University of Essex
School of Mathematics, Statistics and Actuarial Science

MA111: Foundational Mathematics for Data Science

(Autumn Term 2023-24)

Lecturer: Dr. Jianya Lu
3A.524
jianya.lu@essex.ac.uk

Module Information:

Module materials homepage: MOODLE → MA111

Lectures:

- (Everyone) Thursday, 4-6pm, Weeks 2-10, LTB08; Tuesday, 11am-1pm, Week 11, 4.722.
- (Postgraduates) Friday, 9-10am, Weeks 2, 4, 6, 8, 10, 5N.151.

Classes/Labs (Everyone):

- Classes: CLAa01, Friday, 1-2pm, Weeks 2-10, CTC.2.02; Monday, 1-2pm, Week 11, CTC.2.01.
- Classes: CLAa02, Friday, 11-12am, Weeks 2-10, TC2.6+TC2.7; Monday, 2-3pm, Week 11, CTC.2.01.

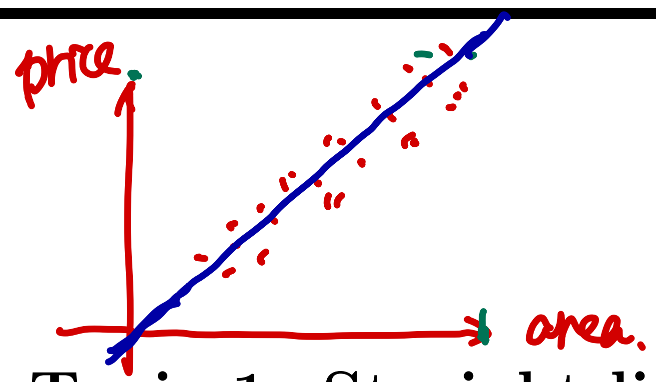
Assessment:

- Exercise sheets: 0% (= 10 × 0% - to be solved in classes)
- Coursework1: 15% (= 1 × 15% - week 5)
- Coursework2: 15% (= 1 × 15% - week 7)
- Coursework3: 15% (= 1 × 15% - week 9)
- End-of-term test: 55% (In-Person, Closed Book, 120 minutes- week 11)

Module Outline

The module develops the foundational mathematical skills needed for students studying Data Science. It supports students from a wide range of educational backgrounds, to give them an understanding of the core mathematical skills needed. It will introduce students to important mathematical functions and equip them with the techniques needed to solve problems from calculus and linear algebra.

- Straight line, Quadratic curves, Simultaneous Equations, Indices
- Functions
- Exponential, Logarithm & Trigonometric Functions
- Series, Summations, Products, Inequalities
- Differentiation
- Basic Integration
- Vectors
- Matrices
- Metrics
- (PG only) Discrete Maths



Topic 1. Straight line, Quadratic curves and Indices

Why? line of best fit, linear regression

In this unit we find the equation of a straight line, when we are given some information about the line. The information could be the value of its gradient, together with the co-ordinates of a point on the line. Alternatively, the information might be the co-ordinates of two different points on the line. There are several different ways of expressing the final equation, and some are more general than others.

By the end of the session, you will be able to find the equation of a straight line and draw it on the x-y plane.

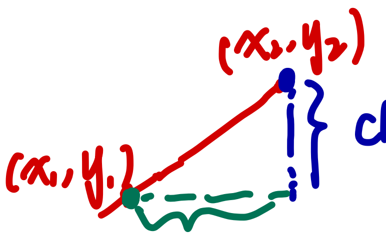
General form of a straight line is given by $y = mx + c$, where

$y = 2x + 1$

x increase 1. y increase 2.

