Topic 3. Direct variation, Exponential, Logarithm and Trigonometric Functions

Why? data transformation, visualisation, analysis

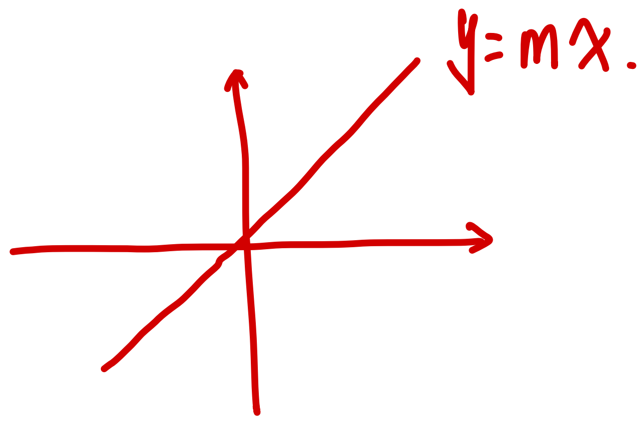
x & y

A variation is a relation between a set of values of one variable and a set of values of other variables.

In the equation $y = mx + b$, if m is a nonzero constant and $b = 0$, then you have the function $y = mx$, which is called a direct variation. That is, you can say that y varies directly as x or y is directly proportional to x. In this function, m is called the constant of proportionality or the constant of variation. The graph of every direct variation passes through the origin.

$y \propto x \quad \Leftrightarrow \quad y = mx.$

· proportional to:



coffee p x

For example, when you buy something, the total you pay is proportional to how many you buy and $m =$ per unit price.

$$p \propto x \iff p = m \cdot x.$$

T

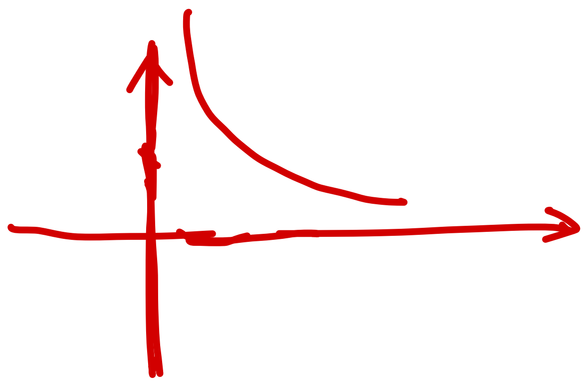
As another example, the time it takes to finish a task is normally inversely proportional to the number of people doing it, i.e., the time decreases as the number of people increases.

N

$$T \propto \frac{1}{N}.$$

point: $xy = m \rightarrow$ fixed.

$$y \propto \frac{1}{x} \iff y = \frac{m}{x}$$



EXAMPLE: Given that y is inversely proportional to the square of x and that when $x = 1.5$, $y = 8$. Find:

x^2

- The formula connecting x and y;
- y value, when $x = \sqrt{3}$.

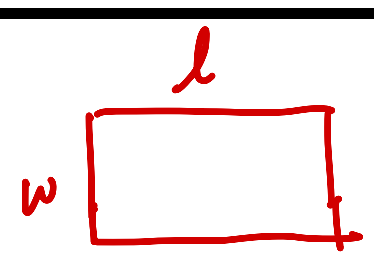
• $y \propto \frac{1}{x^2}$. $\Leftrightarrow$ $y = \frac{m}{x^2}$

$8 = \frac{m}{1.5^2} \Rightarrow m = 8 \times 1.5 \times 1.5 = 18.$

i.e. $y = \frac{18}{x^2}$.

• $y = \frac{18}{(\sqrt{3})^2} = \frac{18}{3} = 6.$

Joint variation



$a(l, w) = l \cdot w.$

As we saw last week, it is possible to have two or more things (variables) to affect our output. In this multivariate case we say the output varies jointly with the input variables.

intersection between x & y.

