Assignment 01

Multivariable Calculus

Sumbitted To :- Sir Kazim Usman

Sumbitted by : Fatima Batool
FA21- BEE-049

Question No: 1

The points A, B, C have position vectors.

$4\hat{i}+4\hat{j}+\hat{k}$, $-4\hat{i}+3\hat{j}-4\hat{k}$, $4\hat{i}-\hat{j}-2\hat{k}$
respectively, relative to the origin O.

a) Find the equation of the plan ABC, giving your answers in the form $ax+by+cz=d$.

sol/

we have $A(4,-4,1)$
$B(-4,3,-4)$
$C(4,-1,-2)$

$$\vec{AB} = \vec{OB} - \vec{OA}$$

$$= \begin{bmatrix} -4 \\ 3 \\ -4 \end{bmatrix} - \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 7 \\ -5 \end{bmatrix}$$

$$= -8\hat{i}+7\hat{j}-5\hat{k}$$

$$\vec{AC} = \vec{OC} - \vec{OA}$$

$$= \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ +3 \\ -3 \end{bmatrix}$$

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$= 3\hat{j} - 3\hat{k}$

$$n = \vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -8 & 7 & -5 \\ 0 & 3 & -3 \end{vmatrix}$$

$$= i(-4+15) - (24-0)\hat{j} + (-24)\hat{k}$$

$$= -6\hat{i} - 24\hat{j} - 24\hat{k}$$

$$= i + 4\hat{j} + 4\hat{k}$$

$$d = a \cdot n = (4\hat{i} - 4\hat{j} + \hat{k}) \cdot (\hat{i} + 4\hat{j} + 4\hat{k})$$

$$= 4 - 16 + 4 = -8$$

Equation of plane

$$r \cdot n = d$$

$$r(\hat{i} + 4\hat{j} + 4\hat{k}) = -8$$

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + 4\hat{j} + 4\hat{k}) = -8$$

$$x + 4y + 4z = -8$$

$$x + 4y + 4z + 8 = 0$$

b) Find ~~p~~ the perpendicular distance from O to the plane ABC

$$\text{Perp distance} = \frac{d}{|n|} = \frac{+8}{\sqrt{1^2 + 4^2 + 4^2}}$$

$$= \frac{8}{3.3} = 1.39$$

c) The point D has position vector $2\hat{i} + 3\hat{j} - 3\hat{k}$. Find the coordinates of the point of intersection of the line OD with the plane ABC.

Line OD: $r = a \times \lambda b$

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$$r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$$

as some value of λ

$$\begin{pmatrix} 2\lambda \\ 3\lambda \\ -3\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} = -8$$

$$= 2\lambda + \cancel{12\lambda} - \cancel{12\lambda} = -8$$

$$\Rightarrow 2\lambda = -8$$

$$\lambda = -4$$

$$r = \begin{bmatrix} 2(-4) \\ 2(-4) \\ -3(-4) \end{bmatrix} = (-8, -12, 12)$$

Question No :3

Let t be a positive constant. The line L_1 passes through ......

a) Find the value of t.

$L_1 = t\hat{i} + \hat{j}$ $\quad\quad -2\hat{i} - \hat{j}$

$L_2 = \hat{j} + t\hat{k}$ $\quad\quad -2\hat{j} + \hat{k}$

The shortest distance between L_1 and L_2 is $\sqrt{21}$

$r_1 = OA + \lambda AB$

$r_2 = OA + \lambda AB$

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$r_1 = t\hat{i} + \hat{j} + \lambda 2\hat{i} - \hat{j}$

$$L_1 = \vec{r_1} = \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

$$L_2 = \vec{r_2} = \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix} + \mu \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore a_2 - a_1 = \text{normal vector}$

$$D = \frac{(b_1 \times b_2)\cdot(a_2 - a_1)}{|b_1 \times b_2|}$$

$$b_1 \times b_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -1 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -1 & 0 \\ -2 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} -2 & 0 \\ 0 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} -2 & -1 \\ 0 & -2 \end{vmatrix}$$

$$= \hat{i}(-1) - \hat{j}(-2) + \hat{k}(4)$$

$$= -\hat{i} + 2\hat{j} + 4\hat{k}$$

$$|b_1 \times b_2| = \sqrt{(-1)^2 + (2)^2 + (4)^2}$$

$$= \sqrt{1 + 4 + 16}$$

$$= \sqrt{21}$$

$$(a_2 - a_1) = \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix} - \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$$

