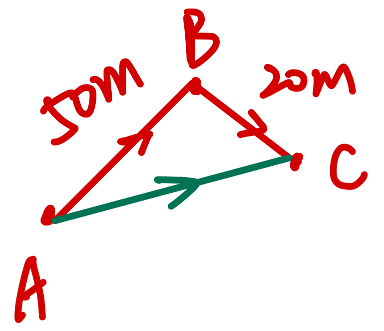
# Topic 7. Vectors

## Vectors




Clearly I've walked 70m in total, although it is not necessarily 70m from A to C. To get from one place to another, we need two bits of information: a direction and a length, such quantities that encode these pieces of information are called vectors.

direction

EXAMPLE: Going from A to B can be denoted by $\overrightarrow{AB}$, B to C, $\overrightarrow{BC}$ and A to C, $\overrightarrow{AC}$.

start end

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

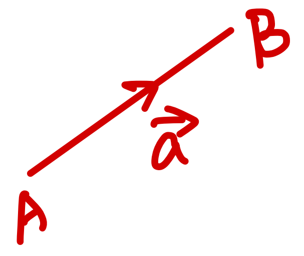
We can denote the length of a vector as $|\overrightarrow{AB}|$. $= 50.$ $|\overrightarrow{BC}| = 20.$

Note: $|\overrightarrow{AB}| + |\overrightarrow{BC}| \neq |\overrightarrow{AC}|$ $|\overrightarrow{AC}| \neq 70.$

$\geq$

## Other notation

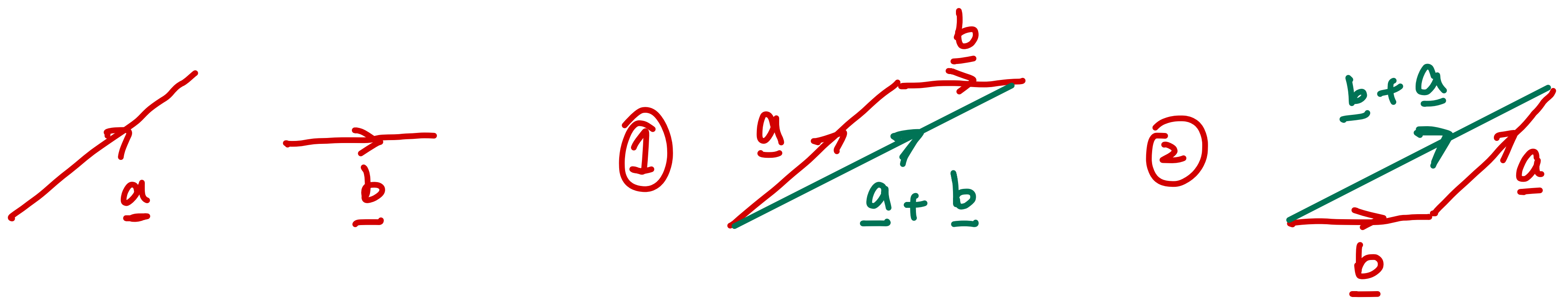
We usually denote a vector using bold or underlined letters, or with an arrow top.




$\vec{AB} = \vec{a} = \mathbf{a} = \underline{a}$ .

## Vector properties

Addition (geometrically):

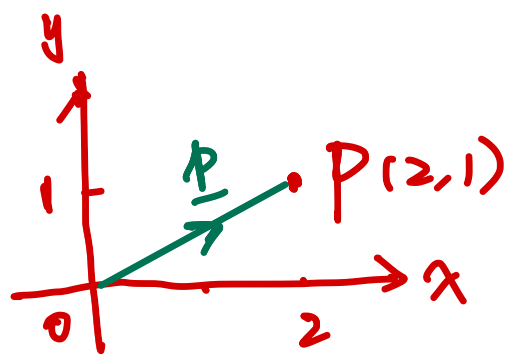



Commutative law: $\underline{a} + \underline{b} = \underline{b} + \underline{a}$

Associative law: $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$

Position vector: Any point in space can be represented by a coordinate. We can also denote a coordinate (P) by a vector, this is called its postion vector.

