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Implicit difference approximation for the time fractional heat equation with the nonlocal condition

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ABSTRACT

In this work, a method for solving inhomogeneous nonlocal fractional heat equations is proposed. The method is based on the modified Gauss elimination method. It is proved by using matrix stability approach that the method is unconditionally stable. Numerical results are provided to illustrate the accuracy and efficiency of the proposed method.

1. Introduction

Ordinary and partial differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. A great deal of effort has been expended over the last 15 years or so in attempting to find robust and stable numerical and analytical methods for solving fractional partial differential equations. Several methods have been introduced to solve fractional differential equations, the popular Laplace transform method [10,21], the Fourier transform method [20], the iteration method [22] and the operational method [8]. Most nonlinear fractional differential equations do not have exact analytic solutions, so approximation and numerical techniques were used in [4,9,11–15,18,19]. One-dimensional linear time fractional diffusion equations were solved by implicit numerical methods in [25] and [17] for initial value problems. They showed that the method is unconditionally stable and convergent. In [26], some techniques were proposed to improve the order of the convergence for the solutions of anomalous sub-diffusion equations. One-dimensional classical heat equation with the nonlocal condition is solved and stability is discussed by the semigroup approach in [1] and the matrix stability is proved in [7]. The stepwise stability of difference schemes for two-dimensional parabolic equations with nonlocal boundary conditions are presented in [5].

The main purpose of this work is to solve the time fractional heat equations with *the nonlocal condition*.

We consider the one-dimensional fractional heat equation with the nonlocal condition:

$$\begin{cases} \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x), & (0 < x < 1, 0 < t < 1), \\ u(0,x) = u(1,x) + \rho(x), & 0 \leqslant x \leqslant 1, \\ u(t,0) = 0, u(t,1) = 0, & 0 \leqslant t \leqslant 1. \end{cases} \tag{1.1}$$

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Here the term $\frac{\partial^\alpha u(t,x)}{\partial t^\alpha}$ denotes α-order Caputo derivative [21], defined by the formula:

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha}=\begin{cases}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u_t(\tau,x)}{(t-\tau)^\alpha}\,d\tau, & \text{if } 0<\alpha<1,\\ \frac{\partial u(t,x)}{\partial t}, & \text{if } \alpha=1.\end{cases}$$

We discretize the time fractional derivative of order α, using Caputo finite difference formula [3,21,22] which is a first order approximation,

$$\frac{\partial^\alpha u(t_k,x_n)}{\partial t^\alpha}=\tau^{-\alpha}\sum_{r=0}^{k} w_r\big(U_n^{k-r}-U_n^0\big)+O(\tau), \tag{1.2}$$

where $w_0=1$, $w_r=(1-\frac{\alpha+1}{r})w_{r-1}$, $r=1,2,\dots$ and $\sum_{r=0}^{\infty} w_r=0$ and U_n^k denotes the numerical approximation to the exact solution $u(t_k,x_n)$, $t_k=k\tau$, $0\leqslant k\leqslant N$, $N\tau=1$ and $x_n=nh$, $0\leqslant n\leqslant M$, $Mh=1$.

Using the approximation at (1.2) for fractional time derivative and centered difference approximation for the spatial second derivative in Eq. (1.1), we obtain the following difference equation which is accurate of order $O(\tau+h^2)$;

$$\begin{cases}\frac{\sum_{r=0}^{k} w_r(U_n^{k-r}-U_n^0)}{\tau^\alpha}-\frac{U_{n+1}^k-2U_n^k+U_{n-1}^k}{h^2}=f(t_k,x_n), & 1\leqslant k\leqslant N,\ 1\leqslant n\leqslant M,\\ U_0^k=0,\quad U_M^k=0, & 0\leqslant k\leqslant N,\\ U_n^0-U_n^N=\rho(x_n), & 1\leqslant n\leqslant M-1,\end{cases} \tag{1.3}$$

We can arrange the system above, to obtain

$$\begin{cases}(-\frac{1}{h^2})U_{n+1}^k\\ \quad+[\frac{-(w_0+w_1+\cdots+w_k)}{\tau^\alpha}U_n^0+(\frac{w_{k-1}}{\tau^\alpha}U_n^1+\frac{w_{k-2}}{\tau^\alpha}U_n^2+\cdots+\frac{w_1}{\tau^\alpha}U_n^{k-1})+\frac{2}{h^2}U_n^k]\\ \quad+(-\frac{1}{h^2})U_{n-1}^k=f(t_k,x_n),\quad 1\leqslant k\leqslant N-1,\ 1\leqslant n\leqslant M-1,\\ U_0^k=U_M^k=0,\quad 0\leqslant k\leqslant N,\\ U_n^0-U_n^N=\rho(x_n),\quad 1\leqslant n\leqslant M-1.\end{cases} \tag{1.4}$$

