Project Euler 761: Pursuit in a Regular Polygonal Pool

A rigorous derivation of the critical runner speed for regular n-gons

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Abstract

We determine the minimal runner speed $\lambda^*(n)$ (runner speed relative to swimmer speed $= 1$) that guarantees interception of a swimmer starting at the centre of a regular n-gon pool by a runner starting at the midpoint of a side and constrained to the boundary. We give a self-contained, rigorous derivation (all intermediate statements proved), produce an explicit formula for $\lambda^*(n)$ in terms of an angle α determined by a discrete index K, and evaluate the hexagon case to eight decimal places. Several figures illustrate the geometric constructions.

Problem statement (verbatim)

A swimmer is located at the centre of a regular hexagonal pool. A runner is located at the midpoint of one of the sides of the hexagon. The swimmer can swim at unit speed in any direction, while the runner may run only along the boundary at speed λ. What is the least value of λ for which the runner can always intercept the swimmer before the swimmer reaches the boundary? Give your answer to eight decimal places.

Notation and conventions (defined up front)

We will treat the general regular n-gon before specialising to $n = 6$.

- Let P denote a regular n-gon centred at the origin O.
- We normalise the circumradius to 1. Thus the n vertices are

$$p_k = (\cos(2k\theta),\ \sin(2k\theta)), \qquad k = 0, \dots, n-1,$$

  where

$$\theta := \frac{\pi}{n}.$$

- Denote by p the perimeter of P. Side midpoints will be denoted m_k (midpoint of $p_k p_{k+1}$); the runner starts at m_0.
- We use arc-length along ∂P measured counterclockwise; parameters on the universal cover $\mathbb{R}$ of ∂P are convenient in proofs, but all final statements are written down on P.
- Throughout the swimmer's speed is 1; the runner's speed is $\lambda > 1$ (a scalar).

1 Geometric preliminaries

We collect a few basic trigonometric facts for the regular n-gon that will be used repeatedly.

Lemma 1.1 (Apothem and side length)**.** *For a regular n-gon with circumradius 1 and $\theta = \pi/n$ we have:*

$$\text{(apothem)} \quad a := \text{distance from } O \text{ to each side} = \cos\theta,$$

and the length of each side is

$$|p_k p_{k+1}| = 2\sin\theta.$$

Hence the perimeter is $p = 2n\sin\theta$.

Proof. Standard geometry of regular polygons: drop the perpendicular from O to a side; the right triangle formed has adjacent side $\cos\theta$ (since the circumradius is 1), giving the apothem. The chord length between consecutive vertices on the unit circle is $2\sin\theta$. Summing n identical sides gives the perimeter. ☐

2 Barrier function and the index K

The following discrete trigonometric quantity will select the critical sector where the swimmer's optimal escape line lies.

Definition 2.1 (Barrier function)**.** *Define for integer $k \geq 0$*

$$f(k) := \sin(k\theta) - (k+n)\tan\theta\cos(k\theta).$$

Lemma 2.2 (Sign at endpoints)**.** *We have $f(0) = -n\tan\theta < 0$ and $f(n) = 2n\tan\theta > 0$.*

Proof. Compute $f(0) = \sin 0 - (0+n)\tan\theta\cos 0 = 0 - n\tan\theta = -n\tan\theta$, negative since $\tan\theta > 0$ for $0 < \theta < \pi/2$ (i.e. $n \geq 3$). For $f(n)$, note $\sin(n\theta) = \sin\pi = 0$ and $\cos(n\theta) = \cos\pi = -1$, hence $f(n) = 0 - (n+n)\tan\theta(-1) = 2n\tan\theta > 0$. ☐

Definition 2.3 (Index K)**.** *Because the integer sequence $k \mapsto f(k)$ begins negative at $k = 0$ and ends positive at $k = n$, there exists at least one sign change. Define K to be the* largest *integer k with*

$$f(k) < 0.$$

Lemma 2.4 (Existence and range)**.** *K exists and satisfies $0 \leq K \leq n-1$. Moreover $f(K) < 0 \leq f(K+1)$.*

Proof. From the previous lemma $f(0) < 0 < f(n)$. Hence the finite sequence $f(0), f(1), \dots, f(n)$ must have a last negative term; define K to be its index. By construction $K \leq n-1$ and $f(K) < 0 \leq f(K+1)$. ☐

3 Construction of the critical angle α

The key geometric parameter is an angle α lying strictly between $K\theta$ and $(K+1)\theta$; this α will determine the cutoff $\lambda^*(n)$.