2 4

- x: the variable (the thing we change)
- m (constant): the slope or gradient (for one unit in x, y goes m unit upwards)
- c (constant): the y-intercept



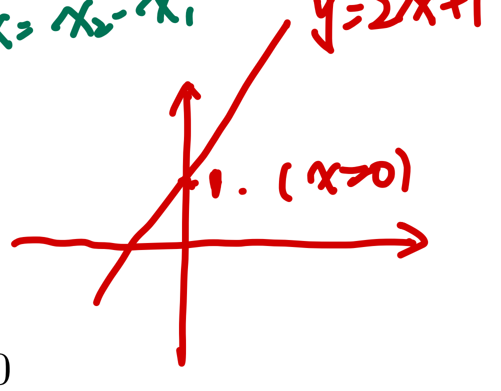
→ delta.

change in y = $\Delta y = y_2 - y_1$

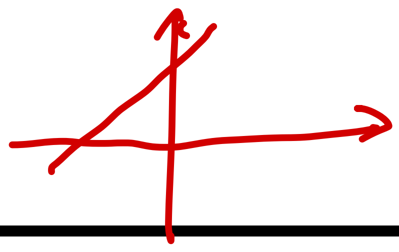
change in x

$\Delta x = x_2 - x_1$

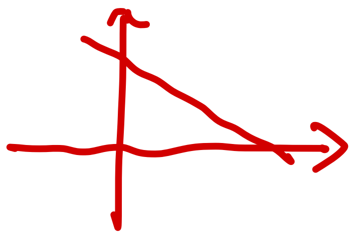
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



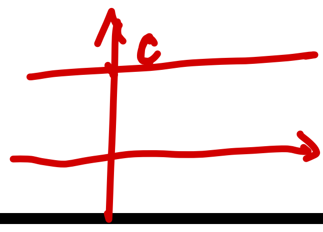
if $m > 0$



if $m < 0$



if $m = 0$ $y = c$

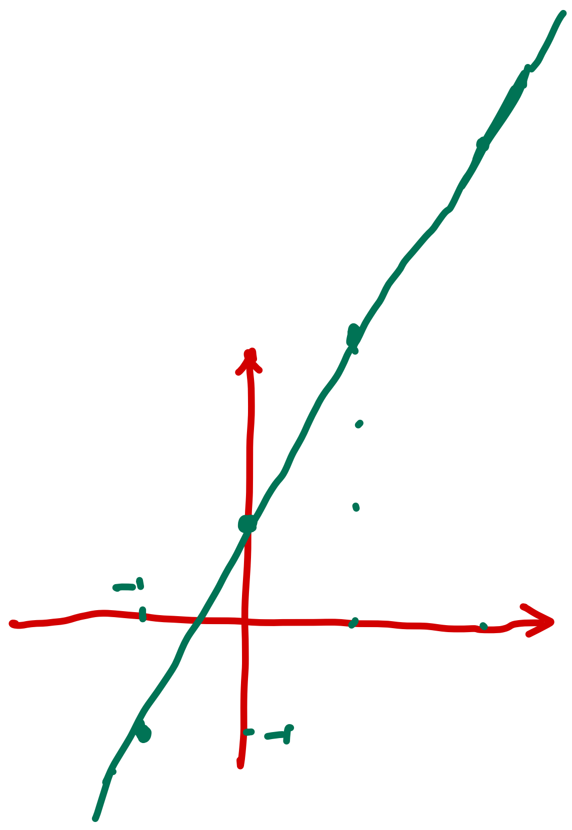


How to sketch a straight line

EXAMPLE: $y = 2x + 1$

1. Working out the points on it:

x	-1	0	1	2
y	-1	1	3	5



2. Interpreting constants m and c:

$y = mx + c$

$m = 2$

$c = 1$

How many points do we need in order to find the equation of a straight line? 2.

EXAMPLE: Find the equation of the straight line passing through $\overset{x_1, y_1}{(0, 2)}$ and $\overset{x_2, y_2}{(2, 4)}$.

To find the equation we need to identify m and c values. $y = mx + c$

To find m:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{2-0} = \frac{2}{2} = 1.$$

Then to find c:

$y = x + 2$

$y = x + c$

Let $y = 2$, & $x = 0$

$2 = 0 + c = c$

Or, alternatively, we can solve simultaneous equations:

$$\begin{cases} 2 = m \times 0 + c \\ 4 = m \times 2 + c \end{cases} \Rightarrow \begin{cases} c = 2 \\ m = 1 \end{cases}$$

$(0,2)$ $(2,4)$

In general, if we have two points (x_1, y_1) and (x_2, y_2), we can find the equation of the straight line using the formula:

$y = mx + c$

$m = \frac{\Delta y}{\Delta x}$

$(x_1, y_1) \to (x, y)$

$$m = \frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

$(x_2, y_2) \to (x_1, y_1)$

EXAMPLE: Find the equation of the straight line connecting $(-1, -1)$ and $(3, 1)$.

x_1 y_1 x_2 y_2

$$\frac{y-(-1)}{x-(-1)} = \frac{-1-1}{-1-3} = \frac{-2}{-4} = \frac{1}{2}$$

$$\Rightarrow \quad y+1 = \frac{1}{2}(x+1) \quad \Rightarrow \quad y = \frac{1}{2}x - \frac{1}{2}$$

Inverse function of a straight line

EXAMPLE: Consider the line $y = 2x + 1$. The calculation involved in this formula (from x to y) is:

- double the value and then add one; $x \to y$ when $x=1$ $y=3$

$\times 2$ $+1$

Now consider the case if we know the y value that is resulted from the function, and we want to derive it backwards to find the corresponding x:

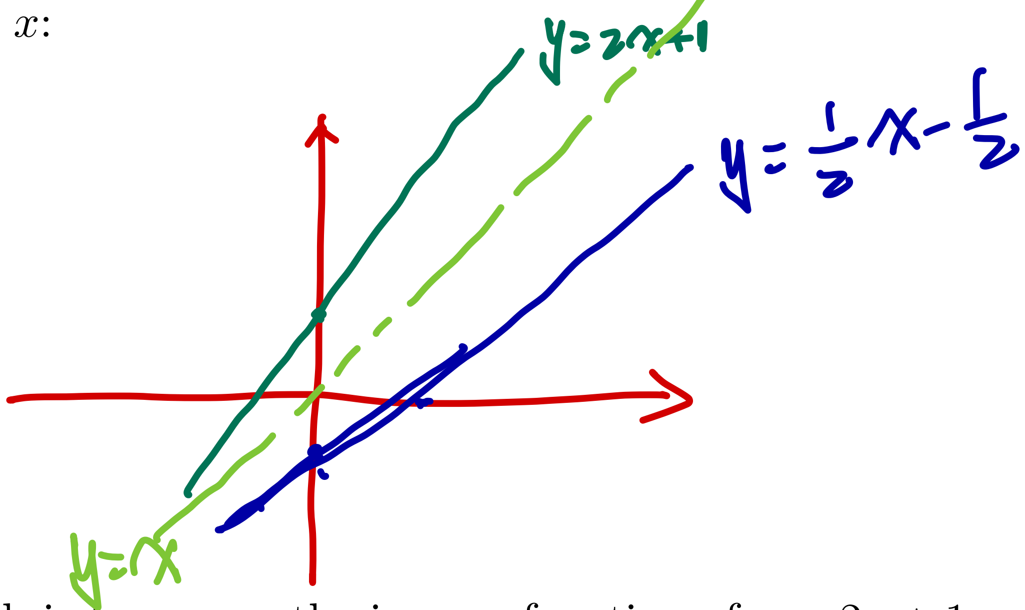
- subtract one and then half it;

-1 $\div 2$

Writing this mathematically, we have:

$$y = \frac{1}{2}(x-1)$$

$$= \frac{1}{2}x - \frac{1}{2}$$



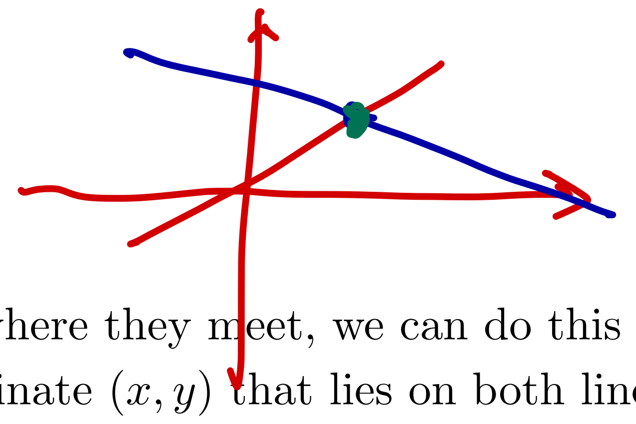
This defines a new straight line function which is known as the inverse function of $y = 2x + 1$.