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$= -t\hat{i} + t\hat{k}$

$D = \frac{(-\hat{i}+2\hat{j}+4\hat{k})\cdot(-t\hat{i}+t\hat{k})}{\sqrt{21}}$

$\sqrt{21} = t + 4t / \sqrt{21}$

$21 = t + 4t$

$21 = 5t$

$\boxed{t = \frac{21}{5}}$

Part b

$\vec{r_1} = \frac{21}{5}\hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j})$

$\vec{r_2} = \hat{j} - \frac{21}{5}\hat{k} + \mu(-2\hat{j} + \hat{k})$

$\pi_1 = r = \vec{OA} + \lambda\vec{AB} + \mu AC$

$\vec{r} = -\frac{21}{5}\hat{i} + \hat{j} + \lambda(-2\hat{i} - \hat{j}) + \mu(-2\hat{j} + \hat{k}$

part c:-

$\lambda_2 = 5x - 6y + 5z = 0$

$\lambda_2 = \frac{x-0}{0}, \lambda_2 = \frac{y-1}{-2}$

$\lambda_2 = \frac{z - 4.2}{1}$

From L_2 direction vector is

$= \{0, -2, 1\}$

From π_2 normal vector is

$= (5, -6, 7)$

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$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|a|\ |b|}$$

$$(a \cdot b) = \begin{bmatrix} 0 \\ 1 \\ \frac{21}{5} \end{bmatrix} \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$$

$$= -6 + 29.4 = 23.4$$

$$|a| = \sqrt{(1)^2 + \left(\frac{21}{5}\right)^2} = \sqrt{1 + 17.64}$$

$$\boxed{|a| = 4.3}$$

$$|b| = \sqrt{(5)^2 + (6)^2 + (7)^2}$$

$$\boxed{|b| = 10.49}$$

$$\theta = \cos^{-1}\left(\frac{23.4}{4.3 \times 10.49}\right)$$

$$\theta = \cos^{-1}\frac{23.4}{45.11}$$

$$\boxed{\theta = 59.34^\circ}$$

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part d

$$\bar{n}_1 = \begin{bmatrix} -21/5 \\ 1 \\ 0 \end{bmatrix}, \bar{n}_2 \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$$

$$\bar{n}_1 \cdot \bar{n}_2 = \begin{bmatrix} -21/5 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$$

$$= -21 - 6 = -27$$

$$|a| = \sqrt{\left(\frac{-21}{5}\right)^2 + (1)^2} = \sqrt{17.64 + 1} = 4.3$$

$$|b| = \sqrt{5^2 + (-6)^2 + 7^2} = 10.49$$

$$\theta = \cos^{-1} \frac{-27}{4.3 \times 10.49}, \quad \theta = 126.78$$

acute angle:- $\phi = 180 - 126.78$

$$\boxed{\phi = 53.23^\circ}$$

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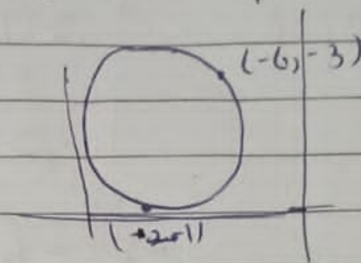
part (a)

If $P = (-2, -1)$ and $Q = (-6, -3)$ are the end point of diameter of a circle, find the equation of the circle.



