CSC413: Homework 3

Tianyu Du (1003801647)

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1 Weight Decay

1.1 Under-parameterized Model [0pt]

Solution. Given

$$\mathcal{J}(\hat{\mathbf{w}}) = \frac{1}{2n}\|X\hat{\mathbf{w}} - \mathbf{t}\|_2^2 + \frac{\lambda}{2}\hat{\mathbf{w}}^T\hat{\mathbf{w}} \tag{1.1}$$
$$\mathcal{J}(\hat{\mathbf{w}}) = \frac{1}{2n}\|X\hat{\mathbf{w}} - \mathbf{t}\|_2^2 + \frac{\lambda}{2}||\hat{\mathbf{w}}||_2^2 \tag{1.2}$$

The gradient descent converges when $\frac{d}{d\hat{\mathbf{w}}}\mathcal{J}(\hat{\mathbf{w}}) = 0$. Altogether with the fact that $\frac{d}{d\mathbf{x}}||\mathbf{x}||_2^2 = 2\mathbf{x}^T$, the training converges if and only if:

$$\frac{d}{d\hat{\mathbf{w}}}\mathcal{J}(\hat{\mathbf{w}}) = \frac{1}{n}(X\hat{\mathbf{w}} - \mathbf{t})^T X + \lambda\hat{\mathbf{w}}^T \tag{1.3}$$
$$= \frac{1}{n}\hat{\mathbf{w}}^T X^T X - \frac{1}{n}\mathbf{t}^T X + \lambda\hat{\mathbf{w}}^T = 0 \quad (\dagger) \tag{1.4}$$

Note that when $d \leq n$, $rank(X) = d$ implies $X^T X$ is invertible. Suppose $X^T X + n\lambda I$ is invertible as well. Therefore,

$$(\dagger) \implies \left(\hat{\mathbf{w}}^T X^T X + n\lambda\hat{\mathbf{w}}^T\right) = \mathbf{t}^T X \tag{1.5}$$
$$\implies \hat{\mathbf{w}}^T \left(X^T X + n\lambda I\right) = \mathbf{t}^T X \tag{1.6}$$
$$\implies \hat{\mathbf{w}}^T = \mathbf{t}^T X \left(X^T X + n\lambda I\right)^{-1} \tag{1.7}$$
$$\implies \hat{\mathbf{w}} = \left(X^T X + n\lambda I\right)^{-1} X^T \mathbf{t} \tag{1.8}$$

■

1.2 Over-parameterized Model

1.2.1 Warmup: Visualizing Weight Decay [1pt]

Solution. Given

$$\mathbf{x}_1 = (2, 1) \text{ and } t_1 = 2. \tag{1.9}$$

From previous homework, the solution of gradient descent without regularization is

