

Part II REL Problem Sheet Solutions

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1 Problem Sheet 1

Special Relativity

Problem 1.1 (Spacetime interval):

- (a) Show that if two events are separated by a timelike interval, then there is a frame in which they occur at the same spatial location.
- (b) Similarly, if two events are separated by a spacelike interval, show there is a frame in which they are simultaneous.

Answer 1.0.1.

- (a) Without loss of generality, align axes of S such that $\Delta y = \Delta z = 0$. Perform standard Lorentz boost to another frame S' :

$$\Delta y' = \Delta z' = 0, \quad \Delta x' = \gamma(\Delta x - \frac{v}{c}\Delta t)$$

To ensure they occur at the same spatial location in frame S' :

$$\Delta x' = 0 \implies \frac{v}{c} = \frac{\Delta x}{c\Delta t} \implies |\Delta x| < c|\Delta t|$$

The spacetime interval is

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (c\Delta t)^2 - (\Delta x)^2 - 0 > 0$$

and hence the events are time-like separated.

- (b) Again without loss of generality, $\Delta y = \Delta z = 0$. Perform a standard Lorentz boost, i.e. $c\Delta t' = \gamma(c\Delta t - \beta\Delta x)$. For two events to be simultaneous in frame S' , $\Delta t' = 0$, we have

$$\beta = \frac{c\Delta t}{\Delta x} \implies \frac{\Delta t}{\Delta x} = \frac{\beta}{c} < 1 \implies c\Delta t < \Delta x$$

The spacetime interval is

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (c\Delta t)^2 - (\Delta x)^2 - 0 < 0$$

and hence the events are space-like separated.

Problem 1.2 (Spacetime interval):

- (a) Show that if an event A precedes an event B in some frame S at the same spatial location, then the event A precedes event B in all frames.
- (b) Two general events A and B are separated in S by a spatial distance Δr . If event A causes event B, determine an inequality for the time difference between the events, $\Delta t = t_B - t_A$. Hence show that the events are causally related in all frames.

Answer 1.0.2.

- (a) In frame S , if two events occur at the same spatial location $\Delta \mathbf{r} = \mathbf{0}$, then $\Delta t = t_B - t_A > 0$, i.e. event A precedes event B. Under standard Lorentz boost,

$$c\Delta t' = \gamma(c\Delta t) > 0$$

Hence $\Delta t' > 0 \forall \gamma$, i.e. for all frames related to S by a Lorentz boost.

- (b) If A causes B, time is needed for a signal to propagate between them, i.e. $c\Delta t > \Delta r \implies \Delta s^2 = (c\Delta t)^2 - (\Delta \mathbf{r})^2 > 0$. By invariance of the spacetime interval, we must have $|c\Delta t'| > |\Delta \mathbf{r}'|$ in any frame. Choose axes in S and S' so in standard configuration,

$$\Delta x \leq \Delta r, \quad c\Delta t > \Delta r \implies c\Delta t' = \gamma(c\Delta t - \beta\Delta x) > 0 \implies c\Delta t' > |\Delta \mathbf{r}'|$$

Hence, the two events are causally related in all frames.

Problem 1.3 (Spacetime diagram):

- (a) On a spacetime diagram with the x and ct axes of an inertial frame S horizontal and vertical, respectively, construct the lines of constant x' and ct' , where these coordinates refer to the frame S' in standard configuration with S (i.e., where S' moves at a speed v along the positive x -direction and the two frames coincide at $t = t' = 0$). Show that the angle between the x - and x' - axes is the same as that between the ct - and ct' - axes and has the value $\tan^{-1}(v/c)$.
- (b) Sketch on your diagram the loci of events separated from the spacetime origin $x = ct = 0$ by a constant invariant interval $\Delta s^2 = c^2 t^2 - x^2$ for positive (timelike) and negative (spacelike) values of Δs^2 . How can these curves be used to calibrate lengths along the axes of the S and S' frames?
- (c) Use your diagram to illustrate graphically why a rod at rest in S' is *contracted* as measured in S , and the time on a clock at rest in S' is *dilated* as observed in S .