The difference scheme above can be written in matrix form,

$$\begin{cases}AU_{n+1}+BU_n+AU_{n-1}=D\varphi_n,\quad 1\leqslant n\leqslant M-1,\\ U_0=\vec{0},\quad U_M=\vec{0}.\end{cases} \tag{1.5}$$

where $\varphi_n=[\varphi_n^0,\varphi_n^1,\varphi_n^2,\dots,\varphi_n^N]^T$, $\vec{0}=0_{(N+1)\times 1}$, $\varphi_n^0=\rho(x_n)$, $1\leqslant n\leqslant M-1$, and $\varphi_n^k=f(t_k,x_n)$, $1\leqslant k\leqslant N$, $1\leqslant n\leqslant M-1$. Here $A_{(N+1)\times(N+1)}$ and $B_{(N+1)\times(N+1)}$ matrices are in the form of

$$A=\begin{bmatrix}0&0&0&0&\dots&0\\ 0&-\frac{1}{h^2}&0&0&\dots&0\\ 0&0&-\frac{1}{h^2}&0&\dots&0\\ 0&0&0&-\frac{1}{h^2}&\dots&0\\ \dots&\dots&\dots&\dots&\dots&\dots\\ 0&0&0&0&\dots&-\frac{1}{h^2}\end{bmatrix},$$

$$B=\begin{bmatrix}1&0&0&\dots&0&-1\\ -\frac{w_0}{\tau^\alpha}&\frac{w_0}{\tau^\alpha}+\frac{2}{h^2}&0&\dots&0&0\\ \frac{-(w_0+w_1)}{\tau^\alpha}&\frac{w_1}{\tau^\alpha}&\frac{w_0}{\tau^\alpha}+\frac{2}{h^2}&\dots&0&0\\ \vdots&\vdots&\ddots&\ddots&\dots&\dots\\ \frac{-(w_0+w_1+\cdots+w_{N-2})}{\tau^\alpha}&\frac{w_{N-2}}{\tau^\alpha}&\dots&\frac{w_1}{\tau^\alpha}&\frac{w_0}{\tau^\alpha}+\frac{2}{h^2}&0\\ \frac{-(w_0+w_1+\cdots+w_N-1)}{\tau^\alpha}&\frac{w_{N-1}}{\tau^\alpha}&\dots&\frac{w_2}{\tau^\alpha}&\frac{w_1}{\tau^\alpha}&\frac{w_0}{\tau^\alpha}+\frac{2}{h^2}\end{bmatrix},$$

$$U_n=\big[U_n^0,U_n^1,U_n^2,\dots,U_n^N\big]^T,$$

$$U_{n-1}=\big[U_{n-1}^0,U_{n-1}^1,U_{n-1}^2,\dots,U_{n-1}^N\big]^T,$$

$$U_{n+1}=\big[U_{n+1}^0,U_{n+1}^1,U_{n+1}^2,\dots,U_{n+1}^N\big]^T.$$

Using the idea on the modified Gauss-Elimination method, we can convert Eq. (1.5) into the following form of difference scheme:

$$U_n=\alpha_{n+1}U_{n+1}+\beta_{n+1},\quad n=M-1,\dots,2,1,0. \tag{1.6}$$

This way, the two-step form of difference scheme in (1.5) is transformed to one-step method as in (1.6). Now, we need to determine the matrices α_{n+1} and β_{n+1} satisfying the last equality.

Since $U_0 = \alpha_1 U_1 + \beta_1 = 0$, we can select $\alpha_1 = O_{(N+1)\times(N+1)}$ and $\beta_1 = O_{(N+1)\times 1}$.