$y \propto x \cdot z \Leftrightarrow y = m x z.$

EXAMPLE: Given that y varies jointly with x and the square root of z, and that $y = 2$ when $x = 1/8$ and $z = 1/4$. Find:

x $\sqrt{z}$

- The equation connecting y with x and z;
- y value, when $x = 3/8$, $z = 1/9$.

- $y \propto x \cdot \sqrt{z}$. $\Leftrightarrow$ $y = m \cdot x \cdot \sqrt{z}$.

$2 = m \cdot \times \frac{1}{8} \times \sqrt{\frac{1}{4}} = m \times \frac{1}{8} \times \frac{1}{2} \Rightarrow m = 32$

$\Rightarrow y = 32 x \sqrt{z}$

- $y = 32 \times \frac{3}{8} \times \sqrt{\frac{1}{9}} = 32 \times \frac{3}{8} \times \frac{1}{3} = 4.$

Partial variation

It is also possible to have a relationship where things vary by different amounts.

$y \propto x_1 + x_2 \iff y = k_1 x_1 + k_2 x_2.$

EXAMPLE: y varies partly with the square root of x_1 and partly with the square of x_2. Given that $y = 11$ when $x_1 = 9$ and $x_2 = 2$, and that $y = 23$ when $x_1 = 25$ and $x_2 = 4$. Find:

$\sqrt{x_1}$

x_2^2

- The equation connecting y with x_1 and x_2;
- x_1 value, when $y = 30$, $x_2 = 6$.

$y \propto \sqrt{x_1} + x_2^2 \iff y = m_1\sqrt{x_1} + m_2 x_2^2.$

Simultaneous functions
1. get rid of variables.
2. substitute

$$\begin{cases} 11 = 3k_1 + 4k_2 \\ 23 = 5k_1 + 16k_2. \end{cases}$$

$\Rightarrow m_1 = 3 \quad m_2 = \frac{1}{2}$

Hence $y = 3\sqrt{x_1} + \frac{1}{2}x_2^2.$

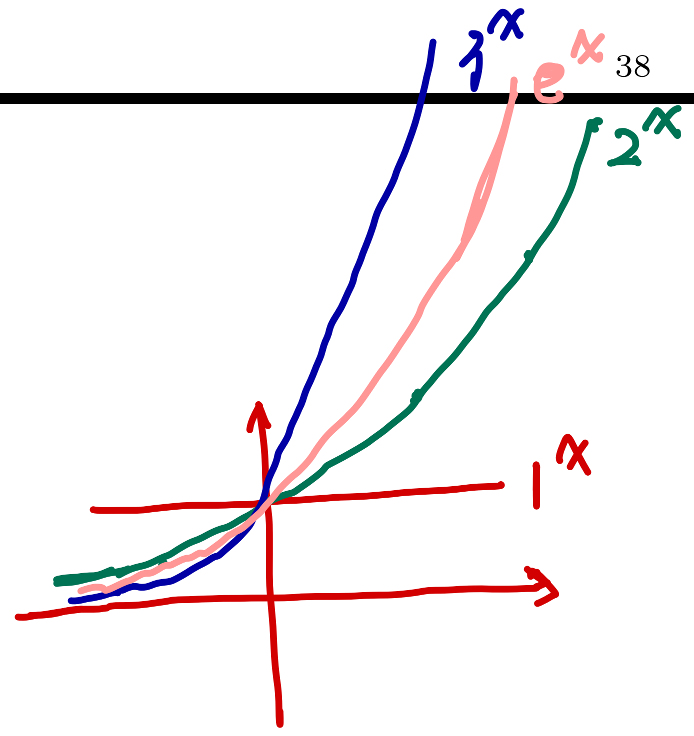
$30 = 3\sqrt{x_1} + \frac{1}{2}\cdot 36 = 3\sqrt{x_1} + 18$

$\Rightarrow 12 = 3\sqrt{x_1} \Rightarrow \sqrt{x_1} = 4 \Rightarrow x_1 = 16.$

Exponential function

power function.

