Vectors

> *There is a great temptation to put vectors tail-to-tail when you go to add them. May all the battles that you wage in your war against that temptation end with your glorious triumph. Vectors add head-to-tail.*

We are about to embark on the study of the motion of a particle which does not move only along a line. To be sure, we will take things a step at a time and next study the motion of a particle that stays in a plane, an imaginary flat (but not necessarily horizontal) sheet. In considering such motion we find that we can no longer specify the direction of a quantity that has direction by means of a simple + or – sign as we could in the case of motion along a line. Now, the direction of a quantity that has direction, such as the velocity of a car moving along a horizontal surface, might be specified in words as, for example, "in a compass direction that is 18 degrees north of due east". A number, with or without a plus or minus sign, is no longer sufficient to specify the magnitude and direction of a velocity, or an acceleration.

Enter the vector. The vector is a **mathematical entity**[1] that has both magnitude and direction. Mathematicians have devised some operations involving vectors that pertain to physical quantities that have magnitude and direction. Knowing about vectors, and the mathematical operations devised for them, comes in mighty handy in the study of physics. We start our discussion of vectors by presenting a couple of different representations of vectors (graphical and magnitude-and-direction).

Graphical Representation of a Vector

Graphically, a vector is represented by an arrow. The magnitude of the vector is represented by the length of the arrow and the direction of the vector is represented by which way the arrow is pointing. A vector variable is represented by a letter with **an arrow over it**[2] as in the vector $\vec{A}$. Once we define a vector, we often need to write about the magnitude of that vector—just how big it is as opposed to both how big and which way it is. The magnitude of the vector $\vec{A}$ can be written two ways, either $|\vec{A}|$ or A. The first way makes it more obvious that we are dealing with the magnitude of a vector (rather than some ordinary variable). The second method is used when it is already clear from the context that we are dealing with the magnitude of a vector. You can never go wrong using $|\vec{A}|$. A is easier to write but the reader might think that it is an ordinary variable rather than the magnitude of a vector.

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[1] An entity is just a "something". Thus, when we say that a vector is a mathematical entity, all we are saying is that a vector is a mathematical "something". I use the word entity here because one typically uses "a something" when one is talking about a solid object, and "an entity" when one is talking about a "non-object" such a thought or a ghost or, as in the case at hand, a vector.

[2] In books, a vector variable is often designated as a vector variable by printing it in boldface, instead of putting an arrow over it.
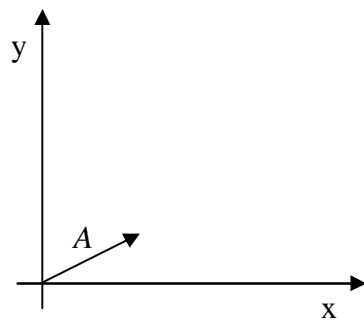
An interesting and convenient thing about the mathematical entity, the vector, is that it has no position. Thus, whether you represent the vector $\bar{A}$ as,
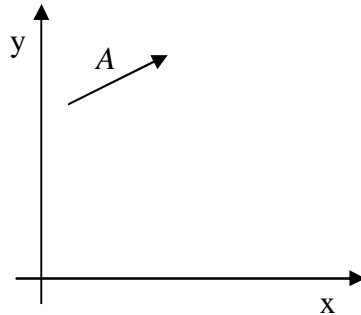

or as,
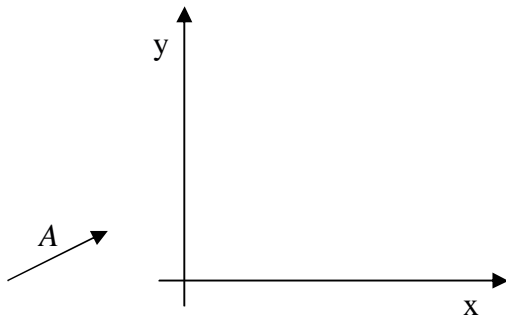
or as,

it is one and the same **vector**[3]. The fact that we can slide a vector around, and still have it be the same vector, is useful when we want to add two vectors.