Graphically, the inverse of a straight line can be found by reflecting in the line $y = x$, or mathematically, by switching x and y in the equation and rearranging.

$$x = 2y + 1$$

$$\Rightarrow \quad 2y = x - 1 \quad \Rightarrow \quad y = \frac{1}{2}(x-1)$$

Simultaneous (linear) equations



Suppose we have two straight lines and we want to find where they meet, we can do this by solving them simultaneously. That is finding values for the coordinate (x, y) that lies on both lines.

EXAMPLE: $y = 2x + 3$ and $y = x + 4$, where do these lines cross?

$$\begin{cases} y = 2x+3 & ① \\ y = x+4 & ② \end{cases}$$

Method 1: Get "rid of" one variable

② × 2 : $2y = 2x + 8$ ... ⊛

① − ⊛ : $y - 2y = 2x+3-(2x+8) = -5$

$\Rightarrow -y = -5$ i.e. $y = 5$

$5 = x+4$

$\Rightarrow x = 1$

pt (1, 5).

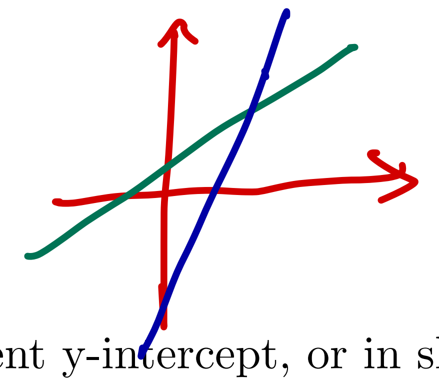
Method 2: Substitution

substitute $y = x+4$ to ①.

$\underbrace{x+4}_{y} = 2x+3 \Rightarrow x = 1$

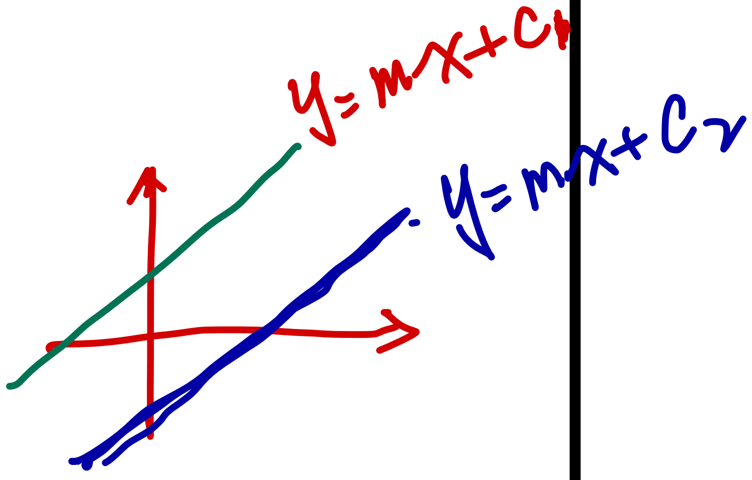
$y = x+4 = 1+4 = 5$

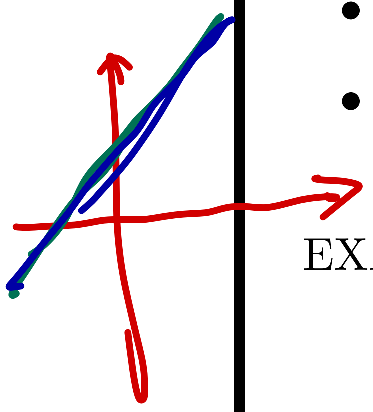
(1, 5)



Two straight lines either:

- meet at precisely one point
- never meet (same gradient, different y-intercept, or in short, parallel)
- meet at infinite points (same line)





EXAMPLE: $2y = 2x - 2$ and $y = x - 2$, where do these lines cross?

