05 June 2018

ISLAMIC UNIVERSITY OF TECHNOLOGY (IUT)
ORGANISATION OF ISLAMIC COOPERATION (OIC)
Department of Computer Science and Engineering (CSE)

SEMESTER FINAL EXAMINATION **WINTER SEMESTER, 2017-2018**
DURATION: 3 Hours **FULL MARKS: 150**

Math 4707: Probability and Stochastic Processes

Programmable calculators are not allowed. Do not write anything on the question paper.
There are 8 (eight) questions. Answer any 6 (six) of them.
Figures in the right margin indicate marks.

---

1. a) Select integrated circuits, test them in sequence until you find the first failure, and then stop. Let N denote the number of tests. All tests are independent with probability of failure $p = 0.1$. Consider the condition $B = [N > 20]$. Find $P_{N|B}(n)$, the conditional PMF of N given that there have been 20 consecutive tests without a failure.
Find $P_{N|B}(n)$, the conditional PMF of N given that there have been 20 consecutive tests without a failure. 7

b) A business trip is equally likely to take 2, 3, or 4 days. After a d-day trip, the change in the traveler's weight, measured as an integer number of pounds, is uniformly distributed between $-d$ and d pounds. For one such trip, denote the number of days D and the change in weight by W. Find the joint PMF $P_{DW}(d, w)$. 9

c) A man invites his fiancée to an elegant hotel for a Sunday banquet. They decide to meet in the lobby of the hotel between 11:30 am and 12 noon. If they arrive at random times during this period, what is the probability that the first person to arrive has to wait at least 12 minutes. 9

2. a) A machine produces photo detectors in pairs. Tests show that the first photo detector is acceptable with probability $\frac{3}{5}$. When the first photo detector is acceptable, the second photo detector is acceptable with probability $\frac{4}{5}$. If the first photo detector is defective, the second photo detector is acceptable with probability $\frac{2}{5}$.
i. Find the probability that exactly one photo detector of a pair is acceptable. 7
ii. Find the probability that both photo detectors in a pair are defective. 8

b) An urn contains five red and three blue chips. Suppose that four of these chips are selected at random and transferred to a second urn, which was originally empty. If a randomly selected chip from the second urn is blue, find the probability that two red and two blue chips were transferred from the first urn to the second urn. 10

3. a) Let the joint probability mass function of X and Y be given by 10

$$P_{XY}(x, y) = \begin{cases} \frac{1}{15}(x + y), & x = 0, 1, 2, \quad y = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

Find $P_{X|Y}(x|y)$ and $P[X = 0|Y = 2]$.

b) An unbiased die is successively rolled. Let X and Y denote, respectively, the number of rolls necessary to obtain a six and a five.
i. Find the expected value of X, $E[X]$. 8
ii. Find conditional expectation of X, $E[X|Y = 1]$ and $E[X|Y = 5]$. 7

4. a) A student's final exam grade depends on how close the student sits to the center of the classroom during lectures. If a student sits r feet from the center of the room, the grade is Gaussian random variable with expected value $80 - r$ and standard deviation r. If r is a sample value of random variable R, and X is the exam grade, then find $f_{X|R}(x|r)$. 9

b) First a point Y is selected at random from the interval $(0,1)$. Then another point is selected at random from the interval $(Y,1)$. Find the probability density function of X. 9

c) Random variables X and Y have the joint PMF 7

$$P_{XY}(x,y) = \begin{cases} cxy, & x = 1,2,4, y = 1,3, \\ 0, & \text{otherwise.} \end{cases}$$

Find $P[Y < X]$.