Answer 1.0.3.

- (a) The lines of constant t' and x' have equations $ct' = \gamma(ct - \beta x) = \text{const.}$ and $x' = \gamma(x - \beta ct) = \text{const.}$. The angle between the x and x' -axes is

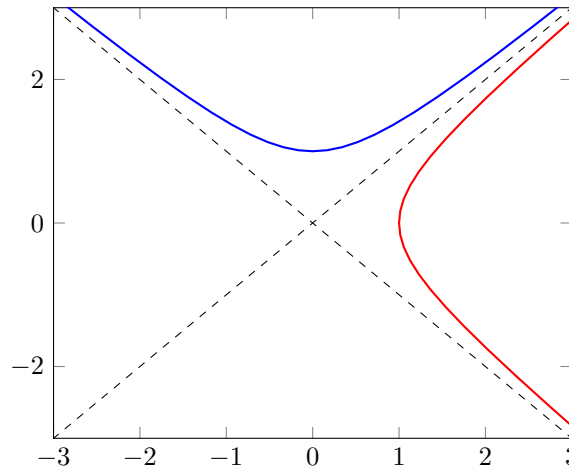
$$\tan^{-1}(x'/ct) = \tan^{-1}(\beta)$$

The angle between the ct' and ct -axes is

$$\tan^{-1}(x/ct) = \tan^{-1}(\beta)$$

where $\beta = v/c$.

- (b) The spacetime interval is $\Delta s^2 = c^2 t^2 - x^2$ and are represented by hyperbolae. On the spacetime diagram below, we have $\Delta s^2 < 0$ and $\Delta s^2 > 0$ to be represented by the red and blue curves respectively. The diagonal asymptotes are the worldlines of photons starting at the common spacetime origin, travelling at $\pm c$.



Consider $\Delta s^2 = -1$, then this hyperbola intersects the x and x' -axes at $x = 1$ and $x' = 1$ respectively.

- (c) The fact that the spacetime interval hyperbolic curves can be used to calibrate lengths along the axes of the S and S' frames, we can demonstrate length contraction and time dilation.
- **Time dilation:** Suppose we have a clock with worldline $x' = 0 \forall t'$ which measures a period $ct' = cT_0$. The spacetime interval curve $\Delta s^2 = c^2 T_0^2$ connects points on the ct' - and ct -axes with length cT_0 . In frame S however, the observer measures the event with period $ct = c\Delta t$ instead, which graphically we have $c\Delta t > cT_0$, i.e. time dilated in S .
 - **Length contraction:** Suppose we have a rod of length ℓ_0 at rest in frame S' . Again, the hyperbola $\Delta s^2 = -\ell_0^2$ connects points on the x' - and x -axes with length ℓ_0 . To measure the length of the rod Δx in S , we need to mark out the positions of both ends simultaneously. We can see graphically that $\ell_0 > \Delta x$, i.e. length contraction in S .

Problem 1.4 (Lorentz boost): An inertial frame S' is related to the frame S by a boost of \vec{v} whose components in S are (v_x, v_y, v_z) . Show that the coordinates (ct', x', y', z') and (ct, x, y, z) of an event are related by

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1 + \alpha\beta_x^2 & \alpha\beta_x\beta_y & \alpha\beta_x\beta_z \\ -\gamma\beta_y & \alpha\beta_y\beta_x & 1 + \alpha\beta_y^2 & \alpha\beta_y\beta_z \\ -\gamma\beta_z & \alpha\beta_z\beta_x & \alpha\beta_z\beta_y & 1 + \alpha\beta_z^2 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

where $\vec{\beta} = \vec{v}/c$, $\gamma = (1 - |\vec{\beta}|^2)^{-1/2}$ and $\alpha = (\gamma - 1)/|\vec{\beta}|^2$. (Hint: resolve the 3-vector position with components (x, y, z) into parallel and perpendicular parts with respect to $\vec{\beta}$, and similarly in the S' frame.)

