Topic 2. Functions

Combining functions; domains, ranges; piece-wise functions

A function is a rule which operates on one number to give another number. However, not every rule describes a valid function. This unit explains how to see whether a given rule describes a valid function, and introduces some of the mathematical terms associated with functions.

By the end of the session, you will be able to

- recognise when a rule describes a valid function,
- be able to plot the graph of a part of a function,
- find a suitable domain for a function, and find the corresponding range.

Definition

$f(x) = mx + c$

A function is a rule which maps a number to another unique number.
In other words, if we start off with an input, and we apply the function, we get an output.

$y = m \cdot x + c.$

what we get (output)

variable (input).

$\boxed{2} \to \boxed{2x+1} \to \boxed{5}$

Think of this as a "machine/process/algorithm":

$\boxed{\text{picture}} \to \boxed{\text{algorithm}} \to 0$

We usually write functions in "short hand" as $f(x)$, where f indicates the function (the rule) and x indicates the variable.

- Doesn't have to be x, can be anything, e.g., y, a, γ, θ... $f(y) = 3y+2$ $f(\theta) = \theta^2+2\theta+1$
- Doesn't have to use f for the function, can be anything, e.g., $g(x), h(x), \phi(x)$...

$g(x) = 3x+2$ $g(\theta) = 3\theta+2$

EXAMPLE: $f(x) = 2x + 1$.

What is $f(2)$?

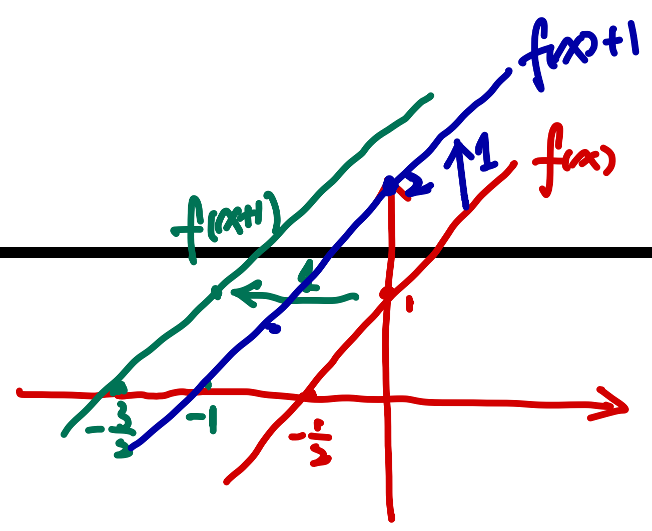
replace x in $f(x)$ to 2. $f(2) = 2\times2+1 = 5$

What is $f(x + 1)$? if $x=0$. $f(1)$. if $x=-1$. $f(0)$.

replace x in $f(x)$ to $x+1$ $f(x+1) = 2(x+1)+1 = 2x+3$

What about $f(x) + 1$?

$f(x)+1 = 2x+1+1 = 2x+2$

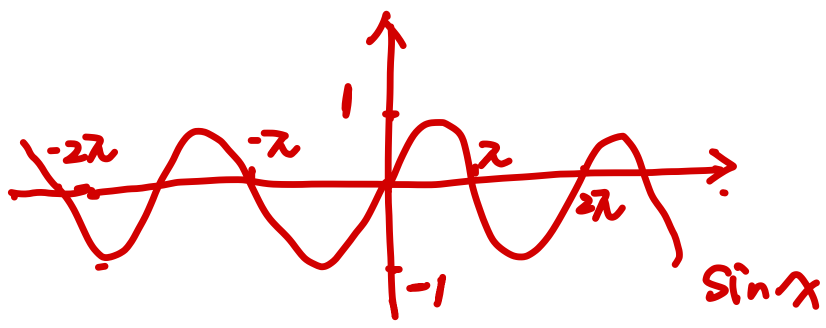


For example (a is a constant),

	$af(x)$	$f(ax)$	$f(x) \pm a$	$f(x \pm a)$
$f(x) = x^2$	$a \cdot x^2$	$(ax)^2$	$x^2 \pm a$	$(x \pm a)^2$
$f(x) = 1/x$				