$$\mathbf{w}^* = \left(\frac{4}{5}, \frac{2}{5}\right) \tag{1.10}$$

From the previous part, the gradient descent converges if and only if

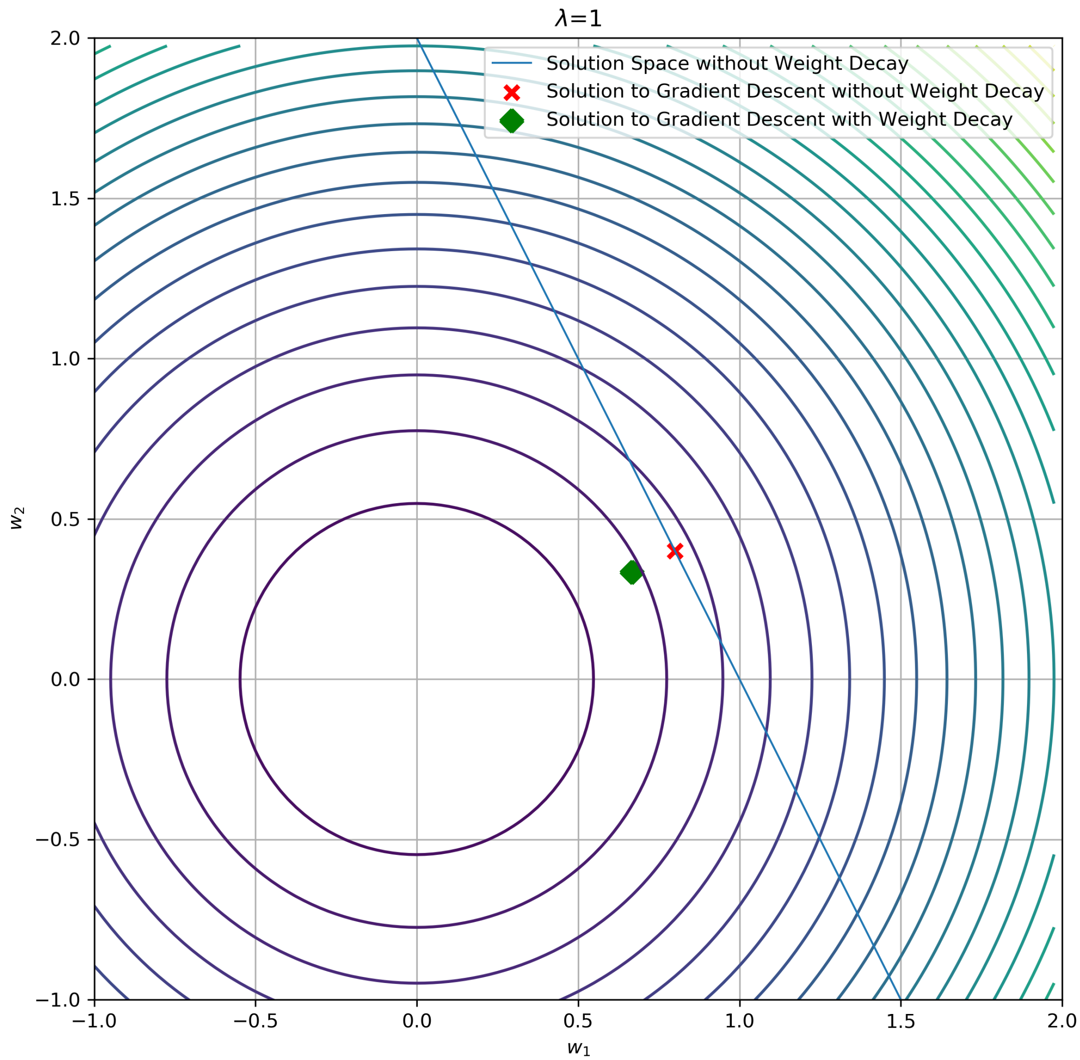
$$\begin{aligned}
\frac{d}{d\hat{\mathbf{w}}}\mathcal{J}(\hat{\mathbf{w}}) &= \frac{1}{n}(X\hat{\mathbf{w}} - \mathbf{t})^T X + \lambda \hat{\mathbf{w}}^T && (1.11)\\
&= ((2,1)\cdot(w_1, w_2) - 2)(2,1) + \lambda(w_1, w_2) && (1.12)\\
&= (2w_1 + w_2 - 2)(2,1) + (\lambda w_1, \lambda w_2) = 0 && (1.13)\\
& && (1.14)
\end{aligned}$$

Therefore, the solution to gradient descent with weight decay is:

$$\begin{cases} 4w_1 + 2w_2 - 4 + \lambda w_1 & = 0 \\ 2w_1 + w_2 - 2 + \lambda w_2 & = 0 \end{cases} \tag{1.15}$$

$$\implies \begin{cases} w_1^* & = \frac{4}{\lambda+5} \\ w_2^* & = \frac{2}{\lambda+5} \end{cases} \tag{1.16}$$

Figure 1.1: Visualization



■

1.2.2 Gradient Descent and Weight Decay [0pt]

Solution. The solution to gradient descent with weight decay at rate λ has been derived in the previous section:

$$\mathbf{w}^{*\lambda}_{\text{weight decay}} = \left(\frac{4}{\lambda+5}, \frac{2}{\lambda+5}\right) \tag{1.17}$$

■

1.3 Adaptive optimizer and Weight Decay [1pt]

Solution. After applying L_2 regularization, the solution found is a scaled down version of the original solution without weight decay for some $\varphi \in \mathbb{R}$ depends on λ: Is this true in general? Show it.

$$\mathbf{w}^{*\lambda}_{\text{weight decay}} = \varphi\ \mathbf{w}^{*}_{\text{without decay}} \tag{1.18}$$