$\underline{p}$

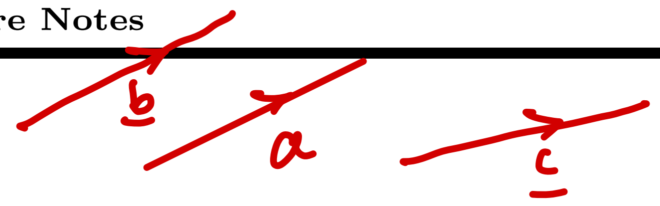



$\begin{pmatrix}2\\1\end{pmatrix}$

A position vector tells us how to get to a point in space when starting at the origin $(0, 0)$.

NOTE:

- A coordinate tells us where we are in space.
- A position vector tells is how to get to a coordinate for the origin
- In general, a vector tells us in which direction and how far we should move from a starting coordinate.




Parallel vector: two vectors are parallel if they are scalar multiples of each other.

EXAMPLE: Let $\underline{a}$ and $\underline{b}$ be vectors then if $\underline{a}\lambda = \underline{b}$, where $\lambda$ is a constant, $\underline{a}$ and $\underline{b}$ are parallel. Let $\underline{a} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$, are $\underline{a}$ and $\underline{b}$ parallel?

$\lambda \underline{a} = \lambda \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2\lambda \\ -4\lambda \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$ $\quad \lambda = 2$. i.e. $2\underline{a} = \underline{b}$

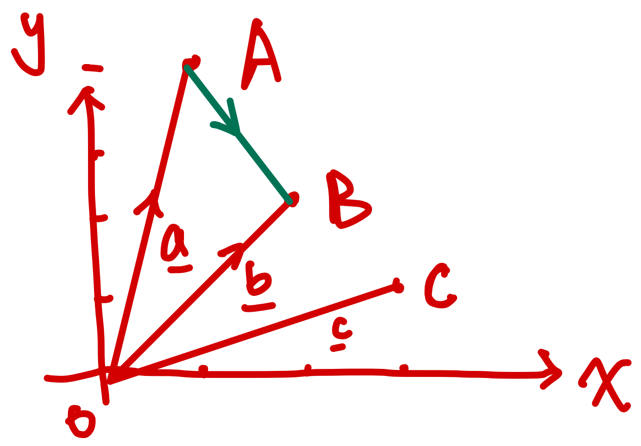
We can add and subtract vectors quite nicely:

EXAMPLE: $\underline{a} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \underline{b} = \begin{pmatrix} -1 \\ -3 \\ 4 \end{pmatrix}$ $\quad \underline{a} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ $\underline{c} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\underline{a} + \underline{b} = \begin{pmatrix} 2-1 \\ 3-3 \\ 5+4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 9 \end{pmatrix}$

## Finding vectors between points

What is the vector $\overrightarrow{AB}$ ? $\underline{a} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \underline{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \underline{c} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$




$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

$\Rightarrow \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

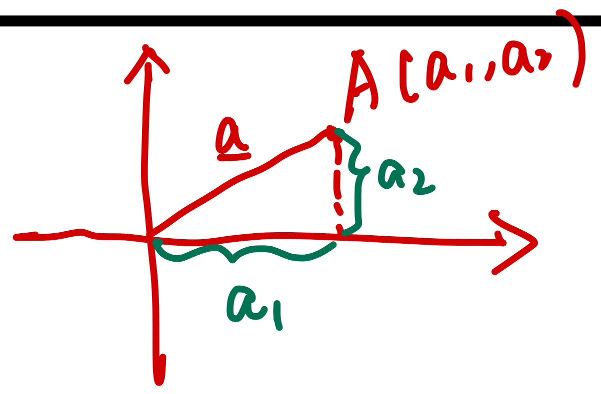
$= \underline{b} - \underline{a}$

$= \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2-1 \\ 2-4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$|\overrightarrow{AB}| = \sqrt{1^2 + (-2)^2} = \sqrt{5}.$

What is the length of vector $\underline{a}$, i.e., $|\underline{a}|$?