Answer 1.0.4. Resolve $\mathbf{r} = (x, y, z)$ into parallel and perpendicular parts with respect to $\vec{\beta}$:

$$\mathbf{r} = (\mathbf{r} \cdot \hat{\vec{\beta}})\hat{\vec{\beta}} + [\mathbf{r} - (\mathbf{r} \cdot \hat{\vec{\beta}})\hat{\vec{\beta}}] := r_{\parallel}\hat{\vec{\beta}} + \mathbf{r}_{\perp}$$

and similarly for $\mathbf{r}' = (x', y', z')$. For a standard Lorentz Boost, the components perpendicular to $\hat{\vec{\beta}}$ are unchanged and that parallel transform as usual, i.e.

$$r'_{\parallel} = \gamma(r_{\parallel} - \beta ct), \quad \mathbf{r}'_{\perp} = \mathbf{r}_{\perp}, \quad ct' = \gamma(ct - \beta r_{\parallel})$$

Hence, we have

$$\mathbf{r}' = r'_{\parallel}\hat{\vec{\beta}} + \mathbf{r} - r_{\parallel}\hat{\vec{\beta}} = -\gamma\beta ct\hat{\vec{\beta}} + \left[Id + (\gamma - 1) \begin{pmatrix} \hat{\beta}_x^2 & \hat{\beta}_x\hat{\beta}_y & \hat{\beta}_x\hat{\beta}_z \\ \hat{\beta}_y\hat{\beta}_x & \hat{\beta}_y^2 & \hat{\beta}_y\hat{\beta}_z \\ \hat{\beta}_z\hat{\beta}_x & \hat{\beta}_z\hat{\beta}_y & \hat{\beta}_z^2 \end{pmatrix} \right] \mathbf{r}$$

but $\gamma - 1 = \alpha|\vec{\beta}|^2$, and so

$$\mathbf{r}' = -\gamma\beta\hat{\vec{\beta}}ct + \begin{pmatrix} 1 + \alpha\beta_x^2 & \alpha\beta_x\beta_y & \alpha\beta_x\beta_z \\ \alpha\beta_y\beta_x & 1 + \alpha\beta_y^2 & \alpha\beta_y\beta_z \\ \alpha\beta_z\beta_x & \alpha\beta_z\beta_y & 1 + \alpha\beta_z^2 \end{pmatrix} \mathbf{r}$$

Together with

$$ct' = \gamma ct - \gamma\beta(x\beta_x + y\beta_y + z\beta_z)$$

We have our desired answer.

Problem 1.5 (Velocity composition): In a given inertial frame, two particles are shot out simultaneously from a given point, with equal speeds v in orthogonal directions. What is the speed of each particle relative to the other?

Answer 1.0.5. Let particle A and B have velocities $v(0, 1, 0)$ and $v(1, 0, 0)$ respectively. In the frame of particle B (Lorentz boost by $v(1, 0, 0)$), particle B observes particle A to have $u_x = 0$, $u_y = v$, $u_z = 0$ and so by usual velocity transformation,

$$u'_x = -v, \quad u'_y = v/\gamma_v, \quad u'_z = 0$$

The relative speed is then $v(1 + \gamma_v^{-2})^{1/2} = v(2 - (v/c)^2)^{-1/2}$.

Problem 1.6 (Velocity composition):

- (a) Frame S' moves with speed v relative to frame S in standard configuration. A rod at rest in frame S' makes an angle θ' with respect to the forward direction of motion. What is the angle θ measured in S ?
- (b) If a bullet is fired in S' at speed u' at an angle θ' with respect to the forward direction of motion, what is the angle θ measured in S ? What if the bullet is a photon?

Answer 1.0.6.