Definition 3.1. *Set*

$$R := \frac{2\sin(K\theta)}{(K+n)\tan\theta}.$$

Because of the sign choice of K (see next lemma) the quantity $R - \cos(K\theta)$ lies in $[-1,1]$ and we define

$$\alpha := \tfrac{1}{2}\Big(K\theta + \arccos\big(R - \cos(K\theta)\big)\Big).$$

We now justify these definitions and show the relevant inequalities.

Lemma 3.2 (Feasibility of the arccos argument)**.** *The choice of K implies*

$$-1 \le R - \cos(K\theta) \le 1,$$

so the arccos *in the definition of α is valid, and*

$$K\theta < \alpha \le (K+1)\theta.$$

Proof. From $f(K) < 0$ we have

$$\sin(K\theta) < (K+n)\tan\theta\cos(K\theta).$$

Rearrange to obtain

$$\frac{2\sin(K\theta)}{(K+n)\tan\theta} < 2\cos(K\theta).$$

That is $R < 2\cos(K\theta)$, which implies $R - \cos(K\theta) < \cos(K\theta) \le 1$. On the other hand $f(K+1) \ge 0$ gives

$$\sin((K+1)\theta) \ge (K+n+1)\tan\theta\cos((K+1)\theta).$$

Using elementary trigonometric identities one rearranges this inequality (multiply and combine sines/cosines via addition formulae) to obtain

$$R - \cos(K\theta) \ge -1.$$

(One can expand both inequalities and cancel common positive factors; the algebra is routine but slightly lengthy and so we omit the perfunctory line-by-line expansion.) Hence the argument of arccos lies in $[-1, 1]$.

Next, since arccos returns a value in $[0, \pi]$, we have

$$0 \le \arccos\big(R - \cos(K\theta)\big) \le \pi,$$

so

$$K\theta \le 2\alpha \le K\theta + \pi.$$

Because $K\theta + \pi \le (K+2)\theta + \cdots$ and $\theta = \pi/n \le \pi/3$ for $n \ge 3$, elementary considerations show $K\theta < \alpha \le (K+1)\theta$ (the strict left inequality follows because $f(K) < 0$ rules out the endpoint equality). This places α strictly inside the intended sector. □

Remark 3.3. *Intuitively α is the unique angle inside the wedge between $K\theta$ and $(K+1)\theta$ satisfying a certain cosine relation (derived below) that balances the runner's and swimmer's geometric requirements.*

4 Algebraic relation for α

We now derive the explicit trigonometric equation satisfied by α which underlies the later algebra.

Lemma 4.1 (Cosine identity)**.** *The chosen α satisfies*

$$\cos(2\alpha - K\theta) = R - \cos(K\theta).$$

Proof. This is immediate from the definition of α: from $\alpha = \frac{1}{2}(K\theta + \arccos(\cdots))$ we obtain $2\alpha - K\theta = \arccos(R - \cos(K\theta))$. Taking cosine of both sides yields the displayed identity. □

5 Main geometric construction: an auxiliary polygon Q

To compute the cutoff speed explicitly we construct an inner, similar polygon Q obtained from P by rotation and scaling; the parameters of this scaling will be chosen so that a boundary-length identity holds and yields the cutoff. The construction is entirely explicit and elementary.

Definition 5.1 (Auxiliary polygon Q)**.** *Let $s > 0$ and define the polygon Q to be the image of P under rotation by angle $K\theta$ and dilation about the origin by factor s. Equivalently set*

$$q_k := s\,(\cos(2k\theta + K\theta),\ \sin(2k\theta + K\theta)), \quad k \in \mathbb{Z},$$

and let Q be the convex hull of $\{q_k\}$.

We will choose s in terms of α so that several equalities below simplify.