In the previous homework, we've shown that $\mathbf{w}^{*}_{\text{without decay}} \notin Row(x)$. Therefore, $\mathbf{w}^{*\lambda}_{\text{weight decay}} \notin Row(X)$ as well. ■

2 Ensembles and Bias-variance Decomposition

2.1 Weight Average or Prediction Average?

2.1.1 [1pt]

Solution. Without loss of generality, assume the bias is zero. This is equivalent to inserting a column of ones to the X, so that $X \in \mathbb{R}^{n\times(d+1)}$, and we can ignore the bias. Suppose there are K different models indexed using $j \in \{1, 2, \cdots, K\}$: Add bias

$$h_j(\mathbf{x}) = \mathbf{w}_j(\mathcal{D}_j)\mathbf{x} \tag{2.1}$$

where $\mathcal{D}_j$ are i.i.d. realization of datasets. Let $\overline{h}(\mathbf{x})$ denote the weight average ensemble:

$$\overline{h}(\mathbf{x}) = \overline{\mathbf{w}}\mathbf{x} \tag{2.2}$$

$$= \sum_{j=1}^{K} \mathbf{w}_j(\mathcal{D}_j)\mathbf{x} \tag{2.3}$$

And the prediction average at dataset point $\mathbf{x}$is simply

$$\sum_{j=1}^{K} h_j(\mathbf{x}) \tag{2.4}$$

Therefore, the prediction from weight-average and prediction-average ensembles are the same. Hence, the expected generalization error should be the same. ■

2.1.2 [0pt]

Solution. ■

2.2 Bagging - Uncorrelated Models

2.2.1 Bias with bagging [0pt]

Solution. Note that

$$\mathbb{E}\left[\bar{h}(\mathbf{x};\mathcal{D})|\mathbf{x}\right] = \mathbb{E}\left[\frac{1}{k}\sum_{i=1}^{k} h\left(\mathbf{x};\mathcal{D}_i\right)|\mathbf{x}\right] \tag{2.5}$$

$$= \frac{1}{k}\sum_{i=1}^{k} \mathbb{E}\left[h\left(\mathbf{x};\mathcal{D}_i\right)|\mathbf{x}\right] \tag{2.6}$$

Since $\mathcal{D}_i$ are drawn from the identical distribution, so that

$$\mathbb{E}\left[h\left(\mathbf{x};\mathcal{D}_i\right)|\mathbf{x}\right]=\mathbb{E}\left[h\left(\mathbf{x};\mathcal{D}_j\right)|\mathbf{x}\right]\quad\forall i,j\in\{1,2,\cdots,k\} \tag{2.7}$$

Hence,

$$\mathbb{E}\left[\bar{h}(\mathbf{x};\mathcal{D})|\mathbf{x}\right]=\frac{1}{k}\sum_{i=1}^{k}\mathbb{E}\left[h\left(\mathbf{x};\mathcal{D}_i\right)|\mathbf{x}\right]=\mathbb{E}\left[h\left(\mathbf{x};\mathcal{D}_i\right)|\mathbf{x}\right]\quad\forall i\in\{1,2,\cdots,k\} \tag{2.8}$$

Since each data point in $\mathcal{D}_i$ is uniformly sampled with replaced from $\mathcal{D}\sim p_{\text{data}}$, therefore, $\mathcal{D}_i\sim p_{\text{data}}$ as well.

$$\mathbb{E}\left[h\left(\mathbf{x};\mathcal{D}\right)|\mathbf{x}\right]=\mathbb{E}\left[h\left(\mathbf{x};\mathcal{D}_i\right)|\mathbf{x}\right]\quad\forall i\in\{1,2,\cdots,k\} \tag{2.9}$$

Therefore,

$$bias=\mathbb{E}\left[\left|\mathbb{E}\left[\bar{h}(\mathbf{x};\mathcal{D})|\mathbf{x}\right]-y_*(\mathbf{x})\right|^2\right]=\mathbb{E}\left[\left|\mathbb{E}\left[h(\mathbf{x};\mathcal{D})|\mathbf{x}\right]-y_*(\mathbf{x})\right|^2\right] \tag{2.10}$$

■

2.2.2 Variance with bagging [1pt]

Solution. Suppose

$$\mathbb{E}\left[\left|h(\mathbf{x};\mathcal{D})-\mathbb{E}\left[h(x;\mathcal{D})|\mathbf{x}\right]\right|^2\right]=\sigma^2 \tag{2.11}$$

