Topic 5. Differentiation

Introduction to differentiation, Maxima and minima, Partial derivatives

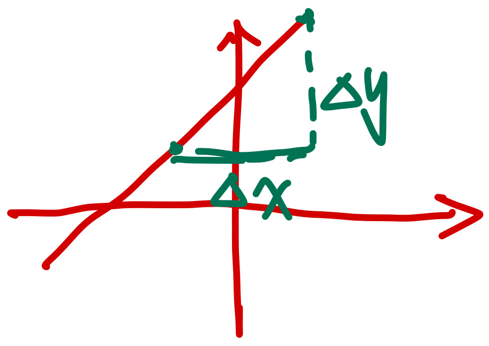
Recall the gradient (change in slope) of a straight line is given by finding the difference in y-coordinates divided by the difference in x-coordinates.

$$y = mx + c$$

$\uparrow$

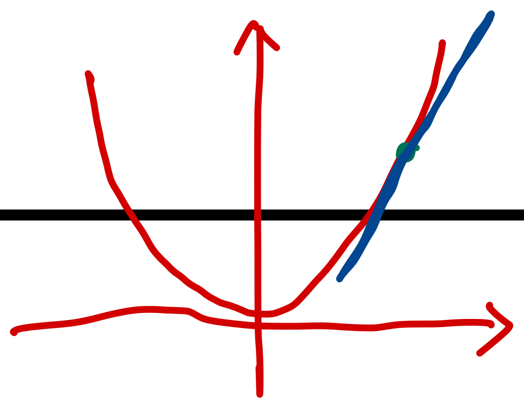
gradient/slope.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2}$$



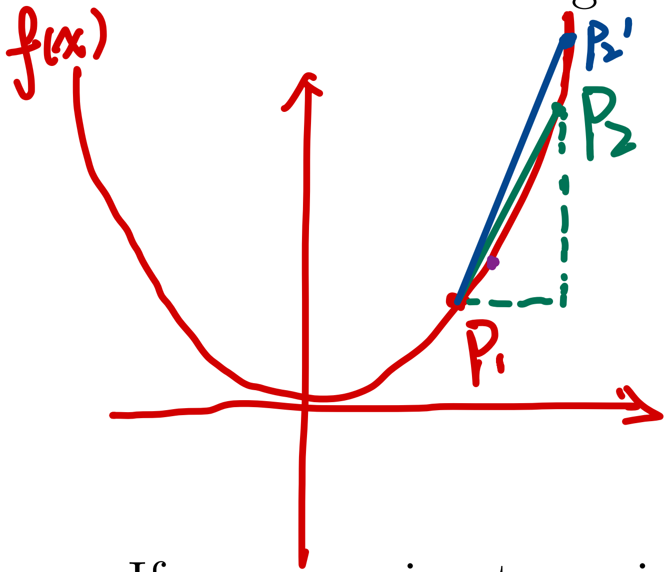
How can I do this for a curve?

- First attempt: Pick a point on the curve, place a straight edge against the curve so that it just touches the curve at this point (tangent) and measure the gradient of the straight line.



gradient of (x_1, y_1).

- Instead we could pick two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the curve, connect them and find the gradient of this line.



- If we zoom in at a point on the curve, we notice the curve 'looks like' a straight line, hence we calculate the gradient to be the gradient of this line. we let $\Delta x = x_2 - x_1$ to get "as small as passible", but not 0! The gradient of the line connecting P_1 and P_2 is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(\overset{x_2}{x_1 + \Delta x}) - f(x_1)}{\Delta x}$$

- But want Δx to be very small: so the gradient of $f(x)$ at x becomes

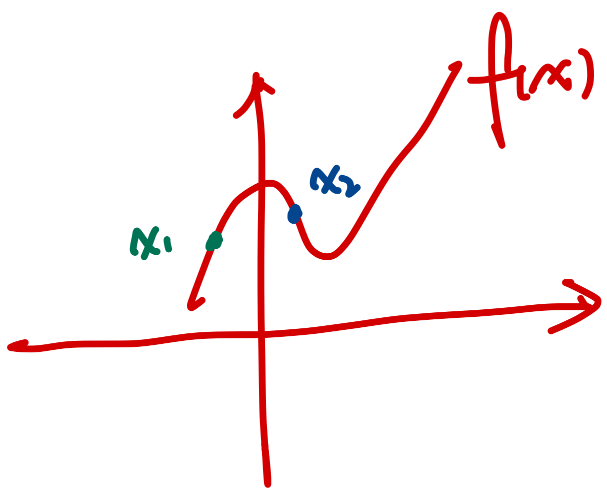
$$f'(x_1) = \lim_{\Delta x \to 0} \frac{f(x_1+\Delta x) - f(x_1)}{\Delta x}$$ gradient of $f(x)$ at x_1.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$ def. of derivative.