Combining the equalities $U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}$, and $U_{n-1} = \alpha_n U_n + \beta_n$ and the matrix equation (1.5), we have

$$(A + B\alpha_{n+1} + A\alpha_n\alpha_{n+1})U_{n+1} + (B\beta_{n+1} + A\alpha_n\beta_{n+1} + A\beta_n) = D\varphi_n.$$

Then, we write

$$\begin{cases} A + B\alpha_{n+1} + A\alpha_n\alpha_{n+1} = 0, \\ B\beta_{n+1} + A\alpha_n\beta_{n+1} + A\beta_n = D\varphi_n, \end{cases}$$

where $1 \leqslant n \leqslant M-1$.

So, we obtain the following pair of formulas:

$$\begin{cases} \alpha_{n+1} = -(B + A\alpha_n)^{-1}A, \\ \beta_{n+1} = (B + A\alpha_n)^{-1}(D\varphi_n - A\beta_n) \end{cases}$$

where $1 \leqslant n \leqslant M-1$.

To show the stability of this method we give some remarks and lemmas.

Remark 1.1. If X, Y, Z, T are square block matrices and the matrices X and S are invertible, then

$$\begin{bmatrix} X & Y \\ Z & T \end{bmatrix}^{-1} = \begin{bmatrix} X^{-1} + X^{-1}YS^{-1}ZX^{-1} & -X^{-1}YS^{-1} \\ -S^{-1}ZX^{-1} & S^{-1} \end{bmatrix}$$

where S is the Schur complement [2] of the block inversion and $S = (T - ZX^{-1}Y)$.

Remark 1.2. The symbol $\|.\|$ denotes the infinity norm which is

$$\|A_{n\times n}\| = \|A_{n\times n}\|_\infty = \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}|a_{ij}|\right\}$$

where $A = [a_{ij}]_{n\times n}$.

Remark 1.3. $Z_{n\times n}$ is called a strictly diagonally dominant if $|Z_{i,i}| > r_i(Z)$, $1 \leqslant i \leqslant n$, where $r_i(Z)$ is the sum of the absolute value of nondiagonal elements on the i-th row of Z [16].

Remark 1.4. If $Z_{n\times n}$ is strictly diagonally dominant matrix then it's not singular [23] and

$$\|Z^{-1}\| \leqslant \frac{1}{\min_{1\leqslant i\leqslant n}(|Z_{i,i}| - r_i(Z))}.$$

Remark 1.5. Let A, B and α_n defined as above. When $\|\alpha_n\| < 1$, $(B + A\alpha_n)$ is strictly diagonally dominant. Invertibility of the matrix of $(B + A\alpha_n)$ follows from Remark 1.4.

2. Stability of the method

Lemma 2.1. *If A and B are matrices given in* (1.5), *then* $\|B^{-1}A\| \leqslant \frac{1}{2}$.

Proof. Let's partition the matrix B into subblocks to find its inverse that is

$$\begin{aligned} B^{-1} &= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & -1 \\ -\frac{w_0}{\tau^\alpha} & \frac{w_0}{\tau^\alpha} + \frac{2}{h^2} & 0 & \dots & 0 & 0 \\ \frac{-(w_0+w_1)}{\tau^\alpha} & \frac{w_1}{\tau^\alpha} & \frac{w_0}{\tau^\alpha} + \frac{2}{h^2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \dots & \dots \\ \frac{-(w_0+w_1+\dots+w_{N-2})}{\tau^\alpha} & \frac{w_{N-2}}{\tau^\alpha} & \dots & \frac{w_1}{\tau^\alpha} & \frac{w_0}{\tau^\alpha} + \frac{2}{h^2} & 0 \\ \frac{-(w_0+w_1+\dots+w_N-1)}{\tau^\alpha} & \frac{w_{N-1}}{\tau^\alpha} & \dots & \frac{w_2}{\tau^\alpha} & \frac{w_1}{\tau^\alpha} & \frac{w_0}{\tau^\alpha} + \frac{2}{h^2} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} X & Y \\ Z & T \end{bmatrix}^{-1} \end{aligned}$$