EXAMPLE: Let $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$. Find $|\underline{a}|$



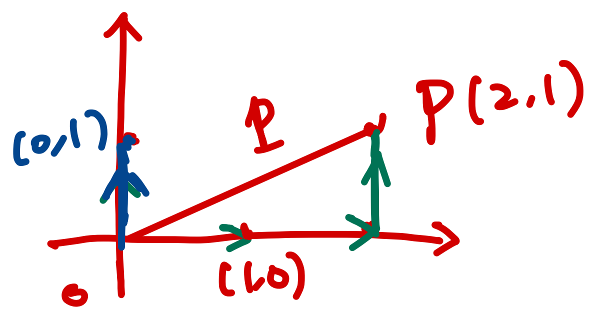

$|\underline{a}| = \sqrt{a_1^2 + a_2^2}.$

In general, for an $n$-dimensional vector $\underline{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \in \mathbb{R}^n.$ $= (p_1, p_2, \ldots, p_n)^{\underline{T}},$ → transpose.

$$|\underline{p}| = \sqrt{p_1^2 + p_2^2 + \ldots \cdot p_n^2} = \sqrt{\sum_{i=1}^{n} p_i^2}$$

## Component form

We can split the vector into 'moves' across space, usually moves 'across' and 'up' for 2 vectors.




$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \underline{p} = 2 \text{ cross} + 1 \text{ up}.$

$= 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$:= 2\,\underline{i} + 1\,\underline{j}.$

## Unit vectors

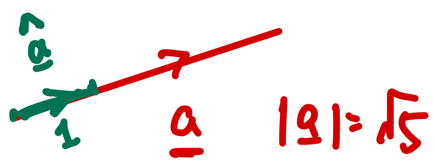
$\underline{a}$ s.t. $|\underline{a}| = 1$.

These are vectors with length 1, and are useful when comparing between directions. We can make any vector a unit vector simply by dividing it by its length.

EXAMPLE: $\underline{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2\mathbf{i} - \mathbf{j}$, where $\mathbf{i}, \mathbf{j}$ are unit vectors in the directions of the $x, y$ axes respectively.

$$\underline{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= 2\underline{i} - \underline{j}.$$



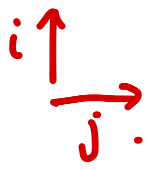

$$|\underline{a}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}.$$

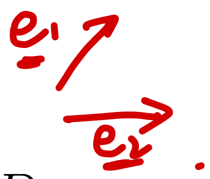
$$\hat{\underline{a}} = \frac{1}{|\underline{a}|} \times \underline{a} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \qquad |\hat{\underline{a}}| = 1.$$

## Base vector and change of basis

Using the vectors $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we can reach any point in space.

$$\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$




In fact, any pair of non-parallel vectors give a base pair in $2D$ space. We usually denote base vectors in $2D$ as $e_1$ and $e_2$.

EXAMPLE: Express $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ as a vector using base vectors $e_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\underline{e}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

① base: $\underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 7 \\ 6 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 6\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 7\underline{i} + 6\underline{j}.$$

② base: $\underline{e}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ & $\underline{e}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, i.e. $\begin{pmatrix} 7 \\ 6 \end{pmatrix} = 3\underline{e}_1 + 1 \cdot \underline{e}_2$

$$\begin{pmatrix} 7 \\ 6 \end{pmatrix} = \lambda_1 \underline{e}_1 + \lambda_2 \underline{e}_2 = \lambda_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2\lambda_1 \\ \lambda_1 \end{pmatrix} + \begin{pmatrix} \lambda_2 \\ 3\lambda_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2\lambda_1 + \lambda_2 \\ \lambda_1 + 3\lambda_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 7 = 2\lambda_1 + \lambda_2 \\ 6 = \lambda_1 + 3\lambda_2 \end{cases}$$

i.e. $\lambda_1 = 3$; $\lambda_2 = 1$.

3D: base: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

## <u>Vector dot product/scalar product/inner product</u>

Let $\underline{a}$ and $\underline{b}$ be two $n$-dimensional vectors with $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

Dot product of $\underline{a}$ and $\underline{b}$:

$$\underline{a} \cdot \underline{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3) + \ldots + (a_n \times b_n) = \sum_{i=1}^{n} a_i b_i$$

EXAMPLE: $\underline{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

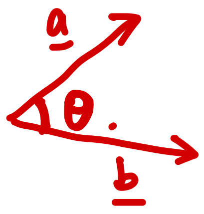
$\underline{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ $\times$.

Note: dot product gives a number (scalar) not a vector.

$\underline{a} \cdot \underline{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 3 \times (-1) + 1 \times 2 = -3 + 2 = -1.$

Geometrically, the dot product has a very nice (and helpful) property as it tells us the angle between two vectors.