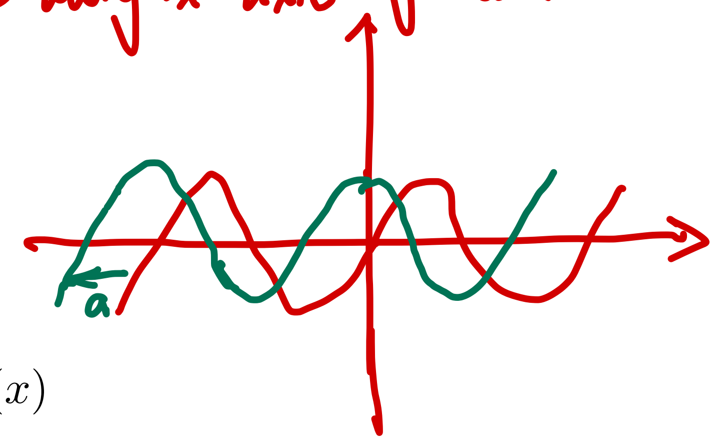
Graphical interpretations

EXAMPLE: $f(x) = sin(x)$



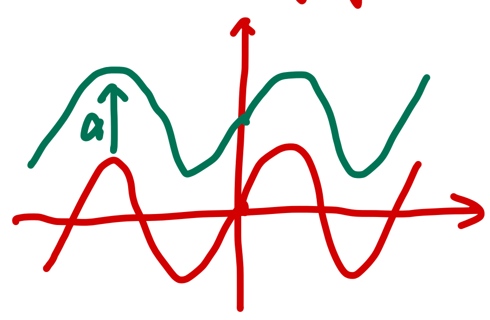
$f(x+a)$

move along x-axis by −a.



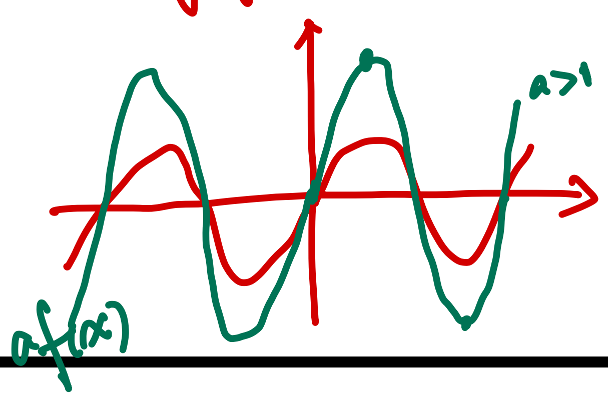
$f(x)+a$

move up along y-axis by a.



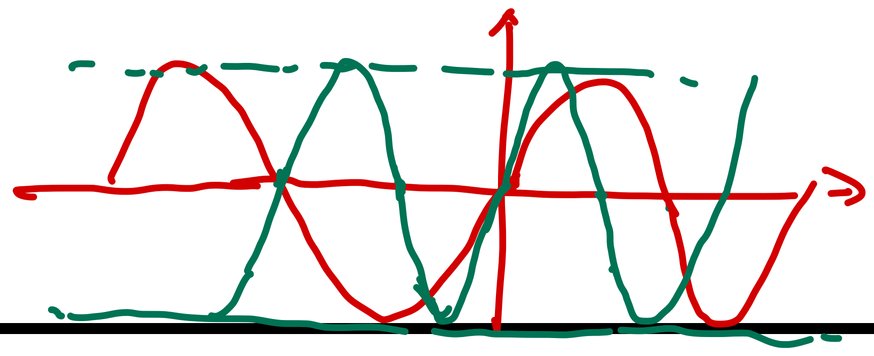
$af(x)$

strech along y-axis by a factor of a.



$f(ax)$

strech along x-axis by a factor of $\frac{1}{a}$



Combining functions

Let $f(x) = 2x + 1$, $g(x) = -3x + 2$.

- $f \pm g(x) := f(x) \pm g(x) = (2x+1) \pm (-3x+2) = \begin{matrix} -x+3 \\ \text{or } 5x-1 \end{matrix}$
- $f \times g(x) := f(x) \times g(x) = (2x+1) \times (-3x+2)$.
- $f/g(x) = \dfrac{f(x)}{g(x)} = \dfrac{2x+1}{-3x+2}$

Swapping order on + and × doesn't change results.

$f + g(x) = g + f(x) \qquad f \times g(x) = g \times f(x)$.

Composition of functions

$\boxed{x} \to \boxed{f} \to \boxed{f(x)} \to \boxed{g} \to \boxed{g(f(x))}$

$$g(f(x)) = g \circ f(x).$$

of   composed with.

Important: Apply functions from "right" to "left".

