MATHforAI Report

Assignment 7: Eigenvalues, Diagonalization, and the Power Method

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7.1 Finding Eigenvalues and Eigenvectors

Given a square matrix A, eigenvalues are found by solving the **characteristic equation**:

$$\det(A - \lambda I) = 0$$

where λ represents the eigenvalues of A.

Once λ is known, we find eigenvectors x by solving:

$$(A - \lambda I)x = 0$$

Example:

Let

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

Then

$$\det(A - \lambda I) = \begin{vmatrix} 4 - \lambda & 2 \\ 1 & 3 - \lambda \end{vmatrix} = (4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0$$

Thus, $\lambda_1 = 5$, $\lambda_2 = 2$.

For $\lambda_1 = 5$:

$$(A - 5I)x = 0 \Rightarrow \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} x = 0 \Rightarrow x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 2$:

$$(A - 2I)x = 0 \Rightarrow \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} x = 0 \Rightarrow x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So the eigenpairs are:

$$(\lambda_1, x_1) = (5, [2, 1]^T), \quad (\lambda_2, x_2) = (2, [1, -1]^T)$$

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7.2 Diagonalization

A matrix A is **diagonalizable** if there exists a matrix P such that:

$$A = PDP^{-1}$$

where D is a diagonal matrix containing the eigenvalues of A.

If the matrix has n linearly independent eigenvectors, then it is diagonalizable.

For the previous example:

$$P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$$

Then:

$$A = PDP^{-1}$$

This simplifies many computations, such as powers of A:

$$A^k = PD^kP^{-1}$$

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7.3 The Power Method

The **Power Method** is an iterative algorithm used to find the largest eigenvalue (dominant eigenvalue) of a matrix and its corresponding eigenvector.

Algorithm:

$$x^{(k+1)} = \frac{Ax^{(k)}}{\|Ax^{(k)}\|}$$

Repeat until convergence. Then:

$$\lambda_{\max} \approx \frac{(x^{(k)})^T Ax^{(k)}}{(x^{(k)})^T x^{(k)}}$$

Example:

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad x^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Iteration 1:

$$Ax^{(0)} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \Rightarrow x^{(1)} = \frac{1}{\sqrt{3^2 + 4^2}} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$$

Approximate eigenvalue:

$$\lambda_1 \approx (x^{(1)})^T Ax^{(1)} = 3.6$$

Further iterations bring λ_1 closer to the true dominant eigenvalue (≈ 3.618).

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Conclusion – What I Learned

In this assignment, I learned:

- How to find eigenvalues and eigenvectors using the characteristic equation.
- The concept and process of matrix diagonalization, which simplifies complex computations.
- How the Power Method approximates the dominant eigenvalue through iteration.

These techniques are essential in many AI and engineering applications, especially in dimensionality reduction, data transformation, and stability analysis.