Mid point

$$\left(\frac{-2-6}{2}\right)\left(\frac{-1-3}{2}\right)$$

$$\frac{-8}{2}, \frac{-4}{2}$$

$$(-4, -2)$$

equation of circle $= (x+h)^2 + (y-k)^2 = r^2$ — (1)

$$(x+4)^2 + (y+2)^2 = r^2$$

$x \quad (x, y) = (-2, -1)$

$$(-2+4)^2 + (-1+2)^2 = r^2$$

$$(2)^2 + (1)^2 = r^2$$

$$4 + 1 = r^2$$

$$5 = r^2$$

$$r^2 = 5$$

put the values

$$(x+4)^2 + (y+2)^2 = 5$$

part (b)

If the circle pass through $(4, 0)$ and $(0, 2)$ and center at y-axis then find the radius of the circle.

sol/

equation of circle

$$(x-h)^2 + (y-k)^2 = r^2$$

let $x = 0,$ $y = b$

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at point $(4, 0)$

$(4)^2 + (-b)^2 = r^2$

$16 + b^2 = r^2$ —— (1)

at point $(0, 2)$

$(0)^2 + (2-b)^2 = r^2$

$(2-b)^2 = r^2$ —— (11)

compare eq (1) and eq (11)

$16 + b^2 = (2-b)^2$

$16 + b^2 = 4 - 4b + b^2$

$16 + \cancel{b^2} - \cancel{b^2} - 4 + 4b = 0$

$12 + 4b = 0$

$4b = -12$

$\boxed{b = -3}$

Put the values in eq (1)

So, $r^2 = (4)^2 + (-3)^2$

$r^2 = 16 + 9$

$r = \pm 5$

~~reo~~ neglect -ve

$\boxed{r = 5}$

part C :-

Find the equation of directrix of parabola $y^2 = 100x$

$y^2 = 100x$

compare with

$y^2 = 4ax$

$4a = 100$

$\boxed{a = 25}$

equation of directrix $= x = -a$

$x = -25$

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Part (d)

Find the equation of the axis of the parabola $x^2 = 24y$

$$x^2 = 24y$$

compare with

$$x^2 = 4ay$$

$$4a = 24$$

$$a = 8$$

So, focus is $F(a, 0) = F(\ , 0)$
and equation of directrix is

$$x = -a$$

$$\boxed{x = -8}$$

part e:-

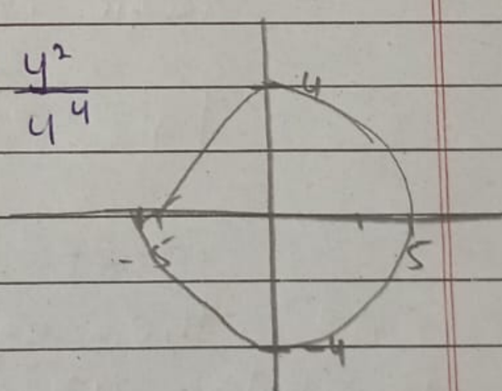
What is the major axis length for ellipse $\left(\frac{x}{25}\right)^2 + \left(\frac{y}{16}\right)^2 = 1$

compare with

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x^2}{5^2} + \frac{y^2}{4^4}$$

$$a = 5$$

$$b = 4$$



$$c = \sqrt{a^2 - b^2} = \sqrt{25 - 16} = \pm 3$$

$$F_1 = (3, 0)$$

$$F_2 = (-3, 0)$$

length of major axis $= 2a$

$$= 2(5) = 10$$

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part (f)

If length of major axis is 10 and minor is 8 and major axis is along x-axis then find the equation of ellipse.

major axis = 10
minor axis = 8

$2a = 10$
$a = 5$
$2b = 8$
$b = 4$

Equation of ellipse along x-axis.

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

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Question No: 2

The position vector of the point

$$A = [7\hat{i} + 4\hat{j} - \hat{k}), \quad B = [11\hat{i} + 3\hat{j})$$

$$C\ [2\hat{i} + 6\hat{j} + 3\hat{k}) \quad D = [2\hat{i} + 7\hat{j} + \lambda\hat{k})$$

a) part a

Given that the shortest distance between the line AB and the line CD is 3 show that $\lambda^2 - 5\lambda + 4 = 0$

$$\vec{r_1} = \overrightarrow{OA} - \overrightarrow{AB}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{bmatrix} 11 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{r_1} = \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

$$r_2 = \overrightarrow{OC} - \overrightarrow{CD}$$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$$

$$= \begin{bmatrix} 2 \\ 7 \\ \lambda \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \lambda - 3 \end{bmatrix}$$

