Topic 6. Basic Integration

Introduction

Simplest way to think about integration is that it is the 'reverse' of differentiation. Suppose we differentiate the function $y = x^2$. We obtain $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$. Integration reverses this process.

$$\frac{d}{dx} x^2 = 2x$$

$$x^2 \underset{\text{integration}}{\overset{\frac{d}{dx}}{\rightleftarrows}} 2x.$$

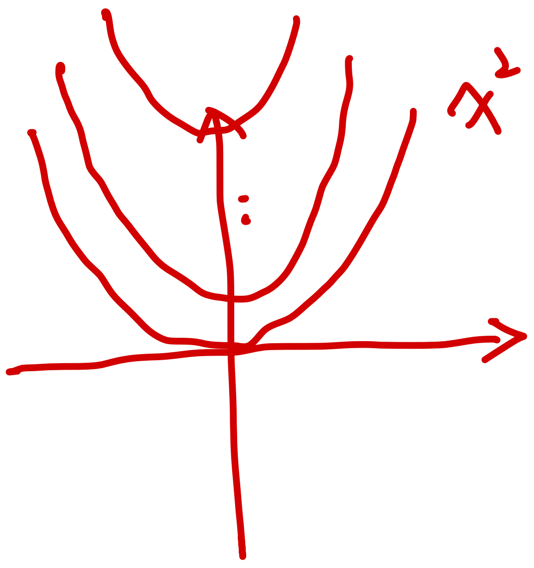
Symbol for integration is $\int f(x)dx$.

what I'm integrating w.r.t.

notation of integration.

the function to integrate.

$$\int 2x\, dx = x^2$$ it is not quite right



$$\frac{d}{dx}(x^2+1) = 2x$$

$$\vdots$$

$$\frac{d}{dx}(x^2+c) = 2x.$$

c is a constant.

$$\int 2x\, dx = x^2 + C.$$

$\triangle \rightarrow$ c is a constant.

$\frac{d}{dx} \cdot x^{n+1} = (n+1) \cdot x^n \Rightarrow \frac{d}{dx}\left(\frac{1}{n+1} \cdot x^{n+1}\right) = x^n.$

$\sin(x) \xrightarrow{\frac{d}{dx}} \cos(x) \xrightarrow{\frac{d}{dx}} -\sin(x) \xrightarrow{\frac{d}{dx}} -\cos(x) \xrightarrow{\frac{d}{dx}} \sin(x)$

Standard integrals

$\int x^n dx = \frac{1}{n+1}x^{n+1} + c$ or $\int ax^n dx = a\int x^n dx = \frac{a}{n+1}x^{n+1} + c.$

$\int \sin(x)dx = -\cos(x) + c, \int \cos(x)dx = \sin(x) + c.$

$\int e^{ax}dx = \frac{1}{a}e^{ax} + c, \int \frac{1}{x}dx = \ln|x| + c.$

$\int \frac{f'(x)}{f(x)} = \ln|f(x)| + c.$

$\frac{d}{dx} e^{ax} = a \cdot e^{ax}.$

$\frac{d}{dx} \ln|x| = \frac{1}{x}.$

$\frac{d}{dx} \ln|f(x)| = \frac{1}{f(x)} \cdot f'(x).$

Some notes of integration

$\int f(x)dx = F(x) + C.$ $\int af(x)dx = a\int f(x)dx, a =$ constant.

$\int[af(x) + bg(x)]dx = \int af(x)dx + \int bg(x)dx = a\int f(x)dx + b\int g(x)dx.$

EXAMPLE: $\int(6x^2 - 4x + 2)dx =$

$F(x) + C \xrightarrow{\frac{d}{dx}} f(x)$, $f(x) \xrightarrow{\int \cdot dx} F(x) + C$

$\int (6x^2 - 4x + 2)\, dx$

$= \int 6x^2 dx - \int 4x\, dx + \int 2\, dx \longrightarrow \int 2 \cdot x^0 dx$

$= 6 \times \frac{x^{2+1}}{2+1} - 4 \times \frac{x^{1+1}}{1+1} + 2x + C.$

$= 2 \cdot x^3 - 2x^2 + 2x + C.$

EXAMPLE: A curve passes though the point $(1, -1)$ and at the point (x, y) the equation of the gradient is given by $2x - 3$. Find the equation of the curve.

$$\frac{dy}{dx} = 2x - 3 \quad \Rightarrow y = \int 2x - 3 \, dx = \frac{2x^2}{2} - 3x + C = x^2 - 3x + C$$

To find C. we use the pt $(1, -1)$ which implies that

$$-1 = 1^2 - 3 \times 1 + C = -2 + C \Rightarrow C = 1 \qquad \text{i.e. } y = x^2 - 3x + 1$$

EXAMPLE: Find the function, $f(x)$, given that $f''(x) = 6x - 6$ and that when $x = 1$, we have $f'(x) = 3$ and $f(x) = 0$.

$$\begin{tikzcd} f(x) \arrow[r, bend left, "\frac{d}{dx}"] & f'(x) \arrow[l, bend left, "\int \cdot \, dx"] \arrow[r, bend left, "\frac{d}{dx}"] & f''(x) \arrow[l, bend left, "\int \cdot \, dx"] \end{tikzcd}$$

$$f'(x) = \int f''(x) \, dx$$

$$= \int 6x - 6 \, dx$$

$$= \frac{6x^2}{2} - 6x + C. = 3x^2 - 6x + C$$

when $x = 1$, $f'(x) = 3$. i.e. $3 \times 1^2 - 6 \times 1 + C = 3$

$$\Rightarrow \quad C = 6.$$

i.e. $f'(x) = 3x^2 - 6x + 6$.