$$\begin{cases} 2y = 2x-2 & \text{①} \\ y = x-2 & \text{②} \end{cases}$$

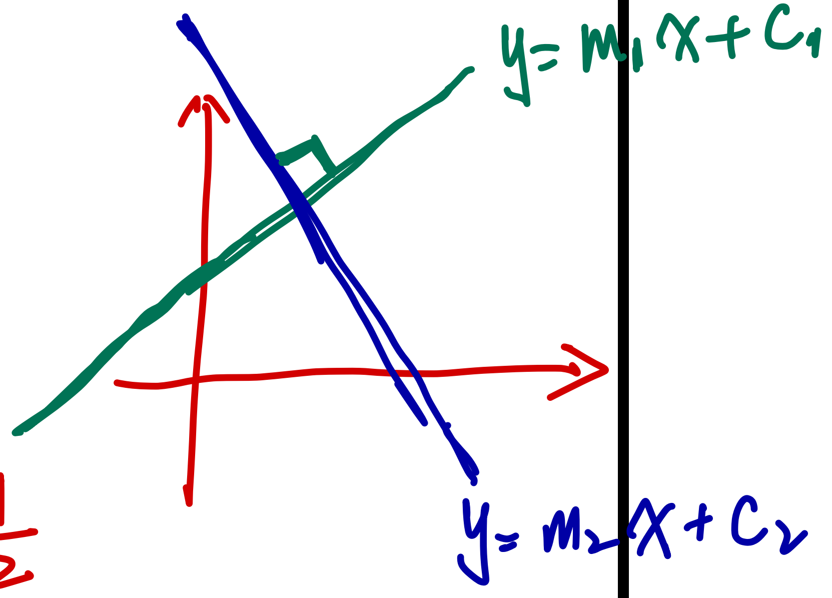
substitute $y = x-2$ to ①

$2(\underbrace{x-2}_{y}) = 2x-2$, but $-4 \neq -2$.

we can not find this point.

Normal

We say a line is normal to another line if they meet at right angles, i.e., they are perpendicular. The gradient of the two lines satisfy:



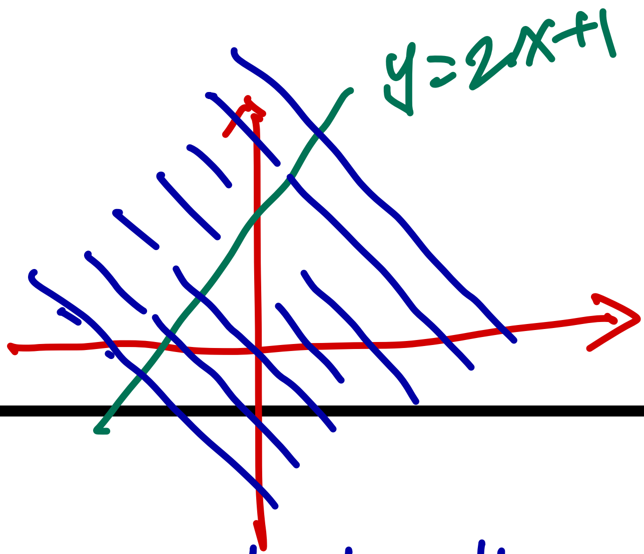
$$m_1 = -\frac{1}{m_2} \to m_1 m_2 = -1$$

EXAMPLE: Find the family of lines that are normal to $y = 2x + 1$.

