Part II REL Problem Sheet Solutions

Tai Yingzhe, Tommy (ytt26)

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1 Problem Sheet 1

Special Relativity

Problem 1.1 (Spacetime interval)**:**

(a) Show that if two events are separated by a timelike interval, then there is a frame in which they occur at the same spatial location.

(b) Similarly, if two events are separated by a spacelike interval, show there is a frame in which they are simultaneous.

Answer 1.0.1.

(a) Without loss of generality, align axes of S such that $\Delta y = \Delta z = 0$. Perform standard Lorentz boost to another frame S':

$$\Delta y' = \Delta z' = 0, \quad \Delta x' = \gamma(\Delta x - v\Delta t)$$

To ensure they occur at the same spatial location in frame S':

$$\Delta x' = 0 \implies \frac{v}{c} = \frac{\Delta x}{c\Delta t} \implies |\Delta x| < c|\Delta t|$$

The spacetime interval is

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (c\Delta t)^2 - (\Delta x)^2 - 0 > 0$$

and hence the events are time-like separated.

(b) Again without loss of generality, $\Delta y = \Delta z = 0$. Perform a standard Lorentz boost, i.e. $c\Delta t' = \gamma(c\Delta t - \beta\Delta x)$. For two events to be simultaneous in frame S', $\Delta t' = 0$, we have

$$\beta = \frac{c\Delta t}{\Delta x} \implies \frac{\Delta t}{\Delta x} = \frac{\beta}{c} < 1 \implies c\Delta t < \Delta x$$

The spacetime interval is

$$\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (c\Delta t)^2 - (\Delta x)^2 - 0 < 0$$

and hence the events are space-like separated.

Problem 1.2 (Spacetime interval)**:**

(a) Show that if an event A precedes an event B in some frame S at the same spatial location, then the event A precedes event B in all frames.

(b) Two general events A and B are separated in S by a spatial distance Δr. If event A causes event B, determine an inequality for the time difference between the events, $\Delta t = t_B - t_A$. Hence show that the events are causally related in all frames.

Answer 1.0.2.

(a) In frame S, if two events occur at the same spatial location $\Delta\mathbf{r} = \mathbf{0}$, then $\Delta t = t_B - t_A > 0$, i.e. event A precedes event B. Under standard Lorentz boost,

$$c\Delta t' = \gamma(c\Delta t) > 0$$

Hence $\Delta t' > 0 \ \forall\gamma$, i.e. for all frames related to S by a Lorentz boost.

(b) If A causes B, time is needed for a signal to propagate between them, i.e. $c\Delta t > \Delta r \implies \Delta s^2 = (c\Delta t)^2 - (\Delta\mathbf{r})^2 > 0$. By invariance of the spacetime interval, we must have $|c\Delta t'| > |\Delta\mathbf{r}'|$ in any frame. Choose axes in S and S' so in standard configuration,

$$\Delta x \le \Delta r,\ c\Delta t > \Delta r \implies c\Delta t' = \gamma(c\Delta t - \beta\Delta x) > 0 \implies c\Delta t' > |\Delta\mathbf{r}|$$

Hence, the two events are causally related in all frames.

Problem 1.3 (Spacetime diagram)**:**

(a) On a spacetime diagram with the x and ct axes of an inertial frame S horizontal and vertical, respectively, construct the lines of constant x' and ct', where these coordinates refer to the frame S' in standard configuration with S (i.e., where S' moves at a speed v along the positive x-direction and the two frames coincide at $t = t' = 0$). Show that the angle between the x- and x'- axes is the same as that between the ct- and $ct'-$ axes and has the value $\tan^{-1}(v/c)$.

(b) Sketch on your diagram the loci of events separated from the spacetime origin $x = ct = 0$ by a constant invariant interval $\Delta s^2 = c^2t^2 - x^2$ for positive (timelike) and negative (spacelike) values of ΔS^2. How can these curves be used to calibrate lengths along the axes of the S and S' frames?

(c) Use your diagram to illustrate graphically why a rod at rest in S' is *contracted* as measured in S, and the time on a clock at rest in S' is *dilated* as observed in S.

Answer 1.0.3.

(a) The lines of constant t' and x' have equations $ct' = \gamma(ct - \beta x) =$ const. and $x' = \gamma(x - \beta ct) =$ const.. The angle between the x and x'-axes is

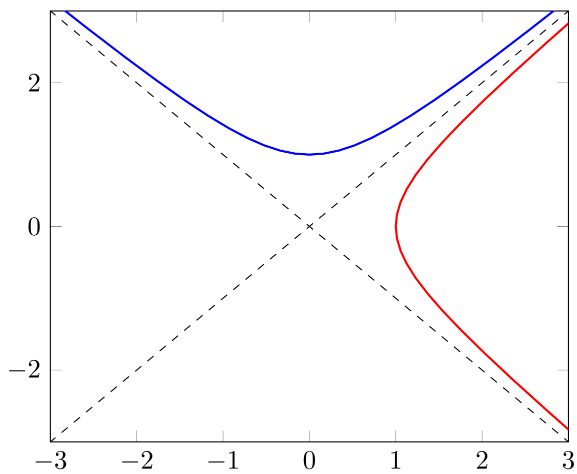
$$\tan^{-1}(x'/ct) = \tan^{-1}(\beta)$$

The angle between the ct' and ct-axes is

$$\tan^{-1}(x/ct) = \tan^{-1}(\beta)$$

where $\beta = v/c$.