For the bagging model,

$$\begin{aligned}
Var(\overline{h})&=\mathbb{E}\left[\left|\overline{h}(\mathbf{x};\mathcal{D})-\mathbb{E}\left[\overline{h}(x;\mathcal{D})|\mathbf{x}\right]\right|^2\right] &(2.12)\\
&=\mathbb{E}\left[\left|\frac{1}{k}\sum_{i=1}^{k}h\left(\mathbf{x};\mathcal{D}_i\right)-\mathbb{E}\left[\overline{h}(x;\mathcal{D})|\mathbf{x}\right]\right|^2\right] &(2.13)\\
&=\mathbb{E}\left[\left(\frac{1}{k}\sum_{i=1}^{k}h\left(\mathbf{x};\mathcal{D}_i\right)-\mathbb{E}\left[\overline{h}(x;\mathcal{D})|\mathbf{x}\right]\right)^2\right] &(2.14)
\end{aligned}$$

Since $\mathbb{E}\left[\overline{h}(x;\mathcal{D})|\mathbf{x}\right]$ is constant for all realizations of datasets $\mathcal{D}_i$ and equals $\mathbb{E}\left[h(x;\mathcal{D})|\mathbf{x}\right]$ by linearity of expectation.

$$\begin{aligned}
&=\mathbb{E}\left[\left(\frac{1}{k}\sum_{i=1}^{k}\left\{h\left(\mathbf{x};\mathcal{D}_i\right)-\mathbb{E}\left[h(x;\mathcal{D}_i)|\mathbf{x}\right]\right\}\right)^2\right] &(2.15)\\
&=\frac{1}{k^2}\mathbb{E}\left[\left(\sum_{i=1}^{k}\left\{h\left(\mathbf{x};\mathcal{D}_i\right)-\mathbb{E}\left[h(x;\mathcal{D}_i)|\mathbf{x}\right]\right\}\right)^2\right]\quad(\dagger) &(2.16)
\end{aligned}$$

Because datasets $\mathcal{D}_i$ are drawn independently,

$$\mathbb{E}\left[\left(h\left(\mathbf{x};\mathcal{D}_i\right)-\mathbb{E}\left[h(x;\mathcal{D}_i)|\mathbf{x}\right]\right)\left(h\left(\mathbf{x};\mathcal{D}_j\right)-\mathbb{E}\left[h(x;\mathcal{D}_j)|\mathbf{x}\right]\right)\right]=Cov(h_i,h_j)=0 \tag{2.17}$$

Hence, after expanding the squared sum in (†),

$$(\dagger) = \frac{1}{k^2}\mathbb{E}\left[\sum_{i=1}^{k}\left(h\left(\mathbf{x};\mathcal{D}_i\right) - \mathbb{E}\left[h(x;\mathcal{D}_i)|\mathbf{x}\right]\right)^2\right] \tag{2.18}$$

$$= \frac{1}{k^2}\sum_{i=1}^{k}\mathbb{E}\left[\left(h\left(\mathbf{x};\mathcal{D}_i\right) - \mathbb{E}\left[h(x;\mathcal{D}_i)|\mathbf{x}\right]\right)^2\right] \tag{2.19}$$

$$= \frac{1}{k^2}\sum_{i=1}^{k}Var(h_i) \tag{2.20}$$

Since $\mathcal{D}_i$ are i.i.d. from the dataset, $Var(h_i) = Var(h)$ for every i, therefore,

$$Var(\bar{h}) = \frac{\sigma^2}{k} \tag{2.21}$$

■

2.3 Bagging - General Case

2.3.1 Bias under Correlation [1pt]

Solution. The bias does not change and is independent from ρ. While deriving the bias, we firstly exchanged the expectation and summation using the linearity of summation operator:

$$\mathbb{E}\left[\bar{h}(\mathbf{x};\mathcal{D})|\mathbf{x}\right] = \mathbb{E}\left[\frac{1}{k}\sum_{i=1}^{k}h\left(\mathbf{x};\mathcal{D}_i\right)|\mathbf{x}\right] \tag{2.22}$$

$$= \frac{1}{k}\sum_{i=1}^{k}\mathbb{E}\left[h\left(\mathbf{x};\mathcal{D}_i\right)|\mathbf{x}\right] \tag{2.23}$$