- Differentiation is the name of the process to find the derivative.

$f(x)$: $f'(x)$, $\frac{df}{dx}$. $\frac{d}{dx}f(x)$.

y : y', $\frac{dy}{dx}$. $\frac{d}{dx}y$.



$f'(x_1) > 0$

$f'(x_2) < 0$

case: $n=1$ $f(x)=ax$
$n=0$ $f(x)=a$
$n=2$ $f(x)=ax^2$.

Rules of differentiation (common functions)

If $f(x) = ax^n, a$ is constant, then $\frac{df}{dx} = f'(x) = anx^{n-1}$ (decrease old power by 1, multiplied by the old power)

EXAMPLE: $f(x) = x^2,\ x^3,\ 3x^4,\ \sqrt{x},\ 2x,\ kx,\ 2,\ k,\ \frac{1}{x^3}$. Find $f'(x)$.

$f(x)=2x$. gradient 2. $a=2, n=1$ $f'(x)=2\times1\times x^0=2$

$f(x)=3$. $f'(x)=0$.

$f(x)=\sqrt{x}=x^{\frac{1}{2}}$. $f'(x)=\frac{1}{2}x^{\frac{1}{2}-1}=\frac{1}{2}x^{-\frac{1}{2}}$.

$f(x)=\frac{1}{x^3}=x^{-3}$. $f'(x)=-3x^{-4}$.

EXAMPLE: Find the gradient at the point (2,16) of the curve $y = x^2 + 7x - 2$.

$y'=(x^2+7x-2)'=(x^2)'+(7x)'-2'$

$=2x+7$

gradient at $(2,16)$: $2\times2+7=11$

Higher derivatives

The derivative, $\frac{dy}{dx}$, is more expressly called the first derivative of y. By differentiating the first derivative, we obtain the second derivative; by differentiating the second derivative we obtain the third derivative and so on. These second and subsequent derivatives are known as higher derivatives.

$(ax^n)' = a \cdot n \cdot x^{n-1}$.

EXAMPLE: $f(x) = x^4$

$f(x) = x^4$.

$f^{(1)}(x) = \frac{df}{dx} = f'(x) = 4 \cdot x^3$.

$f^{(2)}(x) = \frac{d^2f}{dx^2} = f''(x) = (4x^3)' = 4 \times 3 \times x^{3-1} = 12x^2$

$f^{(3)}(x) = \frac{d^3f}{dx^3} = f'''(x) = 24x$.

In general, the nth derivative is given by $f^{(n)}(x)$ or $\frac{d^nf}{dx^n}$.

$f(x)$	$f'(x)$
e^x	e^x
e^{ax}	$a \cdot e^{ax}$
$\ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$.
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$

$$\sin(x) \xrightarrow{f'} \cos(x) \xrightarrow{f'} -\sin(x) \xrightarrow{f'} -\cos(x) \xrightarrow{f'} \sin(x)$$

Rules of differentiation

- $\frac{d}{dx}[af(x)] = a\frac{d}{dx}f(x) = af'(x), a = \text{constant}.$

$$\frac{d}{dx}[2\cdot\cos(x)] = 2\frac{d}{dx}[\cos(x)] = -2\sin(x)$$

- $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x).$

$$\frac{d}{dx}[\underset{\uparrow\atop f}{x^2} + \underset{\uparrow\atop g}{\sin(x)}] = \frac{d}{dx}x^2 + \frac{d}{dx}\sin(x) = 2x + \cos(x).$$

- $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) = f(x)g'(x) + f'(x)g(x).$ product rule.

$$\frac{d}{dx}[\underset{\uparrow\atop f}{x^2}\cdot\underset{\uparrow\atop g}{\cos(x)}] = x^2\cdot\frac{d}{dx}\cos(x) + \cos(x)\cdot\frac{d}{dx}x^2$$

$$= -x^2\sin(x) + 2x\cdot\cos(x).$$

$\frac{d}{dx}\left(f(x)\cdot\frac{1}{g(x)}\right)$

- $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$. quotient rule.

$$\frac{d}{dx}\frac{\cos(x)}{x} = \frac{x\frac{d}{dx}\cos(x) - \cos(x)\cdot\frac{d}{dx}\cdot x}{x^2}$$

f, g

$$= \frac{-x\cdot\sin(x) - \cos(x)}{x^2}.$$

$f\circ g(x)$

- $\frac{d}{dx}f(g(x)) = g'(x)f'(g(x))$. chain rule.