EXAMPLE: $f(x) = 2x + 1$, $g(x) = -3x + 2$.

- $f \circ g(x) = 2(\underbrace{-3x+2}_{g(x)}) + 1 = -6x + 5$

  $= f(g(x))$
- $g \circ f(x) = g(f(x)) = -3(\underbrace{2x+1}_{f(x)}) + 2 = -6x - 1$

Important: Order matters!

DNN.

One can compose as many functions as needed:

$f(g(h(g(f(x))))) = f \circ g \circ h \circ g \circ f(x).$

$\underbrace{f \circ g \circ h \circ g \circ f \circ \cdots}$

One can also compose functions with themselves, denoted as $f^2 = f \circ f$. But $f^2(x) \neq [f(x)]^2$!

EXAMPLE: $f(x) = 2x + 1$

$f^2(x) = f \circ f(x) = 2\underbrace{(2x+1)}_{f(x)} + 1 = 4x+3.$

$[f(x)]^2 = (2x+1)^2 = 4x^2+4x+1.$

$f^n = f \circ f \circ \ldots \circ f$

$\neq$

$[f(x)]^n.$

$\boxed{y} \to \boxed{f^{-1}} \to \boxed{x} \to \boxed{f} \to \boxed{y}$

Inverse functions $f^{-1}(x)$. the inverse function of $f(x)$

$f \circ f^{-1}(y) = y$
$f^{-1} \circ f(x) = x$

$\boxed{x} \longrightarrow \boxed{f} \longrightarrow \boxed{y}$

$\boxed{x} \longleftarrow \boxed{f^{-1}} \longleftarrow \boxed{y}$

$\boxed{x} \to \boxed{f} \to \boxed{y} \to \boxed{f^{-1}} \to \boxed{x}$

We normally indicate the inverse function by $f^{-1}(x)$. Note that $f^{-1}(x) \neq \frac{1}{f(x)}$.

$f^2(x) \neq [f(x)]^2$.

EXAMPLE: $f(x) = 2x + 1$, what is $f^{-1}(x)$?

swap $f(x)$ & x, we have $x = 2f(x) + 1$.

change $f(x)$ to $f^{-1}(x)$. then we have $x = 2f^{-1}(x) + 1$.

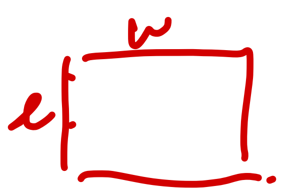
$\Rightarrow f^{-1}(x) = (x-1)/2$.

Multivariate functions

Functions can have multiple variables (inputs).

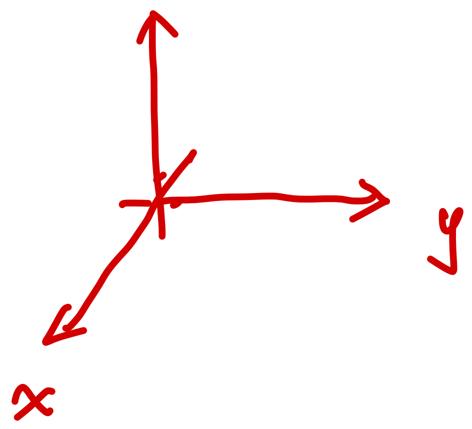
- area of a circle $a(r) = \pi r^2$. univariate.

area of rectangle $a(w, l) = wl$.



EXAMPLE: $f(\cancel{x}) = x^2 + 2xy - y^2$.

$f(x, y)$.



Domains and ranges

$f(x) = \sqrt{x}$. $x \geq 0$.

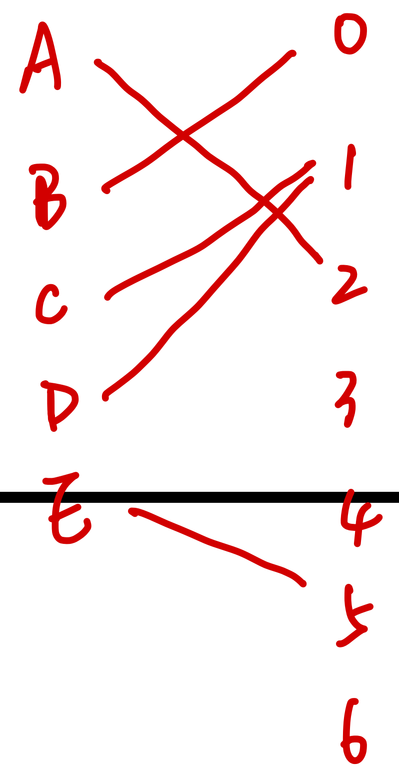
Domain: the values the input can take;
Range: the values the output can take, depending on the domain;
Codomain: the possible values the output can take.