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[3] Note that, in the diagrams, we label the arrow with *A* rather than $\bar{A}$. The arrow in the diagram makes it clear that we are dealing with a vector, and we label that vector with its magnitude.

Addition of Vectors—The Head-to-Tail Method

When you add two vectors, you get another vector. The sum of two vectors, a.k.a. the resultant, can be obtained by sliding the second vector so that its tail is touching the head of the first vector. The resultant is then a vector extending from the tail of the first vector to the head of the second vector.

For instance, suppose you are asked to find the sum (the resultant) of two vectors $\vec{A}$ and $\vec{B}$ represented graphically as
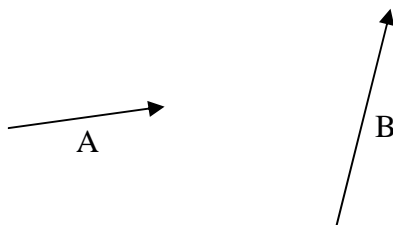
You are being asked to find the $\vec{R}$ in $\vec{R} = \vec{A} + \vec{B}$. Graphically, all you have to do is slide $\vec{B}$ over so that the tail of $\vec{B}$ is touching the head of $\vec{A}$ and then draw the vector that extends from the tail of $\vec{A}$ to the head of $\vec{B}$. That vector is the resultant $\vec{R} = \vec{A} + \vec{B}$.
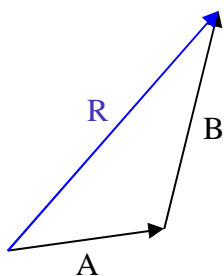
Note that the vector addition diagram yields a triangle as long as the two vectors that you are adding are not **collinear**[4]. In general, the triangle is <u>not</u> a right triangle. Given the magnitudes of $\vec{A}$ and $\vec{B}$ and the angle that they make with each other, it is possible to analyze the triangle to get the magnitude of $\vec{R}$ and the angles that it makes with $\vec{A}$ and $\vec{B}$ through the use of **the law of sines and the law of cosines**[5], but I am going to introduce a more powerful (more powerful because it can be used to add any number of vectors, not just two, all at once) method

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[4] Collinear vectors are vectors that, when placed head to tail, lie along one and the same line.
[5] The law of sines and the law of cosines are some trigonometric formulas that apply to all triangles, not just right triangles. They are straightforward and fairly easy to use but you don't need them for this course.

for determining the magnitude and direction of $\bar{\mathbf{R}}$, below. Thus, unless we actually establish a scale relating the length of the arrow in our diagram to the magnitude of the vector, use a ruler to make each vector just the right length, and use a protractor to properly orient each vector on the page; the graphical method depicted above yields an approximation to the answer that can be used to check the results of the more analytical method below. There is an exception to this.

In the case of **two vectors at right angles to each other**[6], the triangle resulting from the vector addition diagram is a right triangle and it can easily be analyzed directly to obtain the magnitude and direction of the resultant.

Magnitude and Angle Representation of Vectors

Another way of specifying a vector is to give the magnitude of the vector as a value with units and to specify the angle that the vector makes with a reference direction. There are two conventions for the reference direction, the mathematics convention and the compass heading convention.

In the mathematics convention, angles are measured counterclockwise from the positive x direction. Thus a velocity vector represented as 5.4 m/s at 38° in the mathematics convention would have a graphical representation of
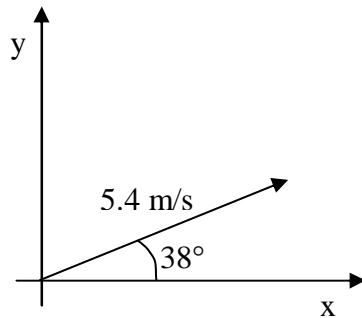

whereas an acceleration vector having a magnitude of 3.5 m/s^2 at -55° would have a graphical representation of
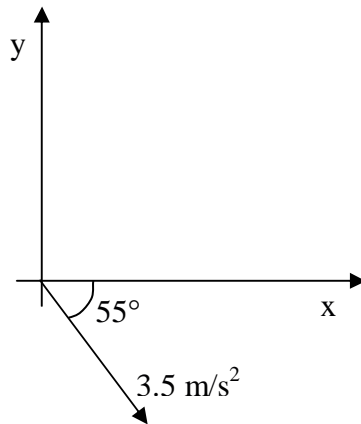
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In the compass heading convention, compass headings are measured clockwise from north, or, as stated explicitly in the specifications of the compass direction. The following examples provide clarification:

The acceleration vector 42 m/s^2 at a compass heading of 108° is represented graphically as
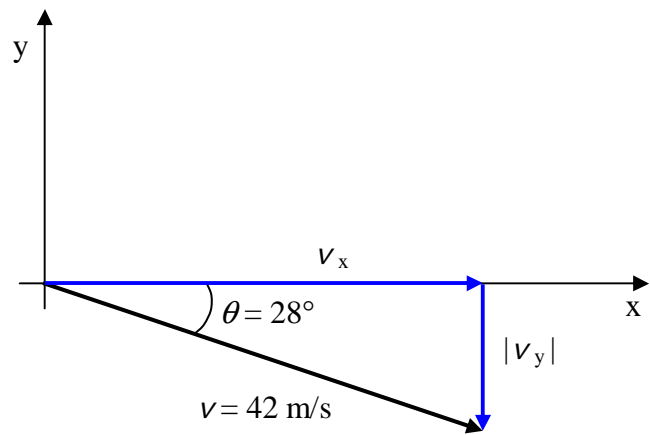
NORTH

108°

WEST ← → EAST
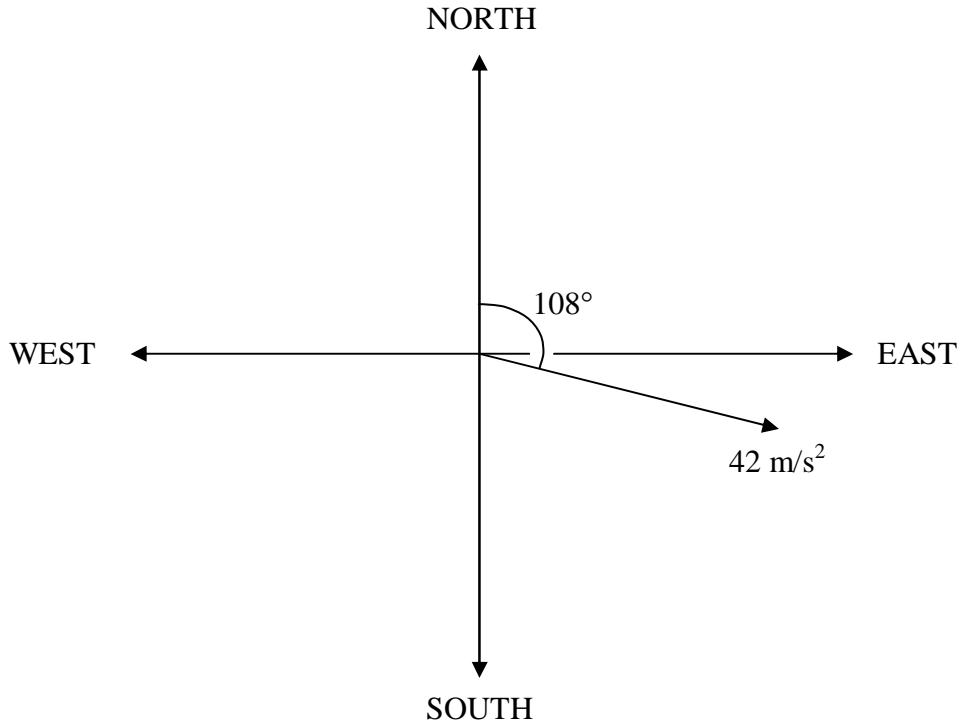
42 m/s^2

SOUTH

That same vector could be specified by writing 42 m/s^2 at 18° south of east, or, 42 m/s^2 at 72° east of south. These two methods are examples of explicitly stating the compass direction relative to a **cardinal direction**[7] which may or may not be north.