- (a) Let the two ends of the rod (with proper length ℓ_0) be A and B where A is at the origin. The worldline of A has coordinates $(ct', 0, 0, 0)$ in frame S' . The worldline of B has coordinates $(ct', \ell_0 \cos \theta', \ell_0 \sin \theta', 0)$. Inverse Lorentz transform back to frame S , the coordinates for A and B are respectively

$$\begin{pmatrix} \gamma ct' \\ \gamma \beta ct' \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} ct \\ \beta ct \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \gamma ct' + \gamma \beta \ell_0 \cos \theta' \\ \gamma \ell_0 \cos \theta' + \gamma \beta ct' \\ \ell_0 \sin \theta' \\ 0 \end{pmatrix} = \begin{pmatrix} ct \\ \gamma \ell_0 \cos \theta' + \beta(ct - \gamma \beta \ell_0 \cos \theta) = \ell_0 \gamma^{-1} \cos \theta' + vt \\ \ell_0 \sin \theta' \\ 0 \end{pmatrix}$$

The back and the front both moves at speed v , as expected. Hence, we must have

$$\tan \theta = \frac{\ell_0 \sin \theta'}{\ell_0 \cos \theta' / \gamma} = \gamma \tan \theta'$$

- (b) Inverse transform the velocity components for u' :

$$u_x = \frac{u' \cos \theta' + v}{1 + (u'v/c^2) \cos \theta'}, \quad u_y = \frac{u' \sin \theta'}{\gamma_v(1 + (u'v/c^2) \cos \theta')}, \quad u_z = 0$$

The angle θ in S satisfies

$$\tan \theta = \frac{u' \sin \theta'}{\gamma_v(u' \cos \theta' + v)}$$

If the object was a photon, then $u = c$ and we recover the light aberration relation $\tan \theta = \frac{\sin \theta'}{\gamma_v(\cos \theta' + \beta)}$.

Problem 1.7: Frame S' moves with speed v relative to frame S in standard configuration. Neutral π -mesons at rest in S' decay into two photons that are emitted isotropically. Show that the angular distribution of photons in S is

$$P(\theta)d\theta = \frac{\sin \theta d\theta}{2\gamma^2(1 - \beta \cos \theta)^2}$$

Answer 1.0.7. From Q6, we have

$$\cos \theta' = \frac{\cos \theta - \beta}{1 - \beta \cos \theta} \implies \frac{d \cos \theta'}{d \cos \theta} = \frac{1 - \beta \cos \theta - (\cos \theta - \beta)(-\beta)}{(1 - \beta \cos \theta)^2} = \frac{1}{\gamma^2(1 - \beta \cos \theta)^2}$$

Photons are emitted isotropically in a solid angle $2\pi d(\cos \theta')$ in S' are emitted in a solid angle $2\pi d(\cos \theta)$ in S . The number of photons emitted in either frame is the same:

$$P(\theta)d\theta = P(\theta')d\theta' = \frac{1}{4\pi} 2\pi d \cos \theta' \implies P(\theta)d\theta = \frac{1}{2} \frac{d(\cos \theta')}{d(\cos \theta)} \sin \theta d\theta = \frac{\sin \theta d\theta}{2\gamma^2(1 - \beta \cos \theta)^2}$$

i.e. the photons are strongly 'beamed' in the x -direction due to aberration.

Problem 1.8 (Acceleration):

- (a) A spaceship travels in a straight line at a variable speed $u(t)$ in some inertial frame S . An observer on the spaceship measures his acceleration to be $f(\tau)$, where τ is his proper time. If at $\tau = 0$ the spaceship has a speed u_0 in S show that

$$\frac{u(\tau) - u_0}{1 - u(\tau)u_0/c^2} = c \tanh \psi(\tau)$$

where $c\psi(\tau) = \int_0^\tau f(\tau')d\tau'$. Show that the speed of the spaceship can never reach c .

- (b) If the spaceship left base at time $t = \tau = 0$ and travelled forever in a straight line with constant acceleration g (for comfort), how long by the spaceship clock does it take to reach a star 10 light years from the base?

Answer 1.0.8.