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos(\theta)$.




$$\Rightarrow \cos(\theta) = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$$

EXAMPLE: $\underline{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ What is the angle between $\underline{a}$ and $\underline{b}$ ?

$$\underline{a} \cdot \underline{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 3 \times (-1) + 1 \times 2 = -1.$$

$$|\underline{a}| = \sqrt{3^2 + 1^2} = \sqrt{10}. \qquad |\underline{b}| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}.$$

$$\Rightarrow \cos(\theta) = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = -\frac{1}{\sqrt{50}}.$$

$$\Rightarrow \theta = \cos^{-1}\left(-\frac{1}{\sqrt{50}}\right) \approx 98^\circ.$$

## Important properties

$\underline{a} \cdot \underline{a} = |\underline{a}||\underline{a}| \overset{=1}{\cos(0)} = |\underline{a}||\underline{a}| = |\underline{a}|^2 = \sum_{i=1}^{n} a_i^2$

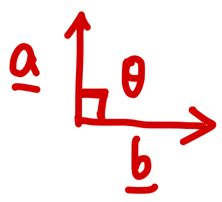
$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$

$(k\underline{a}) \cdot \underline{b} = \underline{a} \cdot (k\underline{b}) = k(\underline{a} \cdot \underline{b}), \quad k =$ constant

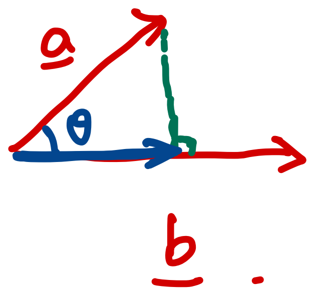
$\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$

If two vectors are perpendicular. i.e. normal, then $\underline{a} \cdot \underline{b} = 0$.

$$\underline{a} \cdot \underline{b} = a_1 \times b_1 + a_2 \times b_2 + \cdots + a_n \times b_n$$
$$= b_1 \times a_1 + b_2 \times a_2 + \cdots + b_n \times a_n$$
$$= \underline{b} \cdot \underline{a}.$$




$$\underline{a} \cdot \underline{b} = |\underline{a}| \times |\underline{b}| \cdot \cos(90°)$$
$$= |\underline{a}| \times |\underline{b}| \times 0$$
$$= 0.$$

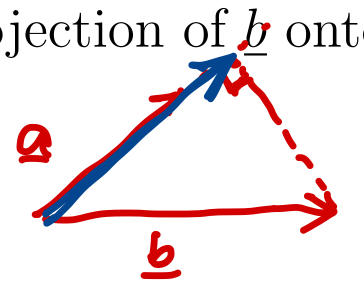



## Projections

Formula for finding the projection of $\underline{a}$ onto $\underline{b}$ is given by

$$\text{proj}_{\underline{b}}\,\underline{a} = \frac{\underline{a}\cdot\underline{b}}{|\underline{b}|^2}\underline{b} = \underbrace{\frac{\underline{a}\cdot\underline{b}}{|\underline{b}|}}_{=\frac{|\underline{a}||\underline{b}|\cdot\cos\theta}{|\underline{b}|} = |\underline{a}|\cdot\cos\theta} \times \underbrace{\frac{\underline{b}}{|\underline{b}|}}_{\text{unit vector.}}.$$

The projection of $\underline{b}$ onto $\underline{a}$ is given by




$$\text{proj}_{\underline{a}}\,\underline{b} = \frac{\underline{b}\cdot\underline{a}}{|\underline{a}|^2}\underline{a} = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}|^2}\underline{a}$$

Note: the projection $\text{proj}_{\underline{b}}\,\underline{a}$ and $\text{proj}_{\underline{a}}\,\underline{b}$ must give a vector.

EXAMPLE: Find the projection of $\underline{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ onto $\underline{a} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

$$\underline{a}\cdot\underline{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = 2\times 1 + 1\times 0 + (-1)\times(-2) = 4.$$

$$|\underline{a}| = \sqrt{1^2+0^2+(-2)^2} = \sqrt{5}$$

$$\text{proj}_{\underline{a}}\,\underline{b} = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}|^2}\underline{a} = \frac{4}{5}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ 0 \\ -\frac{8}{5} \end{pmatrix}.$$