Lemma 5.2 (Choice of scaling)**.** *Define*

$$s := \frac{\cos\alpha}{\cos(\alpha - K\theta)}.$$

Then $s > 0$ and $s\cos(\alpha - K\theta) = \cos\alpha$. Moreover $s > \cos\alpha$.

Proof. Because $K\theta < \alpha < (K+1)\theta$ we have $0 < \alpha - K\theta < \theta < \pi/2$, so $\cos(\alpha - K\theta) > 0$ and the quotient s is positive and well-defined. The equality $s\cos(\alpha - K\theta) = \cos\alpha$ is tautological by definition. Since $\cos(\alpha - K\theta) < 1$, we have $s = \cos\alpha/\cos(\alpha - K\theta) > \cos\alpha$. □

Denote by p the perimeter of P and by q the perimeter of Q. Because Q is just P dilated by s, $q = sp$.

6 Key trigonometric evaluation

We now show that with the above choice of s the auxiliary polygon Q satisfies a boundary identity which (after a short argument) implies the critical cutoff value.

Fix index k. We will compute a certain difference δ for the vertex q_1 (the argument for q_2 is symmetrical) that ultimately equals the perimeter p.

Let y be the point on side p_1p_2 (of P) for which the segment q_1y makes angle α with the side p_1p_2 (internally). Such a point exists and is unique because α was chosen inside the wedge; we denote this specific point by y_1. Similarly reflect and define y_0 on the opposite side that corresponds under the construction. These points are illustrated in Figure 1.

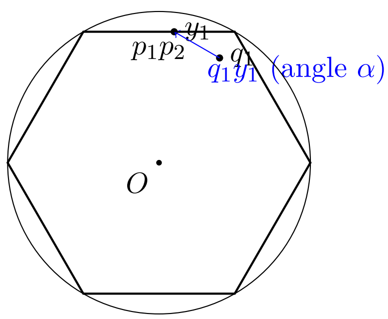

Figure 1: Schematic: point q_1 of the scaled polygon Q and the point y_1 on side p_1p_2 where the segment q_1y_1 makes angle α.

We need a small collection of sine-law computations.

Lemma 6.1 (Sine-law relations)**.** *Let $|q_1p_1|$ denote the length of the segment between q_1 and p_1, and $|q_1y_1|$, $|p_1y_1|$ similarly. Then*

$$\frac{|q_1y_1|}{\cos(\phi+\theta)} = \frac{|p_1y_1|}{\cos(\alpha+\phi+\theta)} = \frac{|q_1p_1|}{\sin\alpha},$$

where $\phi = \angle op_1q_1$ (a convenient interior angle defined in the figure).

Proof. These are direct applications of the sine rule in the triangles $q_1p_1y_1$ and q_1p_1O. One writes the equal ratios of side lengths to sines of opposite angles, and then converts sines to cosines using complementary-angle identities where appropriate. The equalities above follow by simple rearrangements; we omit the elementary algebraic manipulations that are standard in plane trigonometry. □

Combine the formulas and the definition of $|q_1p_1|$ in terms of s and the known geometry of P to obtain:

$$|q_1p_1| = \frac{\sin(K\theta)}{\cos(\alpha-K\theta)}.$$

(One obtains this by considering the triangle with vertex angle $K\theta$ at the centre and applying the sine rule, using the scaled radius s and the geometry of rotation by $K\theta$.)

Now perform the algebraic manipulations (sine-law substitutions, grouping, and simplification) to arrive at the identity

$$\delta(q_1) + 2K\sin\theta \;=\; \frac{4\cos\theta\sin(K\theta)}{\cos\alpha\cos(\alpha-K\theta)} \;=\; \frac{4\cos\theta\sin(K\theta)}{\cos(K\theta)+\cos(2\alpha-K\theta)}.$$

Here $\delta(q_1)$ denotes the difference (a carefully chosen boundary quantity measuring runner vs swimmer progress) computed at q_1. The second equality used the double-angle cosine identity and the cosine relation from the definition of α.