- (a) $f(\tau)$ is the acceleration in the instantaneous rest frame (IRF). At proper time τ , the speed is u and increases by $f(\tau)d\tau$ in the IRF. In S , the speed increases by du . By velocity addition,

$$u + du = \frac{f d\tau + u}{1 + (u/c^2)f d\tau} = u \left(1 - \frac{u}{c^2} f d\tau \right) + f d\tau + \dots = u + \frac{f d\tau}{\gamma_u^2} + \dots \implies \frac{du}{d\tau} = \frac{f}{\gamma_u^2}$$

Let $u/c = \tanh \theta(\tau)$, then

$$\frac{du}{d\tau} = \frac{c}{\cosh^2 \theta} \frac{d\theta}{d\tau} \implies \frac{d\theta(\tau)}{d\tau} = \frac{f(\tau)}{c}$$

since $\gamma_u = \cosh \theta$ (definition of rapidity). We then have

$$c\theta(\tau) = \int_0^\tau f(\tau')d\tau' + c \tanh^{-1}(u_0/c)$$

where the integration constant occurs since $u = u_0$ at $\tau = 0$. Evaluate $\tanh \psi(\tau)$:

$$\tanh \psi = \tanh(\theta - \tanh^{-1}(u_0/c)) = \frac{\tanh \theta - u_0/c}{1 - (u_0/c) \tanh \theta} = \frac{1}{c} \frac{u - u_0}{1 - uu_0/c^2}$$

Upon rearranging, we have

$$\frac{u}{c} = \frac{(u_0/c) + \tanh \psi}{1 + (u_0/c) \tanh \psi}$$

For $0 \leq \psi < \infty$, the RHS increases monotonically from u_0/c to 1, i.e. $u(\tau)$ approaches but never exceeds c .

- (b) By definition of proper time,

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 = c^2 dt^2 (1 - u^2/c^2)$$

We have $d\tau/dt = 1/\gamma_u = 1/\cosh \psi$, so since

$$u = \frac{dx}{dt} \frac{d\tau}{dt} = c \tanh \psi \implies \frac{dx}{d\tau} = c \sinh \psi$$

But $\psi(\tau) = (g/c) \int_0^\tau d\tau = g\tau/c$, so

$$x = \frac{c^2}{g} [\cosh(g\tau/c) - 1]$$

where the integration constant is obtained from $x(\tau = 0) = 0$. We have $x = 10$ light years and so $xg/c^2 = \frac{(365.25)(24)(3600)(10)(9.81)}{3 \times 10^8} = 10.32$ and hence $g\tau/c = \cosh^{-1}(10.32 + 1) = 3.118$. Hence,

$$\tau = \frac{3.118}{9.81} \frac{3 \times 10^8}{24(3600)(365.25)} = 3.02 \text{ years}$$

Manifolds and Coordinates

Problem 1.9: In 3D Euclidean space, coordinates x'^a are related to Cartesian coordinates x^a by

$$x^1 = x'^1 + x'^2, \quad x^2 = x'^1 - x'^2, \quad x^3 = 2x'^1 x'^2 + x'^3$$

Describe the coordinate surfaces in the primed system. Obtain the metric functions g'_{ab} in the primed system and hence show that these coordinates are not orthogonal. Calculate the volume element dV in the primed coordinate system.

Answer 1.0.9. To describe the surfaces associated to the primed coordinates, we rearrange to get

$$x'^1 = 0.5(x^1 + x^2), \quad x'^2 = 0.5(x^1 - x^2), \quad x'^3 = x^3 - 0.5((x^1)^2 - (x^2)^2)$$

