$1, 3, 2, 5, 6, 10, \cdots$

Topic 4. Sequence, Series, Products and Inequalities

A sequence is a list of numbers. This could be defined by a formula, a relationship, or it can just be a list. They can be infinitely finitely long.

$(a_n)_{n=1}^{5} = a_1, a_2, a_3, a_4, a_5.$

Normally a sequence is written as $(a_n)_{n=first}^{last}$, which means there is a sequence called a_n, whose term starts from $n = first$ and goes all the way up to $n = last$ (inclusive).

EXAMPLE: $2, 4, 6, 8, ...$

$a_1 \quad a_2 \quad a_3 \quad a_4$

formula: $a_n = 2n$. sequence: $(2n)_{n=1}^{\infty}$.

EXAMPLE: $0, 1, 0, 1, 0, 1, ...$

$a_1\, a_2\, a_3\, a_4\, a_5\, a_6 \ldots$

$a_n = \begin{cases} 0, & n \text{ is odd.} \\ 1, & n \text{ is even.} \end{cases}$

EXAMPLE: $a_n = n^2 + 1$

$(a_n)_{n=3}^{5} = 10, 17, 26.$

$a_3 \quad a_4 \quad a_5$

There could be a recursive relationship between terms of a sequence, i.e., future terms depend on previous terms. In this case, the first term has to be given.

EXAMPLE: $a_{n+1} = 2a_n$, $a_1 = 1$.

$a_1 = 1.$ $a_2 = 2 \cdot a_1 = 2$

$a_3 = 2 \cdot a_2 = 2 \times 2 = 4$

$a_4 = 2 \cdot a_3 = 2 \times 4 = 8.$

$a_5 = 2 \cdot a_4 = 16.$

alternatively, $a_n = 2^{n-1}.$

EXAMPLE: $a_{n+1} = a_n + n$, $a_0 = 0$.

$a_0 = 0.$ $a_1 = a_0 + 1 = 0 + 1 = 1$

$a_2 = a_1 + 2 = 3$

...

AR-model : price of stock.

$w_n = c_1 w_{n-1} + c_2 w_{n-2} + c_3 w_{n-3} + \varepsilon.$

Series

A series is, generally, the summation of a sequence. It's normally denoted by S_n = the sum of the first n terms.

$a_1, a_2, a_3, a_4, \cdots$. $S_n = a_1 + a_2 + \cdots + a_n$.

EXAMPLE: Sequence a_1, a_2, a_3, ...

$S_1 = a_1$

$S_2 = a_1 + a_2 = S_1 + a_2$

$S_3 = a_1 + a_2 + a_3 = S_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4 = S_3 + a_4$

$S_n = S_{n-1} + a_n$

However, sometimes we might not want to start at the first term of a sequence and to make it clear where we want to start and where we want to finish. In this case we use summation notation:

$$\sum_{n=\text{start}}^{\text{end}} a_n = a_{\text{start}} + a_{\text{start}+1} + a_{\text{start}+2} + \cdots + a_{\text{end}}.$$

Σ : capital sigma. "sum of".

$$\sum_{n=4}^{7} a_n = a_4 + a_5 + a_6 + a_7$$

$S_5 = a_1 + a_2 + a_3 + a_4 + a_5$

EXAMPLE: $a_0 + a_1 + a_2 + ... + a_5$

$\sum_{n=0}^{5} a_n$ //

EXAMPLE: $\sum_{n=3}^{6} a_n = a_3 + a_4 + a_5 + a_6.$

EXAMPLE: $f(1) + f(2) + ... + f(n)$ $a_k = f(k).$ $f(1) + f(2) + \cdots + f(n) = \sum_{k=1}^{n} f(k)$

a_1 a_2 $a_n.$

EXAMPLE: $\sum_{k=0}^{4} k^2 = 0^2 + 1^2 + 2^2 + 3^2 + 4^2 = 30.$

EXAMPLE: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$

a_1 a_2 a_3 a_4 $\cdots$ $a_k = \frac{1}{k}.$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{k}.$

<u>**Arithmetic rules of series**</u>

We can have more than one sequence in my series:

$\sum_{i=0}^{n} x_i y_i = x_0 y_0 + x_1 y_1 + \cdots + x_n y_n.$

We can have more than one summation:

$$\sum_{i=0}^{n} \sum_{j=0}^{m} x_i y_j = \sum_{i=0}^{n} (x_i y_0 + x_i y_1 + \cdots + x_i y_m).$$

$$= x_0 y_0 + x_0 y_1 + \cdots + x_0 y_m$$
$$+ x_1 y_0 + x_1 y_1 + \cdots + x_1 y_m$$
$$+ \cdots$$
$$+ x_n y_0 + x_n y_1 + \cdots + x_n y_m.$$

Things to note:

- $\left(\sum_{i=0}^{n} a_i\right)^2 \neq \sum_{i=0}^{n} a_i^2$

$(a_0+a_1+a_2+\cdots+a_n)^2 \neq a_0^2+a_1^2+a_2^2\cdots+a_n^2$

$(x+y)^2 = x^2+y^2+2xy$

$\neq x^2+y^2$

- $\sum_{i=0}^{n} x_i y_i \neq \left(\sum_{i=0}^{n} x_i\right)\left(\sum_{i=0}^{n} y_i\right)$

- $\sum_{i=0}^{n} c x_i = c \sum_{i=0}^{n} x_i$, where c is a constant.