the collection of possible lines

Let $m_1 = 2$ then we have $m_2 = -\frac{1}{m_1} = -\frac{1}{2}$

$\Rightarrow y = -\frac{1}{2}x + C$



e.g. Find the line is normal to $y = 2x + 1$ and passes $(2, 5)$

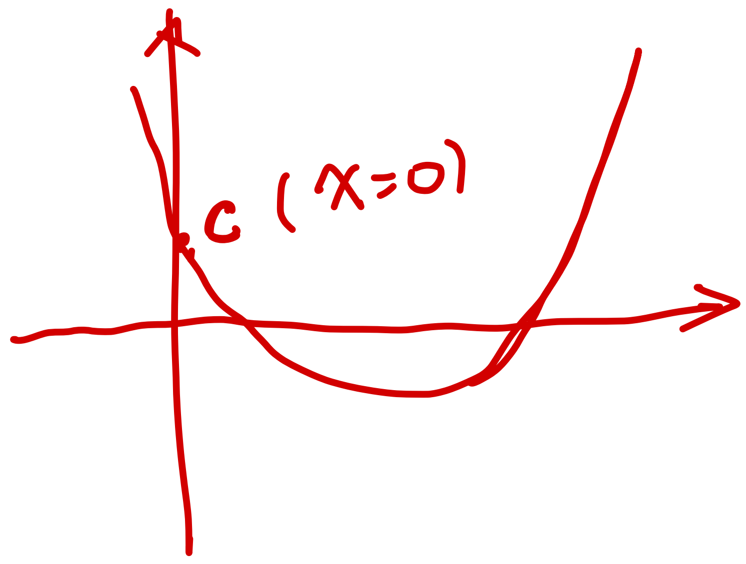
$5 = -\frac{1}{2} \times 2 + C \Rightarrow C = 6$

$\Rightarrow y = -\frac{1}{2}x + 6$

Quadratics

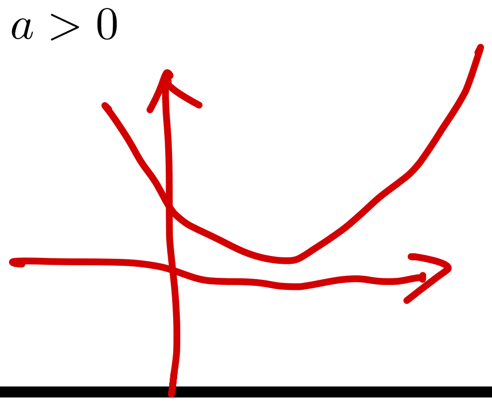
$y = m \cdot x + c$

The general form of a quadratic function is $y = ax^2 + bx + c$, where a, b, c are constants.

