# CSC413: Homework 2

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## 1 Optimization

### 1.1 Stochastic Gradient Descent (SGD)

#### 1.1.1 Minimum Norm Solution

*Answer.* Recall from the previous homework that the solution found by gradient descent in the over-parameterized situation was

$$\mathbf{w}^* = X^T(XX^T)^{-1}\mathbf{t} \tag{1.1}$$

Moreover, the solution $\mathbf{w}^*$ reached by gradient descent was in the row space of $X$ (i.e., the span of rows of $X$), and $\mathbf{w}^*$ is a zero loss solution.
For one single updating using $\mathbf{x}_i$ in the $t^{th}$ iteration in the SGD process, the gradient is

$$\begin{aligned}
\nabla_{\mathbf{w}_t}\mathcal{L}_i &= \frac{\partial}{\partial \mathbf{w}_t}\left\|\mathbf{w}_t^T\mathbf{x}_i - t_i\right\|_2^2 && (1.2)\\
&= \underbrace{2(\mathbf{w}_t^T\mathbf{x}_i - t_i)}_{\in\mathbb{R}}\mathbf{x}_i && (1.3)\\
\implies -\eta\nabla_{\mathbf{w}_t}\mathcal{L}_i &\in Row(X) && (1.4)
\end{aligned}$$

Provided that the starting point $\mathbf{w}_0 = \mathbf{0} \in Row(X)$, for every $t$, $\mathbf{w}_t \in Row(X)$ by an inductive argument.
Because $\mathbf{w}_t \in Row(X)$, let

$$\begin{aligned}
\mathbf{w}_t &= X^T\mathbf{r}_t \quad \mathbf{r}_t \in \mathbb{R}^n && (1.5)\\
\hat{\mathbf{w}} &= X^T\hat{\mathbf{r}} \quad \hat{\mathbf{r}} \in \mathbb{R}^n && (1.6)
\end{aligned}$$

Note that, if $\mathbf{x}_i$ is chosen for the $t^{th}$ iteration of SGD, then only the $i^{th}$ components of $\mathbf{r}_t$ and $\mathbf{r}_{t+1}$ are different, and all other components are the same.
Suppose the solution $\hat{\mathbf{w}}$ reached by SGD is a zero loss solution, that is, $X\hat{\mathbf{w}} = \mathbf{t}$. Then

$$XX^T\hat{\mathbf{r}} = \mathbf{t} \tag{1.7}$$

Because $\hat{\mathbf{w}}$ is a global minimum and SGD converges to this point, then it must be the case that

$$\nabla_{\hat{\mathbf{w}}}\mathcal{L}_i(\mathbf{x}_i, \hat{\mathbf{w}}) = 0 \quad \forall i \in \{1, \cdots, n\} \tag{1.8}$$

That is, one additional iteration of SGD does not improve performance no matter which sample is chosen. This is the same as

$$
\begin{aligned}
& \nabla_{r_i} \left\| XX^T \hat{\mathbf{r}} - \mathbf{t} \right\|_2^2 = 0 \quad \forall i \in \{1, \cdots, n\} && (1.9) \\
\iff & \nabla_{\mathbf{r}} \left\| XX^T \hat{\mathbf{r}} - \mathbf{t} \right\|_2^2 = 0 && (1.10) \\
\implies & (XX^T \hat{\mathbf{r}} - \mathbf{t})^T XX^T = 0 && (1.11) \\
\implies & XX^T (XX^T \hat{\mathbf{r}} - \mathbf{t}) = 0^T && (1.12) \\
\implies & XX^T XX^T \hat{\mathbf{r}} = XX^T \mathbf{t} && (1.13) \\
\implies & \hat{\mathbf{r}} = (XX^T)^{-1} \mathbf{t} && (1.14) \\
\implies & \hat{w} = X^T \hat{\mathbf{r}} = X^T (XX^T)^{-1} \mathbf{t} && (1.15) \\
& = \mathbf{w}^* && (1.16)
\end{aligned}
$$