where

$$X=[1],$$
$$Y=[0,0,\dots,0,-1],$$
$$Z=\begin{bmatrix} -\frac{w_0}{\tau^\alpha} \\ \frac{-(w_0+w_1)}{\tau^\alpha} \\ \vdots \\ \frac{-(w_0+w_1+\dots+w_N-1)}{\tau^\alpha} \end{bmatrix},$$
$$T=\begin{bmatrix} \frac{w_0}{\tau^\alpha}+\frac{2}{h^2} & 0 & 0 & \dots & 0 & 0 \\ \frac{w_1}{\tau^\alpha} & \frac{w_0}{\tau^\alpha}+\frac{2}{h^2} & 0 & \dots & 0 & 0 \\ \frac{w_2}{\tau^\alpha} & \frac{w_1}{\tau^\alpha} & \frac{w_0}{\tau^\alpha}+\frac{2}{h^2} & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \dots \\ \frac{w_{N-2}}{\tau^\alpha} & \dots & \frac{w_2}{\tau^\alpha} & \frac{w_1}{\tau^\alpha} & \frac{w_0}{\tau^\alpha}+\frac{2}{h^2} & 0 \\ \frac{w_{N-1}}{\tau^\alpha} & \dots & \frac{w_3}{\tau^\alpha} & \frac{w_2}{\tau^\alpha} & \frac{w_1}{\tau^\alpha} & \frac{w_0}{\tau^\alpha}+\frac{2}{h^2} \end{bmatrix}.$$

Then

$$B^{-1}=\begin{bmatrix} X^{-1}+X^{-1}YS^{-1}ZX^{-1} & -X^{-1}YS^{-1} \\ -S^{-1}ZX^{-1} & S^{-1} \end{bmatrix}$$
$$=\begin{bmatrix} 1+1YS^{-1}Z1 & -1YS^{-1} \\ -S^{-1}Z1 & S^{-1} \end{bmatrix}.$$

Hence $B^{-1}A=\left[\begin{smallmatrix} 0 & -YaS^{-1} \\ 0 & aS^{-1} \end{smallmatrix}\right]$, where $a=-\frac{1}{h^2}=-M^2$.
$\|B^{-1}A\|=\max\{\|-YaS^{-1}\|,\|aS^{-1}\|\}=\|aS^{-1}\|$, since YaS^{-1} is exactly the last row of aS^{-1}.

$$aS^{-1}=a(T-ZX^{-1}Y)^{-1}=a(T-ZY)^{-1}$$
$$=a\begin{bmatrix} \frac{w_0}{\tau^\alpha}+\frac{2}{h^2} & 0 & 0 & 0 & -\frac{w_0}{\tau^\alpha} \\ \frac{w_1}{\tau^\alpha} & \frac{w_0}{\tau^\alpha}+\frac{2}{h^2} & 0 & 0 & \frac{-(w_0+w_1)}{\tau^\alpha} \\ \frac{w_2}{\tau^\alpha} & \frac{w_1}{\tau^\alpha} & \frac{w_0}{\tau^\alpha}+\frac{2}{h^2} & 0 & \frac{-(w_0+w_1+w_2)}{\tau^\alpha} \\ \vdots & \ddots & \ddots & \dots & \dots \\ \frac{w_{N-1}}{\tau^\alpha} & \frac{w_{N-2}}{\tau^\alpha} & \dots & \frac{w_1}{\tau^\alpha} & \frac{w_0}{\tau^\alpha}+\frac{2}{h^2}-\frac{(w_0+w_1+\dots+w_{N-1})}{\tau^\alpha} \end{bmatrix}^{-1}.$$

Therefore $\|aS^{-1}\|=|a|\|S^{-1}\|=M^2\|S^{-1}\|$. Here S is strictly diagonally dominant, then $\|B^{-1}A\|\leqslant\|aS^{-1}\|=M^2\|S^{-1}\|\leqslant M^2\frac{1}{\frac{2}{h^2}}=M^2\frac{1}{2M^2}=\frac{1}{2}$.

So $\|B^{-1}A\|\leqslant\frac{1}{2}$.

Lemma 2.2. *If $\|\alpha_n\|\leqslant 1$ and $\|B^{-1}A\|\leqslant\frac{1}{2}$ then $(I+\alpha_nB^{-1}A)$ is strictly diagonally dominant and $\|(I+\alpha_nB^{-1}A)^{-1}\|\leqslant\frac{1}{1-\|\alpha_n\|\|B^{-1}A\|}$.*

Proof. Since $\|\alpha_n\|\leqslant 1$ and $\|B^{-1}A\|\leqslant\frac{1}{2}$, we have $\|\alpha_nB^{-1}A\|\leqslant\|\alpha_n\|\|B^{-1}A\|\leqslant\frac{1}{2}$.