$\cos(x^2)$. $\quad f(x) = \cos(x)$. $\quad f'(x) = -\sin(x)$

$g(x) = x^2$. $\quad g'(x) = 2x$.

$$\frac{d}{dx}\cos(x^2) = 2x\cdot(-\sin(x^2))$$

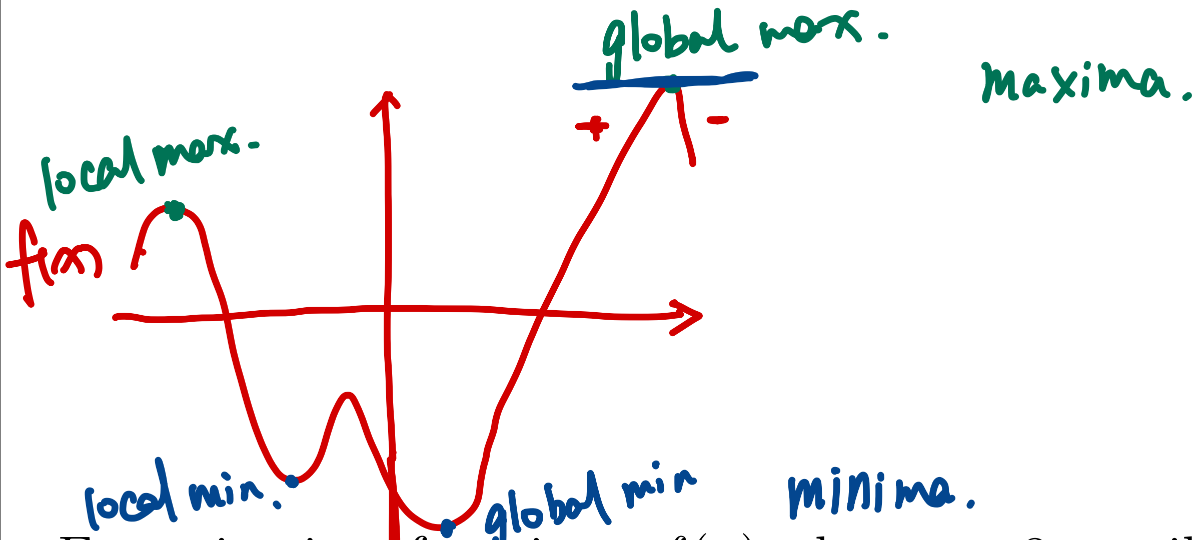
$$= -2x\cdot\sin(x^2).$$

$(e^{ax})' = a\cdot e^{ax}$.

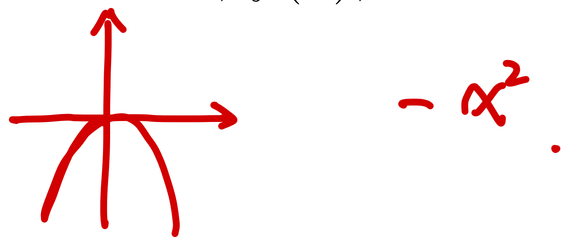
$f(x) = e^x \quad g(x) = ax$.