EXAMPLE: A survey collected following data:

- Ann has 2 children;
- Ben has no children;
- Colin has 1 child;
- Dove has 1 child;
- Eve has 5 children.

5 people $\{A, B, C, D, E\}$.

possible no. of children $\{0, 1, 2, 3, 4, 5, 6\}$.



domain: $\{A, B, C, D, E\}$.

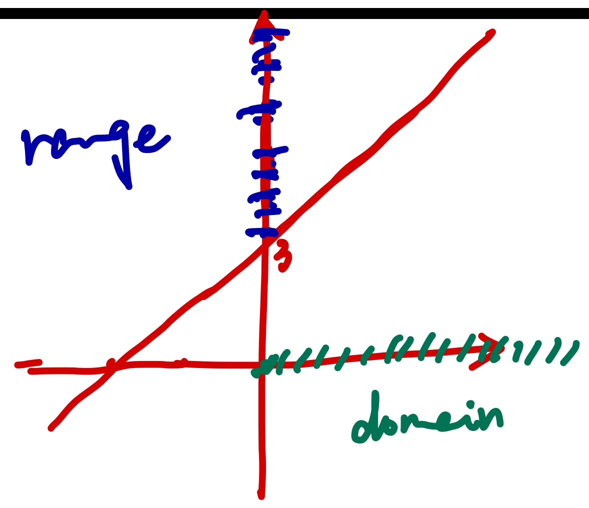
Range : $\{0, 1, 2, 5\}$.

Codomain: $\{0, 1, 2, \ldots 6\}$.

EXAMPLE: $f(x) = x + 3$, $x \geq 0$

domain $x \geq 0$ or $x \in [0, \infty)$

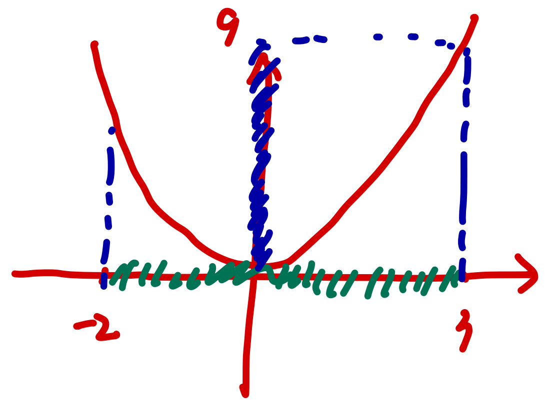
range $f(x) \geq 3$ or $f(x) \in [3, \infty)$



EXAMPLE: $g(x) = x^2$, $-2 \leq x \leq 3$

domain : $-2 \leq x \leq 3$ $\quad x \in [-2, 3]$

range : $0 \leq f(x) \leq 9$ $\quad f(x) \in [0, 9]$



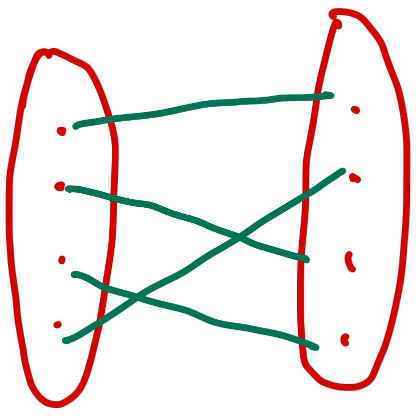
For uni-variate functions:

- the domain is the possible values the x-axis can take;
- the range is the possible values on the y-axis (given the domain).

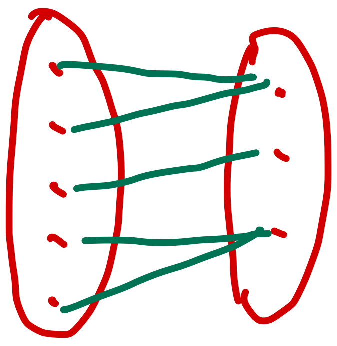
Mapping types and no functions

no $f^{-1}(x)$

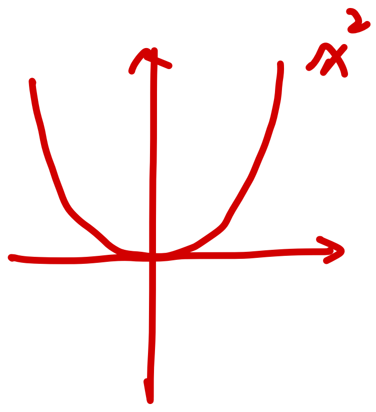
One-to-one



Many-to-one



e.g.