The linearity of expectation holds regardless of the correlation. Then we used the fact that $\mathcal{D}_i$ are identically distributed to show

$$\frac{1}{k}\sum_{i=1}^{k}\mathbb{E}\left[h\left(\mathbf{x};\mathcal{D}_i\right)|\mathbf{x}\right] = \mathbb{E}\left[h\left(\mathbf{x};\mathcal{D}\right)|\mathbf{x}\right] \tag{2.24}$$

this step did not assume independent distribution. The entire proof (please refer to 2.2.1 for a more detailed derivation) did not use any assumption on distributional independence, hence the original proof is still valid in the general case. ■

2.3.2 Variance under Correlation [0pt]

Solution.

$$
\begin{aligned}
Var(\bar{h}) &\equiv Cov\left(h(\mathbf{x};\mathcal{D}_i), h(\mathbf{x};\mathcal{D}_i)\right) && (2.25)\\
&= Cov\left(\frac{1}{k}\sum_{i=1}^{k} h\left(\mathbf{x};\mathcal{D}_i\right), \frac{1}{k}\sum_{i=1}^{k} h\left(\mathbf{x};\mathcal{D}_i\right)\right) && (2.26)\\
&= \frac{1}{k^2} Cov\left(\sum_{i=1}^{k} h\left(\mathbf{x};\mathcal{D}_i\right), \sum_{i=1}^{k} h\left(\mathbf{x};\mathcal{D}_i\right)\right) && (2.27)\\
&= \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k} Cov\left(h\left(\mathbf{x};\mathcal{D}_j\right), h\left(\mathbf{x};\mathcal{D}_j\right)\right) && (2.28)\\
&= \frac{1}{k^2}\sum_{i=1}^{k}\sum_{j=1}^{k} Cov\left(h\left(\mathbf{x};\mathcal{D}_j\right), h\left(\mathbf{x};\mathcal{D}_j\right)\right) && (2.29)\\
&= \frac{1}{k^2}(k\sigma^2 + (k^2-k)\rho\sigma^2) && (2.30)\\
&= \left(\frac{1}{k} + \rho - \frac{\rho}{k}\right)\sigma^2 && (2.31)\\
&= \left(\rho + \frac{1-\rho}{k}\right)\sigma^2 && (2.32)
\end{aligned}
$$

■

2.3.3 Intuitions on Bagging [1pt]

Proof. Solution When $\rho = 1$, that is, all bootstrapped datasets are perfectly correlated. In fact, all datasets are identical, the variance is independent from k, and increasing number of estimators, k, does not help reduce the variance.

However, For any $\rho < 1$, increasing number of estimators in the bagging, k, helps reduce the variance. In particular, when $\rho = 0$, which is the uncorrelated dataset case, the effect is most significant: the variance shrinks linearly in k. ■

3 Generalization and Dropout

3.1 Regression Coefficients

3.1.1 Regression from X_1 [0pt]

Solution. ■

3.1.2 Regression from X_2 [1pt]

Solution. Since we are using X_2 only, equivalently, we can set the weight of X_1 to zero:

$$\begin{aligned}
\mathcal{J} &= \mathbb{E}_{(x_2,y)\sim(X_2,Y)}[(y-\hat{y})^2] && (3.1)\\
&= \mathbb{E}_{(x_2,y)\sim(X_2,Y)}[(y-w_2x_2)^2] && (3.2)\\
&= \mathbb{E}_{(x_2,y)\sim(X_2,Y)}[(y-w_2(y+Gaussian(0,1)))^2] && (3.3)\\
&= \mathbb{E}_{(x_2,y)\sim(X_2,Y)}[((1-w_2)y-w_2Gaussian(0,1))^2] && (3.4)\\
&= \mathbb{E}_{(x_2,y)\sim(X_2,Y)}[((1-w_2)y)^2]+w_2^2\mathbb{E}_{(x_2,y)\sim(X_2,Y)}[Gaussian(0,1)^2] && (3.5)\\
&= (1-w_2)^2\mathbb{E}_{y\sim Y}[y^2]+w_2^2 && (3.6)
\end{aligned}$$