■

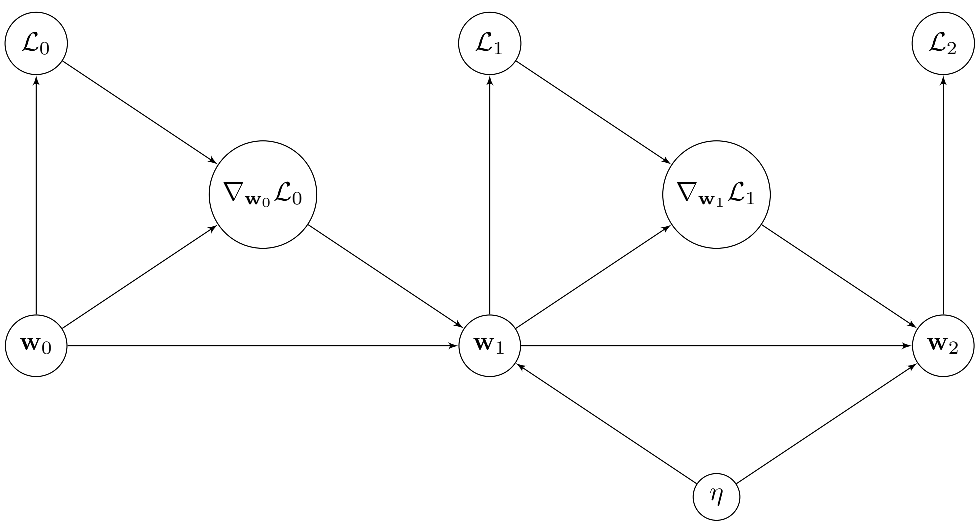
#### 1.1.2 Mini-batch SGD (Optional)

*Answer.* ■

# 2 Gradient-Based Hyper-Parameter Optimization

## 2.1 Computation Graph of Learning Rates

### 2.1.1




### 2.1.2

*Answer.*
(**Forward-Propagation**) For each iteration $t$, the prediction $X\hat{\mathbf{w}} \in \mathbb{R}^n$ and label $\mathbf{t}$ take memory of size $2t$, the loss takes 1 unit of memory. Therefore, the overall memory complexity is $\mathcal{O}(d)$.

(**Back-Propagation**) $\nabla_\eta \mathcal{L}_t$ needs to be computed using chain rule through $\mathbf{w}_\tau$ for every $\tau \leq t$. Note that for every $\tau$, $\nabla_{\mathbf{w}_\tau}\mathcal{L}_\tau \in \mathbb{R}^d$ and $\frac{d}{d\eta}\mathbf{w}_\tau \in \mathbb{R}^d$. The memory complexity is $\mathcal{O}(dt)$. ∎

#### 2.1.3

*Answer.* The memory complexity analysis in the previous question suggests the required memory of tracing gradient with respect to $\eta$ grows linearly in the number of iterations, the memory cost can be prohibitive. ∎

## 2.2 Learning Learning Rates

#### 2.2.1

*Answer.*

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{w}_0 - \eta \nabla_{\mathbf{w}_0}\mathcal{L}_0 && (2.1)\\
&= \mathbf{w}_0 - \frac{2\eta}{n}X^T(X\mathbf{w}_0 - \mathbf{t}) && (2.2)\\
\mathcal{L}_1 &= \frac{1}{n}\left\| X\left[\mathbf{w}_0 - \frac{2\eta}{n}X^T(X\mathbf{w}_0 - \mathbf{t})\right] - \mathbf{t}\right\|_2^2 && (2.3)\\
&= \frac{1}{n}\left\| X\left[\mathbf{w}_0 - \frac{2\eta}{n}X^T\mathbf{a}\right] - \mathbf{t}\right\|_2^2 && (2.4)\\
&= \frac{1}{n}\left\| X\mathbf{w}_0 - \frac{2\eta}{n}XX^T\mathbf{a} - \mathbf{t}\right\|_2^2 && (2.5)\\
&= \frac{1}{n}\left\| \mathbf{a} - \frac{2\eta}{n}XX^T\mathbf{a}\right\|_2^2 && (2.6)
\end{aligned}$$

∎

#### 2.2.2

*Answer.* Note that the an arbitrary norm is convex: let $\lambda \in [0, 1]$

$$\begin{aligned}
\|\lambda x + (1-\lambda)y\|_p &\leq \|\lambda x\|_p + \|(1-\lambda)y\|_p \ \text{(triangle inequality)} && (2.7)\\
&= |\lambda|\,\|x\|_p + |1-\lambda|\,\|y\|_p && (2.8)\\
&= \lambda\,\|x\|_p + (1-\lambda)\,\|y\|_p && (2.9)
\end{aligned}$$

