MATHforAI Report

Assignment 8: Linear Transformations, Matrix Representation, and SVD

Prepared by: Fatima Alibabayeva

Course: MATHforAI

8.1 Linear Transformations

A **linear transformation** (or linear map) is a function

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

that satisfies, for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars $c \in \mathbb{R}$:

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T(\mathbf{u}) + T(\mathbf{v}) \quad \text{(Additivity)} \\ T(c\mathbf{u}) &= c\,T(\mathbf{u}) \quad \text{(Homogeneity)} \end{aligned}$$

These properties mean that T preserves vector addition and scalar multiplication.

Example:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 2y \\ 3x - y \end{bmatrix}.$$

Then T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, and can be written as

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \mathbf{x}.$$

8.2 The Matrix of a Linear Transformation

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then there exists a unique $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

The columns of A are given by the images of the standard basis vectors:

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n) \end{bmatrix}.$$

Example. Let

$$T(x, y, z) = (x + y,\, 2y - z).$$

Then

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T(\mathbf{e}_3) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Thus the standard matrix of T is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}.$$

Therefore,

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ 2y - z \end{bmatrix}.$$

8.3 Singular Value Decomposition (SVD)

Every real $m \times n$ matrix A can be decomposed as

$$A = U\Sigma V^T,$$

where:

- U is an $m \times m$ orthogonal matrix ($U^T U = I_m$),
- V is an $n \times n$ orthogonal matrix ($V^T V = I_n$),
- Σ is an $m \times n$ diagonal matrix with nonnegative real numbers $\sigma_1, \sigma_2, \ldots, \sigma_r$ (singular values) on the diagonal.

The nonzero singular values σ_i are the square roots of the nonzero eigenvalues of $A^T A$:

$$A^T A \mathbf{v}_i = \sigma_i^2 \mathbf{v}_i.$$

The corresponding left singular vectors satisfy

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i.$$

Hence,

$$A = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T,$$

where $r = \text{rank}(A)$.
Geometric interpretation:

- V^T rotates the input space in $\mathbb{R}^n$,
- Σ stretches or compresses along the coordinate axes,
- U rotates the result in $\mathbb{R}^m$.

Example: If

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix},$$

then

$$A^T A = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}.$$

Let λ_1 and λ_2 be the eigenvalues of $A^T A$, then the singular values are

$$\sigma_i = \sqrt{\lambda_i}.$$

Finally, U and V are formed from the corresponding eigenvectors, giving the full decomposition

$$A = U\Sigma V^T.$$

Conclusion

In this assignment, I learned how linear transformations connect algebraic operations with geometric intuition. I understood that every linear transformation can be expressed as a matrix that transforms vectors between spaces. Moreover, through the concept of Singular Value Decomposition (SVD), I discovered how any matrix can be broken down into simpler components that describe rotation, stretching, and reflection.

This topic helped me realize how powerful linear algebra is — not only for theoretical understanding but also for real-world applications in machine learning, image processing, and data science. It strengthened my foundation in matrix theory and gave me deeper insight into how transformations truly work behind the scenes.