Breaking a Vector up into its Components

The component vectors of a vector are two vectors, one in the $\pm x$ direction and one in the $\pm y$ direction, that add up to the vector. To find the components of a vector we draw a vector component diagram. Suppose, for instance, that we are asked to find the components of the velocity vector $\bar{v}$ = 42 m/s at -28°. We draw the graphical representation of the vector, and then we add two vectors to the diagram, one in the $\pm x$ direction and one in the $\pm y$ direction, drawn such that the two vectors add up to the given vector.

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[7] The cardinal directions are North, South, East, and West.

$\dfrac{v_x}{v} = \cos\theta$	$\dfrac{\left\|v_y\right\|}{v} = \sin\theta$
$v_x = v\cos\theta$	$\left\|v_y\right\| = v\sin\theta$
$v_x = 42\dfrac{\text{m}}{\text{s}}\ \cos 28°$	$\left\|v_y\right\| = 42\dfrac{\text{m}}{\text{s}}\ \sin 28°$
$v_x = 37.1\dfrac{\text{m}}{\text{s}}$	$\left\|v_y\right\| = 19.7\dfrac{\text{m}}{\text{s}}$
	$v_y = -19.7\dfrac{\text{m}}{\text{s}}$

Note that the vector component diagram always yields a right triangle. As shown above, we use the properties of right triangles to determine the absolute values of the components, treating the vectors which form the sides of the triangle as ordinary triangle sides (always having positive length, which, in the case of a vector component diagram, we treat as being the absolute value of the corresponding vector component). The component is the length of the corresponding triangle side if the component vector points in the positive axis direction, and, it is the negative of the length of the corresponding triangle side if the component vector points in the negative axis direction.

The converse to the "given a vector, find its components" problem is the "given the components of a vector, find the vector (find its magnitude and direction)". The latter problem is the second-

easiest[8] kind of vector addition problem there is. The graphical vector addition diagram yields a right triangle which can be analyzed directly to find the magnitude and direction of the resultant.

The Vector Addition Recipe

There is an awful lot in physics that involves serious hard thinking. Adding vectors does not fall into that category—its as easy as following a recipe. Here's the **recipe**[9]:

1. Sketch the vector addition diagram to give yourself an idea of the answer. This sketch is not part of the analysis but provides you with a rough check on your final answer. Draw the first vector. Starting at the head of the first vector, draw the second vector. Draw the resultant vector from the tail of the first vector to the head of the second vector. If there are more than 2 vectors, draw them all, always starting the next vector at the head of the preceding vector. Do not omit arrowheads! (Shortcut: If there are only two vectors and the vectors are at right angles to each other, and you are sure that the angle between them really is 90 degrees rather than just appears to be 90 degrees in your sketch, then you can analyze the vector addition diagram directly using the trigonometric functions, because, the vector addition diagram is a right triangle. This eliminates the rest of the steps in this recipe. The trivial case in which the vectors all lie along the same line does not require the steps below either.)

2. Break each vector up into its components. For each vector in the sum:
   a) Draw a separate vector component diagram for that vector.
   b) Either by inspection (in the case in which the vector is along either the x or the y axis) or by means of trigonometric functions, determine and write the x and y components of the vector.

3. Add the x components of the individual vectors to determine the x component of the resultant. Add the y components of the individual vectors to determine the y component of the resultant.

4. Draw the vector addition diagram corresponding to the addition of the x component vector of the resultant, to, the y component vector of the resultant, and analyze the diagram to determine the magnitude and direction of the resultant.

Note: In the process of adding vectors, you should not take the cosine, sine, or tangent of a negative angle, nor should you take the cosine, sine, or tangent of an angle that is greater than 90 degrees.

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[8] The easiest kind of vector addition problem is the one in which the vectors to be added lie along one and the same line because then one can determine the magnitude of the resultant by adding or subtracting the magnitudes of the given vectors. We're talking ordinary addition and subtraction of numbers!

[9] This recipe assumes that the vectors are neither collinear nor at right angles to each other—when they are, vector addition is much easier.