Therefore $I+\alpha_nB^{-1}A$ is strictly diagonally dominant. Put $Z=I+\alpha_nB^{-1}A$. Define $r_i(Z)$ as in Remark 1.4. Then, using Remark 1.4, we have

$$\left\|\left(I+\alpha_nB^{-1}A\right)^{-1}\right\|=\left\|Z^{-1}\right\|\leqslant\frac{1}{\min_i\{|Z_{i,i}|-r_i(Z)\}}$$
$$\leqslant\frac{1}{1-\|\alpha_nB^{-1}A\|}$$
$$\leqslant\frac{1}{1-\|\alpha_n\|\|B^{-1}A\|}.\qquad\Box$$

Theorem 2.3. *The matrix equation* (1.6) *which we use to solve the differential equation* (1.1) *is unconditionally stable.*

Proof. α_{n+1} is the iteration matrix of the system

$$U_n = \alpha_{n+1}U_{n+1} + \beta_{n+1}$$

and to prove the stability we will show that $\|\alpha_{n+1}\| \leqslant 1$ for all $0 \leqslant n \leqslant M-1$ as in [24].

We prove it by induction.

Since $\alpha_1 = \vec{0}$ then $\|\alpha_1\| \leqslant 1$.

Moreover, $\alpha_2 = -(B + A\alpha_1)^{-1}A = -(B + A\ \vec{0})^{-1}A = -B^{-1}A$, from Lemma 2.1 we already know $\|\alpha_2\| = \|-B^{-1}A\| \leqslant \frac{1}{2} \leqslant 1$.

Now, assume $\|\alpha_n\| \leqslant 1$.

On the other hand, from the Kailath theorem [6], we can write $(B + A\alpha_n)^{-1} = B^{-1} - B^{-1}A(I + \alpha_n B^{-1}A)^{-1}\alpha_n B^{-1}$. Then we obtain,

$$\begin{aligned}
\|\alpha_{n+1}\| &= \left\|-(B + A\alpha_n)^{-1}A\right\| \\
&= \left\|(B + A\alpha_n)^{-1}A\right\| \\
&= \left\|\left[B^{-1} - B^{-1}A\left(I + \alpha_n B^{-1}A\right)^{-1}\alpha_n B^{-1}\right]A\right\| \\
&= \left\|B^{-1}A - B^{-1}A\left(I + \alpha_n B^{-1}A\right)^{-1}\alpha_n B^{-1}A\right\| \\
&\leqslant \left\|B^{-1}A\right\| + \left\|B^{-1}A\right\|\left\|\left(I + \alpha_n B^{-1}A\right)^{-1}\alpha_n\right\|\left\|B^{-1}A\right\| \\
&\leqslant \left\|B^{-1}A\right\| + \left\|B^{-1}A\right\|\left\|\left(I + \alpha_n B^{-1}A\right)^{-1}\right\|\|\alpha_n\|\left\|B^{-1}A\right\| \\
&\leqslant \left\|B^{-1}A\right\| + \left\|B^{-1}A\right\|^2\left\|\left(I + \alpha_n B^{-1}A\right)^{-1}\right\|\|\alpha_n\| \\
&= \left\|B^{-1}A\right\|\left(1 + \|\alpha_n\|\left\|B^{-1}A\right\|\left\|\left(I + \alpha_n B^{-1}A\right)^{-1}\right\|\right).
\end{aligned}$$

If we combine the last inequality and Lemma 2.2, then we have

$$\begin{aligned}
\|\alpha_{n+1}\| &\leqslant \left\|B^{-1}A\right\|\left(1 + \frac{\|\alpha_n\|\|B^{-1}A\|}{1 - \|\alpha_n\|\|B^{-1}A\|}\right) \\
&\leqslant \left\|B^{-1}A\right\|\left(\frac{1}{1 - \|\alpha_n\|\|B^{-1}A\|}\right).
\end{aligned}$$

Since $\|\alpha_n\| \leqslant 1$ and $\|B^{-1}A\| \leqslant \frac{1}{2}$, we obtain

$$\|\alpha_{n+1}\| \leqslant \left\|B^{-1}A\right\|\left(\frac{1}{1 - \|B^{-1}A\|}\right) \leqslant 1.$$

Therefore $\|\alpha_{n+1}\| \leqslant 1$ for all $0 \leqslant n \leqslant M-1$. □

3. Numerical analysis

Example 1.

$$\begin{cases}
\frac{\partial^{0.5}u(t,x)}{\partial t^{0.5}} - \frac{\partial^2 u(t,x)}{\partial x^2} = 3.009011112t^{3/2}\sin(\pi x)\cos(\pi x) + 4t^2\sin(2\pi x)\pi^2, \quad (0 < x < 1, 0 < t < 1), \\
u(0,x) = u(1,x) - \sin(2\pi x), \quad 0 \leqslant x \leqslant 1, \\
u(t,0) = 0, \quad u(t,1) = 0, \quad 0 \leqslant t \leqslant 1.
\end{cases} \tag{3.1}$$

Exact solution of this problem is $U(t,x) = t^2\sin(2\pi x)$.