Recall that $2^x = \underbrace{2 \times 2 \times ... \times 2}_{x \text{ times}}$. Let's plot $1^x, 2^x, 3^x, ...$.



A very important function is the exponential function, e^x or $exp(x)$, where $e = 2.718...$ (Euler's number). e^x has many useful characters.

- $e^x > 0$; it is never negative nor precisely zero;
- As x gets larger, e^x gets large very quickly (exponential growth);
- As x gets more negative, e^x gets closer to zero ;
- Every possible real positive number can be written in the form of e^x; i.e., $r = e^x$ for any $\cancel{r \in \mathbb{R}}$ $r>0$, and there is a one-to-one correspondence between r and x.
- Remember indices rules: $e^x e^y = e^{(x+y)}$, $e^x/e^y = e^{(x-y)}$, $(e^x)^2 = e^{2x}$.

Logarithm function

Logarithm function is inverse function of exponential. It allows us to write $2^x = 5$ as $x = \log_2 5$

log of 5 in base of 2.

EXAMPLE:

x is between 2 & 3.

x	0	1	2	3
y	1	2	4	8

5

x	2.0	2.1	2.2 .. 3
y			

$a^x = b \Rightarrow x = \log_a b$.

Rules:

power rules

$a^x \cdot a^y = a^{x+y}$

$a^x / a^y = a^{x-y}$

$a^0 = 1$

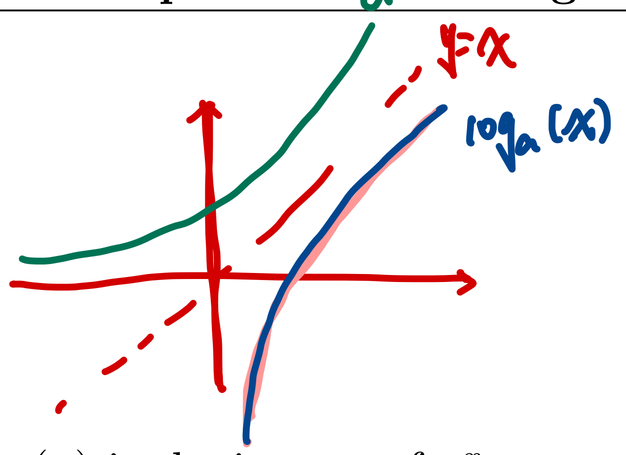
$a^1 = a$.

- $log_a(xy) = log_a(x) + log_a(y)$;
- $log_a(x/y) = log_a(x) - log_a(y)$;
- $log_a(1) = 0$;
- $log_a(a) = 1$;
- $log_a(x^k) = klog_a(x)$.

$\log_a(\underbrace{x \cdot x \cdot x \cdots x}_{k}) = \log_a x + \log_a x^{k-1} = \cdots = k\log_a x$.

If the base of a logarithm function is e, i.e., $log_e(x)$, it's called the natural log and is written as $ln(x)$.