Taking the gradient and solve the first order condition:

$$\begin{aligned}
&\nabla_{w_2}(1-w_2)^2\mathbb{E}_{y\sim Y}[y^2]+w_2^2=0 && (3.7)\\
&\implies -2(1-w_2)\mathbb{E}_{y\sim Y}[y^2]+2w_2=0 && (3.8)\\
&\implies \mathbb{E}_{y\sim Y}[y^2]-w_2\mathbb{E}_{y\sim Y}[y^2]-w_2=0 && (3.9)\\
&\implies \mathbb{E}_{y\sim Y}[y^2]+w_2(1-\mathbb{E}_{y\sim Y}[y^2])=0 && (3.10)\\
&\implies w_2=\frac{\mathbb{E}_{y\sim Y}[y^2]}{\mathbb{E}_{y\sim Y}[y^2]+1} && (3.11)
\end{aligned}$$

The expectation of y^2 is

$$\begin{aligned}
\mathbb{E}_{y\sim Y}[y^2] &= \mathbb{E}_{x_1\sim X_1}(x_1+Gaussian(0,\sigma^2))^2 && (3.12)\\
&= 2\sigma^2 && (3.13)\\
\implies w_2 &= \frac{2\sigma^2}{2\sigma^2+1} && (3.14)
\end{aligned}$$

■

3.1.3 Regression from (X_1, X_2) [1pt]

Solution. Let G_1, G_2, G_3 denote the three Gaussian distributions respectively, so that

$$\begin{aligned}
X_1 &\leftarrow G_1 && (3.15)\\
Y &\leftarrow X_1+G_2 && (3.16)\\
X_2 &\leftarrow Y+G_3 && (3.17)
\end{aligned}$$

So that,

$$\begin{aligned}
\mathcal{J} &= \mathbb{E}_{(x_1,x_2,y)\sim(X_1,X_2,Y)}[(y-\hat{y})^2] && (3.18)\\
&= \mathbb{E}[G_1+G_2-w_1G1-w_2(G_1+G_2+G_3)]^2 && (3.19)\\
&= \mathbb{E}[(1-w_1-w_2)G_1+(1-w_2)G_2-w_2G_3]^2 && (3.20)\\
&= (1-w_1-w_2)^2\sigma^2+(1-w_2)^2\sigma^2+w_2^2 && (3.21)
\end{aligned}$$

For w_1:

$$\frac{\partial}{\partial w_1}\mathcal{J} = -2(1 - w_1 - w_2)\sigma^2 = 0 \tag{3.22}$$

For w_2:

$$\frac{\partial}{\partial w_2}\mathcal{J} = -2(1 - w_1 - w_2)\sigma^2 - 2(1 - w_2)\sigma^2 + 2w_2 = 0 \tag{3.23}$$

Solving two equations:

$$w_1 = \frac{1}{\sigma^2 + 1} \tag{3.24}$$

$$w_2 = \frac{\sigma^2}{\sigma^2 + 1} \tag{3.25}$$

■

3.1.4 Different σs [0pt]

Solution. ■

3.2 Dropout as Data-Dependent $L2$ Regularization

3.2.1 Expectation and variance of predictions [0pt]

Solution. Let

$$\tilde{y} = 2\left(m_1 w_1 x_1 + m_2 w_2 x_2\right) \tag{3.26}$$

Then

$$\mathbb{E}\left[\tilde{y}\right] = \mathbb{E}\left[2\left(m_1 w_1 x_1 + m_2 w_2 x_2\right)\right] \tag{3.27}$$

■

3.3 Effect on Dropout [1pt]

Solution. Using bias-variance decomposition of the generalization error while assuming zero irreducible error:

$$\begin{aligned}
\mathbb{E}[\tilde{\mathcal{L}}] &= \mathbb{E}[(\tilde{y} - y)^2] && (3.28)\\
&= \mathbb{E}\left[(\mathbb{E}_m[\tilde{y}] - y)^2\right] + Var(\hat{y}) && (3.29)\\
&= \mathbb{E}\left[(\hat{y} - y)^2\right] + Var(2\left(m_1 w_1 x_1 + m_2 w_2 x_2\right)) && (3.30)\\
&= \mathbb{E}\left[(\hat{y} - y)^2\right] + 4Var\left(m_1 w_1 x_1 + m_2 w_2 x_2\right) && (3.31)\\
&= \mathbb{E}\left[(\hat{y} - y)^2\right] + 4Var\left(m_1 w_1 x_1\right) + 4Var\left(m_2 w_2 x_2\right) && (3.32)\\
&= \mathbb{E}\left[(\hat{y} - y)^2\right] + Var(x_1)w_1^2 + Var(x_2)w_2^2 && (3.33)\\
&= \mathbb{E}\left[(\hat{y} - y)^2\right] + \sigma^2 w_1^2 + (2\sigma^2 + 1)w_2^2 && (3.34)
\end{aligned}$$