The errors for various values of time and space nodes are listed at Table 1 (see also Figs. 1 and 2). The formula for calculating errors is

$$\|error\| = \max_{0 \leqslant n \leqslant M,\ 0 \leqslant k \leqslant N}\left|u(t_k, x_n) - U_n^k\right|.$$

Example 2.

$$\begin{cases}
\frac{\partial^{\alpha}u(t,x)}{\partial t^{\alpha}} - \frac{\partial^2 u(t,x)}{\partial x^2} = \frac{2t^{2-\alpha}\ln(1+x(1-x))}{(2-\alpha)(1-\alpha)\Gamma(1-\alpha)} + \frac{t^2(2x^2-2x-1)}{(x^2-x+1)^2}, \quad (0 < x < 1, 0 < t < 1), \\
u(0,x) = u(1,x) - \ln(1 + x(1-x)), \quad 0 \leqslant x \leqslant 1, \\
u(t,0) = 0, \quad u(t,1) = 0, \quad 0 \leqslant t \leqslant 1.
\end{cases} \tag{3.2}$$

Exact solution of this problem is $U(t,x) = t^2\ln(1 + x(1-x))$, if $0 < \alpha < 1$.

The errors for various values of time and space nodes are listed at Tables 2 and 3 (see also Figs. 3 and 4).

Table 1
The norms of the errors for the problem (3.1).

M # space nodes	N # time nodes	$\|error\|$
4	4	0.229769597243842216
8	8	0.053837931524415184
16	16	0.013918008554363204
32	32	0.003843610763920369
64	64	0.001152111099004927
128	128	0.000384422489498104
256	256	0.000144775623174720

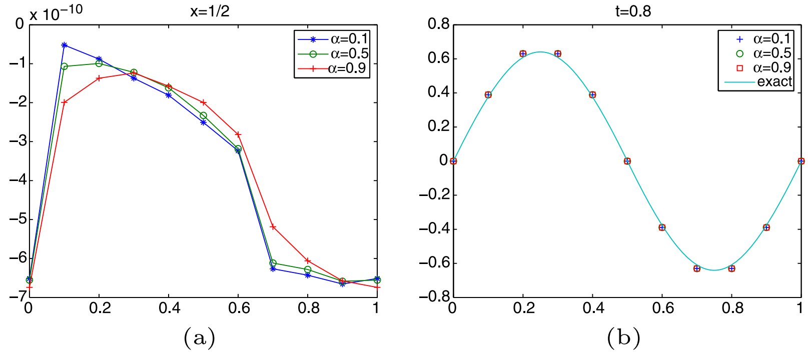

Fig. 1. The solutions at $x = 1/2, 0 \leqslant t \leqslant 1$ given in (a) and the solutions at $t = 0.8, 0 \leqslant x \leqslant 1$ given in (b) by the proposed method for the problem (3.1), when $N = 10$, $M = 10$, where $\alpha = 0.1$, $\alpha = 0.5$ and $\alpha = 0.9$.

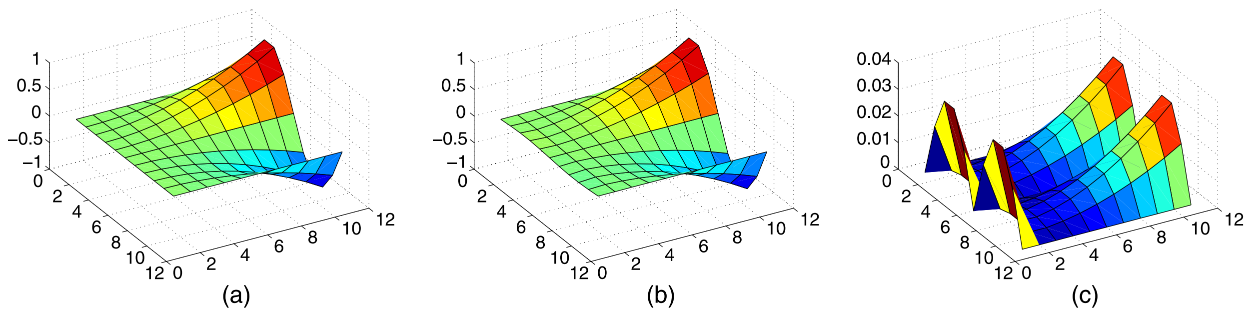

Fig. 2. The exact solution (a), the solution by the proposed method (b), the errors (c) for the problem (3.1), when $N = 10$, $M = 10$ and $\alpha = 0.5$.

