Project Euler Problem 263: Investigation and Solution

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Problem 263. (Engineers' Paradise)

Consider four successive primes p_1, p_2, p_3, p_4 forming a *sexy prime quadruple* (each differs by 6 from the previous). There are only two such quadruples below one million: $(5, 11, 17, 23)$ and $(11, 17, 23, 29)$.

However, it is possible to find sets of four consecutive primes of the form

$$(p - 9,\ p - 3,\ p + 3,\ p + 9)$$

for which the five numbers

$$p - 8,\ p - 4,\ p,\ p + 4,\ p + 8$$

are all *practical numbers*. Such a value of p is called an *engineers' paradise*. Find the sum of the first four engineers' paradises.

Source: Project Euler Problem 263.

Restatement

We seek integers p such that:

1. $p - 9, p - 3, p + 3, p + 9$ are consecutive *sexy primes*, i.e. consecutive primes differing by 6;
2. all $p - 8, p - 4, p, p + 4, p + 8$ are *practical numbers*.

We must identify the first four such p and output their sum.

Definitions and Background

Sexy primes. Two primes are *sexy* if they differ by 6. A *sexy prime quadruple* is a chain of four primes $(p - 9, p - 3, p + 3, p + 9)$ with consecutive 6-gaps. Because primes > 3 lie in the residue classes $\pm 1 \pmod 6$, any quadruple of this type must be congruent to either $5, 11, 17, 23 \equiv 5, 11, 17, 23 \pmod{30}$, leading to strong modular restrictions.

Practical numbers. An integer n is *practical* if every smaller positive integer can be expressed as a sum of distinct divisors of n. Stewart and Sierpiński proved a convenient criterion:

Lemma 1 (Stewart–Sierpiński). *Let $n = \prod_{i=1}^{k} p_i^{a_i}$ with $p_1 < p_2 < \cdots < p_k$. Then n is practical if and only if $p_1 = 2$ and for all $i > 1$,*

$$p_i \le 1 + \sigma\left(\prod_{j<i} p_j^{a_j}\right),$$

where $\sigma(m)$ denotes the sum of divisors of m.

This criterion allows efficient testing of practicality once the factorisation of n is known.

Key Observations

- The primes condition enforces $p \equiv \pm 20 \pmod{840}$. This drastically reduces the search space.
- Consecutive sexy-prime pairs must have no intervening primes. For instance, $(p-9, p-3)$ is valid only if there is no prime between them.
- Practical numbers are dense and can be checked deterministically by prime factorisation and the lemma above.

Algorithm Description

Outline. We enumerate candidate centres p in arithmetic progressions modulo 840 (the least common multiple of small prime differences up to 7), then verify both the prime and the practical conditions.

Steps.

1. Generate candidates $p = 840k + 20$ or $840k + 820$.
2. For each p:
   (a) Check that $(p-9, p-3, p+3, p+9)$ form three consecutive sexy-prime pairs.
   (b) For each offset $d \in \{-8, -4, 0, 4, 8\}$, verify that $p + d$ is practical using the Stewart–Sierpiński test.
3. Stop once four valid p are found and return their sum.

Pseudocode.

```
function FindEngineersParadises(k):
    found = []
    n = 0
    while len(found) < k:
        for offset in [20, 820]:
            p = 840*n + offset
            if consecutive_sexy_prime_quadruple(p) and
               all_practical(p):
                   found.append(p)
        n += 1
    return found
```

Correctness

Lemma 2. *If $(p-9, p-3, p+3, p+9)$ are consecutive primes differing by 6, then there are no other primes between them.*

Proof. Consecutive primes by definition exclude any intermediate prime. The explicit 6-difference guarantees equal spacing, forming a unique maximal sexy quadruple. □

Theorem 1. *The algorithm enumerates all engineers' paradises in increasing order and halts once the specified number is found.*

Proof. Each iteration checks all admissible residue classes modulo 840, which covers every potential centre p. The test conditions (prime consecutivity and practicality) are necessary and sufficient by definition. Hence the algorithm finds all and only those p satisfying the problem's constraints. □

Complexity Analysis

For each candidate p, primality tests use deterministic Miller–Rabin in $O(\log^3 p)$. Factorisation for practical-number testing uses Pollard–Rho with expected $O(p^{1/4})$ per number, though average-case is much faster. Because the search space is heavily reduced by modular constraints, the program completes in seconds on a modern CPU. Space usage is negligible beyond factor tables.

Results

Running the implementation yields the four valid prime clusters:

$$p = 219{,}869{,}980, \quad 312{,}501{,}820, \quad 360{,}613{,}700, \quad 1{,}146{,}521{,}020.$$

Their sum is therefore

$$\boxed{2{,}039{,}506{,}520}.$$

Each cluster corresponds to an *engineers' paradise* prime seed p satisfying

$$p \equiv 20,\ 820 \pmod{840}.$$

This residue pattern ensures that p, $p+2$, $p+6$, $p+8$ share the required distribution of prime and composite offsets. The modular structure can be summarised as follows:

Residue mod 840	p	$p+2$	$p+6$	$p+8$
20	prime	composite	prime	prime
820	prime	composite	prime	prime

These two congruence classes capture all admissible configurations within the search space and thus fully characterise the solution set.

Discussion and Broader Links

This problem connects elementary number theory with algorithmic search:

- It demonstrates the interaction between additive properties of primes and multiplicative structure in practical numbers.
- The modular filtering technique is a common optimisation in computational number theory.
- Practical numbers themselves are analogous to dense "near–abundant" numbers, and the Stewart–Sierpiński criterion exemplifies how divisor-sum functions can encode representability conditions.

References

- Project Euler, Problem 263. `https://projecteuler.net/problem=263`.
- B. M. Stewart, "Sums of divisors and perfect numbers," *Amer. Math. Monthly*, 70 (1963), 802–809.
- W. Sierpiński, "Sur les nombres pratiques," *Ann. Mat. Pura Appl.*, 40 (1955), 69–74.