Therefore, adding the dropout is equivalent to adding a regularization term in which the level of plenty (λ_j) for each w_j depends on the variance of x_j. In this case, w_1 is more regularized. And such regularization would help the model achieve a better generalization error. ■

4 Hard-Coding Recurrent Neural Networks [1pt]

Solution. Let $\sigma = \frac{1}{1+\exp(-z)}$, and $\mathbf{x}_t = (x_1^t, x_2^t)$ denotes the input feature at time t. Note that when weights are sufficient large in scale, σ behaves like hard threshold function. Consider the following recurrent network:

$$\hat{y}_t = \sigma(\mathbf{w}_{hy}\mathbf{h}_t + b_y) \tag{4.1}$$

$$\mathbf{h}_t = \sigma\left(\mathbf{w}_{xh}\mathbf{x}_t + \mathbf{w}_{hh}\mathbf{h}_{t-1} + \mathbf{b}_h\right) \tag{4.2}$$

with the following parameters:

$$\mathbf{w}_{xh} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \mathbf{w}_{hh} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathbf{b}_h = \begin{pmatrix} -0.5 \\ -1.5 \\ -2.5 \end{pmatrix} \tag{4.3}$$

$$\mathbf{w}_{hy} = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \quad b_y = -0.5 \tag{4.4}$$

$$\mathbf{h}_t = \begin{pmatrix} \mathbb{1}\{x_1^t + x_2^t + h_{t-1} \geq 1\} \\ \mathbb{1}\{x_1^t + x_2^t + h_{t-1} \geq 2\} \\ \mathbb{1}\{x_1^t + x_2^t + h_{t-1} \geq 3\} \end{pmatrix} \tag{4.5}$$

Justification:

$$\mathbf{w}_{xh}\mathbf{x}_t = \begin{pmatrix} x_1^t + x_2^t \\ x_1^t + x_2^t \\ x_1^t + x_2^t \end{pmatrix} \quad \mathbf{w}_{hh}\mathbf{h}_{t-1} = \begin{pmatrix} \mathbb{1}\{x_1^{t-1} + x_2^{t-1} + h_{t-2} \geq 2\} \\ \mathbb{1}\{x_1^{t-1} + x_2^{t-1} + h_{t-2} \geq 2\} \\ \mathbb{1}\{x_1^{t-1} + x_2^{t-1} + h_{t-2} \geq 2\} \end{pmatrix} \tag{4.6}$$

Let c_t denote the carry from the previous significant figure. Therefore, elements in $\mathbf{w}_{hh}\mathbf{h}_{t-1}$ are one only if $c_t = 1$. Then,

$$\mathbf{h}_t = \sigma\left(\mathbf{w}_{xh}\mathbf{x}_t + \mathbf{w}_{hh}\mathbf{h}_{t-1} + \mathbf{b}_h\right) = \begin{pmatrix} \mathbb{1}\{x_1^t + x_2^t + c_t \geq 1\} \\ \mathbb{1}\{x_1^t + x_2^t + c_t \geq 2\} \\ \mathbb{1}\{x_1^t + x_2^t + c_t \geq 3\} \end{pmatrix} \tag{4.7}$$

For the output layer,

$$\hat{y}_t = \sigma\left(\mathbf{w}_{xh}\mathbf{x}_t + \mathbf{w}_{hh}\mathbf{h}_{t-1} + \mathbf{b}_h\right) = \mathbb{1}\{x_1^t + x_2^t + c_t \geq 1\} \vee \mathbb{1}\{x_1^t + x_2^t + c_t \geq 3\} \tag{4.8}$$

Therefore, let $c \in \{0, 1\}$ denote the carry, $\hat{y}$ whenever $x_1 + x_2 + c$ is one or three, and $\hat{y} = 0$ otherwise. ■