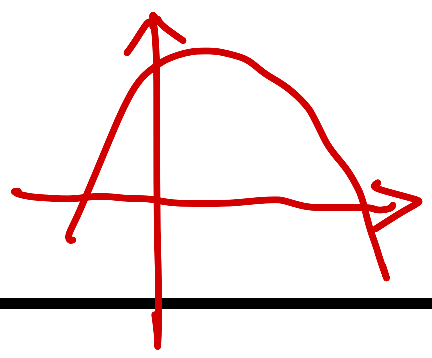


$y = a \cdot 0^2 + b \cdot 0 + c = c.$

if $a > 0$

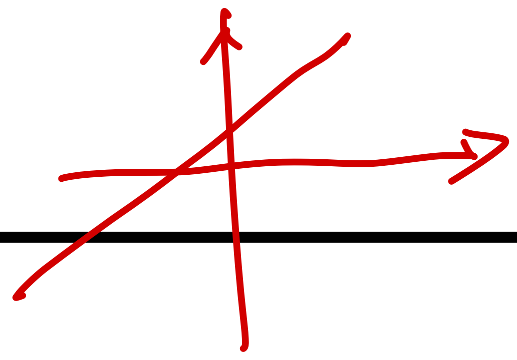


if $a < 0$



if $a = 0$

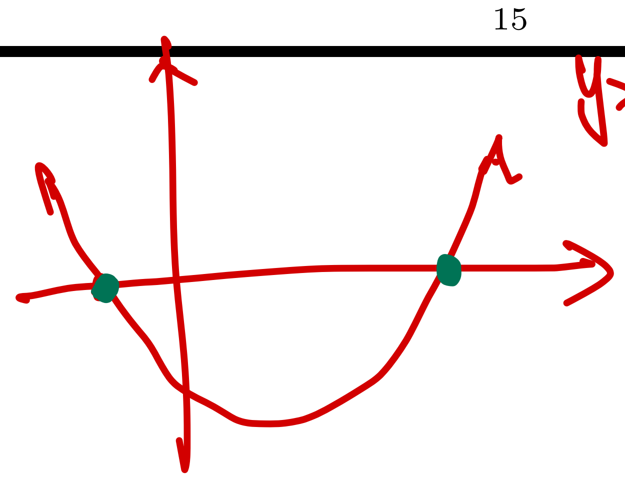
$y = b \cdot x + c$



$y = ax^2 + bx + c.$

$y=0.$

How to find the x intercepts of the function



Let $y=0.$

$\Rightarrow ax^2+bx+c = 0.$

$(x+2)(x-1)=0.$

$x_1=-2$

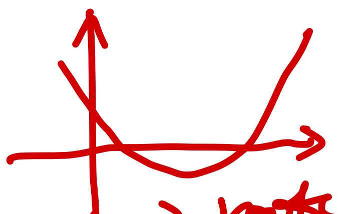
$x_2=1.$

$\to ax^2+bx+c=0$

These are also known as "roots" of the function. Two ways to find the roots:

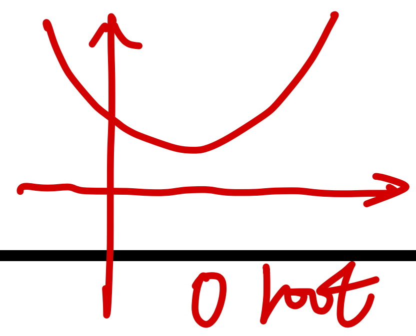
- Factorise into $(x \pm r_1)(x \pm r_2) = 0$ $\quad x_1 = \mp r_1, \; x_2 = \mp r_2$
- Use quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, where $b^2 - 4ac$ is known as discriminant

if $b^2 - 4ac > 0$



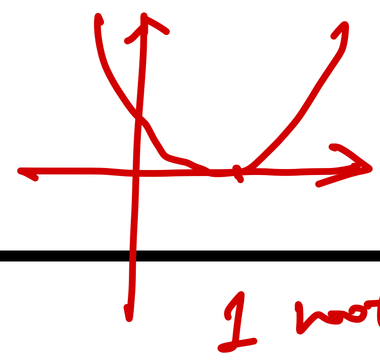
2 roots

if $b^2 - 4ac < 0$



0 root

if $b^2 - 4ac = 0$



1 root.

$x^2+3x+1=0.$

we have $a=1 \quad b=3 \quad c=1$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-3 \pm \sqrt{9-4\times1\times1}}{2} = \frac{-3\pm\sqrt{5}}{2}$$

$$x_1 = \frac{-3+\sqrt{5}}{2}. \qquad x_2 = \frac{-3-\sqrt{5}}{2}$$

Intersection of curves and straight lines

EXAMPLE: Find intersection of $y = x^2$ and $y + 3x = 4$.

$$\begin{cases} y = x^2 & (1) \\ y + 3x = 4 & (2) \end{cases}$$

substitute $y = x^2$ to (2), we have

$x^2 + 3x = 4$ i.e. $x^2 + 3x - 4 = 0$

$ax^2 + bx + c = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Here $a = 1$, $b = 3$ $c = -4$

$b^2 - 4ac = 3^2 - 4 \times 1 \times (-4) = 9 + 16 = 25$.

$$x = \frac{-3 \pm \sqrt{25}}{2}$$

$x_1 = \frac{-3+5}{2} = 1$ $x_2 = \frac{-3-5}{2} = -4$.

Sub $x_1 = 1$ into $y + 3x = 4$, we have $y_1 = 1$

$x_2 = -4$ $y_2 = 16$.

Then we have the intersection pts $(1, 1)$ & $(-4, 16)$

Rules of indices/power

x^k. x: base k: order/index. $x^2 = x \cdot x$.

- $x^{1/n} = \sqrt[n]{x}$ $9^{\frac{1}{2}} = \sqrt{9} = 3$.
- $x^{m/n} = (\sqrt[n]{x})^m$
- $x^{-n} = \frac{1}{x^n}$
- $x^1 = x$
- $x^0 = 1$
- $x^m \times x^n = x^{m+n}$ same base
- $\frac{x^m}{x^n} = x^{m-n}$
- $(x^m)^n = x^{mn}$ $\underbrace{x^m \cdot x^m \cdot x^m \cdots x^m}_{n \text{ times}} = x^{\overbrace{m+m+\cdots+m}^{n \text{ times}}} = x^{m \cdot n}$