$x'^1 = \text{const.}$ and $x'^2 = \text{const.}$ are diagonal planes in the x -space, while $x'^3 = \text{const.}$ are hyperbolic paraboloids with principal axes along x^1 and x^2 directions. The spacetime invariant is

$$\begin{aligned} ds^2 &= (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \\ &= (dx'^1 + dx'^2)^2 + (dx'^1 - dx'^2)^2 + (2x'^1 dx'^2 + 2dx'^2 dx'^1 + dx'^3)^2 \\ &= 2(dx'^1)^2 + 2(dx'^2)^2 + 4(x'^1)^2 (dx'^2)^2 + 4(x'^2)^2 (dx'^1)^2 + (dx'^3)^2 \\ &\quad + 8x'^1 x'^2 dx'^1 dx'^2 + 4x'^1 dx'^2 + dx'^3 + 4x'^2 dx'^1 dx'^3 \\ &= 2(1 + 2(x'^2)^2)(dx'^1)^2 + 2(1 + 2(x'^1)^2)(dx'^2)^2 + x'^1 x'^2 dx'^1 dx'^2 + 4x'^1 dx'^2 dx'^3 + 4x'^2 dx'^1 dx'^3 + (dx'^3)^2 \end{aligned}$$

with the metric tensor as

$$g'_{ab} = \begin{pmatrix} 2(1 + 2(x'^2)^2) & 4x'^1 x'^2 & 2x'^2 \\ 4x'^1 x'^2 & 2(1 + 2(x'^1)^2) & 2x'^1 \\ 2x'^2 & 2x'^1 & 1 \end{pmatrix}$$

Since the metric tensor is not diagonal, it is not an orthogonal coordinate system. The determinant and hence volume element is

$$\begin{aligned} \det g &= 2(1 + 2(x'^2)^2)[2(1 + 2(x'^1)^2 - 4(x'^1)^2)] - 4(x'^1)(x'^2)[4x'^1 x'^2 - 4x'^1 x'^2] + 2x'^2[8(x'^1)^2(x'^2) - 4(x'^2)(1 + 2(x'^1)^2)] \\ &= 4 \implies dV = \sqrt{\det g} dx'^1 dx'^2 dx'^3 = 2dx'^1 dx'^2 dx'^3 \end{aligned}$$

Problem 1.10: Show that the line element of a 3-sphere of radius a embedded in 4D Euclidean space can be written in the form

$$ds^2 = a^2[d\chi^2 + \sin^2 \chi(d\theta^2 + \sin^2 \theta d\phi^2)]$$

Hence, in this 3D non-Euclidean space, calculate the area of the 2-sphere defined by $\chi = \chi_0$. Also find the total volume of the 3D space.

Answer 1.0.10. The 3-sphere $x^2 + y^2 + z^2 + w^2 = a^2$ is parametrized by

$$w = a \cos \chi, \quad x = a \sin \chi \sin \theta \cos \phi, \quad y = a \sin \chi \sin \theta \sin \phi, \quad z = a \sin \chi \cos \theta, \quad 0 \leq \chi \leq \pi$$

where θ, ϕ are the usual spherical polar coordinates. The infinitesimal elements are

$$dw = -a \sin \chi d\chi, \quad dx = a \cos \chi \sin \theta \cos \phi d\chi + a \sin \chi d(\sin \theta \cos \phi)$$

$$dy = a \cos \chi \sin \theta \sin \phi d\chi + a \sin \chi d(\sin \theta \sin \phi), \quad dz = a \cos \chi \cos \theta d\chi + a \sin \chi d \cos \theta$$

The interval is

$$\begin{aligned} dw^2 + dx^2 + dy^2 + dz^2 &= a^2 \sin^2 \chi d\chi^2 + a^2 \cos^2 \chi d\chi^2 + a^2 \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) + 2a^2 \sin \chi \cos \chi d\chi [\sin \theta \cos \phi d(\sin \theta \cos \phi) \\ &\quad + \sin \theta \sin \phi d(\sin \theta \sin \phi) + \cos \theta d \cos \theta] \\ &= a^2 d\chi^2 + a^2 \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \end{aligned}$$