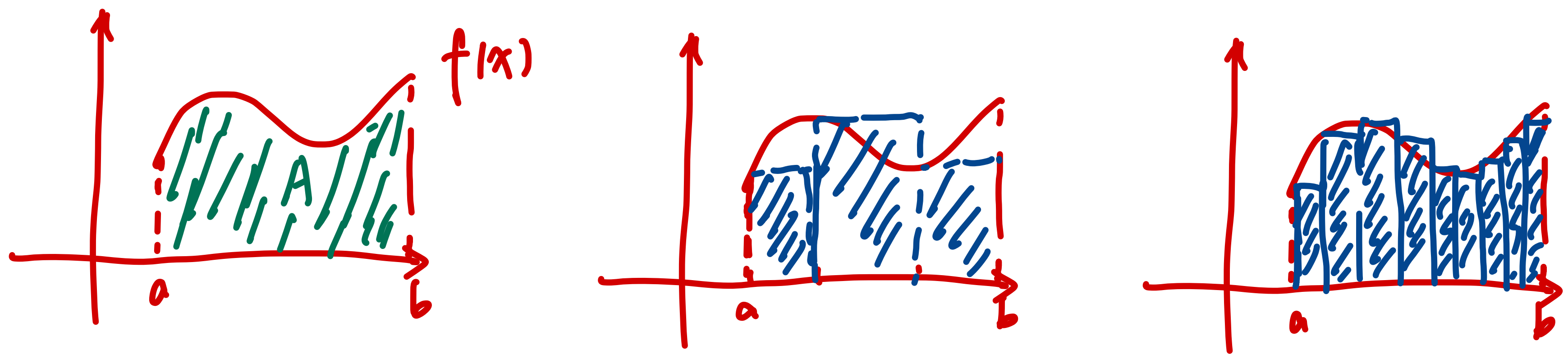
indefinite integration.

EXAMPLE: $\int \frac{1}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} dx = \frac{1}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1} + C$

$= 2 \cdot x^{\frac{1}{2}} + C$

Area under curve

What is the area A? We can split the area into little rectangles and find the area sum of these rectangles. By splitting into very small rectangles, we get a better approximation. (Note that area of rectangle = height × base)



area of A ≈ Σ area of rectangle.

$= \Sigma \underbrace{\Delta x}_{\text{base}} \cdot \underbrace{f(x)}_{\text{height}}$

$= \int_a^b f(x)\, dx$

$\int f(x)dx = F(x) + C$

We can think of integration as a method to find the area under a curve.

integration limit

$$\int_a^b f(x)dx = F(b) - F(a)$$

no constant "C" here

$$:= F(x)\Big|_{x=a}^{b}$$

EXAMPLE: Find $\int_1^3 (4x-2)dx$ $= 2x^2 - 2x \Big|_{x=1}^{3}$

$= 2\times3^2 - 2\times3 - (2\times1^2 - 2\times1)$

$= 18 - 6 - 2 + 2 = 12$

EXAMPLE: Find $\int_0^3 x(x-3)dx$

$$= \int_0^3 x^2 - 3x \, dx$$

$$= \underbrace{\frac{1}{3}x^3 - \frac{3}{2}x^2}_{F(x)} \Big|_{x=0}^{3}$$

$$= \left(\frac{1}{3}\cdot 3^3 - \frac{3}{2} 3^2\right) - \left(\frac{1}{3} 0^3 - \frac{3}{2}\cdot 0^3\right)$$

$$= \ldots$$

Note: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

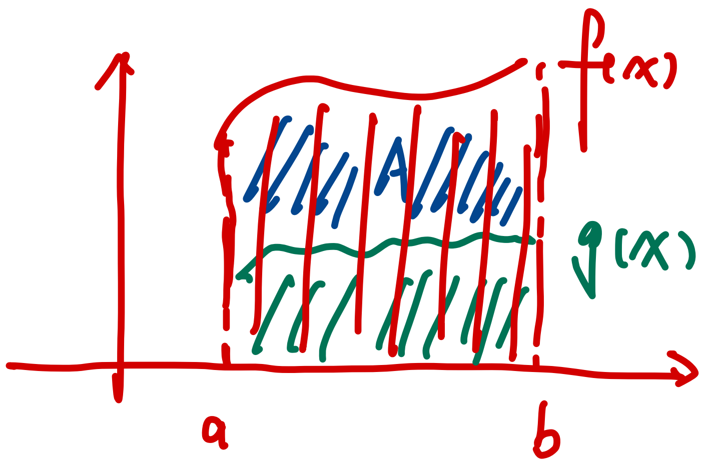
$F(b) - F(a)$ $\qquad$ $-[F(a) - F(b)]$

Area between curves

We can also use integration to find the area between two curves.