(b) The spacetime interval is $\Delta s^2 = c^2t^2 - x^2$ and are represented by hyperbolae. On the spacetime diagram below, we have $\Delta s^2 < 0$ and $\Delta s^2 > 0$ to be represented by the red and blue curves respectively. The diagonal asymptotes are the worldlines of photons starting at the common spacetime origin, travelling at $\pm c$.



Consider $\Delta s^2 = -1$, then this hyperbola intersects the x and x'-axes at $x = 1$ and $x' = 1$ respectively.

(c) The fact that the spacetime interval hyperbolic curves can be used to calibrate lengths along the axes of the S and S' frames, we can demonstrate length contraction and time dilation.

- *Time dilation: Suppose we have a clock with worldline $x' = 0\ \forall t'$ which measures a period $ct' = cT_0$. The spacetime interval curve $\Delta s^2 = c^2T_0^2$ connects points on the ct'- and ct-axes with length cT_0. In frame S however, the observer measures the event with period $ct = c\Delta t$ instead, which graphically we have $c\Delta t > cT_0$, i.e. time dilated in S.*
- *Length contraction: Suppose we have a rod of length ℓ_0 at rest in frame S'. Again, the hyperbola $\Delta s^2 = -\ell_0^2$ connects points on the x'- and x-axes with length ℓ_0. To measure the length of the rod Δx in S, we need to mark out the positions of both ends simultaneously. We can see graphically that $\ell_0 > \Delta x$, i.e. length contraction in S.*

Problem 1.4 (Lorentz boost)**:** An inertial frame S' is related to the frame S by a boost of $\vec{v}$ whose components in S are (v_x, v_y, v_z). Show that the coordinates (ct', x', y', z') and (ct, x, y, z) of an event are related by

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+\alpha\beta_x^2 & \alpha\beta_x\beta_y & \alpha\beta_x\beta_z \\ -\gamma\beta_y & \alpha\beta_y\beta_x & 1+\alpha\beta_y^2 & \alpha\beta_y\beta_z \\ -\gamma\beta_z & \alpha\beta_z\beta_x & \alpha\beta_z\beta_y & 1+\alpha\beta_z^2 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

where $\vec{\beta} = \vec{v}/c$, $\gamma = (1-|\vec{\beta}|^2)^{-1/2}$ and $\alpha = (\gamma-1)/|\vec{\beta}|^2$. (Hint: resolve the 3-vector position with components (x, y, z) into parallel and perpendicular parts with respect to $\vec{\beta}$, and similarly in the S' frame.)

Answer 1.0.4. *Resolve* $\mathbf{r} = (x, y, z)$ *into parallel and perpendicular parts with respect to* $\vec{\beta}$*:*

$$\mathbf{r} = (\mathbf{r}\cdot\hat{\vec{\beta}})\hat{\vec{\beta}} + [\mathbf{r} - (\mathbf{r}\cdot\hat{\vec{\beta}})\hat{\vec{\beta}}] := r_{\parallel}\hat{\vec{\beta}} + \mathbf{r}_{\perp}$$

and similarly for $\mathbf{r}' = (x', y', z')$*. For a standard Lorentz Boost, the components perpendicular to* $\hat{\vec{\beta}}$ *are unchanged and that parallel transform as usual, i.e.*

$$r'_{\parallel} = \gamma(r_{\parallel} - \beta ct), \quad \mathbf{r}'_{\perp} = \mathbf{r}_{\perp}, \quad ct' = \gamma(ct - \beta r_{\parallel})$$

Hence, we have

$$\mathbf{r}' = r'_{\parallel}\hat{\vec{\beta}} + \mathbf{r} - r_{\parallel}\hat{\vec{\beta}} = -\gamma\beta ct\hat{\vec{\beta}} + \left[Id + (\gamma-1)\begin{pmatrix} \hat{\beta}_x^2 & \hat{\beta}_x\hat{\beta}_y & \hat{\beta}_x\hat{\beta}_z \\ \hat{\beta}_y\hat{\beta}_x & \hat{\beta}_y^2 & \hat{\beta}_y\hat{\beta}_z \\ \hat{\beta}_z\hat{\beta}_x & \hat{\beta}_z\hat{\beta}_y & \hat{\beta}_z^2 \end{pmatrix}\right]\mathbf{r}$$

but $\gamma - 1 = \alpha|\vec{\beta}|^2$*, and so*

$$\mathbf{r}' = -\gamma\beta\hat{\vec{\beta}}ct + \begin{pmatrix} 1+\alpha\beta_x^2 & \alpha\beta_x\beta_y & \alpha\beta_x\beta_z \\ \alpha\beta_y\beta_x & 1+\alpha\beta_y^2 & \alpha\beta_y\beta_z \\ \alpha\beta_z\beta_x & \alpha\beta_z\beta_y & 1+\alpha\beta_z^2 \end{pmatrix}\mathbf{r}$$

Together with

$$ct' = \gamma ct - \gamma\beta(x\beta_x + y\beta_y + z\beta_z)$$

We have our desired answer.

Problem 1.5 (Velocity composition)**:** In a given inertial frame, two particles are shot out simultaneously from a given point, with equal speeds v in orthogonal directions. What is the speed of each particle relative to the other?