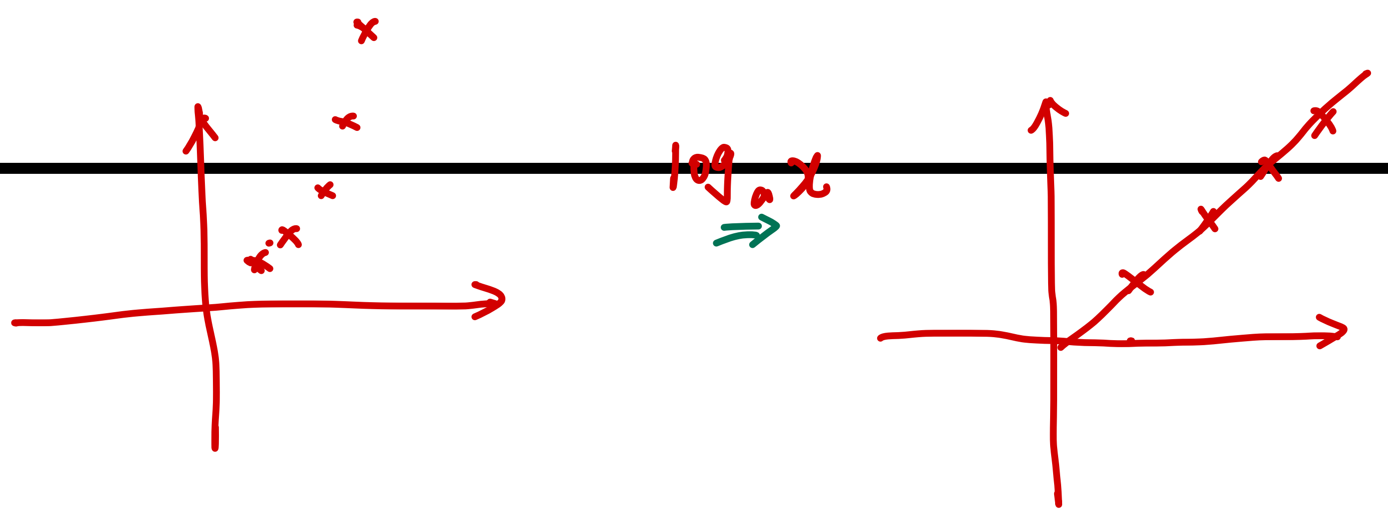
Sketch of Exponential and Logarithm functions



- $log_a(x)$ is the inverse of a^x;
- $log_a(x)$ can be positive, negative or zero;
- $log_a(x)$ doesn't exist when $x \leq 0$, $log_a(x)$ never crosses $y-$axis.

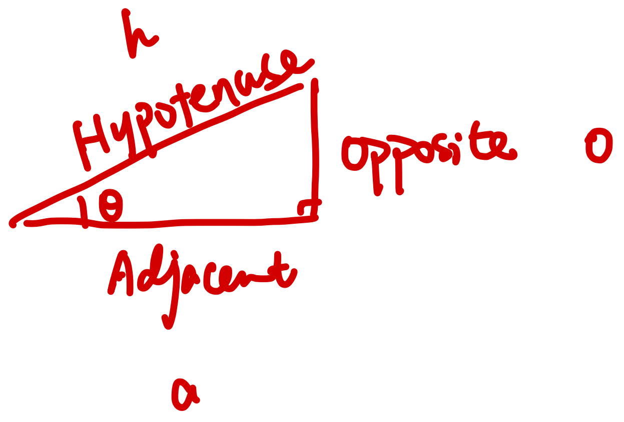
domain $(0, \infty)$

Usage: Logarithmic transformation is a convenient means of transforming a highly skewed variable into a more normalized dataset. By applying the logarithm to your variables, there is a much more distinguished and or adjusted linear regression line through the base of the data points, resulting in a better prediction model.



Trigonometry

The sine ($\sin(x)$), cosine ($\cos(x)$) and tangent ($\tan(x)$) of an angle are all defined in terms of trigonometry, but they can also be expressed as functions.



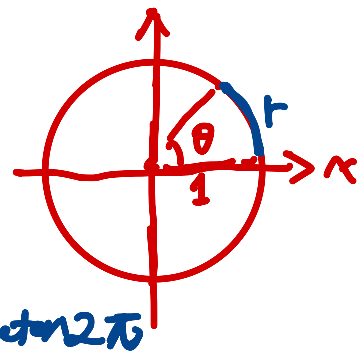
$\sin(\theta) = \frac{o}{h}$ SOH

$\cos(\theta) = \frac{a}{h}$ CAH

$\tan(\theta) = \frac{o}{a}$ TOA.