$A = \int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

Note: this only works when $f(x) \geqslant g(x)$ for all values between a and b.



$\int_a^b f(x)\,dx$

$\int_a^b g(x)\,dx$

chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$.

Integration by substitution

inverse of chain rule.

EXAMPLE: $\int 2x\cos(x^2)\,dx =$

Let $u = x^2$. then $\frac{du}{dx} = 2x$. $\Rightarrow$ $dx = \frac{1}{2x} du$

$\int 2x\cos(x^2)\,dx = \int \cancel{2x} \cos(u) \frac{1}{\cancel{2x}} \cdot du$

$= \int \cos(u)\,du = \sin(u) + C \overset{u=x^2}{=} \sin(x^2) + C$

$\int \frac{x}{x^2+1} dx =$ $\left(\frac{x}{x^2+1} = x \cdot \frac{1}{x^2+1}\right)$

Let $u = x^2+1$. then $\frac{du}{dx} = 2x$. $\Rightarrow$ $du = 2x \cdot dx$ i.e. $\frac{1}{2} du = x \cdot dx$

$\int \frac{x}{x^2+1} dx = \int \frac{1}{u} \cdot \frac{1}{2} du = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|x^2+1| + C$

Let $u = x^2$, then $\frac{du}{dx} = 2x$

$\int \frac{x}{x^2+1} dx = \int \frac{1}{u+1} \frac{1}{2} du = \frac{1}{2} \ln|u+1| + C$

EXAMPLE: $\int_1^3 (x+1)^3 dx =$

$x: \quad 1 \to 3$

Let $u = x+1$. $\frac{du}{dx} = 1$. i.e. $du = dx$.

$u: \quad 2 \to 4$

$\int_2^4 u^3 \, du = \frac{1}{4} u^4 \Big|_2^4 = \frac{1}{4} \times 4^4 - \frac{1}{4} \times 2^4 = \cdots$

$\int \sin(x^2)\, dx =$

Let $u = x^2$. $\frac{du}{dx} = 2x$

$\int \sin(x^2)\, dx = \int \sin(u) \frac{1}{2x}\, du$. $\times$

$\sin(x^2)$ cannot be integrated and written as a "nice" function.

product rule: $\frac{d}{dx} f\cdot g = f\cdot g' + f'\cdot g$

Integration by parts

Formula is given by $\int f(x)dg(x) = f(x)g(x) - \int g(x)df(x)$

$\Rightarrow \int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$

EXAMPLE: $\int x(3x-2)^4dx =$

Let $f(x) = x$ $\quad g'(x) = (3x-2)^4 \Rightarrow f'(x) = 1 \quad g(x) = \frac{1}{15}(3x-2)^5$

Then $\int \underbrace{x}_{f}\,\underbrace{(3x-2)^4}_{g'}dx = \underbrace{x}_{f}\cdot\underbrace{\frac{1}{15}(3x-2)^5}_{g} - \int \underbrace{1}_{f'}\times\underbrace{\frac{1}{15}(3x-2)^5}_{g}dx$

$= \frac{1}{15}x(3x-2)^5 - \frac{1}{15\times6\times3}(3x-2)^6 + C$

$= \frac{x}{15}(3x-2)^5 - \frac{1}{270}(3x-2)^6 + C$

EXAMPLE: $\int_2^4 x\ln x dx =$

Let $f(x) = \ln x$, $g'(x) = x \Rightarrow f'(x) = \frac{1}{x}$, $g(x) = \frac{1}{2}x^2$

$\int_2^4 \underbrace{x}_{g'}\,\underbrace{\ln x}_{f}\,dx = \underbrace{\frac{1}{2}x^2}_{g}\,\underbrace{\ln x}_{f}\Big|_{x=2}^{4} - \int_2^4 \underbrace{\frac{1}{x}}_{f'}\cdot\underbrace{\frac{1}{2}x^2}_{g}dx$

$= \frac{1}{2}\times4^2\times\ln4 - \frac{1}{2}\times2^2\times\ln2 - \int_2^4 \frac{1}{2}x\,dx$

$= 8\ln4 - 2\ln2 - \frac{1}{4}x^2\Big|_{x=2}^{4}$

$= 8\ln4 - 2\ln2 - 3$

Similar example: $\int_1^2 x\cdot e^x dx$.

EXAMPLE: $\int [e^{2x} + \sin(2x)]dx =$

EXAMPLE: $\int_2^3 (\frac{1}{\sqrt{x}} + 4x^2 + 1)dx =$