Answer 1.0.5. *Let particle A and B have velocities* $v(0,1,0)$ *and* $v(1,0,0)$ *respectively. In the frame of particle B (Lorentz boost by* $v(1,0,0)$*), particle B observes particle A to have* $u_x = 0$, $u_y = v$, $u_z = 0$ *and so by usual velocity transformation,*

$$u'_x = -v, \quad u'_y = v/\gamma_v, u'_z = 0$$

The relative speed is then $v(1+\gamma_v^{-2})^{1/2} = v(2-(v/c)^2)^{-1/2}$.

Problem 1.6 (Velocity composition)**:**

(a) Frame S' moves with speed v relative to frame S in standard configuration. A rod at rest in frame S' makes an angle θ' with respect to the forward direction of motion. What is the angle θ measured in S?

(b) If a bullet is fired in S' at speed u' at an angle θ' with respect to the forward direction of motion, what is the angle θ measured in S? What if the bullet is a photon?

Answer 1.0.6.

(a) Let the two ends of the rod (with proper length ℓ_0) be A and B where A is at the origin. The worldline of A has coordinates $(ct', 0, 0, 0)$ in frame S'. The wordline of B has coordinates $(ct', \ell_0 \cos\theta', \ell_0 \sin\theta', 0)$. Inverse Lorentz transform back to frame S, the coordinates for A and B are respectively

$$\begin{pmatrix} \gamma ct' \\ \gamma\beta ct' \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} ct \\ \beta ct \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \gamma ct' + \gamma\beta\ell_0\cos\theta' \\ \gamma\ell_0\cos\theta' + \gamma\beta ct' \\ \ell_0 \sin\theta' \\ 0 \end{pmatrix} = \begin{pmatrix} ct \\ \gamma\ell_0\cos\theta' + \beta(ct - \gamma\beta\ell_0\cos\theta) = \ell_0\gamma^{-1}\cos\theta' + vt \\ \ell_0\sin\theta' \\ 0 \end{pmatrix}$$

The back and the front both moves at speed v, as expected. Hence, we must have

$$\tan\theta = \frac{\ell_0 \sin\theta'}{\ell_0\cos\theta'/\gamma} = \gamma\tan\theta'$$

(b) Inverse transform the velocity components for u':

$$u_x = \frac{u'\cos\theta' + v}{1 + (u'v/c^2)\cos\theta'}, \quad , u_y = \frac{u'\sin\theta'}{\gamma_v(1 + (u'v/c^2)\cos\theta')}, \quad u_z = 0$$

The angle θ in S satisfies

$$\tan\theta = \frac{u'\sin\theta'}{\gamma_v(u'\cos\theta' + v)}$$

If the object was a photon, then $u = c$ and we recover the light aberration relation $\tan\theta = \frac{\sin\theta'}{\gamma_v(\cos\theta' + \beta)}$.

Problem 1.7: Frame S' moves with speed v relative to frame S in standard configuration. Neutral π-mesons at rest in S' decay into two photons that are emitted isotropically. Show that the angular distribution of photons in S is

$$P(\theta)d\theta = \frac{\sin\theta d\theta}{2\gamma^2(1 - \beta\cos\theta)^2}$$

Answer 1.0.7. *From Q6, we have*

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta} \implies \frac{d\cos\theta'}{d\cos\theta} = \frac{1 - \beta\cos\theta - (\cos\theta - \beta)(-\beta)}{(1 - \beta\cos\theta)^2} = \frac{1}{\gamma^2(1 - \beta\cos\theta)^2}$$

Photons are emitted isotropically in a solid angle $2\pi\ d(\cos\theta')$ in S' are emitted in a solid angle $2\pi\ d(\cos\theta)$ in S. The number of photons emitted in either frame is the same:

$$P(\theta)d\theta = P(\theta')d\theta' = \frac{1}{4\pi}2\pi\ d\cos\theta' \implies P(\theta)d\theta = \frac{1}{2}\frac{d(\cos\theta')}{d(\cos\theta)}\sin\theta d\theta = \frac{\sin\theta d\theta}{2\gamma^2(1 - \beta\cos\theta)^2}$$

i.e. the photons are strongly 'breamed' in the x-direction due to aberration.

Problem 1.8 (Acceleration)**:**

(a) A spaceship travels in a straight line at a variable speed $u(t)$ in some inertial frame S. An observer on the spaceship measures his acceleration to be $f(\tau)$, where τ is his proper time. If at $\tau = 0$ the spaceship has a speed u_0 in S show that

$$\frac{u(\tau) - u_0}{1 - u(\tau)u_0/c^2} = c \tanh \psi(\tau)$$

where $c\psi(\tau) = \int_0^\tau f(\tau')d\tau'$. Show that the speed of the spaceship can never reach c.

(b) If the spaceship left base at time $t = \tau = 0$ and travelled forever in a straight line with constant acceleration g (for comfort), how long by the spaceship clock does it take to reach a star 10 light years from the base?

Answer 1.0.8.