- $\sum_{i=0}^{n} x_i + \sum_{i=0}^{n} y_i = \sum_{i=0}^{n} (x_i + y_i)$

$\boxed{\begin{matrix} x_0 + \\ + y_0 \end{matrix}} \boxed{\begin{matrix} x_1 + \\ + y_1 \end{matrix}} \boxed{\begin{matrix} x_2 + \\ + y_2 \end{matrix}} \cdots \boxed{\begin{matrix} x_n \\ y_n \end{matrix}}$

- If c is a constant, $\sum_{i=1}^{n} c = nc$

$= \underbrace{c+c+c+\cdots+c}_{n \text{ times}}$

Taylor Series (optional). Write functions as series.

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k .$$

$k! = k \times (k-1) \times (k-2) \times \cdots \times 2 \times 1.$

factorial of k.

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} \cdot x^{2k+1} .$$

Products of sequence

Similar to a summation we can multiply the terms in our sequence to get a product. This is denoted by $\prod_{i=1}^{n} a_i$. $= a_1 \times a_2 \times a_3 \times \cdots \times a_n$.

capital pi.

EXAMPLE: $\prod_{i=1}^{5} i^2 = 1^2 \times 2^2 \times 3^2 \times 4^2 \times 5^2 = \_\_$.

$= ca_1 \times ca_2 \times ca_3 \times \cdots \times ca_n$

- $\prod_{i=1}^{n} ca_i$, c is a constant

$= c^n \prod_{i=1}^{n} a_i$

Series

$\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i$.

- $\prod_{i=1}^{n} (x_i + y_i) \neq \prod_{i=1}^{n} x_i + \prod_{i=1}^{n} y_i$

$\sum_{i=1}^{n} (x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$

- $\prod_{i=1}^{n} (x_i y_i) = \prod_{i=1}^{n} x_i \prod_{i=1}^{n} y_i$

$\sum_{i=1}^{n} (x_i y_i) \neq \sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} y_i$.

- $\prod_{i=1}^{n} i = 1 \times 2 \times 3 \times \ldots \times n = n!$

- $\prod_{i=1}^{n} x_i = \left(\prod_{i=1}^{k} x_i\right) \left(\prod_{i=k+1}^{n} x_i\right)$, $1 \leq k \leq n-1$

$$\prod_{i=1}^{n} x_i = \underbrace{x_1 \times x_2 \times \cdots x_k}_{\prod_{i=1}^{k} x_i} \times \underbrace{x_{k+1} \times \cdots \times x_n}_{\prod_{i=k+1}^{n} x_i}.$$

$$= \left(\prod_{i=1}^{k} x_i\right)\left(\prod_{i=k+1}^{n} x_i\right).$$

Inequalities

- If $a > b$, is $a + c > b + c$ true for all $c \in \mathbb{R}$? ✓
- If $a > b$, is $ac > bc$ true for all $c \in \mathbb{R}$? ×

$a=5$ $b=2$ 1. $c=2$ $ac=10$ $>$ $bc=4$

2. $c=0$ $ac=0$ $=$ $bc=0$

3. $c=-1$ $ac=-5$ $<$ $bc=-2$

The rule is

- If $a > b$ and $c > 0$, we have $ac > bc$;
- If $a > b$ and $c < 0$, we have $ac < bc$.
- If $a>b$ and $c=0$ we have $ac=bc=0$.

To remember: if we multiply or divide by a negative number, we must flip/switch the inequality.

EXAMPLE: Divide $10 > -2$ by -2.

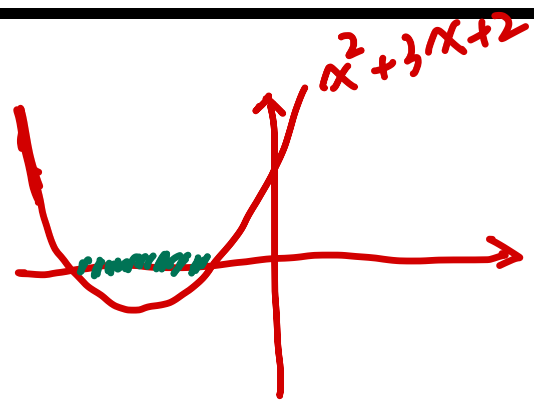
$$\frac{10}{-2} < \frac{-2}{-2} \qquad \frac{10}{-2} = -5 < 1 = \frac{-2}{-2}$$

EXAMPLE: Find all x such that $x^2 + 3x + 2 < 0$.

$x^2+3x+2 = (x+2)(x+1) < 0$.

① $x+2>0$ & $x+1<0$

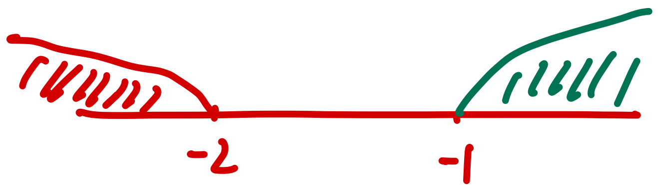
② $x+2<0$ & $x+1>0$.



For ①: $x>-2$ and $x<-1$.

i.e. $-2<x<-1$

For ②: $x<-2$ & $x>-1$



impossible.

Thus $-2<x<-1$.

EXAMPLE: Find all x such that $\frac{x+2}{x+1} < 2$.

$$\frac{x+2}{x+1} < 2.$$

① $x+1>0$ & $x+2 < 2(x+1)$

② $x+1<0$ & $x+2 > 2(x+1)$

EXAMPLE: Find all x such that $|3x - 2| < 4$.

$-4 < 3x - 2 < 4$

$\Rightarrow -2 < 3x < 6$

$\Rightarrow -\frac{2}{3} < x < 2$