Euler Problem 69: Maximizing $n/\varphi(n)$

We are asked to find the integer $n \le 10^6$ for which the ratio $n/\varphi(n)$ is largest.

Step 1. The totient function on prime powers

For a prime power, we know that

$$\varphi(p^k) = p^{k-1}(p-1).$$

This follows because among the p^k integers from 1 to p^k, exactly p^{k-1} of them are multiples of p and hence not coprime to p^k. Removing those leaves $p^k - p^{k-1} = p^{k-1}(p-1)$.

Step 2. Why φ is multiplicative

A number-theoretic function f is called *multiplicative* if

$$f(mn) = f(m)f(n) \quad \text{whenever } \gcd(m, n) = 1.$$

Euler's totient function has this property. Intuitively, if m and n share no common factors, then a number is coprime to mn if and only if it is coprime to both m and n. This correspondence shows that the counts multiply.

For example, $\varphi(8) = 4$, $\varphi(9) = 6$, and since $\gcd(8, 9) = 1$ we have

$$\varphi(72) = \varphi(8 \cdot 9) = \varphi(8)\,\varphi(9) = 24.$$

Step 3. Product formula for $\varphi(n)$

Because n factors uniquely as a product of prime powers,

$$n = \prod_p p^{a_p},$$

with each p^{a_p} coprime to the others, multiplicativity gives

$$\varphi(n) = \prod_p \varphi(p^{a_p}).$$

Substituting the prime power formula,

$$\varphi(n) = \prod_p p^{a_p-1}(p-1).$$

Factoring out $n = \prod_p p^{a_p}$, we obtain

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Step 4. Expressing the ratio

From this we see immediately that

$$\frac{n}{\varphi(n)} = \prod_{p|n} \frac{p}{p-1}.$$

Step 5. Reducing the problem

At this point a simplification occurs. The ratio depends only on the distinct primes dividing n, since replacing p by p^k leaves the factor unchanged. So the problem is the same as choosing a set of primes S so that the product $\prod_{p\in S} p$ does not exceed 10^6 and the value of $\prod_{p\in S} \frac{p}{p-1}$ is maximized.

Each new prime adds a factor $p/(p-1)$ to the ratio but also multiplies n by p. Because $p/(p-1) = 1+1/(p-1)$ decreases with p, smaller primes add a greater factor to the total. Suppose a set contains a larger prime q but does not a smaller prime $p < q$. Exchanging q for p makes the product of primes smaller while strictly increasing the ratio. Hence the optimal set must be the initial block of consecutive primes $2, 3, 5, 7, \ldots$ until the product would exceed the limit.

Step 6. Carrying it out

Multiplying consecutive primes,

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = 510510.$$

The next prime is 19, but $510510 \cdot 19 = 9699690 > 10^6$, so we stop. Thus the maximizing integer is $n^* = 510510$.

Step 7. Verification

Computing its totient gives

$$\varphi(510510) = 510510 \prod_{p\leq 17} \left(1 - \tfrac{1}{p}\right) = 92160.$$

The ratio is then

$$\frac{510510}{92160} = \prod_{p\leq 17} \frac{p}{p-1} \approx 5.539388.$$

Conclusion

Therefore the answer is that the maximal ratio is attained when $n = 510510$. The reasoning shows why the greedy product of consecutive small primes is the correct choice, and the explicit calculation confirms both $\varphi(n)$ and the numerical value of the ratio.

References

- Project Euler, Problem 69. `https://projecteuler.net/problem=69`.

- Wikipedia, *Euler's totient function.* https://en.wikipedia.org/wiki/Euler%27s_totient_function.
- T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.