(a) $f(\tau)$ is the acceleration in the instantaneous rest frame (IRF). At proper time τ, the speed is u and increases by $f(\tau)d\tau$ in the IRF. In S, the speed increases by du. By velocity addition,

$$u + du = \frac{fd\tau + u}{1 + (u/c^2)fd\tau} = u\left(1 - \frac{u}{c^2}fd\tau\right) + fd\tau + \cdots = u + \frac{fd\tau}{\gamma_u^2} + \cdots \implies \frac{du}{d\tau} = \frac{f}{\gamma_u^2}$$

Let $u/c = \tanh\theta(\tau)$, then

$$\frac{du}{d\tau} = \frac{c}{\cosh^2\theta}\frac{d\theta}{d\tau} \implies \frac{d\theta(\tau)}{d\tau} = \frac{f(\tau)}{c}$$

since $\gamma_u = \cosh\theta$ (definition of rapidity). We then have

$$c\theta(\tau) = \int_0^\tau f(\tau')d\tau' + c\tanh^{-1}(u_0/c)$$

where the integration constant occur since $u = u_0$ at $\tau = 0$. Evaluate $\tanh\psi(\tau)$:

$$\tanh\psi = \tanh(\theta - \tanh^{-1}(u_0/c)) = \frac{\tanh\theta - u_0/c}{1 - (u_0/c)\tanh\theta} = \frac{1}{c}\frac{u - u_0}{1 - uu_0/c^2}$$

Upon rearranging, we have

$$\frac{u}{c} = \frac{(u_0/c) + \tanh\psi}{1 + (u_0/c)\tanh\psi}$$

For $0 \le \psi < \infty$, the RHS increases monotonically from u_0/c to 1, i.e. $u(\tau)$ approaches but never exceeds c.

(b) By definition of proper time,

$$c^2d\tau^2 = c^2dt^2 - dx^2 = c^2dt^2(1 - u^2/c^2)$$

We have $d\tau/dt = 1/\gamma_u = 1/\cosh\psi$, so since

$$u = \frac{dx}{d\tau}\frac{d\tau}{dt} = c\tanh\psi \implies \frac{dx}{d\tau} = c\sinh\psi$$

But $\psi(\tau) = (g/c)\int_0^\tau d\tau = g\tau/c$, so

$$x = \frac{c^2}{g}[\cosh(g\tau/c) - 1]$$

where the integration constant is obtained from $x(\tau = 0) = 0$. We have $x = 10$ light years and so $xg/c^2 = \frac{(365.25)(24)(3600)(10)(9.81)}{3\times10^8} = 10.32$ and hence $g\tau/c = \cosh^{-1}(10.32 + 1) = 3.118$. Hence,

$$\tau = \frac{3.118}{9.81}\frac{3\times10^8}{24(3600)(365.25)} = 3.02 \text{ years}$$

Manifolds and Coordinates

Problem 1.9 (Metric tensor, volume coordinate)**:** In 3D Euclidean space, coordinates x'^a are related to Cartesian coordinates x^a by

$$x^1 = x'^1 + x'^2, \quad x^2 = x'^1 - x'^2, \quad x^3 = 2x'^1x'^2 + x'^3$$

Describe the coordinate surfaces in the primed system. Obtain the metric functions g'_{ab} in the primed system and hence show that these coordinates are not orthogonal. Calculate the volume element dV in the primed coordinate system.

Answer 1.0.9. *To describe the surfaces associated to the primed coordinates, we rearrange to get*

$$x'^1 = 0.5(x^1 + x^2), \quad x'^2 = 0.5(x^1 - x^2), \quad x'^3 = x^3 - 0.5((x^1)^2 - (x^2)^2)$$

$x'^1 = const.$ *and* $x'^2 = const.$ *are diagonal planes in the* x*-space, while* $x'^3 = const.$ *are hyperbolic paraboloids with principal axes along* x^1 *and* x^2 *directions. The spacetime invariant is*

$$\begin{aligned} ds^2 &= (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \\ &= (dx'^1 + dx'^2)^2 + (dx'^1 - dx'^2)^2 + (2x'^1dx'^2 + 2dx'^2dx'^1 + dx'^3)^2 \\ &= 2(dx'^1)^2 + 2(dx'^2)^2 + 4(x'^1)^2(dx'^2)^2 + 4(x'^2)^2(dx'^1)^2 + (dx'^3)^2 \\ &\quad + 8x'^1x'^2\,dx'^1\,dx'^2 + 4x'^1\,dx'^2 + dx'^3 + 4x'^2\,dx'^1\,dx'^3 \\ &= 2(1 + 2(x'^2)^2)(dx'^1)^2 + 2(1 + 2(x'^1)^2)(dx'^2)^2 + x'^1x'^2\,dx'^1\,dx'^2 + 4x'^1\,dx'^2\,dx'^3 + 4x'^2\,dx'^1\,dx'^3 + (dx^3)^2 \end{aligned}$$

with the metric tensor as

$$g'_{ab} = \begin{pmatrix} 2(1+2(x'^2)^2) & 4x'^1x'^2 & 2x'^2 \\ 4x'^1x'^2 & 2(1+2(x'^1)^2) & 2x'^1 \\ 2x'^2 & 2x'^1 & 1 \end{pmatrix}$$

Since the metric tensor is not diagonal, it is not an orthogonal coordinate system. The determinant and hence volume element is

$$\begin{aligned} &\det g \\ &= 2(1 + 2(x'^2)^2)[2(1 + 2(x'^1)^2 - 4(x'^1)^2] - 4(x'^1)(x'^2)[4x'^1\,x'^2 - 4x'^1\,x'^2] + 2x'^2[8(x'^1)^2(x'^2) - 4(x'^2)(1 + 2(x'^1)^2)] \\ &= 4 \implies dV = \sqrt{\det g}dx'^1\,dx'^2\,dx'^3 = 2dx'^1\,dx'^2\,dx'^3 \end{aligned}$$

Problem 1.10 (Line element, volume)**:** Show that the line element of a 3-sphere of radius a embedded in 4D Euclidean space can be written in the form

$$ds^2 = a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]$$

Hence, in this 3D non-Euclidean space, calculate the area of the 2-sphere defined by $\chi = \chi_0$. Also find the total volume of the 3D space.