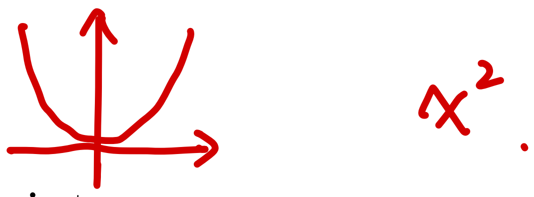
Finding maxima, minima and turning points

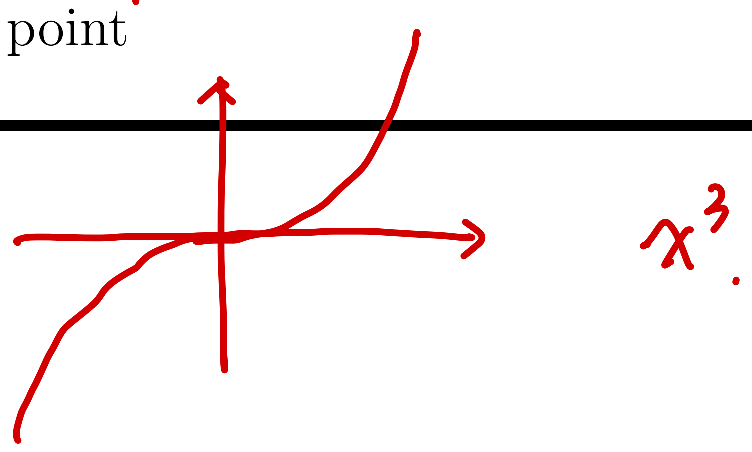
One of the most common and useful reasons for calculating the derivative of a function is that it lets us find maximum and minimum points.



For univariate functions, $f(x)$, there are 3 possible cases for $\frac{dy}{dx} = 0$ (called stationary points)

- maximum 

- minimum 

- inflection point 

In the max or min cases we can further classify as being 'local' or 'global'.

- Local minima are points that are less than all 'nearby' points.
- Global minima are points that are less than all other point.

EXAMPLE: Find all turning points of the function $g(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3$.

Stationary pst. : $\frac{df}{dx} = 0$.

$$\frac{d}{dx} g(x) = \frac{d}{dx}\left(\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3\right).$$

$$= \frac{d}{dx}\left(\frac{1}{3}x^3\right) - \frac{d}{dx}\left(\frac{1}{2}x^2\right) - \frac{d}{dx}(2x) + \frac{d}{dx}3.$$

$$= x^2 - x - 2$$

Let $\frac{d}{dx} g(x) = x^2 - x - 2 = 0$.

$$= (x-2)(x+1) = 0$$

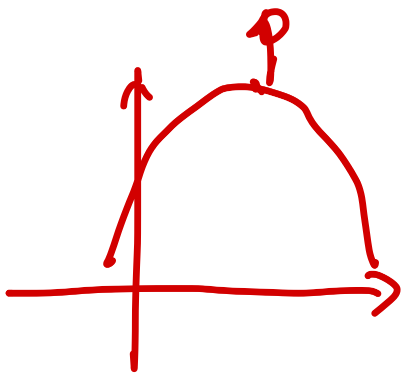
Then $x_1 = 2$. $x_2 = -1$.

Case 1: $x = 2$, $g(2) = \frac{1}{3} \times 2^3 - \frac{1}{2} \times 2^2 - 2 \times 2 + 3 = \cdots$

Classifying turning points

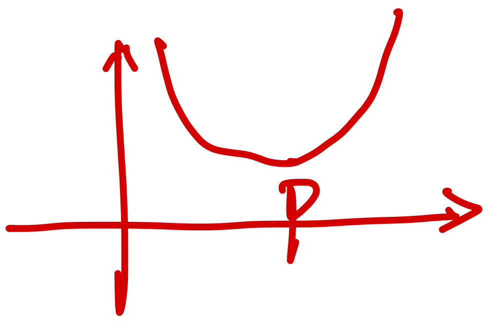
Consider a maximum and minimum at point P on $f(x)$.

max.



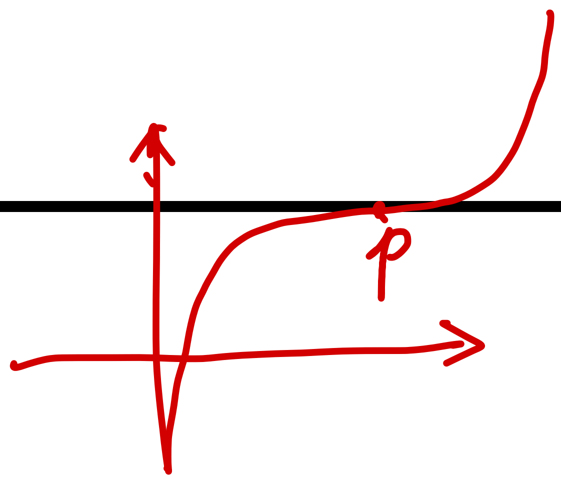
x	$x<p$	$x=p$	$x>p$
$f'(x)$	$+$	0	$-$

min



x	$x<p$	$x=p$	$x>p$
$f'(x)$	$-$	0	$+$

inflection:



x	$x<p$	$x=p$	$x>p$
$f'(x)$	$+$	0	$+$

One way to classify turning points is to find the gradient either side of the turning point, but this is not precise! We actually can use the second derivative (in most cases) at the turning point.