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$$b_1 \times b_2 = \begin{bmatrix} i & j & k \\ 4 & -1 & 1 \\ 0 & 1 & \lambda-3 \end{bmatrix}$$

$$= \hat{i}\begin{vmatrix} -1 & 1 \\ 1 & \lambda-3 \end{vmatrix} - \hat{j}\begin{vmatrix} 4 & 1 \\ 0 & \lambda-3 \end{vmatrix} + \hat{k}\begin{vmatrix} 4 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= \hat{i}\,[-(\lambda-3)-1] - \hat{j}\,[4(\lambda-3) + \hat{k}(4)$$

$$= (2-\lambda)\hat{i} - \hat{j}(4\lambda-12) + 4\hat{k}$$

$$a_2 - a_1 = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix} - \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 4 \end{bmatrix}$$

$$|b_1 \times b_2| = \sqrt{(2-\lambda)^2 + (4\lambda-12)^2 + (4)^2}$$

$$= \sqrt{4+\lambda^2-4\lambda+16\lambda^2+144-9\lambda+16}$$

$$= \sqrt{17\lambda^2 - 100\lambda + 164}$$

$$(b_1 \times b_2)\cdot(a_1 - a_2) = [(2-\lambda)\hat{i} - \hat{j}(4\lambda-12) + 4\hat{k}]\cdot(-5\hat{i} + 2\hat{j} + 4\hat{k})$$

$$= -5(2-\lambda) - 2(4\lambda-12) + 16$$

$$= -10 + 5\lambda - 8\lambda + 24 + 16$$

$$= 30 - 3\lambda$$

$$d = \frac{(b_1 \times b_2)\cdot(a_1 - a_2)}{|b_1 \times b_2|}$$

$$3 = \frac{30 - 3\lambda}{\sqrt{17\lambda^2 - 100\lambda + 164}}$$

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Taking square on both side.

$$9 = \frac{(30 - 3\lambda)^2}{(\sqrt{17\lambda^2 - 100\lambda + 164})^2}$$

$$9 = \frac{900 + 9\lambda^2 - 180\lambda}{17\lambda^2 - 100\lambda + 164}$$

$$9(17\lambda^2 - 100\lambda + 164) = 900 + 9\lambda^2 - 180\lambda)$$

$$153\lambda^2 - 900\lambda + 1476 - 900 - 9\lambda^2 + 180\lambda = 0$$

$$144\lambda^2 - 720\lambda + 576 = 0$$

$$4(36\lambda^2 - 180\lambda + 144) = 0$$

$$36\lambda^2 - 180\lambda + 144 = 0$$

$$4(9\lambda^2 - 45\lambda + 36) = 0$$

$$9\lambda^2 - 45\lambda + 36 = 0$$

$$9(\lambda^2 - 5\lambda + 4) = 0$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda^2 - 4\lambda - \lambda + 4 = 0$$

$$\lambda(\lambda - 4) - 1(\lambda - 4) = 0$$

$$(\lambda - 4)(\lambda - 1) = 0$$

$$\lambda - 4 = 0 \quad | \quad \lambda - 1 = 0$$

$$\boxed{\lambda = 4} \quad | \quad \boxed{\lambda = 1}$$

put $\lambda = 4$ in eq (1)

$$(4)^2 - 5(4) + 4 = 0$$

$$16 - 20 + 4 = 0$$

$$0 = 0$$

put $\lambda = 1$ in eq (1)

$$(1)^2 - 5(1) + 4 = 0$$

$$1 - 5 + 4 = 0$$

$$0 = 0$$

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i)

(ii)

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Part 2

Let π_1 be the plane ABD when $\lambda = 1$
let π_2 be the plane ABD when $\lambda = 4$

i) Write down an equation of π_1 giving your answer in the form
$r = a + sb + tc$

$\lambda)$
$80\lambda = 0$

$$\pi_1 = \bar{r}_1 = \vec{OA} + s\vec{AB} + t\vec{AD}$$

when $\lambda = 1$

$$\vec{AD} = \vec{OD} - \vec{OA}$$
$$= \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}$$

$$\pi_1 = 7\hat{i} + 4\hat{j} - \hat{k} + s(4\hat{i} - \hat{j} + \hat{k}) + $$
$$+ (-5\hat{i} + 3\hat{j} + 2\hat{k})$$