Answer 1.0.10. *The 3-sphere* $x^2 + y^2 + z^2 + w^2 = a^2$ *is parametrized by*

$$w = a\cos\chi, \quad x = a\sin\chi\sin\theta\cos\phi, \quad y = a\sin\chi\sin\theta\sin\phi, \quad z = a\sin\chi\cos\theta, \quad 0 \le \chi \le \pi$$

where θ, ϕ *are the usual spherical polar coordinates. The infinitesimal elements are*

$$dw = -a\sin\chi d\chi, \quad dx = a\cos\chi\sin\theta\cos\phi d\chi + a\sin\chi d(\sin\theta\cos\phi)$$

$$dy = a\cos\chi\sin\theta\sin\phi d\chi + a\sin\chi d(\sin\theta\sin\phi), \quad dz = a\cos\chi\cos\theta d\chi + a\sin\chi d\cos\theta$$

The interval is

$$\begin{aligned} &dw^2 + dx^2 + dy^2 + dz^2 \\ &= a^2\sin^2\chi d\chi^2 + a^2\cos^2\chi d\chi^2 + a^2\sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) + 2a^2\sin\chi\cos\chi d\chi[\sin\theta\cos\phi d(\sin\theta\cos\phi) \\ &\quad + \sin\theta\sin\phi d(\sin\theta\sin\phi) + \cos\theta d\cos\theta] \\ &= a^2 d\chi^2 + a^2\sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned}$$

On the 2-sphere $\chi = \chi_0$*, the induced line element is*

$$ds^2 = a^2\sin^2\chi_0(d\theta^2 + \sin^2\theta d\phi^2)$$

as expected. The 2-volume and 3-volume respectively are

$$dV_{(2)} = a^2\sin^2\chi_0\sin\theta\,d\theta\,d\phi \implies V_{(2)} = 4\pi a^2\sin^2\chi_0, \quad dV_{(3)} = a^3\sin^3\chi\,d\chi\,d\cos\theta\,d\phi \implies V_{(3)} = 2\pi^2 a^3$$

2 Problem Sheet 2

Vector and Tensor Algebra

Problem 2.1 (Metric tensor, dual basis)**:** In 3D Euclidean space, coordinates x'^a are related to Cartesian coordinates x^a by

$$x^1 = x'^1 + x'^2, \quad x^2 = x'^1 - x'^2, \quad x^3 = 2x'^1x'^2 + x'^3$$

(a) Express the coordinate basis vectors $\mathbf{e'_a} = \frac{\partial}{\partial x'^a}$ for the primed coordinates in terms of those for the Cartesian coordinates. How are these related to the intersections of the coordinate surfaces that you sketched in Question 9 of Examples Sheet 1? By considering $\mathbf{g}(\mathbf{e'_a}, \mathbf{e'_b})$ obtain the components of the metric g'_{ab}. (Hint: since the original coordinates are Cartesian, $\mathbf{g}(\mathbf{e_a}, \mathbf{e_b}) = \delta_{ab}$.)

(b) Let the vector $\mathbf{v} = \mathbf{e_1}$. Write down the components v^a and those of the associated dual vector v_a. Calculate the components of the same vector $\mathbf{v}$ and its associated dual vector in the primed coordinates.

Answer 2.0.1.

(a) Using chain rule,

$$e'_1 = \frac{\partial x^1}{\partial x'^1}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial x'^1}\frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial x'^1}\frac{\partial}{\partial x^3} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + 2x'^2\frac{\partial}{\partial x^3} = e_1 + e_2 + (x^1 - x^2)e_3$$

$$e'_2 = \frac{\partial x^1}{\partial x'^2}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial x'^2}\frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial x'^2}\frac{\partial}{\partial x^3} = \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2} + 2x'^1\frac{\partial}{\partial x^3} = e_1 - e_2 + (x^1 + x^2)e_3$$

$$e'_3 = \frac{\partial x^1}{\partial x'^3}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial x'^3}\frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial x'^3}\frac{\partial}{\partial x^3} = e_3$$

$e'_3 = e_3$ is the derivative with respect to x'^3 at fixed x'^1 and x'^2, so along the directions of the line of intersection at constant x'^1 and x'^2. We have the metric tensor to be

$$g(e'_a, e'_b) = g'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}g_{cd}$$

$$g'_{11} = g_{11} + g_{22} + g_{33}(2x'^2)^2 = 2 + 4(x'^2)^2, \quad g'_{22} = 2 + 4(x'^2)^2$$

$$g'_{12} = g_{11}\frac{\partial x^1}{\partial x'^1}\frac{\partial x^1}{\partial x'^2} + g_{22}\frac{\partial x^2}{\partial x'^1}\frac{\partial x^2}{\partial x'^2} + g_{33}\frac{\partial x^3}{\partial x'^1}\frac{\partial x^3}{\partial x'^2} = 1 - 1 + 2x'^2 2x'^1 = 4x'^1x'^2 = g'_{21}$$

$$g'_{13} = g_{11}\frac{\partial x^1}{\partial x'^1}\frac{\partial x^1}{\partial x'^3} + g_{22}\frac{\partial x^2}{\partial x'^1}\frac{\partial x^2}{\partial x'^3} + g_{33}\frac{\partial x^3}{\partial x'^1}\frac{\partial x^3}{\partial x'^3} = 2x'^2 = g'_{31}$$

$$g'_{33} = g_{11}\frac{\partial x^1}{\partial x'^3}\frac{\partial x^1}{\partial x'^3} + g_{22}\frac{\partial x^2}{\partial x'^3}\frac{\partial x^2}{\partial x'^3} + g_{33}\frac{\partial x^3}{\partial x'^3}\frac{\partial x^3}{\partial x'^3} = 1$$

where we get the same result as that in Q9 of the previous example sheet.