On the 2-sphere $\chi = \chi_0$, the induced line element is

$$ds^2 = a^2 \sin^2 \chi_0 (d\theta^2 + \sin^2 \theta d\phi^2)$$

as expected. The 2-volume and 3-volume respectively are

$$dV_{(2)} = a^2 \sin^2 \chi_0 \sin \theta d\theta d\phi \implies V_{(2)} = 4\pi a^2 \sin^2 \chi_0, \quad dV_{(3)} = a^3 \sin^3 \chi d\chi d \cos \theta d\phi \implies V_{(3)} = 2\pi^2 a^3$$

2 Problem Sheet 2

Vector and Tensor Algebra

Problem 2.1: In 3D Euclidean space, coordinates x'^a are related to Cartesian coordinates x^a by

$$x^1 = x'^1 + x'^2, \quad x^2 = x'^1 - x'^2, \quad x^3 = 2x'^1x'^2 + x'^3$$

- (a) Express the coordinate basis vectors $\mathbf{e}'_a = \frac{\partial}{\partial x'^a}$ for the primed coordinates in terms of those for the Cartesian coordinates. How are these related to the intersections of the coordinate surfaces that you sketched in Question 9 of Examples Sheet 1? By considering $\mathbf{g}(\mathbf{e}'_a, \mathbf{e}'_b)$ obtain the components of the metric g'_{ab} . (Hint: since the original coordinates are Cartesian, $\mathbf{g}(\mathbf{e}_a, \mathbf{e}_b) = \delta_{ab}$.)
- (b) Let the vector $\mathbf{v} = \mathbf{e}_1$. Write down the components v^a and those of the associated dual vector v_a . Calculate the components of the same vector \mathbf{v} and its associated dual vector in the primed coordinates.

Answer 2.0.1.

(a)

(b)

Problem 2.2:

- (a) If the tensor A_{ab} is an antisymmetric tensor, S_{ab} is a symmetric tensor and T_{ab} is a general tensor, show that $A^{ab}T_{ab} = A^{ab}T_{[ab]}$ and $S^{ab}T_{ab} = S^{ab}T_{(ab)}$.
- (b) If v_a are the components of a dual vector, show that in an arbitrary coordinate system $A_{ab} = \partial_b v_a - \partial_a v_b$ are the components of a type-(0, 2) tensor. Show further, for a general antisymmetric tensor A_{ab} , that $B_{abc} = \partial_c A_{ab} + \partial_a A_{bc} + \partial_b A_{ca}$ are the components of a type-(0, 3) tensor. What are the symmetry properties of B_{abc} ?

Answer 2.0.2.

(a)

(b)

Vector and tensor calculus on manifolds

Problem 2.3:

- (a) If $g = \det(g_{ab})$ is the determinant of the metric, show that $\partial_c g = g g^{ab} (\partial_c g_{ab})$.
- (b) Verify directly, in a general coordinate system, that $\nabla_c g_{ab} = 0$ for the covariant derivative constructed with the metric connection.
- (c) For a diagonal metric g_{ab} , show that the connection coefficients are given by (with $a \neq b \neq c$ and no summation over repeated indices)

$$\Gamma_{bc}^a = 0, \quad \Gamma_{aa}^b = -\frac{1}{2g_{bb}} \frac{\partial g_{aa}}{\partial x^b}, \quad \Gamma_{ba}^a = \Gamma_{ab}^a = \frac{\partial}{\partial x_b} (\ln \sqrt{|g_{aa}|})$$

Answer 2.0.3.

- (a)
- (b)
- (c)

Problem 2.4: In 2D Euclidean space, the line element in plane-polar coordinates is

$$ds^2 = d\rho^2 + \rho^2 d\phi^2$$

- (a) Obtain the non-zero connection coefficients

$$\Gamma_{\rho\phi}^\phi = \Gamma_{\phi\rho}^\phi = 1/\rho, \quad \Gamma_{\phi\phi}^\rho = -\rho$$

- (b) If the coordinate components v^a of a vector \mathbf{v} are written as v^ρ and v^ϕ , show that the divergence of \mathbf{v} is

$$\nabla_a v^a = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho v^\rho) + \frac{\partial v^\phi}{\partial \phi}$$

What would be the equivalent result in terms of the components of \mathbf{v} in an orthonormal basis aligned with the coordinate directions?