(ii) Find an equation of π_2 giving your answer in the form $ax + by + cz = d$

$$\pi_2 = \vec{r_2} = \vec{OA} + \lambda\vec{AB} + \mu\vec{AD}$$

when $\lambda = 4$

$$\vec{AD} = \vec{OD} - \vec{OA}$$
$$= \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} - \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 5 \end{bmatrix}$$

$$\pi_2 = \vec{r_2} = \vec{OA} + \lambda\vec{AB} + \mu\vec{AD}$$
$$= \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} + \lambda\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} + \mu\begin{bmatrix} -5 \\ 3 \\ 5 \end{bmatrix}$$

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Part C

$\bar{n}_1 = \vec{r_1} = \vec{OA} + \lambda \vec{AB} + \mu \vec{AD}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix}$$

$x = 7 + 4\lambda - 5\mu$ — (I)

$y = 4 - \lambda + 3\mu$ — (II)

$z = -1 + \lambda + 2\mu$ — (III)

eq (II) + eq (III)

$y = 4 - \cancel{\lambda} + 3\mu$

$z = -1 + \cancel{\lambda} + 2\mu$

$y + z = 3 + 5\mu$ — (iv)

Multiply eq (II) by 4

$4y = 16 - 4\lambda + 12$ — (V)

Add eq (I) and eq (V)

$x = 7 + \cancel{4\lambda} - 5\mu$

$4y = 16 - \cancel{4\lambda} + 12$

$x + 4y = 23 + 7\mu$

$\frac{x + 4y - 23}{7} = \mu$

$y + z = 3 + 5\left(\frac{x + 4y - 23}{7}\right)$

$y + z = 3 + \frac{5x + 20y - 115}{7}$

Multiply by both side by 7

$7y + 7z = 21 + 5x + 20y - 115$

$7y + 7z - 21 - 5x - 20y + 115 = 0$

$-5x - 13y + 7z + 94 = 0$

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$x = 7 + 4\lambda - 5\mu$ — (i)

$y = 7 - \lambda + 3\mu$ — (ii)

$z = -1 + \lambda + 5\mu$ — (iii)

eq (ii) + eq (iii)

$y = 4 - \lambda + 3\mu$

$z = -1 + \lambda + 5\mu$

$y + z = 3 + 8\mu$ — (iv)

Multiply eq (ii) by 2

$y = 4 - \lambda + 3\mu$

$\times 4$

$4y = 16 - 4\lambda + 12\mu$ — (v)

eq (i) + eq (v)

$x = 7 + 4\lambda - 5\mu$

$4y = 16 - 4\lambda + 12\mu$

$x + 4y = 23 + 7\mu$

$\frac{x + 4y - 23}{7} = \mu$

put the values in eq (iv)

$y + z = 3 + 8\left(\frac{x + 4y - 23}{7}\right)$

$y + z = 3 + \frac{8x + 32y - 184}{7}$

Multiply both side by 7

$7y + 7z = 21 + 8x + 32y - 184$

$7y + 7z - 21 - 8x - 32y + 184 = 0$

$-8x - 25y + 7z + 163 = 0$

$-(8x + 25y - 7z - 163) = 0$

$8x + 25y - 7z = 163$

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$$-(5x + 13y - 7z - 94) = 0$$

$$5x + 13y - 7z = 94$$

$$\theta = \cos^{-1} \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1||\bar{n}_2|}$$

$$\bar{n}_1 \cdot \bar{n}_2 = \begin{bmatrix} 5 \\ 13 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 25 \\ -7 \end{bmatrix}$$

$$= 40 + 325 + 49 = 414$$

$$|n_1| = \sqrt{5^2 + 13^2 (-7)^2} = 9\sqrt{3}$$

$$|n_2| = \sqrt{8^2 + 25^2 + (-7)^2} = 3\sqrt{82}$$

$$\theta = \cos^{-1}\left(\frac{414}{9\sqrt{3} \times 3\sqrt{82}}\right)$$

$$\theta = \cos^{-1}\left(\frac{414}{27\sqrt{246}}\right)$$

$$\boxed{\theta = 12.15^{\circ}}$$