(b) The vector is $v = v^a e_a \implies v^a = \delta^a{}_1 = (1,0,0)^T$. The dual vector is

$$v_b = g_{ab}v^a = \delta_{ab}\delta^a{}_1 = \delta_{b1} = (1,0,0)$$

where $g_{ab} = \delta_{ab}$ in the coordinate basis and it is used to lower the index. Now in the primed coordinates,

$$v'^a = \frac{\partial x'^a}{\partial x^b}v^b = \frac{\partial x'^a}{\partial x^1} = \begin{pmatrix} 1/2 \\ 1/2 \\ -(x'^1 + x'^2) \end{pmatrix}$$

where $x'^1 = 0.5(x^1 + x^2)$, $x'^2 = 0.5(x^1 - x^2)$ and $x'^3 = x^3 - 2x'^1x'^2 = x^3 - 0.5((x^1)^2 + (x^2)^2)$. Similarly,

$$v'_a = \frac{\partial x^b}{\partial x'^a}v_b = \frac{\partial x^1}{\partial x'^a} = (1,1,0)$$

Problem 2.2 (Tensor)**:**

(a) If the tensor A_{ab} is an antisymmetric tensor, S_{ab} is a symmetric tensor and T_{ab} is a general tensor, show that $A^{ab}T_{ab} = A^{ab}T_{[ab]}$ and $S^{ab}T_{ab} = S^{ab}T_{(ab)}$.

(b) If v_a are the components of a dual vector, show that in an arbitrary coordinate system $A_{ab} = \partial_b v_a - \partial_a v_b$ are the components of a type-(0, 2) tensor. Show further, for a general antisymmetric tensor A_{ab}, that $B_{abc} = \partial_c A_{ab} + \partial_a A_{bc} + \partial_b A_{ca}$ are the components of a type-(0, 3) tensor. What are the symmetry properties of B_{abc}?

Answer 2.0.2.

(a) We are given $A_{ab} = -A_{ba}$ and $S_{ab} = S_{ba}$, so

$$A^{ab}T_{[a,b]} = \frac{1}{2}A^{ab})(T_{ab} - T_{ba}) = \frac{1}{2}(A^{ab}T_{ab} + A^{ab}T_{ab}) = A^{ab}T_{ab}$$

where $A^{ba}T_{ba} = A^{ab}T_{ab}$. Similarly,

$$S^{ab}T_{(a,b)} = \frac{1}{2}S^{ab}(T_{ab} + T_{ba}) = S^{ab}T_{ab}$$

(b) The covariant derivative of a dual vector is given as

$$\nabla_b v_a = \partial_b v_a - \Gamma^c_{ba}v_c \implies \nabla_b v_a - \nabla_a v_b = \partial_b v_a - \partial_a v_b$$

since in GR, the connection is torsion free, i.e. $\Gamma^c_{ba} = \Gamma^c_{ab}$. Hence, it follows that

$$A_{ab} = \partial_b v_a - \partial_a v_b = \nabla_b v_a - \nabla_a v_b$$

is a type (0,2) tensor. Now, the covariant derivative for a general antisymmetric tensor A_{ab} is

$$\nabla_c A_{ab} = \partial_c A_{ab} - \Gamma^d_{ca}A_{db} - \Gamma^d_{cb}A_{ad}$$

Do permutations of the indices to get two other equations. Add all three up and we have

$$\nabla_c A_{ab} + \nabla_a A_{bc} + \nabla_b A_{ca} = \partial_c A_{ab} + \partial_a A_{bc} + \partial_b A_{ca} + 0 = B_{abc}$$

where the zero is as a result of terms like

$$-\Gamma^d_{ab}A_{dc} - \Gamma^d_{ba}A_{cd} = +\Gamma^d_{ab}(-A_{dc} + A_{dc})$$

since A is anti-symmetric and that Γ^c_{ab} is symmetric in the bottom indices a, b. Hence, B_{abc} is a type (0,3) tensor. Now, anti-symmetrize:

$$B_{[a,b,c]} = \frac{1}{6}(B_{abc} + B_{bca} + B_{cab} - B_{bac} - B_{acb} - B_{cba}) = \frac{1}{2}(B_{abc} - B_{acb})$$

where B_{abc} is evidently symmetric for any even permutations of a, b and c. Since A is anti-symmetric, it follows that

$$B_{acb} = \partial_b A_{ac} + \partial_a A_{cb} + \partial_c A_{ba} = -\partial_b A_{ca} - \partial_a A_{bc} - \partial_c A_{ab} = -B_{abc}$$

i.e. B_{abc} is anti-symmetric in the indices c and b. Hence, we must have B_{abc} to be totally anti-symmetric, i.e. $B_{[a,b,c]} = \frac{1}{2}(B_{abc} + B_{abc}) = B_{abc}$.