$f(x)$

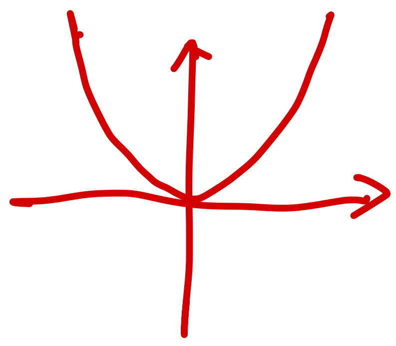
$f''(x) > 0$. minima.

$f''(x) < 0$ maxima.

$f''(x) = 0$ we don't know.

EXAMPLE: $f(x) = x^2$.

$f'(x) = 2x$.



Let $f'(x)=0$, we have $x = 0$.

$f''(x) = 2 > 0$.

$f(x)$ has a minima at $x=0$.

Partial derivatives

Recall a multivariate function is one that depends on multiple variables.

EXAMPLE: $f(x, y) = x^2 + y^2$, $f(x_1, x_2, x_3, x_4) = x_1 + 2x_2 - x_3^2 + \cos(x_4 x_1)$.

We still may want to know how these functions change as one of variables changes, or where these functions are maximised/ minimised or where they are stationary. To do this we need to differentiate. But unlike in the univariate case we need to pay attention to the variable we are differentiating with respect to. We treat any variable we are not differentiating w.r.t., as a constant, and proceed with the differentiation as in the univariate case. We call this process partial derivatives.

EXAMPLE: $f(x, y) = 2x + y^2$.

diff. w.r.t. x. $\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}(2x + \underbrace{y^2}_{\text{consider as constant}}) = 2$

diff w.r.t. y: $\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}(\underbrace{2x}_{\text{consider as constant}} + y^2) = 2y$

EXAMPLE: $f(x, y) = x^4y^3$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^4 y^3\right) = y^3 \frac{d}{dx} x^4 = 4x^3y^3$$
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^4 y^3\right) = x^4 \frac{d}{dy} y^3 = 3x^4y^2$$

EXAMPLE: $g(x, y) = x^2y + y\cos(x)$.

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(x^2y + y\cos(x)\right) = \frac{d}{dx}x^2y + \frac{d}{dx}y\cdot\cos(x) = 2xy - y\sin(x)$$
$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(x^2y + y\cos(x)\right) = x^2 + \cos(x)$$

EXAMPLE: $h(x, y) = \sin(x^2y)$.

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}\left(\underbrace{\sin}_{f}\,(\underbrace{x^2y}_{g})\right)$$
$$= \cos(x^2y)\cdot 2xy$$

$$\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}\left(\sin(x^2y)\right)$$
$$= \cos(x^2y)\cdot x^2.$$

recall: $\frac{d}{dx} f(g(x)) = g'(x)\cdot f'(g(x))$

Higher order partial derivatives

We can partially differentiate multiple times to give higher order derivatives. But there is a slight different compared to the univariate case:

$$f(x,y) \xrightarrow{\partial/\partial x} \frac{\partial f}{\partial x} \xrightarrow{\partial/\partial x} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}.$$

$$f(x,y) \xrightarrow{\partial/\partial y} \frac{\partial f}{\partial y} \xrightarrow{\partial/\partial y} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}.$$

$$f(x,y) \xrightarrow{\partial/\partial x} \frac{\partial f}{\partial x} \xrightarrow{\partial/\partial y} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}.$$

$$f(x,y) \xrightarrow{\partial/\partial y} \frac{\partial f}{\partial y} \xrightarrow{\partial/\partial x} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}.$$

Notation:

$$\frac{\partial f}{\partial x} = f_x, \frac{\partial f}{\partial y} = f_y.$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}, \frac{\partial^2 f}{\partial y^2} = f_{yy}, \frac{\partial^2 f}{\partial x \partial y} = f_{yx}, \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$