5. A sequence of Bernoulli trials consists of choosing components at random from a batch of components. A selected component is classified as either defective or non-defective. A non-defective component is considered as success, while a defective component is considered as a failure. If the probability that a selected component is non-defective is 0.8, determine the probabilities of the following events.
   a) The first success occurs on the fifth trial. 5
   b) The third success occurs on the eighth trial. 5
   c) There are 4 successes by the tenth trial, and there are 10 successes by the eighteenth trial (of which the first four successes occur by the 10^{th} trial). 10
   d) Find the expected number of trials required to get the 4^{th} success. 5

6. a) A computer device can be either in a busy mode (state 1) processing a task, or in an idle mode (state 2), where there are no tasks to process. Being in a busy mode, it can finish a task and enter an idle mode any minute with the probability 0.2. Being in an idle mode, it receives a new task any minute with the probability 0.1 and enters a busy mode. Assume the scenario is set up as a discrete time Markov chain.
   i. Find the n-th step transition probability matrix of the chain for $n = 4$. 10
   ii. Find the steady state transition probability vector for the chain. 10
   iii. If the initial state of the chain is idle, then find the probability that the chain will be in idle state after 4 transitions. 5

7. a) A node in a computer network sends data packet to another node one after another. Each packet is successfully sent with probability 0.75 and is dropped with probability 0.25. What is the probability that 5 packets are dropped before 10 packets are successfully sent? 10
   b) Passengers are making reservations for a particular flight on a small commuter plane 24 hours a day at a Poisson rate of 3 reservations per 8 hours. If 24 seats are available for the flight, find the probability that by the end of the second day all the plane seats are reserved. 7
   c) A doctor has five patients with migraine headaches. He prescribes for all five a drug that relieves the headaches of 82% of such patients. What is the probability that the medicine will not relieve the headaches of two of these patients? 8

8. a) In test of stopping distance for automobiles, cars traveling 30 miles per hour before the brakes were applied tended to travel distances that appeared to be uniformly distributed between two points a and b. Find the following probabilities:
   i. One of these automobiles, selected at random, stops closer to a than to b. 5
   ii. One of these automobiles, selected at random, stops at a point where the distance to a is more than 9 times the distance to b. 5
   iii. If three automobiles (which behave independently) are used in the test, then one of the three travels past the midpoint between a and b. 8
   b) A point is selected at random on a line segment of length l. What is the probability that none of two segment is smaller than $\frac{l}{3}$? 7

Appendix A: PMF/PDF and the expected values of some Random Variables

Distribution	PMF/PDF	Expected value	Variance
Bernoulli	$P_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & otherwise \end{cases}$	$E[X] = p$	$Var[X] = p(1-p)$
Geometric	$P_X(x) = \begin{cases} p(1-p)^{x-1} & x \geq 1 \\ 0 & otherwise \end{cases}$	$E[X] = 1/p$	$Var[X] = (1-p)/p^2$
Binomial	$P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 1, \dots, n \\ 0 & otherwise \end{cases}$	$E[X] = np$	$Var[X] = np(1-p)$
Pascal	$P_X(x) = \begin{cases} \binom{x-1}{k-1} p^k (1-p)^{x-k} & x = k, k+1, \dots \\ 0 & otherwise \end{cases}$	$E[X] = k/p$	$Var[X] = k(1-p)/p^2$
Poisson	$P_X(x) = \begin{cases} \frac{(\lambda T)^x e^{-(\lambda T)}}{x!} & x \geq 0 \\ 0 & otherwise \end{cases}$	$E[X] = \alpha$ $\alpha = \lambda T$	$Var[X] = \alpha$
Uniform (discrete)	$P_X(x) = \begin{cases} \frac{1}{b-a+1}, & x = a, a+1, a+2, \dots, b \\ 0, & \text{otherwise} \end{cases}$	$E[X] = \frac{a+b}{2}$	$Var[X] = \frac{(b-a)(b-a+2)}{12}$
Exponential	$f_X(x) = \begin{cases} ae^{-ax} & x \geq 0 \\ 0 & otherwise \end{cases}$	$E[X] = 1/a$	$Var[X] = 1/a^2$
Gaussian	$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} & \sigma > 0 \\ 0 & otherwise \end{cases}$	$E[X] = \mu$	$Var[X] = \sigma^2$
Uniform (Continuous)	$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x < b \\ 0, & otherwise \end{cases}$	$E[X] = \frac{a+b}{2}$	$Var[X] = \frac{(b-a)^2}{12}$