Radians radians

Angels can also be measured in radians instead of degrees. To do this consider a unit circle:



$$\frac{\theta^\circ}{360^\circ} = \frac{r}{2\pi}$$

perimeter 2π

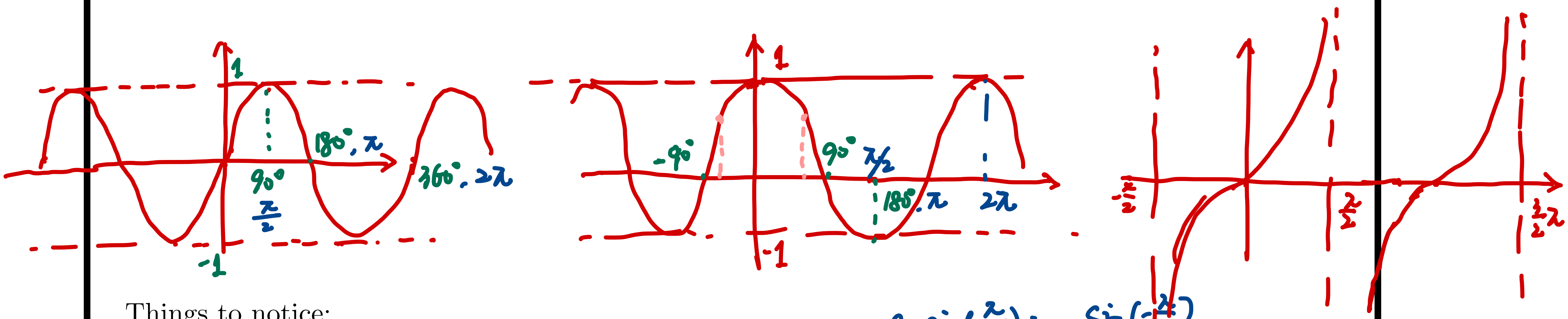
Radians are the length of the perimeter of the unit circle measured anti-clockwise from the $x-$axis , for a given angle θ, i.e., it is the proportion of the perimeter covered by the angle θ.

Whole perimeter of an unit circle is 2π.

- If the angle is $\theta = 360^\circ$: $r = \frac{360^\circ}{360^\circ} \cdot 2\pi = 2\pi$
- If the angle is $\theta = 90^\circ$: $r = \frac{90^\circ}{360^\circ} \cdot 2\pi = \frac{1}{2}\pi$
- In general, if the angle is θ°, it covers $\theta/360$ of the ~~perimeter~~: perimeter.

Sketching trigonometry graphs

$f(x) = sin(x)$ $\quad$ $f(x) = cos(x)$ $\quad$ $f(x) = tan(x)$



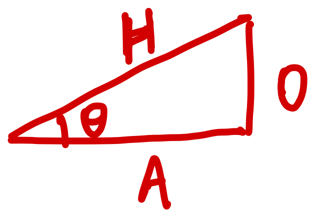
e.g. $\sin(\frac{\pi}{2}) = -\sin(-\frac{\pi}{2})$

Things to notice:

- $\sin(\theta)$ has rotational symmetry of 180° about the origin, $\sin(-\theta) = -\sin(\theta)$;
- ~~sin~~ cos(θ) has reflective symmetry through the line $\theta = 0$ ($y-$axis), $\cos(-\theta) = \cos(\theta)$;
- Both sin and cos repeat themselves every 2π (or 360°, $\sin(\theta) = \sin(\theta + 2n\pi), n \in \mathbb{Z}$).
- Moving sin curve $\pi/2$ to the left will give you cos, $\cos(\theta) = \sin(\theta + \pi/2)$;
- $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.

Inverse Trig functions

Convert ratios back to angels/radians:



$$\sin(\theta) = \frac{O}{h} \Rightarrow \theta = \sin^{-1}\left(\frac{O}{h}\right)$$

$$\cos(\theta) = \frac{A}{H} \Rightarrow \theta = \cos^{-1}\left(\frac{A}{h}\right)$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \Rightarrow$$

- $\sin^{-1}(x) = \arcsin(x)$;
- $\cos^{-1}(x) = \arccos(x)$;
- $\tan^{-1}(x) = \arctan(x)$;

In geometric terms,

- Trigonometric (trig) functions take an angle/radian as input and give out a "length/ratio";
- Inverse trig functions take a "length/ratio" as input and give out an angle/radian.