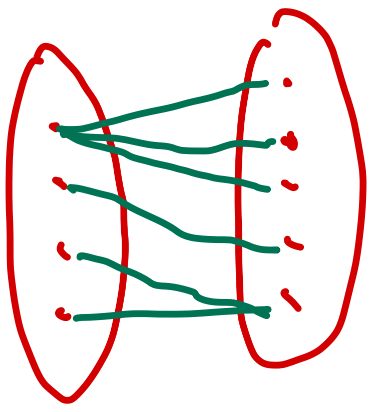


$f(2) = 4$ what is $f^{-1}(4)$?

$= -2$ or 2 ✗

not a function.

One-to-many (not a function!)

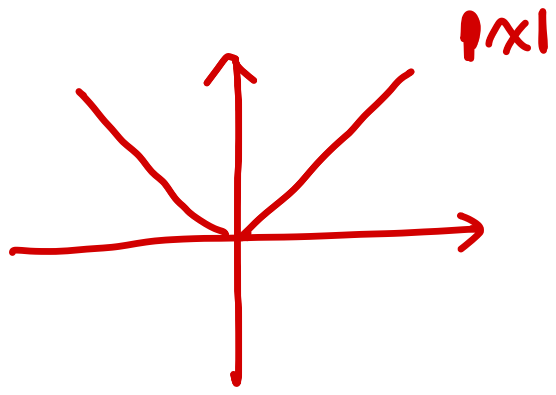


Piecewise functions

A piecewise function is a function that is defined on a sequence of intervals.

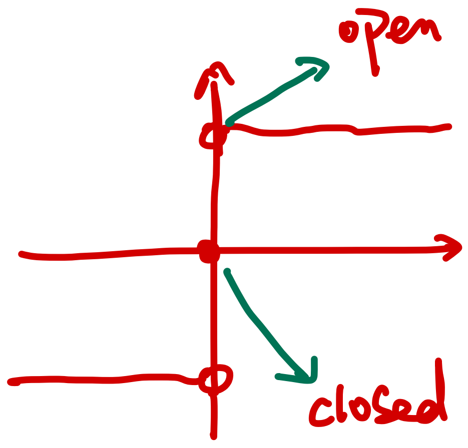
A common example is the absolute value function:

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



Or the signum function:

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

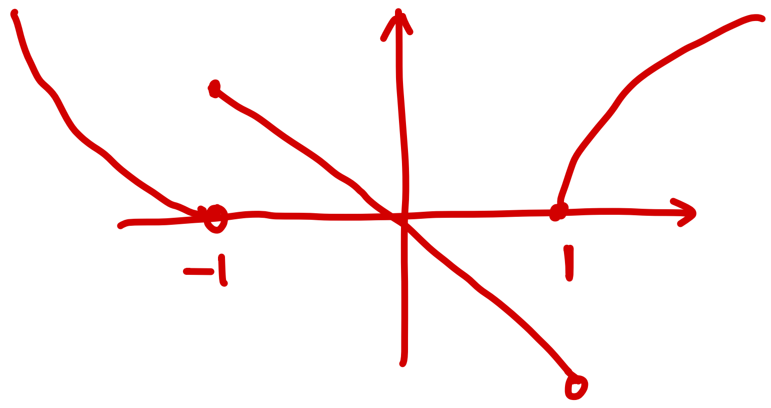


open dot: as close to 0 as possible but not at 0

closed dot: this value is used at precisely this x value

Or anything else having exactly one output at every node in the domain:

$$f(x) = \begin{cases} (x+1)^2 & x < -1 \\ -x & -1 \le x < 1 \\ \sqrt{x-1} & x \geqslant 1 \end{cases}$$



Alternatively with the interval notation:

$$f(x) = \begin{cases} (x+1)^2 & x \in (-\infty, -1) \\ -x & x \in [-1, 1) \\ \sqrt{x-1} & x \in [1, \infty). \end{cases}$$

- Square bracket - closed interval - including the boundary points [ ]
- Circle bracket - open interval - excluding the boundary points ( )