Table 2
The norms of the errors for the problem (3.2) when $M = 16$.

N # space nodes	$\|error\|$ when $\alpha = 0.95$	$\|error\|$ when $\alpha = 0.45$
2	0.0146819359116160954	0.00745788994416868302
4	0.0075951945455153824	0.00406080289552948616
8	0.0040330751012416788	0.00230404368301584395
16	0.0022457955672617602	0.00141025796582411323
32	0.0013504788345075069	0.00095946340155034227
64	0.0009023965861451599	0.00073308854279580116
128	0.0006782522407356995	0.00061965684210099625
256	0.0005661529941335952	0.00056288132411858038

4. Conclusion

A method is proposed to solve parabolic fractional differential equations with nonlocal condition. The unconditional stability of the difference scheme for fractional parabolic differential equations with nonlocal condition is proved. Numerical results are provided to show that the numerical results are in good agreement with our theoretical analysis.

The technique described in this work can be applied to a variety of partial differential equations. It gives a very practical way of analyzing the stability of nonlocal fractional parabolic differential equations.

Table 3
The norms of the errors for the problem (3.2) when $N = 16$.

M # space nodes	$\|error\|$ when $\alpha = 0.95$	$\|error\|$ when $\alpha = 0.45$
2	0.0388470060672256728	0.0430064964698151919
4	0.0095182570840683933	0.0094746077282939212
8	0.0036508228606439985	0.0029620062230540922
16	0.0022457955672617602	0.0014102579658241132
32	0.0018981180618916071	0.0010267454656332498
64	0.0018114183496316660	0.0009311362296762216
128	0.0017897323651612229	0.0009073058083821639
256	0.0017845654666746591	0.0009013354956686759

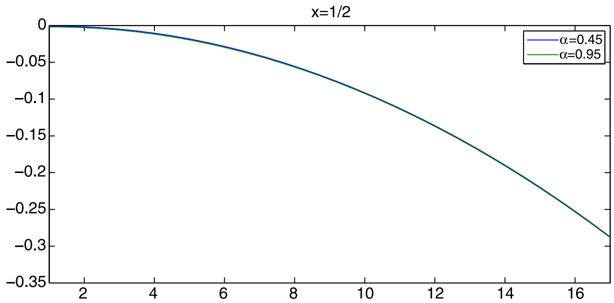

Fig. 3. The solutions at $x = 1/2$ by the proposed method for the problem (3.2), when $N = 16$, $M = 32$, where $\alpha = 0.45$ and $\alpha = 0.95$.

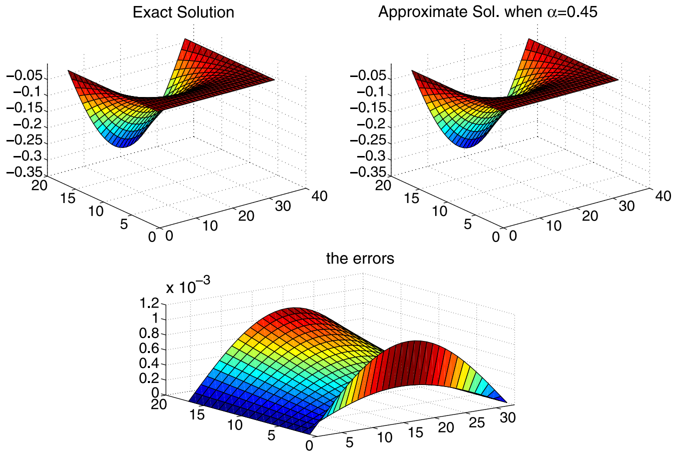

Fig. 4. The exact solution (a), the solution by the proposed method (b), the absolute value of errors (c) for the problem (3.2), when $N = 16$, $M = 32$ and $\alpha = 0.45$.

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