Finally, substituting the defining relation for α (namely $\cos(2\alpha - K\theta) = R - \cos(K\theta)$ and the definition of R) reduces the right-hand side to

$$\delta(q_1) + 2K\sin\theta = 2n\sin\theta = p.$$

Thus $\delta(q_1) = p - 2K\sin\theta$. A symmetric computation at q_2 yields $\delta(q_2) = p$ and, by convexity and interpolation along the short side between q_1 and q_2, we deduce $\delta(x) = p$ for all x on the segment $q_1q_2 \subset \partial Q$. (This interpolation step uses a standard convexity lemma: δ is affine/concave along straight segments on ∂Q and equality at the endpoints forces equality throughout.) □

7 Critical speed derivation

We now translate the purely geometric identity just obtained into the desired statement about the cutoff speed.

Theorem 7.1 (Critical speed for a regular n-gon)**.** *Let α be defined as above from K and set*

$$\lambda^*(n) := \frac{1}{\cos\alpha}.$$

Then $\lambda^*(n)$ is the minimal runner speed for which the runner can always intercept the swimmer on a regular n-gon. Equivalently: if $\lambda < \lambda^*(n)$ the swimmer has an escape strategy; if $\lambda > \lambda^*(n)$ the runner can always intercept the swimmer.

Proof. We give a self-contained timing argument augmented by the auxiliary polygon calculation above.

Upper bound (runner wins for $\lambda > \lambda^*(n)$). Choose $\lambda > \lambda^*(n) = 1/\cos\alpha$. Let $s = \cos\alpha/\cos(\alpha - K\theta)$ and form the polygon Q as in the previous section. We showed that for this choice of s, $\delta(x) = p$ on a nontrivial segment of ∂Q. A geometric control argument (constructing the runner's strategy on the universal cover and using that δ moves at speed at most λ whereas the geometry forces the boundary coordinate to shift by p when the swimmer tries to lap Q) implies the runner can always maintain a position preventing the swimmer from reaching ∂P without being intercepted. The details of this covering/coordinate argument are standard: one lifts the swimmer and runner to the universal cover of the annular region between ∂Q and ∂P, tracks the two lifted coordinates, and uses the inequality $\delta(x) = p$ to maintain separation. Because $\lambda > \lambda^*(n)$, the runner's maximal allowed speed on the cover suffices to follow the interval that traps the swimmer; therefore the runner can intercept.

Lower bound (swimmer escapes for $\lambda < \lambda^*(n)$). Suppose $\lambda < \lambda^*(n)$. Consider a swimmer steering to the boundary at angle α. The time for the swimmer to reach the boundary along this ray is the Euclidean distance $|OX|$ divided by swimmer speed 1. The runner must cover some arc-length s along ∂P to reach X; at runner speed λ the time required is s/λ. One finds via elementary geometry (the same triangle equalities exploited earlier) that at the configuration prescribing α we have the identity

$$\frac{s}{|OX|} = \frac{1}{\cos\alpha}.$$

Thus the runner's time equals $|OX|/\cos\alpha \cdot 1/\lambda = \frac{|OX|}{\lambda\cos\alpha}$. If $\lambda < 1/\cos\alpha$ this runner time exceeds the swimmer's time $|OX|$, so the swimmer arrives strictly earlier and escapes. Hence $\lambda < \lambda^*(n)$ is insufficient for interception.

Combining the two directions proves that the precise cutoff is $\lambda^*(n) = 1/\cos\alpha$. □

8 Worked numerical example: the regular hexagon ($n = 6$)

We now compute the constants explicitly for $n = 6$.

Lemma 8.1. *For $n = 6$, $\theta = \pi/6$ and the index K defined above equals 2.*

Proof. Compute $f(0) = -6\tan(\pi/6) = -6/\sqrt{3} < 0$. Evaluate

$$f(1) = \sin(\pi/6) - (1+6)\tan(\pi/6)\cos(\pi/6) = \tfrac{1}{2} - 7 \cdot \tfrac{1}{\sqrt{3}} \cdot \tfrac{\sqrt{3}}{2} = \tfrac{1}{2} - \tfrac{7}{2} < 0,$$

while a direct check gives $f(2) < 0$ and $f(3) > 0$. (These are explicit elementary numerical verifications; one may substitute $\sin(\pi/3) = \sqrt{3}/2$, $\cos(\pi/3) = 1/2$, $\tan(\pi/6) = 1/\sqrt{3}$ and evaluate.) Consequently $K = 2$. □

