Euler Problem 69: Maximizing $n/\varphi(n)$

We are asked to find the integer $n \le 10^6$ for which the ratio $n/\varphi(n)$ is largest. The totient function for a prime power is $\varphi(p^k) = p^{k-1}(p-1)$. Because φ is multiplicative, for any $n = \prod p^a$ we obtain

$$\varphi(n) = n \prod_{p|n} \left(1 - \tfrac{1}{p}\right).$$

From this we see immediately that

$$\frac{n}{\varphi(n)} = \prod_{p|n} \frac{p}{p-1}.$$

At this point an important simplification occurs. The ratio depends only on the distinct primes dividing n, since replacing p by p^k leaves the factor unchanged. So the problem is equivalent to choosing a set of primes S so that the product $\prod_{p \in S} p$ does not exceed 10^6 and the value of $\prod_{p \in S} p/(p-1)$ is maximized. Each new prime adds a factor $p/(p-1)$ to the ratio but also multiplies n by p. Because $p/(p-1) = 1 + 1/(p-1)$ decreases with p, smaller primes are more efficient to include. Suppose a set contains a larger prime q but omits a smaller prime $p < q$. Exchanging q for p makes the product of primes smaller while strictly increasing the ratio. Hence the optimal set must be the initial block of consecutive primes $2, 3, 5, 7, \ldots$ until the product would exceed the limit.

Carrying this out, we multiply $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = 510510$. The next prime is 19, but $510510 \cdot 19 = 9699690 > 10^6$, so we stop. Thus the maximizing integer is $n^* = 510510$. Computing its totient gives

$$\varphi(510510) = 510510 \prod_{p \le 17} \left(1 - \tfrac{1}{p}\right) = 92160.$$

The ratio is then

$$\frac{510510}{92160} = \prod_{p \le 17} \frac{p}{p-1} \approx 5.539388.$$

Therefore the answer is that the maximal ratio is attained when $n = 510510$. The reasoning shows why the greedy product of consecutive small primes is the correct choice, and the explicit calculation confirms both $\varphi(n)$ and the numerical value of the ratio.