Vector and tensor calculus on manifolds

Problem 2.3 (Metric connection):

(a) If $g = \det(g_{ab})$ is the determinant of the metric, show that $\partial_c g = g g^{ab}(\partial_c g_{ab})$.

(b) Verify directly, in a general coordinate system, that $\nabla_c g_{ab} = 0$ for the covariant derivative constructed with the metric connection.

(c) For a diagonal metric g_{ab}, show that the connection coefficients are given by (with $a \neq b \neq c$ and no summation over repeated indices)

$$\Gamma^a_{bc} = 0, \quad \Gamma^b_{aa} = -\frac{1}{2g_{bb}}\frac{\partial g_{aa}}{\partial x^b}, \quad \Gamma^a_{ba} = \Gamma^a_{ab} = \frac{\partial}{\partial x_b}(\ln\sqrt{|g_{aa}|})$$

Answer 2.0.3.

(a) We have $g = \det g \implies \ln g = \mathrm{Tr}(\ln g)$, *so*

$$\frac{1}{g}\partial_c g = \mathrm{Tr}(g^{-1}\partial_c g) \implies \frac{1}{g}\partial_c g = g^{ab}\partial_c g_{ab}$$

where we used definition of trace.

(b) The covariant derivative of the metric tensor is

$$\nabla_c g_{ab} = \partial_c g_[ab] - \Gamma^d_{ca} g_{db} - \Gamma^d_{cb} g_{ad}$$

where the metric connection is

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc})$$

Hence,

$$\begin{aligned}\nabla_c g_{ab} &= \partial_c g_{ab} - \frac{1}{2}(\partial_c g_{ab} + \partial_a g_{bc} - \partial_b g_{ca}) - \frac{1}{2}(\partial_c g_{ba} + \partial_b g_{ac} - \partial_a g_{cb}) \\ &= 0\end{aligned}$$

(c) Now, g_{ab} *is diagonal.*

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) = 0$$

if $a \neq b \neq c$, *all the* ∂g *terms are zero since* g *is diagonal tensor. If two indices are the same, we contract:*

$$\Gamma^a_{bb} = \frac{1}{2}g^{ad}(\partial_b g_{bd} + \partial_b g_{db} - \partial_d g_{bb}) = -\frac{1}{2}g^{aa}\partial_a g_{bb} = -\frac{1}{2g_{aa}}\frac{\partial g_{bb}}{\partial x^a}$$

$$\Gamma^a_{ba} = \frac{1}{2}g^{ad}(\partial_b g_{ad} + \partial_a g_{db} - \partial_d g_{ba}) = \frac{1}{2}g^{aa}(\partial_b g_{aa}) = \frac{1}{2g_{aa}}\frac{\partial g_{aa}}{\partial x_b} = \frac{\partial}{\partial x^b}\ln\sqrt{|g_{aa}|}$$

where we bear in mind g *is diagonal.*

Problem 2.4 (Connection)**:** In 2D Euclidean space, the line element in plane-polar coordinates is

$$ds^2 = d\rho^2 + \rho^2 d\phi^2$$

(a) Obtain the non-zero connection coefficients

$$\Gamma^{\phi}_{\rho\phi} = \Gamma^{\phi}_{\phi\rho} = 1/\rho, \quad \Gamma^{\rho}_{\phi\phi} = -\rho$$

(b) If the coordinate components v^a of a vector $\mathbf{v}$ are written as v^ρ and v^ρ, show that the divergence of $\mathbf{v}$ is

$$\nabla_a v^a = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v^\rho) + \frac{\partial v^\phi}{\partial\phi}$$

What would be the equivalent result in terms of the components of $\mathbf{v}$ in an orthonormal basis aligned with the coordinate directions?

(c) Show that the Laplacian of a scalar field f is

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}$$

Answer 2.0.4.

(a) From the line element, we have $g_{\rho\rho} = 1$ and $g_{\phi\phi} = \rho^2$. From Q3,

$$\Gamma^{\phi}_{\rho\phi} = \frac{\partial}{\partial\rho}\ln\sqrt{|g_{\phi\phi}|} = \frac{\partial\ln\rho}{\partial\rho} = \frac{1}{\rho}, \quad \Gamma^{\rho}_{\phi\phi} = -\frac{1}{2g_{\rho\rho}}\frac{\partial g_{\phi\phi}}{\partial\rho} = -\frac{2\rho}{2} = -\rho$$

(b) The covariant derivative is

$$\begin{aligned}
\nabla_a v^a &= \partial_a v^a + \Gamma^a_{ab} v^b \\
&= \partial_\rho v^\rho + \partial_\phi v^\phi + \Gamma^a_{a\rho} v^\rho + \Gamma^\phi_{a\phi} v^\phi \\
&= \partial_\rho v^\rho + \partial_\phi v^\phi + (\Gamma^\rho_{\rho\rho} + \Gamma^\phi_{\phi\phi}) v^\rho + (\Gamma^\rho_{\rho\phi} + \Gamma^\phi_{\phi\phi}) v^\phi \\
&= \partial_\rho v^\rho + \partial_\phi v^\phi + \frac{1}{\rho} v^\rho \\
&= \frac{1}{\rho}\partial_\rho(\rho v^\rho) + \partial_\phi v^\phi
\end{aligned}$$