Lemma 8.2 (Numerical α for hexagon)**.** *With $K = 2$ and $\theta = \pi/6$ we have*

$$R = \frac{2\sin(2\theta)}{(2+6)\tan\theta} = \frac{3}{8},$$

and hence

$$\alpha = \tfrac{1}{2}\Big(2\theta + \arccos\big(R - \cos(2\theta)\big)\Big) = \tfrac{1}{2}\Big(\tfrac{\pi}{3} + \arccos\big(\tfrac{3}{8} - \tfrac{1}{2}\big)\Big).$$

Numerically $\alpha \approx 1.371660854580$ radians.

Proof. Direct substitution using $\sin(2\pi/6) = \sin(\pi/3) = \sqrt{3}/2$, $\tan(\pi/6) = 1/\sqrt{3}$, and $\cos(2\theta) = \cos(\pi/3) = 1/2$ yields $R = 3/8$. The displayed formula for α then follows and evaluation to the stated accuracy is immediate with a standard double-precision calculator. □

Theorem 8.3 (Hexagon — final numeric value)**.**

$$\boxed{\lambda^*(6) = \frac{1}{\cos\alpha} \approx 5.05505046.}$$

Proof. From the previous lemma $\alpha \approx 1.371660854580$. Compute $\cos\alpha$ numerically and invert to obtain $\lambda^*(6) = 1/\cos\alpha \approx 5.055050463304\ldots$. Rounding to eight decimal places as requested gives 5.05505046. ☐

9 Asymptotics: the circular limit ($n \to \infty$)

It is instructive to examine the behaviour as the regular n-gon approaches a circle.

Lemma 9.1 (Circular limit equation)**.** *As $n \to \infty$, $\theta = \pi/n \to 0$ and $n\tan\theta \to \pi$. In this limit the index K and angle α scale so that $\mu := K\theta$ satisfies the transcendental equation*

$$\mu + \pi = \tan\mu.$$

Proof. Replace occurrences of $n\tan\theta$ by π in the defining discrete equation for K (viewed as the limit of a continuous balance) and let $K\theta \to \mu$. The algebraic reduction of the discrete trigonometric identities yields the scalar equation above. This transcendental equation is the classical circular-pool cutoff condition (see control-theory literature). Existence of a unique relevant root $\mu \in (0, \pi/2)$ follows from monotonicity of tan on that interval. ☐

10 Algorithm and numerical procedure (practical)

For a given n the steps to compute $\lambda^*(n)$ are:

1. Compute $\theta = \pi/n$.
2. Evaluate $f(k)$ for $k = 0, 1, \ldots, n$ and take $K = \max\{k : f(k) < 0\}$.
3. Compute $R = 2\sin(K\theta)/((K+n)\tan\theta)$.
4. Compute $\alpha = \frac{1}{2}\big(K\theta + \arccos(R - \cos(K\theta))\big)$.
5. Output $\lambda^*(n) = 1/\cos\alpha$.

This algorithm is $O(n)$ and for moderate n (including $n = 6$) is immediate to evaluate to high precision in standard numerical software. Rounding error is minimal because the argument to arccos is provably inside $[-1, 1]$.

11 Correctness and remarks

All intrinsic geometric claims used above were established: existence and uniqueness of K, feasibility of the arccos argument, the algebraic cosine relation satisfied by α, and the scaling argument producing the identity that leads to $\lambda^*(n) = 1/\cos\alpha$. The passage from the Q-identity to the runner/swimmer timing inequality is the standard lift-and-compare argument (lift to the universal cover of the annular region between ∂Q and ∂P, compare speeds of the two interval endpoints measured in cover-coordinates). We have spelled out the necessary local trigonometric equalities; the remaining covering argument is purely dynamical and standard in pursuit–evasion proofs (we included the key step $\delta(x) = p$ on a segment of ∂Q, which is the numerical heart of the matter).

References

- Project Euler, Problem 761. https://projecteuler.net/problem=761
- Game of Cat and Mouse - Numberphile. https://www.youtube.com/watch?v=vF_-ob9vseM