- (c) Show that the Laplacian of a scalar field f is

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2}$$

Answer 2.0.4.

- (a)
- (b)
- (c)

Problem 2.5: On the surface of a unit sphere $ds^2 = d\theta^2 + \sin^2 \theta d\phi^2$.

- (a) Calculate the connection coefficients in the (θ, ϕ) coordinate system directly from the metric.
- (b) By considering the ‘Lagrangian’ $L = g_{ab}\dot{x}^a\dot{x}^b$, derive the equations for an affinely-parameterised geodesic on the surface of a sphere in the coordinates (θ, ϕ) and thereby verify your answer to (a). Hence show that, of all the circles of constant latitude on a sphere, only the equator is a geodesic.
- (c) A vector \mathbf{v} of unit length is defined at the point $(\theta_0, 0)$ and is parallel to the circle $\phi = 0$. Calculate the components of \mathbf{v} after it has been parallel transported around the circle $\theta = \theta_0$. Hence show that, in general, after parallel transport, the direction of \mathbf{v} is different, but its length is unchanged.

Answer 2.0.5.

- (a)
- (b)
- (c)

Problem 2.6: A hypersurface \mathcal{H} within a manifold \mathcal{M} contains a non-null curve \mathcal{C} . Give a geometric argument showing that if \mathcal{C} is a geodesic in \mathcal{M} , it is also a geodesic in \mathcal{H} . Give an example to show that the converse is not necessarily true.

Answer 2.0.6.

Problem 2.7 (Optional):

- (a)

Answer 2.0.7.

- (a)
- (b)

Minkowski spacetime and particle dynamics

Problem 2.8: In Minkowski spacetime, two uniformly-moving observers \mathcal{E} and \mathcal{R} have 4-velocities u and v , respectively.

- (a) Show that $u^\mu v_\mu = c^2 \gamma_V$, where V is their relative speed.
- (b) If \mathcal{E} emits a photon that is subsequently received by \mathcal{R} , show that the ratio of the emitted and received photon frequencies is given by

$$\frac{v_{\mathcal{E}}}{v_{\mathcal{R}}} = \frac{u^\mu p_\mu}{v^\nu p_\nu}$$

where \mathbf{p} is the photon 4-momentum.

Answer 2.0.8.

(a)

(b)

Problem 2.9: Suppose an observer \mathcal{O} begins to accelerate in Minkowski spacetime such that, at some instant, his 3-velocity and 3-acceleration in an inertial frame S are \vec{u} and \vec{a} , respectively. Show that the (proper) acceleration α measured by \mathcal{O} at this instant is given by

$$\alpha^2 = \frac{\gamma_u^6 (\vec{u} \cdot \vec{a})^2}{c^2} + \gamma_u^4 \vec{a} \cdot \vec{a}$$

Find an expression for α if the motion in S is circular with radius r .

Answer 2.0.9.

3 Problem Sheet 3

Problem 3.1:

(a)

Answer 3.0.1.

(a)

(b)

Problem 3.2:

(a)

Answer 3.0.2.

(a)

(b)

Problem 3.3:

(a)

Answer 3.0.3.

(a)

(b)

Problem 3.4:

(a)

Answer 3.0.4.

(a)

(b)

Electromagnetism

Problem 3.5:

(a)

Answer 3.0.5.

(a)

(b)

Spacetime Curvature

Problem 3.6:

(a)

Answer 3.0.6.

(a)

(b)

Problem 3.7:

(a)

Answer 3.0.7.

(a)

(b)

Problem 3.8:

(a)

Answer 3.0.8.

(a)

(b)

Problem 3.9:

(a)

Answer 3.0.9.

(a)

(b)

Problem 3.10:

(a)

Answer 3.0.10.

(a)

(b)

4 Problem Sheet 4