{\ln(x)}dx = \ln|a+1| \tag{3.5}$$

Now that a general expression for an integral of the form in the question has been obtained, $a = e - 1$ can be inputted in the expression to obtain a solution, shown in **Eq. 4**.

$$\int_0^1 \frac{x^{e-1}-1}{\ln(x)}dx = \ln(e-1+1) = \ln(e) = 1 \tag{4}$$