PG Topic 4. Permutations and Combinations

Permutations

the order is important.

Consider the letters A, B, C. How many distinct ways are there to arrange these 3 letters? 6

ABC, ACB, BAC, BCA, CAB, CBA. Each of these is a permutation of {A, B, C}.

How can we calculate without having to manually list?

We can think about this as having 3 spaces. (1)(2)(3)

- How many letters could be in space 1? 3
- Then, has many letters can be in space 2? 2
- Finally, how many letters can be in space 3? 1

Thus, total number of options is

= 3×2×1 = 6, i.e. 3!

In general, if we wanted to place n items in a row, how many options we would have?

= n!

What about if we want to choose a subset of the list of items?

EXAMPLE: I have the letters A, B, C, D, E, if I want to arrange two of these letters, how many ways are there? (order is important)

AB. AC, AD, AE, BA, BC, BD, BE, CA, CB, CD, CE, DA, DB, DC, DE, EA, EB, EC, ED. } 20

To work this out we can use the formula:

$$^nP_r \stackrel{\text{def}}{=} \frac{n!}{(n-r)!}, = n \times (n-1) \times (n-2) \times \cdots (n-r+1).$$

where n = total number in the list, r = number of items I'm choosing.

$n=5,\ r=2 \quad {}^5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5\times4\times3\times2\times1}{3\times2\times1} = 20$

① ②
For ①: 5.
②: 4. 5×4

EXAMPLE: 10 people in a race. How many different ways can 1st, 2nd and 3rd be filled?

$n=10 \quad r=3$

${}^{10}P_3 = \frac{10!}{(10-3)!} = \frac{10\times9\times8\times7\times\cdots1}{7\times6\times5\times\cdots1} = 10\times9\times8 = 720.$

$g_1 g_2$
$= g_2 g_1$

EXAMPLE: In how many ways can 4 red, 3 yellow and 2 green disks be arranged in a row if the same colour disks are indistinguishable?

$n = 9$

$$\#: \frac{9!}{4!\,3!\,2!} = \frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1 \times 3\times2\times1 \times 2\times1} = 9\times4\times7\times5 = 1260.$$

In general, the number of arrangements of n items, with p of one kind, q of another kind, r of another kind ... is $\frac{n!}{p!q!r!\cdots}$.

Combinations

Consider A, B, C, D, E, how many combinations (i.e. order is not important) of picking 2 letters are there? 10

n = 5

r = 2

AB, AC, AD, AE, BC, BD, BE, CD, CE, DE

~~BA, CA, DA, EA CB, DB, EB DC EC, ED.~~

Notice that the order is not important, i.e. "AB = BA". we can ignore the 2 line.

We can calculate the # of combination using

${}^nC_r \overset{\text{def}}{=} \frac{n!}{r!(n-r)!} \overset{\text{def}}{=} \binom{n}{r}$ n choose r.

EXAMPLE: How many selections of 4 letters can be made from A, B, C, D, E, F?

$n=6 \quad r=4.$

$${}^6C_4 = \frac{6!}{4!(6-4)!} = \frac{6\times5\times\cancel{4\times3\times2\times1}}{\cancel{4\times3\times2\times1}\times 2\times1} = \frac{30}{2} = 15.$$

EXAMPLE: How many different after school clubs, each consisting of 3 boys and 2 girls can be chosen from 7 boys and 5 girls?

Consider boys: How many ways of choosing 3 boys from 7? $n=7 \; r=3$ ${}^7C_3 = \frac{7!}{3!(7-3)!}$

$$= \frac{7\times6\times5\times\cancel{4\times3\times2\times1}}{\cancel{3\times2\times1}\times\cancel{6\times3\times2\times1}} = 35$$

Consider girls: How many ways of choosing 2 girls from 5?

$n=5, \; r=2$ $\quad {}^5C_2 = \frac{5!}{2!(5-2)!} = \frac{5\times\cancel{4\times3\times2\times1}}{2\times1\times\cancel{3\times2\times1}} = 10$

$\# = 35\times10 = 350.$

$b_1.b_2,\cdots b_{35}$

$g_1, g_2, \cdots g_{10}$

$b_1g_1, \; b_1g_2, \cdots b_1g_{10}.$

$b_2g_1, \quad \cdots \quad b_2g_{10}.$

$\vdots$

$b_{35}g_1, \quad \cdots \quad b_{35}g_{10}$

EXAMPLE: How many ways can the letters in the word "ALGEBRA" be arranged?

If I assume the two As are distinguishable:

permutation.

$n=7$ $\quad\#=7!$

If I assume the two As are indistinguishable:

$$\#: \frac{7!}{2!}$$