And $f(x) = x^2$ is strictly convex as well. Moreover, $\mathbf{a} - \frac{2\eta}{n}XX^T\mathbf{a}$ is linear in $\eta$. $\mathcal{L}_1(\eta)$ is a composite of linear, convex and strictly convex functions. Therefore, $\mathcal{L}_1(\eta)$ is strictly convex with respect to $\eta$. ∎

### 2.2.3

*Answer.* The derivative of $\mathcal{L}_1$ w.r.t. $\eta$ is

$$\nabla_\eta \mathcal{L}_1 = \nabla_\eta \frac{1}{n} \left\| \mathbf{a} - \frac{2\eta}{n} X X^T \mathbf{a} \right\|_2^2 \tag{2.10}$$

$$= \frac{2}{n} \left( \mathbf{a} - \frac{2\eta}{n} X X^T \mathbf{a} \right)^T X X^T \mathbf{a} \tag{2.11}$$

$$= \frac{2}{n} \left( \mathbf{a}^T - \frac{2\eta}{n} \mathbf{a}^T X X^T \right) X X^T \mathbf{a} \tag{2.12}$$

$$= \frac{2}{n} \left( \mathbf{a}^T X X^T \mathbf{a} - \frac{2\eta}{n} \mathbf{a}^T X X^T X X^T \mathbf{a} \right) \tag{2.13}$$

Set derivative to zero and solve for the optimal learning rate $\eta^*$:

$$\nabla_\eta \mathcal{L}_1 = 0 \tag{2.14}$$

$$\implies \frac{2}{n} \left( \mathbf{a}^T X X^T \mathbf{a} - \frac{2\eta}{n} \mathbf{a}^T X X^T X X^T \mathbf{a} \right) = 0 \tag{2.15}$$

$$\implies \left\| X^T \mathbf{a} \right\|_2^2 = \frac{2\eta^*}{n} \left\| X X^T \mathbf{a} \right\|_2^2 \tag{2.16}$$

$$\implies \eta^* = \frac{n \left\| X^T \mathbf{a} \right\|_2^2}{2 \left\| X X^T \mathbf{a} \right\|_2^2} \tag{2.17}$$

■

# 3 Convolutional Neutral Networks

## 3.1 Convolutional Filters

*Answer.*

$$\mathbf{I} * \mathbf{J} = \begin{bmatrix} -1 & 2 & 2 & -2 & 0 \\ -2 & 1 & 0 & 2 & -1 \\ 3 & 0 & 0 & 1 & -1 \\ -2 & 2 & 0 & 2 & -1 \\ 0 & -2 & 3 & -2 & 0 \end{bmatrix} \tag{3.1}$$

This convolutional filter detect edges. ■

## 3.2 Size of ConvNets

*Answer.* The table below presents numbers of parameters in all layers.

| Layer | Dim. In | Dim. Out | Num. Weights | Num. Bias | Total Params. |
|---|---|---|---|---|---|
| Conv3-64 | 112*112*3 | 112*112*64 | 3*3*3*64 | 64 | 1,792 |
| Max Pool | 112*112*64 | 56*56*64 | 0 | 0 | 0 |
| Conv3-128 | 56*56*64 | 56*56*128 | 3*3*64*128 | 128 | 73,856 |
| Max Pool | 56*56*128 | 28*28*128 | 0 | 0 | 0 |
| Conv3-256 | 28*28*128 | 28*28*256 | 3*3*128*256 | 256 | 295,168 |
| Conv3-256 | 28*28*256 | 28*28*256 | 3*3*256*256 | 256 | 590,080 |
| Max Pool | 28*28*256 | 14*14*256 | 0 | 0 | 0 |
| FC-1024 | 14*14*256 | 1024 | 14*14*256*1024 | 1024 | 51,381,248 |
| FC-100 | 1024 | 100 | 1024*100 | 100 | 102,500 |
| Softmax | 100 | 100 | 0 | 0 | 0 |
| Total | | | | | 52,444,644 |

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