Set $\boldsymbol{e_\rho} = \frac{\partial}{\partial\rho}$, $\boldsymbol{e_\phi} = \frac{\partial}{\partial\phi}$, it follows that the unit vectors are

$$\boldsymbol{\hat{e}_a} = \frac{1}{g_{aa}}\boldsymbol{\hat{e}_a} \implies \boldsymbol{\hat{e}_\rho} = \boldsymbol{\hat{e}_\rho}, \ \boldsymbol{\hat{e}_\phi} = \frac{1}{\rho}\boldsymbol{\hat{e}_\phi}$$

So written in terms of the unit vectors

$$\mathbf{v} = v^\rho \boldsymbol{e_\rho} + v^\phi \boldsymbol{e_\phi} = v^{\hat{\rho}}\boldsymbol{\hat{e}_\rho} + \frac{v^{\hat{\phi}}}{\rho}\boldsymbol{\hat{e}_\phi} \implies v^{\hat{\rho}} = v^\rho, \ \frac{v^{\hat{\phi}}}{\rho} = v^\phi$$

Hence, the divergence is given as

$$\nabla_a v^a = \frac{1}{\rho}\partial_\rho(\rho v^{\hat{\rho}}) + \frac{1}{\rho}\partial_\phi(v^{\hat{\phi}})$$

(c) The Laplacian is $\nabla^2 f = \nabla_a \nabla^a f$ but $\nabla^a f = g^{ab}\partial_b f$, and we have $\boldsymbol{\nabla} f = (\partial_\rho f, \rho^{-2}\partial_\phi f)$, and so

$$\nabla^2 f = \rho^{-1}\partial_\rho(\rho\partial_\rho f) + \rho^{-2}\partial_\phi(\partial_\phi f)$$

Problem 2.5 (Geodesic)**:** On the surface of a unit sphere $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$.

(a) Calculate the connection coefficients in the (θ, ϕ) coordinate system directly from the metric.

(b) By considering the 'Lagrangian' $L = g_{ab}\dot{x}^a\dot{x}^b$, derive the equations for an affinely-parameterised geodesic on the surface of a sphere in the coordinates (θ, ϕ) and thereby verify your answer to (a). Hence show that, of all the circles of constant latitude on a sphere, only the equator is a geodesic.

(c) A vector $\mathbf{v}$ of unit length is defined at the point $(\theta_0, 0)$ and is parallel to the circle $\phi = 0$. Calculate the components of $\mathbf{v}$ after it has been parallel transported around the circle $\theta = \theta_0$. Hence show that, in general, after parallel transport, the direction of $\mathbf{v}$ is different, but its length is unchanged.

Answer 2.0.5.

(a)

(b)

(c)

Problem 2.6 (Geodesic)**:** A hypersurface $\mathcal{H}$ within a manifold $\mathcal{M}$ contains a non-null curve $\mathcal{C}$. Give a geometric argument showing that if $\mathcal{C}$ is a geodesic in $\mathcal{M}$, it is also a geodesic in $\mathcal{H}$. Give an example to show that the converse is not necessarily true.

Answer 2.0.6.

Problem 2.7 (Optional)**:**

(a)

Answer 2.0.7.

(a)

(b)

Minkowski spacetime and particle dynamics

Problem 2.8: In Minkowski spacetime, two uniformly-moving observers $\mathcal{E}$ and $\mathcal{R}$ have 4-velocities u and v, respectively.

(a) Show that $u^\mu v_\mu = c^2\gamma_V$, where V is their relative speed.

(b) If $\mathcal{E}$ emits a photon that is subsequently received by $\mathcal{R}$, show that the ratio of the emitted and received photon frequencies is given by

$$\frac{v_\mathcal{E}}{v_\mathcal{R}} = \frac{u^\mu p_\mu}{v^\nu p_\nu}$$

where $\mathbf{p}$ is the photon 4-momentum.

Answer 2.0.8.

(a)

(b)

Problem 2.9: Suppose an observer $\mathcal{O}$ begins to accelerate in Minkowski spacetime such that, at some instant, his 3-velocity and 3-acceleration in an inertial frame S are $\vec{u}$ and $\vec{a}$, respectively. Show that the (proper) acceleration α measured by $\mathcal{O}$ at this instant is given by

$$\alpha^2 = \frac{\gamma_u^6(\vec{u}\cdot\vec{a})^2}{c^2} + \gamma_u^4\vec{a}\cdot\vec{a}$$

Find an expression for α if the motion in S is circular with radius r.

Answer 2.0.9.

3 Problem Sheet 3

Problem 3.1:

(a)

Answer 3.0.1.

(a)

(b)

Problem 3.2:

(a)

Answer 3.0.2.

(a)

(b)

Problem 3.3:

(a)

Answer 3.0.3.

(a)

(b)

Problem 3.4:

(a)

Answer 3.0.4.

(a)

(b)

Electromagnetism

Problem 3.5:

(a)

Answer 3.0.5.

(a)

(b)

Spacetime Curvature

Problem 3.6:

(a)

Answer 3.0.6.

(a)

(b)

Problem 3.7:

(a)

Answer 3.0.7.

(a)

(b)

Problem 3.8:

(a)

Answer 3.0.8.

(a)

(b)

Problem 3.9:

(a)

Answer 3.0.9.

(a)

(b)

Problem 3.10:

(a)

Answer 3.0.10.

(a)

(b)

4 Problem Sheet 4